YOUNG CHILDREN'S UNDERSTANDING OF MULTIPLICATIVE CONCEPTS

A PSYCHOLOGICAL APPROACH

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ABSTRACT

This thesis investigates the origins of children's understanding of multiplication and division and how they progressively become coordinated in young children. The hypothesis of the study is that the origins of these operations are in children's schemas of action. This leads to the prediction that children can understand about multiplicative relations before they can solve computational multiplicative problems. This hypothesis is contrasted with Fischbein's et al. (1985) hypothesis that multiplication originates from repeated addition and quotitive division from repeated subtraction, which leads to the prediction that children must be able to quantify in order to understand multiplicative relations. To test these predictions a series of studies was carried out analysing children's performance in relational, non computational problems and in computational problems, involving discontinuous and continuous quantities which could not be quantified.

The first study explored children's use of one-to-many correspondence reasoning to solve a variety of problems. Children aged 4 to 7 were asked: a) to order different sets on the basis of correspondence relations; and b) to indicate the size of the corresponding sets. Even the 5 year olds were able to order the size of the sets when they could use correspondence relations. The same children had great difficulty in indicating the size of the sets. This demonstrated that the ability to quantify is not a prerequisite for understanding multiplicative relations.

The second and third studies explored children's understanding of the inverse relation between the number of the quotas and their size in sharing problems. The children (aged 4 to 7) were tested in partition and quotition problems. In partitive problems more than half of the 6 year olds and the majority of the 7 year olds showed an understanding of the inverse divisor-quotient relation. In the quotitive problems, although there was a drop in the level of performance, it was not found to be significant across the same age groups. Children's's ability to reflect on the relations involved in a sharing situation before being able to quantify the problems challenges the hypothesis that quantification is the origin for understanding division.

The fourth study explored whether multiplication and division develop as independent or coordinated operations. Children were asked to quantify a set of multiplication and division problems in which it was possible to model the problem directly using their schemas of action and a second set where this direct modelling was not possible because a crucial piece of information was missing. It was expected that children who have coordinated their multiplicative schemas of action would be able to deploy an action usually associated with division to solve a multiplication problem and vice versa. More than half of the children were able to quantify the problems that matched their actions directly, but their performance decreased in the missing value multiplication problems that were solved by an action related to division. The discrepancy in the children's performance suggests that the two operations have different roots and initially develop independently of each other.

The findings support the hypothesis that the understanding of multiplication and division is constructed from children's schemas of action. The two operations have distinct roots and develop independently before they become coordinated at a later stage.

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To my parents

Nikos & Eleni

The limit of their love is infinity

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CHAPTER 1

INTRODUCTION

1.1 Overview

The aims of the thesis

This study was designed to investigate the *origin(s)* of the concept of multiplication and division, whether the two concepts have the same or a distinct root, and how they progressively develop and become coordinated in young children.

Where does the understanding of arithmetic operations stem from?

Although there are a variety of controversies regarding children's development of mathematical reasoning, there is largely a consensus that children's understanding of arithmetic operations stems from their *schemas of actions* that is "*generalizable and structured actions which can be applied to a variety of objects and which centre on the relations between objects and transformations rather than on the objects per se*" (Nunes, 1996a, p.242). Children construct their schemas of action on the basis of their everyday experience of the world where quantities are reasoned about and acted on and is not dependent on formal instruction (Resnick and Singer, 1993). Action schemas are quite different from number operations. Number operations involve the understanding of the conventional system of numbers, which are a compressed representation of objects (8 is a compressed representation of eight individual objects), and the use of mathematical signs such as +, -, x, : and =. Why then do number operations derive from children's schemas of actions? The reason that schemas of actions provide the first meanings for understanding arithmetic operations is because what is *invariant* in an arithmetic operation also has to be invariant in the schema of action that the child has for this operation. Therefore, the schemas of actions that children have form the knowledge basis for the understanding of number operations. This also suggests that children can reflect on the relations between objects and their transformations *before* being able to deal with the strictly numerical aspect of the situation. Bryant (1974) has argued that children can reason on a situation on the basis of relations - what he terms *relative codes* - before they make similar deductions using *absolute codes*. Resnick and Singer (1993) have called these early schemas of action *"protoquantitative schemas"* (p. 109) because the children reason non-numerically on the relations between amounts of physical material.

The hypothesis that the origin of children's understanding of operations is in their action schemas was initially put forward by Piaget (1965) and has been strengthened by a number of studies on addition and subtraction. These studies have shown that young children are able to solve a number of addition and subtraction problems by modelling the actions in the problem, that is by joining or separating objects, long before they are able to name which arithmetic operation is adequate to calculate the result formally (Carpenter and Moser, 1982; Hudson, 1983; Hughes, 1986, Riley, Greeno and Heller, 1983).

What do we not know about the origin of multiplication and division?

This study hypothesizes that the origins of multiplication and division are to be sought in children's schemas of action. Little is known, though, about the relationship between actions schemas and the origin and development of the concept of multiplication and division which are the operations that are focused on in this thesis.

An alternative view about the origins of multiplication and division has been proposed by Fischbein, Deri, Nello and Marino (1985). According to them the origins of multiplication and division are to be sought in additive structures. Multiplication is intuitively attached to a repeated addition model and quotitive division to a repeated subtraction model. Fischbeinet al's hypothesis implies that the understanding of multiplicative concepts stems from quantification. Consequently children who cannot solve multiplicative problems are not expected to have any understanding of multiplicative relations. It is argued that a) these are problem solving procedures rather than the core of multiplication and division, and b) the *invariants* of additive and multiplicative situations are considerably different.

The aim of this section is a) to discuss theoretically the invariants of multiplication and division which are hypothesized to be preserved in children's schemas of action where the origin of these concepts is to be sought and b) show that these two operations are qualitative different from addition and subtraction.

Looking at the invariants:

(a) Multiplication

Nunes and Bryant (1996) pointed out that the most salient invariant in a multiplicative situation is that *two sets are in a constant one-to-many correspondence relation* (1 chair has 4 legs). This constant one-to-many correspondence relation is the invariant of the situation, an invariant which is uniquely found in multiplication and is not present in any additive situation. The one-to-many correspondence relation lays the basis for understanding a new mathematical concept, the concept of *ratio.* In order to maintain the relations between "chair-legs" constant, that is the ratio 1:4, each time a chair is added to the set of chairs, 4 legs have to be added in the set of legs. The action that is carried to maintain the ratio constant is *replication* and its inverse and involves adding to each set the corresponding unit for the set so that the invariant one-to-many correspondence is preserved. That means that each time a chair is removed from the set of chairs 4 legs are removed from the corresponding set of legs. The number of times a replication is carried out is known as *scalar factor*. If, for example, we want to replicate the situation 1 chair - 4 legs 7 times, 7 is the scalar factor and refers neither to the chairs nor to the legs, but to the number of replications relating the sets that are in correspondence in order to maintain the ratio constant.

In order to understand multiplication, the child has to master a whole new set of invariants that are not present in an additive situation. Nunes and Bryant (1996) argued that additive situations are about *part-whole relations* in which objects or sets of objects that are not related to each other are either put together or separated. In contrast, multiplicative situations are about a constant relationship, expressed as a ratio, between two corresponding sets. The multiplication learner must deal with a new type of quantity, the "intensive quantity" which is not part of their engagement with additive relations (Schwartz, 1988). In addition and subtraction the children work with extensive quantities, that is quantities that can be counted or measured directly (e.g. the number of sweets, the height of a person). In multiplication, though, the children work with intensive quantities which refer to the relation between two extensive quantities, rather than to their actual amounts. For example, "1 pound per chocolate" is an intensive quantity relating two extensive quantities. This intensive quantity is like a third variable connecting the money with the chocolates and refers neither to the money nor to the chocolates, but to the relation between them. Schwartz argues that multiplication and division are "referent transforming" operations, because they take two quantities with different referents as input and output a third quantity whose referent is different from either of the first two. In the above example chocolates are multiplied by pounds per chocolate and the outcome is number of pounds. Schwartz argues that the notion of multiplication as repeated addition is problematic because addition is referent preserving operation whereas multiplication is referent transforming.

For Piaget, Grize, Szeminska and Bang (1977, as cited in Clark and Kamii, 1996) the difference between addition and multiplication also lies in the number of the levels of abstraction and the number of inclusion relations that the child has to consider *simultaneously.* This is shown clearly in Figure 1 below (from Clark and Kamii, p. 42). Additive thinking requires only one level of abstraction, each unit of three that the child adds is made of ones and there are three ones; inclusion relations are only on one level:

The child includes one in two, two in three, until twelve. That means that the groups are constructed *successively*. In contrast, multiplication requires two kinds of relations that are not present in addition: the one-to-many correspondence between the three units of ones and the one unit of three and the composition of inclusion relations on more than one level, as it is shown in Figure 1. The construction of three units of ones into one unit of three requires a higher level of abstraction than thinking only of single units as in addition. Inclusion relations are also more complicated: There are inclusion relations *horizontally* at the level of single units - 1 is included in 2, 2 is included in 3 - and at the level of units of three - 1 three in 2 threes, 2 threes in 3 threes and 3 threes in 4 threes - as well as *vertically* because 3 ones are included in each unit of 1 three, and 4 units of three are included in the total sum. All these relations are made *simultaneously*. In multiplication, the children should think, as stated by Steffe (1994), about *composite units*, that is units of units.

FIGURE 1.1

(a) Additive thinking (3+3+3+3) compared with (b) multiplicative thinking (4x3).

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(a) Additive

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(b) Multiplicative

Note. From Clarke and Kamii, (1996), p.42.

Because the invariants of multiplication are different from the invariants of addition, claims that multiplication can be seen as a repeated addition situation (Fischbein, Deri, Nello and Marino, 1985), must be questioned. The invariants that are present in a multiplicative situation suggest that the basic meaning for children's understanding of multiplication has to be sought in an action schema quite different from the schemas present in addition, the schema of one-to-many correspondence.

(b) Division

Division, as pointed out by Nunes and Bryant (1996), requires the understanding of another new set of invariants which are present neither in the schemas of addition nor of multiplication. Addition problems are about part-whole relations. The whole is the sum of the parts, which do not have to be of equal size. In division, though, the size of the shared quotas always has to be the same. In a sharing situation the child has to consider the relations between three quantities: the size of the whole (known as the dividend), the number of recipients (known as the divisor) and the size of the shared parts (known as the quotient). For example, if there are 12 biscuits (dividend) and 3 children to share them (divisor), each child will receive 4 biscuits (quotient). According to Correa (1995) and Nunes and Bryant (1996) there is a new set of relations that the child has to consider in a sharing situation: There is a *direct* relationship between the dividend and the divisor, which means that the more biscuits there are, the more each child would receive in the above situation; and an *inverse* relations between the divisor and the size of the quota, which means that the more children sharing the fewer each would get when the number of biscuits is kept the same.

Because the invariants of a sharing situation are different from the invariants of subtraction Fischbein et al's claim that quotitive division originates from repeated subtraction can be question. Although quotitive problems can be quantified by repeated subtraction, this is a quantification procedure and not the essence of the operation. The hypothesis of the study that division originates from children's schemas of action is based on the fact that the invariants of division as an operation are preserved in children's schemas of action.

There are two schemas of action that can be implemented in sharing both of which observe the same invariant, the inverse divisor-quotient relationship. In partitive division problems, when a quantity is shared among a number of recipients, the schema takes the form of sharing in a one-for-you, one-for-me fashion. It is true that young children adopt the sharing procedure to distribute objects amongst themselves quite early and can recognise the equality of the shared sets (Davis and Pitkethly, 1990; Desforges and Desforges, 1980; Frydman and Bryant, 1988; Miller, 1984). Frydman (1990) has also shown that they are even able to adjust the routine of sharing when the shared units are not of equal size in order to form equal shares. For example, they would give two single sweets to a doll receiving only single sweets and one double sweet to a doll receiving only double units. In quotitive division problems, when quotas are shared among the recipients, then the schema takes the form of forming equal quotas until the quantity to be shared is exhausted. For example, in order to determine how many friends can be invited to a party if each child is to be given 3 sweets and there is a total of 12 sweets to be shared, children often form quotas of 3s until the sweets are exhausted.

It has been proposed (Fischbein, Deri, Nello and Marino, 1985; Dickson, Brown and Gibson, 1984; Kouba, 1989) that division and sharing are conceptually close because in both cases the child has to share a quantity into equal sized quotas. It is, therefore, reasonable to assume that when children have an action schema to attain equal distribution they already understand division.

However, Correa, Nunes and Bryant (1998) suggested, that there are good reasons not to treat division the same as sharing, although sharing is the action schema from which it originates. In a sharing situation children's consideration is to give equal amounts to each recipient following one-to-one correspondence procedures (one-for-me, one-for-you). In contrast, in division the equality of the shared quotas is a relationship that is assumed and has to be respected and the child has to understand that there is a direct relationship between the dividend and the quotient and the inverse relationship between the divisor and the quotient. Correa (1995) presented convincing evidence that sharing is a necessary but not sufficient condition for the understanding of the relations between the dividend, the divisor and the quotient. Not all the children who could do sharing were able to reflect on the effect that the size of the divisor had on the size of the quotient.

Because the invariants of a sharing situation are qualitatively different from the invariants of additive situation, as well as the schemas of action involved in each, it is unlikely that the understanding of division will stem from the additive structures.

The existing evidence suggests that the action schema of sharing is the origin of children's understanding of division, but division itself is a more complex concept (Correa, Nunes and Bryant, 1998).

Do multiplication and division have the same or distinct roots?

The literature has not yet been able create a clear picture of whether multiplication and division have the same or distinct roots. For example, Piaget (1965) suggested that division is the inverse of multiplication and that the two operations are discovered simultaneously. In contrast, Fischbein, Deri, Nello and Marino (1985) proposed that the two operations have distinct origins: multiplication originates from addition, while partitive division originates from sharing and quotitive division from subtraction.

This study aims to investigate whether the two operations have the same or different roots. Our hypothesis is that multiplication and division have distinct roots and that children discover their inverse relation at a later stage.

This hypothesis is based on two propositions: Firstly, it is possible that the development of multiplication and division would be similar to the development of addition and subtraction. Subtraction might be the inverse of addition but the research findings suggest that initially children have different schemas of actions for the two operations. There is evidence (Carpenter and Moser, 1982; Carraher and Bryant, 1987; Hudson, 1983; Marton and Neuman, 1990; Riley, Greeno and Heller, 1983) that children are equally successful in solving addition and subtraction problems by modelling the actions described in the situation, even before being able to tell which arithmetic operation is needed to solve the problem, but they only understand the inverse relations of the two operations later on.

Secondly, the invariants of the two operations are different. Nunes and Bryant (1996) pointed out that in a one-to-many correspondence situation the ratio between children and biscuits, for example, is fixed from the beginning and the child is asked how many biscuits there are in total. In contrast, in a sharing situation the child has to share a total and establish the ratio between children and biscuits. Confrey (1994) also distinguishes between sharing and one-to-many correspondence situations. She suggested that in one-to-many correspondence situations there is a constant ratio between the corresponding sets which is not affected by the number of successive replications; 1 chair has 4 legs, 2 chairs have 8, 3 chairs have 12 etc., which means that the number of legs will develop in an arithmetic progression. Division situations are, though, different. The sequence in the number of pieces of cake, for example, after successive splits would develop in a geometric progression, 2, 4, 8, 16 etc. The actions that are carried out in multiplication and division situations are different, therefore, it is unlikely that the child would discover their inverse relation early on.

What statements can be made regarding the origin of multiplication and division and how are they going to be tested?

The theoretical discussion on the relations between children's schemas of action and the origin of the arithmetical operations of multiplication and division leads to certain statement, which are tested in this thesis: (a) the origin of children's understanding of multiplication is to be sought in the schema of one-to-many correspondence, (b) the origin of children's understanding of division is to be sought in children's understanding of the relations involved in a sharing situation; that is the relations between three values: the dividend, the divisor and the quotient, and (c) the two concepts are likely to have different roots and develop independently before becoming coordinated.

The hypothesis of the study that the origins of multiplication and division are in children's schemas of action is contrasted with the alternative hypothesis proposed by Fischbein et al (1985) that their understanding originates from addition and subtraction. Fischbein's hypothesis implies that the understanding of multiplicative concepts stems from quantification. Therefore, children who cannot quantify multiplicative problems are not expected to have any understanding of multiplicative concepts. Quantification proceeds the ability to reflect on multiplicative relations.

The proposed hypothesis that the origins of multiplication and division are to be sought in children's schemas of action implies that children will be able to reflect on multiplicative relations before being able to deal strictly with the numerical aspects of the situation because schemas of action are about the transformation of objects and not about computation.

To test the predictions that result from these two hypotheses a series of studies were carried out analysing children's performance in relational, non-computational multiplicative problems and in computational problems, involving not only discontinuous but also continuous quantities that the children could not quantify.

The first study explores children's understanding of a significant invariant in multiplication, the concept of ratio, the second and the third, examines children's understanding of sharing relations in partitive and quotitive division situations and the third examines whether multiplication and division develop as independent or as coordinated operations.

1.2. The organization of the thesis

The following chapter (Chapter 2), the Literature Review, is a presentation of the different theories which have been developed to answer the question of the origin of multiplication and division. Two streams of theories are presented, analysed and evaluated. The theories that sought the origin of multiplication and division in children's schemas of actions, and alternative theories that sought their origin in other arithmetic operations, such as addition and subtraction.

Chapter three is about the study of children's understanding of the invariants of multiplication. It starts with the aim and the rationale of the study and continues with the methods and the results obtained. The findings are discussed in relation to previous research and to the hypotheses of this study.

Chapter four presents two experiments on children's understanding of sharing relations, one in the context of partitive and the other in the context of quotitive division problems. Both experiments investigate children's understanding of a significant invariant in division, the inverse relation between the divisor and the quotient.

Chapter five presents the final study on whether multiplication and division develop as coordinated operations from the beginning or whether they develop in parallel before becoming coordinated. The study explores the coordination of multiplicative relations within and across the operation of multiplication and division.

Chapter six is on the conclusions of the study. The findings are discussed in relation to the research questions, the educational implications of the study are presented, its limitations are illuminated and suggestions are made for further research.

CHAPTER 2

THE ORIGINS OF CHILDREN'S UNDERSTANDING OF MULTIPLICATION AND DIVISION.

The aim of this chapter is to present and evaluate the theories that have been developed to search for the origin of multiplication and division. Two streams of theories have been developed: the origins of multiplication and division have been sought either in children's schemas of actions or in another operation like addition and subtraction.

Both perspectives and the relative research evidence are described in the following sections. The theories that looked for the origin of multiplication and division in children's schemas of actions will be presented first, followed by the alternative views.

It has to be pointed out that the understanding of multiplication and division is not an instant acquisition, but must be seen as a long process that gradually develops. This thesis focuses on the first steps that children take towards this understanding.

Children's *actions schemas* as the origin of multiplication and division

2.1. The role of ratio and one-to-many correspondence in the understanding of multiplication

2.1.1. The onset of multiplicative reasoning

The concept of ratio has been proposed to be the key to the development of multiplicative reasoning. Freudenthal (1983) proposed that the first meaning of ratio is understood as "relatively". He said that children can make judgements about which shoe fits which doll. They understand for example that a pair of shoe is relatively too large to belong to doll A, or relatively too small to belong to doll B. Similar to Freudenthal's idea of relativeness is Resnick and Singer's (1993) idea about children's fittingness schema, that "two things go together because their sizes and amounts are appropriate for one another" (p.107). In the Goldilocks story the children have no difficulty in recognizing which bed, chair and porridge bowl belong to Daddy Bear, to Mummy Bear and to Baby Bear. It can be said that they form some kind of equations that reads like "the bear size, relates to the bed size". These concepts of relativeness and fitness are important for the emergence of the concept of ratio, because the children are establishing a relation between two variables.

More systematic work on children's understanding of the relations between two variables was done by Piaget (1965). He did a series of studies on children's understanding of one-to-many correspondence where he believed the origin of multiplication lies. According to him the simultaneous use of one-to-one correspondence across several sets involves numerical multiplication. The discovery of multiplication is the result of the coordination of the relations of equivalence across sets, which is initially achieved by understanding the principle of one-to-one correspondence. He stated that:

> "...since arithmetical multiplication is an equi-distribution, equivalence through one-to-one correspondence between two or n sets A is a multiplicative equivalence indicating that one of the sets A is multiplied by two or by n. From the psychological point of view this simply means that a one-to-one correspondence is an implicit multiplication, so that when the child has established the correspondence between several sets, he will sooner or later become aware of this multiplication and use it as an explicit operation." (Piaget, 1965, pp.203-204).

The composition of relations of equivalence presupposes that children can construct a lasting correspondence between two groups whatever the distribution of the elements that were in previous correspondence. The composition of relations of equivalence involves additional difficulties because it involves correspondence between two or three sets that were not put previously in correspondence. The child should recognize that if A=B and B=C then A=C (transitive inference).

The technique used by Piaget to investigate the composition of relations of equivalence was simple. He first tested children's understanding of relations of equivalence in a situation where transitive inference was required. Children were required to put 8 blue flowers (A) into 8 vases (B). Then the blue flowers were removed from the vases and arranged into a single bunch. In order to test whether the child regarded the equivalence between the vases and the blue flowers as lasting he asked the children whether the number of flowers was still equal to the vases when the flowers were bunched together or spaced out. It was expected that the children who did not regard the correspondence between the two sets as lasting, would not be able to compose relations of equivalence. Then the children were asked to put another set of 8 pink flowers (c) into the vases. The pink flowers were then removed and bunched together. The child had to decide whether there was the same number of blue and pink flowers (A=C) even when one of the sets was spaced out or bunched together.

Piaget identified three stages in the development of the composition of relations of equivalence. At the first stage there is failure in both the construction of correspondence and the composition of equivalence. During the second stage, the children can make a one-to-one correspondence between two sets, but their equivalence is not lasting. Regarding the composition of equivalence, they succeed only if the sets remained opposite each other and have the same perceptual characteristics. At this stage the child relies only on perceptual intuition and compares directly the blue and the pink flowers without doing so by means of the number of vases. This is because the composition ability is intuitive and not yet operational. Finally, at the third stage the children recognize the equality of the blue and pink flowers and do so on the basis of the correspondence of each set to the number of vases. The composition of relations of equivalence has now become operational.

Piaget went on to study how the "composition of equivalence can be generalized in the form of one-to-one correspondence between *n* sets ... and numerical multiplication" (Piaget, 1965, p. 213). In Piagetian terms multiplication is conceptualized as equal grouping, that is as a sequence of *n* equivalent sets. The children who understand one-to-one correspondence and transitivity should also be able to understand one-to-many correspondence. The task he presented to the children was the succession of the above and was based on the reasoning that if A=2B and A=C, then C=2B. In the previous task the children were asked first to put the 8 blue flowers into the 8 vases, remove them, and then put in the 8 pink flowers, remove them and reason whether the two bunches of blue and pink flowers were equal. When the children verified the equality of the two sets of flowers the children were asked to think how many flowers would go into each vase if they wanted to share all the blue and pink flowers equally into them. If the children could not say that two flowers were to go into each vase they were allowed to find out by putting the flowers into the vases. Then, the two bunches of the two flowers were put aside but the vases remained in sight. The children were then asked to pick up from a box as many plastic tubes (c) as they needed to place all the blue and pink flowers in, under the condition that only one flower could be fitted into each tube (C=A). Piaget wanted to see if the children would pick two tubes for each vase, given that two flowers went into each vase before.

The children's performance was classified into three stages. The children in the first stage were those who in the previous task could not recognize that there was the same number of blue and pink flowers. The children who could not compose relations of equivalence by transitive inference were equally unable to make the two sets of flowers correspond simultaneously with the vases or pick up as many tubes as there were flowers. Their behaviour showed that they were incapable of composing multiplicative relations. They made an arbitrary estimate of the increase of flowers without perceiving the necessity of duplication. They did not understand that if the blue and pink flowers corresponded simultaneously to the vases, then two flowers and not only one would correspond to each vase.

The children in the second stage did not anticipate that there would be two flowers in each vase, but managed to grasp this correspondence when they discovered that there were remaining flowers, after attempting to put one flower in each vase. In the second part of the task where they had to pick up the correct number of tubes, they firstly put one tube beside each vase and then realised that they needed two tubes to each vase. According to Piaget these children could not be regarded as having an understanding of multiplicative composition, because they had not yet mastered the composition of relations of equivalence and mostly because the multiple correspondence is not generalizable. When these children were presented immediately afterwards with a similar problem that involved one-to-three or four correspondence they were not able to anticipate the relations between the sets but proceeded by trial and error.

In the third stage, all the children who were able to compose relations of equivalence were also able to understand the relations of multiple correspondence involved in the problem. These children did not judge intuitively but had an anticipatory schema of action that they could also generalize from "two-to-one" correspondence to three, four and five. According to Piaget, two conclusions can be drawn from this fact:

> "Firstly, the transition from the intuitive to the operational method of procedure entails the possibility of generalization,... Secondly, the operation of correspondence is revealed in its true light, as being a multiplicative composition. In the various correspondences, one to one, two to one, three to one, etc. the value of each new set is no longer regarded only as an addition, but as a multiplication, "1 x *n*", "2 x *n*", "3 x *n*", etc." (Piaget, 1965, p. 219).

In the same schema of one-to-many correspondence Piaget (1965) sought the origin of division. What is being suggested in his book the "Child's Concept of Number" is that the concept of division originates in the concept of multiplication and that the two operations are discovered simultaneously. He suggested that the two operations are intrinsically related, because one is the inverse of the other. Therefore, if children understand multiplication they should also be able to understand its inverse which is division. As shown in the vase and flowers task the children who understand that the blue and pink flowers correspond simultaneously to the vases in a one-to-two correspondence, could also pick up the right number of tubes in which to place each flower in. This shows that the children discover both operations at the same time. Piaget did not study division itself, therefore, we cannot draw any conclusions about its development.

Piaget's work has shown that the origin of one-to-many correspondence can be traced in the understanding of one-to-many correspondence schema. His findings on multiplication suggest that children at around the age of 5 to 6 can deal quite well with one-to-many correspondence situations. The way the children reasoned on the flower task suggests that they treated the situation as a multiplicative one and considered the relations between the two variables that were in correspondence. This type of reasoning is distinct from additive reasoning. The flower task suggests that multiplication is about the constant one-to-many correspondence relation between two variables. Additive situations are about part-whole relations, where sets of objects are either joined or separated from each other. Children's solution strategies in the flower task suggest that multiplication is not another, more complicated form of addition, but a new operation with a new sets of invariants and number meanings.

Piaget's primary work can be regarded as the very beginning of the concept of multiplication. There is a need to replicate Piaget's research on young children's understanding of one-to-many correspondence situations in a more systematic way. There is also more to be understood about this form of correspondence as the children grow older. One significant step towards the understanding of multiplication is the use of the concept of ratio as a way to express quantitative comparison and build equal sets. Can the children build equal sets by correspondence procedures or order different one-to-many correspondence ratios? In the following session some evidence is presented on the first question.

2.1.2 Young children's understanding of the concept of ratio and their ability to build equal sets by correspondence procedures

Piaget, Kaufmann and Bourquin (1977)[1] in their book "Recherches sur l' Abstraction Reflechissante" investigated children's understanding of the concept of ratio as a way to express quantitative comparison. They examined the development of multiplicative relations starting from children's initial comprehension of the concept of equivalence by means of one-to-one correspondence passing through the operations of one-to-many correspondence and the constitution of ratio as a general way of expressing quantitative comparison.

[1] Please note that all the information on Piaget, Kaufmann and Bourquin's (1977) work are extracted from Correa's (1995) and Frydman's (1990) thesis, because the original book has not yet been translated into English.

In their study the children were presented with a series of tasks all using the same type of material: two sets of blocks of different colour and size. The size of the blocks ranged from one to five units. The first three tasks referred to children's ability to infer numerical equivalence between two sets through one-to-one correspondence.

In task 1, they were presented with two sets of blocks of the same size but of different colour (blue and red). They were asked to put the two sets in one-to-one correspondence and were questioned about the numerical equivalence of the two sets. The children had to justify their answers.

When the equivalence was recognised the children proceeded to task 2 where they were asked to build two walls one with the blue blocks and one with the red blocks. The children had to find out whether the two walls would be of the same length.

In task 3 one more question was added to the above. The children had to think about whether the equality between the two sets would be conserved if the actions of correspondence were continued indefinitely.

In task 4 the children had to quantify the difference between one-to-many correspondence ratios. They had to estimate the relative length of two walls built with the same number of blocks but of different size. Each blue block consisted of five single units (b=5) and each red of one (r=1). The children could initially compare the blocks. Then the experimenter put the two blocks aside as a model and built a wall with some other blue blocks. After that he asked the children where the red wall would finish if it was built with the same number of blocks.

The children reacted in three different ways in the solution of the above tasks. The first category of children, at the age of 5 to 6, could easily solve the first three tasks. They recognized the equality of the sets put in one-to-one correspondence and could generalize conservation in the case where the action of correspondence was to be carried out indefinitely. However, in task 4, the children could not quantify the difference in the height of the two walls. According to Piaget et al. (1977) the children were just starting to process the differentiation between number and length.

The second category of children (age 7 - 8) succeeded in the task either by applying a visual correspondence between each blue block and the difference (b-r) or by touching each blue block and simultaneously counting the number of red blocks that corresponded to it. This stage is characterized by the beginning of quantification. However, according to Piaget et al. (1977) the procedure remained additive because children could not deduce the general relation and its multiplicative formulation. The children tried to find the difference between the blue and the red walls by the repetitive addition of the difference between the blue and the red blocks.

Finally, the third category of children (age 9-10) was led to the correct solution by means of the a multiplicative procedure illustrated in the following protocol:

> The child was presented with 10 blocks of each colour " In each blue block there are five red ones; then five and five equals ten" He immediately pointed at the second blue block as the limit for the red wall" (Piaget, Kaufmann and Bourquin, 1977, p.21).

With this last task Piaget showed how the children use the concept of ratio as a way to make comparisons between different corresponding sets and quantifying their differences. He found that the quantification of the difference between the two walls was not possible before the age of 9. The study, though, does not give any information about whether the children anticipated any difference in the height of the wall or could even order the height of the two wall on the basis of the size of blocks. There is evidence (Lawrenson and

Bryant 1972; Bryant, 1974) that children are able to reflect on a situation on the basis of relations before they are able to compute the corresponding sums. Quantification is difficult not only because it presupposes the understanding of relations, but also because it requires the organization of the counting ability. Multiplication is a concept that develops over a period of time and it is possible that the children are able to order different one-to-many correspondence ratios before they are able to quantify their sums.

Piaget et al's (1977) study on children's ability to quantify the difference between two one-to-many correspondence sets leads to a second issue. The children might not be able to quantify the difference between two sets that are in correspondence, but they might be able to build equal sets by correspondence procedures. To do so the same ratio should be applied for equality to be obtained.

Piaget et al. (1977) were the first to investigate children's use of ratio as a way to achieve equality. Children's ability to build equal sets was examined in situations where the construction of common multiples was required.

The children were presented with one collection of blue and another collection of yellow counters of equal size. They had to create two equal amounts of counters. They were instructed to take the blue counters two at a time and the yellow ones three at a time. That meant that to form two equal sets of 12 counters, the children had to pick up 6 times 2 blue counters and 4 times 3 yellow counters.

In a situation like the above the procedure to achieve equality is clearly multiplicative and illustrates a fundamental difference between addition and multiplication. Piaget et al (1977) proposed that addition is counting the number of objects whereas multiplication is also counting the number of operations (n) whereby objects are gathered in classes of x. A significant component of multiplication is that the equivalence of two numerical products presupposes an inverse relation between the multiplier (n) and the multiplicand (x) of each product. They proposed that children lack an understanding of these relations *before the operational stage*.

The children's performance in the common multiples task was classified into three stages. In stage I (age 6-7) the children dealt out the units of each collection on a one-to-one basis as if they were equivalent. Each time they picked up a set of counters from one collection they picked up another set from the other collection. After a while they all realized that it was impossible to reach equivalence and in some cases, if they reached equivalence by trial and error, they were not aware of the number of times they took each of the two units. According to Piaget et al (1977) in the first stage children do not understand that the equivalence of two numerical products presupposes an inverse relation between the multiplier and the multiplicand. For them the construction of two equivalent collections means adding quantities on a one-to-one correspondence basis.

In stage II (age 7-8) the children understood that the inequality of the quantities had to be compensated by the number of actions. Once they had obtained two equal amounts of each unit they could not say how many actions they had performed in each collection although they could express it in terms of packs rather than in terms of times.

In stage III children knew from the beginning how many units they had to pick up from each collection in order to achieve numerical equivalence.

Piaget et al's (1977) study illustrates that multiplicative situations are quite different from the additive ones and that they involve a whole new sets of invariants and relations that are not present in addition and subtraction. It is clear that multiplication is about the constant relation between two variables and that to achieve equality not only do you have to consider the number of objects joined together but also how many times the objects are gathered. Because of these fundamental differences between addition and multiplication it is unlikely that the first meaning of multiplication will stem from addition.

Regarding children's ability to use ratio as a way to achieve equality it can be agreed that the situation of common multiples chosen by Piaget et al (1977) was very hard for the children. Their failure to indicate the number of actions they performed on each set to obtain equality might be due to the fact that they focused their attention on the distribution of counters. The second component of multiplication, that the equivalence of two numerical products presupposes an inverse relation between the multiplier and the multiplicand of each product, actually refers to the principle of commutativity in multiplication which is regarded as a late acquisition (Nunes and Bryant, 1996). The fact that children had considerable difficulty in the construction of common multiples does not mean that they cannot use the concept of ratio as a way to express numerical equivalence. It is possible that they could reflect on the concept of ratio in correspondence situations before they could understand the commutativity rule.

Frydman and Bryant (1988) investigated the progressive development of the concept of ratio in situations simpler than those requiring the construction of common multiples examined by Piaget, Kaufmann and Bourquin (1977). They designed a number of sharing situations of ascending difficulty to investigate whether pre-operational children were able to understand the relationship between different one-to-many ratios and build equivalent sets by correspondence procedures.

In their study 4 and 5 year old children were asked to distribute blocks (pretend-sweets) between two dolls. The distribution took place under two different conditions. In condition I, the children had to share units of equal quantity in the form of either single or double and triple blocks on the basis of one-to-one correspondence (one single for A and one single for B, or one double for A and one double for B, etc). The results revealed that even the youngest children had no difficulty in forming equal sets of sweets when they were distributing blocks of the same size between the dolls.

In condition II, the children had to build again equal sets of sweets for the two dolls, but this time they had to deal different sized units to the two recipients. The children were told that one doll wanted her sweets to be in single units, while the other wanted them either in double or in triple units (the double and the triple units consisted of single units joined together) and despite this difference both dolls wanted to end up with the same total amount. In this case an object-based one-to-one correspondence sharing was inappropriate. The results showed that this condition was very hard for the 4 year olds, while three quarters of the 5 year olds provided a correct response. The majority of the children who failed shared the sweets on an object-based one-to-one correspondence by giving each doll one quantity at a time, paying no attention to the difference between singles, doubles and triples. As a result one doll ended up with double or triple number of sweets in term of units. The fact that the 4 year olds persisted in responding in a simple one-to-one correspondence without taking into account the ratio difference between the shared quantities, suggests that for them the number of actions or the number of discrete perceptual units rather than the amounts dealt to each recipient, was the important variable. However, there was another possibility. The children got it wrong, not because they did not understand the one-to-many relationship, but because they did not understand that the double sweets were equal to two singles sweets.

In order to rule out this possibility, Frydman and Bryant (1988) designed another experiment, where they tried to make the children more aware of the fact that the double and the triple units were equivalent to two and three singles respectively. This was accomplished by using colour cues. The double and the triple units were constructed with blocks of different colours. This manipulation allowed children to account for numerosity through colour-based one-to-one correspondence and to see whether this type of one-to-one correspondence would help the young children to understand how to cope with units of different values in an equal sharing situation. During the pretest, the training situation and the post-test the children were presented with the same tasks as in the previous experiment. The difference was that the pre-test and the post-test involved only blocks of the same colour while the training session involved two different colours. The double-sweets were presented in two colours, containing one blue and one yellow block joined together, while the singles came in yellow and blue. There was also a control group that was presented with the same tasks but without any colour cues. The results of the pretest confirmed the trends observed in the previous experiment. The experimental group did a great deal better in the intervening colour task, while the control group stayed at the same level as in the pretest. There was no performance difference between doubles and triples units. The most striking thing was that the difference between the two groups was maintained in the post-test. All the children in the experimental group, who had failed in the pre-test, not only performed well in the intervening colour task, but continued to do so in the post-test where colour cues were no longer available. However, the children in the control group maintained their low performance in the post-test.

The fact that all the children who improved in the training situation also performed well in the post test showed that they started taking the ratio difference into account. In a short time the children were alerted to the nature of the double and triple units, which shows that young children have no particular difficulty with this aspect of one-to-many correspondence.

However, there was not yet evidence that children were aware of the number of operations involved in this adjustment. Frydman (1990) proposed that Piaget, Kaufmann and Bourquin's (1977) claim that pre-operational children lack an understanding of the number of operations involved in multiplication was in a way premature because it was based only in children's difficulty with the construction of common multiples, that is with problems involving an m-to-n correspondence rule, and did not look at children's performance in problems in which the construction of two equal quantities could be achieved by using a one-to-n correspondence. Moreover, in Piaget et al's experiment the children had to build two equal amounts of counters by taking 2 yellow and 3 blue blocks at a time. Perhaps the children were not aware of the number of actions performed because their attention was concentrated on the number of blocks they were taking each time. Because the counters were presented in single units and the children had to pay a lot of attention to counting two or three counters each time their attention to the task might have been distracted.

Frydman (1990) suggested that there might be several developmental stages in the construction of equal sets by means of ratio in the context of a sharing situation. He designed another set of experiments were he looked at children's ability to build equal sets while sharing unequal quantities, when one of the quantities was the multiple of the other and when the children had to construct a common multiple of the two quantities. He also wanted to see whether the children who did well in those tasks were also aware of the number of operations performed. It was expected that the problems requiring the construction of common multiples would be more difficult for the younger children compared with the problems where one quantity was the multiple of the other.

In order to test his assumption he asked 5 and 6 year old children to share blocks (pretending to be chocolate bars) that had been broken into different size units and make sure that the recipients would end up with the same total despite receiving different sized pieces. Before performing the task the children were given the opportunity to see how many units were stuck together in the units of each set. In the first two tasks (the one-to-n tasks) the children had to share out unequal sets with ratios such as 1:2 versus 1:4 and 1:2 versus 1:6 where the units of one set were multiples of the units of the other. In this case children could form equal sets by repetition of the gestures of the distribution. For example, in the 1:2 versus 1:4 ratio the correct answer could be reached by giving two pieces with 2 units to one recipient while giving one piece of 4 units to the other. In the m-to-n tasks the children had to share out more complex ratios such as 1:2 versus 1:3 and 1:3 versus 1:4. In the latter task no simple repetition of the gesture of repetition was possible. The children could solve the task only by finding a common multiple. For example, in the 1:2 versus 1:3 ratios the children could build an equal share only by giving three doubles to one recipient while giving two triples to the other. During the experiment the participants were allowed to assemble the doubles and the triples but were not allowed to break up the pieces into units. Initially the participants were given some easier sharing tasks where they had to build equal sets when both sets were in singles and when one set was in singles and the other was either in doubles or in triples. The aim was to familiarize the subjects with the sharing problems and ensure that they could solve them.

