

Children's Understanding of Number in the Primary School Years:

A unifying view from early counting to knowledge of place value

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*To Gordi and Xuga,
the perfect excuse for improvements in my Soul*

ABSTRACT

Previous research has tended to focus on the development of separate number components (e.g. counting, addition, written numbers) and so, cannot comment on how development in one component affect development in others. The purpose of this thesis was to provide preliminary evidence towards a unifying view about the development of children's number competence, from early counting skills, at age four, to knowledge of place value, at age seven. To accomplish that aim 152 children from three different cohorts (Reception, Year 1 and Year 2) were given thirteen maths tasks, three times along one school year, assessing their understanding of four separate number components: counting and knowledge of the number-word sequence; generation of verbal number-words and the understanding of the structure of the numeration system; understanding of the arithmetical operations; and the ability to read and write numbers and understanding of the principles underlying place value. Beyond the assessment of these various number components, special emphasis was given to the separate role of each component and the developmental inter-relations amongst components in the child's development of progressively more complex ideas about number.

Based on the children's performance on these tasks and the exploration of their relationships along time, it was possible to outline a preliminary proposal about children's number development. The evidence suggests that each number component plays a significant role at key times. For example, no children could develop the counting-on strategy or succeed in the arithmetical operation tasks without prior knowledge of continuation of counting. The data also showed that no development is possible without the inter-related development of several components, at other times. For example, no child could understand the structure of the decade numeration system without previous combined understanding of continuation of counting, addition and multiplication. Between 93% and 97% of the children fitted the model proposed in the various assessments.

Although limited by the constraints of a correlational design, these findings suggest that the present inter-relational approach is relevant and worth further investigation through the introduction of intervention studies and the rigorous examination of causality.

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1

INTRODUCTION

Abundant research in mathematical cognition has consistently shown that (1) children's knowledge about numbers involve several components, (2) that each of these components emerge and develop at different times, from age two well into adolescence, and (3) that the grasp of some complex components require knowledge of the simpler ones. Yet, most of this research has tended to focus on the development of a few aspects of number development and so cannot comment on how developments in one component affect development in others.

There is, however, evidence suggesting that the developmental study of children's number competence would benefit from the use of a unifying conceptual framework, where relations between different number components might be observed. Examples of this approach, considerably less in number, are a few studies that have looked at 'multi-component arithmetics' in adults from the perspective of cognitive psychology and psychometrics (Geary and Widaman, 1992), or at the relation between calculation reasoning in addition and subtraction, background and psychometric measures in five to nine year-olds (Dowker, 1998), or at the relation between enumeration and

knowledge of addition, at different stages of development, in low attaining seven to nine year-olds (Denvir and Brown, 1986a; 1986b).

Quite encouragingly, the evidence has supported two main ideas justifying the expansion of this line of inquiry. Firstly, that normal arithmetical development and functioning implies the functional autonomy of different components, which seem to go beyond the traditional two modules of procedural and conceptual knowledge¹ (Dowker, 1998). Also, that low attaining children follow *individual* routes in their number acquisitions, which can only be clarified by a multi-component analysis. These data, in turn, can be used beneficially to help children expand and consolidate their numerical knowledge (Denvir and Brown, 1986a; 1986b). More, however, remains to be understood about the developmental relations between a larger set of number components.

The issue of multi-component number processing has received considerably more attention in the neuropsychological literature, where most results are based on the performance of *adult* brain-lesioned patients with acalculia. Interest in this area resurged in the early 1980's, beginning with the studies by Warrington (1982) and Deloche and Seron (1982). Based on these and other studies, McCloskey et al. (1985) proposed an influential general architecture for number processing, postulating the existence of three *independent* number modules (i.e. number comprehension, number production, and a calculatory system), and an abstract internal representation of numbers that supports both calculation and the communication between the modules.

However, whilst authors agree about the existence of a multi-component number architecture, they disagree in terms of the nature and the structure of these mental

¹ – For a review about the relationship between conceptual and procedural knowledge in learning mathematics see Rittle-Johnson and Siegler (1986).

representations (Deloche and Seron, 1987; Campbell and Clark, 1988; McCloskey, 1992; Dehaene, 1992; Noel and Seron, 1993; Butterworth, 1999), and the network of brain areas implied in this processing (Dehaene and Cohen, 1995; Dehaene et al., 1999).

There is, for instance, evidence that the modules inter-depend in their functioning (Clark and Campbell, 1991), rather than function with complete independence (McCloskey, 1992). Also, there is evidence that no abstract internal representation of number is required for processing (Deloche and Seron, 1987), an argument that has also been consistently supported with data from developmental studies (Deloche and Seron, 1987; Seron and Noel, 1995; Power and Dal Martello, 1997).

However, beyond the confirmation that arithmetical development involves the development of different components, the neuropsychological data seems to have limited importance to developmental studies (e.g. Ashcraft, 1992) including the present study. Several reasons support this argument: one, is that single-case data usually does not include information about the patient's pre-morbid state, which makes it difficult to assess whether impairments are due to the lesion only, to mere individual differences (Deloche et. al, 1994) or to the combination of the two. Deloche et al. (1994) showed that many "normal" adults performed below the expected levels in many number component tasks of a standardised testing battery for the evaluation of brain-damaged adults.

Another reason, is the growing evidence that the mathematical recovery of brain-lesioned adults may differ qualitatively from the expected normal development of numerical knowledge in children (Power and Dal Martello, 1990; Hittmair-Delazer et al., 1994). Finally, a fundamental limitation of the proposed neuropsychological

models for number processing relates to the purely functional framing they present, which is seen to lack a conceptual component in their architectures (Ashcraft, 1992; Hittmair-Delazer et al., 1994; Dehaene and Cohen, 1995). Such lack of specificity about the mathematical concepts that are being processed in these models limit inferences about mathematical functioning (Hittmair-Delazer et al., 1994; 1995) and its development.

In view of this, little is still known about the developing relationships between the various number components in the same children, during the primary school years. Such a unifying conceptual framework would be helpful to teachers in their everyday classroom activities (e.g. Denvir and Brown, 1986a; 1986b).

Possibly, one of the difficulties in proposing a unified view for the child's number development, relates to the lack of agreement on learning targets in early mathematics. The school curriculum is constructed by key attainment stages which are not necessarily related, from the child's point of view. Agreement upon one such target in the case of number development, could hopefully unify the efforts of both school teachers and cognitive researchers, as it would help the exchange of information between these two perspectives.

At another level, it would be quite helpful for teachers if this target - to be attained after two or three years of schooling - could take into account at least some of the mathematical knowledge children develop before entering school. This would provide teachers with important *materia prima* from which to begin teaching children about numeracy.

The choice of an important learning target in early mathematics is not a problem of availability but, rather, one of convincing argumentation. This is not an easy task considering that educational policy and objectives change in time and place. Amongst several and equally important learning targets, this thesis focuses on one that has been widely recognised as the most important instructional task in mathematics in the primary school years (e.g. Resnick, 1986). Such a target has the advantage of being curriculum-related, as it considers the needs of educationalists. It has, however, the limitation implicit in the impossibility of covering all aspects of the number curriculum for the primary school in one study alone. Furthermore, mathematics is more than only number knowledge.

With these limitations in mind, the target this thesis will focus on is children's understanding of the decade numeration system and the positional system based on it, also known as place value — a convention defining that each digit signifies a unit of different size according to the position occupied in the number. In the number 222, for example, the first digit means two-hundred, the second means twenty and the last means two.

Place value is a fundamental milestone in children's number development (Resnick, 1986). It is, on the one hand, the basis for the understanding of written multi-digit numbers and, on the other, the fundamental developmental step in their ability to compute written algorithmical calculations (e.g. addition and subtraction by columns) correctly. Place value represents an abstract convention without which children cannot develop their mathematical knowledge further without serious problems.

