WHAT IS A DEFINITION FOR IN SCHOOL MATHEMATICS?

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Abstract: This paper discusses the place of definitions in school mathematics, considering official UK curriculum guidance, literature related to definitions in advanced mathematical thinking and to experimental teaching focused on student development of definitions. A two dimensional framework is suggested for considering their functions, the ways in which students are expected to relate to them and their didactic purposes. Two contrasting examples of definitions from textbooks are analysed using systemic-functional linguistic tools.

Keywords: definitions; systemic-functional linguistics; textbooks; student positioning; discourse analysis

Definition of mathematical concepts has been a topic of interest in mathematics education research for some years. This interest arises primarily from the commonly observed difficulties met by students entering advanced levels of study as they are asked to use definitions in formal mathematical reasoning. Yet students also encounter definitions of mathematical concepts much earlier in their educational experience. Recent government guidance for teachers in English primary and secondary schools recommends classroom use of mathematical dictionaries by teachers and students (DfES, 2000, 2001). This guidance, including a list of ‘key words’ for each of Years 1 to 9, constructs an official curriculum discourse that privileges vocabulary over other characteristics of mathematical language. Central to this discourse is the notion that mathematical words are unambiguous and that their meaning can be clarified by using a dictionary definition. This and other assumptions about the nature of mathematical language and approaches to learning it are discussed more widely in a critique of this official guidance by Barwell, Leung, Morgan and Street (2005). In this paper I consider critically the roles played by definitions in school mathematics, in the light of curriculum guidance and the place of definition in mathematical activity, presenting analyses of some examples of definitions occurring in secondary school textbooks.

ARE MATHEMATICAL DEFINITIONS ‘SPECIAL’?

In discussing the characteristics of mathematical definitions, Borasi identifies two functions they must fulfil. A definition of a given mathematical concept should:

> 1. Allow us to discriminate between instances and non-instances of the concept with certainty, consistency, and efficiency (by simply checking whether a potential candidate satisfies *all* the properties stated in the definition).
>
> 2. “Capture” and synthesise the mathematical essence of the concept (*all* the properties belonging to the concept should be logically derivable from those included in its definition). (Borasi, 1992, pp.17-18)

The first requirement does not seem peculiar to mathematics; though definitions of everyday concepts may be ‘fuzzy’, precision characterises the definition of scientific concepts in many specialist domains.[i] Borasi’s second criterion, however, hints at a role for definitions within mathematical practice that goes beyond both the record of usage of standard dictionaries and the technical taxonomising of common-sense phenomena in natural and social sciences (Wignell, 1998). Definitions in mathematics form a basis for logical derivation not only of properties already known (perhaps in a common-sense way) to belong to the concept but of new properties.

Vinner (1991) claims that, while definitions in everyday contexts have little relationship to development of concepts (Fodor et al., 1980), they are essential for technical concepts. By providing examples of mathematical situations in which use of a formal definition appears vital to overcoming the limitations of students’ intuitive ‘concept images’, he distinguishes advanced mathematics as a technical context. This seems uncontroversial. Definition is distinguished from description by a number of mathematics education researchers working in the area of advanced mathematical thinking (e.g., Barnard, 1995; Tall, 1991), with the use of definitions presented as characteristic of advanced mathematics. Alcock and Simpson (2002) identify this distinction between the functions of ‘dictionary definitions’ and of mathematical definitions as a root cause of breakdown in communication between lecturers and undergraduate students:

> what eludes the students is the distinction between a dictionary definition as a *description* of pre-existing objects and a mathematical definition as the chosen basis for deduction, one which serves to *determine* the nature of the objects. (p.33, original emphasis)

Here Alcock and Simpson also hint at another characteristic of the ways mathematicians use definitions – the element of choice. While dictionary definitions describe the ways a word is actually used in practice, mathematical definitions are chosen *in order that they may be used* for deduction and proof of theorems.