The results of the study showed that nearly all the children in both age groups did well on the one-to-n tasks. However, most children failed one or both of the m-to-n tasks. Only four 5 year olds (out of 18) and six 6 year olds gave a correct response in both problems. An analysis of variance revealed that there was a significant effect of condition, indicating that the one-to-n tasks elicited more correct solutions. Age was not found to have a significant effect. The children's failure suggested that the tasks requiring the construction of common multiples were much more difficult. The children who failed could not match the larger units by assembling smaller ones together. They either stopped in the face of this difficulty or ignored the difference between the quantities and distributed them on a "one for A" and a "one for B basis". Additive reasoning was not sufficient for the solution of the ratios where common multiples had to be found, as there was no simple conversion of one unit into the other. Equal totals could only be built by anticipation. For example, in the case of 1:2 versus 1:3 it seems unlikely that children could work this out by adding up the three doubles given to one recipient, which makes six, and by adding up the two triples that also results to six. Such a calculation required a double counting of pieces and units. The common multiples tasks could easily be solved if the children had understood the idea of commutativity in multiplication where 2x3 equals the same as 3x2. However, commutativity in multiplication appears to be hard for young children. The children who solved the task either used a correspondence based strategy, that means that they assembled for example two 3s together and three 2s together, or used a counting strategy. Children's performance in constructing equal sets by correspondence procedures highlights the need to distinguish between additive and multiplicative reasoning. To achieve equality in addition the focus is on the size of the sets joined, while in multiplication equality is achieved by counting the size of the sets and also the number of the sets.

The children who performed well on the tasks were tested on their awareness of the temporal correspondence rule which they used to solve the problem. The participants were asked to count the number of pieces of chocolate taken at each "go" to give the same to both recipients. Almost all the children answered the "how many times" question correctly when the ratio was 1:2 versus 1:3 , while only one five year old (out of 4) and 3 six year olds (out of 7) managed to give the correct response when the ratio was 1:3 versus 1:4.

The results of Frydman's (1990) study give support to the Piaget, Kaufmann and Bourquin's (1977) assumption that the construction of common multiples is difficult for the young children. Contrary to Piaget et al's (1977) claim, the majority of the 5 year olds could differentiate the total number of blocks distributed from the number of actions performed to obtain it. Piaget had underestimated children's understanding of the concept of ratio. Frydman showed that there are different developmental stages in the construction of equal sets by means of ratio. This suggests that young children are not totally unaware of multiplicative relations and that they can deal with them at a young age.

2.1.3 Summary

The research findings suggest that: (a) Multiplicative thinking is distinct from additive thinking because the invariants of the two situations are different. Addition is about part-whole relations. Objects or sets of objects that do not have any relation to each other are either joined or separated. Multiplication is about two variables that are in a constant one-to-many correspondence relation expressed as a ratio. Addition is about counting the objects that are put together, while multiplication is also about counting the number of times that the objects are put together; (b) Multiplication is a complex concept that progressively develops over a long period of time. Five year old children are able to solve a variety of problems that involve one-to-many correspondence relations. They can make transitive inferences based on one-to-many correspondence and infer that if A=2B and C=A then C=2B. By the age of 6 most of the children are able to build equal sets by correspondence procedures, even when they are sharing unequal quantities that have a simple ratio relation to each other.

The aim of this study is to explore further some aspects of children's understanding of one-to-many correspondence that lays the basis for the understanding of multiplication. There is evidence that young children can build equal sets by correspondence procedures (Frydman, 1990) and that they can quantify the difference between one-to-many correspondence ratios by the age of 9 to 10 (Piaget et al., 1977), but there is no evidence that the children can order different one-to-many correspondence ratios before quantifying the corresponding sums. Ordering different one-to-many correspondence ratios is important because the product in a multiplicative situation depends both on the number of elements in the basic set and on the ratio, given the same basic set. Investigating further children's understanding of ratio as a way to make multiplicative comparisons helps the formation of a more accurate picture of children's understanding of one-to-many correspondence situations as the origin of the concept of multiplication.

2.2 The sharing schema of action and its importance in the understanding of division

2.2.1 How does children's understanding of division begin?

It has often been proposed that the understanding of division originates from children's schemas of action in sharing situations (Correa, Nunes and Bryant, 1998; Dickson, Brown and Gibson, 1984;). Although division is more than sharing, it can be claimed that the origin of division can be found in the context of sharing, because in both sharing and division the child has to form equal quotas, equal to the number of recipients.

There is much evidence suggesting that 4 and 5 year old children are efficient in sharing when they share equal sized units (Davis and Hunting, 1990; Davis and Pitkethly, 1990; Davis and Pepper, 1992; Frydman, 1990), able to adjust their sharing procedures when they share different size units (Frydman and Bryant, 1988; Frydman, 1990) and successful when they have to deal with remainders (Desforges and Desforges, 1980).

But what do children learn anything about division from the action schema of sharing? Is division the same as sharing? Correa, Nunes and Bryant (1998) pointed out that division as an operation is different from sharing. In a sharing situation the child focuses on giving equal amounts to each recipient in a one-for-me, one-for-you fashion. In division, though, the equality of the shared quotas is assumed and is not a relation that has to be respected. The invariants are more complex as the child has to grasp the set of relations between three elements: the size of the whole, the number of parts/recipients and the size of the parts (quotas) which have to be the same for all the recipients. That means that the children have to understand that there is a direct relationship between the dividend and the quotient, when the divisor is kept constant, and an inverse relationship between the divisor and the quotient when the dividend is kept constant. In other words, the bigger the shared quantity is, the bigger the quotas are, and the more recipients there are the less each would get.

The question that follows is whether children who are good at sharing also understand the set of relations between the divisor, the dividend and the quotient. Does efficiency in sharing guarantee such an understanding? This question was explored by Correa (1995). She devised a set of experiments to test whether children who are able to share and infer the equivalence of the cardinal values of the shared sets can also understand that there is an inverse relationship between the divisor and the size of the quotas in non-computational division problems.

The participants of her first experiment in partitive division were children aged 5 to 7 who during the pretest were found to be able to share and infer the numerical equivalence of the shared sets. These children had never received any formal instruction on division. They were presented with a number of situations where the same number of sweets was to be shared by a number of rabbits attending two different parties. Each rabbit was carrying a small basket on his back into which the experimenter placed the shared sweets. The distribution of the sweets was carried out behind a screen, so that the children could not see how many sweets she was giving to each rabbit. The children had to anticipate the relative size of the shared sweets received by the rabbits attending each of the two parties. The question posed to the child was: "are the rabbits at one party going to receive the same number of sweets as the rabbits at the other party?". Under those methodological precautions there was no ambiguity about what the questions was about. Because the children could not see the result of the sharing their response was entirely based of their anticipation schemas.

There were two experimental conditions set in the above problem. In the first condition there was the same number of rabbits at both parties (same condition); in the other condition, which was expected to be harder, the number of rabbits differed between the parties (different condition). In all the cases the number of sweets to be shared remained the same. The children had to show whether the rabbits at both parties got the same number of sweets or different, and if not, at which party the rabbits received more sweets. This type of sharing situation where the number of recipients is known and the children are asked about the size of the shared quotas is known as a partitive division problem.

The results of the study confirmed Correa's prediction. It was found that children of all age groups performed nearly at ceiling level when the number of the rabbits was the same at both parties and justified their responses by drawing attention to the equivalence between the divisor and the quotient. However, the level of correct responses dropped significantly in the different condition tasks. It was found that the percentage of children performing above chance level at the age of 5 was 30%, at the age of 6, 55 % and at the age of 7, 85%. The results indicated that the youngest children had the greater difficulty, but more than half of the 6 year olds and the majority of the 7 year olds could give the correct answer. The analysis of the children's errors showed that most of the 5 and 6 year olds thought that no matter how many rabbits are sharing they would get the same number of sweets because the number of sweets to be shared at the two parties was the same. The majority of the 7 year olds though, who gave a false response in the different condition tasks made a different type of error. These children applied a direct relationship between the number of sweets and the number of recipients. They thought that the more the rabbits there were, the more each would receive. The 7 year olds' errors might indicate their attempt to take into consideration the number of recipients in the sharing situation, which was ignored by the younger children who only focused their attention on the number of sweets to be shared. However, instead of applying the correct inverse relationship between the number of rabbits and the size of the shared sets of sweets the older children applied a direct relation, or as Correa (1995) describes it the "more-is-more" rule: the more rabbits that have to share, the more each will get.

Correa (1995) also investigated the understanding of the inverse relation between the divisor and the quotient in the context of a different situation known as quotitive division. In quotitive division problems the divisor indicates the size of a quotient that is measured against an original unit (dividend) to determine how many times the quotient is contained in the original unit. The sample of the study consisted of children aged 5 to 7 who had no previous instruction on division. They were told that the experimenter wanted to invite pink and blue rabbits to a picnic, but she did not know how many rabbits to invite. Then, for each collection of rabbits a picture of some pretend sweets (2, 3 or 4) on a plate was presented, which corresponded to the amount of sweets that she wanted to give respectively to each rabbit in the two groups. The total amount of sweets to be shared was the same in both groups (either 12 or 24). The children were asked to make their judgements about the relative number of pink and blue rabbits to be invited to the picnic in two conditions that were designated as the same and different condition. In the same condition the number of sweets to be given to each rabbit in the two groups was the same (2 sweets to each blue and 2 sweets to each pink rabbit), while in the different condition it was different (2 sweets to the pink and 3 to the blue rabbits).

The results of this study verified once again that the different condition questions were harder for the children. The children made very few mistakes when the same number of sweets was given to each group of rabbits. In contrast, the percentage of children who gave correct responses was 15% at the age of 5, 40% at the age of 6 and 45% at the age of 7. Once more, the majority of wrong responses at the ages of 6 and 7 was due to the application of the direct relation between the size of the quota and the number of recipients. The children thought that the more the sweets there were to be shared, the more rabbits should be invited. The 5 year olds either applied the "more-is-more" rule or suggested that the same number of rabbits would be invited to the two parties based on the equality of the number of sweets available at the two parties.

A comparative analysis of the children's performance in the above non-computational partitive and quotitive division tasks revealed that 5 year olds had more difficulty in solving the quotitive than the partitive tasks. Their performance difference was apparent even at the age of 7.

Although all the children who took part in the study were able to reason on the quantitative equivalence of the shared quotas, still, not all of them were able to reflect on the relations between the dividend, the divisor and the quotient. This finding reinforces the conceptual distinction between sharing and division.

Correa's study focused on children's ability to reason on the inverse relation between the divisor and the quotient when the quantity to be shared was discontinuous. The is also some evidence on children's understanding of sharing relations when the quantity to be shared is continuous.

Desli (1997), carried out a study to investigate whether 6, 7 and 8 year olds could compare the relative size of the shared quotas in partitive problems when the quantity to be shared was a bar of chocolate. The children had to judge whether two groups of children would receive the same or a different amount of chocolate. The amount of chocolate to be shared was the same but the number of children varied across the groups.

The results revealed that 75% of the 6 year olds, 85% of the 7 year olds and 95% of the 8 year olds gave a correct response. The analysis of children's justifications showed that at the age of 6 many children focused only on the equality of the amount to be shared and stressed that the children in both groups - no matter the size difference between them -

would receive the same amount of chocolate. At the age of 7 though the majority of unsuccessful children applied the "more-is-more" rule indicating that the children at the party with the more children would get more chocolate.

The similarity between the results of Correa's (1995) study with discontinuous quantities and Desli's (1997) study with continuous quantities is apparent. The percentage of children applying the inverse relation between the divisor and the quotient was similar, which suggests that reflecting on the relations that are involved in the sharing of continuous and discontinuous quantities is of the same level of difficulty. However, a direct comparison of the two studies is not possible because they were carried out under different experimental conditions and in different countries: Correa's study was conducted in Oxford and Desli's study in Thessaloniki, Greece.

In the same line of reasoning Sophian, Garyantes and Chang (1997) carried out a series of studies to investigate children's understanding of the effect of partitioning a quantity into different numbers of recipients when the quantity to be shared was continuous. Children 5 to 7 were introduced to the "Pizza Monster", a stuffed toy fed exclusively with bite-sized pizzas (orange lentils). The Monster always had to share the pizzas with his friends. The children had to decide which of two sharing alternatives would give the Pizza Monster a greater amount of pizza. The children were asked to make their judgements in two situations. In one situation the amount of pizzas to be shared was the same in both alternatives, but the number of recipients was different (contrasting recipients problems). For example, in one instance the Monster was sharing with 2 while in the other with 3 friends. In the second situation the number of friends sharing with the Monster was the same in both alternatives, but the amount of pizzas to be shared was different (contrasting totals problems). The last situation was expected to be easier and all the children were expected to understand that the larger the quantity to be shared was the larger the size of the quotas would be. For this reason it was used as a reference point.

The results of the study verified their prediction. In the contrasting totals problems children's performance was almost errorless. However, their performance in the contrasting recipients problems was significantly lower. The majority of the children incorrectly expected the size of the shared quotas to be positively related to the number of recipients. Even the 7 year olds did not perform better than the 5 year old ones. Only 6 out of 20 7 year olds gave more correct than erroneous answers. The children also appeared to be inconsistent in giving their answers. They did not have a systematic wrong notion but were uncertain in choosing one of the two alternatives.

Sophian et al's (1997) study, contrary to Correa's (1995) study, suggests that young children have considerable difficulty in understanding the inverse divisor-quotient relation. It is possible that the discrepancy in the level of success in these two studies is due to differences in the design of the study and to the difference in the types of quantities used. Correa's study involved discontinuous quantities while Sophian et al presented their tasks with continuous quantities. In order to investigate the effect that the type of quantities used had on children's reasoning the children should be assigned randomly to a continuous versus a discontinuous quantities sharing task or ask the same children to reflect on sharing relations both in discontinuous and continuous quantities.

In order to explore further children's difficulty to understand the inverse divisor-quotient relation Sophian et al (1997) contrasted 5 year olds reasoning about inverse relations in a fractional versus a subtractive context. The fractional situation was identical to the contrasting recipients and contrasting totals problems presented above. In the subtractive situation the Pizza Monster had to take aside some cups of pizzas from the quantity to be shared for the baby monster (because the baby monster always ate its pizzas from a cup) and then eat what was left. The children were asked to choose between two alternatives from which a different number of cups were subtracted. The results of the study suggested that the children had no difficulty to recognize the effect that the amount taken out had on the size of the quantity left in the subtractive situations. However, the children failed to recognize the effect that the number of recipients had on the size of quotas in the fractional situations. Their findings reflected the limited understanding that the children have on sharing relations. Commenting further on their findings it can be said that subtractive reasoning does not facilitate children's reasoning on sharing relations. Understanding the effect that the size of the subtracted quantity has on the size of the quantity left does not aid the understanding of the relations in a sharing situation. That means that sharing situations are different from additive situations and is unlikely that the children would learn anything about division from their experience to add and subtract.

Sophian et al (1997) raised the question whether the inclusion of both contrasting totals and contrasting recipients problems confused the children in their answers. For this reason in a subsequent study the children were presented only with contrasting recipients problems. They also examined the effect that the size of the numerical contrasts had on children's performance. It was possible that the children would be encouraged to think of the effect that the number of monsters sharing would have on the size of the shares when they were asked to choose between an alternative where 2 versus 5 monsters were sharing. The findings suggested that including only contrasting recipients problems improved the performance of the 7 but not of the 5 year olds. Children's performance was not affected though by the specific numerical contrasts used in the problems.

Another issue explored be Sophian et al (1997) was whether the children would be helped in making judgements on the inverse divisor-quotient relationship, if they were given the opportunity to observe the effect that the number of recipients had on the size of the quotas. For this reason 5 year olds were randomly assigned to an experimental or to a control group. Both groups went through a pre-test, a training session and a post-test where the children had to make judgements on various contrasting recipients problems. In the training session the children of the experimental group first made their judgement about the alternative that would give the Pizza Monster a larger share. Then the experimenter divided the pizzas among the recipients and the child had the opportunity to check his/her answer and find out the effect of the number of splits had on the size of the shares. In the control group, though, only the alternative that the children selected was partitioned. Therefore, they did not compare the results of dividing the quantity into different shares. The findings of the study showed that although the children of the experimental group performed poor in the pre-test after the small training they received their performance improved significantly in the post-test. The children of the control groups did not show any significant improvement from the pre-test to the post-test.

Sophian et al. (1997) proposed that the understanding of partitioning relations can be an important sources of early fractional knowledge. Their findings suggest that young children have difficulty to understand fractional relations, but their misconceptions about the effect of the number of quotas on their size are not strong and can easily be corrected in a brief training session.

In another study Correa (1995) also examined whether the children would be helped to think of the inverse divisor-quotient relation if they themselves were asked to share one of the quantities not only in the context of partitive problems as Sophian et al (1997) did, but also in the context of quotitive problems. The assumption was that by asking children to share one of the quantities they would be encouraged to think about the relation between the divisor and the quotient.

Correa explored her assumption in the context of an experiment similar to her previous ones. There was a control condition in which the children were asked to judge the relative numerosity of the shared sets in the same and different condition in partitive and quotitive problems as described before and an experimental condition in which they had to make their judgements after sharing one of the quantities. For example, in partitive division the children were initially asked to divide 12 sweets between 2 pink teddies. Then the experimenter said that she also intended to share 12 sweets among 4 teddies. The child had to judge the relative numerosity of the amount of sweets received by each group of teddies.

The results of the study showed that there was a significant superiority in the performance of 5 year olds in the experimental partitive tasks compared with that of the same age group in the control condition. However, no significant difference was found between the control and the experimental group in quotitive tasks in this age group. On the other hand, a major difference was found in the performance of 6 and 7 year olds. Both the number of correct responses and the justifications they gave demonstrated that they took advantage of using sharing procedures to estimate the relative size of the quotient. Overall, the children achieved better scores in the experimental than in the control condition.

Both Correa's and Sophian et al's studies suggest that children's understanding of sharing relations can be improved if the children are exposed to sharing situations where they can observe the effect of partitioning a quantity into a different number of shares.

2.2.2 Summary

The research findings suggest that children at the age of 6 have a good understanding of the relations that are involved in a sharing situation between the dividend, the divisor and the quotient when the quantity shared is discontinuous. Most of them can apply the inverse relationship between the divisor and the quotient when asked to judge the relative size of the shared quotas in partitive division situations. Children found it more difficult to reflect on the sharing relations in the context of quotitive division problems. The understanding of the relations between the dividend, the divisor and the quotient is a significant step beyond the simple activity of sharing to the understanding of division. This understanding originates from children's experience of sharing, but sharing itself is not sufficient. As shown by Correa (1995) proficiency in sharing does not lead immediately to the comprehension of the relations between the quantities involved in division. This finding reinforces the conceptual distinction between sharing and division.

In Correa's study (1995) the children were asked to reflect on sharing relations in partitive and quotitive problems with discontinuous quantities only. Sophian et al. (1997) asked the children to order sharing relations in the context of partitive problems only with continuous quantities. The aim of this study is to extend both studies and compare children's understanding of sharing relations in partitive and quotitive division problems both with discontinuous and continuous quantities. Reflecting on sharing relations with continuous quantities places the problem in the domain of fractions, which are characterized as rather difficult for the children. The research evidence suggests that it is not easy for the children to order different fractions and they usually assume that the larger the denominator, the larger the fraction (Gelman, 1991; Mack, 1990; Post, 1981). If the children have a genuine understanding of sharing relations then the difference in the types of quantities used was not expected to have an impact in their reasoning. By introducing continuous quantities we could ensure that the children would reason on the situation on the basis of relations only because the quantification of a problem involving continuous quantities would have been beyond their grasp.

2.3 Evidence of children's use of schemas of action in the *quantification* of multiplication and division problems

The studies presented up to here focused mostly on the schemas of action that children employ to reflect on the *relations* involved in one-to-many and sharing situations. There is a lot more research on young children's ability to *quantify* multiplication and division problems with discontinuous quantities. There is also evidence on children's ability to quantify fractions. This section of the literature review focuses on the studies that investigated children's ability to quantify multiplication and division problems involving discontinuous and continuous quantities, before receiving school instruction, when provided with materials to model the situation.

There is a growing body of research which investigates the effect that the types of numbers (whole or decimal number), the context (rate problems, change of size etc), the nature of the quantities involved (intensive or extensive) as well as the misconceptions regarding the operation of multiplication and division have in recognizing and solving multiplication problems. Because these studies are dealing with older children they are not going to be reported in the present literature review.

2.3.1 Quantifying multiplication problems

Steffe (1994) proposed that for a situation to be established as multiplicative, it is necessary to at least coordinate two composite units, i.e. units that are themselves composed of other units, in such a way that one unit is distributed over the elements of the other composite unit. Steffe's definition of multiplication as a coordination of two composite units does not differ from the description of multiplication as a one-to-many correspondence situation because both definitions highlight the fact that multiplication is about the relation between two variables. He conducted a case study to investigate whether an 8 year old boy was able to carry out the double counting required in replicating one-to-many correspondence situations. Zachary was shown objects that were laid in one-to-many correspondence. For example, the child was presented with six rows, each row having three blocks, but only one row was visible and the other five were hidden. Zachary had to quantify how many blocks there were altogether. That means that he had to represent the hidden objects and count them. Zachary using his index finger traced a segment on the table and then touch the table three times, where his points of contact formed a row. He made six such rows and gave the correct answer, eighteen. In order to quantify Zachary represented the hidden rows and created units of threes that were as real for him as the row of three in his visual field. Steffe's study provides evidence that young children are able to carry out the double counting required for the quantification of one-to-many correspondence situations. Because Steffe's study was a case study conclusions cannot be draw for the population because we do not know to what extent young children are able to quantify one-to-many correspondence problems.

Anghileri (1989) in a case study describes the effort of a young child to calculate with her fingers the number of coins in a 6x3 array that she had seen but which were afterwards hidden.

> "... She now started again with three fingers on her left hand, "One, two, three". She clasped these together saying, "One lot". Now she extended the remaining two fingers of her left hand and one from her right hand saying, "One, two, three ... Two lots". She proceeded in this manner working across both hands, counting in ones all the fingers she extended, "One, two, three ... Six lots". Now she went back to the beginning and successfully counted in ones all her fingers she had extended for grouping, "One, two, three, four, five, six, ..., sixteen, seventeen, eighteen". [She must have memorised all the fingers she had raised.]. In her final attempt, J.F. had kept a tally of the sets she had constructed and then used a unitary counting process in which all eighteen items were represented." (Anghileri, 1989, p.373).

As shown in the above description the child tried to distribute the elements of one unit over the elements of the other. She respected the ratio relation between the two variables (number of units per row) and tried to coordinate three pieces of information: (a) the number of elements in each set, (b) the number of such sets and (c) the quantification procedure to find the total.

Both Steffe's (1994) and Anghileri's (1989) studies reveal that children can think simultaneously of the relation between two variables and are able to carry out the double counting that is required to quantify the situation.

The ability to quantify multiplication problems through action schemas has not only been reported with young children but has also been documented in work with adult foremen and fishermen with little or no school training (Nunes, Schliemann and Carraher, 1993). When foremen, for example, reason about converting the size of a wall from a scale drawing to the life size construction, they really speak about what value on paper corresponds to what value on the construction. Although they solved many multiplication problems using repeated addition they did not confuse repeated addition and multiplication, and continuously referred to the correspondence between the variables during their calculations. For example, an illiterate foreman, was presented with a situation where 9 cm in the blueprint corresponded to 3 metres in the live construction and was asked to find how many metres 15 cm would correspond to. This is the solving procedure he followed:

"Foreman: Nine centimetres, three metres. Right. This is easy.

Interviewer: Why?

F: Because it is just, you just take three centimetres for each metre.

I: How did you come up with this so quickly?

F: Isn't nine equal to three times three? Then, if the wall is three metres, three times three, nine. The other one here (on paper) is fifteen, the wall will be five metres. Because three times five, fifteen, this is an easy one." (Nunes, Schliemann and Carraher, 1993, p. 94-95)

The foreman found the relation, take 3 cm for each metre, and transferred it to the second pair by multiplication. The strength of the connection made by the foremen to the one-to-many correspondence schema is demonstrated by their persistence in using complex scalar reasoning when a functional solution would have been rather simple for the same problem.

In another study with street vendors Nunes, Schliemann and Carraher (1993) asked a 12 year old boy who had only studied up to grade three in school what the price of 10 coconuts would be if the price of each coconut was 35 cruzeiros (Brazilian currency). This was his answer:

"Customer: How much is one coconut?

Boy: Thirty-Five

C: I would like ten. How much is that?

B: Three will be one hundred and five; with three more. That will be two hundred and ten. (Pause) I need four more. That is ... (Pause) three hundred and fifteen ... I think it is three hundred and fifty" (Nunes, Schliemann and Carraher, 1993, p.19)

As suggested by Nunes (1996a) this type of reasoning is best described as "replications" of the original one-to-many correspondence relation, and replication is a procedure that preserves the ratio which is the most salient invariant in multiplicative situations. In contrast, simple addition schemas do not involve two variables in the same fashion.

2.3.2 Quantifying division problems

There is a large body of research showing that young children have their own schemas of actions that enable them to quantify division problems when provided with the material to model the situation. Correa (1995) designed a series of experiments to study whether 5 and 6 year olds would be able to quantify partitive and quotitive division problems before receiving any school instruction. The children were presented with two groups of rabbits and were allowed to play with them for a while. Then they were told that the rabbits were tired and wanted to sleep. The rabbits were piled up in a corner of the table. The children were asked to prepare sweets (small blocks) as treats for the rabbits. The rabbits were piled up in groups of either 2, 3, 4, or 6 while they were sleeping. Because the rabbits were piled up the children could not establish a perceptual correspondence between the sweets and the rabbits. They had to anticipate that each location, in which sweets were placed represented a rabbit and tell the experimenter how many sweets they gave to each rabbit. In quotitive problems the children had to prepare the right number of plates for a picnic giving to each rabbit either 2, 3, 4 or 6 sweets (taking them out of a pile of 8 or 12 sweets) and afterwards tell the experimenter how many rabbits had a share.

The results demonstrated that 5 year olds had great difficulty in solving the quotitive tasks. Only one-fifth of those children was successful. In contrast, almost half of them gave the correct answer in the partitive division situation. At the age of 6 children performed quite well in quotitive division tasks, although their performance in partition was still better.

The analysis of children's strategies in solving the problems showed that successful children used one-to-one correspondence procedures in partition and one-to-many procedures in quotition. Five year olds were more likely to solve the tasks when they had to share sweets between two rabbits. In this case they did the distribution in a one-to-me and one-to-you basis and then counted the sweets in one of the subsets. However, even in this case some children counted the whole lot and gave the wrong response. In quotitive problems the majority of the successful children repeatedly took the number of blocks that corresponded to the size of the divisor from the pile and placed them on different sides of the table to correspond to each rabbit's place, and then counted the number of groups formed. However, even in this case some children, instead of counting the number of groups formed, counted the number of the sweets in the set and gave a wrong answer.

The results of this study suggest that a considerable number of 6 year olds can organize their actions and reason about division well when provided with concrete material. But would the children be equally successful in computing the size of the shared quotas when no material is provided? Bryant (1974) on the basis of a rather different set of experiments has suggested that children are able to reflect on the basis of relations before dealing with absolute values. Would this also be the case with division problems? Correa (1995) provided evidence that children can deal with sharing relations before receiving school instruction. But would the children also be successful with division computational problems?

In order to find an answer to this question Correa (1995) designed a study which aimed to test the development of children's computational solutions and their ability to work out the cardinal values in partitive and quotitive division problems in situations parallel to those that the children had faced in the previous situations.

In partition, children aged 6, 7, 8 and 9 were presented with a certain amount of sweets which were to be shared by the experimenter between a specific number of teddy bears. The children had to work out the number of sweets given to each teddy without manipulating the material.

The results of the study showed that the older children performed significantly better than the younger ones in those tasks. Their performance was considerably affected by the size of the dividend and the divisor. The increase in the size of the numbers caused considerable difficulty for the children, especially the young ones. In order to examine whether the children dealt with partition primarily in terms of relative rather than absolute codes, the percentage of their correct responses in the computational condition was compared with the percentage of correct responses in the relative value condition in which they had to compare the relative size of two shared quotas (see section 2.2.1). It was found that 6 and 7 year old children were able to make a non-computational estimate of the size of the quotient in partitive division tasks, and yet found it difficult to state the exact size of the quotient.

In quotitive division problems, the children had to estimate the number of teddies that could be invited to a picnic in a situation in which they knew the total number of sweets available and the number of sweets to be given to each teddy. Once more it was found that the percentage of 6 and 7 year olds children who worked out the task successfully was below the percentage of children of the same age group who succeeded in the relational problems where they had to reason on the relative number of recipients based on the inverse relation between the divisor and the quotient. A direct comparison between the computational partitive and quotitive tasks followed the general pattern observed in the relative tasks. Generally, children scored better in the partitive than the quotitive tasks. Regarding the strategies that the children employed to reach the answer it was found that they used the repeated addition strategy frequently in partitive division problems, while in quotitive problems they relied mostly on double counting. A possible explanation for this seems to be a need in children to achieve answers by using a procedure that is more closely related to the action implicit in the structure of the problems.

Until now evidence has been presented on children's schemas of action in situations where they have to quantify exhausted division problems. But how would children perform in quantifying division problems when there is a remainder? Burton (1992) presented 7 year old children with 12 partitive and quotitive division problems with and without remainders, with remainders used and not used, when provided with manipulatives that either matched the problems precisely (such as eggs, in a problem involving eggs) or not (such as counters in a problem with cookies).

The results of the study showed that there were more correct responses in the problems without remainders. Children were capable of solving division problems when manipulatives were supplied and they were especially successful if the objects matched the problem situation. Problems with remainders used or not used were more difficult for children but still a significant number of children managed to give the correct answer. No significant differences were found between partitive and quotitive problems although there were more correct responses given in partitive problems. Burton's findings suggests that children can solve a variety of division problems with the help of manipulatives before having any instruction at school.

Children's ability to deal with remainders in a sharing situation has also been examined by Brown (1992). She studied how second grade children, who had not received formal instruction in division, cope with division problems with no remainders, with remainders used or not. The participants were provided with counters that they could manipulate.

In her study quotitive problems were found to be easier (87%) compared with the partitive ones (70%). The children performed nearly the same in quotitive (47%) and partitive problems (45%) when the remainder was not used. The most difficult problems were the partitive ones which had a remainder. Only 43% of the children achieved a correct solution compared with 55% in the corresponding quotitive problems. Regarding the analysis of the strategies that children followed to obtain the answer it was found that grouping with multiples was more frequently used than sharing. The results of her study suggest that children are able to solve a variety of division problems, even with remainders, when provided with concrete materials, well before receiving any instruction on the operation of division.

2.3.3 Comparison studies on children's efficiency to quantify multiplication and division problems

All the above mentioned studies focused on the quantification of either multiplication or division problems. There are also some studies which investigated the solution strategies the same children employ in solving both multiplication and division problems. These studies provide useful information on whether children can solve multiplication and division problems at the same time or whether the understanding of one operation precedes the other.

Children strategies in solving multiplication and division problems were reported in detail in Kouba's (1989) study. Six to 8 year olds were asked to compute a number of multiplication and division problems, mixed with addition and subtraction problems. Manipulatives to model the situation were provided.

The children were presented with problems, both in multiplication, partitive and quotitive division as shown in Table 2.1.

TABLE 2.1

Multiplication and division word problems

Multiplication - Grouping
You are having soup for lunch. There are ___ bowls. If you put ___ crackers in each bowl, how many crackers do you need altogether?
Multiplication - Matching
Pretend you are a squirrel. There are ___ trees. If you find ___ nuts under each tree, how many nuts do you find altogether?
Measurement Division - Grouping
You are making hot chocolate. You have ___ marshmallows to use up. If you put ___ marshmallows in each cup, how many cups do you need?
Measurement Division - Matching
You are making lunch. You have ___ carrots to use up. If you put ___ carrots with each apples, how many apples do you need?
Partitive Division - Grouping
You are having a party. You have ___ cookies and ___ plates. You put all the cookies on the plates so that there is the same number of cookies on each plate. How many cookies are on one plate?
Partitive Division - Matching
You are shopping. You paid ___ pennies altogether for ___ toys. You paid the same number of pennies for each toy. How many pennies did you pay for one toy?

Note. From Kouba (1989), p. 150

Just above a quarter of the first graders, less than half of the second graders and two thirds of the third graders followed the correct strategy in the multiplication problems, while similar results were observed for the division problems. However, it is not mentioned in the study how many children actually gave a correct response apart from using the correct strategy. The strategies that led to a correct solution or could have led to one if the child had not made a counting mistake were characterised as appropriate strategies.

Children's strategies, as characterized by the degree of abstraction, were classified into five categories.

(a) Direct representation: The children processed the information in a sequential way that reflected the structure of the problem. They used physical objects to model the situation and some form of one by one counting to calculate the answer. For example, in the problem: "There are 6 cups and you put 5 marshmallows in each cup. How many marshmallows are there in the cups altogether?" The children might have created 6 groups with 5 objects in each and then counted the total number of objects one by one. They could set out 5 objects to represent the set of the 5 marshmallows and count this set 6 times. Another possibility was to use the objects as tallies. In this case the child counted "one...five"; and set out an object, "six...ten" and set out a second object, and so on until six objects were set out. There was variation in children's counting strategies and some children used counting-on for the first set while others followed a counting-all strategy.

(b) Transitional counting: In this case children calculated the answer by using a counting sequence based on multiples of a factor in the problem. For example in the above problem they counted six groups of five by saying "five, ten, fifteen, twenty, twenty-five, thirty" or "five, ten, fifteen, twenty, twenty-five, twenty- six, twenty-seven, twenty-eight, twenty-nine, thirty".

(c) Additive or subtractive counting: The children clearly identified the use of repeated addition or subtraction to obtain the answer. For example they said: five plus five is ten, ten plus five is fifteen ... twenty five plus five is thirty"

(d) Recalled number facts: In this case the children recalled the appropriate multiplication combination from the times tables. For example the child could have said 5x6=30 or 5x5=25 and 5 more makes 30.

(e) Double counting: This strategy was observed only in division problems. When the children used double counting strategy for quotitive division they did not start by representing the size of the dividend. The dividend was used as an indicator of when to stop the action of forming groups. The children kept a running count of the total number of objects in the groups while also counting out the objects to form groups. When the count of the total reached the size of the dividend the child stopped and reported the numbers of groups formed as an answer. For partitive division double counting was done while sharing out the objects one by one.

The percentage of children using each strategy in multiplication and division problems is shown in the following tables.

TABLE 2.2

Percent of children using each type of solution strategy on multiplication problems

		Type				
Grade	N	Direct Representation	Double Count	Transitional Count	Additive or Subtractive	Recalled Facts
			Grouping			
1	43	25	0	2	2	0
2	35	9	0	6	24	3
3	50	8	0	10	8	40
			Matching			
1	43	23	0	4	0	0
2	35	18	0	12	18	0
3	50	8	0	4	10	34

Note. From Kouba (1989), p.154

TABLE 2.3

Percent of children using each type of solution strategy on measurement division problems

		Type				
Grade	N	Direct Representation	Double Count	Transitional Count	Additive or Subtractive	Recalled Facts
			Grouping			
1	43	25	0	0	0	2
2	35	26	6	6	0	6
3	50	12	4	10	0	38
			Matching			
1	43	25	0	0	0	0
2	35	35	17	3	3	0
3	50	16	4	6	2	32

Note. From Kouba (1989), p.154

TABLE 2.4

Percent of children using each type of solution strategy on partitive division problems

		Type				
Grade	N	Direct Representation	Double Count	Transitional Count	Additive or Subtractive	Recalled Facts
			Grouping			
1	43	25	0	0	0	0
2	35	30	0	3	9	0
3	50	12	4	8	2	34
			Matching			
1	43	11	0	0	0	0
2	35	23	0	3	0	3
3	50	4	2	8	0	32

Note. From Kouba (1989), p.154

Children's strategies varied according to the problem, depending on whether it was a multiplication or a division one. Multiplication problems were mainly solved by direct representation or by number facts, while measurement/quotitive problems were solved by double counting, recalled facts and by assembling equivalent groups until the pile of objects representing the dividend was exhausted. For partitive division children solved the problems by dealing out objects and by trial-and-error grouping.

What is not stated in Kouba's research report is whether the children participating in the study had received any instruction in school which was likely to affect the strategies that they employed to solve the problem.

Kouba offers a very detailed presentation of the *computational strategies* that young children employed to quantify the multiplication and division problems when they were provided with manipulatives. Kouba however did not distinguish between the way the problem is computed and the way the problem is represented. It has to be made clear that the fact that the children used different solution strategies to compute the multiplication problems does not mean that they represented the problems in a different way. In all the computing solutions the children came up with in multiplication problems the underlining schema was that of the one-to-many correspondence. The underlining schema of action in partitive division was that of sharing no matter whether the children dealt the objects or grouped them by trial and error in equal quotas. In quotitive division the underlining schema of action was that of forming equal quotas.

The available evidence suggests that children are able to solve a wide range of multiplication and division problems early on. The question that follows is how much more difficult are the multiplication and division problems compared with the addition and subtraction problems.

Evidence of British children's ability to solve word problems is provided by the Durham Project (Aubrey, 1997) which aimed to establish what mathematical knowledge pre-school children bring into class. One hundred and sixty eight children were given simple addition, subtraction, multiplication and division problems. They were also provided with concrete material to model the situation. It was found that in addition 51% gained a score of 4 or 5 out of 5, 39% scored 1 to 3 and 10% scored nil. In subtraction 63% scored 4 or 5 out of 5, 24% scored between 1-3 and 13% scored nil. Children did even better in multiplication: 66% scored 4 or 5 out of 5, 13% scored 1 and 21% scored nil. Division yielded even higher scores: 73% scored 4 or 5 out of 5, 25% scored 1 to 3, and 2% scored nil.

Aubrey (1997) suggests that children have a surprising range and diversity of competences that the existing school curriculum may not be able to support and develop further. Children were able to deal not only with additive, but also with multiplicative problems that are only introduced at the end of grade 2, with great ease. The report of the project however does not give any information about the strategies that children followed.

The comparative difficulty of multiplication and division problems versus addition and subtraction was also studied by Carpenter, Ansell, Franke, Fennema and Weisbeck (1993) who gave word problems on all four operations to kindergarden pupils. The children who took part in their study had spent a year in the nursery solving a variety of basic word problems involving all the basic operations, although they had never received formal instructions on how to solve them. Their aim was: (a) to investigate the problem solving processes that kindergarden children use in solving all the basic operation word problems and (b) given the claimed difficulty of multiplication and division problems, to see if the children can solve them when given appropriate exposure, or whether these problems are beyond their abilities. In their tasks they included addition and subtraction situations (join, compare and separate problems), multiplication (correspondence problems), division (partitive and quotitive), multi-step problems (division with a remainder, multiplication with subtraction) and non-routine problems as shown in Table 2.5 .

The 70 children in the study were interviewed and were provided with counters, paper and pencils that they could use to solve the problems. Responses were coded both in terms of the strategy used and whether the answer given was correct. A child was coded as following a valid strategy when s/he used a strategy that would have resulted in a correct response when there was no counting error. They were classified as following (a) "direct modelling" strategy when they used counters, tally marks and fingers to model directly the actions or relations described in the situation, (b) "counting strategy" when they count up or back from a given number or skip counted to calculate the answer, (c) "derived facts" when they recalled number facts to quantify the answer, (d) "others"when they used counters in a way that did not directly represent the situation and (e) "uncodable" if the answer was correct but the child's actions could not be identified. The computational strategies that the children used to quantify the multiplication problems suggest that they reasoned on the basis of a one-to-many correspondence schema.

TABLE 2.5

Interview problems

Problem Type	Order given	Problem
Addition and Subtraction		
Separate (result unknown)	1	Paco had 13 cookies. He ate 6 of them. How many cookies does Paco have left?
Join (Change unknown)	3	Carla has 7 dollars. How many more dollars does she have to earn so that she will have 11 dollars to but a puppy?
Compare	5	James has 12 balloons. Amy has 7 balloons. How many more balloons does James have than Amy?
Multiplication and Division		
Multiplication	2	Robin has 3 packages of gum. There are 6 pieces of gum in each package. How many pieces of gum does Robin have altogether?
Measurement Division	4	Tad had 15 guppies. He put 3 guppies in each jar. How many jars did Tad put guppies in?
Partitive Division	6	Mr Gomez put 20 cupcakes into 4 boxes so that there were the same number of cupcakes in each box. How many cupcakes did Mr Gomez put in each box?
Multistep and nonroutine		
Division with remainder	9	19 children are going to the circus. 5 children can ride in each car. How many cars will be needed to get all 19 children to the circus?
Multistep	8	Maggie had 3 packages of cupcakes. There were 4 cupcakes in each package. She ate 5 cupcakes. How many are left?
Nonroutine	7	19 children are taking a minibus to the zoo. They will have to sit either 2 or 3 to a seat. The bus has 7 seats. How many children will have to sit three to a seat and how many can sit two to a seat?