Without this understanding most children resort to idiosyncratic rote-procedures while attempting to write multi-digit numbers (Ginsburg, 1977; Fuson, 1990; Nunes and

Bryant, 1996) or performing multi-column operations (VanLehn, 1990), leading to errors and frustration. A final crucial criterion for this choice of educational target relates to the fact that such a relevant instructional task has, simultaneously, not been understood by nearly half of all nine year-olds, according to the available studies (e.g. Kamii, 1980; Brown, 1981; Bednarz and Janvier, 1982; Kamii, 1986). This, in itself, makes it an urgent matter for further investigation.

Research on the development of separate number components has identified the most important ones as: (1) counting and knowledge of the number-word sequence; (2) the ability to generate verbal number-words and the understanding of the structure of the numeration system; (3) arithmetical operations, and (4) the ability to read and write numbers and the understanding of the principles underlying place value.

The research into individual number components as separate entities has had important advantages, which will be used by this thesis to further explore their relationships. Firstly, it has enabled the refinement of specific assessment procedures, and the development of these into widely used tasks such as Gelman and Gallistel's (1978) counting tasks or Carpenter and Moser's (1983) classification of addition and subtraction word-problems. Secondly, the application of these tasks has led to a detailed description of children's typical achievements in practically all the mentioned number-components, in function of their ages. Finally, and due to these previous investigations, it is now possible to trace the development of children's abilities in each separate number component at different ages throughout their primary school years.

Although a lot is known about the development of each separate component, little is still understood about the developmental interactions amongst components. The main limitation has been that the data made available in the literature relates to the

performance of *different groups of children* in different tasks. This indicates that further research is needed to clarify ways in which the same children relate different areas of number knowledge; and how these interrelations may help them in the understanding of more complex ideas about number, such as place value.

To illustrate this point, Gelman and Gallistel (1978), for instance, have provided insightful hypotheses to account for the development of children's counting skills, the first component, but have not explained the influence of this knowledge in later number acquisitions - any of the remaining components. This relationship is not investigated either in their book "The child's understanding of number" (1978) or in later studies (e.g. Gelman and Meck, 1983; 1986). Likewise, although Fuson (1988) has provided an explanation for the development of children's number-word sequence from age two to seven, the relationship between this progress and the child's understanding of the decade numeration system and the combination of ones and tens - another crucial component - has not yet been clarified.

The studies to be reported here were designed to investigate the simultaneous development of the children's understanding of the various number components, in the same subjects, along the first primary years of primary schooling. A special emphasis is given to the inter-relation between these number components and their separate roles in the construction of more complex number concepts, such as the understanding of the principles underlying the decade numeration system, and the correct use of place value.

Place value is recognised to involve a fundamental pre-requisite, the previous understanding of the structure of the numeration system or, in Ginsburg's (1977) words, a "theory" formed by the child before they can understand the more complex

conventions about number (see also, Fuson, 1990; Nunes and Bryant, 1996). This thesis will examine the development of this "theory", or understanding, in the same children throughout the first three years of primary school.

At another level, however, there is evidence that children's understanding of the structure of the numeration system involves previous understanding of counting ones (Gelman and Gallistel, 1978; Kamii, 1986), as well as addition and multiplication (Ross, 1989; Seron and Fayol, 1994; Power and Dal Martello, 1990; Nunes and Bryant, 1996). However, little is still known about the relation between all these number components in the same children, and about the relevance of their separate roles in the understanding of an important target, in this case, place value.

Counting is seen as both crucial (Gelman and Gallistel, 1978; Fuson, 1988) and as secondary (Piaget, 1952; Resnick, 1986; Miller and Stigler, 1987; Nunes and Bryant, 1996) to children's understanding of the principles underlying the decade numeration system. However, further data is required to clarify children's development from the early counting skills to their understanding of the numeration system.

The main limitation of this situation is that lack of data on the interrelated development of number components does not help to clarify children's acquisition of more complex number concepts, such as the emergence of additive composition of number, and the contribution that each separate number component may have in this acquisition. For example, little is known about which counting skills predict better understanding of any of the remaining number components. At the moment, it can only be speculated that children's ability to count-up to higher numbers in the number word-sequence correlates with an increased chance of teasing out the principles underlying the

structure of the numeration system and its units of different denominations; i.e. ones, tens, hundreds, and so on.

Also, further data is required to clarify children's development from early counting — and their handling of the number-line — to their grasp of addition and multiplication. Most of the data available relates to the development of children's counting strategies whilst solving addition word-problems (e.g. Carpenter and Moser, 1982). This thesis investigates the relevance of the first important development in children's handling of the number-line at age three and four, i.e. continuation of counting (Fuson, 1988), in their understanding of addition and multiplication, as well as the principles underlying the decade system.

Finally, attention will be given to the child's development from knowledge of early addition and multiplication to their understanding of additive composition of number, a measure of their understanding of the structure of the decade system (Fuson, 1990; Nunes and Bryant, 1996). Considering that the numeration system involves additive and multiplicative properties (Power and DalMartello, 1990; Seron and Fayol, 1994), is counting ones sufficient to teach children about tens and hundreds, or must they also master addition and multiplication, as suggested by Piaget (1952) ? No research has yet related the development of these number components in the same children.

Assuming that development brings children progressively new ways of looking at numbers, it seems worthwhile to attempt to map these changes longitudinally. It may be possible to clarify which types of achievements can contribute to the child's acquisition of further sophisticated skills. The wealth of data produced by previous studies has laid the foundations for the more global approach to number development proposed in this thesis. By giving the same group of tasks to the same children, it will

be possible to explore the relationships amongst the different components involved in the development of numeracy. On the other hand, the longitudinal analysis of the results will enable the investigation of the various sequences of development within each task.

The need to limit the scope of this investigation, dictates that the studies here included will concentrate on what happens in the classroom in terms of children's mathematical achievements in school tasks throughout their first three years of primary school. Rather than focusing on children's logical development (Piaget, 1952), or on the cultural transmission of mathematical knowledge (Vygotsky, 1978; Luria, 1976), this thesis will explore the relevance of tasks that can be used by teachers in the context of the classroom.

The literature to be reviewed is discussed in the three chapters that follow. Chapter two briefly describes the various number components that will be investigated. It also makes distinctions between components and describes particular aspects that the components may have, as in the case of addition problems where different types of problems can be found. It finally briefly outlines the importance of each component in terms of a global picture of early number competence.

In Chapter three, studies that have been carried out investigating the development of each component are reviewed and the choices of methods in the specific assessments are briefly discussed. The chapter briefly reviews suggestions by previous studies about children's development of competence in individual components. I also argue about the preference of some methods over others, as these will be used in the studies.

In Chapter four, a review of the theoretical debate and previous empirical studies concerning the role competence in some components may play in the development of other components, is made. Chapter five presents the methodology used in this study with greater detail, and the results obtained. Chapter six summarises the findings, interprets and discusses the evidence and presents some conclusions and educational implications. An outline of a model towards a unified view about number development is suggested, intended as a starting point for further investigations.

2

NUMBER COMPONENTS AND THEIR DESCRIPTION

2.1 INTRODUCTION

This chapter describes each of the number components that are included in this study. Each of them is outlined in terms of what defines them, what children must know to use them, and the conceptual importance of each number component for the whole study. It highlights the importance of the development of counting, both as a tool that enables children to be accurate in their determination of numerosity and also, as a way to grasp the realm of addition and subtraction. But the importance of counting continues as new and more complex units are counted, such as the ones, tens and hundreds, leading to the child's understanding of the decade numeration system. However, the specific development in early counting that enable new understandings about number are not yet clear.