> The research mathematician may come to his results starting from special cases, which will appear as corollaries in the final version, from which he gets his ideas, which is worked with until he has a proof. Then the theorem is what has been proved. At this point he formulates his definitions so as to make the theorem and proof as neat as possible. (Burn, 2002, p.30)

At first, the concepts the mathematician works with may be more or less intuitive, derived from special cases. The construction of the formal definition and consequent creation of a technical term is thus purposeful and creative, aiming not simply to describe or “capture” a pre-existing concept but to shape that concept in a way that lends itself to particular purposes. Of course, this definition may subsequently be used to generate deductive sequences leading to the discovery of further theorems.

The idea of choice and purposeful formulation of definitions constructs an active role for the mathematician him/herself, not simply as a user of correct mathematical vocabulary but as one who chooses between alternative definitions or creates new ones. This role is very different from that constructed for school students by the official discourse of the English curriculum. Here the booklet *Mathematical Vocabulary* focuses on students' development of understanding of the meaning of words, "using the correct mathematical terminology" and "learning to read and write new mathematical vocabulary" (DfES, 2000, p.2), using a dictionary "to look up the meaning of words" (p.36).

Rather more active student roles in relation to definition are proposed elsewhere. In particular, activities that engage students in forming and critically evaluating their own definitions have been described with middle school (Keiser, 2000; Lin & Yang, 2002) and high school students (Borasi, 1992). Keiser's students developed their own definitions of 'angle'. While the discussions she describes seemed to support the students' development of the ability to distinguish between examples and non-examples of the concept, the notion of 'definition' in this case was descriptive of an independently existing object rather than purposeful design of a definition for theory building. Lin & Yang's study involved a problem solving activity in which students were encouraged to develop minimal definitions of rectangle and square. In this case, some of the students were able to make logical connections between the two, suggesting that their understanding of the nature of definition was going beyond the purely descriptive. At a higher level, Borasi's students, as well as working with the idea of minimal definition, explored the consequences of using alternative definitions of the same object (e.g. the different approaches to solution of a problem that might arise when using metric or analytic definitions of a circle), thus being introduced to the idea of choice and purposeful definition.

A FRAMEWORK FOR CURRICULAR APPROACHES TO DEFINITION

For learners of mathematics, definitions function in several ways. On the one hand, using a definition to distinguish between instances and non-instances of the defined concept is one approach to developing awareness and understanding of the concept itself as well as learning correct application of the language. This is the purpose of definition assumed by English curriculum guidance for teachers at primary and secondary level. At the same time, however, if one of the aims of mathematics education is to develop participation in the discipline of mathematics itself and in mathematical ways of thinking, then negotiation of definitions, choice between alternative definitions, deduction from agreed definitions and arguments traceable back to definitions also need to feature in the experiences offered to students.

The two functions of definitions for learners of mathematics outlined above may be characterised as a content-process dichotomy. Most current curriculum thinking recognises the need for aims related both to the learning of specific content and to more general processes of using and applying mathematics, though there may be differences in emphasis (and in implementation). A second dichotomy relates to the positioning of the mathematician/ student in relation to mathematics in general and to definitions and the act of defining in particular. This may be characterised as the opposition between seeing mathematics as a ‘given’ body of knowledge to be discovered or acquired and allowing that mathematicians (in general and students in particular) themselves play an active part in constructing mathematical knowledge. Table 1 suggests a framework for thinking about the ways in which definitions may feature in mathematics classrooms; the four cells identify the types of activity that might utilise definitions and the didactic purposes these might have. (The types of activity and purpose suggested here are indicative rather than exhaustive.)

Table 1: Framework for definition-related activity in the classroom

		nature and function of definition	
	Definitions …	… distinguish between instances and non-instances of a concept	… are used as the foundation of logical argument
positioning of user in relation to definition	… are pre-existing/ given by authority	A: to apply criteria to test examples or to create examples that match criteria *purpose: develop the concept itself*	C: to deduce further properties and to construct proofs *purpose: develop connected knowledge within the domain of study; develop proof skills; engage in mathematical deductive reasoning*
	… may be constructed by the user	B: to ‘pin down’ the user’s concept image (and through debate, counter-examples etc. refine the concept image to become closer to that of the mathematical community) *purpose: develop the concept; engage in mathematical reasoning and debate*	D: to create a new concept that yields interesting or useful results *purpose: engage in ‘authentic’ mathematical practice*