Note. From Carpenter et al. (1993), p.434

A quantitative analysis of the strategies that the children followed showed that the majority of strategies used can be characterized as directly representing or modelling the actions of the relations described in the problem.

The kindergarden children in this study showed remarkable success in solving word problems as shown in Table 5.6.

TABLE 5.6

Number of children correctly solving each problem and the number and kind of valid strategies

Problem	Number Correct	Valid Strategies	Direct Modelling	Counting	Derived Facts	Other	Uncodable
Separate result unknown	51	62	54	5	2	0	1
Join change unknown	52	56	39	12	1	1	3
Compare	17	50	34	7	3	6	0
Multiplication	50	60	46	14	0	0	0
Measurement division	50	51	50	1	0	0	0
Partitive division	49	49	39	1	1	6	2
Division remainder	45	45	42	1	2	0	0
Multistep	45	47	44	0	3	0	0
Nonroutine	36	41	40	1	0	0	0

Note. From Carpenter et al. (1993), p.435.

Surprisingly multiplication and division problems were not found to be more difficult compared with addition and subtraction problems for the children.

According to Carpenter et al. (1993) the results of this study suggest that children can solve a wider range of problems, including multiplication and division. This study also provides evidence that young children can learn to solve a broad range of problems by directly modelling the actions and relations in the problem. However, Carpenter et al (1993) pointed out that they cannot claim that all the children would solve these problems in the same way because instruction did encourage the use of direct modelling to solve the problems and it is possible that if instruction had a different focus, the strategies adopted by them might have been different.

Carpenters et al's (1993) findings suggest that children's problem solving abilities have been seriously underestimated and that children can solve a variety of problems quite early. They proposed that children can be helped to develop problem solving abilities if adults build upon and extend their intuitive modelling skills.

There is a big discrepancy in the level of success of the children that took part in Carpenter et al's (1993) and the children in Correa's (1995) and Kouba's (1989) study. Although the children in Carpenter et al's investigation were in nursery class they were doing much better than the first grade children in Kouba's (1989) study and they were as successful as the third graders. Although Carpenter et al's (1993) study was not an experimental one, we cannot overlook the fact that their children had some experience with word problem solving. The discrepancy between Correa's and Kouba study can be explained by the difference in the administration of the problems.

Carpenter et al's findings support the idea that children have their own schemas of action that enable them to deal with both multiplication and division problems at a young age. Based on these schemas of action the children will later on build their knowledge on multiplication and division as operations.

2.3.4 Is the understanding of the schema of one-to-many correspondence the key for understanding all multiplicative relations?

Quantifying proportional and Cartesian problems

The aforementioned studies have shown that children can solve a variety of multiplication problems as soon as they recognize the one-to-many correspondence relation between the two variables.

For example, "One shelf has 4 books. How many books are held in 3 shelves?". In this example the child can proceed by scalar reasoning: "1 shelf 4 books, 2 shelves, 8 books, 3 shelves 12 books".

There are, though, problems where this type of reasoning would not lead to the right solution. Consider, for example, the problem: "3 kilograms of coffee cost £15. What is the price of 7 kilograms?". This proportional problem is clearly a one-to-many correspondence problem, but *a scalar solution* is not appropriate, because 7 is not in the replication (3 kg cost £15, 6 kg cost £30, 9 kg cost £45). A problem like the above needs *a functional solution*. There is a constant ratio relation between weight and price (1 kg cost £5). That means that the price per kilogram must be found first in order to find the price of 7 kilograms. A functional solution requires the establishment of a multiplicative relation not only *within* the variables but also *across* the corresponding variables. Although, there is no evidence on young children's performance on this type of problems it is unlikely that young children would find them easy to quantify, especially when they involve complex computational procedures.

What has been well documented is that the understanding of one-to-many correspondence does not necessarily help the children to distinguish the situations where they have to establish a proportional relation between the variables. As a result the children carry on applying scalar solutions that can be labouriously intensive to problems that could have been more easily quantified by a functional solution. A genuine multiplicative reasoning requires competence both in scalar and functional reasoning.

In the onion soup problem presented by Hart (1984) 13 to 15 year olds were given the recipe for a soup for 8 people and had to find the quantity of the ingredients they had to use to make the onion soup for 4 and 6 people. She found that the children had no difficulty in finding how much water and chicken cubes they would need to make the soup for 4 people. More than 90% of the participants obtained the correct solution by halving the original amounts. This was a scalar solution based on inverse replications. When the children had to estimate the amount of ingredients needed for 6 people they also applied replications. Because 6 people were equal to 4 people plus 2 the children went on halving the amount they needed for 4 people and added the two results together. Again this solution was based on inverse replications and the majority of the children were successful in the task. The preference of scalar solutions was surprising because the same problem could have been solved easily by a functional solution where the amount of each ingredient would have been a function of the number of persons.

Vergnaud (1988) gave 11 to 15 year olds a correspondence problem "The consumption of my car is 7.5 litres of gas for 100 km; how much gas will I use for a vacation trip of 6580 km?". He found that although the functional solution required a simple arithmetic procedure even the older children preferred a scalar solution. Similar findings on the prevalence of scalar over functional solution have been reported to a number of other studies (Karplus, Pulos and Stage, 1983; Kaput and Maxwell-West, 1994; Nunes, Schliemann and Carraher, 1993).

There is also another type of multiplication problem which is regarded as being rather difficult for young children. This type of problems is known as the Cartesian product problem (Anghileri, 1989; Brown, 1981; Hervey, 1966; Mulligan and Mitchelmore, 1997; Nesher, 1992). An example of a Cartesian problem is: "Anna has 4 skirts and 3 shirts. How many different combinations of outfits can she make?". Nesher (1988) suggested that Cartesian problems are complex because a problem like the above consists of three strings: the first two refer to two independent sets of objects (shirts and skirts) and the third one, the outfits, which is the product of a cross multiplication between each of the skirts and each of the shirts. Moreover, in a Cartesian problem the one-to-many correspondence between each skirt and the shirt is not clearly indicated in the verbal expression of the problem and it is up to the child to establish this relation.

In order to examine if children treat Cartesian problems as one-to-many correspondence situations and what strategies they employ to solve them, Bryant, Morgado and Nunes (1993) set up a study to investigate how children come to master them. Children aged 8 and 9 from Oxford were given four multiplication problems, two one-to-many correspondence problems and two Cartesian product problems so that they could contrast their level of difficulty. The children were also provided with the materials that they could use to quantify the problems. In one Cartesian product problem the children were given miniatures of the different shorts and t-shirts that they could manipulate to solve the problem. It was expected that the procedure the children would follow to solve the problem would reveal what the children think about the situation. The problem with this design, though, was that the children could just imitate the problem situation, combine the shorts with the t-shirts and count the different outfits. In order to make sure the children would solve the problem by reasoning mathematically, they were randomly split into two groups. In one group the children were provided with all the materials described in the problem. The children had to think how they were going to organize their actions to form the number of outfits. In the second group the children were only presented with a subset of the materials which could be used to create a model of thinking that could help them to solve the problem.

It was expected that the simple one-to-many correspondence problems would be easier for the children no matter whether they had all the materials or just a sample of them. The children assigned to the group presented with all the materials were expected to be more successful overall than the children presented with a sample of it. It was also assumed that in the Cartesian product problems the children would benefit more when they were given all the materials.

The results of the study verified the above expectations. Most of the children gave the correct solution when presented with all the materials in the one-to-many correspondence problems, but only half of the 8 year olds could respond correctly when presented with a sample of the materials. Regarding the Cartesian product problems just above half of the 9 year olds and a few of the 8 year olds gave the right response after manipulating all the materials. However, none of the 8 year olds and less than half of the 9 year olds could solve the situation given only a sample of the materials.

The strategies that the successful children followed while manipulating the materials showed that they recognize the need to establish a one-to-many correspondence between the number of shorts and the t-shirts. However, because Cartesian problems are complex, only 9 year olds recognized the implicit one-to-many correspondence.

Bryant, Morgado and Nunes's (1993) study reveals that children are successful in Cartesian problems only if they identify the correspondence that is implied between the two sets.

The difficulty of Cartesian problems was also highlighted in Mulligan and Mitchelmore (1997) study. On four occasions, they gave grades 2 and 3 children who had not previously been instructed on multiplication and division, a number of problems. These problems differed in semantic structure. The purpose was to assess the relative difficulty of the problems and the strategies the children used to solve them. The children were read the following different types of multiplication problems: (a) equivalent groups (e.g., 2 tables, each with 4 children), (b) multiplicative comparison (e.g., 3 times as many boys as girls), rectangular arrays (e.g., 3 rows of 4 children) and Cartesian products (e.g., the number of possible boy girls pairs).

Despite the use of manipulative materials Cartesian problems were the most difficult, averaging 1% correct responses over interviews 1 - 3 to 14% at interview 4. In contrast, in the equivalent-groups, the success rate for the rate and array problems increased from the average of 45% at interview 1 to 86% at interview 4. The authors argue that the Cartesian problems are hard because the recognition of the one-to-many correspondence between the sets is not straightforward.

The research findings suggest that the difficulty of Cartesian problems is attributed to the semantic structure of the problem situation which does not reveal the one-to-many correspondence relation between the two sets. It could also be the case that in order to solve the Cartesian problems the development of an entirely different schema of action is required. However, there is no research evidence to support this.

Commutativity in multiplication

The work of Piaget, Kaufmann and Bourquin (1977) and Frydman (1990) that was discussed previously showed that the children had a very poor understanding of the principle of commutativity in multiplication. The children had great difficulty in the common multiplies tasks that would have been easily solved if they had understood the idea of commutativity.

This finding has been verified by a number of other studies. Pettito and Ginsburg (1982) asked unschooled adults in Africa to compute a number of exercises. Some of these exercises examined their understanding of commutativity. They were asked to compute 10x7 and immediately after 7x10 and 6x100 and 100x6. The results showed that the participants were accurate in their calculations but they rarely used commutativity in multiplication. When the adults were asked whether or not the two problems in the commuted pairs were the same, only 38% agreed that 10x7 and 7x10 were the same and 68% agreed for the 6x100 and 100x6 pair. But even the ones who agreed about the equality of the two pairs did not always solve the two problems by means of commutativity. The study does not report the participants reasons for judging commutativity problems to be the same of different. Thus, it provides a limited insight into the way adults come to understand the principle of commutativity. In contrast, US college students were shown to have a good understanding of commutativity in multiplication. About 93% of them used this principle to compute the problems.

Nunes and Bryant (1995) proposed that children's difficulty in understanding the property of commutativity in multiplication can be explained by the problem situation. They suggested that correspondence problems might be the easiest to quantify but the situation itself might not help the children to understand the mathematical principle of commutativity because it raises conceptual difficulties for the children. In a computational level 30x5 is the same as 5x30, but buying 30 oranges for 5 pence each is a different problem than buying 5 oranges for 30 pence each, because the values of the measures must be changed. It is more likely that the children would understand commutativity better in situations where the two variables have a spatial relation to each other. For example, a chocolate bar that is 4 squares long and 2 squares wide is of the same size as one that is 2 squares long and 4 wide. The values of the measure are not changed and the children can realize their equality by rotating the bars of chocolate. Nunes and Bryant hypothesized that commutativity can be more easily understood in spatial rearrangement problems where the children do rotation than in one-to-many correspondence problems where they have to rearrange the size of the corresponding sets.

In one of the studies they carried out the children were given the sum of the total in one multiplication problem, for example 12 chairs x 8 rows = 96 chairs, and then they were presented with a similar problem where there were 8 chairs in 12 rows. If the children could infer the total number of chairs in the second problem using the previous information and not the calculator then they had an understanding of commutativity. Two types of problems were used, rotation and rearrangement problems in order to test whether the understanding of commutativity varied as a result of the problem situation. These problems were mixed with non-commutative problems to test whether the children had developed a set way to respond in all the problems or if they could recognise the situations where they could apply the commutativity rule. The 10 year olds participants of the study were assigned in 4 groups. Two groups were given rotation problems and the other two rearrangement problems. In one of each of these groups the children were presented with small numbers and in the other with large numbers.

The results revealed that the only problems where the children decided not to use the calculator to compute the sum were the commutativity problems. Thus, they could distinguish between the commutative and the non-commutative problems. There was a clear difference between the rotation and the rearrangement problems. The children performed significantly better in the rotation problems. The children could infer the size of the set without using the calculator. More than half of the children quantified the spatial rearrangment commutativity problems (56.7%) with small numbers by means of the cues they were given compared with 39.5% of the one-to-many correspondence rearrangement problems. Another result was that the children were more likely to use commutativity with small than with large numbers. It seems that as soon as the children see large numbers they think that the problem is a hard one and turn immediately to the calculator without considering the possibility of avoiding it.

Nunes and Bryant's study demonstrates that children's understanding of mathematical properties can vary across situations. Although one-to-many correspondence problems are the easiest one to be quantified they do not provide the best context for the understanding of the principle of commutativity.

2.3.5 Quantifying Fractions

Evidence has been presented on children's ability to quantify multiplication and division problems that involved discontinuous quantities. There is also a large body of research on children's ability to quantify fractions, while there is no evidence on young children's ability to solve multiplication problems with continuous quantities. The aim of this section is to present the evidence available on young children's ability to quantify fractions.

2.3.5.1 The role of half in the division of continuous quantities

In contrast to Piaget, Inhelder and Szeminska's (1960) assumption that part-whole relations are the starting point for understanding fractions, Bryant (1974) suggested that children, even before being able to understand part-whole relations, are still able to understand some more elementary relations when they first encounter continuous quantities in fractions: the part-part relations. He argued that if a quantity is divided into two parts young children are able to judge which part is bigger, smaller or whether they are equal. He suggested that the understanding of these first logical relations can be used as the beginning of quantifying fractions. Because these relations are used in situations where a whole is divided into two parts, he suggests that "half" plays a special role in the origin of quantification of fractions, as the half boundary defines whether the two parts are equal or one is bigger than the other. In order to check for the effect of the half boundary in children's equivalence judgements Spinillo and Bryant (1991) asked children aged 5 to 8 years old to look at a picture that represented a box with a particular proportion of white and blue bricks inside. Then the children were shown two boxes only one of which had the same proportion of white to blue bricks. The picture was actually smaller than the boxes and the bricks in the box had been rearranged. In the picture the stripe of the blue blocks was horizontal while in the box it was vertical or vice versa. The children had to find which box had the same proportion of white-to-blue stripes as the picture. The perceptual modifications were done to ensure that the children would not carry out the comparison on a perceptual box-to-picture match. They predicted that the children would perform better in those tasks where they could use the half boundary or cross the half boundary, than in those where the half boundary could not be used.

They found that the majority of the children performed significantly better in the tasks where they could use the half boundary to make their judgements, which suggests that the half boundary might represent the first step in children's use of relations to quantify. Less than half of the 5 year olds, half of the 6 year olds and the majority of the 7 and 8 year olds performed significantly above chance level in the tasks where the half boundary could be used as a reference to match the picture with one of the boxes.

These findings suggest that the concept of half represents the first step in children's use of relations to quantify fractions.

2.3.5.2 Comparison of children's ability to share discontinuous and continuous quantities

There is a growing body of research that aims to compare the relative difficulty of sharing discontinuous and continuous quantities.

Hiebert and Tonnessen (1978), studied the development of the fraction concept within two physical contexts: continuous and discrete quantities. The sample of their study was quite small comprising 9 children aged between 5;4 to 8 years. The children were presented with an area (clay pie) and a length task (piece of licorice), which they were required to divide into halves, thirds and quarters. They were also asked to share penny candies (discontinuous quantities) among two, three and four recipients in order to assess the relative difficulty of the two types of quantities.

The results of the study revealed that the discrete objects task was considerably easier than the continuous cases of length and area tasks. Six children were successful in the candies task, but only two succeeded in both area and length tasks. The factors responsible for these results could be that for discrete quantities children can solve the task without anticipating the final solution beforehand. Discrete quantities tasks were also solvable on the basis of number strategies, but the continuous quantities tasks of area and length required a subdivision into equal groups, which requires familiarity with area metric properties and certain measurement skills.

Regarding the developmental sequence in understanding fractional numbers it was found that for area problems halves and fourths were successfully constructed by some children but thirds were not. For the length task, fourths were not found to be easier than thirds. No order of difficulty sequence was observed in the discrete quantities tasks and the children solved the situation following a one-to-one partitioning procedure.

The overall results suggest that the sharing of continuous quantities is harder than the sharing of discontinuous quantities and that children employ different strategies with each quantity.

The relative difficulty of sharing discontinuous and continuous quantities was also studied by Miller (1984). He asked children aged 3 to 9 to share a number of different kinds of material among two, three and four turtles. The turtles were supposed to enjoy a snack consisting of materials emphasizing number (candies), length (strips of clay spaghetti), area (clay squares of fudge) and volume (glasses of kool-aid) as shown in the following picture.

PICTURE 2.1

Materials used to investigate spontaneous measurement procedures.

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Note. From Miller, (1984), p.198.

In the above scene an extra plate was set for dinner, with the explanation that another turtle was invited but could not make it. Therefore, the children had to help the two turtles to share the spare portion of snack evenly. For doing this the children were provided with various measurement devices (rulers and cups of different sizes) that they could use.

The analysis of the strategies used to share the candies showed that the vast majority of children in all age groups employed the strategy of distributive counting, in which the candies were distributed one at a time among the turtles, in many cases accompanied by statements such as "one for you and one for you ...". For sharing the glass of drink (volume) most children visually compared the levels of fluid to determine relative volume. More complicated methods like measuring the height of the column, or using a unit from a measuring cup were employed by a small percentage of older children. Only a small number of preschoolers used non-quantitative procedures to share the materials.

In sharing the stripes of spaghetti (length) children exhibited different solution strategies. Most preschooler cut the spaghetti into arbitrary numbers of pieces. They took care to make sure that the same number of pieces was distributed, although the pieces were often of different sizes. The use of strategies in which children cut the material directly into fractions of approximately equal size increased with age. In this case the children either estimated the size of each piece and then cut it into fractions or folded the piece into halves or thirds. The choice of this strategy indicates that for those children equality involves considering the size as well as the number of the shared pieces. The strategies observed for the sharing of the fudge cake (area) were similar to those in the length problem. Again, preschoolers had the tendency to cut the cake into arbitrary pieces, whereas strategies involving the use of units of constant size were limited to older children.

Further evidence on children's ability to share discontinuous and continuous quantities was provided by Hunting and Sharpley (1988a, 1988b). Their aim was to explore what kinds of behaviour preschoolers display when solving partitioning problems with discrete and continuous quantities, what mental processes seem to govern such behaviour and what they understand about the fractions ½ 1/3 and 1/4. The sample of their study consisted of 206 children, mean age 4.5. In the first set of tasks the children were required to share different types of material between two, three and four dolls, in a way that the material was all used up and each doll had an even share. The children had to share skipping ropes (continuous quantities) and crackers (discrete quantities). The second set of problems required the children to represent the fraction ½, and if successful fractions 1/3 and 1/4, by halving a sausage (continuous quantity) and putting half of a number of cards (discontinuous quantities) in an envelope.

The results of sharing the rope between 2 dolls showed that most of the children cut the string just once but the length difference between the pieces was in many cases 60mm or less. When asked if they considered their dolls to be happy with their share the majority of the children agreed and less than half of them employed a checking behaviour. When the rope had to be shared among 3 dolls, one third of the children made one cut, the other one third made two cuts and the remaining made three, four or more cuts. Various procedures were observed in the behaviour of children who made more than three cuts, such as selecting three pieces to give to the 3 dolls, measuring the second and the third piece off against the first piece, adjusting by trimming as the pieces were being cut etc. Only four children were observed to fold the rope before cutting. In answer to the question as to whether the dolls were happy with their shares 190 children answered "yes". Checking behaviour was observed only in one fourth of the cases.

In sharing crackers among 3 dolls 44% of the children showed a systematic procedures by sharing out the crackers one by one. Sixteen percent of the children adopted a one-to-many correspondence procedure for at least one cycle and 10% of them made equal shares as a result of non-systematic methods. The remainder of the children followed non-systematic procedures and ended up with unequal shares. When the children were asked how they could be confident about the fairness of their sharing the most common checking methods were numerical justification without overt counting, point counting of the piles of crackers and visual comparison. When the children had to re-apportion the shares of the 3 dolls for a fourth doll 94 children employed successful methods.

Regarding the second set of problems where the children had to cut a sausage into halves, thirds and quarters, 73 children cut the sausage near the mid-point when halving it and 118 executed a succession of cuts. Of the successful children only 10 managed to cut the sausage in quarters and none of them was successful in cutting it into thirds. Before executing the cutting the children were asked how many pieces and how many cuts they were expected to make. It was found that the vast majority of the children did not have an anticipatory schema.

In the swap-cards tasks where the children were asked to put half of the 12 cards into an envelope, a substantial number of children (35%) placed all of them in the envelope provided. Only 23 children (11%) responded appropriately. However, the vast number of children's responses cluster around six which suggests a qualitative or approximate notion of half. None of the successful children was able to divide the cards into quarters.

The overall results of Miller's (1984) and Hunting and Sharpley (1988a, 1988b) study suggest that children perform better in the context of discrete rather than continuous quantities. Children's difficulty with fractions can be attributed to the lack of anticipatory schema and checking procedures, which are important for efficient problem solving. This could be due either to the complete lack of schemas or to the fact that the problems were quite dissimilar to children's experiences therefore, they could not draw on prior knowledge.

It was found that cutting continuous quantities into halves is easier than cutting them into quarters and thirds. It is possible though, that children had great difficulty with quarters and thirds due to lack of understanding of the fractional language used. Many children in the study showed a qualitative conception of half which preceded the understanding of the other fractions. This finding strengthens Spinillo and Bryant's (1991) assumption that the concept of half plays a significant role in the quantification of fractions.

Despite the methodological problems these studies show that the potentiality for learning fractions is present in children at an early age. Hunting and Sharpley suggested that instructional activities which encourage exhaustive distribution, together with discussions about the merits of systematic sharing procedures for forming equal shares, would prepare children for further learning. They suggest that the fraction vocabulary (halves, thirds, fourths) should be introduced when the children become proficient in allocating shares. It is not, however, clear whether the children should start from the sharing of discontinuous quantities before being introduced to the continuous ones.

2.3.5.3 Summary

The results of the aforementioned studies on children's ability to quantify multiplication and division problems involving discontinuous quantities are rather encouraging. All the studies agree that the vast majority of children are able to quantify a variety of multiplication and division problems by employing their schemas of action before the introduction of multiplication and division as an operation at school. This understanding is a long process and develops gradually with age. The children are initially able to reason in the situation on the basis of relations and later on they are able to deal with absolute values.

The strategies that children adopt to solve multiplication problems reveal that they reason in the situation on the basis of the one-to-many correspondence schema. Because multiplication involves a constant relation of one-to-many correspondence between two sets, problems in which this correspondence is explicit can be solved relatively easily. However, the understanding of one-to-many correspondence is not the key for solving all multiplication problems. For example, proportional problems often require a functional solution and are difficult for the children. Cartesian problems are also found to be hard, either because the one-to-many correspondence relation between the values is not recognized in the structure of the problem or because they require an entirely different schema of action to be solved. One-to-many correspondence situation do not teach the children much about a significant property of multiplication, the commutativity rule. The children might be efficient in quantifying correspondence problems, but have no understanding of the axb=bxa relation between the corresponding variables.

The research evidence suggests that the children have no difficulty in quantifying correspondence problems. Setting sets into correspondence is a solution strategy that can lead to successful quantification. What needs to be explored further is whether the children can use their correspondence schema of action to reflect on multiplicative situations. Would for example the children use the concept of ratio to order the size of different corresponding sets even in situations that do not provide sources for quantification, like the ones with continuous quantities?

Similarly, there is evidence that 6 year olds are doing well when they quantify division problems that involve the sharing of discrete quantities (Correa, 1995) There is also evidence (Sophian et al., 1997) suggesting that by the age of 7 children can order the size of different quotas when the shared quantity is continuous. There is not, though, a study comparing children's understanding of sharing relations across discontinuous and continuous quantities. If there are schemas of action to support their reasoning there should not be a difference between the two types of quantities.

Alternative views about the origin(s) of multiplication and division

In the previous section the theories that sought the origin of children's understanding of multiplication and division in children's schemas of action were presented. There is an alternative hypothesis according to which the origin of multiplication and division is to be found in other operations, that is in addition and subtraction. There is a large consensus that the origin of division is to be sought in the context of sharing situations, but there is less consensus about the origin of multiplication. For example Dickson, Brown and Gibson (1984) suggested that there is no schema of action related to multiplication; Fischbein and his colleagues (1985) proposed that multiplication originates in the understanding of repeated addition situations. The aim of this section is to present the alternative views and the relative research evidence.

2.4 Dickson, Brown, and Gibson's hypothesis: Multiplication is not related to any schema of action

Dickson, Brown and Gibson (1984) suggested that sharing is the schema of action related to the origin of division but that there is no schema of action related to multiplication. They also proposed that the understanding of division might precede the understanding of multiplication. These conclusions were drawn from Brown's (1981) study where she tested the relative difficulty of the four basic operations. She asked 11 year old children to make up stories to match various arithmetic expressions in all the four operations. It was found that the percentage of success in multiplication was the lowest, 45% when small numbers were involved and 31% when large numbers were involved. Multiplication was also found to be the hardest operation when the children were asked to identify the operation that best fitted a given problem: Addition was easily recognized by 88% of the children, subtraction by 67%, division by 63% and multiplication only by 53%.

This is how Dickson, Brown and Gibson (1984) explain the relative difficulty of multiplication:

> "... whereas "adding", "taking away" and "sharing" are concrete actions easily visualised, "times" has no such obvious active reference. This may explain why children appear to have particular difficulty with attaching any meaning to the operation of multiplication ..." (p.233).

The children who took part in Dickson, Brown, and Gibson's study were 11 years old and had received a significant amount of instruction in all the operations. The study highlights the difficulties that the children had with the basic operations in the context of word problems, and it does not give an answer to the origins of the concept of multiplication and division. Dickson, Brown, and Gibson's proposal that multiplication is not related to any schema of action contradicts a vast number of previous studies

(Burton, 1992; Carpenter, et. al, 1993; Frydman, 1990; Piaget, 1965) which have shown that children understand the invariants that underline the operation of multiplication early on and have the schemas of action to solve a number of multiplication problems at a young age. It is likely that the children's difficulty in constructing multiplication and division situations in Brown's study represents the failure of the school to teach the operations in a meaningful context. As a result the children are not able to produce meaningful multiplicative situations when presented with the symbolic expression 9x3.

2.5 Fischbein, Deri, Nello and Marino's theory: the role of addition and subtraction in the understanding of multiplication and division

Fischbein, Deri, Nello and Marino (1985) proposed the theory of primitive intuitive models about the origins of multiplication and division. According to their theory *"Each fundamental operation of arithmetic generally remains linked to an implicit, unconscious, and primitive intuitive model. Identification of the operation needed to solve a problem with two items of numerical data takes place not directly but as mediated by the model"* (Fischbein et al., 1985, p.4).

Fischbein et al. (1985) suggested that the concept of *multiplication* is intuitively attached to a *repeated addition model*. Thus, 3x5 can be viewed as 5+5+5. The basis for their hypothesis was the finding that children have difficulty in solving problems in which the multiplier is a decimal or a fraction and hence not readily interpreted as "lots of" or "times". In the repeated addition interpretation of multiplication, the multiplier (or operator) must be a whole number, the product must be bigger than the multiplicand (or operand), while the multiplicand can be any positive number. "One cannot intuitively conceive of taking a quantity 0.63 times or 3/7 times whereas one can easily conceive 3 times 0.63 as 0.63+0.63+0.63, even if one cannot perform the operations. Because the operator is always a whole number, multiplication necessarily "makes bigger" (p.6). Fischbein et al. (1985) argued that if the numerical data of the problem do not fit the constraints of the model, the children may not choose the correct operation and the solution effort may be diverted or blocked.

They also proposed that the primitive intuitive model for partitive division is sharing and for quotitive division is repeated subtraction. The partitive division model is where an object or a collection of objects is divided into a number of equal fragments or subcollections, originated from sharing. The constraints are that the dividend must be bigger than the divisor; the divisor must be a whole number and the quotient must be smaller than the dividend. Quotitive division situations are where the object is to determine how many times a given quantity is contained in a larger one and can be seen as *repeated subtraction* if the quotient is a whole number. The constraint is that the dividend must be larger than the divisor.

In order to verify their hypothesis on primitive intuitive models they carried out a study that involved 628 pupils, aged between 10 to 13, from 13 different schools in Italy. The children were given a list of 42 problems, 12 multiplication and 14 division problems, mixed with addition and subtraction problems. They were not asked to perform the actual calculation but only to indicate the operation used to solve the problem.

All the 12 multiplication problems referred to situations where the concept of multiplication as repeated addition was applicable. What varied across the problems was the nature of the number used. In two problems the multiplier and the multiplicand were whole numbers, while in the other problems either the multiplier or the multiplicand were expressed in decimals. They expected that the problems which violated the constraint that the multiplier must be a whole number would be hard for the children.

There were five partitive problems which violated the intuitive rule that the dividend has to be larger than the divisor. In two of those problems the dividend and the divisor were both whole numbers and in the remaining three the dividend was a decimal number. There was also one problem where the divisor was a decimal. The problems where the divisor was bigger than the dividend, where the divisor or the dividend was a decimal, were expected to be hard for the children.

In two of the quotitive division situations, both the dividend and the divisor were whole numbers with the dividend being bigger than the divisor. In the other three quotitive problems the divisor was a decimal number and these were predicted to be difficult for the children.

Almost all the children were successful in the situation involving multiplication where both numbers were integers according to the constraints of the model. Children's performance was affected by the presence of the decimal numbers and tended to vary according to whether they were used as multipliers or multiplicands. Children tended to perform better with a decimal multiplicand than with a decimal multiplier. Nevertheless, the effect of the decimal tended to diminish when the decimal was familiar to the children or when the whole part of it was larger than the fractional part.

In division problems children performed better in the partitive than in the quotitive problems. The pupils performed at ceiling level with whole numbers when the dividend was bigger than the divisor. Performance level dropped significantly when the divisor was bigger than the dividend and in this case they found it easier to deal with problems that had a decimal dividend and a smaller whole number divisor. The children also performed poorly in the partitive problems where the divisor was a decimal number. Once again the children did better in quotitive problems that involved whole numbers according to the constraints of the model. Their performance was lower in the situations where the divisor was a decimal number, especially when it was less than 1.

The results of their study with division led them to the conclusion that *"there is only one intuitive primitive model for division problems* - the partitive model. By instruction only, pupils acquire a second intuitive model - the quotitive model*"* (p.14). This conclusion was justified by children's difficulty in recognising division as the correct operation to solve quotitive division problems.

Fischbein et al's (1985) research on children's understanding of multiplication and division has a number of methodological weaknesses. The problems that were classified as partitive did not always correspond to what is defined as a partitive problem by Fischbein et al (1985). For example, the following problem: "I spent 1500 lire for 3 kg of nuts. What is the price of 1 kg?". The operation used to solve this problems is division, but the structure of the problem is that of a correspondence problem, where one of the terms in correspondence is missing. The child has to find the ratio between the weight and the price. If it was a partitive division problem there should have been a quantity to be shared among a number of recipients. Fischbein et al (1985) also presented partitive division problems where the divisor was a fraction. We can argue that recipients in partitive problems cannot be fractions.

The contrast between the partitive and the quotitive problems was not very convincing. The suggested finding that quotitive problems are harder than partitive is confounded because there was no control of the size of the numbers used. The numbers used for partitive situations were smaller than those used for the quotitive ones, which could have affected children's performance. Furthermore, in one of the quotitive problems the size of the divisor was a decimal smaller than 1 (0.75), but no parallel item was included in the partitive problems. Besides, the story context of the problems was not comparable as in some cases and the quotitive problems were presented within a more complex context.

For example, one of the problems that Fischbein refers to as partitive and one quotitive were phrased in the following way:

"I spent 1500 lire for 3 kg of nuts. What is the price of 1 kg?" (partitive problem)
"The walls of the bathroom are 280 cm high. How many rows of tiles are needed to cover the walls if each row is 20 cm wide?" (quotitive problem)

The context of the partitive problems was familiar to children's experience, easy to interpret and similar to the problems that children were accustomed to solving in school.

In summary, it can be said that the design of the study did not adequately control for all the variables that could have interfered with and affected the children's responses, and the variables that are contrasted (partitive and quotitive) were not designed to be comparable. Under these circumstances it should not be surprising that children found partitive problems easier than the quotitive ones. However, the suggestion that intuitive models have a strong influence on children's understanding of the operation of multiplication and division was an issue that opened new ground for research.

In my view Fischbein et al's (1985) conclusion that repeated addition forms the basis for understanding multiplication and repeated subtraction for understanding quotitive division problems is not well founded. In their study the children were asked to *indicate* the correct operation for solving a problem. It is possible though, that if the children were given the opportunity to *solve* the problem they would have reacted differently. Research on addition and subtraction has shown that children are able to solve a wide range of addition and subtraction by joining and separating sets long before they are able to name which arithmetic operation is adequate to calculate the results formally (Carpenter and Moser, 1982; Hughes, 1986; Riley, Greeno and Heller, 1983; Marton and Neuman, 1990). Hudson (1983) has shown that young children are able to solve a number of problems that involve comparison between sets, when they can use their counting schema, before they are able to indicate which arithmetic operation would be appropriate to solve the problem. This mismatch between solving problems through schemas of action and through formal representation has been documented not only in research with children but also with unschooled adults whose schemas had not been socialized into the ways of knowing used in schools (Nunes, 1992, 1996b). These adults who were capable of solving a wide range of arithmetic problems in everyday life situations, had remarkable difficulty in solving a missing addend problem using a calculator. That is because the calculator cannot work out solutions that are based on counting up from the first addend to the total value. The use of a calculator requires the formal representation of a series of commands (eg. 27-12) that are similar to those that school encourages children to develop in order to solve a missing addend problem.

This is also the case with multiplication and division. In the previous section (2.3) it was shown that young children and unschooled adults who had never been introduced to multiplication and division had the schemas to solve a wide range of multiplication and division, partitive and quotitive, word problems by modelling the actions in the problem (Aubrey, 1997; Brown, 1992; Burton, 1992; Carpenter, et. al, 1993; Correa, 1995; Kouba, 1989; Nunes, Schliemann and Carraher, 1993). So then, why are there children, as in Fischbein and his colleagues' study, who have received formal instruction on multiplication and division and yet are confused and not able to choose the correct operation for a simple one step multiplication or division problem?

As pointed out by Nunes (1996a, 1996b) this is because *there is not a perfect match between schemas of action and operations*. When children are taught the four operations at school their old meanings, derived from their schemas of action, are reshaped when they are connected to the new system of signs that they are taught. As a result of this reshaping "the child has to compress different action schemas into a simplified system of representations with fewer options" (Nunes, 1996a, p.244). That means that all the action schemas that the child had related to addition and subtraction have to be reduced to the expression a+b=c and the implied forms c-b=a and c-a=b. The job done by a number of schemas of action has to be done now by two arithmetic operations: addition and subtraction. An additional difficulty to this is posed by the fact that in the school the child has to solve a problem by acting on numbers. Nunes (1996a) suggests that "schemas of action cannot be used on numbers because numbers offer a compressed representation of objects (8 is one representation for eight individual objects which can be counted, separated into sets etc, when they are individualized" (p.244)).

Because reasoning is not identical to operations, it is possible that Fischbein et al's subjects would have preformed differently if they had been given the chance to activate their schemas of actions in the multiplication and division problems.

Fischbein et al's suggestion that multiplication originates from repeated addition reflects the way multiplication is taught in school rather than the initial development of the concept in children's minds. Because the children who took part in their study had received a considerable amount of instruction in multiplication and division, the study investigated children's understanding of multiplication and division after their schemas of actions had been reshaped into the conventional system of school mathematics. When children enter school they have a good grasp of the concept of ratio in one-to-many correspondence situations (see sections 2.1 and 2.3), which is an invariant present in the operation of multiplication and forms the basis for its understanding. When multiplication is introduced in the school the teacher has to find a context that would facilitate the connection between the old meanings that the child has and the new concept. For the teaching of multiplication the most common practice is to connect it with repeated addition, which is familiar to the children. Children have no difficulty in understanding that instead of adding labouriously 6 times 3, this can be simplified into "six threes" or "six times three". According to Nunes (1996a) this connection allows children to relate multiplication with something familiar, that is addition and also gives them a strategy to calculate a number of multiplication sums. She stresses though, that the price for this particular path in the socialization of children's thinking may be that the meaning that young learners connect with multiplication will need much more redescription for them to progress beyond the initial stage, because there are invariants in the operation of multiplication that are not present in the concept of addition. Additive situations are about *part-whole relations* where objects or sets of objects are either joined or separated. Multiplication, on the other hand, is about a *constant relation, expressed as a ratio, between two variables,* an invariable that is not present in additive situations. The child has also to learn a new set of number meanings. In multiplicative situations quantities do not only refer to actual amounts as in addition and subtraction, but also to relations (intensive quantities). For example, when we are talking about the "price per kilo" we refer to the relation between weight and price, rather than to the amounts of sugar and money. If multiplication were introduced as repeated addition the child would be able to solve problems that require simple scalar solutions, but would have great difficulty in solving the ones that require functional solutions. Because multiplication involves a whole new set of invariants and number meanings that are not present in addition, the child might never represent them if multiplication is introduced as just another more complicated form of addition (Nunes, 1996a, 1996b; Nunes and Bryant, 1996).

Fischbein et al's theory of the primitive intuitive model is itself inconsistent. The origins of operations are sought, in the case of partitive division in children's schemas of action (sharing schema) and in the case of multiplication and quotitive division in an operation, that is repeated addition and repeated subtraction respectively.

The theory of primitive intuitive models accounts more for the difficulties and the misconceptions that upper primary and high school children have about multiplication and division after instruction and practice at school and it does not give an answer neither on how multiplicative reasoning starts, nor on how it develops.

2.6 An overall summary: What is known and what can be explored further

Two contrasting approaches have been developed about the origins of children's understanding of multiplication and division. One approach that was originally put forward by Piaget (1965) sought the origin of multiplicative thinking in children's schemas of action. The second approach proposed by Fischbein et al (1985) sought the origin of these two operations on additive structures. Each of these hypothesis generates different predictions. If multiplication and division originates from addition and subtraction respectively, then the children are not expected to be able to reflect on multiplicative relations before being able to quantify multiplicative problems. If multiplication and division originates from children's schemas of action then the children are expected to be able to reflect on multiplicative relations before being able to deal with the absolute values of the situation.

The aim of the thesis is to test these different predictions by studying children's performance in relational non-computational multiplicative problems and in computational problems involving both discontinuous and continuous quantities; the latter do not provide any source for quantification.

The hypothesis of the study is that the origin of multiplication and division are in children's schemas of action. Schemas of action provide the first meaning for these operations, because they preserve the same invariants that are present in the operation of multiplication and division.

The origin of multiplication is sought in the one-to-many correspondence schema of action. There is evidence that children can understand a wide range of correspondence relations early on. Children as young as 5 and 6 have a good understanding of transitive inferences based on one-to-many correspondence and are able to develop procedures to obtain equal totals when the shared quantities consist of different units and the ratios bear a simple relation to each other. They can also quantify multiplication problems when the one-to-many correspondence relation between the terms is explicit.