Beyond the acquisition of the counting principles, other developments such as the ability to continue counting from an arbitrary number in the counting-list has been

proposed as relevant for the understanding of other number components. Some authors see it as an indicator that children have acquired more flexible and abstract strategies to deal with number (Davydov, 1969; Fuson, 1988; Cobb and Wheatley, 1988; Aubrey, 1993). However, no studies have yet tested this hypothesis.

Beyond the child's understanding that units can have different denominations (i.e. ones, tens and so on), they will soon realise that these units of different sizes can be combined to form any number. Any number, such as 45, involves sum and product relations: $40=4 \times 10$; $5=1+1+1+1+1$. What then is the importance of children's previous understanding about addition and multiplication in their construction of the decade system ?

The emergence of multiplication implies a shift from operating with single units to operating with composite units. What development in counting helps them to count with composite units ? And what is the role of these two abilities combined in children's understanding of the decade system ?

Finally, how do children learn about written multi-digit numbers and place value ? Do they need previous understanding about the structure of the numeration system, independently of knowing how to write number, or do they need them in order to grasp the rules of place value ?

2.2 COUNTING AND KNOWLEDGE OF THE NUMBER-WORD LIST

2.2.1 What is Counting ?

Children use counting and the number-word sequence to determine the cardinal number of a set of objects; i.e., to determine how many objects are in small sets. Counting implies the use of *units of the same size* (one) and is defined by the mathematical expression of $n+1$, where the child progressively adds one more unit to the ones already counted, bearing in mind that all units must have different tags. Counting requires a set of specific skills that take children some time to master. To count correctly children must grasp the logico-mathematical properties that define the concept, also known as the invariants of counting.

2.2.1.1 What children must know to Count

Children's counting is seen to be dependent on the correct use of several principles (Gelman and Gallistel, 1978). The first three principles are known as the 'how-to-count' principles and represent the rules of counting: Stable-order, one-to-one correspondence and cardinality. The *stable-order principle* implies that each number-tag stands one after the other sequentially, and that each number has a unique label. Failing to observe the stable order principle, leads children to end up with different quantities everytime they count. For example, counting 1, 3, 2, 5 or 1, 2, 4, 3 makes a definite difference.

The *one-to-one correspondence principle* implies that every item in a display of objects must be named with only one tag and none of the items counted should be skipped or counted twice. Gelman and Gallistel (1978) argue that this ability implies the recognition of the unit and helps children to determine numerosity and to establish equalities between sets of objects. If children miss one object or count it twice, they end up with different quantities every time they count. By using this principle, children soon start associating numbers (which they are used to utter) with objects.

The third principle, *cardinality*, enables children's use of verbal counting to determine quantity. This is done by associating the last number-tag counted in a sequence (termed cardinal number) to the number of objects included in that set. This principle allows children the comparison of sets and to establish relations of order — it is therefore a significant step in mathematical thought. Cardinality allows children to make up their minds about whether there are more oranges on the left than lemons on the right, for example. It also enables the measurement of quantities, and broadly, the quantification of the child's surroundings.

The other two principles involved are the *order irrelevance* principle and the *abstraction principle*. Order irrelevance involves knowledge that the direction in which the units are counted (e.g. from left to right or vice-versa) will not change the final number of units counted (cardinality), as long as the one-to-one correspondence and the stable order principles are observed. The abstraction principle implies that various kinds of objects can be put together for the purpose of counting. In other words, the abstraction principle tells children that counting is valid to determine any numerosity; i.e. can be used to count any set of discrete entities.

Greeno, Riley and Gelman's (1984) distinction between conceptual, procedural and utilisational competence is useful for understanding why children make mistakes in counting. According to these authors *conceptual* competence implies that children understand the principles and are able to use them in planning their counting activity. *Procedural* competence refers to knowledge of the steps involved in determining an action; in this case, determining numerosity. *Utilisational* competence implies understanding the relations between the features of a task setting and requirements of performance; it refers to the ability to apply the knowledge of the principles and procedures to particular tasks.

Gelman (1982) argued that children's difficulties in counting do not relate to a possible lack of conceptual competence, but are caused by difficulties in their procedural or utilisational competence. Gelman and Gallistel's central thesis then, is that although children understand the basic principles of counting, they still have difficulties because they lack the skills to perform the appropriate actions involved in counting. In other words, children try progressively to improve their counting but, in the process, they make mistakes.

2.2.1.2 Why is Counting important ?

From the teacher's point of view, the importance of counting is threefold. Firstly, it is the earliest specific activity that enables children to be accurate in their determination of numerosity. What may seem like a simple improvement from the adult eye, represents for children a major development in the cues they use to enumerate. The difference is that a child who is able to count correctly has gained a more precise strategy for

comparing numerosities then perceptual cues such as length and density (e.g. Michie, 1984; Cowan et al., 1993). This, most authors agree, represents a revolutionary development in children's ideas about number. And these are the characteristics that make counting a commonly used starting point for the exploration of children's mathematical development, which is reflected by its inclusion as the first item in the national curriculum for mathematics.

Secondly, children's understanding of counting puts them in the powerful position of being able to enter the realms of addition and subtraction (as well as multiplication), which in themselves constitute faster ways of counting. Children who have difficulties in remembering addition facts, or cannot derive new facts from an existing repertoire, can only find sums of numbers (or differences between them), by counting. This ability, which will be known to some children before the beginning of school, will soon be mastered by many others and quickly become a reliable way to obtaining answers to addition and subtraction operations throughout the whole of primary school.

Thirdly, there is an important conceptual difference between counting units of the same denomination — i.e. ones — and counting units of different denominations, such as ones, tens, hundreds and so on. Whilst counting units of the same size is limited to the handling of small numbers, the combined utilisation of ones, tens and hundreds allows the use of larger numbers, through the abstraction of generative rules which underlie the ordering of the number-word list and the structure of the decade numeration system. Further studies are needed to clarify children's development from counting ones to using the decade system.

2.2.2 Understanding of the number-word list

Beyond the counting principles that are used to count sets of objects, children must also learn to count higher quantities. To do this they must learn the number-word list. Children's learning of the number-word list develops in two distinct phases: a first acquisition phase, in which children learn to say the conventional sequence correctly, and a later elaboration phase, in which equivalence and order relations and operations on sequence words are constructed and the sequence can be produced in more complex ways (Fuson et al., 1982; Fuson, 1988). These phases are overlapping as the early part of the sequence may be undergoing elaboration while later parts are still being acquired and cannot yet be said correctly.

According to Fuson (Fuson et al., 1982; Fuson, 1988), children's sequences during the acquisition level (i.e. before they have learned the standard sequence) have a characteristic structure. For sequences up to thirty, children tend to produce a first portion consisting of an *accurate number-word sequence*, followed by a *stable incorrect portion* of from 2 to 6 words that are produced with some consistency over repeated trials, followed by a final *nonstable incorrect portion* that may vary in length. The first portion consists of the first x words said in the conventional order and varies with age.

Most of the *stable incorrect portions* include words in the conventional order, but words are omitted (e.g. 12, 14, 18, 19). *Nonstable incorrect portions* are composed of three types of elements (1) forward runs, from two to five contiguous words said in the conventional sequence (e.g. sixteen, seventeen, eighteen); (2) forward runs with

omissions (e.g. twelve, fourteen, seventeen); (3) single, unrelated words (e.g. twelve, thirty, nine, sixty). According to Fuson, these incorrect sequences seem to result from the irregularities found in the English system of number-words for the words between ten and twenty and for the decade words. These irregularities suggest also that

"one of the first experiences of English-speaking children with a mathematical structure (the English sequence of number-words) is that it is complex, irregular, and must be memorised laboriously rather than that it has a clear and obvious pattern that is easy to learn" (Fuson, 1988; p. 58).