Cells B and D might be further sub-divided according to whether the active agent of construction is the student him/herself or whether any such creative mathematics is the activity of a more distant mathematician. The examples described above suggest that several of these cells can be identified with school curricular discourses involving definition in mathematics. The official discourse of the English curriculum is clearly located in cell A; the examples offered by Keiser and Lin & Yang fall within cell B, constructing definition as primarily descriptive but positioning the student actively and powerfully. Borasi’s course included elements within both cells B and C.

ANALYSIS OF TEXTBOOK DEFINITIONS

Recent curriculum developments in the UK have paid considerable attention to the need to develop students’ understandings and capabilities in relation to mathematical proof, but little has been said about the nature or function of mathematical definition at primary or secondary school level beyond the simplistic assumptions of the DfES booklet already mentioned. While students certainly encounter definitions throughout their mathematical education, the difficulties reported at university level suggest that their earlier experiences may not provide a basis for using definitions in ways that go beyond the development of concepts.

Table 2: Analytic Tools.

Descriptive questions:	Grammatical tools:	*Illustrative interpretations**
Who or what are the actors and where does agency lie?	What objects and humans are present? How are active or passive voice used?	*Human agency, especially in mental processes (e.g. think, decide), tends to position mathematicians more actively in relation to definition. (Cells B/D)*
What are the processes?	Relational, material, mental/behavioural?	*A preponderance of relational processes (e.g. be, have) tends to characterise definitions used to distinguish between instances and non-instances. (Cells A/B)*
What are the roles of the author and reader and what is the relationship between them?	How are personal pronouns used? In what kinds of processes are author and reader actors?	*This can distinguish the way in which the student is positioned or not as a potential creative mathematician. (further sub-dividing cells B and D)*
Is the modality absolute or contingent?	Modal verbs, adverbs, adjectives	*Contingent modality allows the possibility of alternative definitions and choices (distinguishing between cells A/C and B/D).*

**These illustrations refer to the framework presented in Table 1. Further illustration is provided in the analysis of Examples 1 and 2 below. The illustrations should not be interpreted deterministically as any analysis has to take into account the broader text and the context of its use.*

In this section, I use the framework outlined in Table 1 to consider examples of definitions taken from secondary school textbooks published in the UK, analysing the nature and function of the definition as it is presented in the text and the positioning of the student/ mathematician in relation to it. The analysis uses tools drawn from systemic functional grammar (Halliday, 1985) selected to illuminate the ways in which the nature of mathematics and mathematical activity may be constructed through the texts presented to students. A fuller discussion of this approach and its applications in mathematics education research may be found in

(Morgan, 1998; in press). Table 2 identifies the questions used to interrogate each text and the grammatical tools that operationalise the resulting description. These are a subset of the tools described and used in (Morgan, 2005). The first two questions in the table are related to the ideational function of language, concerned with the nature of our experience of the world, the next two to the interpersonal function, concerned with the identities of the participants and relationships between them. The description thus constructed[ii] allows us to address critical questions that help to locate each occurrence of definitions within the framework presented above, in particular: *What is the function of definition?* and *How is the student/ mathematician positioned in relation to definition?*

Example 1: (extract from Bostock & Chandler, 1978, pp.134-135)

> For any acute angle θ there are six trigonometric ratios, each of which is defined by referring to a right angled triangle containing θ. …
>
> Since we are now regarding an angle as the measure of rotation from a given position of a straight line about a fixed point, it is clear that the size of an angle is unlimited, as the line can keep on rotating indefinitely. The meaning of the six trigonometric ratios is, as yet, restricted to acute angles, since the definition used so far for each ratio refers to an angle in a right angled triangle. If we wish to extend the application of trigonometric ratios to angles of any size, they must be defined in a more general way.