There are, though, some aspects of children's understanding of correspondence relations that should be explored further. There is evidence that young children have a good understanding of one-to-many correspondence situations, but we do not know whether they can order different one-to-many correspondence ratios. The ability to order different one-to-many correspondence ratios is important, because the product in a multiplicative situation of one-to-many correspondence type depends both on the number of elements in the basic set and on the ratio (given the same basic set). If the children can order the size of the two corresponding sets on the basis of their relations then it can be claimed that they have an understanding of the multiplicative relations that stem from their schemas of action. If the children though need to quantify the two sets first to order their size, then it can be claimed that Fischbein's suggestion that multiplication originates from addition has some ground. To ensure that children's multiplicative thinking is independent of their ability to quantify, the children would also be asked to order corresponding ratios with continuous quantities that are beyond their quantification ability.

By exploring the previously undocumented evidence on children's ability to order different one-to-many correspondence ratios this study will contribute to the formation of a more elaborated picture of children's understanding of one-to-many correspondence situations.

The origin of division is sought in the sharing situation. Sharing and division are conceptually close to each other but division is more than sharing. In division the child has to consider a whole new set of relations between the terms and particularly the effect that the number of quotas has of their size. Correa (1995) has provided evidence that even the 5 year olds understand the relations that are involved between the dividend, the divisor and the quotient in the context of a sharing situation. They are able to apply the inverse relation between the divisor and the quotient and make judgements on the relative size of the shared quotas not only in partitive problems, but also in quotitive problems which have been characterized as rather difficult for young children before they are able to quantify the size of the shared quotas. Her findings suggest that children have some understanding of sharing relations before being able to quantify the size of the shared quotas.

To test whether the understanding of division stems from children's sharing schemas of action or from subtraction as suggested by Fischbein, this study will investigate children's understanding of sharing relations with two types of quantities. It seeks to replicate Correa's findings with discontinuous quantities and also to study children's reasoning on sharing relations with continuous quantities. The introduction of continuous quantities should ensure that the children reason on the situation on the basis of relations only because the quantification of the task is beyond their ability, since their sharing results in fractions. If the children can reflect on sharing relations with continuous quantities that they cannot quantified, then it can be claimed that their reasoning on sharing relations stems from their schemas of action and not from their ability to quantify the size of the quotas.

Most of the research on multiplication and division has focused on when and how the children quantify different multiplication and division problems, but there is restricted evidence on whether the children understand the complementary relation of the two operations. The research evidence on whether multiplication and division develop independent of each other or in a coordinated fashion is inconclusive. Piaget (1965) suggested that division is the inverse of multiplication and that the two operations are discovered simultaneously, but he did not carry any research to verify his assumption. In contrast, Fischbein et al. (1985) proposed that the two operations have distinct origins: multiplication originates from addition, while division originates from children's sharing schemas, but they did not provide any evidence on how the children come to coordinate the two operations.

This study aims to examine whether multiplication and division develop independently of each other or as coordinated operations. The hypothesis of this study is that multiplication and division have distinct roots and that children discover their inverse relation at a later stage. The reasons for this are firstly because the invariants of the two operations are different and secondly because multiplication and division are likely to follow the same developmental path as addition and subtraction, which are discovered simultaneously but are coordinated only at a later stage. The study explores the coordination of multiplicative relations not only across the operation of multiplication and division, but also within each operation. Division is described as one operation but the children have two schemas of action, that is, sharing for partitive problems and forming quotas for the quotitive problems. Have the children coordinated the relations between these two sharing schemas of action? Have they coordinated the multiplicative relations in correspondence situations that require the understanding of the commutativity rule?

The experimental chapters that follow give a detailed account of the aims, the rationale and the findings of each study.

CHAPTER 3

EXPLORING CHILDREN'S UNDERSTANDING OF MULTIPLICATION

3.1 The aim and the rationale of the study

The aim of this study is to investigate the origins of children's understanding of multiplication. Two possible sources of this origin have been proposed; addition (Fischbein et al, 1985) and one-to-many correspondence (Piaget, 1965). This experiment is designed to investigate these two possible origins in situations where young children would have to order the size of two corresponding sets. If multiplication originates from addition, then it can be predicted that the children will be able to order the two sets only after quantifying each of these two totals. If multiplication originates from the understanding of one-to-many correspondence then the children should be able to establish a multiplicative relation between the two sets and order their size sets on the basis of this relation. For example, would the children understand that 5 vases with 3 flowers in each will have a total of *more* flowers than 5 vases with 2 just by reflecting on the ratio difference of the corresponding sets or do they have to quantify the two sums to give the correct answer? Note that thinking additively in this problem would mean that each set would be thought of in isolation from the other. The child would need to quantify the sum of each set and then compare the two totals. Thinking multiplicatively would mean that the child would establish a relation between these sets and would make a judgement on the basis of this relation. If the children are able to order the product of different ratios without having to quantify, then it can be claimed that they have an understanding of the relations involved in a multiplicative situation and particularly of the concept of ratio. They also understand a core relation, that the total in an multiplicative situation of one-to-many correspondence depends both on the number of elements in the basic set and on the ratio, given the same basic set. If children can make their judgements only after quantifying the number of flowers then it can be claimed that children's first understanding of multiplication stems from addition.

The hypothesis of the study is that the origins of multiplication should be sought in children's correspondence schema of action. If the hypothesis that children's first understanding of multiplication comes from the one-to-many correspondence schema of action is correct, then the children should be able to show an understanding of the mathematical property of ratio and order the relative size of the corresponding sets on the basis of multiplicative relations. If children can make judgements on the size of the corresponding sets by means of their correspondence relations, then their performance should not vary across the situations that provide cues for quantification and those which do not. To ensure that children's reasoning is based on their understanding of multiplicative relations and not on quantification strategies the children in this study were asked to make judgements not only in tasks involving discontinuous quantities, such as the flower task which provides cues for quantification, but also in tasks involving continuous quantities that cannot be easily quantified by young children. If children order the size of corresponding sets by quantification, then there should be a difference in children's performance between the ordering of discontinuous and continuous quantities.

In order to examine whether the children could distinguish the situations that could be ordered by means of multiplicative relations they were presented with a) a set of problems that could be ordered on the basis of correspondence relations and b) another set of problems that could not be ordered on the basis of correspondence relations and where they had to quantify the sums to order the sets.

The contrast between children's ability to make judgements about multiplicative quantities on the basis of correspondence relations or on the basis of quantification was explored further by asking the children to quantify the sets. If the children could make correct judgements on the basis of correspondence relations but yet could not quantify the sets then it could be claimed that their reasoning was genuinely based on the understanding of the properties of multiplication.

3.2 METHODS

3.2.1 Design

Children's ability to order the product of different one-to-many correspondence ratios was tested both with *discontinuous* and *continuous* quantities that did not provide any source for quantification.

With discontinuous quantities the children were asked to *order* the size of the corresponding ratios a) in situations where this was possible on the basis of correspondence relations and b) in situations where they had to quantify the sets. Thus, it was possible to examine whether the children could distinguish between the situations where they could order the sets on the basis of correspondence relations and those were they had to quantify.

The children were also asked to *quantify* some problems. Thus, whether their quantification ability proceeded or followed their ability to reflect on one-to-many correspondence relations could be examined.

With continuous quantities the children were also asked to reflect on *ordering* relations and also to *quantify* correspondence problems.

A detailed account of the structure of the study is given below.

3.2.1.1 Discontinuous Quantities

Relational Tasks

In the case of discontinuous quantities the children were presented with two rows of hutches which had rabbits in them. The child had to decide which row of hutches had more rabbits in total in a number of conditions which either the number of rabbits per hutch or the number of hutches per set or both were varied.

The rabbits and hutches task was chosen because the two variables in correspondence, that is the number of hutches and the number of rabbits in each hutch, were clearly represented. The child could see the two rows of hutches and think about the rabbits that were in each hutch. Note that the rabbits were in the hutch and therefore were not visible. This precaution was taken to avoid solutions based on visual perception. In order to encourage the children to think of the rabbits in each row as a sum, they were asked to order the size of the two sets after the rabbits in each row had moved into their house.

The children had to order the totals of rabbits in two sets of problems (Table 3.1). In the first set the children could order the sets on the basis of their multiplicative relations because either the number of rabbits per hutch or the number of hutches per row was kept constant. Because only one variable varied each time, these problems were called *one variable problems*. In the second set of problems the children could not make any judgements about the relative size of the sets because neither the number of rabbits per hutch nor the number of hutches per row was kept constant. In these problems ordering was impossible without quantification. Because both the number of rabbits per hutch and the number of hutches varied in the two sets these problems were called the *two variable problems*.

The structure of the one and two variable problems is presented below.

One Variable Problems

The one variable problems consisted of three groups of problems depending on which variable varied each time.

a. The Same Number - Same Ratio Problems

The number of hutches and the number of rabbits per hutch was the same in both groups For example

[2] [2] [2] [2]

[2] [2] [2] [2]

This was a control problem to see whether the children had an understanding of the situation with which they were presented.

b. The Same Number - Different Ratio Problems

The number of hutches was the same in both sets, but the number of rabbits per hutch was different in the two sets. For example

[3] [3] [3] [3]

[2] [2] [2] [2]

The children had to consider the ratio difference in the two sets to give the correct answer.

c. Different Number - Same Ratio Problems

The number of rabbits per hutch was the same in both sets, but one set had more hutches. For example

[3] [3] [3] [3]

[3] [3] [3] [3] [3]

Because the ratio was kept constant in the two sets the children could give the correct response considering only the difference in the number of elements in the two sets.

Each set of problems had 4 items in order to test the consistency of children's responses.

Two Variables Problems

The two variable problems consisted of two groups of problems.

d) The Different Number - Different Ratio Problems

In this set of problems ordering on the basis of relations was not possible, because none of the variables, neither the number of rabbits per hutch nor the number of hutches per row was kept constant. The children had to use quantification strategies to give the correct answer.

There was, though, the chance of children finding the correct response by focusing merely on the number of rabbits per hutches. For example

[4] [4] [4]

[2] [2] [2] [2]

In other items the children could give the correct answer by focusing merely on the number of hutches. For example

[5] [5]

[3] [3] [3] [3]

In order to be able to detect the children who gave a correct response following the above wrong strategies both types of items were included in the design.

The Different Number - Different Ratio problems had 8 items in total, 4 in which the correct answer was the row that had more rabbits per hutch but less hutches and 4 in which the correct answer was in the row that had less rabbits per hutch but more hutches.

e. Commutativity

The difference between the Commutativity and the Different Number - Different Ratio problems was that in principle the children could give the correct answer without computing the precise cardinals, because the number of hutches was compensated by the number of rabbits per hutch. For example

[4] [4]

[2] [2] [2] [2]

There were 4 items designed with respect to the commutativity rule in multiplication.

It was expected that the one-variable problems, that is the Same Number - Same Ratio, Same Number - Same Ratio and Different Number - Same Ratio would be easier for the children than the two variable problems, that is those that involved the Different Number - Different Ratio and the Commutativity.

Quantification Tasks

In order to contrast the children's ability to *order* the relative size of the different totals on the basis of ratio relations with their ability to *quantify* the total number of rabbits in each set, they were also asked to pick up a number of pellets equal to the number of rabbits in six items, four from the one variable and two from the two variable problems (Table 3.1).

3.2.1.2 Continuous Quantities

In the Continuous quantities tasks the children had to judge the sweetness of two cakes on the basis of the amount of sugar put into each cake. They had to consider the number of sugar spoons and the size of the spoons (teaspoons or soup spoons) used to pick up the sugar. The children were told that 1 soup spoon corresponded to teaspoons.

Continuous quantities were included because quantifying the amount of sugar put in the two bowls would have been very difficult for children so young. Ordering the sweetness of the two cakes on the basis of relations would have been the only strategy available for the children.

Two sets of tasks were designed: one relational task where the children were asked to *order* the sweetness of the two cakes on the basis of correspondence relations and a *quantification* task where they were asked to quantify the amount of sugar in the cakes. Thus, it was possible to examine whether they reasoned on the situation on the basis of multiplicative relations or by quantification. More details about the two sets of tasks are given below.

Relational Tasks

The children were asked to order the sweetness of the cakes in two sets of problems (Table 3.1).

a. The Same Number - Same Ratio Problems.

Both cakes had the same number of same sized spoons. For example:

Cake A: 3 teaspoons

Cake B: 3 teaspoons

This was a control condition to see whether the children had an understanding of the situation with which they were presented.

b. The Same Number - Different Ratio Problems

This condition examined children's understanding of correspondence relations. Both cakes had the same number of spoons, but different sized spoons were used for each cake. For example

Cake A: 3 teaspoons

Cake B: 3 soup spoons

In this case the child had to consider the ratio difference in the two sets to give the correct answer.

If the children responded on the basis of correspondence relations then ordering the sweetness of the cakes was expected to be a relatively easy task. But if the children reasoned on the situation following a quantification strategy then this was expected to be a difficult condition because it involved complex counting procedures.

Two variable problems were not presented with Continuous quantities due to the complication they involved. Testing whether the children could distinguish between the situations where the could order the ratios on the basis of relations and the ones where they had to quantify was done with Discontinuous quantities only.

Quantification Task

In order to test whether the children had a genuine understanding of correspondence relations and to ensure that their judgments were made on the basis of multiplicative relations only their performance was observed in quantification tasks.

The children were asked to put in a bowl an amount of sugar equal to the amount picked up by the researcher using a different sized spoon. For example, if the experimenter put 3 soup spoons of sugar into cake A the child had to pick up the same amount of sugar for cake B using the tea spoon or vice versa. The children were told that 2 teaspoons corresponded to 1 soup spoon.

The quantification task was expected to be more difficult, not only because it required the use of the principle that 1 soup spoon was equal to 2 teaspoons but also because it required the organization of the counting activity.

To test the consistency of children's responses each condition with continuous quantities had 4 items.

TABLE 3.1 The design of the study

	Discontinuous		Continuous
Condition	Ratio of rabbits per hutch		Ratio of sugar per spoon
Same Number of Hutches	2 (5:1) vs 2 (5:1)		5 teaspoons vs 5 teaspoons
Same Number of Rabbits per Hutch	3 (4:1) vs 3 (4:1)		4 soup spoons vs 4 soup spoons
	4 (3:1) vs 4 (3:1)		3 soup spoons vs 3 soup spoons
	5 (2:1) vs 5 (2:1)		2 teaspoons vs 2 teaspoons
Same Number of Hutches	2 (5:1) vs 2 (3:1)		2 soup spoons** vs 2 teaspoons
Different Number of Rabbits per Hutch	3 (3:1) vs 3 (4:1)*		3 soup spoons** vs 3 teaspoons
	4 (2:1) vs 4 (4:1)*		4 teaspoons** vs 4 soup spoons
	5 (2:1) vs 5 (3:1)		5 teaspoons vs 5 soup spoons
Different Number of Hutches	2 (2:1) vs 3 (2:1)		
Same Number of Rabbits per Hutch	2 (3:1) vs 4 (3:1)*		
	3 (4:1) vs 4 (4:1)*		
	3 (5:1) vs 5 (5:1)		
Different Number of Hutches	3 (4:1) vs 5 (2:1)	2 (5:1) vs 5 (3:1)*	
Different Number of Rabbits per Hutch	2 (4:1) vs 3 (2:1)	2 (4:1) vs 4 (3:1)	
	2 (6:1) vs 3 (2:1)	3 (3:1) vs 5 (2:1)	
	3 (4:1) vs 4 (2:1)*	2 (4:1) vs 5 (2:1)	
Commutativity	2 (4:1) vs 4 (2:1)		
	3 (5:1) vs 5 (3:1)		
	2 (5:1) vs 5 (2:1)		
	2 (3:1) vs 3 (2:1)		

* the pairs where the children were also asked to pick up a number of bowls equal to the number of rabbits (quantification task)

* * In the quantification task the children were given the indicated amount of sugar and were asked to pick up an equal amount with the other spoon.
The trial "6 teaspoons= x soup spoons?" was also included

3.2.2 Participants

The participants of the experiment were (a) 30 4-year olds (19 male and 11 female), mean age 4.5; range 4.1 to 4;10, (b) 35 5-year olds (16 male and 19 female), mean age 5.7; range 5.1 to 5.11, (c) 35 6-years olds (18 male and 17 female); mean age 6.6; range 6 to 6.10, and (d) 35 7-year olds (17 male and 18 female); mean age 7.7; range 7 to 7.10.

The children were from two state schools in North East London. According to the information given by their teachers, none of them had received formal instruction in multiplication at school. Only the 7 year olds who had been taught the multiplication tables up to 5.

3.2.3 Material

The material consisted of 10 paper-made hutches (5 red and 5 blue), 2 paper-made houses (1 red and 1 blue), 50 pellets of rabbit food, 1 soup spoon and 1 teaspoon, 2 opaque plastic bowls (1 blue and 1 red) and 1 packet of sugar. All the above materials, except the spoons, were of identical size and features and differed only in their colour.

3.2.4 Procedure

Discontinuous quantities

The Relational Tasks

The children were presented with two sets of hutches, the blue and red ones, which were laid down in two rows (basic sets). For example, they were shown 3 red and 3 blue hutches. At the end of each row there was a big house. The blue house was at the end of the blue row of hutches and the red house was at the end of the red row of hutches (Picture 3.1). The children were told that the hutches were carrying some rabbits that lived in the big houses. In each blue hutch there were 2 rabbits and in each red hutch there were 2 rabbits. Then the children were told that all the rabbits from the blue hutches had to go into the blue house and all the rabbits from the red hutches had to go into the red house. The experimenter removed the rabbits into their houses behind a screen. The question posed to the children was: *"Does the blue house have the same or different number of rabbits as the red house? Which house has more rabbits?"*.

To make the situation understandable the children were initially presented with an exemplary situation. In this situation there was a blue and a red hutch each having 2 rabbits. Each hutch was opened to show the children the rabbits that were in. Then the rabbits were moved into their houses in the full view of the child. The question posed to the child was: "Does the blue house have the same or different number of rabbits as the red house?". Then they were presented with a similar exemplary situation. This time there were 3 rabbits in the blue hatch and 2 rabbits in the red hutch. If the children were not able to provide the correct answer in both exemplary situations their participation in the experimental tasks was terminated.

It was only in the exemplary situation that the children saw how many rabbits were in the hutch. In all the other trials they were merely told how many rabbits each hutch had. The houses as well as the hutches were opaque and the children could not see the rabbits inside. The moving of the rabbits from the hutches to the houses was done behind a screen to avoid the questions being answered on perceptual cues. When the rabbits were moved the children were reminded how many rabbits were previously in each hutch.

Quantification task

After the rabbits had been moved into the houses and the child had indicated which house had more rabbits the child was told that the rabbits were hungry and each of them wanted to have a pellet of food. The child was shown a pile of pellets and was asked to pick up the right number to feed the rabbits in each house. *"Now look: we want to feed the rabbits that are in the blue house. Each rabbit wants to have a pellet. Can you pick up the right number so that each rabbit will have one?*. During this task the hutches that used to hold the rabbits were in the complete view and the children were reminded how many rabbits were previously in each hutch.

Continuous Quantities

Relational Task

Because the task involved taking sugar with different size spoons the child was initially shown the quantitative difference between a soup spoon and a teaspoon. Each soup spoon contained two teaspoons of sugar. The children were then presented with two bowls one blue and one red, which were going to be used to prepare two cakes (Picture 3.2). The experimenter took the soup spoon and placed 3 spoons of sugar into the red bowl. For the other cake the experimenter took the teaspoon and put 3 spoons of sugar into the blue bowl. The experimenter counted with the children the number of spoonfuls of sugar put into each cake. The amount of sugar placed in the bowls was not visible to the child. The question posed to the child was: *"Will the cake in the blue bowl taste as sweet as the cake in the red bowl or differently? Which one do you think is going to be sweeter? Why?"* Because the spoons of sugar were put in continuously and the children might lose track of the number of spoons placed in the bowl, they were shown a card with dots corresponding to the number of teaspoons or soup spoons placed in each bowl. The type of spoon used to put the sugar in the bowl was put next to the card with the dots.

Quantification Task

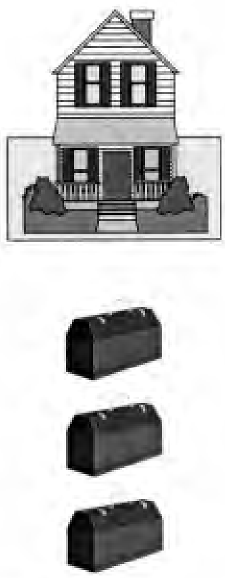
In the quantification task the experimenter put a few spoons of sugar either with the big or the small spoon into one bowl and asked the children to put the same amount of sugar into the other bowl - in order to make the two cakes taste the same - using a different sized spoon from the one used by the experimenter. If for example the experimenter put

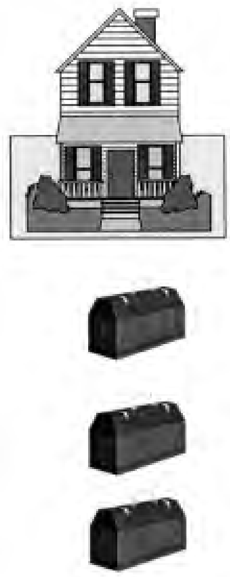
2 soup spoons into the bowl the child had to pick up the equal amount using the teaspoon. In that case the child had to consider that 2 teaspoons corresponded to 1 soup spoon and organize his/her counting activity. A card with dots next to the bowl displayed how many spoonfuls were put in by the experimenter.

PICTURE 3.1

Discontinuous Quantities

Ordering the size of the multiplicative product





Each hutch carries 2 rabbits

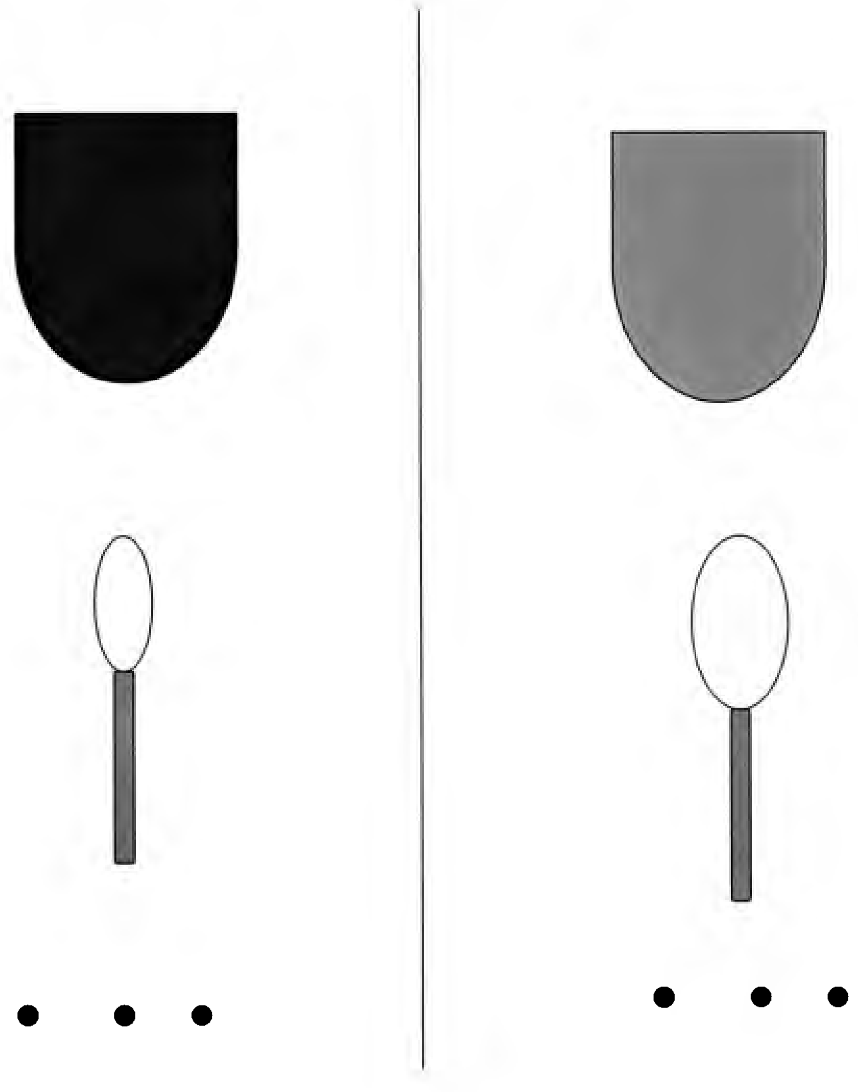
Each hutch carries 3 rabbits

Does the blue house have the same number of rabbits as the red house?
Which house has more?

PICTURE 3.2

Continuous Quantities

Ordering the size of the multiplicative product



3 teaspoons of sugar in the blue bowl

3 soup spoons of sugar in the red bowl

Will the cake in the blue bowl taste as sweet as the cake in the red bowl?
Which one is going to be more sweet?

3.3 RESULTS

3.3.1 Preliminary analysis

The screening procedure

There were 8 4 year olds who failed to give a correct response in the exemplary situation.

Scoring children's responses

One point was awarded every time the child indicated correctly which house had more rabbits. The scores in each group of problems ranged from 0 (minimum) to 4 (maximum), because each condition had 4 items. Only in the Different Number - Different Ratio group the maximum score was 8, because there were 8 items.

The distribution of children's scores in each group revealed that they were not normally distributed. The children gave either the correct or the wrong response in most of the items in the ordering and the quantification task, therefore, the distributions were skewed as it is shown in figures 3.1. to 3.9. It was also possible that the children may have given correct responses by chance. For this reason the children were grouped into two categories: those who gave correct responses by chance and those who gave correct responses in a significant above chance level. The children had to choose between three possible answers a) the red house has the same number of rabbits as the blue house, (b) the red house has less rabbits than the blue house, and (c) the red house has more rabbits than the blue house. For this reason, the probability of correct responses by chance was estimated by means of Binomial Distribution. The children had to achieve a score of 4 out of 4 ($p<.01$) in the group of problems that had 4 items to be considered as scoring *above chance level*, or a minimum score of 6 out of 8 ($p<.001$) in the Different Number - Different Ratio group that had 8 items.

The majority of the 4 year-olds scored at chance level. Only 7 children gave corect responses at above chance level in the one variable problems. Therefore, they were not included in the analysis.

FIGURE 3.1
The distribution of children's scores in the relational tasks in the Same Number - Same Ratio situation with discontinuous quantities

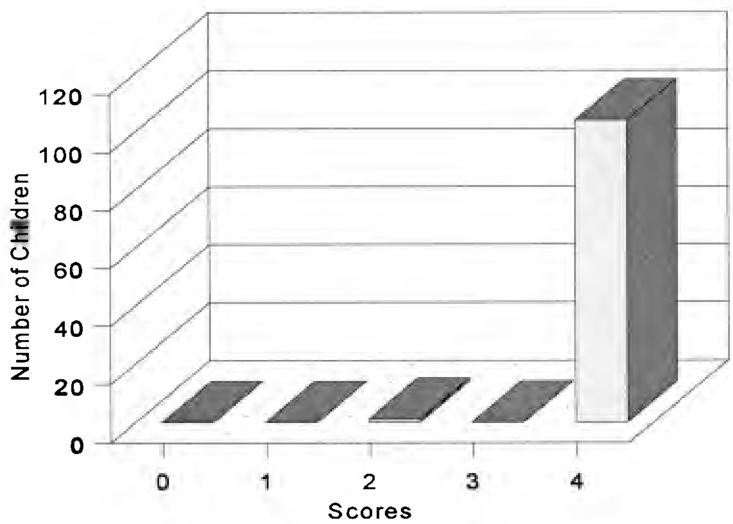


FIGURE 3.2
The distribution of children's scores in the relational task in the Same Number - Different Ratio situation with discontinuous quantities

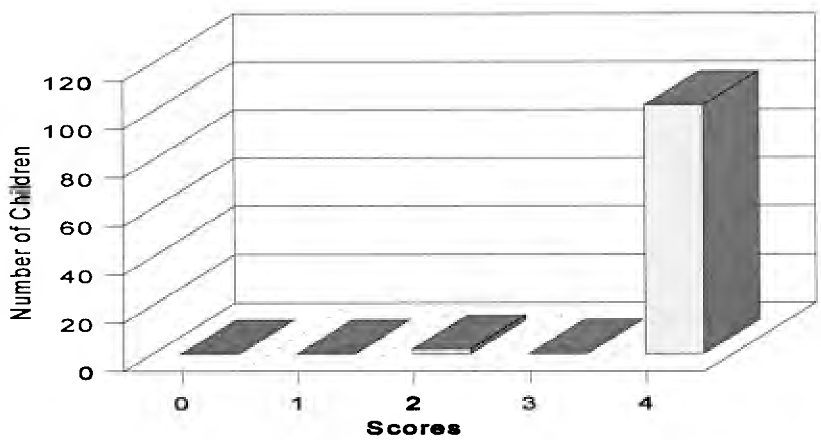


FIGURE 3.3
The distribution of children's scores in the relational task in the Different Number - Same Ratio situation with discontinuous quantites

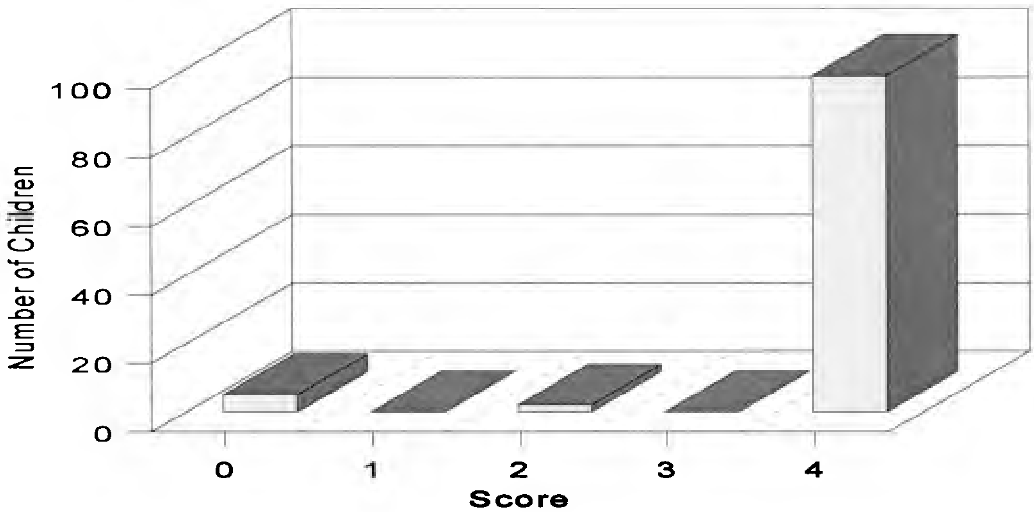


FIGURE 3.4
The distribution of children's scores in the relational task in the Different Number - Different Ratio situation with discontinuous quantities

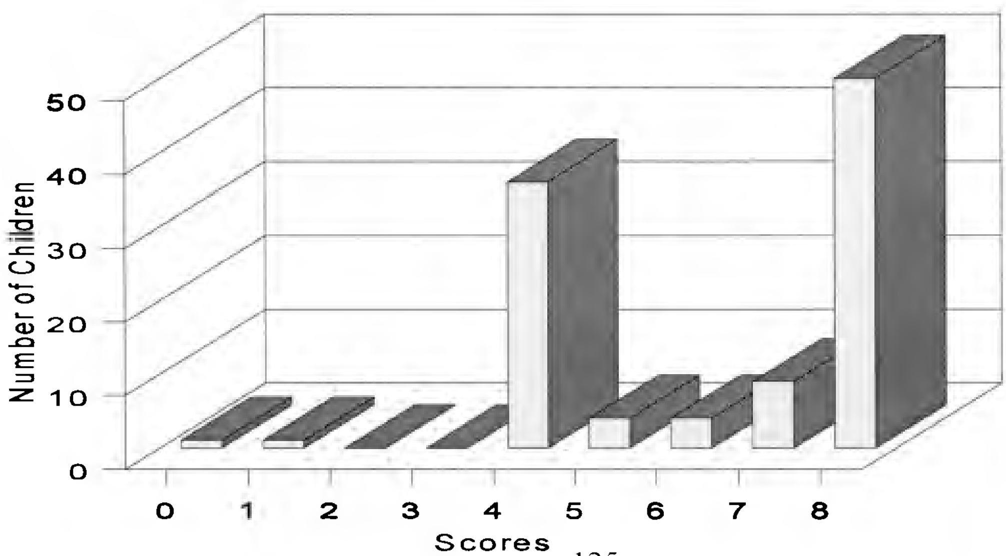


FIGURE 3.5
The distribution of children's scores in the relational task in the commutativity situation with discontinuous quantities

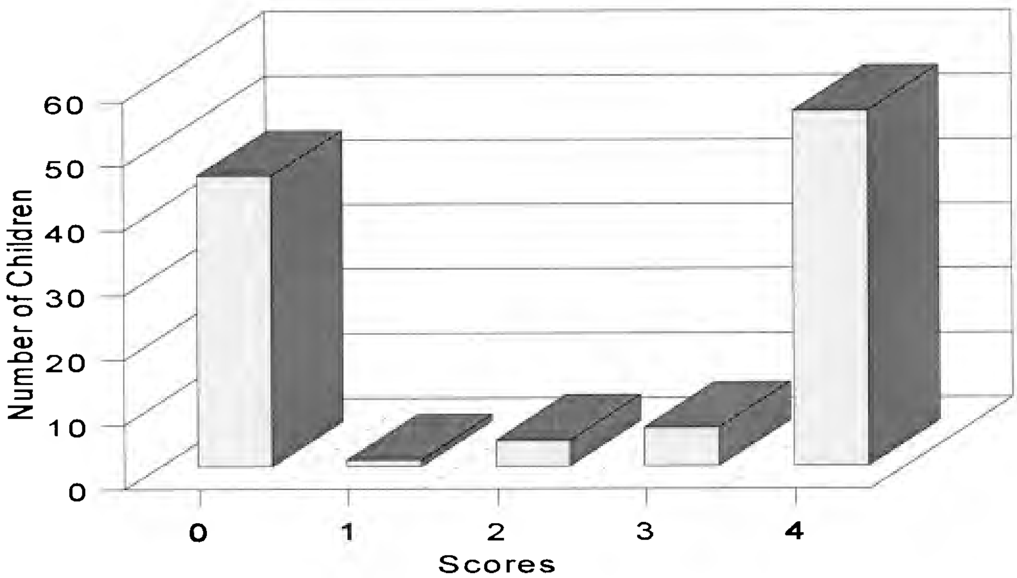


FIGURE 3.6
The distribution of children's scores in the relational task in Same Number - Same Ratio situation with continuous quantities

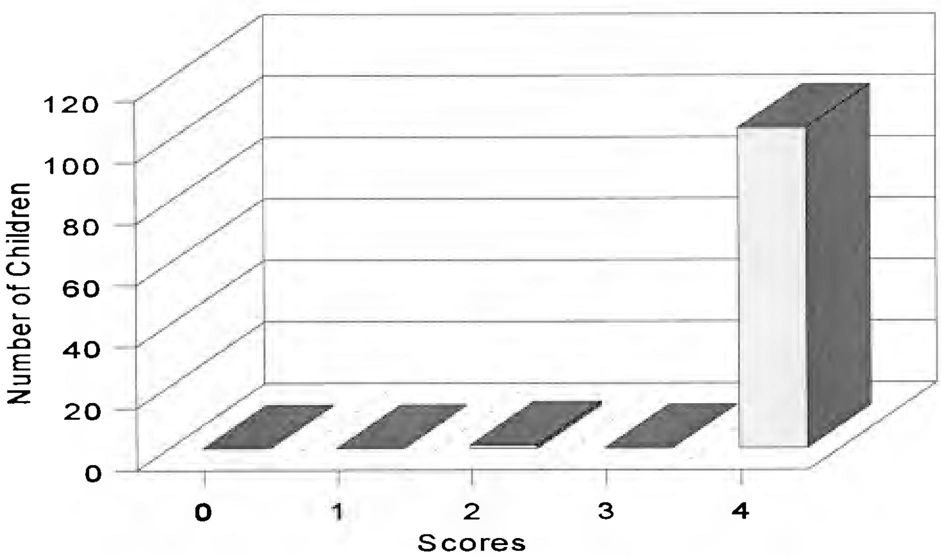


FIGURE 3.7
The distribution of children's scores in the relational task in the Same Number - Different Ratio situation with continuous quantities

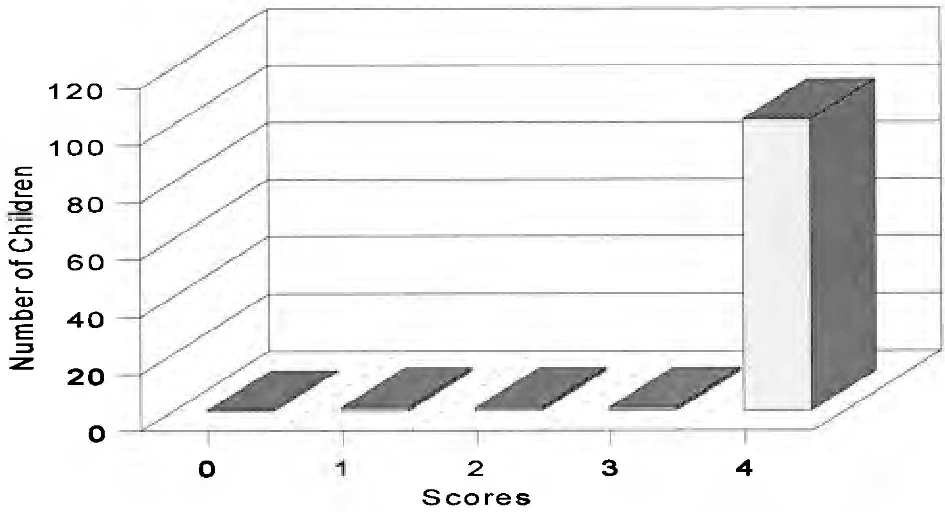


FIGURE 3.8
The distribution of children's scores in the quantification task with discontinuous quantities

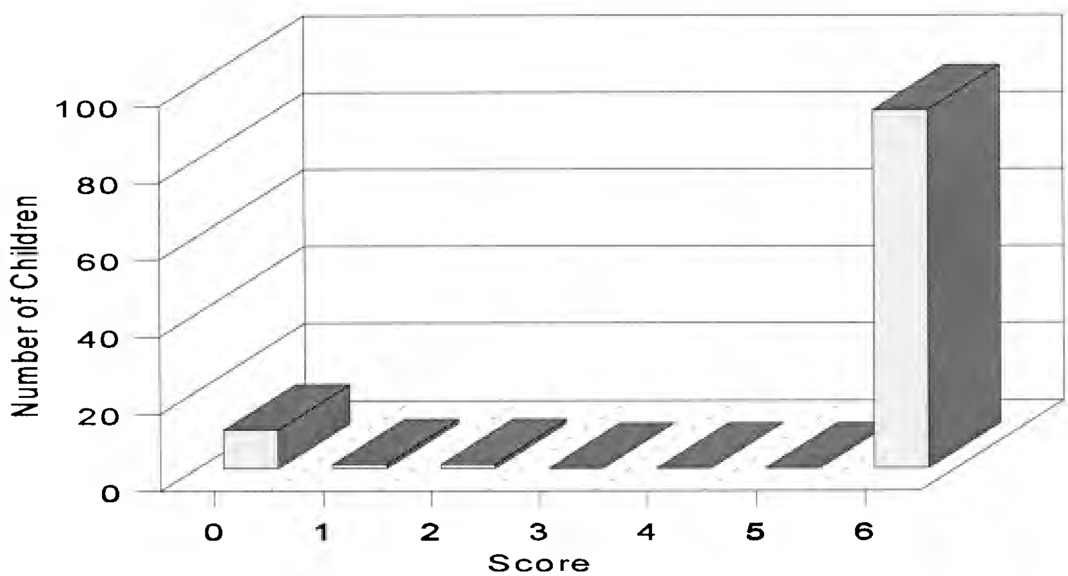
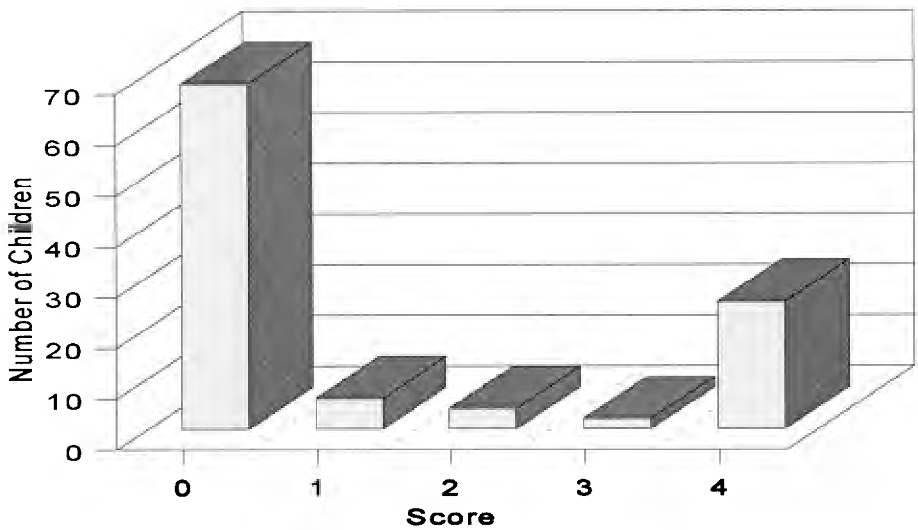


FIGURE 3.9

The distribution of children's scores in the quantification task with continuous quantities



3.3.2 Discontinuous Quantities

3.3.2.1 Relational tasks

The aim of the relational tasks with discontinuous quantities was to test whether the children could order the totals of two sets of rabbits on the basis of correspondence relations. In the one variable problems either the number of rabbits per hutch or the number of hutches per row was constant. Therefore, the children could order the sets by reflecting on the one-to-many correspondence relations. In the two variable problems, though, ordering the ratios on the basis of relations was not possible because neither the number of rabbits per hutch nor the number of hutches per row was constant.