Children's learning of the number-word sequence continues beyond the acquisition phase, i.e. the point when they can produce the list correctly. The representation of the number-word sequence at the "elaboration of the sequence level" differs qualitatively from the previous level, where several new abilities can now be displayed. Different parts of the sequence can be at different elaborative levels at the same time, and the elaboration level is a lengthy process that develops between age 4 and 8 (Fuson, 1988).

Regarding the elaboration level, Fuson (1988) defined five sequential levels of competence in children's counting according to the way they deal with the number-word sequence and display more elaborate counting skills. These are the string level; the unbreakable chain level; the breakable chain level; the numerable chain level and the bidirectional chain level.

Initially, children are at a *string level*, where the words are not yet objects of thought. Instead, they are produced but not reflected upon as separate words. Then, the sequence moves into an *unbreakable chain level*, where separate words become differentiated and intentional one-to-one correspondence can be established, although children must count all the objects without stopping; any interruption in the counting sequence implies starting from one again.

According to Fuson (1988), children at this level may already answer the question "how many blocks are there ?" with the last count word, showing, therefore, numerical competence. Also, they may use counting for cardinal addition and subtraction operations through a strategy known as 'counting-all' (e.g. Carpenter and Moser, 1982), which will be discussed in later sections.

The next conceptual level within the sequence is the *breakable chain level*, where children can start counting from any given point in the number-list, rather than from the beginning. At the next level the sequence becomes a *numerable chain level*, where the level of number abstraction enables children to regard number-words as units and to establish relations between them. The number words can be used as sequence unit items in counting and can simultaneously represent the sum and the addends embedded within the sum. Children at this level are capable of using the 'counting-on' strategy in addition problems.

Finally, the sequence becomes a *bidirectional chain level*, where words can be produced easily and flexibly in either direction, forward or backward. At this level, no entities are required to define cardinality, and the child can now conceptually operate on and relate specific cardinal numbers. Children at this level are capable of using the 'recalled-facts' strategy in Addition and Subtraction problems.

The first important conceptual development of children's counting, therefore seems to be the passage from the unbreakable chain level to the breakable chain level, when children become able to continue counting from an arbitrary number in the list. This happens before they start to enumerate sets of objects (numerable chain level), where counting-on is possible.

2.2.2.1 The importance of knowledge of the number-word list and children's understanding of continuation of counting.

The importance of children's understanding of the number-word list relates to the finer developments involved in children's passage from the *unbreakable* to the *breakable* chain level - i.e. when they become able to continue counting from any number, as suggested by Fuson (1988).

The idea that children's mastery of continuation of counting is not a trivial development was suggested by several authors (Davydov, 1969; Siegler and Robinson, 1982; Secada et al., 1982; Cobb and Wheatley, 1988; Steffe, 1992, Fuson, 1988). Children who cannot continue counting need to count from one every time because they cannot create numbers such as six in a purely conceptual manner. For these children, once the counting episode is completed, that number ceases to exist. Hence, continuation of counting is seen as an indicator that children have acquired more flexible and abstract strategies to deal with number (Cobb and Wheatley, 1988; Steffe, 1992; Aubrey, 1993).

Also, Fuson (1988) and Davydov (1969) have highlighted the conceptual relevance of continuation of counting in a different way. According to these authors, once children become able to understand counting and cardinal situations simultaneously, the perceptual unit items become capable of simultaneously representing a sum and an addend embedded within that sum:

"this sequence ability combined with the simultaneous perceptual unit items allows children to carry out new, more efficient solution procedures in addition and subtraction situations: *counting on*, counting up, and counting down with entities" (Fuson, 1988; p. 407).

Secada et al. (1982), in particular, has shown that continuation of counting is a relevant precursor of more efficient counting strategies such as counting-on. Despite several views suggesting the relevance of continuation of counting, there remains the need to clarify its importance in furthering children's ideas about the arithmetical operations and knowledge of the decade numeration system.

2.3 GENERATING VERBAL NUMBER-WORDS & USING NUMERATION SYSTEMS

2.3.1 What are numeration systems ?

Numeration systems are cultural inventions devised to enable a powerful and more accurate utilisation of number (e.g. Luria, 1969; Saxe, 1991; Nunes and Bryant, 1996). This is made possible through the use of conventional rules that are comparatively more abstract than knowledge of the counting principles. These rules allow the production of any number in the numeration system. In order to understand the generative rules of a numeration system, children must first learn a base sequence of numberlogs, and the length of this base may vary across linguistic environments. According to Zaslavsky (1973), the base may be as small as 2 in some cultures and as large as 20, in others, although the most common bases are in between 5 and 15 items long. Most Western countries have conventionalised the use of 10 as a base.

2.3.2 What children must know to use numeration systems ?

Whilst children's counting entails the use of units of the same denomination (i.e. ones), their grasp of the conventional invariants of the decade numeration system further entails the understanding of more complex knowledge. Firstly, that units may have different denominations - ones, tens and so on -, and secondly, that these units can be counted and combined in order to form larger numbers. In other words, that all numbers are compositions of other smaller numbers that came before them in the

number-line; i.e. additive composition of number (Resnick, 1983; Fuson, 1990; Nunes and Bryant, 1996).

At a very simple level, the idea underlying the verbal number-word system is that whenever a group of 10 units are added together, or regrouped, they become *one* of the next unit; i.e. 1 ten. Subsequently, a group of 10 tens, become one-hundred; 10 hundreds become a thousand and so on. In short, through the use of addition and multiplication the system reproduces the same base (10) over and over, allowing faster movements forwards and backwards in the system. This idea is quite different from the requirements of dealing with units of the same size (i.e. ones), where numbers can be thought of as part of a number-line - the higher the numbers are, the further down the line they will be placed. As units of the same size, numbers can only be seen in terms of relative size - i.e. bigger or smaller than others - but not in terms of compositions of other numbers (Resnick, 1983).

2.3.3 The conceptual importance of using numeration systems

The use and understanding of the decade numeration system has a fundamental conceptual importance in children's arithmetical cognition. As a complex thinking tool, it avoids the need to memorise an endless number-word list and provides the child with a much faster and more complex counting system. The need to reproduce unrelated tags for each number implied in counting, entails a memory-load problem, which is solved by the generativity rules implied in the use of numeration systems. This aim has been sought by primitive and modern numeration systems (e.g. Menninger, 1969; Skemp, 1971; Hughes, 1986).

The Oksapmin tribe, for example, reduced the number of memorable tags to a 'round' of twenty-seven. Each tag is represented by a body-part from, say, the left thumb, each of the fingers, knuckles, up the elbow, some parts of the face, down to the right arm, and back to the right-hand thumb (Saxe, 1981; 1982). This solution, however, is only partial and still does not enable the generation of larger numbers, in the hundreds and the thousands. The Hindu-Arabic system, on the other hand, solved the problem through the use of a generative rule with base-10.

The understanding of the complexity of the numeration system, through the use of the generativity rule and the ability to count and combine units of different denominations, means that there is no need to know all the numbers involved in the counting up to 5275, for instance. Hence, the child may be able to count up to that number, without previous knowledge, if asked to do so. In other words, the number of number-words to be memorised is almost insignificant when compared to those implied in the whole number-line. All the numeration system requires for its use is the understanding of the logic of regrouping of ten units into the next unit and the memorisation of some thirty number-tags.

Some languages require the memorisation of some number-tags beyond the 1-9 digits, due to the irregularities of the system, mostly in the teens, but this does not change the fundamental importance of generativity rule. In the case of the English language, up to number twenty the number-tags used are short of cues for the young learner who has no choice but to memorise them. Numbers such as eleven or twelve give nothing away about the structure of the system, when compared with the clarity of *ten-one* or *ten-two* used in Japanese. Here, the young learner finds that the decade structure is faithfully reproduced throughout the system, and the system relies only on 10 basic number-tags. After ten, comes ten-one, ten-two, ten-three, and so on. This way, the user knows which decade s/he is dealing with.