Actors & Agency	Human actors "we" are present as decision makers. However, at other points, agency in the process of definition is obscured by use of the passive voice: *meaning … is … restricted*; *they must be defined …* As well as more or less concrete objects such as angles and lines, *meaning* and *definition* are themselves actors in this text. This produces a meta-discourse about definitions in addition to introducing a new definition of trigonometric ratios.
Processes	Mental processes *regard* and *wish* construct mathematics as an intellectual activity involving choices Trigonometric ratios are to be applied, a material process, although agency in this is obscured by the nominalization *application*.
Author & Reader	It is not clear whether the use of *we* is exclusive or inclusive, though it could certainly be read as an invocation of solidarity, calling upon the reader to share in the new way of thinking about angles and the desire to extend the application of trigonometric ratios to take account of this.
Modality	There are several temporal modifications: "we are *now* regarding"; "The meaning … is, *as yet*, restricted"; "the definition used *so far*". These emphasise the contingent nature of definition and, further, suggest progression for the student-reader from an earlier, basic or elementary, understanding of angle and trigonometric ratio, to a more advanced one. The high modality of "Since we …, *it is clear*" and "If we …, they *must* be

	defined" ascribes authority to the argument rather than primarily to the author as each occurrence appears as the consequence of a premise that the reader has been called into sharing.

There is only space here to present two examples, taken from texts for university-bound (though not necessarily intending to study mathematics at university) students (aged 17-18). Elsewhere (Morgan, 2005) I have presented analyses of examples from texts aimed at intermediate and higher attaining students aged 15-16, showing marked differences between the ways in which definitions were presented to the different groups of students. The intermediate text constructed definition simply as naming pre-existing objects while the higher text demonstrated the purposeful construction of an alternative definition, opening up the possibility that the student-reader would make active choices about the usefulness or applicability of alternatives.

Example 2: (extract from Martin et al., 2000, pp.89-90)

Right-angled triangles are used to define the three basic trigonometric functions for some acute angle θ; sine, cosine and tangent.

$$\sin\theta = \frac{a}{c} = \frac{\text{side opposite } \theta}{\text{hypotenuse}} \quad \cos\theta = \frac{b}{c} = \frac{\text{side adjacent to } \theta}{\text{hypotenuse}} \quad \tan\theta = \frac{a}{c} = \frac{\text{side opposite } \theta}{\text{side adjacent to } \theta}$$

This principle can be used to define the sine, cosine and tangent of any angle θ.

Draw perpendicular axes Ox and Oy, and a circle centred on the origin, with radius 1 unit. Then θ will fix some point P on the circle.

[diagram]

The coordinates of P (x,y) are then $(\cos\theta, \sin\theta)$. Now adopt the convention that θ is measured anti-clockwise from the positive x-axis. …

Actors & Agency	The passive voice is used, obscuring agency, especially in the act of definition, though a human agent is implicitly present in the imperative instructions to *draw* and to *adopt the convention.* In addition to concrete objects, *principle* and *convention* are included as mathematical objects.
Processes	The mental process of defining is presented as a mathematical activity, yet, as its agency is obscured, it is distanced from the student-reader. Material processes (*draw, fix, measure*) construct a mathematics which is about practical activity.
Author & Reader	The imperative constructs an active role for the student-reader – but this role involves material activity (drawing) and following conventions (whose origins are obscured) rather than decision making. The author is absent from the text, again distancing them from the reader and

	placing authority in mathematics rather than in human mathematicians.
Modality	The modality is generally absolute, presenting the content as unquestionable. The temporality (then… now…) sequences the argument rather than suggesting contingency.

The two examples both address the issue of re-defining trigonometric ratios (previously defined for acute angles only) to apply to general angles. The idea that mathematical definitions can be changed seems likely to be new or at least unusual for students at this level and the extracts of text considered introduce this notion. This context gives us a particularly good opportunity to consider how the nature of definition itself and the role of mathematicians in the construction of knowledge are presented to the students, though it may not lend itself to considering other aspects of definition, such as its use in constructing proofs.