The findings revealed that in the one variable problems the children performed almost at ceiling level. As shown in Table 3.3 all the children apart from one 5 year old succeeded in the control task (Same Number - Same Ratio). That means that the children had no difficulty in understanding the situation they were presented. The Same Number - Different Ratio problems were equally easy for the children. In order to achieve a correct answer the children had to think of the difference in the ratio of rabbits per hutch in the two rows. The children considered this difference and only two 5 year olds failed. The Different Number - Same Ratio problems were slightly more difficult. In order to achieve a correct score the children had to consider the difference in the number of hutches, since the ratio of rabbits per hutch was the same in both rows, but even in this task 31 out of 35 children 5 year olds gave the correct responses.

The children were given an overall passing score in the one variable problems when they answered correctly in all the three one-variable conditions. This was to ensure that the successful children were those who had considered both the number of rabbits per hutch, as in the Same Number -Different Ratio problems *and* the number of hutches per row, as in the Same Ratio - Different Number problems for making their judgements. As shown in Table 3.2 the vast majority of children ordered the corresponding sets on the basis of correspondence relations taking into account both the number of rabbits per hutch and the number of hutches per row.

TABLE 3.2

Number of children passing or failing in one variable problems with discontinuous quantities by age

Age	n	*Pass*	*Fail*
5	35	31	4
6	35	33	2
7	35	34	1
Total	105	98	7

Variation in children's performance was observed only in the two variable problems, that is at the Different Number - Different Ratio set where the children had to employ quantification strategies (Table 3.3). In these problems only 7 5 year old, 22 6 year old and 30 7 year old gave the correct responses. A Chi-Square test showed that there was a significant association between age and performance (X^2=31.64, df=2, p<.001). The older the children were the better they performed in this task.

Children's performance was slightly lower in the Commutativity task where, in principle, they did not have to quantify in order to give the correct answer.

TABLE 3.3

Number of children passing or failing in the ordering task with discontinuous quantities by age and condition

Age	n	Condition	*Fail*	*Pass*
5 yrs	35	Same Number - Same Ratio	1	34
		Same Number - Different Ratio	2	33
		Different Number - Same Ratio	4	31
		Different Number - Different Ratio	28	7
		Commutativity Task	30	5
6 yrs	35	Same Number - Same Ratio	-	35
		Same Number - Different Ratio	-	35
		Different Number - Same Ratio	2	33
		Different Number - Different Ratio	13	22
		Commutativity Task	14	21
7 yrs	35	Same Number - Same Ratio	-	35
		Same Number - Different Ratio	-	35
		Different Number - Same Ratio	1	34
		Different Number - Different Ratio	5	30
		Commutativity Task	6	29

3.3.2.2 Quantification tasks

The children were presented with a quantification task in which they had to infer the number of rabbits in each row. This condition was included for two reasons: a) to test whether they could quantify the one-to-many correspondence situation and b) contrast their ability to order quantities on the basis of relations with their ability to quantify them.

If the children could order the totals of corresponding ratios on the basis of relations before being able to quantify them then it could be claimed that the understanding of one-to-many correspondence relations precedes children's ability to quantify and therefore the understanding of one-to-many correspondence relations and not quantification forms the basis for the understanding of multiplication.

In the quantification task when they were asked to pick up the right number of pellets for the rabbits the children had to consider two things: the number of hutches in each row and the number of rabbits per hutch. The children were considered to perform significantly above chance if they quantified correctly five out of the six trials presented ($p<.01$). The results showed that the majority of the 5 year olds and all the 6 and 7 year old children were able to pick up pellets equal to the number of rabbits (Table 3.4). A McNemar test showed that ordering the two sets on the basis of relations was significantly easier for the 5 year olds ($p<.01$) than quantifying the number of rabbits. There were 8 5 year olds who could order the size of the total number of rabbits in the two rows but could not quantify the number of rabbits. The difference between the ordering and the quantification tasks was not significant in the other age groups.

TABLE 3.4

Children's performance in one-variable ordering and quantification tasks by age

Age	n	Fail Ordering Fail Quantification	Pass Ordering Fail Quantification	Fail Ordering Pass Quantification	Pass Ordering Pass Quantification
5 yrs	35	4	8	-	23
6 yrs	35	-	-	2	33
7 yrs	35	-	-	1	34
Total	105	4	8	3	90

However, there were children who did *not* succeed in the Different Number - Different Ratio ordering task and yet could quantify the number of pellets necessary to feed the rabbits (Table 3.5). The above evidence suggests that the quantification strategy was available, but the children did not use it for ordering the size of the corresponding sets.

TABLE 3.5

Children's performance in two-variable ordering and quantification tasks by age

Age	n	Fail Ordering Fail Quantification	Pass Ordering Fail Quantification	Fail Ordering Pass Quantification	Pass Ordering Pass Quantification
5 yrs	35	11	1	17	6
6 yrs	35	-	-	13	22
7 yrs	35	-	-	5	30
Total	105	11	1	35	58

3.3.3 Continuous Quantities

3.3.3.1 Relational tasks

Children's abilities to order sets on the basis of relations was tested with Continuous quantities which do not provide any source of quantification. If the children could order the sweetness of the two cakes, it could be inferred that they did so on the basis of relations only because quantifying the amount of sugar used was beyond the computational abilities of children so young.

The results of the study revealed that all but one of the 5 year olds had no difficulty to order the sweetness of the two cakes in the control task. In the experimental condition the vast majority of children was able to order the sweetness of the cakes and only 3 5 year olds failed in the task (Table 3.6).

TABLE 3.6

Number of children passing or failing in the ordering task with continuous quantities by age and condition

Age	n	Condition	*Fail*	*Pass*
5 yrs	35	Same Number of Spoons Same Size	1	34
		Same Number of Spoons Different Size	3	32
6 yrs	35	Same Number of Spoons Same Size	-	35
		Same Number of Spoons Different Size	-	35
7 yrs	35	Same Number of Spoons Same Size	-	35
		Same Number of Spoons Different Size	-	35

3.3.3.2 Quantification tasks

The quantification task was designed to contrast children's ability to order quantities on the basis of relations with their ability to quantify these quantities.

Although the vast majority of children could *order* the sweetness of the two cakes they had great difficulty in quantifying the amount of the sugar. When, for example, the experimenter put 2 soup spoons of sugar into the red bowl the 5 and 6 year olds could not put the same amount in the blue bowl using the tea spoon, although they were instructed that 1 soup spoon corresponded to 2 teaspoons.

The results (Table 3. 7) revealed that fewer than the 1/4 of the 6 year olds and fewer than half of the 7 year olds were able to quantify the amount of sugar, whereas even the 5 year old, were performing at ceiling level when asked to order the sweetness of the two cakes. The difficulty of the quantification task can be attributed to the complex counting procedures that were involved. When the children had to convert the big spoons to small ones, they had to count each big spoon as two small spoons and vice versa. This required a higher level of abstraction than imagining and counting x number of rabbits in each hutch. Children's quantification scores improved significantly with age (X^2=21.33, df=8, p<.01)

TABLE 3.7

Number of children obtaining each score (maximum=4) in the quantification task with continuous quantities by age

Age	**n**	***0***	***1***	***2***	***3***	***4***
5 yrs	35	32	1	-	-	2
6 yrs	35	22	2	2	1	8
7 yrs	35	14	3	2	1	15
Total	105	68	6	4	2	25

3.3.4 Children's justifications and strategies

In order to provide further evidence on whether the children reasoned on the situation on the basis of relations or by following quantification strategies, it was necessary to see if they distinguished the items where quantification was needed from those where it was not necessary. Asking the children about their strategies and justifications for their answers could provided such an insight.

The children's justifications and strategies were classified into five different categories derived from the observations in this study. Note that the order in which the justifications are presented does not correspond to any order of sophistication.

I. *Irrelevant or no justifications*

This type of justification does not involve any mathematical reasoning relevant to the solution of the task. This category comprises the absence of justification, "I don't know" responses, personal preferences and socially desirable behaviour.

II. *Justifications focusing on one variable only*

Many children failed to give the correct response because they assumed that the house which corresponded to the row with the more hutches would have more rabbits, overlooking the number of rabbits in each hutch. For example, Alex aged 6.6, when he saw the row of 3 red hutches that had 4 rabbits in each and the row of 5 blue hutches with 2 rabbits in each concluded without any hesitation that the blue row had more rabbits "because there are more blue hutches".

Some other children were misled by the number of rabbits in each hutch neglecting the number of hutches. For example, in the trial where there were 2 red hutches with 4 rabbits and 5 blue hutches with 2 rabbits, Holly, aged 5.4, said that the red house had more rabbits because "each red hutch has 4 rabbits, but these (pointing to the blue) have only 2".

III. *Justifications based on quantification*

This justification was mostly observed in the Different Number - Different Ratio and the Commutativity problem. Because neither the ratio of rabbits per hutch nor the number of hutches per set was kept constant, the only way to order the size of the sets was to quantify the number of rabbits in each row and compare the final product. The children displayed a number of quantification strategies. Counting in ones was frequently observed. For example, in the case where there were 3 hutches with 4 rabbits each, the children tapped on the cage counting simultaneously the number of rabbits: 1,2,3,4 (one hutch), 5,6,7,8 (the second hutch), 9, 10, 11, 12 (the third hutch). Other children represented the rabbits with their fingers and established a visual correspondence between the hutches and their finger. They looked at the first hutch and extended 4 finger, then they looked at the second hutch and extended 4 fingers on the other hand and said "4+4 is 8". Then they looked at the third hutch and said " and 4 more is ... (counting with the fingers) 12! Other children added the number of rabbits in the 3 hutches by number facts. For example, 4+4=8, 8+4=12, while a few of the 7 year olds responded on the basis of number facts based on their knowledge of the times tables (3x 4=12).

IV. *Commutativity*

This strategy was observed only in the Commutativity problems. The children recognized that both rows of hutches had the same number of rabbis without having to quantify. For example, in the trials were there were 2 red hutches with 4 rabbits and 4 blue hutches with 2 rabbits, Chelsea, a 7.5 year old girl, responded that there is the same number of rabbits because "2 4s is the same as 4 2s". It was difficult though to distinguish whether the children reached the conclusion on the equality of the corresponding sets after quantifying the situation (2x4=8, 4x2=8) or by means of the commutativity rule only.

V. *Correspondence relationships*

In the one-variable problems the children were able to find out which house had more rabbits on the basis of correspondence relations: the set with more hutches had a total of more rabbits, when the number of rabbits per hutch was the same in both sets; and the set which had more rabbits in each hutch had a total of more rabbits when the two compared sets had an equal number of hutches.

For example, in the situation where there were 4 red hutches with 2 rabbits in each hutch and 4 blue hutches and 4 rabbits in each hutch, Francesca, a 5.8 year old girl said that the blue house had more rabbits because "this and this and this and this hutch (pointing to each hutch) have 4 rabbits, while this and this and this and this (pointing to the red hutches) have only 2".

In the situation where there were 3 red hutches with 4 rabbits in each and 4 blue hutches with 4 rabbits in each, Junior, a 6.5 year old boy, said that the blue house had more rabbits because “there is one more 4 there".

Two independent judges analysed children’s justifications. Interjudge agreement for the two judges was 96.3%. Discrepant judgements were presented to a third judge. Because in all the cases the evaluation made by the third judge coincided with one of the first two judges’ evaluation, this judgement was taken as final.

3.3.4.1 Types of justification across age

As shown in Tables 3.8 and 3.9 in the one variable problems both with Discontinuous and Continuous quantities the children gave their responses on the basis of one-to-many correspondence relations.

However, in the Different Number - Different Ratio and the Commutativity problems where neither the number of hutches nor the number of rabbits was kept constant, children’s strategies varied. Many 5 year olds focused their attention either on the number of hutches or on the number of rabbits per hutch. The number of those children declined with age, but there were still some children at the age of 7 who focused on only one variable of the situation. The number of children who tried to quantify in this condition increased with age.

Regarding the Commutativity problems the majority of the children did not recognize the equality of the total number of rabbits in the two rows by means of commutativity and the vast majority of the successful children ordered the relative size of the two rows by means of quantification.

TABLE 3.8

The proportion of justifications with discontinuous quantities by of age and condition

Age	Condition	Justifications: Irrelevant	Focus on one variable	Quantification	Correspondence	Commutativity	Total
5	Same Number/Different Ratio	.07	-	.01	.92	-	1
	Different Number/ Same Ratio	.06	.06	.04	.84	-	1
	Different Number / Different Ratio	.03	.78	.19	-	-	1
	Commutativity	.02	.81	.17	-	-	1
6	Same Number/Different Ratio	.01	-	.03	.96	-	1
	Different Number/ Same Ratio	-	.06	.07	.87	-	1
	Different Number / Different Ratio	.02	.36	.62	-	-	1
	Commutativity	-	.37	.60	-	.03	1
7	Same Number/Different Ratio	-	-	-	1	-	1
	Different Number/ Same Ratio	-	.03	-	.97	-	1
	Different Number / Different Ratio	-	.14	.86	-	-	1
	Commutativity	-	.14	.77	-	.09	1

TABLE 3.9

The proportion of justifications in the Same Number - Different Ratio problems with continuous quantities by age

Age	*Irrelevant*	*Correspondence*	Total
5	.05	.95	1
6	-	1	1
7	-	1	1

The fact that the children ordered the two sets on the basis of correspondence relations for discontinuous as well as for continuous quantities suggests that the children treated the two situations in the same way. This shows that the continuous quantities did not impose any further difficulty for the children when they are treated on the level of relations.

3.3.4.2 Types of justification and overall success

There was a strong association between the types of justification and children's performance in all the conditions, both with discontinuous and continuous quantities (Tables 3.10 and 3.11). In the one variable problems where either the number of hutches or the number of rabbits per hutch was kept constant the children ordered the total number of rabbits on the basis of correspondence relations. In the two-variable problems with discontinuous quantities (Different Number - Different Ratio and Commutativity) where neither the number of hutches nor the number of rabbits per hutch was constant, the successful children were those who tried to quantify the two sets. The majority of the children who failed in this condition was those who paid attention only to one variable of the situation, either to the number of hutches or to the number of rabbits per hutch.

In the Commutativity condition the majority of the children did not recognize that the number of rabbits per hutch could be compensated for by the number of hutches per row. None of the 5 year olds applied the commutativity principle and only 2 of the 6 year olds and 5 of the 7 year olds thought in terms of commutativity in some of the items. The strategies that the children applied suggest that Commutativity was treated in the same way as the Different Number - Different Ratio problems. It might be surprising that some of the children who followed quantification strategies failed to order the sets correctly. This is because some children made counting errors. The children had to be accurate in their counting because they had to come up with the same total number in order to give the correct answer.

TABLE 3.10

The proportion of justifications across each of the conditions of the ordering task with discontinuous quantities as a function of children's performance

	Same Number - Different Ratio				
Performance	*Irrelevant*	*Focus on one variable*	*Quantification*	*Relations*	*Total*
Children who Failed	.64	-	-	.36	1
Children who Succeeded	.01	-	.01	.98	1

	Different Number - Same Ratio				
Performance	*Irrelevant*	*Focus on one variable*	*Quantification*	*Relations*	*Total*
Children who Failed	.29	.71	-	-	1
Children who Succeeded		-	.04	.96	1

	Different Number - Different Ratio				
Performance	*Irrelevant*	*Focus on one variable*	*Quantification*	*Relations*	*Total*
Children who Failed	.03	97	-	-	1
Children who Succeeded	.01	-	.99	-	1

	Commutativity				
Performance	*Irrelevant*	*Focus on one variable*	*Quantification*	*Commutativity*	*Total*
Children who Failed	.03	.93	0.3	-	1
Children who Succeeded	-	-	.93	.07	1

TABLE 3.11

Proportion of justifications in the continuous quantities experimental task as a function of children's performance

	Same Number - Different Ratio				
Performance	*Irrelevant*	*Focus on one variable*	*Quantification*	*Relations*	*Total*
Children who Failed	.58	.42	-	-	1
Children who Succeeded	-	-	-	1	1

3.4 DISCUSSION AND CONCLUSIONS

The aim of this study was to investigate the origins of children's understanding of multiplication. Two possibilities were considered: a) that multiplication originates from the understanding of additive structures and b) that multiplication originates from the understanding of one-to-many correspondence situations. If the first assumption was correct then the children were expected to order the corresponding ratios by following quantification strategies. The were expected to have difficulty to quantify the continuous quantities tasks that were beyond their quantification ability. If the second assumption was correct then the children were expected to order the sets that bore a correspondence relation to each other on the basis of multiplicative relations. The fact that some tasks involved discontinuous or continuous quantities was not expected to impose extra difficulty.

The findings of the study revealed that young children ordered the corresponding sets on the basis of relations. This is supported by the following findings:

a) There was a discrepancy in the children's performance in the tasks where ordering the size of the sets on the basis of correspondence relations was possible and the tasks where they had to quantify. This difference in the level of performance suggests that the children could use the correspondence relations to order the sets that bore a multiplicative relation to each other, but failed in the tasks that required a quantification strategy. Even the 5 year olds were able to consider the effect that the ratio of rabbits per hutch or the number of hutches per row had on the total number of rabbits. If children's judgements in the one and the two variable problems were made on the basis of quantification then children's performance would not have varied across the problem situations.

b) There were 5 year old children who could order the size of the sets on the basis of relations, but could not quantify their size. This finding suggests that their ability to reflect on correspondence relations did not stem from their quantification ability, but from their schemas of action.

c) The finding that children are able to order sets on the basis of one-to-many correspondence relations is strengthened by children's performance on the tasks involving continuous quantities. Continuous quantities could not easily be quantified even by the 7 year olds, but even the 5 year olds had no difficulty in ordering their size.

d) Further evidence is provided by children's justification. The justifications of the successful children in the one variable problems with discontinuous and continuous quantities suggest that they ordered the sets on the basis of multiplicative relations. The children, in their judgements, referred either to the ratio difference or to the difference in the elements in the two sets. It was only in the two variable problems where they attempted to quantify in order to find out the size of each set.

This study showed that young children who have not received instruction on multiplication at school have a good understanding of the mathematical properties of multiplication. The children exhibited a good understanding of one-to-many correspondence relations and of the idea that the sum in a multiplicative situation depends both on the number of elements in the set and on the ratio. Their understanding of the invariants of multiplication was not dependent on their ability to quantify. The children used quantification strategies only in the problems where the sets did not bear a simple correspondence relation to each other. The children performed at ceiling level even with continuous quantities that did not provide any source for quantification. These findings question Fischbein's hypothesis that the origin of multiplication is to be found in addition, because the children had an understanding of multiplicative relations before they could quantify the multiplicative problems.

One-to-many correspondence relations form the basis for the understanding of multiplication as an operation but do not, though, teach the children everything about multiplication. The participants in this study showed a poor understanding of the principle of commutativity in multiplication which is an important property of the operation.

In conclusion, this study showed that the origin of multiplication is to be found in children's one-to-many correspondence schema of action and not in their ability to quantify multiplicative problems. The one-to-many correspondence schema of action supports children's reasoning even in situations that are beyond their quantification ability.

CHAPTER 4

EXPLORING CHILDREN'S UNDERSTANDING OF DIVISION

4.1 Overview of the aims of the experiments on partitive and quotitive division

The aim of this study is to investigate children's understanding of the mathematical properties of division. There is evidence that young children are efficient in using sharing to distribute a quantity between a number of recipients (Davis and Pitkethly, 1990; Desforges and Desforges, 1980; Frydman and Bryant, 1988; Miller, 1984). They are competent even in complex sharing situations. For example, they can adjust the sharing process to take into account differences in the size of the shared units in order to obtain equal quotas (Frydman, 1990). But do they learn anything about division as an operation from their sharing schemas of action?

Sharing and division are conceptually close because in both cases the child has to share a quantity into equal size quotas. However, as suggested by Correa, Nunes and Bryant (1998) sharing and division are not the same thing. In sharing the child's only consideration is to give equal amounts to each recipient following a procedure in which each recipient is given one item and then another one and so on, until the total quantity is exhausted. In division, though, not only has the equality of the shared quotas to be considered, but also the child has to understand a whole new set of relations: the direct relation between the dividend and the quotient and the inverse relation between the divisor and the quotient. That means that in division the child has to understand that the more the quantity to be shared is, the more each recipient will get, and the more recipients there are, the less each will get if the shared quantity remains constant.

The aim of this set of studies is to investigate young children's understanding of the invariants of a sharing situation. It focuses on children's understanding of the relations between the three terms: the dividend, the divisor and the quotient. Do the children understand that there is a direct relation between the dividend and the quotient and an inverse relation between the divisor and the quotient? The understanding of the sharing relations between these terms constitutes a significant step beyond the simple activity of sharing towards the understanding of division as an operation.

Previous evidence provided by Correa (1995) suggested that by the age of 6 most children have an understanding of the inverse relation between the divisor and the quotient when the shared quantity is discontinuous, such as sweets. In another study involving only continuous quantities Sophian et al (1997) showed that 7 year olds can reflect on sharing relations but the level of success in their study with continuous quantities was lower than the level of success in Correa's study suggesting that the children have more difficulty to reflect on situations that result in fractions. This study aims to investigate children's understanding of sharing relations both with discontinuous and continuous quantities. Continuous quantities were included for two reasons. Firstly, because the sharing of continuous quantities results in fractions, which constitute a major source of difficulty for children (Behr, Harel, Post and Lesh, 1993; Post, 1981). Children's most common error, when asked to order fractions like ½, 1/4, 1/5, is to conclude that increasing the size of the denominator increases the value of the fraction (Gelman, 1991; Mack, 1990; Post, 1981). Would the children make the same error in relational problems when asked to order the relative size of the shared quotas or do they have an anticipatory schema based on their understanding of the relations between the terms? Can they judge that a cake shared between 3 children would result in bigger shares than the same cake shared between 5 children?

Secondly, the introduction of continuous quantities should ensure that the children would reason on the situation on the basis of relations only, because the quantification of a fractional problem like the above would be beyond the grasp of young children.

The hypothesis of the study is that the origins of children's understanding of division can be traced to children's sharing schemas because they preserve the same mathematical properties as the operation of division. If this hypothesis is correct then the children should be able to reflect on the *sharing relations* between the dividend, the divisor and the quotient both with discontinuous and continuous quantities, before being introduced to division at school at around the age of 8. If the alternative hypothesis that children come to understand division through quantification (Fischbein et al., 1985) is correct, then the children are expected to be able to reflect on the sharing relations in situations involving discontinuous quantities that they might be able to quantify when the computation is simple, but in the situations involving continuous quantities that are beyond their quantification ability this would not be possible.

The understanding of the direct relation between the dividend and the quotient was not examined because this relation was not considered to be a good indicator of children's understanding of sharing relations.In Sophian et al's study (1997) the children who could not apply the inverse relation between the divisor and the quotient when the number of recipients varied had no difficulty to apply the direct relation between the dividend and the quota in the situations where the amount of the shared quantity differed but the number of recipients remained the same. All the children had to understand was additive relations: the bigger the whole, the bigger the parts and vice versa ignoring the effect of the divisor. The study investigated children's understanding of the inverse relation between the number of recipients and the size of the quotas by varying the size of the divisor in situations where the dividend was kept constant.

The understanding of sharing relations was investigated in two situations: the partitive and the quotitive. The reasons for exploring sharing relations in these two situations is because each of them evokes a different schema of action. In partitive division problems the schema takes the form of sharing in a one-for-me, one-for-you fashion, whereas in quotitive division problems the schema involves constructing equal quotas until the total to be shared is exhausted. The two studies on partitive and quotitive division were designed in a parallel way to make possible the comparison of their results.

STUDY I

PARTITIVE DIVISION

The first study was designed to investigate whether young children who had not received any instruction in division had an understanding of the inverse relation between the size of the shares and the number of the shares in the context of partitive division problems both with discontinuous and continuous quantities. In partitive problems the size of the quantity to be shared as well as the number of recipients is known and the question refers to the size of the shared quotas. For example, with discontinuous quantities the children had to decide which group of cats would eat more fish if both groups had 12 fish, but in one group there were 2 cats sharing and in the other 4. A similar decision had to be made when the cats were sharing a continuous quantity like a fish-cake.

4.2 METHODS

4.2.1 Design

The children had to make their judgements on the relative size of the shared quotas both with Discontinuous (fish) and Continuous (fish-cakes) quantities in two Conditions that varied the number of recipients. The task conditions are presented in detail below:

The Same Condition

The amount of fish to be shared was the same in both groups of cats as well as the number of cats sharing. This condition was a control task to test whether the children had an understanding of the situation with which they were presented. For example, the children were shown two groups of 3 cats in each, sharing 12 fish. This trial was coded as trial 12 3(3) and means that a group of 3 cats and another group of the same size (3) are sharing 12 fish each.

[12 fish]	[12 fish]
[cat] [cat] [cat]	[cat] [cat] [cat]

The Different Condition

The amount of fish to be shared was the same in both groups but the number of cats sharing varied. This was the experimental condition. If the children understood that the fewer cats sharing the more they would get, then it can be claimed that they have an understanding of the inverse relation between the number of the shares and their size. For example, a group of 2 cats and another group of 4 cats were sharing 12 fish each. This trial was coded as 12 2(4).

[12 fish]	[12 fish]
[cat] [cat]	[cat] [cat] [cat] [cat]

The Different Condition was expected to be harder because it required the understanding of the inverse relation between the number of the quotas and their size. The more cats sharing, the less each would get.

Both conditions were presented with Continuous quantities as well but this time the cats were sharing a fishcake(s).

The size of the shared quantity varied. The number of fish to be shared was either 12 or 24 with Discontinuous quantities, and 1, 2 or 3 fishcakes with Continuous quantities. Varying the size of the dividend should indicate whether children's reasoning was affected by the size of the shared set. Each condition had four items with each dividend which were sufficient to test the consistency of children's responses.

Table 4.1 gives a detailed view of the experimental design.

The Rationale of the Numbers Used

For discontinuous quantities numbers 12 and 24 were chosen as the most appropriate for dividends, because each has four divisors and thus, it was possible to check eight times the consistency of children's responses. The introduction of large numbers like 24 was done to ensure that the children would order the size of the quotas on the basis of relations between the terms because the computation of the tasks where there were 24 fish to be shared would have been difficult especially for the younger children. With fractional problems both unitary and non-unitary fractions were included because there is evidence to suggest that children have a better understanding of the former (Goldblatt & Raymond, 1996).

The study also raised the question of whether the specific numerical comparisons used in each problem would affect children's performance. Spinillo and Bryant (1991) and Parrat Dayan and Voneche (1992) have suggested that half plays an important role in children's understanding of fractions and that children can solve problems more easily where the values cross the half boundary. To test the effect that half might play in children's reasoning about sharing relations, each condition had four pairs that crossed the half boundary. That means that in one group there were always 2 cats sharing as for example in trial "12 fish 2(3) cats". The other four pairs were within the half boundary. That means that the cats in both groups received less than half of the quantity of fish, as for example in trial "12 fish, 3(4) cats".

The study also controlled for the effect of increasing the numerical contrast between the alternatives. There were pairs that had a difference of one cat as in trial "12 fish, 2(3) cats" and pairs that had a bigger difference as in trial "12 fish, 3(6) cats". It was expected that the situations where the numerical contrast was greater the children would be encouraged to think about the inverse relation between the number of recipients and the size of the quotas.

TABLE 4.1

The design of the study

	Discontinuous Quantities		**Continuous Quantities**			
	Discontinuous		*Unit fractions*		*Non Unit fractions*	
Condition	Dividend	Divisors	Dividend	Divisors	Dividend	Divisors
	12 fish	2(2) cats	1fish-cake	2(2) cats	2 cakes	4(4) cats
		3(3)		3(3)		5(5)
		4(4)		4(4)		6(6)
Same		6(6)		6(6)		7(7)
	24 fish	2(2) cats			3 cakes	6(6) cats
		3(3)				7(7)
		4(4)				8(8)
		6(6)				9(9)
	12 fish	2(3) cats	1fish-cake	2(3) cats	2 cakes	4(6) cats
		2(6)		2(6)		4(5)
		3(4)		3(4)		5(7)
Different		3(6)		3(6)		6(7)
	24 fish	2(3)			3 cakes	6(8) cats
		2(6)				6(7)
		3(4)				9(7)
		6(3)				9(8)

4.2.2 Participants

The participants were (a) 32 4-year olds (16 male and 16 female), mean age 4.6; range 4 to 4.11; (b) 32 5-year olds (16 male and 16 female), mean age 5.5; range 5 to 5.11; (c) 32 6-years olds (18 female and 14 male); mean age 6.6; range 6 to 6.11; and (d) 32 7-years olds (15 male and 17 female), mean age 7.6; range 7 to 7.11.

This age range was chosen for the study because there is evidence suggesting that some 4 year olds are successful in distributing counters of the same value among recipients (Frydman, 1990). It is not, though, until the age of 5 that children are confident in sharing and have an understanding of the numerical equivalence of the shared quotas (Frydman and Bryant, 1988; Hunting and Davis, 1991).

The children were from two state schools in North East London and none of them had received any formal instruction in division at school according to the information given by the class teachers.

4.2.3 Materials

The material consisted of 18 cats (9 brown and 9 white), 48 identical grey fish and 6 identical round cakes all made from paper. All the cats were identical in size and features and differed only in their colour.

4.2.4 Procedure

The children were interviewed individually on their school premises in two sessions. The division of the interview into two sessions was necessary because the experiment lasted for a long time and it would have been tiring for the children to go through all the items in one session. Each child worked with the discontinuous quantities in one session and the continuous quantities in another session. The order of the sessions was systematically varied among the subjects; half of the subjects started with the discontinuous and the other half with the continuous quantities. The interval between the sessions was half a day: one session was given in the morning and the other in the afternoon of the same day.

In the Discontinuous quantities tasks each child was presented with two groups of cats, one composed of white and the other of brown cats, placed at different sides of the table. The children were allowed to play with the cats for a while and then, with the help of the experimenter, counted how many cats were at each side. The children were told that the cats were going to have their favourite dinner which was fish! The experimenter told them that there were 12 (or 24) fish to be shared fairly among each group of cats (See Picture 4.1). The experimenter actually counted with the children the number of fish assigned to each group of cats to avoid doubts about their equality, and put them in a pile to avoid responses based on correspondence procedures. The children were told that the cats were going to share the fish fairly among them and eat it all up. Then the experimenter pointed to one white and one brown cat and asked the children whether the white and the brown cats will received the same or different amount of fish . The children were told: *"Look at this white and this brown cat: do you think they will eat the same or a different amount of fish?"*. If the answer was "different" the children were asked to indicate which cat will eat more. After each response the children were asked to justify their answer independently of whether it was correct or not. An alternative way of setting the question in case the children could not understand it was: *"Look at this white cat and this brown cat: do you think that they will put the same or a different amount of fish in their tummies?"* The children were not allowed to manipulate the material and no feedback was given regarding the correctness of their answers.

The same procedure was followed in the continuous quantities tasks. The only difference was that the quantity to be shared among the cats was a fishcake(s).

The order in which the trials were presented as well as the position of the sets on the table (more cats on the right or on the left) were systematically varied across trials.

To ensure that the children understood the situation, they were initially presented with an exemplary situation in which they were allowed to manipulate the material. In the discontinuous quantities situation the exemplary problem was 2 fish to be shared by 2 white cats and 2 fish to be shared by 2 brown cats. The material was in the full view of the children and the question was: “Look at this white and this brown cat: do you think they will eat the same amount of fish or will one of them eat more?”. Then they were presented with a similar exemplary situation where 2 brown and 1 white cat sharing 2 fish each. If the children were not able to provide the correct answer to the above problems, their participation in the experimental tasks was terminated.

PICTURE 4.1

Partitive Division

Sharing Discontinuous Quantities

2 cats sharing 12 fish

3 cats sharing 12 fish

Which cats will eat more?

4.3 RESULTS

4.3.1 Preliminary Analysis

The screening procedure

The exemplary situation was used as a screening procedure. There were 9 4 year olds and 3 5 year olds excluded from the experimental procedure after failing in the exemplary situation.

Scoring children's responses

One point was given each time the child indicated correctly which cat would receive more fish or fishcake.

The distribution of children's scores in each condition shows that they were not normally distributed. The majority of children either gave the correct or the wrong response in all the items of each condition. The distribution of the children's scores in the Same Condition was skewed to the right indicating that the children were getting the maximum score and bimodal in the Different Condition showing that the children either got all the items correct or all items wrong. The distribution of children's scores is presented in Figures 4.1 to 4.4.

FIGURE 4.1

The distribution of children's scores in the same condition with discontinuous quantities

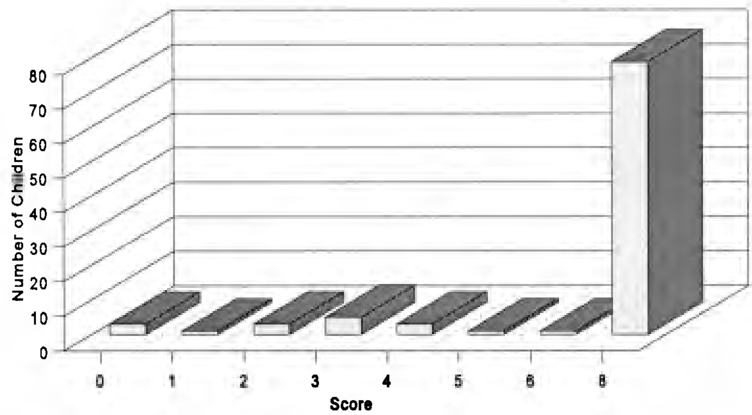


FIGURE 4.2

The distribution of children's scores in the different condition with discontinuous quantities

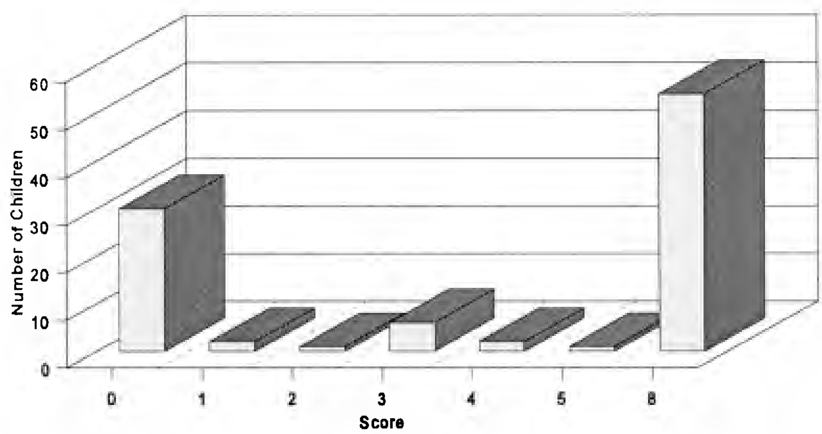


FIGURE 4.3

The distribution of children's scores in the same condition with continuous quantities

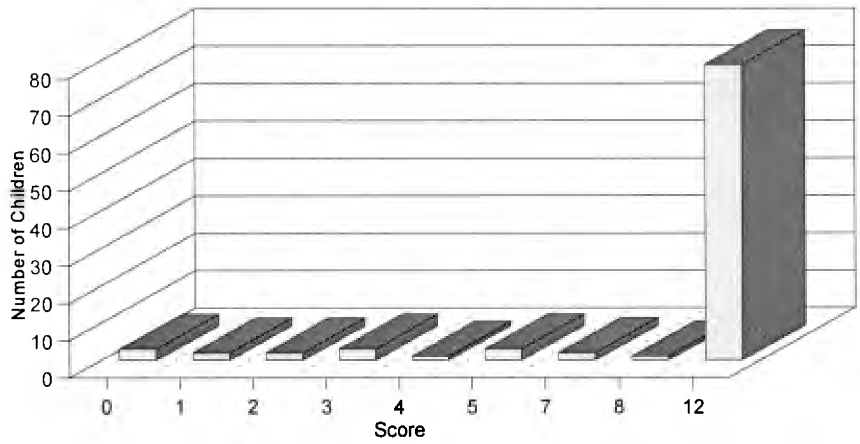
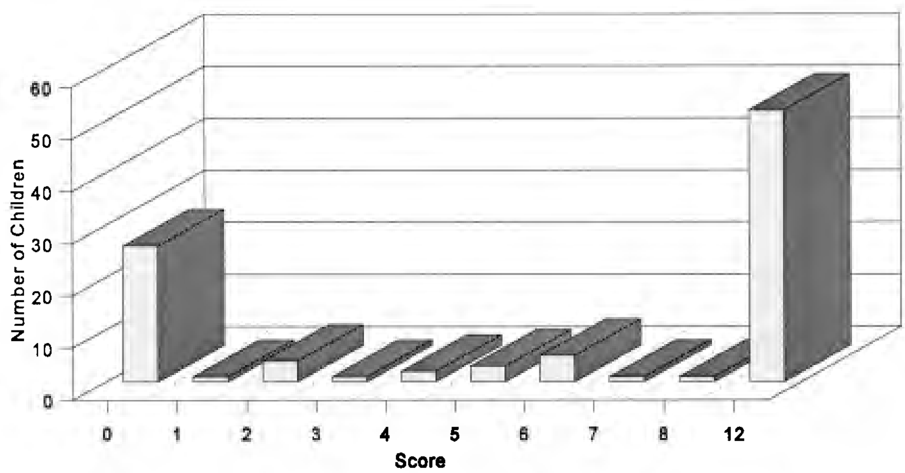


FIGURE 4.4

The distribution of children's scores in the different condition with continuous quantities



Children's scores as shown in Figures 4.1 to 4.4 are not a meaningful source of information on whether the children had an understanding of the relation between the dividend, the divisor and the quotient. If for example the child had a score of 2 out of 8 it had to be clarified whether this score was by chance or not. It could have been possible that the children got some answers correct by chance. The probability of correct answers occurring by chance was calculated in each condition. In each trial the children had to choose between three possible answers: (a) the white cat would get the same number of fish as the brown cat, (b) the white cat would get more fish than the brown cat, and (c) the white cat would get less fish than the brown cat. The probability of correct answers was estimated by means the Binominal Distribution. In order to be confident that a child did not answer correctly by chance (s)he had to answer correct a minimum of 6 or more out of the 8 items ($p<.01$). When there were twelve trials, as in the case of Continuous quantities it was necessary to achieve at least 9 correct answers out of 12 ($p<.01$).