In French, the difficulties are similar, but extended to the complication of dealing with numbers such as *soixante-dix* (sixty-ten meaning seventy), *soixante-douze* (sixty-twelve; meaning seventy-two), or even *quatre-vingt-douze* (four-twenty-twelve; meaning four times 20 (80) plus twelve, meaning 92 !). These constructions follow no specific pattern other than a linguistic style — which, incidentally, are not shared by all French speaking countries. So, while the Japanese rely on the recombination of tens and ones, English and French also rely on that structure, but have some exceptions that have to be known by the child beforehand. Although these exceptions can be an obstacle, the use of the system clearly makes number generation much easier.

At another level, the use of the numeration system, such as the base-10, entails the compatibility between notation, measurement and the currency systems. In other words, counted, written and measured numbers share the same conventional structure. When written, numbers acquire a place-value according to their position and any computation on base 10 becomes more efficient and economic when compared to systems without a base (e.g. Saxe and Posner, 1983).

The use of the decade numeration system implies, from the child's point of view,

"a break with simpler concepts of the past, and a reconceptualization of number itself" (Hiebert and Behr, 1988; p. 9).

Children can only use this tool after having understood a particular aspect of the principle underlying the numeration system, i.e. additive composition of number. The idea that any number is equal to the sum of any other two numbers that precede it in the counting-list is common to numeration systems of any base. However, the specific aspect of additive composition that helps children to understand the decade numeration system in particular is the idea that any number can be seen as a sum of tens and ones, or as a sum of hundreds, tens and ones and so on.

However, little is known about the conceptual requisites of this development. What specific number knowledge must the child know before this reconceptualisation takes place ? In other words, what are the number components involved in the development from counting one to counting units of different denomination (i.e. ones and tens) and the emergence of additive composition of number ? Furthermore, considering that knowledge of the decade system entails the previous understanding of sum and product relations, what is the role of early addition and multiplication in the development of the decade system ?

2.4 ARITHMETICAL OPERATIONS

2.4.1 Addition and Subtraction

2.4.1.1 What are Addition and Subtraction ?

Addition and subtraction represent a relation between two sets of numbers, and a third set, the sum. The consequence of this relation is the change in numbers either by an increase or a decrease of the quantities. In other words, two distinct sets of objects, with no members in common, are put together and the child is required to compute the cardinal number of the new set. In the case of subtraction, a set of objects is partitioned (e.g. some of the elements of the set are taken away) and the child is required to find out how many are left. The relation established between the sets in addition is more complex than the relation of larger and smaller or increase/decrease (e.g. English and Halford, 1995; Haylock and Cockburn, 1997).

2.4.1.2 What children must know to Add and Subtract

The same basic model of adding two distinct sets, say $4+3$, may be applied to a range of different situations. Several stories can be created such as "Mary has 4 sweets and Peter gave her another 3 sweets. How many does Mary have altogether ?", or "there are 4 girls and 3 boys. How many children altogether ?", or "a plant was 4 inches tall on Monday. A week later it had grown another 3 inches. How tall is the plant now ?" However, to succeed in each case, children are required to interpret the numbers in each set in the cardinal sense, before proceeding to their union. In time, they will also

see that these situations share a common structure, one that implies the aggregation (or partitioning, in the case of subtraction) of sets:

"Aggregation is an important structure that has to be linked with addition. It forms part of the network of connections that constitutes the concept of addition" (Haylock and Cockburn, 1997; p.34).

According to the same authors, children can also grasp addition through the structure of aggregation, as situations that incorporate the ideas of counting-on. For example, start at 6 and count-on 4.

"This idea often relates most strongly to the ordinal aspects of the numbers and is experienced most clearly in making moves on a number line " (Haylock and Cockburn, 1997; p.35).

2.4.1.3 The importance of addition and subtraction

Practice with addition and subtraction shows children the new possibilities in the realm of number, in two ways. Whereas in the earlier stages counting led to enumeration and cardinality, to the ability to relate quantities and to an entry into the realms of addition and subtraction, addition and subtraction will now show children that numbers can be transformed and operated at will.

Secondly, practice with addition will show children much faster ways of counting and the possibility of handling the decade numeration system. Addition, therefore, becomes a central number component to assess whether children can handle the more complex transformations of number.

At another level, the emergence of addition remains unclear. It has been widely assumed that children use counting strategies to progress to more sophisticated ways of doing certain types of sums. This excludes the more difficult addition problems such as start-unknown word problems (inversion; $?+5=8$), which cannot be solved by counting, considering that it is difficult to represent the first addend with fingers.

2.4.2 Multiplication

2.4.2.1 What is Multiplication ?

The introduction of the multiplication operation represents a significant change in children's dealings with number. They must now contend with two forms of change in the nature of the unit, namely, changes in what the numbers are and changes in what they are about (Hiebert and Behr, 1988). The emergence of multiplication implies a shift from operating with single units to operating with composite units (Steffe, 1988). The same can be seen when children count in 2's, 5's or 10's.

Multiplication is generally introduced in second year of school and treated as a faster way of doing repeated addition (Fennell et al., 1991; Hoffer et al., 1991; Newmark, 1991; Nichols and Behr, 1982). Although multiplication is easily understood by some children, others struggle with it throughout primary school. Some of the reported problems in the learning of multiplication have been that children tend to add instead of multiplying. For instance, in the problem "if a carton has four yoghurts, how many yoghurts are there in five cartons ?", children write $4+5=9$ (Hart, 1981; Kamii and Livingston, 1994).

Also, children find it much easier to derive answers to addition problems from other addition facts than to derive answers to multiplication problems from other multiplication facts (Kamii and Livingston, 1994). Knowledge that $4 \times 4 = 16$ is less helpful for children to figure out how much 4×5 is. However, $5+4$ can be easily derived from $4+4=8 \dots +1=9$. Other children have problems with the meaning of

multiplication, and mistake the "x" sign, for a "+" sign. O'Brien and Casey (1983) asked children to interpret 6×3 . Forty-four percent of fifth graders said

"there are 6 ducks swimming in the pond...Then a while later 3 more ducks come, so how many are there ?" (O'Brien and Casey, 1983; p. 248).

2.4.2.2 What children must know to multiply

Although authors generally agree, regarding the classification of multiplication word-problems in terms of isomorphism of measures, product of measures and multiple proportions (Vergnaud, 1983), they have disagreed in relation to the development of children's grasp of multiplication. One view, is that multiplication develops from addition (i.e. sequentially) and these operations are seen as conceptually similar. Another perspective takes the opposite view, that is, addition and multiplication are discontinuous operations (i.e. develop at the same time). Although some multiplications can be solved using repetitive addition, that does not mean that the whole of multiplication procedures can be explained by addition.

Alternatively, Steffe (1988; 1994) has extended Piaget's work by further detailing the development of children's multiplicative thinking. Steffe based his research on the development of the counting scheme, which as mentioned in earlier sections, is not limited to counting by ones, but can include composite units (units of more than one).

According to Steffe's (1994) view,

"for a situation to be established as multiplicative, it is necessary to at least coordinate two composite units in such a way that one of the composite units is distributed over the elements of the other composite unit" (Steffe, 1994; p.19).

In other words, evidence that children are able to think multiplicatively should come from their ability to think simultaneously about units of one and about units of more than one, as was suggested by Piaget (1952) with the vase (and flowers) experiments.

2.4.2.3 The importance of Multiplication

Children's understanding of multiplication may help them see the number word system in terms of composition of units that bear product relations (e.g. 600 and 6 hundred; 50 and 5×10). It is important to assess the significance of early multiplication in counting, and whether this understanding expands children's understanding of the decade numeration system.