As these examples involve the extension of definition of terms to new contexts, it might be considered that they should be located in cell D, creating a new concept. However, neither text contains a clear purpose for this extension. Example 2 merely states that it can be done "This principle can be used to define …"; thus the extended definition is derived from the original but there is little sense of why it might be worthwhile doing so. Example 1 suggests that "we" might "wish to extend the application …", hinting at some motivation for doing so but still not stating an explicit purpose. The function of definition in both examples, therefore, seems to be located in the left-hand column of Table 1, allowing instances of the concept to be distinguished.

There are, however, significant differences between the two texts in the positioning of the student/ mathematician in relation to the definition and to mathematical activity more generally. Example 1 constructs an important role for human mathematicians in making decisions. The student may consider him/herself to be invited to share in this intellectual activity and to be engaged in and persuaded by argument (though, as Pimm (1984) suggests, there are alternative ways the use of *we* might be interpreted by the student reader). In contrast, Example 2 constructs a less powerful student role. Rather than being invited to share in decision-making activity, the student is instructed to carry out material tasks; rather than being persuaded by argument, s/he is presented authoritatively with a procedure to follow. Example 2, therefore, may be located in cell A of the framework with the limited didactic purpose of developing the new or extended concept, while Example 1 is located in cell B with the additional purpose of engaging the student in mathematical reasoning.

DISCUSSION

I do not wish to claim too wide a scope for the results of the analysis presented in this paper. The examples clearly represent a very limited sample of the texts, both written and oral, that students encounter during their school mathematics experience.

In textbooks we will find definitions of different kinds of mathematical concepts, some of which lend themselves more (or less) fully to the various activities and purposes identified in the proposed framework. We will also find definitions making use of a wider range of semiotic systems, especially algebraic notation, that have meaning potentials not immediately addressed by the analytic tools used in this paper. It may further be argued that students' experience is affected more by their teachers' practices than by their textbooks. While agreeing that this is so, I would also argue that teachers themselves are strongly influenced by the resources available to them in textbooks and curriculum guidance. Such texts provide ways of structuring and sequencing the subject matter and also construct emphases and values that, while they may be resisted and revised by some teachers, are nevertheless likely to be influential in shaping classroom practices.

In preparing students to study advanced mathematics, I suggest that they not only need to have opportunities to learn to appreciate the roles definitions play in mathematical reasoning but also to begin to see that doing mathematics involves more than following procedures or reproducing standard arguments. Neither of the examples presented here, nor the examples from school texts discussed in (Morgan, 2005)[iii], hints at the function of definitions as a basis for logical deduction. It may be that the topic does not lend itself to this function, particularly as it is primarily about extending an existing conceptual structure rather than creating and using a new concept. On the other hand, if a clear reason were identified for needing to extend the concept of trigonometric ratios to be applicable to general angles then the activity of creating a definition suitable for such a purpose would involve logical reasoning and could be located in cell D of the proposed framework.

The analysis of the two examples displays a sharp contrast in the ways in which the student-reader is positioned in relation to mathematics: as a potentially active participant in decision making and reasoning or as a rule follower. Both of these roles may be necessary parts of learning and doing mathematics. However, students whose predominant experience constructs definitions as dictionary entries – authoritative but author-less – seem likely to find more difficulty in adapting to the demands of advanced mathematics. The discourse of vocabulary in the UK curriculum thus needs to be addressed critically. More generally, the framework of types of definition-related activity suggested here, while no doubt incomplete, provides a starting point for thinking about the purposes and effects of various approaches to definitions in the classroom. The analysis of textbook extracts provides concrete tools for anticipating the meanings, both substantive and positional, that students may construct from interacting with such texts. This analytic method could be developed to offer guidelines for writing or choosing texts for students. It has potential to be applied more widely beyond the study of definition to inform critique of other aspects of students' experiences of mathematical discourse.

[i] Leung (2005) argues that some mathematical concepts also have core and non-core meanings and hence some ‘fuzziness’.

[ii] Only partial descriptions are presented here, focusing on those aspects most relevant to definition.

[iii] An example from a research paper discussed in (Morgan, 2005) demonstrates the purposeful creation of a new definition for an existing concept.

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