For each condition the children were given either a passing score if they answered correctly significantly above chance level, or otherwise, a failing score. The number of children who had a passing and failing score in each condition, with discontinuous and continuous quantities is presented in Tables 4.2 and 4.3

TABLE 4.2

Number of children succeeding in the Same and the Different condition with discontinuous quantities by age and size of the dividend

Age	n	Dividend	*Same Condition*	*Different Condition*
4	32	12	6	2
		24	6	2
5	32	12	20	11
		24	20	11
6	32	12	27	17
		24	27	17
7	32	12	32	26
		24	32	26

TABLE 4.3

Number of children succeeding in the Same and Different condition with continuous quantities by age and size of the dividend

Age	n	Dividend	*Same Condition*	*Different Condition*
4 yrs	32	1	6	2
		2	5	2
		3	5	2
5 yrs	32	1	20	11
		2	20	10
		3	20	10
6 yrs	32	1	27	17
		2	27	16
		3	27	16
7 yrs	32	1	32	26
		2	32	26
		3	32	26

Because the majority of the 4 year olds obtained failing scores this age group was not included in the further analysis of the results.

As shown in the above tables the size of the dividend did not have any effect on children's responses. The children who were successful with dividend 12 were equally successful with dividend 24. Similarly, with Continuous quantities the children who were successful when the number of fishcakes to be shared was 1 were also successful when the number to be shared was 2 and 3. For this reason the data with the different dividends were merged and no distinction is made from here on between the discontinuous quantities tasks with dividend 12 and 24 and the unitary and non-unitary fractions.

Whether the compared pairs were within or crossed the half boundary did not affect children's performance. This finding contrasts with the finiding of other studies (Desli, 1997; Spinnilo and Bryant, 1991) that suggest that the presence of the half boundry can improve children's performance. A possible reason for not verifying this well established finding can be that the children of this study had to go through a large number of tasks, that encouraged the children to operate on the basis of an either a more-is-more or a more-is-less rule instead of focusing of the effect of the specific numbers used. Similarly, the numerical contrasts between the alternatives did not affect children's performance either. Children's performance did not differ in the situations where there were 2 cats versus 3 cats sharing and in the situations where there were 2 versus 6 cats sharing. (Appendix 4.1 and 4.2).

4.3.2 The effect of the condition and the type of quantity on children's performance

Because an overall analysis of variance was not possible, as there is not such a non-parametric test that allows a within and between subjects analysis with the number of conditions, each factor was worked out separately in each age group.

The frequency of correct responses in Tables 4.2 and 4.3 shows that there were more children succeeding in the Same than in the Different condition. When the performance of the children was compared across the two situations it was found that not all the children who did well in the Same condition performed well in the Different condition. Tables 4.4 and 4.5 show that there were 25 and 27 children who ordered the size of the quotas correctly in the Same condition with discontinuous and continuous quantities respectively, but failed in the Different condition. This finding means that the children had no difficulty in understanding the situation, but were not employing the inverse relation between the number of recipients and the size of the quotas to give a successful answer in the Different condition. All the children who answered correctly in the Different condition had no problem in the control condition, but not the other way round.

It is possible that the 17 children who failed in both conditions had no understanding of the situation or did not have any anticipatory schema to cope with it. The number of those children decreased with age and had disappeared at the age of 7. The number of children who succeeded in both conditions improved with age. One third of 5 year olds, half of the 6 years olds and around 80% of the 7 years olds were successful in the task both with Discontinuous and Continuous quantities. A Chi-Square test showed that there was a significant association between age and performance with Discontinuous (X^2=14.47, df=2, p<.001) and Continuous quantities (X^2=16.44, df=2, p<.001).

TABLE 4.4

Children's performance across conditions with discontinuous quantities by age

Age	n	*Fail Both Conditions*	*Pass Same Fail Different*	*Pass Both*
5 yrs	32	12	9	11
6 yrs	32	5	10	17
7 yrs	32	-	6	26
Total	96	17	25	54

TABLE 4.5

Children's performance across conditions with continuous quantities by age

Age	n	*Fail Both Conditions*	*Pass Same Fail Different*	*Pass Both*
5 yrs	32	12	10	10
6 yrs	32	5	11	16
7 yrs	32	-	6	26
Total	96	17	27	52

Tables 4.4 and 4.5 show that the total number of children giving correct responses with discontinuous and continuous quantities was not very different. Success with the continuous quantities that could not be quantified was the criterion for saying that the children's reasoning was based on sharing relations and not the quantification of the task. The children's scores were submitted to a further analysis in order to examine the importance of the types of quantities shared (discontinuous versus continuous) on their performance.

A Chi-Square test showed that there was a significant association between performance in the type of quantity presented in the problems (X^2=120.09, df=1, p<.0001). The children who were successful with discontinuous quantities, were the same, except two, who were successful with continuous quantities (Table 4.6). A McNemar test showed that this difference was not significant (p<.5). The consistency in the children's responses suggests that the children had a genuine understanding of sharing relations that was not affected by the type of quantities used in the task.

TABLE 4.6

Number of children succeeding and failing in the different condition across discontinuous and continuous quantities

	Discontinuous Quantities		
Continuous Quantites	*Pass*	*Fail*	Total
Pass	52	-	52
Fail	2	42	44
Total	54	42	96

4.3.3 Children's strategies and justifications

Children's ability to reflect on the effect of the number of splits on the size of the quotas especially with the continuous quantities that they could not quantify suggests that their ability to reflect on sharing relations did not stem from their ability to quantify. In order to provide more evidence on *how* the children reasoned on the relation between the divisor and the quotient, they were asked to justify their responses after indicating which cat was going to have more fish to eat. Children's justifications in the Different Condition were classified into five different categories. The order in which their strategies are presented does not indicate any levels of sophistication.

I. *Irrelevant justifications*

This justification does not involve any mathematical reasoning relevant to the solution of the tasks. It comprises the absence of justification, "I don't know" responses, personal preferences and socially desirable answers.

For example, when where there were 12 fish to be shared between 2 white cats and 12 fish to be shared among 3 brown cats many different responses were given:

Phoebe, a 5.2 year old girl, answered that "the white cat will eat more because it will cheat the others".

Jude, a 5.8 years old girl, answered that "the brown cat will eat more because it is more hungry".

Abdulraman, a 5.1 years old boy, answered that "both groups will eat the same amount because they are good friends and they do not want to fight".

Lee, a 5.4 years old boy, answered that "the brown cat will eat more because it will run faster and get more fish".

Emmanuel, a 5.7 years old boy, answered in the same question that "the white cat will eat more because I like it most".

This kind of justifications was frequently observed among the younger age group.

II. *Justifications focused on the dividend*

In this case children focused their attention on the size of the dividend, that is the amount of fish to be shared, without taking into account the number of cats sharing.

For example, when there were 12 fish to be shared between 2 white cats and 12 fish to be shared among 3 brown cats Peter, a 5.5 years old boy, said that all the cats will get the same amount of fish because "both groups have 12 fish".

III. *Justifications based on a direct relation between the divisor and the quotient*

In this case the children established an incorrect direct relation between the number of recipients and the size of the quota. That means that the children applied the "more-is-more" rule. They thought that the more cats there were, the more fish they would get. For example, when there were 12 fish to be shared between 2 white cats and 12 fish to be shared among 3 brown cats Claret, a 6.3 years old girl, said that the brown cats will eat more fish because "there are more brown cats".

IV. *Attempt to quantify*

Very few of the children justified their answers by quantifying the situation. For example, when there were 12 fish to be shared between 2 white cats and 12 fish to be shared among 3 brown cats Phil, a 7.2 years old boy, said that "the white cats will eat more because if you share 12 fish to 2 white cats each of them will get 6, because 6 and 6 equals 12. But if you share 12 fish to 3 brown cats each of them will get 4, because 4 and 4 and 4 equals 12". Alicia, a 7.3 years old girl, said that "the white cat will get more because it will eat half of the fish. The brown cat will eat less". The fact that some children attempted to quantify the situation does not necessarily mean that they gave the correct result.

With the Continuous quantities when there was 1 cake to be shared between 2 white cats and 1 cake to be shared among 3 brown cats Alexia, a 6.8 years old girl, answered "the white cats will get more because they will get half cake each, but the brown cats will get a smaller piece".

V. *Justification based on the inverse relation between the divisor and the quotient*

Children spoke in terms of an inverse relation between the values: the more cats there are, the fewer fish they would receive. For example, when there were 12 fish to be shared between 2 white cats and 12 fish to be shared among 3 brown cats Richard, a 6.9 years old boy, said that "the white cat will eat more because there are less cats here and they will get a bigger portion".

Two independent judges analysed children's justifications. Interjudge agreement for the two judges was 98.9%. Discrepant judgements were presented to a third judge. Because in all the cases the evaluation made by the third judge coincided with one of the first two judges' evaluation, this judgement was taken as final.

4.3.3.1 The effect of age on the type of justification given

Because the main interest of the study was in children's responses in the Different Condition tasks their justifications in this condition were analysed in relation to the age of the participants.

Tables 4.7 and 4.8 present the proportion of the types of justification given by each age group in the Different Condition with Discontinuous and Continuous quantities. As age increased children tended to give correct justifications. This finding was expected because as was shown earlier, there was an increase of correct responses in the Different Condition tasks with age. One third of the 5 year olds, half of the 6 year olds and 80% of the 7 years olds focused on the inverse relation between the number of recipients and the size of the shared quotas in their justifications. Many 5 year olds gave either no justification or an irrelevant justification (justification type I), but this type of reasoning decreased dramatically with age. Five and 6 year olds often applied the "more-is-more" rule between the values (justification type III). There was a small increase in the application of the "more-is-more" rule at the age of 6 with Discontinuous quantities. There was also an increase with age in children's attempt to quantify (justification type IV) their answers.

TABLE 4.7

The proportion of justifications with discontinuous quantities by age

Age	n	*Irrelevant*	*Focus on the Dividend*	*Direct Relation*	*Attempt to Quantify*	*Inverse Relation*	*Total*
5 yrs	32	.36	.04	.29	.01	.30	1
6 yrs	32	.12	.03	.35	.02	.48	1
7 yrs	32	.04	.02	.13	.06	.75	1

TABLE 4.8

The proportion of justifications with continuous quantities by age

Age	n	*Irrelevant*	*Focus on the Dividend*	*Direct Relation*	*Attempt to Quantify*	*Inverse Relation*	*Total*
5 yrs	32	.32	.04	.34	.01	.29	1
6 yrs	32	.09	.03	.33	.02	.53	1
7 yrs	32	.06	.01	.13	.03	.77	1

4.3.3.2 Types of justification and overall success

Further analysis demonstrated that there was also a strong association between the types of justification given and children's performance (Tables 4.9 and 4.10). The majority of the successful children pointed out the inverse relation between the divisor and the quotient (justification type V). Some of the successful children also tried to quantify (justification type IV). The majority of children who failed to indicate correctly which cat was going to eat more fish applied a direct relation between the divisor and the quotient. Some of the children who failed also gave no response or an irrelevant justification (justification type I) and, more rarely, focused their attention on the equality of the size of the dividends (justification type II). The distribution of the types of justification across wrong and correct responses was similar in discontinuous and continuous quantities.

TABLE 4.9

Proportion of justifications with discontinuous quantities as a function of children's performance

Performance	*Irrelevant*	*Focus on the Dividend*	*Direct Relation*	*Attempt to Quantify*	*Inverse Relation*	*Total*
Children who Failed	.32	.07	.58	-	.02	1
Children who Succeeded	.05	-	-	.06	.89	1

TABLE 4.10

Proportion of justifications with continuous quantities as a function of children's performance

Performance	*Irrelevant*	*Focus on the Dividend*	*Direct Relation*	*Attempt to Quantify*	*Inverse Relation*	*Total*
Children who Failed	.28	.06	.65	-	.01	1
Children who Succeeded	.06	-	.01	.03	.90	1

As it can be seen in the above tables some of the successful children's justifications are under the Irrelevant category. These children - 3 in total - gave no justification for their answer. They were successful both with Discontinuous and Continuous quantities but were too shy to speak and justify their response.

4.4 DISCUSSION AND CONCLUSIONS

The aim of this study was to investigate the origins of children's understanding of division. It was hypothesized that the origin of division is to be sought in children's sharing schemas of action. If this was correct then the children were expected to have an understanding of the relations involved in a sharing situation. They were expected to be able to understand a core relation: that the size of quotas is inversely related to the number of the quotas. If the alternative hypothesis was correct, i.e. that the origin of division lies in children's ability to quantify sharing problems then the children were not expected to be able to reflect on the inverse relation of the sharing terms in the problems involving continuous quantities that they could not quantity.

The results of the study revealed that young children have a good understanding of the inverse relation between the divisor and the quotient and that this understanding stems from their schemas of action and is independent of their ability to quantify. This was supported by the following findings:

a) The majority of children could order the size of the shared quotas with continuous quantities that were beyond their quantification ability. If understanding of sharing relations stemmed from quantification then the children were not expected to have any understanding of sharing relations in these tasks.

b) Children's understanding of the effect of partitioning a quantity among different number of recipients was not affected by the size of the shared quantity. This implies that the children reasoned on the situation on the basis of relations. If they had applied quantification strategies then they were expected to have some difficulties with the discontinuous quantities tasks where the dividend was large and it would have certainly been impossible to deal with fractions which would have been difficult even for older children who had already been taught them at school (Behr, Harel, Post and Lesh, 1993;

Goldblatt & Raymond, 1996).

c) Further positive evidence in favour of the hypothesis is provided by children's justifications. The vast majority of the successful children pointed out the inverse relation between the divisor and the quotient and consistently applied this relation in all the experimental tasks. Contrarily, those who failed applied a direct relation between the divisor and the quotient.

These findings confirm the hypothesis of the study, i.e. that the children have an understanding of the properties of division and that this understand arises from their schemas of action and not from their ability to quantify sharing situations.

The finding that children are able to reflect on an arithmetical situation on the basis of relations before they are able to compute sums are in accordance with Bryant's (1974) and Correa's (1995) previous findings.

Regarding the children who failed in the Different Condition but succeeded in the Same Condition their failure can be attributed to their difficulty in being able to apply the inverse relation between the number of recipients and the size of the quota. Those children had an understanding of the situation, because they did well in the tasks of the Same Condition, but did not have an anticipatory schema for the effect that the number of splits have on the size of the portions. The majority of the participants were led to wrong answers by assuming a direct relationship between the number of the recipients and the size of the quota. This type of error increased at the age of 6, before decreasing at the age of 7. This deterioration in children's answers was also observed by Correa (1995) and might reflect their effort to take into account both the number of the recipients in the sharing situation and the size of the shared set. This shift in children's focus occurred the moment that the children's irrelevant responses decreased and they started to view the situation as a mathematical one.

It is possible that children's experience with additive situations, where the more elements you add to a set the more you get, interfered and led them to conclude that the more sharing the more they would get.

These results are in accordance with Correa's (1994) findings who observed approximately the same level of success with children with discontinuous quantities. The 5 year old children of this study did, though, better in the continuous quantities tasks than the 5 year olds in Sophian et al's (1997) study, while there is not a difference between the two studies in the performance of the 7 year olds. The direct comparison between the findings of this study and Sophian's study should be done with caution because the two studies were carried in different countries and had a different design.

Age was found to have a significant effect on children's performance. It seems that the older the children get, the better understanding of partitive division relations they have. Age is treated here as a descriptive and not explanatory factor. Children's improvement with age could result from their everyday experiences with sharing or from school learning of related concepts. The design of the study does not allow us to select one possibility over the others.

To conclude, this study has provided strong evidence that before being introduced to division young children have a good understanding of a significant invariant of division: the inverse relation between the divisor and the quotient. This understanding stems from their sharing schemas of action and not from their ability to quantify the size of the quotas. Their schemas of action support their reasoning even in situation that are beyond their quantification ability. The understanding of the mathematical properties of division is a significant step from sharing towards the understanding of division as an operation.

STUDY II

QUOTITIVE DIVISION

The second study also investigated children's understanding of the inverse relation between the divisor and the quotient, but this time in the context of quotitive division problems. In quotitive problems the divisor is the size of the shared quotas. For example, "there are 12 fish and each gets 4. How many cats can have a share?". In quotitive problems the size of the shared quantity as well as the size of the shared quotas are given and the question refers to the number of quotas that can be formed. In this case, if the dividend is kept constant, the bigger the quotas are, the fewer there would be.

Children's understanding of the inverse relation between the divisor and the quotient was studied both with discontinuous and continuous quantities, in situations where they were asked to compare the relative number of quotas formed. For example, in the case of discontinuous quantities the children were presented with a situation where two cats had a number of 12 fish each. One cat was serving its fish in portions of 2s and the other in portions of 3s. The children had to decide which cat was going to invite more friends for dinner. The children had to recognize that an increase in the size of the quotas resulted in a decrease in the number of recipients.

4.5 METHODS

4.5.1 Design

The children had to make their judgements of the relative number of recipients both with discontinuous (fish) and continuous (fishcakes) quantities, in two conditions that varied the number of recipients. The conditions are presented in detail below:

The Same Condition

The amount of fish to be shared was the same in both groups of cats as well as the size of the shared quotas. This was a control condition to test whether the children understood the situation with which they were presented. For example, two cats had 12 fish each. Both cats shared their fish by giving 2 fish to each guest coming for dinner "trial 12 2(2)".

[12 fish]	[12 fish]
to be shared in 2s	to be shared in 2s

The Different Condition

The amount of fish to be shared was the same in both groups of cats, but the size of the shared quotas was different. For example, one cat shared its lot of fish in 2s and the other cat in 3s "trial 12 2(3)".

[12 fish]	[12 fish]
to be shared in 2s	to be shared in 3s

The children were asked to compare the relative number of recipients that could be invited in the two situations, that is which cat would be able to invite more friends. This was the experimental condition. If the children understood that the smaller the quota is, the more the recipients there could be, then it can be said that they have an understanding of the inverse relation between the divisor and the quotient.

The Different Condition was expected to be harder because it required the understanding of the effect of the size of the quotas on their number.

Both conditions were presented also with continuous quantities. With continuous quantities the cats were sharing pieces of fish-cake instead of fish.

The size of the shared quantities varied in both conditions. In some trials the cats had either 12 or 24 fish (discontinuous quantities) or 1, 2 or 3 fish-cakes (continuous quantities) to share. The size of the shared quotas was varied to see whether children's reasoning was affected by the size of the dividend. Each condition had 4 items with each dividend to account for the consistency of children's responses.

Because the sharing of continuous quantities resulted in fractions there was a procedural difference in the two conditions that might have affected children's responses. With discontinuous quantities the subjects were told the size of the quota, for example: "each guest will receive 3 fish". In contrast, with continuous quantities the children were only presented with the size of the quota and were not given any numerical information about its size because the quantity could only be described as a fraction, for example, a quarter of a cake. That would have been rather confusing of the children because they were not familiar with the fractional language. In order to make the two situations comparable an additional situation with discontinuous quantities was designed in which the subjects were merely presented with the size of the quota, without being told its cardinality. The order in which the two situations were presented was systematically varied across the subjects.

Table 4.11 gives a detailed view of the design of the study.

TABLE 4.11

The design of the study

	Discontinuous Quantities		**Continuous Quantities**			
	Discontinuous		*Unit fractions*		*Non Unit fractions*	
Condition	Dividend	Divisor	Dividend	Divisor	Dividend	Divisor
	12 fish	2(2) fish	1fish-cake	½(½) cake	2 cakes	½(1/2) cake
		3(3)		1/3(1/3)		1/3(1/3)
Same		4(4)		1/4(1/4)		1/4(1/4)
		6(6)		1/8(1/8)		1/8(1/8)
	24 fish	2(2) fish			3 cakes	½(1/2) cake
		3(3)				1/3(1/3)
		4(4)				1/4(1/4)
		6(6)				1/8(1/8)
	12 fish	2(3) fish	1fish-cake	½(1/4) cake	2 cakes	½(1/4) cake
		2(6)		½(1/8)		½ (1/8)
		3(4)		1/3(1/4)		1/3(1/4)
		3(6)		1/3(1/8)		1/3(1/8)
Different						
	24 fish	2(3) fish			3 cakes	½(1/4) cake
		2(6)				½(1/8)
		3(4)				1/3(1/4)
		3(6)				1/4(1/8)

The rationale of the numbers used

The numbers chosen and the contrasting pairs formed complied to the same criteria as in study I in partitve division.

4.5.2 Participants

The subjects of the experiment were (a) 32 5-year old (16 male and 16 female), mean age 5.7; range 5 to 5.11; (b) 32 6-year old (15 male and 17 female), mean age 6.7; range 6.1 to 6.11; (c) 32 7-years old (16 female and 16 male); mean age 7.6; range 7 to 7.10.

The children were from two state schools in North East London from the same Local Educational Authority as the children who participated in the partitive division experiment. None of them had received formal instruction in division at school according to the information given by their teachers.

4.5.3 Material

The material consisted of 2 cats (1 brown and 1 white), 2 dogs (1 brown and 1 white), 48 identical fish, 48 identical biscuits, 6 identical round cakes and pictures of plates having either 2, 3, 4, or 6 fish or biscuits or ½, 1/3, 1/4, 1/8 pieces of cakes, all made out of paper. All the cats and dogs were of identical size and features and differed only in their colour.

4.5.4 Procedure

The children were interviewed individually in a quiet area of their schools in two sessions. It was necessary to split the interview into two sessions because the experiment lasted a long time. The children worked with discontinuous quantities in one session and with continuous quantities in another and vise versa. The interval between the two sessions was half a day: One session was given in the morning and the other in the afternoon of the same day. The order the sessions were presented was varied systematically across the subjects.

In the discontinuous quantities tasks each child was presented with two cats the white and the brown cat, placed at different sides of the table. Then the children were told a little story about these cats. They had their birthday and therefore, each of them wanted to give a party - the brown party and the white party - and invite friends. Each cat had either 12 or 24 fish to give as a treat to its guests. The fish was placed in a pile on the table in front of each cat. The experimenter counted the fish in each pile with the children to make sure that there was no doubt regarding the equality of the two amounts. Then the children where told that the cats had a problem: they did not know how many friends to invite, because they did not know if there was going to be enough fish for all. The brown cat then decided to serve, for example, 2 fish to each friend coming to the party. A picture of some fish in a plate (2, 3, 4 or 6 fish) which corresponded to the amount of fish that the brown cat wanted to give as a treat to each visitor was presented (see Picture 4.2). The experimenter explained that the brown cat would take the 2, 3, 4 or 6 fish from her pile, put them on a plate like the one shown in the picture and invite a friend to have it. Then, she would take again the same number of fish out of the pile put them on another plate and invite another friend to have it. She will keep on doing this until no fish will be left. The same procedure was followed by the white cat. The experimenter then asked the child whether the two cats will invite the same or different number of guests (*Will the brown cat invite the same number of friends as the white cat, or different? Which cat will invite more?).*

A similar procedure was used for the discontinuous quantities situation where no numbers were mentioned. This time the children were presented with a parallel story: the dog and the bone shaped biscuits task. The number of biscuits that each dog was willing to give to its friends was presented in pictures with perceptual patterns so that the children could easily compare which picture had more elements. For example (::) for 4 biscuits and (:::) for 6 biscuits.

The same procedure was followed with the continuous quantities items. The difference this time was that the quantity to be shared was a cake and the children were presented with portions of this cake that corresponded to the size of pieces that each cat was willing to cut from the cake and treat each guest. Although the cakes were identical, the size of the pieces of cake varied: they were small (1/8 of the cake), medium (1/4 of the cake), big pieces (1/3 of the cake) or halves (½ of the cake). Again the children were asked to judge the relative number of guests that could be invited.

The children were not allowed to manipulate the materials. After each response the children were asked to justify their answers, independently of whether it was correct or not. The order in which the trials were presented and the position of the sets on the table (bigger shares on the right or on the left) was systematically varied across trials.

To ensure that the children understood the experimental situation they were presented with an exemplary situation in which they were allowed to manipulate the material. In the discontinuous quantities situation the example was: The brown and the white cat had a total of 4 fish each and wanted to give a dinner to their friends. The white cat wanted to share its lot in 2s as did the brown cat. The question was whether the two cats would be able to invite the same or a different number of friends. Then they were presented with another situation where the brown cat wanted to give to each friend 4 fish and the white cat wanted to give 2 to each. If they were not able to provide the correct answer their participation in the experimental tasks was terminated. In the exemplary situations the children were allowed to manipulate the material.

PICTURE 4.2

Quotitive Division

Sharing Discontinuous Quantities

The brown cat will share its lot of 12 fish in 2s

The white cat will share its lot of 12 fish in 3s

Which cat will be able to invite more friends?

4.6 RESULTS

4.6.1 Preliminary Analysis

The Screening Procedure

The exemplary situation was the screening procedure for this task. There were 15 4 year olds and 8 5 year olds excluded from the experimental procedure for failing in the exemplary situation.

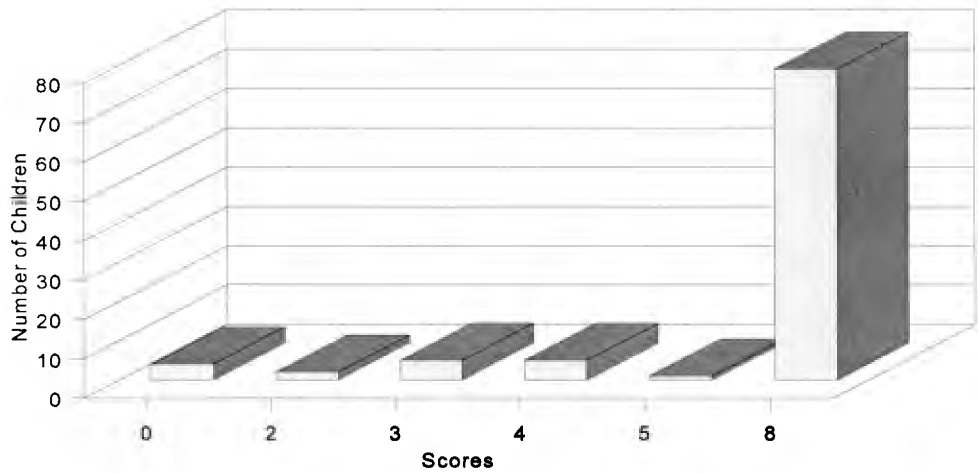
Scoring children's responses

One point was given each time the child indicated correctly which cat was going to invite more friends.

The distribution of children's scores in each condition showed that they were not normally distributed. The majority of children either gave the correct or the wrong response in all the items of each condition. The distribution of children's scores in the Same Condition was skewed to the right indicating that the children were getting the maximum score and bimodal in the Different Condition suggesting that the children either got all the items correct or wrong. The distributions are presented in figures 4.5 to 4.8.

FIGURE 4.5

The distribution of children's scores in the same condition with discontinuous quantities



FIFGURE 4.6

The distribution of children's scores in the different condition with discontinuous quantities

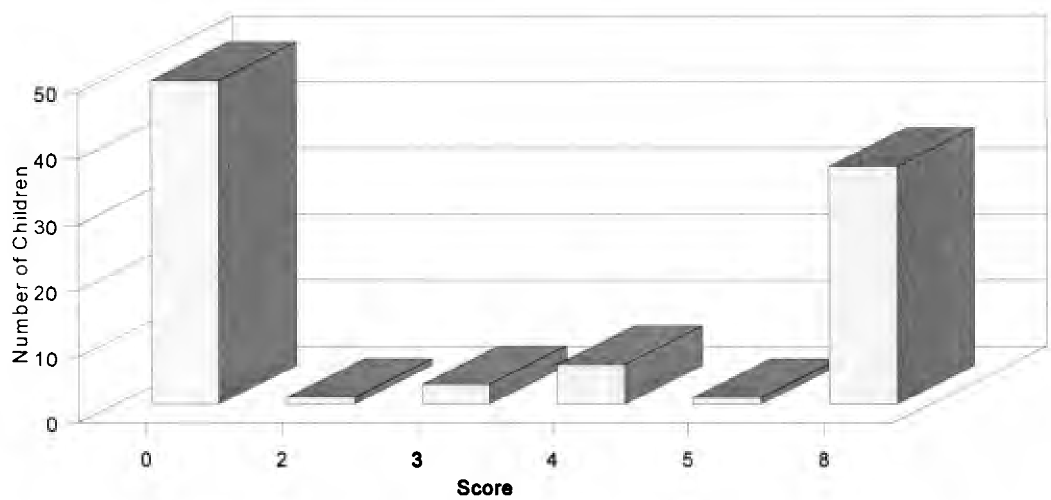


FIGURE 4.7

The distribution of children's s scores in the same condition with continuous quantities

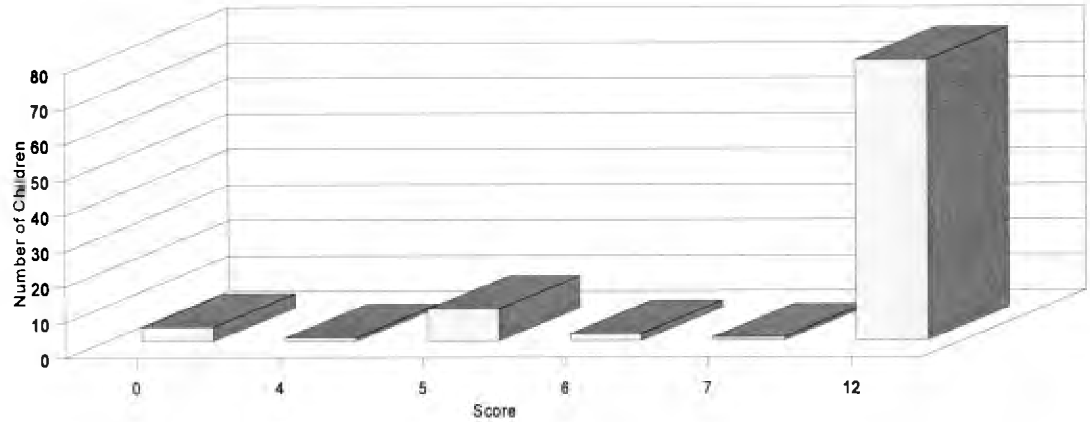
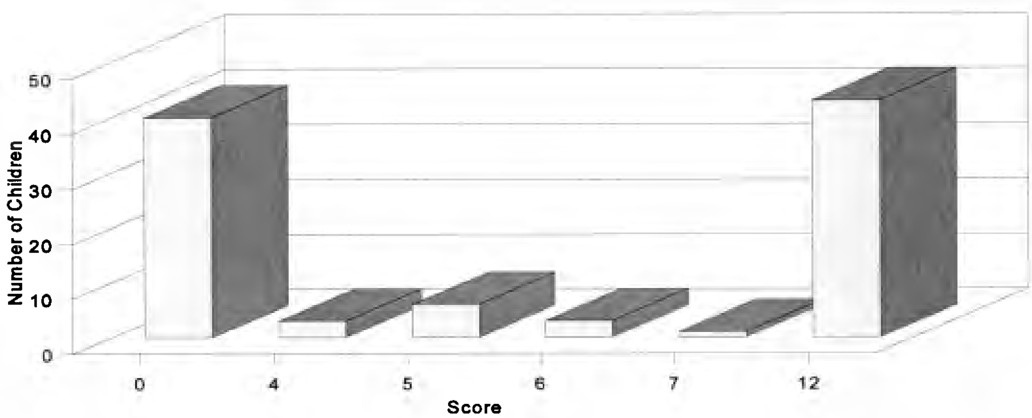


FIGURE 4.8

The distribution of children's scores in the different condition with continuous quantities



Children's scores as shown in Figures 4.5 to 4.8 are not a meaningful source of information on whether they had an understanding of the relation between the dividend, the divisor and the quotient. If, for example, the child had a score of 3 out of 8 in the different condition with discontinuous quantities it had to be clear whether this score was by chance or not. For this reason the probability of correct responses occurring by chance was calculated for each condition. In each trial the children had to choose between three possible answers: (a) the white cat would invite more friends than the brown cat, (b) the white cat would invite fewer friends than the brown cat, and (c) the white cat would invite the same number of friends as the brown cat. The probability of correct responses was estimated by means the Binominal Distribution. In order to be confident that a child did not respond by chance (s)he had to answer correct a minimum of 6 or more out of 8 items ($p<.01$). When there were twelve trials, as in the case of Continuous quantities it was necessary to achieve at least 9 correct answers out of 12 ($p<.01$).

In each condition the children were given either a passing score if their correct responses were at above chance level, or otherwise, a failing score.

The number of children who had a passing or a failing score in each condition, with discontinuous and continuous quantities is presented in Tables 4.12 and 4.13.

TABLE 4.12

Number of children succeeding in the Same and Different condition with discontinuous quantities by age and size of the dividend

Age	n	Dividend	*Same Condition*	*Different Condition*
5 yrs	32	12	16	5
		24	16	5
6 yrs	32	12	27	12
		24	27	12
7 yrs	32	12	32	22
		24	32	22

TABLE 4.13

Number of children succeeding in the Same and Different condition with continuous quantities by age and size of the dividend

Age	n	Dividend	*Same Condition*	*Different Condition*
5 yrs	32	1	16	7
		2	16	7
		3	16	7
6 yrs	32	1	27	15
		2	27	14
		3	27	14
7 yrs	32	1	32	23
		2	32	23
		3	32	23

With discontinuous quantities the children were tested in two conditions, one where numbers were mentioned and one where they were not. Children performed identically in the two situations. The children who succeeded when the size of the shared quota was described quantitatively were those who succeeded when no numbers were mentioned. For this reason the following analysis is based on children's responses in the situation where the size of the quotas was not mentioned.

As shown in the above tables the size of the dividend did not have any effect on children's responses. The children who were successful with dividend 12 were equally successful with dividend 24. The same was true with continuous quantities. For this reason no distinction is made between the discontinuous quantities tasks where the dividend varied and between unitary and non-unitary fractions.

Also, whether the compared pairs were within or crossed the half boundary or not did not affect children's performance neither did the difference in the size of the compared quotas (Appendix 4.3 and 4.4).

4.6.2 The effect of the condition and the type of quantity on children's performance

Because an overall analysis of variance was not possible, as there is no such non parametric test that would allow a within and between subjects analysis with the number of conditions, each factor was worked out separately in each age group.

Tables 4.12 and 4.13 show that there were less children successful in the Different condition both with discontinuous and continuous quantities. In order to examine how the children who succeeded in the Same condition performed in the Different condition their performance was compared across the two conditions.

Tables 4.14 and 4.15 show that not all the children who succeeded in the Same condition performed equally well in the Different condition. There were 36 children in discontinuous and 35 in continuous quantities who gave the correct response in the Same condition tasks but failed in the Different condition. There were no children, though, who did well in the Different condition without being equally successful in the Same condition. Children's success in the Same condition excludes the possibility that the children did not understand the question in the Different condition.

There were also children who failed in both situations. These children either had difficulty in understanding the situation or did not have the schemas to deal with it. The number of those children decreased dramatically with age and there were none at the age of 7.

The number of children who succeeded in both conditions improved with age. As can be seen in Tables 4.14 and 4.15 16% of the 5 year olds, 38% of the 6 year olds and 69% of the 7 year olds were successful in both conditions with discontinuous quantities, while 22% of the 5 year olds, 44% of the 6 year olds and 72% of the 7 year olds did well with continuous quantities. However, even by the age of 7 not all the subjects had acquired this understanding. A Chi-Square test showed that there was a significant association between age and performance in the discontinuous (X^2=18.91, df=2, p<.001) and continuous quantities tasks (X^2=14.23, df=2, p<.001).

TABLE 4.14

Children's performance across conditions with discontinuous quantities by age

Age	n	*Fail Both Conditions*	*Pass Same Fail Different*	*Pass Both*
5 yrs	32	16	11	5
6 yrs	32	5	15	12
7 yrs	32	-	10	22
Total	96	21	36	39

TABLE 4.15

Children's performance across conditions with continuous quantities by age

Age	n	*Fail Both Conditions*	*Pass Same Fail Different*	*Pass Both*
5 yrs old	32	13	12	7
6 yrs old	32	4	14	14
7 yrs old	32	-	9	23
Total	96	17	35	44

The above Tables show that the number of children succeeding in the Different condition was nearly the same with discontinuous and continuous quantities. Children's success with continuous quantities indicates that their schemas of action supported their reasoning even in the situations that they could not quantify. In order to examine how the children who could ordered the number of recipients with discontinuous quantities performed with continuous quantities their performance was compared across the two situations.

A Chi-Square test showed that there was an association between the type of quantity presented and children's performance (X^2=77.62, df=1, p<.0001). As it can be seen in Table 4.16 all the children who succeeded with discontinuous quantities were equally successful with continuous quantities. There were 5 children in total who succeeded in the continuous quantities task but failed with discontinuous quantities. A McNemar test showed that there was no significant difference in children's performance across the two types of quantities (p<.06). This finding suggests that the children can reflect on sharing relations even in situations that they cannot quantify.

TABLE 4.16

Number of children succeeding and failing in the different condition across discontinuous and continuous quantities

	Discontinuous Quantities		
Continuous Quantites	*Pass*	*Fail*	Total
Pass	39	5	44
Fail	-	52	52
Total	39	57	96

4.6.3 Children's strategies and justifications

In order to provide more evidence on whether the children ordered the relative number of the quotas on the basis of relations or by quantifying the situation the justifications of their responses were analysed. Their justifications are classified into five different categories derived from the observations of this study.

I. *Irrelevant justification*

This category includes justifications that did not consider any mathematical facts relevant to the situation. It also includes the absence of justification, "I do not know" responses, personal preferences and socially desirable behaviour.

For example, when there were 12 fish to be shared in 2s by the brown cat and 12 fish to be shared in 3s by the white cat Amit, a 5.2 year old boy, answered that "the brown cat will invite more friends because it is more popular". Hellene, a 5.3 year old girl, answered that "the white cat will have more guests because her friends are more hungry". Chris a 5.5 year old boy, answered that "both cats will invite the some number of friends because that is fair". Javier, a 5.7 year old boy, answered that "the brown cat will have more friends because it is a nice cat".

II. *Justification focused on the dividend*

In this case children focused their attention on the size of the dividend, that is on the number of fish to be shared, ignoring the size of each shared portion. For example, in the same example as above Nathaniel, a 5.8 year old boy, said that both cats will invite the same number of friends because "both cats have 12 fish".

III. *Justification based on the direct relation between the divisor and the quotient*

In this case children established an incorrect direct relationship between the size of the shared quota and the number of recipients. That means that the children applied the "more-is-more" rule. They thought that the bigger the quota is, the more the recipients would be. In the above example, Fabio, a 6.2 year old boy, said that the white cat will have more guests because it is sharing its fish in bigger portions.

IV. *Attempt to quantify*

In this type of justification children justified their answers by attempting to execute in their mind the operation of division and find out the result of sharing. In the example describe above Manoj, a 7.5 year old boy, said that "the brown cat will have more guests because if you have 12 fish and you give 2 to each friend you will have enough for 6. But the white cat gives 3 so it have enough for 4 cats. In the case of continuous quantities when there was 1 cake to be shared in portions of half by the brown cat and 1 cake to be shared in quarters by the white cat Nicola, a 6.8 year old girl, answered "the white cat will have more guests because it cuts the cakes in small pieces so it will have enough to treat more friends, while the brown cat cuts the cake in half and will be able to treat only two friends".

For the children who tried to quantify the situation it is not clear whether they first did the quantification and then gave their responses, or whether they quantified to verify their initial prediction or to explain to the experimenter why their response is the correct one.