2.5 WRITTEN NUMBER AND THE UNDERSTANDING OF PLACE VALUE

2.5.1 Writing single-digit numbers

The reading of single-digit numbers is, for the child, a basic associative learning task, where s/he sees the numeral and recalls its correspondent number word (Fuson, 1988). According to Bialystok (1992), symbol recognition represents the second step in a three-step development of a symbolic number representation. In the first step, the child recites the correct name for each element in the number sequence. In the second step —symbol recognition - children are able to recognise, produce and name the written notations. Here, they represent the written numbers as objects with particular visual characteristics but not as meaningful symbols. The understanding of the individual symbolic forms will only emerge on the third stage, where the child is finally able to associate the written form of a number with the quantity it represents.

The mapping of this process starts with a number 5, for example, to its verbal representation (“five”), and then to a model of this number, either a mental, physical or pictorial representation. Once children have grasped the meaning of symbol recognition, they need practice to transfer its meaning through the different representations (Post et al., 1993).

Children’s progression to written multi-digit numbers involves a totally new dimension, that has been described by several authors (e.g. Menninger, 1969; Ginsburg, 1977; Brown, 1981; Sinclair et al., 1983; Ross, 1989; Miura and Okamoto, 1989; Bergeron and Herscovics, 1990; Sinclair et al., 1992; Seron and Fayol, 1994;

Yang and Cobb, 1995; Nunes and Bryant, 1996; Sinclair and Scheuer 1993; Miura et al., 1993; Power and Dal Martello, 1990). The Arabic written system constitutes a strict positional system and its lexicon is reduced to a small set of ten symbols - the digits from 1 to 9 and 0. The position of a digit in the numeral determines the power of the ten-base by which it must be multiplied (Fuson, 1990; Nunes and Bryant, 1996).

2.5.2 What is place value ?

The way numbers are written involves the concept of units of different sizes. In the number 136, for example, the 1 indicates the number of hundreds, the 3 the number of tens and 6 the number of ones. This system represents a major evolution, compared with other historically previous systems - such as the Roman - by having introduced the concept of place value.

Place value, is a convention defining that each digit signifies a unit of a different size, according to the position occupied in the number. In the number 222, for example, the first digit means 200, the second digit means 20 and the last digit means 2. Finally, whenever any of the units, either the ones, tens or the hundreds, has no value, a zero is used as a place holder. For example, the number 305; there are no units in the tens.

"Thus, some aspects of the written numeration system require the understanding of the same principles as the oral system but other aspects - namely, place value and the use of zero as a place holder - are specific to the written system" (Nunes and Bryant, 1996; p. 67).

2.5.3 What do children have to know to write multi-digit numbers ?

The verbal number-word system (based on the decade numeration system in the case of the English language) is different from the Arabic system, which is a written representation of numbers (Fuson, 1990; Seron and Fayol, 1994; Nunes and Bryant, 1996). The literature has shown some confusion between the development of these two conceptual structures which tend to be assessed as one [see, for example, the works of Luria (1969), Sinclair et al. (1992) and Sinclair and Scheuer (1993)].

However, several authors support the view that these are two separate number components in the form of 'generating verbal number-words and using numeration systems' (presented earlier) and 'written numbers and the understanding of the principles underlying place value' (Resnick, 1983; Fuson, 1990; Nunes and Bryant, 1996).

The verbal number-word and the Arabic systems differ in their lexicon and in their syntactic structures (Fuson, 1990; Seron and Fayol, 1994; Nunes and Bryant, 1996). The verbal English system, for example, is composed of several classes of numerical quantities; the unit words, the teens words and the tens words. There are also multiplier words like 'hundred', 'thousand' and 'million' which according to their position in a word sequence enter in sum or product relations with the basic numbers. For example, 'four-hundred' corresponds to a product relation, whereas 'hundred and four' corresponds to a sum relation (Fuson, 1990; Seron and Fayol, 1994).

The Arabic written system, on the other hand, is comparatively simpler, since it constitutes a strict positional system. Its lexicon is reduced to a small set of ten

symbols – the digits from 1 to 9 and 0. The position of a digit in the numeral determines the power of the ten-base by which it must be multiplied and the '0' serves to indicate the absence of a given power of the base in the number (Fuson, 1990; Nunes and Bryant, 1999).

There are two views on how children develop their understanding of place value. One group of authors argue that children learn about place value from experience with written numbers, i.e. by observing the relations between different digits. Another group suggests that knowledge of place value can only be developed after children have understood the structure of the numeration system (e.g. Ginsburg, 1977; Nunes and Bryant, 1996).

2.5.4 Why is place value important ?

Teaching children about the decimal system and the positional system based on it, is the most difficult and important instructional task in mathematics in the early school years (e.g. Resnick, 1983; 1986). Considering that understanding of place value is necessary for both the writing of multi-digit numbers and subsequent success in the computation of algorithms (e.g. addition and subtraction by columns), its understanding represents a basic milestone, in children's mathematical learning (Resnick, 1983).

Without this understanding, most children resort to idiosyncratic rote-procedures while attempting to write multi-digit numbers (Ginsburg, 1977; VanLehn, 1990) and performing multi-column operations, leading to errors and frustration. This situation clearly needs further investigation, considering that about half of nine year-old

children, still have not understood place value (e.g. Kamii, 1980; Brown, 1981; Bednarz and Janvier, 1982; Kamii, 1986). It remains of importance to clarify whether children's understanding of the structure of the decade numeration system is a requisite for their grasp of place value.

3

REVIEW OF RESEARCH ASSESSING INDIVIDUAL NUMBER COMPONENTS

3.1. INTRODUCTION

This chapter briefly reviews previous research assessing the individual number components included in this thesis. It briefly discusses the methodologies used in these studies and argues for the choice of methods to be used in the present study. It also briefly reviews what previous studies suggest about the development of competence in the individual components.

3.2. COUNTING AND KNOWLEDGE OF THE NUMBER-WORD LIST

3.2.1 Counting principles

Children's knowledge of counting has been assessed by several authors (e.g. Gelman and Gallistel, 1978; Gelman and Meck, 1983; Meck and Church, 1983; Gelman and Meck, 1986; Briars and Siegler, 1984; Fuson, 1988; Wynn, 1990; Frye et al., 1989;

Greeno, 1991; Wynn, 1992), the majority immersed in the debate of whether understanding precedes counting skill (innate principles) or whether it follows skill acquisition (acquired principles). As this debate is beyond the scope of this thesis, I will concentrate on the data provided by two influential papers by Briars and Siegler (1984) and Fuson (1988), which clarify the ages at which children can be expected to count by making correct use of the counting principles.

3.2.1.1 Assessment, methodology and children's performance

Research has shown that there is a developmental gap between counting and displaying actual knowledge of the counting principles (e.g. Piaget, 1952; Bialystok, 1992; Briars and Siegler, 1984; Fuson, 1988). The reason is that children could simply be repeating a list of number-words, unaware of its meaning. This has made the assessment of the counting principles a complex issue, since it cannot be assessed directly.

Briars and Siegler (1984), for instance, assumed that children's judgements about a puppet's counts reflected their knowledge of the principles. Still, they found that although even the 3 year-olds could count correctly in 75% of the trials, no children had consistently rejected the puppet's counting errors. Briars and Siegler also found that although their 3, 4 and 5 year-olds could accept correct counts from a puppet, they could not discriminate the puppet's unusual counts nor judge the puppet's incorrect counts.

Fuson (1988) also showed that children could count without awareness of its meaning by using a different methodology. She reasoned that if the counts were guided by

principles, children could still apply them in slightly more difficult situations. Initially, children were asked to count objects displayed in straight lines and with regular intervals between the objects to be counted. She then asked children to count the same objects in scattered-irregular rows.