V. *Justification based on the inverse relation between the divisor and the quotient*
Children spoke in terms of an inverse relation between the values: the bigger the portion the fewer guests could be invited. For example, Steven, a 6.7 year old boy, said that "the brown cat will have more guests because it is giving less fish to each guest so it will have more left, while the white cat will soon run out of fish".

Two independent judges analysed children's justifications. Interjudge agreement for the two judges was 97.1%. Discrepant judges were presented to a third judge. Because in all the cases the evaluation made by the third judge coincided with one of the first two judges' evaluations, this judgement was taken as final.

4.6.3.1 The effect of age on the type of justification given

Tables 4.17 and 4.18 present the types of justifications given by each age group in the Different condition with discontinuous and continuous quantities. As age increased children referred to more sophisticated types of justification. The five the and the six year olds applied more frequently the "more-is-more" rule. The percentage of children who applied a direct relation between the size of the quota and the number of recipients was still high at the age of 7. A few of the 7 year olds also tried to justify their responses by quantification. Fifteen percent of the of 5 year olds, 36% of the 6 year olds and 63% of the 7 years olds pointed to the inverse relations between the size of the quota and the number of recipients with discontinuous quantities. Regarding the continuous quantities 18% of the 5 year olds, 42% of the 6 year olds and 65% of the 7 year olds applied the inverse relation between the values. The increase of the justification referring to the inverse divisor-quotient relation in the continuous quantities tasks is due to the fact that there were more successful children with continuous quantities.

TABLE 4.17

The proportion of justifications with discontinuous quantities by age

Age	**n**	*Irrelevant*	*Focus on the Dividend*	*Direct Relation*	*Attempt to Quantify*	*Inverse Relation*	*Total*
5	32	.27	.03	.55	-	.15	1
6	32	.09	-	.53	.02	.36	1
7	32	.03	-	.28	.06	.63	1

TABLE 4.18

The proportion of justifications with continuous quantities by age

Age	n	*Irrelevant*	*Focus on the Dividend*	*Direct Relation*	*Attempt to Quantify*	*Inverse Relation*	*Total*
5 yrs	32	.29	.03	.48	.02	.18	1
6 yrs	32	.06	-	.49	.03	.42	1
7 yrs	32	.02	-	.25	.07	.66	1

4.6.3.2 Types of justification and overall success

Further analysis of the protocols demonstrated that there was a strong association between children success in the tasks and the types of justifications they gave. Tables 4.19 and 4.20 show that the children who gave the correct response were mainly those who applied the inverse relation between the size of the quota and the number of recipients. The children who gave wrong responses were mostly applying the "more-is-more" rule or giving an irrelevant justification.

TABLE 4.19

The proportion of justifications with discontinuous quantities as a function of children's performance

Performance	*Irrelevant*	*Focus on the Dividend*	*Direct Relation*	*Attempt to Quantify*	*Inverse Relation*	*Total*
Children who Failed	.20	.02	.78	-	-	1
Children who Succeeded	.01	-	-	.05	.94	1

TABLE 4.20

The proportion of justifications with continuous quantities as a function of children's performance

Performance	*Irrelevant*	*Focus on the Dividend*	*Direct Relation*	*Attempt to Quantify*	*Inverse Relation*	*Total*
Children who Failed	.21	.02	.77	-	-	1
Children who Succeeded	.04	-	-	.04	.92	1

4.7 DISCUSSION AND CONCLUSIONS

The aim of this study was to investigate the origins of children's understanding of division in their understanding of sharing relations in the context of quotitive problems. The hypothesis that the origins of division are to be found in children's schemas of action was contrasted with the alternative hypothesis suggesting that the origins of division are in children understanding of additive structures. If the former hypothesis was correct the children were expected to be able to reflect on sharing relations in situations they could not quantify. If the latter hypothesis was correct then the children were not expected to be able to reflect on sharing relations in situations that were beyond their quantification ability.

The results of the study revealed that young children have a good understanding of the inverse relation between the divisor and the quotient in the context of quotitive division problems. Their ability to reflect on sharing relations derived from their schemas of action that supported their reasoning even in the situations that they could not quantity. The fact that children's understanding of sharing relations stemmed from their schemas of action and not from their ability to quantify the division problems is supported by the following evidence:

a) By the age of 6 one third of the participants and by the age of 7 more than half of the participants were able to apply the inverse relation between the divisor and the quotient, although non of these children had been taught division at school.

b) Many children were able to order the relative number of quotas in the problems that involved continuous quantities. These problems were beyond the quantification ability of these young children. There is well documented evidence that quantifying fractions is difficult even for the children who have been formally taught them (Behr, Harel, Post and

Lesh, 1993; Goldblatt & Raymond, 1996). These children's reasoning was supported by knowledge derived from their schemas of action.

c) The ordering of the relative number of recipients was not affected either by the size of the shared quantity or by the size of the quotas. This was a common finding both with discontinuous and continuous. If the children had used quantification strategies to order the number of the quotas, then their performance would have varied as a result of the numbers presented in each problem. The children would have performed poorly in the tasks that involved larger numbers, but well in the tasks that involved smaller numbers. The fact that the successful children did well across all the items suggests that they reflected on the situation on the basis of relations only.

d) Additional evidence in favour of the hypothesis is obtained by children's justifications. The vast majority of the successful children referred to the inverse relation between the size and the number of the quotas and consistently applied this relation in all the experimental items. Conversely, those who failed consistently applied a direct relation between the divisor and the quotient.

Age was found to have a significant effect on children's performance. It seems that the older the children get, the better understanding of sharing relations they have. Note that age is treated here as a descriptive and not explanatory factor.

The results of this study regarding discontinuous quantities are in accordance with Correa's (1995) findings who also found that children were able to reflect on sharing relations in the context of quotitive situations long before receiving school instruction.

In conclusion, this study has shown that children have an understanding of sharing relations and particularly of the effect that the size of the quotas has on the number of quotas that can be formed. The children were found to have a good understanding of the properties of division even in the situations that they could not quantify. This finding is in favour of the hypothesis that the origins of division are in children's schemas of action and not in their ability to quantify division problems.

4.8 COMPARATIVE ANALYSIS OF PARTITIVE (STUDY I) AND QUOTITIVE (STUDY II) PROBLEMS

The aim of both studies was to investigate children's understanding of the inverse relation between the divisor and the quotient in two sharing situations: In partitive division problems (study I) where the divisor was the number of recipients and in quotitive division problems (study II) where the divisor was the size of the quota.

The two experiments were designed in a similar way in order to make the comparison between them possible. The subjects in both studies were between 5 to 7 years old and were at state schools in the same Local Educational Authority in North East London. It can be claimed that the two samples were similar, therefore, it is possible to compare them.

4.8.1 Response correctness

The frequency of correct responses in partitive and quotitive problems was compared in order to identify the effect that the type of problem had on children's reasoning (Table 4.21). The two studies were compared across conditions and types of quantities. A Chi Square test showed that there was a significant association between the type of problem presented and children's performance (X^2=4.69, df=1, p<.03). The children performed significantly better in the partitive than in the quotitive problems in the Different condition when the quantity to be shared was discontinuous. The difference between partitive and quotitive division was not significant across the age groups. There was not though a significant difference in children's performance in the Different condition when the quantity to be shared was continuous (X^2=1.68, df=1, p<.19).

TABLE 4.21

The number of correct responses in the partitive (N=96) and quotitive problems (N=96) across discontinuous and continuous quantities

	Type of situation	
Type of Quantity	Partitive	Quotitive
Discontinuous	54	39
Continuous	52	44

4.8.2 Discussion and Conclusions

The comparison of the number of children who showed an understanding of the inverse divisor-quotient relation in partitive and quotitive division situations showed that the children's ability to reflect on sharing relations was dependent upon the type of the sharing situation. On the whole the children could reason about the relations involved in a sharing situation more easily in partitive than in quotitive division problems when the quantities were discontinuous.

No differences, though, were found in children's performance when the quantities were continuous. This was a surprising finding and cannot be easily interpreted. What is important is that children had an anticipation schema about the relative size of the fractions based on the relations between the terms. This finding is even more surprising given that children have great difficulty in ordering and computing fractions at school. Children's success with continuous quantities that do not provide a source for quantification reinforces the hypothesis of the study that the origins of children's understanding of division are to sought in their schemas of action, and not in their ability to compute the number or the size of the shares.

These results are also in accordance with those of Correa (1995) who also found a difference in children's performance in favour of the partitive problems with discontinuous quantities.

The difference in children's performance in partitive and quotitive division situations can be attributed to the different schemas of action that children have for these situations. In partitive division the schema of action takes the form of sharing a quantity in an one-for-a, one-for-b fashion, while in quotitive division the schema involves the formation of equal quotas until the shared quantity is exhausted. Many studies have shown that children have no difficulty to act out sharing problems (Correa, 1995; Davis and Pitkethly, 1990; Desforges and Desforges, 1980; Frydman and Bryant, 1988; Miller, 1984), but find it more difficult in forming quotas to quantify quotitive problems (Correa, 1995). When Correa compared children's ability to act out partitive and quotitive problems in situations where they could manipulate the material she found that the quantification of quotitive problems was significantly harder than the quantification of partitive problems. It can be possible, then, that the difference in reflecting on sharing relations in partitive and quotitive situations is due to the fact that the children develop the sharing schema of action earlier on.

CHAPTER 5

THE COORDINATION OF MULTIPLICATION AND DIVISION

5.1 The aim and the rationale of the study

This study has two aims. The first aim is to investigate the coordination of multiplication and division *across* the multiplicative situations. It examines a) whether the two operations develop in a coordinated fashion or whether their understanding develops in parallel before they become coordinated and b) if the children have an understanding of the cancelling effect of one operation over the other. The second aim is to investigate the coordination of multiplicative relations *within* the operation of multiplication and division, i.e. the coordination of the sharing and forming quotas schemas of action in division and the understanding of commutativity in multiplication.

The coordination of multiplication and division across the multiplicative situations

The hypothesis of the study is that the two operations develop in parallel at the beginning and become co-ordinated at a later stage. This hypothesis is based on two ideas.

Firstly, the origins of multiplication and division are different because the two operations describe different situations and the schemas of action that the children have to deal with each situation are also different.

Multiplication is rooted in one-to-many correspondence situations. The schema of action that children have to deal with multiplication problems is to replicate correspondences. For example, if there are 4 chocolates in a box and there are 3 boxes, the child can quantify the total number of chocolates by replicating 3 times the ratio 1:4 and count the total.

Division is rooted in sharing situations. There are two schemas of action that can be implemented in sharing depending on whether it is a partitive or a quotitive situation. In partitive division, where the size of each quota is unknown, the schema of action is sharing in a one-for-me, one-for-you fashion. For example, if there are 8 sweets to be given to 2 children, in order to find how many sweets each recipient would get the child would give one-to-a, one-to-b, one-to-a, one-to-b and so on, until the total is used up. In quotitive division, where the number of quotas is unknown, the schema of action involves constructing equal quotas until the total to be shared is exhausted. For example, if there are 12 sweets and each child gets 3 in order to find out how many friends can be invited the child would make groups of 3s until the total of 12 sweets is exhausted and then count the number of groups formed.

Because multiplication and division originate from different situations and the actions performed are different, it is likely that the children do not understand their inverse relation from the beginning.

Secondly, it is possible that the development of multiplication and division is similar to the development of addition and subtraction. Although, subtraction is the inverse of addition, initially children have different and independent schemas of actions for the two operations. There is evidence that children are equally successful in solving addition and subtraction problems by modelling the actions described in the situation, but they only understand the inverse relations of the two operations later on (Carraher and Bryant, 1987; Carpenter and Moser, 1982; De Corte and Verschaffel, 1987; Hudson, 1983; Marton and Neuman, 1990; Riley, Greeno and Heller, 1983; Vergnaud, 1983, 1997). Missing addend problems, that have been used to investigate children's understanding of subtraction as the inverse of addition, are initially solved by additive strategies. For example, the problem " Anna has 4 chocolates. Mummy gave her some more. Now Anna has 7 chocolates. How many chocolates did mummy give her ?" describes an additive situation where one of the addends is missing. This problem can be solved in two different ways. One is an additive way that directly represents the actions described in the problem situation: the child counts out 4 counters (pauses), goes on counting up to 7 and then counts the counters that where added on to the 4 to get up to 7. The other way is by inverting the transformation, that is by subtracting the initial stage of 4 chocolates from the final stage of 7 chocolates. This solution does not represent directly the actions described in the problem situation, because an additive situation is quantified with subtraction. This strategy is only observed at a later stage when the children have co-ordinated the two operations. Vergnaud (1997) pointed out that the difficulty that the children have with missing addend problems is conceptual because it requires the distinction between the initial state, the final state, the direct transformation, the inverse transformation and the application of the inverted transformation to the final state.

Following the same line of reasoning it is possible to study whether multiplication and division develop in parallel and independent from each other or in a co-ordinated fashion. There is evidence (Carpenter et al. 1993; Kouba, 1989) that young children are equally successful in solving multiplication and division problems before receiving school instruction by modelling the actions described in the problem situation, but there is no research evidence on whether they understand their inverse relation. If children's schemas of action for the two operations develop in a co-ordinated fashion then they are expected to be able to deal with problem situations that do not match directly their schemas of actions.

Missing factor correspondence problems, that is problems where one of the terms in correspondence is missing, can be used to investigate children's understanding of the inverse relation between multiplication and division. In the problem "Three children came to my party and they all brought me the same number of flowers. I got 12 flowers. How many flowers did each child give me?" one of the terms in correspondence, that is the number of flowers per child, is missing. There are two possible solutions for this problem. Solutions based on correspondence procedures are feasible by trying different correspondences 1:2, 1:3, 1:4 until the total of 12 flowers is achieved. This trial and error correspondence strategy represents directly the actions described in the problem situation. The other way is by inverting the transformations. If the child has co-ordinated the schemas of actions for multiplication and division, then he would invert mentally the transformations and apply one of the two action schemas of sharing to solve the correspondence problem. S/he would either share the 12 flowers among the 3 recipients and count the size of the shared sets or make groups of 3s and count the number of groups formed. Such a solution strategy does not represent the actions described in the problem situation and shows that the child has made a connection between multiplication and division, because a correspondence situation is quantified with sharing.

Similarly, the understanding of the inverse relation between the two operations can be examined with sharing problems where one of the terms is missing. For example, the problem "Four children came to my party. I shared all my balloons among them. Each child got 3. How many balloons had I bought?" describes a sharing situation, but the size of the shared quantity is missing. Again, there are two possible solutions. The problem can be quantified by sharing procedures that match directly the actions described in the problem situation. The child can share balloons in a one-for-a, one-for-b fashion among the four children until each has 3 and then count the total number of the balloons shared. If the child has co-ordinated the action schemas for multiplication and division then (s)he would invert the transformations and use correspondence procedures to quantify the number of balloons. He would form 4 sets of 3s and count the total. Such a strategy does not represent directly the actions described in the problem situation and is based on the inversion of the transformations.

The second question on children's understanding of the relations *across* multiplicative situations examines the understanding of the inverse relation of the two operations in a different paradigm. It investigates the cancelling effect of one operation over the other. Do the children understand that a number multiplied and divided by the same number remains the same? Studies in additive situations explored children's understanding of the cancelling effect of subtraction over addition by asking them to compute problems like 5+7-7=?. Reaction time was used to indicate whether the child carried out the computation or if they quantified the problems based on the fact that the same number was added and then subtracted. It was found that children understood the cancelling relation of the two operations at a later stage. For this reason it was expected that the understanding of the inverse relation between multiplication and division would be a late acquisition.

The coordination of multiplicative relations within division and multiplication

Apart from studying the co-ordination of the sharing and multiplying schemas of action *across* the multiplicative situations, the study also aims to investigate the coordination of multiplicative relations *within* each operation.

In mathematical terms partitive and quotitive division are equivalent situations because they are both quantified by division. The children, though, have different schemas of action for every situation. In partitive division problems the schema involves sharing in a one-for-me, one-for-you fashion, whereas quotitive division problems are about constructing equal quotas until the total to be shared is exhausted. Do these two schemas develop in parallel, independent from each other, or do the children establish any relation between them? A sharing problem such as "I have 12 sweets and I want to give them all to 3 children. How many sweets will each child get?" can be solved either by acting out the problem situation and sharing the sweets among the 3 recipients in a one-to-a, one-to-b fashion and counting the size of the shared quotas or by forming quotas of 3s and counting the number of the quotas. Similarly, a quotitive problem such as "I have 12 sweets and I want to give 3 to each of my friend. How many friends can I invite?" can be solved either by modelling the actions described in the problem situations, that is, by forming quotas of 3s until the total is exhausted and counting the number of sets formed or by sharing the 12 sweets to 3 parts and counting the size of each quota. The alternative way to quantify the above mentioned sharing and quotitive problems does not match directly the actions described in the problem situation and requires the coordination of the sharing and the quotitive schemas of action. It was expected that the two schemas of action develop independently from each other and become coordinated at a later stage.

The coordination of the relations within multiplication is investigated by examining children's understanding of commutativity. Children have been documented to be competent in employing their correspondence schema of action to quantify one-to-many correspondence problems (Carpenter et al, 1993; Kouba, 1989). But do they learn anything about the property of commutativity from quantifying one-to-many correspondence problems? There is evidence suggesting that the concept of commutativity is a late acquisition (Frydman, 1990; Nunes and Bryant, 1995; Piaget, Kaufmann and Borquin, 1977; Pettito and Ginsburg, 1982). Although one-to-many correspondence problems are easily quantified by young children Nunes and Bryant (1995) have shown that the understanding of commutativity is harder in the context of these problems than in the context of spatial rearrangement problems. The sample of the aforementioned studies on commutativity consisted of adults or children who had received considerable amounts of instruction about multiplication and perhaps about the property of commutativity at school. This study aims to investigate the understanding of commutativity in the context of one-to-many correspondence problems with younger children. It was expected that the understanding of commutativity would be related to the coordination of partitive and quotitive division, because they both require the coordination of the relations between the multiplicative terms.

5.2 METHODS

5.2.1 Design

In order to test the coordination of multiplicative relations *across* situations two conditions were designed: the *Acting* and the *Reflecting* condition. The coordination of multiplicative relations *within* multiplication and division was examined in the *Display* Condition.

5.2.1.1 The coordination of multiplicative relations *across* multiplication and division

The Acting Condition

This condition aimed to study whether multiplication and division develop in a parallel or in a coordinated fashion. For this reason the children were presented with multiplication, partitive and quotitive division problems that they could act out with the help of the counters. There were two sets of problems (see Table 5.1). The first set consisted of the *Direct Situation Problems.* That means that it was possible to model the problems directly using the correspondence, the sharing and the forming quotas schemas of action. In the second set this direct modelling was not possible because a crucial piece of information was missing (e.g. the dividend in a sharing problem and the value of one of the corresponding variables in a multiplication problem). These were the *Inverse*

Situation Problems. It was expected that the children who have coordinated the three different schemas of action (sharing, forming quotas and setting into correspondence) would be able to invert the transformation and deploy an action usually connected with division to solve a multiplication problem and vice versa.

It has to be noted that in the sharing problems the word "share" or "sharing" was not used. The word "give" was used instead. This precaution was taken because in the pilot study when the children were told to share, for example, their sweets among 3 children, some children wanted to keep some sweets for themselves, but when they were told to give all the sweets to 3 children then it was clear that they could not include themselves.

Each child was presented with one problem of each type: one direct and one inverse partitive, quotitive and correspondence problem.

The study controlled for the size of the numbers presented in the problems to ensure that they were within the counting range of the young age group. The number of recipients and the size of the quotas/sets never exceeded number 8, while the size of the whole in correspondence problems never exceeded 24. The order in which the different sets of numbers were presented was systematically varied across problems. For example, the set of numbers 3 - 4 - 12 was given to one participant in a direct partitive problem, to another participant to an inverse partitive problem, to another in a correspondence problem etc. This was done to control for the effect that the size of the numbers presented could have had on the children's performance. The child was never presented the same set of numbers within the same condition.

TABLE 5.1

The direct and inverse situation problems of the acting condition

	DIRECT	INVERSE
PARTITIVE		
	I have 15 sweets. I want to give all the sweets to 3 children. How many sweets will each receive?.	I had a party. Three children came Each child brought me the same number of flowers. I got 15 flowers. How many flowers did each child bring?
QUOTITIVE		
	I have 15 sweets. I want to give to each friend coming to my party 3. How many friends can I invite?	I had a party. Each child that came brought me 3 flowers. I got 15 flowers. How many children came?
CORRESPONDENCE		
	I bought 3 boxes. Each box had 4 chocolates in it. How many chocolates do I have?	I had a party. Four children came. I gave all my balloons to them and each got 3. How many balloons did I have?

The Reflecting Condition

This condition aimed to study children's understanding of the inverse relation between multiplication and division in situations where one operation cancelled the effect of the other. Research in addition has presented the children with computation problems like 5+7-7=?. The current study could not follow this trend because an expression like 5x7:7=? would have been an inconceivable sequence of numbers for the young participants who had no received any instruction on multiplication and division. For this reason the children were presented with word problems where a number was multiplied and then divided by the same number.

Two situations were introduced. Both involved setting sets in correspondence and then sharing. In one situation the number of recipients was the same as the number of sets, for example:

> " I have 4 boxes. Each box has 3 biscuits in it. I want to give all my biscuits to 4 friends. How many will each friend get?"

In the second situation the number of recipients was the same as the size of the corresponding sets. For example:

> " I have 5 bags. Each bag has 3 lollies. I want to give all my lollies to 3 friends. How many will each get?"

The problems where the number of recipients were equal to the size of the quotas and the problems where the number of recipients was equal to number of the quotas were mathematically equivalent. They both obeyed the rule that a number multiplied and divided by the same number remains the same. The children were encouraged to think about the problem first and try to infer the size of the quotas *without* the help of the counters. If the children could not quantify the problem then they had the option to act out the situation with the help of the counters.

In the last problem above number *3* (3 lollies in each bag and 3 friends) was presented consecutively. To control for the order in which the numbers were presented in each situation each child was presented with one situation where the identical numbers were presented consecutively and one where they were not. For example, the above problem was also presented as:

"I bought bags of lollies. Each bag had 3 lollies. I bought 5 bags. I want to give all my lollies to 3 friends. How many will each get?"

The children were presented with four problems in total. Two where the number of recipients was the same as the number of the quotas and two where the number of recipients was the same as the size of the quotas. The sets of numbers presented were systematically varied across the participants. The children were never presented the same number set twice within the same condition.

5.2.1.2 The coordination of multiplicative relations *within* multiplication and division

The Display Condition

This Display condition aimed to investigate a) the understanding of commutativity in one-to-many correspondence situations and b) the co-ordination of the two different schemas of action that the children have for partitive and quotitive division. Two situations were designed: The *Direct* and the *Indirect Situation.* In both situations the problem was acted out by the experimenter and the child was asked to look at the arrangement of the counters to quantify the problem. The tasks used in this condition were adopted from Squire's (in progress) study.

The Direct Situation

In the Direct Condition the partitive, quotitive and correspondence problems were acted out with the counters in a way that gave a direct match to children's schemas of action.

For example, in the partitive division problem "I have 20 chocolates and I want to give them all to 5 children. They will all receive the same number of sweets. How many will each child get?" the children were shown the 20 chocolates shared out in 5 sets.

xxxx xxxx xxxx xxxx xxxx

The children could quantify the problem without carrying out any sharing action because the number of quotas was equal to the number of recipients.

In the quotitive problem "I have 12 lollies and I will give 3 to each friend coming to my party. How many friends can I invite?" the children were shown the 12 lollies sorted out in quotas of 3s.

xxx xxx xxx xxx

Similarly, in the multiplication problem "Seven children came to my birthday party. Each child gave me 3 flowers. How many flowers did I get?" the children were shown 7 sets of 3s.

xxx xxx xxx xxx xxx xxx xxx

The Direct Situation was a control condition to examine whether the children could identify a display relevant to the problem.

The Indirect Situation

The Indirect Situation aimed to investigate the coordination between partitive and quotitive division concepts and the understanding of commutativity in multiplication. In order to investigate the co-ordination of the sharing and forming quotas schemas of action the children were presented with partitive problems that were acted out by forming quotas and quotitive problems that were acted out as sharing problems.

For example, in the problem "I have 12 flowers and I want to give them to 3 children. How many flowers will each child receive?" the children were shown 4 sets of 3s. Such a display would have been helpful if the problem was a quotitive one.

xxx xxx xxx xxx

The children could quantify the problem without performing any sharing action if they had coordinated the relations between the multiplicative terms. They could reflect on the internal relation between the number of recipients and the size of the quotas, that is each child would take 4, because (s)he gets one flower from each set.

Similarly, in the quotitive problem "I have 15 chocolates and I will give 3 to each friend coming to my party. How many friends can I invite?" the children were shown 3 sets of 5s. The problem was displayed with a sharing solution.

xxxxx xxxxx xxxxx

The problem could be solved without performing any actions if the children could reflect on the relation between the number of the quotas and their size; each friend will get one from each set, therefore, five friends can be invited.

Children's understanding of commutativity was investigated in one-to-many correspondence situations where the corresponding sets were rearranged with respect to the commutativity rule. For example, in the problem " I bought 4 boxes. Each box had 3 crayons. How many crayons do I have" the crayons were rearranged in 3 sets of 4s as shown below

xxxx xxxx xxxx.

If the children had an understanding of commutativity then they could quantify the problem by means of the display.

The sets of numbers used in the problems were systematically varied across problems and across the participants. The children were never presented the same set of numbers twice in the same condition.

5.2.2 Participants

The participants of the study were (a) 30 5-year olds (14 male and 16 female), mean age 5.6; range 5.1 to 5.11, (b) 30 6-years olds (17 male and 13 female); mean age 6.7; range 6 to 6.11, (c) 30 7-year olds (15 male and 15 female); mean age 7.7; range 7 to 7.11, and (d) 30 8-year olds (15 male and 15 female), mean age 8.6; range 8.1 to 8.11.

The children were from three state supported play-scheme groups in North East London. According to the information given by the children, none of the 5 and 6 year olds had received instruction on multiplication and division. Many 7 year olds said that the had done the times tables at schools, but they had not done division. It was only the 8 year olds who had been introduced both to multiplication and division.

5.2.3 Materials

The only material used were 60 square plastic red and yellow counters.

5.2.4 Procedure

The children were interviewed individually in their play-scheme premises in two sessions. The division of the interview into two sessions was necessary because the experiment lasted approximately for 45 minutes and it would have been tiring for the children. This division was facilitated by the fact that the study itself had two sections, one examining the coordination of relations across and the other within multiplicative situations. The children were tested on the tasks of one situation in the morning and the tasks of the other situation in the afternoon of the same day. The order of presenting the tasks of the across and within situations was systematically varied across the children.

The children were told that they were going to play some number games. Each problem was initially read aloud slowly to them. Then the children were asked if the problem was clear. In order to make sure that they could remember the problem they were asked to say the problem to the experimenter. The experimenter repeated the problems to the child as many times as was necessary.

In the Acting Condition the children had the option of using counters to act out the situation. The were told that they could take counters from the pile, that was on the table, if they thought it was going to be helpful for them.

In the Display Condition the experimenter was the one who manipulated the counters. For the division problems she read the problem to the child, then counted aloud as many counters as needed from the pile to represent the quantity to be shared and then without giving any more explanations of what she was doing she arranged them into quotas. The quotas were presented in a way that the children could easily see that they were all equal. Then she read the problem again to the child. The children were asked to look carefully at the display on the table, because they were told that there were some cues that could help them find the answer. If they did not find the display helpful, then they could manipulate the counters to find the answer.

If child responded immediately by guessing s/he was encouraged by the experimenter to think again, look at the display or use the counters, depending on the experimental condition, to make sure that s/he would give the correct response. After responding, each child was asked to explain to the experimenter how (s)he had worked out the answer.

The pilot study had shown that some children, especially the younger ones, needed some encouragement and probing in order to reach a solution. For example, in the partitive division task where the children had to share the counters some children did not exhaust the quantity to be shared, although it was stressed that they had to give all their sweets to their friends. In this case the experimenter had to remind the children that they had to give all of them to their friends. If the children stopped in the middle of the process either because they could not remember what they wanted to do next or what the question was about, the experimenter probed them by repeating the problem and by asking them what they thought they could do next.

5.3 RESULTS

5.3.1 Introduction

The results section is divided in two parts. The first part examines the coordination of multiplicative relations *across* the multiplicative situations. The analysis of children's responses in the Acting and the Reflecting Condition provides evidence on whether the children have coordinated the schemas of action they have for multiplication and division and whether they have an understanding of the cancelling effect of one operation over the other. The second part looks at the coordination of multiplicative relations *within* multiplication and division. The analysis of children's performance in the Display Condition examines the coordination of the two schemas of actions that can be implemented in a sharing situation, that of sharing and forming quotas and the understanding of commutativity in multiplication.

5.3.2 The Coordination of Multiplicative Relations *Across* Multiplication and Division

5.3.2.1 The Acting Condition

5.3.2.1.1 A Preliminary quantitative analysis

The aim of this condition was to examine the coordination of the operation of multiplication and division. If multiplication and division develop in a coordinated fashion then the children are expected to use their schemas of action to quantify problems that give a direct match to them (direct problems), and also use the *same* schemas to quantify problems that involved multiplication and division as the inverse operation (inverse problems). In order to quantify inverse problems the children should mentally invert the transformations described in the problem situation. It was expected that the children would have difficulty in quantifying problems that involve multiplication and division as inverse operations because it is hypothesized that initially the two operations develop independently of each other.

In order to examine whether the two operations develop in a coordinated fashion or not, children's performance was compared in situations where multiplication and division was employed as a direct and inverse solution. Three pairs of problems were compared: a) problems where sharing was the direct and the inverse solution, b) problems where forming quotas was the direct and the inverse solution and c) problems where setting elements into correspondence was the direct and the inverse solution.

Table 5.2 suggests that the children were more successful in the problems where they could implement sharing and form quotas as a direct solution than as an inverse solution. That means that although the children had the strategy to tackle inverse problems they did not use it.

TABLE 5.2

The number of children succeeding in the direct and inverse problems by age

		Sharing		Forming Quotas		Setting into Correspondence	
Age	n	Direct	Inverse	Direct	Inverse	Direct	Inverse
5	30	16	3	9	5	11	11
6	30	22	9	15	11	21	20
7	30	27	17	24	19	26	26
8	30	30	24	25	24	30	29
Total	120	95	53	73	59	88	86

There was a clear age trend in children's performance. The older the children were the better they performed. A Chi-Square test on the effect of age on children's performance showed that the older children were more competent in solving direct sharing (X^2=22.78, df=3, p<.0001), direct quotitive (X^2=24.44, df=3, p<.0002) and correspondence problems (X^2= 23.43, df=3, p<.0001). This age trend was also observed in the inverse sharing (X^2= 34.16, df=3, p<.0001), inverse quotitive (X^2= 28.37, df=3, p<.0001) and inverse correspondence (X^2= 31.02 df=3, p<.0001) problems.

A small parenthesis. This was the first time young children were asked to quantify partitive and quotitive division problems. Therefore, it was possible to compare their performance across the experiments on relational and computational division problems. This comparison is discussed in detail in Appendix 5.1.

5.3.2.1.2 Children's Strategies

In order to form a more elaborated picture of how the children quantified the direct and the inverse problems their strategies were described.

A distinction is made between *Active Strategies* and *Verbal Strategies*. In the Active strategies the children used the counters to quantify the problem situation. In the Verbal Strategies they did not use the counters, but solved the problem orally. Children's wrong strategies as well as the sources of error in implementing their schemas are also presented. The solution strategies that were observed in the problems of the Acting Condition are presented in detail below.

Active Strategies

I. Sharing

In sharing the children equally distributed a quantity between a number of recipients.

Sources of error

A number of sources of error was observed while the children implemented sharing. Children's errors were classified in two categories.

a) *Counting errors*: This category includes the children who had the sharing schema of action, but who failed to give the correct response because they made a counting mistake while counting the size of the shared quotas. A few children, in counting the size of the quotas, included the counters that stood for the recipients, because they were not different from the other counters.

b) *Conceptual errors:* This category comprises two types of errors. a) Errors due to the fact that the children did not know how to share systematically. There were children who did not give a share to some of the recipients or who gave different amounts to each recipients. b) Errors made by children who knew how to share, but when asked to quantify the size of the shared quotas either counted the number of the quotas or counted the total number of counters shared out. They had not yet coordinated the action of sharing and counting.

II. Forming Quotas

In this schema of action the children formed equal sized quotas and from the number of quotas formed they inferred the number of recipients.

Sources of Error

The errors that were observed when the children formed quotas were classified into two categories depending on the nature of the errors.

a) *Counting errors*: The children who made counting errors had the schema of action that could led them to quantify the problem correctly, but failed to give the correct responses because they did a counting mistake while counting the number of the quotas formed.

b) *Conceptual errors:* This category comprises the children who lacked coordination between the action of forming quotas and counting. These children formed the quotas, but when asked to quantify the number of recipients they either counted the size of the quotas or the number of the counters they shared.

III. Setting into Correspondence

By setting into correspondence the children replicated two corresponding sets x times to build up the size of the whole.

The mistakes that were observed were *counting mistakes,* while the children counted the counters to determine the size of the whole.

It has to be stressed that the children implemented the above schemas of action in various ways depending on the number of variables they represented with the counters (the size of the quantity to be shared, the number of recipients, the number of the corresponding sets or the size of corresponding sets) and the way they acted out the situation (sharing in one-for-me, one-for-you fashion, sharing in small sets, sharing in a double counting fashion). Because these variations in acting out the problems were not found to be related to the coordination of the two operations they are not discussed in detail in this study. More information on the variations of the main schemas of action presented here are given in Appendix 5.2.

Verbal Strategies

The children solved the problems without the help of the counters but by following oral counting strategies and number facts. Children's verbal strategies were classified in two categories that are described in detail below.

I. Counting strategies based on addition and number sequences

This category includes the children who quantified the problems by counting or uttering a string of numbers. For example, in a correspondence problem where 5 children brought 3 sweets each, a child could have said: 1, 2, 3 (pause), 4, 5, 6 (pause) ... 13, 14, 15! while other children explained that they added 3s like 3, 6, 9, 12, 15.

II. Number facts based on times tables

There were children who quantified the problems based on number facts they derived from the times table. Because multiplication is sometimes taught as a string of numbers like 3, 6, 9, 12, 15 it was difficult to distinguish whether the children used addition or the times table. For this reason only the children who explicitly used the times expression (5x3=15) or referred to the times table ("3, 6, 9, 12, 15, because that is how it goes in the times table") were classified under this strategy.

It has to be noted that it was not always observable how the children did the counting. In some cases they did their thinking aloud while in other cases all the thinking and quantification was done silently. In the second case, although the children explained to the experimenter what they did, it is possible that what they reported might not always be the same as what they really did. For example, in quantifying a correspondence problem where 3 children came to the party each bringing 5 sweets, a child might had initially add 5+5=10 and another 5, equal 15. This might have triggered his/her memory of the times table where 3x5=15. When the child was asked how (s)he quantified the problem (s)he might just have only said that it is in the times table that 3x5=15.

Sources of Error

In verbal strategies most mistakes were *counting mistakes.* Even when the children referred to number facts based on times table they made mistakes because they did not always know their tables well.

Strategies leading to wrong responses

I. False Strategies

The most common false strategy was that the children represented the variables of the problem and added them together. This strategy was observed both when the children used the counters to quantify the situation and when they used verbal strategies.

II. Lack of Strategy

This category includes all the children who did not have a strategy to quantify the problem. The children in this category gave the following types of responses:

a. Some children merely said "I do not know".

b. Other children repeated some information already present in the problem situation.

c. Some children represented one or both variables described in the problem situation with counters and could not proceed further.

5.3.2.1.3 Children's schemas as direct and inverse solutions

Table 5.3 on the frequency of the children who succeeded in the different problem situations suggested that the majority of the children had a strategy to quantify the problems that gave a direct match to their schemas of action, but there were fewer successful children in the problems that involved the inverse operation. This section examines a) how the children who were successful in the direct match problems performed in the inverse situation problems and b) the specific strategies that they employed to quantify the direct and the inverse problems. The importance of presenting children's strategies is justified by the fact that the children could quantify the inverse problems without necessarily inverting the transformations. As research in additive structures has shown (Carraher and Bryant, 1987; Marton and Neuman, 1990) many children who have not coordinated addition and subtraction quantify the missing addend problems not by subtraction, but by using additive strategies. Similarly, some of the children who quantified the inverse multiplication and division problems could have given the correct answer, without inverting the transformations.

The sharing schema as a direct and an inverse solution

The children in the study could employ the sharing schema of action to quantify two problems: a) the partitive division problem that gave a direct match to their schemas of action and b) the missing factor correspondence problem where the size of the corresponding sets was missing (inverse sharing problem). To quantify the latter problem by sharing, the children had to invert the transformations mentally. If the children had not coordinated the two operations of multiplication and division they were expected either not be able to quantify the inverse problem or to quantify it by correspondence procedures.

A Chi-Square test showed that there was a significant association between the type of problem presented - direct sharing versus inverse sharing - and performance (X^2=20.66, df=1, p<.0001). However, as Table 5.3 shows, that not all the children who quantified the problem that gave a direct match to their schemas of action were able to quantify the problem where sharing was the inverse operation. There were 43 children who quantified the direct situation, but failed to quantify the inverse situation, although both problems could be acted out by the same strategy. A McNemar test showed that the difference in the level of performance in the two problems was significant (p<.0001). The ability to quantify the direct situation problem was a prerequisite to being able to quantify the inverse situation problem. There was only one child who quantified the inverse situation having failed the direct sharing problem but this was due to a counting error.

TABLE 5.3

The number of children succeeding and failing across the direct and inverse sharing problems

	Direct Sharing Problem		
Inverse Sharing Problem	Fail	Pass	Total
Fail	24	43	67
Pass	1	52	53
Total	25	95	120

Success in the inverse sharing situation could also be achieved without inverting the transformations. Table 5.4 presents all the strategies that the successful children used in the direct and the inverse sharing situation. The majority of the children who quantified the direct problem reached the solution by sharing (62%) and number facts (21%). But not all the children who quantified the direct problem by sharing did sharing to quantify the inverse situation. Out of the 35 children who used sharing in the direct situation only 21 did sharing in the inverse situation. There were 12 children who used sharing in the direct situation but failed to invert the transformations and use sharing as an inverse solution. These children reached the correct solution by applying correspondence strategies. There were 6 additional children who used sharing in the inverse situation, although in the direct situation they had used verbal strategies. For the children who employed verbal strategies, such as counting and number facts, no conclusions can be drawn on whether they had coordinated the two operations. There is no way of knowing if they inverted the transformations mentally, (18 flowers : 3 children =?) and carried out the computational part with the help of times table (3x?=18) or if they treated the situation as a correspondence one and tried different numbers that multiplied by 3 give 18 (3x4=12/ 3x5=15/ 3x6=18).

Regarding the children who failed to quantify the inverse sharing problem although they had quantified the direct situation, Table 5.5 suggests that the majority of them did not have any strategy to quantify the problem and gave "I do not know" responses. Their success, though, in the direct situation suggests that they had a schema to quantify the inverse situation, but they could not mentally invert the transformations to interpret the situation.

TABLE 5.4

The frequency of the successful children's strategies in the direct and inverse sharing problems

Strategies in the Inverse Sharing Problem	Strategies in direct sharing problem				
	Sharing	Forming Quotas	Counting	Number Facts	Total
Sharing	21	-	2	4	27
Forming Quotas	-	-	-	1	1
Correspondence	12	1	-	-	13
Counting	1	1	2	-	4
Number Facts	1	-	-	6	7
Total	35	2	4	11	52

TABLE 5.5

The frequency of the strategies employed by the children who were successful in the direct sharing problems but failed in the inverse sharing problems

Strategies in the Inverse Sharing Problem	Strategies in direct sharing problem			
	Sharing	Counting	Number Facts	Total
No strategy	34	1	-	35
Correspondence	2	1	-	3
Counting Errors	1	-	1	2
Conceptual Errors	3	-	-	3
Total	40	2	4	43

The forming quotas schema as a direct and an inverse solution

The schema of forming quotas could be used to quantify a) quotitive division problems that gave a direct match to children's schemas of action and b) missing factor correspondence problems where the number of the corresponding sets was unknown (inverse quotitive problems) if they could invert the transformations mentally to interpret the problem situation.

A Chi-Square test showed that there was a significant association (X^2=56.58, df=1, $p<.0001$) between performance and the type of quotitive problems presented, direct versus inverse. When the performance of the children who were successful in the direct quotitive problem was compared with their performance in the inverse quotitive problem it was found that not all the children who quantified the direct situation were able to quantify the inverse situation. Table 5.6 shows that there were 17 children who quantified the direct quotitive problem, but, nevertheless, failed to quantify the inverse quotitive problem, although both problems could be acted out by the same schema of action. A McNemar test showed that the children were performing significantly better in the direct quotitive problems ($p<.002$). The ability to quantify the direct problems was a prerequisite for quantifying the inverse problems. There were only three children who failed to quantify the direct situation due to a counting error, but were successful in quantifying the inverse quotitive situation.

TABLE 5.6

The number of children succeeding and failing across the direct and inverse quotitive problems

	Direct Quotitive Problem		
Inverse Quotitive Problem	Fail	Pass	Total
Fail	44	17	61
Pass	3	56	59
Total	47	73	120

When children's strategies in quantifying the direct and the inverse quotitive situations were described it was found that not all the children who quantified the inverse situation did so by inverting their schemas of action. Table 5.7 shows that out of the 49 children who formed quotas in the direct quotitive problem, 33 formed quotas in the inverse problem. There were 11 children who had formed quotas when the problem gave a direct match to their schemas of action, but did not invert the transformations in the inverse situation. Instead, they employed correspondence procedures to quantify the problem. Regarding the children who used verbal strategies to quantify the inverse quotitive problem, there is no way of knowing whether they inverted the transformations and used division to quantify the problem (18:3=?) or if they tried different numbers that multiplied by 3 give 18.

All the children who failed to quantify the inverse situation although they had succeeded in the direct situation said that they did not know how to tackle the problem. Seven of them represented with the counters the size of the total and the size of the quotas, but could not proceed further.

TABLE 5.7

The frequency of the successful children's strategies in the direct and inverse quotitive problems

Strategies in the inverse quotitive problem	Strategies in Quotitive Problem			
	Forming Quotas	Counting	Number Facts	Total
Forming Quotas	33	1	-	34
Correspondence	11	-	-	11
Counting	2	1	1	4
Number Facts	3	-	4	7
Total	49	2	5	56

The correspondence schema as a direct and an inverse solution

The setting into correspondence schema could be applied to quantify a) correspondence problems that gave a direct match to children's schemas of action and b) sharing problems where the size of the shared quantity was unknown. In the latter case, if the children had coordinated the two operations they could invert the transformations and use correspondence instead of sharing procedures to quantify the problem.

Children's performance in the multiplication problem was compared with their performance in the problem where multiplication was the inverse operation. A Chi-Square test showed that there was a significant association between performance and the type of problem presented (X^2=83.38, df=1, p<.0001). Table 5.8 suggests that the majority of the children who solved the direct multiplication problem were able to quantify the inverse problem. A McNemar test revealed that there was no significant difference in children's performance in the two problems (<.73).

TABLE 5.8

The number of children succeeding and failing across the direct and inverse correspondence problems

	Direct Correspondence Problem		
Inverse Correspondence Problem	Fail	Pass	Total
Fail	29	3	32
Pass	5	83	88
Total	34	86	120

In order to examine whether the children quantified the inverse situation by correspondence or sharing procedures their strategies were described. Table 5.9 shows that the majority of the children applied correspondence procedures both in the direct and the inverse problems. There were only 4 children who did not invert the transformations and used sharing instead.

TABLE 5.9

The frequency of the successful children's strategies across the direct and inverse correspondence problems

	Strategies in the direct correspondence problem			
Strategies in the inverse correspondence problem	Correspondence	Counting	Number Facts	Total
Sharing	4	-	-	4
Correspondence	48	11	1	60
Counting	3	4	-	7
Number Facts	-	3	9	12
Total	55	18	10	83

For the 12 children who based their responses on the times table, it is not possible to say whether they had a strategy based on correspondence or sharing procedure. For the 7 children who counted it is not clear whether they were counting the sets (3, 6, 9, 12) or whether they were double counting while sharing: 1 2 3 4 / 5 6 7 8 / 9 10 11 12.

5.3.2.1.4 Conclusions

The aim of this experiment was to test the coordination of the operations of multiplication and division in young children. The hypothesis of the study was that the two operations develop independently from each other and become coordinated at a later stage.

The findings suggest an asymmetry in the coordination of the two operations. The children were able to quantify partitive and quotitive problems that gave a direct match to their schemas of action by sharing and forming quotas respectively. However, not all the successful children in the direct partitive and quotitive problems were able to quantify the inverse sharing and quotitive problems, although, these problems could also be quantified by the sharing and the forming schemas of action respectively. The difficulty in the inverse problems was conceptual because the children had to invert the transformations mentally in order to interpret the situation. In these situations the children exhibited a poor coordination of the operations of multiplication and division.

The children, though, had no difficulty in quantifying the problems where the correspondence schema of action was employed as a direct or an inverse operation. The efficiency that the children displayed in using the correspondence schema as an inverse operation, might be due to the transparent language used in the inverse correspondence problem. In the inverse correspondence problem the one-to-many correspondence relation between the number of recipients and the size of sets was explicitly presented in the problem. Therefore, the children were encouraged to use the correspondence schema of action.

The findings on children's difficulty in coordinating the operations of multiplication and division are analogous to those on the coordination of the operations of addition and subtraction. Research on the development of addition and subtraction has found that children are successful in quantifying addition and subtraction problems, but have difficulty in working out missing addend problems where they have to invert the transformations and do subtraction (Carraher and Bryant, 1987; Marton and Neuman, 1990). Instead, many children carry on using additive procedures to find the size of the missing addend. Similarly, the majority of the children in this study could solve multiplication and division problems that gave a direct match to their schemas of action but found it difficult to quantify missing factor correspondence problems that required an inversion of the transformations. Instead of inverting the transformations many children employed correspondence instead of sharing procedures to find a solution.

The difficulty in coordinating the relations across the multiplicative situations leads to the conclusion that the two operations of multiplication and division develop independently of each other. The coordination of the relations across the multiplicative situations is a late acquisition.

5.3.2.2 The Reflecting Condition

The aim of this condition was to test children's understanding of the cancelling effect of one operation over the other. It investigated children's understanding of the rule that, a number multiplied and divided by the same number remains the same, (axb):b=a or (axb):a=b. It was expected that this understanding would be a late acquisition.

5.3.2.2.1 Quantitative and Qualitative Analysis

The children were asked to infer the size of the shared quotas by reflecting on the situation. If the children could not find the answer by reflecting on the problem they had the option of acting out the situation.

The total number of successful children in Table 5.10 shows that there was no difference in their performance in the problems where the number of recipients was equal to the number of the quotas or the size of the quotas. The order of presenting the problem terms (first the number or the size of the sets and then the number of recipients or vice versa) did not affect the children's performance.

Table 5.10 also shows that there was a difference in the quantification strategy that the children employed depending on whether the number of recipients was equal to the size or the number of the corresponding sets.

TABLE 5.10

The strategies of the children succeeding in the reflecting condition problems

	Types of Problem			
	Number of sets equal to the number of recipients (dividing with the multiplier)		Size of sets equal to the number of recipients (dividing with the multiplicand)	
Strategies	(axb):a=b	(bxa):a=b	(axb):b=a	(bxa):b=a
Reflecting on the relations	28	34	4	2
Acting out part of the situation	23	28	2	1
Acting out the whole situation	27	18	74	76
Number Facts	2	2	2	2
Total	80	82	82	81

About a third of the successful children made an inference about the size of quotas by reflecting on the situation when the number of recipients was equal to the number of the quotas, while only 4 made an inference on the basis of relations when the number of recipients was the same as the size of the quotas. The fact that a third of the children made an inference about the size of the quotas in the first situation does not necessarily mean that they displayed a genuine understanding of the cancelling affect of one operation over the other. What they did was actually a short-cut in their thinking. When the problem was about 4 children who were to share the biscuits of 4 packages with 5 in each, the participants established a one-to-many correspondence relation between the recipients and the packages and said that there is a package for each child.

The two thirds of the successful children in the (axb):a situations could not make an inference of the size of the quotas and went on acting out the situation with the help of the counters. Half of these children set the biscuits into correspondence and then shared them while the other half did not do the sharing because they realized that there was a package for every child.

In the (axb):b situation most of the children acted out the problem first by setting the packages of biscuits in correspondence and then sharing the biscuits among the recipients. The fact that in this situation only a couple of children acted out part of the situation is because only these two had coordinated the relations between the sharing and the forming quotas schemas of action.

There was only a couple of children who quantified the problem on the basis of number facts based on the times table. These children first multiplied the number of packages with the number of biscuits and then divided the total number of biscuits to the recipients. For example, they said: 4 packages x 3 biscuits each = 12 biscuits; 12 biscuits to be given to 4 children is ... 3 because 3x4=12. It seems that even these children who were familiar with the times table could not make a direct inference to the size of the quotas, but like the children who acted out the situation, first multiplied and then divided the biscuits.

It has to be noted that the children were not always consistent in the strategies they employed to quantify the problems. For example, in the two problems where the number of recipients was equal to the number of the quotas, some children solved one of those problems on the basis of correspondence relations and the other by acting out fully or partially the situation, or the other way round.

5.3.2.2.2 Conclusions

The aim of this condition was to test children's understanding of the complementary relation of multiplication and division and especially their understanding of the rule that, a number multiplied and divided by the same number remains the same [(axb):b=a or (axb):a=b].

It was found that the children had no understanding of the cancelling effect of one operation over the other. It is possible though, that the design of the study was not sensitive enough to detect such an understanding. Perhaps the problems were lengthy and the children had to retain to much information in their mind. As a result their focus moved away from the relations described in the problem.

5.3.2.3 The Display Condition

The aim of this condition was to look at children's understanding of multiplicative relations *within* the multiplicative situations of division and multiplication.

More specifically, it looked at the relations *within division* and investigated the coordination of partitive and quotitive division. If the children have coordinated these two schemas of action, then they should be able to quantify partitive problems that have been acted out by forming quotas and quotitive division problems that have been acted out by sharing. It was expected that the two schemas of action develop independently from each other and become coordinated at a later stage.

The relations *within multiplication* were also investigated by examining children's understanding of the commutativity rule. It was expected that the understanding of commutativity would be related to the coordination of partitive and quotitive division, because they both required the coordination of the relations between the multiplicative terms.

5.3.2.3.1. Quantitative and Qualitative Analysis

In order to investigate the coordination of partitive and quotitive division the children were asked to quantify partitive and quotitive problems in two situations: a) the direct situation where the display matched their schemas of action and b) the indirect situation where the display did not match directly their schemas of action and the children had to coordinate the sharing and the forming quotas schemas of action to quantify the problem. The coordination of multiplicative relations was examined in a situation where the arrangement of the counters was done with respect to the commutativity rule. If the children did not find the display helpful they could act out the situation with the help of counters.

The total number of successful children shown in Table 5.11 demonstrates that the vast majority of the participants gave a correct response in all the problems where the display matched their schemas of action. In partitive and quotitive problems 93% and 95% respectively of the successful children found the display relevant and only a few used sharing or formed quotas to quantify the problem. Similarly, in the direct match correspondence problems 92% of the successful children quantified the problem by means of the display.

Children's performance was, though, lower in the situations where the display did not match their schemas of action directly. A McNemar test showed that there was a significant difference in children's performance in the direct and the indirect match partitive ($p<.0001$) and quotitive ($X^2=15.75$, $p<.0001$) problems where they had to coordinate the sharing and the forming quotas schema of action and the direct and indirect correspondence problem that required the understanding of the commutativity rule in multiplication (X^2=24.325, p<.0001).

TABLE 5.11

The frequency of the solution strategies of the succeeding children in the direct and inverse display problems

	Partitive Problems		Quotitive Problems		Correspondence Problems	
Strategies	Direct Match	Indirect Match	Direct Match	Indirect Match	Direct Match	Indirect Match
Cues	102	9	90	3	93	4
Sharing	8	75	-	-	-	-
Forming Quotas	-	-	5	69	-	-
Correspondences	-	-	-	-	8	36
Counting	-	1	-	-	-	14
Number Facts	-	2	-	1	-	6
Total	110	87	95	73	101	70

Not only did the children's performance drop in the problems where the display did not match their schemas of action directly, but also their quantification strategies leading to a correct answer were different. For the partitive and quotitive problems that did not match the children's schemas of action directly and which required the coordination of the sharing and forming quotas schemas of action the children did not find the display relevant to the problem. Only 10% of the successful children quantified the partitive problem by coordinating partitive and quotitive division and 86% did sharing, that is, they rearranged the counters to give a direct match to their schemas. Even fewer children were able to coordinate the sharing and forming quotas schemas of action in quotitive problems that did not match their schemas of action. Only 4% of the successful children found the display helpful while the remaining children quantified the situation by acting out the situation and forming quotas. Similarly, when the display observed commutativity the children found it irrelevant to the problem and rearranged the counters to give a direct match to their correspondence schema of action. Out of 70 successful children only 4 quantified the problem with respect to commutativity.

The profile of the successful children

It has to be noted that the majority of the children who had coordinated the relations between sharing and forming quotas in the context of partitive problems were from the older age groups - 2 7 year olds and 7 8 year olds. From the 3 children who coordinated the same relations in the context of quotitive problems 1 was aged 7 and 2 were aged 8. All the children who had coordinated the sharing and the forming quotas schemas of action in the context of quotitive division had also coordinated the same relations in the context of partitive division, but not the other way round. Out of the 4 8 year olds who displayed an understanding of the commutativity rule only two had coordinated the relations between partitive and quotitive division in at least one problem. A common characteristic of all these children was that they had quantified the problems of the Acting Condition where multiplication and division were used as inverse operations by inverting the transformations. However, not all the children who had inverted the transformations in that condition were equally able to coordinate the relations between the multiplicative terms within division and multiplication.

5.3.2.3.2 Conclusions

The aim of the display condition was to examine the coordination of multiplicative relations within the situations of multiplication and division.

The findings supported the hypothesis that the coordination of the relations within division is a late acquisition. The children had the ability to recognize a display as relevant to a partitive and a quotitive problem when the arrangement of the counters matched their schemas of action. The same children, though, had great difficulty in quantifying the problems where the display did not matched their schemas of action directly and required an understanding of the intrinsic relation between sharing and forming quotas. Children's difficulty in coordinating the different schemas of action they have for sharing situations suggests that these schemas initially developed independently of each other. Their coordination requires a genuine multiplicative thinking where the children should consider simultaneously the relations between the number of quotas, the size of quotas and the number of recipients.

The coordination of the relations between the multiplicative terms is also the key to the understanding of commutativity in multiplication. Even the 5 year olds were able to quantify the one-to-many correspondence problems when the arrangement of the counters matched their schemas of action directly. The same children, though, showed a poor understanding of the property of commutativity in the context of the same problems. That means that the children develop a limited understanding of the mathematical properties of multiplication by quantifying one-to-many correspondence problems. This finding is in accordance with previous evidence showing that the understanding of the property of commutativity in multiplication is a late development (Frydman, 1990; Nunes and Bryant, 1995 Piaget, Kaufmann and Borquin, 1977; Pettito and Ginsburg, 1982). The finding that children have difficulty in understanding commutativity in the context of one-to-many correspondence problems supports further Nunes and Bryant's (1995) claim about the effect of the problem situation on the understanding of mathematical properties. They found that one-to-many correspondence situations raise conceptual difficulties for the understanding of commutativity, because for example, buying 10 apples for 30 pence each is not equivalent to buying 30 apple for 10 pence each. Such a situation requires the children to consider commutativity as a property related to numbers and not to measures. The fact that the children in this study exhibited such a poor understanding of commutativity might be attributed to the difficulty raised in the specific situation. It is possible that the children would have performed differently in rotation situation problems. Nunes and Bryant (1995) have shown that rotation problems foster the understanding of commutativity because the measures remain the same. A chocolate bar 4 square longs and 2 squares wide still contains the same number of squares if it is rotated. For this reason the finding that in this study the children had no understanding of the property of commutativity must be expressed with caution.

Due to the low frequency of success in this condition, the results of the study do not suggest a clear pattern in the development of commutativity and the coordination of sharing and quotitive division.

CHAPTER 6

DISCUSSION AND CONCLUSIONS

The aim of this thesis was to investigate the origins of children's understanding of multiplication and division and how the two operations progressively become coordinated in young children.

The hypothesis of the study was that the origins of these operations are to be sought in children's schemas of action. Schemas of action preserve the same invariants as the operations themselves and, therefore, form the conceptual basis on which the children can build a more elaborate understanding of multiplication and division as operations. Because schemas of action are about objects and their transformations and not about computations the children were expected to be able to reflect on multiplicative relations before dealing with the strictly numerical aspects of the situation.

The hypothesis of the study was contrasted with an alternative hypothesis proposed by Fischbein et al (1985) according to which the origins of multiplicative structures are to be found in other operations, that of addition and subtraction. Therefore, in their theory the ability to compute multiplicative problems has a key role for the understanding of multiplicative relations.

In order to test whether there were schemas of action that support the development of multiplication and division the participants of the study were asked to reflect on multiplicative relations in relational, non-computational problems. The advantage of designing a study where the children were not asked to give numerical answers and compute sums was that it was possible to work with young children who had never been taught multiplication and division at school. The children were asked to reflect on multiplicative relations in situations involving not only discontinuous, but also continuous quantities. The introduction of continuous quantities excluded the possibility of children using quantification strategies to make judgements about the situation. For these reasons ***children's reasoning on multiplicative relations relied entirely on their schemas of actions***. Therefore, it was possible to identify their schemas of actions and examine what they learn about the properties of multiplication and division from them.

The findings of the study supported the hypothesis that the origins of multiplication are to be sought in the one-to-many correspondence schema of action. Even the 5 year olds had no difficulty to order the size of different one-to-many correspondence ratios. They showed a good understanding of the concept of ratio and took into account the ratio differences in making their ordering comparisons. The same children though, had difficulty in quantifying the sums of the two sets. The finding that children have schemas of action that support their reasoning when reflecting on correspondence situations was strengthened even more by their performance in ordering the size of corresponding sets with continuous quantities that were beyond their quantification ability. Children's performance with discontinuous and continuous quantities and their justifications suggest that children's reasoning stems from their correspondence schema of action. The children had a good understanding of a significant invariant of multiplication, the concept of ratio, and were able to apply their understanding to make relational judgements about situations that they could not quantify. If the understanding of the properties of multiplication was related to the understanding of addition as suggested by Fischbein et al (1985) then the children would not have been able to make judgments on situations before dealing with absolute values.

The findings of the study on multiplication suggest that the children can use their correspondence schema of action to reflect on multiplicative relations. It has to be noted, though, that the correspondence schema of action does not teach the children everything about multiplication. It facilitates children's understanding of the concept of ratio, but the findings suggest that it does not teach them much about another important property of multiplication, that of commutativity.

The finding that children have the schemas of action that support their reasoning when reflecting on multiplicative relations was also verified in the context of sharing situations where the children were asked to make judgements about the relative size of the shared quotas (partitive problems) and the relative number of recipients (quotitive problems). By the age of 6 many children were able to reflect on the effect that the number of the quotas had on their size, not only with discontinuous, but also with continuous quantities that resulted in fractions. Children's ability to order the size of different fractions - the computation of which has been well documented to be a major source of difficulty even for the children who have been taught fractions at schools- suggests that children have a powerful understanding of sharing relations that supports their reasoning even in situations that are beyond their computation ability.

The aforementioned studies revealed that young children before being introduced to multiplication and division at school have the schemas of action to deal with multiplicative situations. Their schemas of action have equipped them with the understanding of some of the significant invariants of multiplication and division which allow them to make judgements and reflect on multiplicative relations. But do their multiplicative schemas of action develop in parallel or in a coordinated fashion?

This study took a step further to provide some evidence on the undocumented field of the development and the coordination of multiplication and division. The last experiment showed that the children might have the schemas of action to act out solutions on multiplication and division problems, but when they were presented with inverse problems that required the coordination of their schemas their performance dropped significantly. Their difficulty with this sort of problems was conceptual because they had to mentally invert the transformations to interpret the situation. Mental inversion of the transformations was found to be difficult because it required the coordination of the different multiplicative schemas of actions *across* the multiplicative situations.

The coordination of multiplicative relations *within* each situation was also found to be a late achievement. The children were efficient in sharing and forming quotas to quantify partitive and quotitive division problems respectively, but the sharing and the forming quotas schemas of action were not related to each other. Similarly, in multiplication the children had not coordinated the relations between the number and the size of the corresponding sets that have a key role for the understanding of commutativity. The children showed once again a poor understanding of commutativity, although they were proficient in setting sets into correspondence to quantify one-to-many correspondence situations.

The evidence suggests that the operation of multiplication and division follows the same developmental path as the operation of addition and subtraction. The two operations initially develop independently of each other until becoming coordinated. The finding that multiplication and division initially develop independently is providing further positive evidence for the hypothesis of the study that the two operations have distinct origins.

The research findings on children's understanding of the mathematical properties of multiplication and division and their development and coordination suggest that the development of multiplicative thinking is not an instant achievement but a long process over a period of years, the onset of which is at a very young age. Children's correspondence and sharing schemas of action lay the basis for the development of a wide range of multiplicative relations on which the children themselves and school instruction can build a more elaborate knowledge.

The educational implications of the study

The finding that children have the schemas of action that support their understanding of multiplicative concepts before being introduced to multiplication and division at school has significant implications for the school practice and especially on *how* multiplication and division should be taught.

Today the most common practice adopted by the schools is to teach multiplication as a repeated addition. The popularity of the model of teaching multiplication as repeated addition is reflected in a number of teachers textbooks. Deboy and Pitt (1987) suggest that "as soon as the children are able to add together small numbers the idea of multiplication can be introduced as repeated addition" (p. 116). Thyer and Maggs (1994) also point out the importance of addition for the understanding of multiplication. "A preliminary to the introduction of multiplication would be to practice a variety of repeated addition examples" (p. 56). Biggs and Sutton (1983) suggest that multiplication is nothing more than repeated addition and as soon as the children understand the relation between addition and multiplication a significant aim has been achieved: "General Aim: The children will understand multiplication both as an effective way of adding equal numbers and as magnification" (pp. 52-53). Multiplication has been so strongly related to addition that correspondence problems are often referred in the literature as "repeated addition problems" (Hart, 1981; Luke, 1988; Peled and Nesher, 1988).

When the teacher introduces multiplication for first time in the classroom (s)he has to find a familiar context to introduce the new operation. There is no doubt that teaching multiplication as repeated addition helps the children to connect multiplication to a situation already familiar to them and also provides them with a strategy sufficient to calculate a number of multiplication problems. However, the price for teaching multiplication as repeated addition as pointed out by Nunes (1996a), is that *children's thinking would need much more reshaping in order to progress beyond this initial stage*. The reason is that the invariants of the concept of multiplication are not present in the concept of addition. Teaching multiplication as repeated addition fails to focus on the relation between the corresponding variables which is the most important invariant of multiplication. This constant relation between the two variables is not present in addition because addition is about part-whole relations in which objects or sets of object that do not have any internal relation to each other are put together. The result of teaching multiplication as repeated addition is that the idea of correspondence is overlooked and is not connected to the operation of multiplication. English and Halford (1995) point out that by using addition to interpret multiplicative situations is questionable whether the children would ever come to see multiplicative situations in terms of multiplication.

It is possible that the number of misconceptions concerning multiplication and division that have been widely reported with children as well as with adults (Bell, Fischbein & Greer, 1984; Fischbein et al., 1985; Graeber & Baker, 1991) is a result of the teaching they received at school. The paradox is that children enter the school with a good grasp of some of the invariants of multiplicative situations and after receiving a considerable amount of school instruction they develop a series of misconceptions concerning the use of the operations and an inadequate understanding of multiplicative relations. It can be the case that teaching multiplication as repeated addition prevents the children from relating their early schemas of action to multiplication in order to develop a genuine understanding of the operation.

The school practice of teaching multiplication as repeated addition also implies that children have no understanding of multiplicative concepts so far. The evidence provided in this study as well as in other studies (Correa, 1995; Frydman, 1990) suggests that children are not empty minds but do bring into the classroom a rich understanding of multiplicative relations stemming from their schemas of action. *The schools should therefore call upon children's schemas of action and anchor the operation of multiplication to the correspondence schema of action, which in contrast to addition, preserves the invariants of multiplication as an operation.*

Relating multiplication to the one-to-many correspondence schema of action has the advantage that the new mathematical operation is connected to children's exiting meanings. At the same time the invariants of multiplication are respected and their understanding will facilitate the process of socialization of the old meanings into mathematical concepts (Nunes, 1996a). Multiplication should be taught not as a quicker way to do addition but as an operation that relates two corresponding variables. This relation between variables is the most important invariant in multiplication and should be the foundation on which schools should build their teaching.

There is a larger consensus today that division should be related to children's sharing schema of action (Anghileri, 1997; English and Halford, 1995). Teaching division as sharing helps the children to anchor the new operation on a familiar schema of action and as a result the understanding of the new operation develops in a more meaningful context. It has to be stressed though, that division is not the same as sharing. School teaching should focus not only on the action of sharing but also on sharing relations. The children should be encouraged to reflect on the effect that the number of recipients has on the size of the quotas and vice versa and to connect their sharing and the forming quotas schemas of action. The understanding of the net of relations between the dividend, the divisor and the quotient is a significant step beyond the activity of sharing towards the understanding of division as a multiplicative operation.

The findings of the study suggest that children have the schemas of action that support their reasoning on multiplicative situations, but it is a long process until these schemas are coordinated. The school practice should provide the children with situations that would help them relate and coordinate their schemas of action within and across the multiplicative situations. It is not until the children understand the inverse relation of the two operations that the operations are seen in their true multiplicative light. The schools will have to support the understanding of the properties of multiplication like that of commutativity that the children do not come in grips with from their correspondence schemas of action.

In conclusion, the evidence provided by this study suggests that the children have a good understanding of some of the fundamental invariants of multiplication and division and that their meanings stem from their schemas of action. Schools should make use of children's early understandings and connect the new operations to their existing schemas of action. Connecting multiplication and division to the correspondence and sharing schemas of action has the advantage that the new operations are introduced in a familiar and meaningful context, but most importantly the invariants of the operations are stressed and preserved.

Suggestions for further research

There is still a lot to be understood about children's development of multiplicative thinking.

This study focused on children's understanding of the concept of ratio and their understanding of commutativity in the context of one-to-many correspondence situations. Children's early understanding of multiplicative relations should also be examined in different situations such as those that involve the covariation of two or more variables either directly or inversely. Reflecting on these situations requires a genuine multiplicative thinking that relies on the establishment of a functional relation between the variables.

There is more to be understood about children's ability to invert the transformations and solve missing factor correspondence problems by division. In the last experiment of this study there were many children who quantified the inverse problems by means of times tables. For these children the results were inconclusive about whether they had a genuine understanding of the inverse relations of the two operations and did the computational part by means of the times tables or if they used correspondence procedures. Future studies can clarify the inversion of the transformations by presenting children with problems involving large numbers that cannot be quantified by using time tables but through a calculator. The calculator restricts children's choices to the 4 operations and blocks trial and error procedures. Thus, it can easily checked whether the children solved the problems by using the inverse operation or by trying different possible correspondences.

This study was rather descriptive about how the children come to coordinated the two operations. Studies with longitudinal designs as well as training studies are needed in order to provide evidence on how the children eventually come to coordinate the two operations.

The study also showed that the coordination of the relations within multiplication and division is a late achievement. Commutativity in multiplication was not easily understood and the sharing and forming quotas schemas of action were not related to each other. Although this study hypothesized that there should be some kind of relation between commutativity and the coordinations of the two sharing schemas of action, because they both required the coordination of the relations between the number of the quotas / sets and their size, the study failed to test this hypothesis due to the low level of success in both tasks. A more elaborated intervention study is needed to test whether teaching the children about commutativity would help them to coordinate their sharing schemas or the other way round.

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APPENDICES

Appendix 4.1

The number of correct responses in each trial with discontinuous quantities

			Age		
Condition	**Dividend**	**Trial**	5-yr	6-yr	7-yr
		2(2)	24	29	32
Same	12	3(3)	22	27	32
		4(4)	21	26	32
		6(6)	23	27	32
		2(2)	22	27	32
	24	3(3)	21	28	32
		4(4)	23	29	32
		6(6)	22	28	32
		2(3)	13	18	26
Different	12	2(6)	11	17	26
		3(4)	12	17	27
		3(6)	11	16	26
		2(3)	11	17	27
	24	2(6)	12	16	26
		3(4)	10	17	28
		3(6)	12	17	27

Appendix 4.2

The number of correct responses in each trial with continuous quantities

Condition	Dividend	Trial	Age 5-yr	6-yr	7-yr
		2(2)	23	27	32
Same	1	3(3)	24	29	31
		4(4)	21	28	32
		5(5)	22	27	32
		2(3)	12	16	26
Different	1	2(6)	11	18	26
		3(4)	11	17	27
		3(6)	12	16	28
		4(4)	22	28	32
Same	2	5(5)	23	29	32
		6(6)	20	27	32
		7(7)	22	28	32
		4(6)	10	17	27
Different	2	4(5	9	16	26
		5(7)	11	16	26
		6(7)	10	17	26
		6(6)	23	28	32
Same	3	7(7)	20	27	32
		8(8)	22	29	32
		9(9)	21	27	32
		6(8)	10	18	26
Different	3	6(7)	10	17	26
		9(7)	11	16	26
		9(8)	12	16	27

Appendix 4.3

The number correct responses in each trial with discontinuous quantities

Condition	Dividend	Trial	Age		
			5-yr	6-yr	7-yr
		2(2)	18	30	32
Same	12	3(3)	16	27	32
		4(4)	17	28	32
		6(6)	16	29	32
		2(2)	17	28	32
	24	3(3)	18	26	32
		4(4)	16	27	32
		6(6)	17	28	32
		2(3)	6	13	22
Different	12	2(6)	7	12	23
		3(4)	8	12	21
		3(6)	7	13	22
		2(3)	9	12	24
	24	2(6)	7	12	22
		3(4)	7	13	22
		3(6)	8	11	23

Appendix 4.4

The number of correct responses in each trial with continuous quantities

Condition	Dividend	Trial	Age		
			5-yr	6-yr	7-yr
Same	1	½(½)	17	29	32
		1/3(1/3)	18	26	32
		1/4(1/4)	18	27	32
		1/5(1/5)	16	28	32
Different	1	½(1/4)	8	17	24
		½(1/8)	7	16	23
		1/3(1/4)	7	15	25
		1/3(1/8)	9	15	23
Same	2	½(½)	16	28	32
		1/3(1/3)	15	30	32
		1/4(1/4)	16	26	32
		1/8(1/8)	18	27	32
Different	2	½(1/4)	7	14	23
		½(1/8)	6	13	24
		1/3(1/4)	8	15	23
		1/3(1/8)	8	15	23
Same	3	½(½)	15	26	32
		1/3(1/3)	16	29	32
		1/4(1/4)	18	27	32
		1/8(1/8)	17	28	32
Different	3	½(1/4)	7	13	24
		½(1/8)	7	14	23
		1/3(1/4)	9	15	25
		1/4(1/8)	8	14	23

Appendix 5.1

Comparing children's performance across relational and quantificational division problems

The last study was the only study where the children were asked to *quantify* problems in partitive and quotitive division by acting out the situation. In the studies reported in Chapter 4 the children were presented with *relational* division problems which they could solve by reflecting on the sharing relations.

It was considered appropriate to look at children's performance across the relational and quantification division problems. Correa, Nunes and Bryant (1998) have shown that children's ability to do sharing and infer the numerical equivalence of the shared quotas does not necessarily mean that the children have an understanding of the sharing relations between the dividend, the divisor and the quotient. Based on their findings they suggested that sharing and division are not the same thing. It was therefore, expected that the children would find it easier to act out partitive and quotitive problems that matched their schemas of action with the help of counters than to reflect of sharing relations.

In order to find whether the quantification of division problems would be easier or harder than the reasoning on sharing relations children's performance across relational and quantification problems was compared.

Children's scores were compared across the relational and quantification problems (Table A.1). A Chi Square test showed that there was a significant association (X^2=5.14, df=1, p<.02) between performance and the type of problem presented (partitive relational versus partitive quantificational). Quantifying a partitive problem by acting it out was found to be easier than reflecting on sharing relations. Although there were more children succeessful in quantificational quotitive problems the association between performance and type of quotitive problem (relational versus quantificational) was not find to be significant (X^2=3.01, df=1, p<.08).

TABLE A.1

The percentage of correct responses across relational and quantificational partitive and quotitive division problems

	Type of Division Problem	
Type of Question Asked	Partitive	Quotitive
Relational	56%	41%
Quantificational	72%	48%

The trend that these findings suggest is that the ability to share and form quotas does not guarantee the conceptual understanding of the mathematical relations in division. That means that the operation of division is not the same as sharing, although sharing is the schema of action from which it originates.

Someone, though, should be very careful with the interpretation of these findings because the children were not randomly assigned in the relational versus the computational tasks. Although the children who took part in the studies were under the same Local Educational Authority in North East London in the computational task the children were attending a summer play scheme group in the same area. It is not known whether the children attending the play schemes differ from the children who do not. More reliable results would have been obtained with a repeated measures study in which the same children would be asked to reflect on sharing relations and quantify division problems.

Appendix 5.2

How the children implemented their schemas of action

The children who used counters to quantify the problems where multiplication and division was employed as a direct or inverse operations implemented their schemas in a variety of ways depending on the number of variables they represented with the counters and on whether they did double counting or not. The following paragraphs give a detailed account of the variations observed within the sharing, the forming quotas and the correspondence schema of action.

I. Sharing

Sharing by representing two variables

The children represented with counters both the quantity to be shared and the number of recipients. For example, when they had 15 sweets to share among 3 children, they took 15 counters from the pile to represented the number of sweets they had and another 3 counters to represent the recipients. Then they shared the sweets among the recipients.

Child	Child	Child
x	x	x
x	x	x
x	x	x
x	x	x
x	x	x

Sharing by representing one variable

The children took counters from the pile to represented one variable which was always the number of the sweets they had. Then they did the sharing by placing the sweets at three different places at the table that represented the 3 recipients.

x	x	x
x	x	x
x	x	x
x	x	x
x	x	x

Regardless of whether the children represented one or two variables, some children distributed the sweets to the children one-by-one until the 15 sweets were exhausted, while other children shared the sweets in small groups of 2s, 3s or 4s rather than one at a time.

Double counting by representing two variables

In double counting the children did not start by representing the number of sweets they had. Instead, the number of sweets (15) was used to indicate when sharing should stop. Some children, before sharing in a double counting fashion, took some counters to represent the number of recipients.Then they took counters from the pile one-by-one and shared them. At the same time they controlled for the whole that was to be shared and formed the quotas. For example, the above sharing problem was quantified in the following way :

child	child	child
1	2	3
4	5	6
7	8	9
10	11	12
13	14	15

Double counting by representing one variable

Some children who shared the counters in the double counting fashion did not represent the recipients, but placed the counters they were sharing at distinctive places on the table.

1	2	3
4	5	6
7	8	9
10	11	12
13	14	15

II. Forming Quotas

Forming quotas by representing two variables

The children represented both the quantity to be shared and the number of the recipients. When there were 15 sweets to be shared in quotas of 3s they took 15 counters from the pile to represented the number of sweets they had. Then they put a few counters in the row to represent the friends they could possibly invite. After that they repeatedly subtracted quotas of 3s and matched each quota with a friend until the shared quantity was exhausted.

Friend	Friends	Friend	Friend	Friend	Friend
xxx	xxx	xxx	xxx	xxx	(No sweets)

Then the children counted the number of friends who got sweets and answered how many they could invite.

Other children did not place the counters that represented the friends in a row before sharing the quotas, but every time they formed a quota, they paired it with a friend.

Forming quotas by representing one variable only

The children took 15 counters from the pile to represent one variable only which always was the whole to be shared. Then they repeatedly subtracted quotas of 3s and placed them at different places on the table. The number of the quotas formed was equal to the number of recipients.

xxx xxx xxx xxx xxx

Double Counting by representing one variable only

As in sharing, the child did not start by representing the quantity to be shared. Instead, the number of sweets (15) was used to indicate when the child should stop forming quotas. The child took quotas of 3s from the pile and placed them at different places on the table. At the same time the child controlled for the whole that was to be shared and the size of the quotas. The number of the quotas formed was equal to the number of number of the recipients. For example, the above problem was quantified in the following way:

1 - 2 - 3 4 - 5 - 6 7 - 8 - 9 10 - 11 - 12 13 - 14 - 15

None of the children who formed the quotas in a double counting fashion represented both variables.

III. Setting into Correspondence

Setting into correspondence by representing two variables

In this implementation the children represented both corresponding variables. They took 5 counters from the pile to represented the number of children that came to the party, for example, and then placed 3 counters in front of each child to represent the chocolates each child had brought. Other children took 1 counter at the time and paired it with 3 chocolates until they had built up 5 sets of 3s. In both cases there was a clear one-to-many correspondence between the children and the chocolates.

Child	Child	Child	Child	Child
xxx	xxx	xxx	xxx	xxx

Then the children counted the total number of chocolates given.

Setting into correspondence by representing one variable

The children took counters from the pile to represent only one variable which in some cases was the number of chocolates given. They formed 5 sets of 3s and placed them at different places on the table. Then they counted the total number of chocolates.

xxx xxx xxx xxx xxx

It has to be stressed that there is a similarity between forming quotas and setting sets in correspondence, because in both situations the shared quotas or sets are into correspondence with a recipient. However, in an *activity level* the two schemas are different. The starting point in the forming quotas schema is to form the quantity to be shared the size of which is priory defined. After forming the shared quantity then the child repeatedly subtracts quotas until the shared quantity is exhausted. The activity of forming quotas ends as soon as the total is exhausted. In multiplication, thought, the starting point is not the formation of the quantity to be shared, because this quantity serves as the unknown. The starting point is the replication of the corresponding ratios the number of which is priory defined. The action of setting sets in correspondence ends as soon as the number of replications is accomplished.

Tables A.2, A.3 and A.4 present the frequency that they above variations were used to quantify the problems where the schemas of sharing, forming quotas and setting into correspondence were employed as a direct or as an inverse operation.

TABLE A.2

The frequency of the strategy variations observed within the sharing schema of action in the direct and inverse problems by the successful children

Direct Solution				**Inverse Solution**			
Sharing 2 variables	Sharing 1 variable	Double Counting 2 variables	Double Counting 1 variable	Sharing 1 variables	Sharing 2 variable	Double Counting 1 variable	Double Counting 2 variables
31	41	2	4	16	10	1	1

TABLE A.3

The frequency of the strategy variations observed within the quotitive schema of action in the direct and inverse problems by the successful children

Direct Solution				Inverse Solution			
Forming Quotas 2 variables	Forming Quotas 1 variable	Double Counting 2 variables	Double Counting 1 variable	Forming Quotas 1 variables	Forming Quotas 2 variable	Double Counting 2 variables	Double Counting 1 variable
22	35	2	7	14	9	1	10

TABLE A.4

The frequency of the strategy variations observed within the correspondence schema of action in the direct and inverse problems by the successful children

Direct Solution		Inverse Solution	
Correspondence 1 variables	Correspondence 2 variable	Correspondence 1 variables	Correspondence 2 variable
34	21	35	25