By comparing 3 to 6 year-olds' results in two conditions where 'straight-regular' and 'scattered-irregular' rows of objects were counted, Fuson (1988) found that children invariably made more errors in the scattered trials which confused their strategies. This showed that the modification of the array shape represents a serious obstacle in children's counting, especially considering that a significant part of the 5 and-a-half to 6 year-olds still made counting errors in the scattered condition (i.e. 38 errors over 100 counts). Fuson concluded that although it is possible that children lose their counts because they find it hard to keep track of the scattered blocks, or they simply forget, it seems more plausible to consider that the counting principles are not innate.

Alternatively, Nunes and Bryant (1996) have also assessed children's ability to count units of the same denomination in shopping tasks, where children were required to buy toys from a shop and pay with play-money worth one penny each. This task implies the understanding of cardinality, as the amount to be paid, but avoids the question of "how many objects are in there?". Children are simply requested to match the price with an equal numbers of tokens. Nunes and Bryant reported that both Brazilian and British 5- and 6- year-old children obtained ceiling-level results in amounts up to twenty.

The present study used a counting task similar to the one used by Fuson (1988). Children were required to count scattered rows of tokens and in order to do this they were required to make correct use of the one-to-one correspondence and stable-order

principles. As ceiling-level results were expected with five year-olds, a counting units of the same denomination task was also used.

3.2.2 Knowledge of the number-word list

Children's knowledge of the number-word list has been assessed by several authors with different purposes (Fuson et al., 1982; Siegler and Robinson, 1982; Fuson, 1988; Miller and Stigler, 1987). Siegler and Robinson's (1982) assessment was primarily concerned with making explicit children's underlying representation of the number string. Fuson et al's. (1982) assessment focused on a clearer understanding of how children develop length and accuracy. Miller and Stigler (1987) clarified the relevance of linguistic cues in children's learning of the number-word list. All these studies make important methodological contributions to the present research.

3.2.2.1 Assessment and methodology

Siegler and Robinson (1982) proposed three models to account for the development of the underlying representation of the number string. Their assessment focused on the analysis of children's behaviours in terms of stopping points, omissions and repetitions. One of the key assumptions in their investigation was that children's stopping points were a good indicator of their level of competence.

They argued that children who cannot count beyond 20 generate numbers according to a "next" rule. Those who count between 20 and 99 recall the next number from

memory until reaching 20. Beyond this number, counting within a decade involves concatenating a decade name with the next digit. Transitions between decade, involve the mastery of specific connections between each number ending in nine (e.g. 29) and the first number of the next decade (30). Finally, children who count beyond 100 do so by prefixing number names above 100 with the appropriate hundred's name.

Siegler and Robinson (1982) reported that children who counted beyond 29 were likely to stop their counts with a number ending in "nine", whereas there was no such regularity in the stopping points for children who stopped before 29. This data, suggested these authors, supported the view that in learning the number-word list, children detected and used the relatively transparent structure that features the number string beyond number 20, but not the less obvious structure that features the teens numbers.

The problem, however, is that inferences about stopping points are limited. Asking children to recite numbers maybe an important indicator of knowledge of the count word list but it is not a sufficient assessment of children's knowledge of the counting word system.

Evidence to substantiate the idea that children's stopping points are a poor, rather than a good indicator of their competence was provided by three independent studies (Fuson et al., 1982; Siegler and Robinson, 1982; Miller and Stigler, 1987) which have showed divergent results on children's stopping points in abstract counting.

Whereas Siegler and Robinson found that 69% of children who could count as high as 20 stopped with a number ending in 9, Fuson et al. reported that only 31% of the children they observed did this. Miller and Stigler, on the other hand, reported that

44% of American children who counted up to a number between 20 and 99 stopped with a number ending in 9, and 32% of Chinese children showed the same pattern. Such divergence indicate that where a child chooses to stop counting may be a poor indicator of her level of counting competence and that other important indicators are also involved.

Alternatively, Fuson (Fuson et al., 1982; Fuson, 1988) assessed the length of children's correct sequences (and the structure of their incorrect sequences), and its development from age three to age five. According to Fuson (1988), children within a given age group show considerable variability in the length of the correct sequence that they can produce. Their ability to say the correct sequence of number words is strongly affected by any opportunity they may have to learn and practise this sequence, from exposure to relevant television programs, to older siblings or the aid of parents.

Meanwhile, it has been widely accepted that children's learning of number-words is dependent on the features of the number-word list (Ginsburg, 1977; Fuson et al., 1982; Miller and Stigler, 1987; Song and Ginsburg, 1988; Fuson and Kwon, 1992; Nunes and Bryant, 1996). This has raised the possibility that children's experience of counting with a regular system may help them to understand the properties of a base-ten system, in comparison with the more irregular systems.

This hypothesis was investigated by Miller and Stigler (1987) who compared 4-to 6-year-old Taiwanese and American children by asking both groups to count objects (either in rows or arranged randomly) and to count abstractly as far as they could. The Taiwanese children had learned the Chinese system, which is regular and transparent, whereas the American children used the English system, which is irregular in the numbers between 10 and 20. They reported that the Chinese children were

significantly better at both kinds of counting, either in rows or randomly. The Taiwanese children also differed in their ability to produce the conventional count words as they pointed to objects, producing significantly higher results.

Regarding the abstract counting task, Miller and Stigler also reported that the American children made significantly more mistakes in their sequences between 10 and 20. Hardly any of the Taiwanese children went wrong at this stage whereas a large number of the American children did.

This evidence clearly suggests that there is a relation between the characteristics of a language's number-word system and children's acquisition of that system. A helpful set of words in the form of a regular number-word system such as the Chinese one plays an important role in children's learning of the number-word sequence. Children learning this system will benefit from grasping early on that the numeration system repeats the same structure, whereas children learning the English system will have to recall the meaning of irregular number-words such as "twelve" or "thirteen". This, suggested Miller and Stigler (1987), limits the induction of the underlying rules for number formation. Miura and Okamoto (1989) and Miura et al. (1993) bring further confirmation to this argument in studies that compare the populations of six different countries (People's Republic of China, France, Japan, Korea, Sweden and the United States).

The present study assessed children's knowledge of the number-word list similarly to what has been done in previous studies. Children were invited to count until they made two successive mistakes. The older children were asked how far they thought they could count (n), and were invited to count from n-12, to avoid fatigue. The next

section briefly outlines the results obtained by Fuson (1988) in the assessment of children's production of the number-word list.

3.2.2.2 Children's performance

Fuson's (1988) data on children's production of the English sequence of the number-word list indicate that most children below age 3 and a half attempt to learn the sequence to ten, most children between 3 and a half and 4 and a half are working on the sequence of number-words between ten and twenty, and a significant proportion of children between 4 and a half and 6 have difficulty with the sequence between fourteen and twenty, although many are already working on the decades between twenty and seventy (Table 1).

TABLE 1

Percentage of age groups producing accurate sequences of various lengths (from Fuson, 1988)

Age	n<10	10—14	14—20	20—30	30—72	72—101
3,6 to 3,11*	17	44	22	17		
4 to 4,5		41	35	12	12	
4,6 to 4,11		12	47	18	12	12
5 to 5,5		6	25	13	44	13
5,6 to 5,11		6	22	17	44	11

* years, months

The table also shows that over 50% of children in kindergarten, which is equivalent in age to the English Year 1 class, count over 30. Furthermore, over a half of the First grade (equivalent to Year 2 class) children count over 100. There remains a need to clarify the influence of these developments on other number components.

3.2.3 Continuation of Counting

Gelman and Gallistel's (1978) principles-first counting model does not discriminate the subsequent stages of counting beyond the acquisition of the counting principles themselves (Gelman and Meck, 1992). The conceptual importance of the different stages of development of counting was highlighted by Fuson's (Fuson et al., 1982 and Fuson, 1988) characterisation of the various stages of development in the acquisition of the number-word sequence, and the development from the unbreakable to the breakable chain level at age 3 and 4. This framework is central for a finer investigation of the relationship between counting and other number components, as it highlights continuation of counting as the first significant development after the acquisition of the counting principles.

3.2.3.1 Assessment and methodology

Children as young as 2 or 3 display continuation of counting. Fuson et al's. (1982) evidence that children go through a stage where they must always count from one (i.e. *unbreakable* chain level), to another stage where they are able to continue counting from an arbitrary point (breakable chain level), comes from a study assessing the ability of 24 three and four year-olds to continue counting. In their task children were asked to continue counting after being prompted with either: (1) a single word; (2) two and (3) three successive words from the counting-list.

By giving the child three successive words of the counting-list, the authors intended to provide evidence that the recitation could only continue if induced by the experimenters' sequence and not by children on their own. They reasoned that superiority of the results in the recitation context (i.e. 3) would indicate that children's word sequence does go through an unbreakable chain level, as they cannot continue to count unless they are aided. Children were given number-words of two sizes: single-digits and teens.

In this study children were asked to continue counting from numbers 10 and 20. Both numbers have the advantage of relating to the decade numeration system but do not involve numbers in the teens, which are problematic for children (Siegler and Robinson, 1982; Fuson, 1988).

3.2.3.2 Children's performance

The results (reproduced in Table 2) indicate that generally, the three year-olds had more difficulty in producing sequences in the teens, which did not happen in the 4 year-old group, in the three conditions. Considering the single-digit sequences in the 3 year-old group, 63% of the children gave correct responses to the three-word stimulus, whereas already 39% succeeded in the one-word stimulus - i.e. showed ability to continue counting. There was no difference between the 2-word and 3-word conditions.

TABLE 2

Percentage of correct responses in Fuson et al.'s (1982) study

Age	One-word		Two-words		Three-words	
	Digit	Teens	Digit	Teens	Digit	Teens
3 year-olds	39	15	62	48	63	33
4 year-olds	64	63	82	83	83	81

The 4 year-old group also showed differences between the three-word stimulus (83% correct) and the one-word stimulus (64% correct), but not between the 2 and 3-word conditions. Again, results revealed that the majority of four year-olds could continue counting from an arbitrary list. The higher number of correct responses in both the two and three-word stimulus, confirmed these author's hypothesis that there is a difference between asking children to continue to count and having to provide help for them to do so. Fuson et al., (1982) concluded that those who needed to be induced were at the unbreakable chain level.

Siegler and Robinson's (1982) data supports the unbreakable chain level in children's production of number word sequences. It also suggests that the development from one level to the next is not trivial. Siegler and Robinson (1982) reported that pre-school children made significantly more errors when they were asked to start producing number-words from an arbitrary point within their accurate counting range than when they were allowed to start counting from one. When asked to start from a number other than one, children tended to stop at the end of the decade or make decade transition errors (e.g. going from twenty-nine to forty). Those who started from one made significantly fewer errors of this type. Finally, Secada et al. (1983) reported that



only 6 out of 63 six and a half year-old children could not start from an arbitrary point in the teens.

Aubrey (1993) used a similar task in a study with fourteen children aged four and five (4:4 to 5:0; mean age 4:6), who were assessed within their first weeks in a Reception class. The purpose of this task was to see whether these children were able to count-up or down from any number in the number-line. The children were asked what number came after/before randomly presented numbers 1 to 12, 14, 16 and 20, in a total of 15 items. Aubrey's (1993) results, reporting that half of the children were able to say what number came after randomly presented digits up to 10, support the idea that as early as four some children develop more flexible and abstract counting strategies. However, no data has been produced on the development of continuation of counting or the effect of this development on other number components, after the beginning of schooling.

3.3 GENERATING VERBAL NUMBER-WORDS AND UNDERSTANDING OF THE STRUCTURE OF THE DECADE NUMERATION SYSTEM

3.3.1 Counting with units of the same denomination

The reader is referred to section 3.2.1.1

3.3.2 Counting with units of different denomination

Children's ability to count units of different denominations (i.e. ones, tens, hundreds and so on) has been assessed by different authors, using different contexts (Russell and Ginsburg, 1984; Kamii, 1986; Nunes and Bryant, 1996). Russell and Ginsburg (1984) investigated whether difficulties with base ten concepts may be responsible for the poor performance of children with Maths difficulties in school arithmetic. The same relation was investigated in main-stream children, aged 6 to 10 (Bednarz and Janvier, 1982).

Kamii (1986), on the other hand, tried to clarify ways in which children developed an understanding of the decade system by constructing the structure of tens on the structure of ones. Finally, Nunes and Bryant (1996) researched the influence of cultural tools such as coins on children's understanding of the decade numeration system.

3.3.2.1 Assessment and methodology

To investigate the relevance of knowledge of base ten concepts in children's maths difficulties in school arithmetic, Russell and Ginsburg (1984) asked (1) "normal" fourth grade children, (2) fourth graders with maths difficulties and (3) "normal" third graders to count groups of dots arranged in horizontal rows of ten, presented in cards. The four set sizes presented were 100, 50, 120 and 80 and children were scored for accuracy (i.e. whether children enumerated the sets correctly; total score was four) and strategy used (enumeration by ones, enumeration in groups except tens, enumeration by tens and other).

In another task to verify children's understanding of base ten concepts, Russell and Ginsburg gave each child four piles of play money (coins). They argued that children's grasp of tens and hundreds could be tapped by asking them to count the amounts of \$430 (4-100's, 3-10's), \$660 (6-100's, 1-50, 1-10), \$1530 (3-500's, 3-10's), and \$3020 (5-500's, 5-100's, 2-10's).

Russell and Ginsburg (1984) reported that all three groups of children responded correctly to 3 out of 4 trials, and no significant differences were found between the three groups, in the first task (counting dots). Also, no significant differences in the use of a counting by tens strategy was found. However, the results of the counting money task were much more discriminating with the same groups of children, as they showed that there were significant differences between both groups of fourth graders (i.e. "normal" and with math difficulties).

This data suggests that counting money may represent a slightly more difficult, but also a more accurate assessment of children's concepts of units of different denominations (i.e. ones, tens and hundreds). Furthermore, some children may see no need at all to count the dots on the cards by tens, although they may be able to do so. In other words, counting by tens becomes a choice strategy, which might bias the results.

Bednarz and Janvier (1982) assessed whether 8 to 10 year-old children understood the idea of ones, tens and hundreds in terms of groupings. In their task they showed children written numbers such as 402 and 513 (they also read the numbers) and asked them to write down a number somewhere in between the two previous ones. The child was also asked to use number tags (placed in front of the child; see Table 3) to make the number.

TABLE 3

Number tags used in Bednarz and Janvier's (1982) groupings task

0 ones; 1 one; 2 ones; 3 ones; 4 ones; 5 ones
 10 ones; 11 ones; 12 ones
 3 tens; 4 tens; 5 tens; 40 tens; 41 tens; 42 tens
 43 tens; 45 tens; 51 tens
 3 hundreds; 5 hundreds

Bednarz and Janvier (1982) reported that only 27% of eight year-olds and 44% of ten year-olds showed an understanding of ones, tens and hundred in terms of groupings. These are very low results compared with other assessments of children's understanding of ones and tens. Kamii (1986) reported that 83% of the second graders

included in her study showed some understanding of tens, by grouping heaps of tokens in tens. Also, that 64% of her fourth graders could count in ones and tens (making heaps).

One possibility is that Bednarz and Janvier's (1982) task was too demanding by asking children to make sense of a complex situation where they were required to mix digits with number-words in a quite unconventional way, such as is the case of "12 ones" or "42 tens". This situation does not resemble coins or notes, nor does it describe conventional ways in which numbers are written, constituting a limitation of the study.

Kamii (1986), on the other hand, used coins (play-money) of "one" to investigate ways in which children construct the structure of tens on the structure of ones, and therefore construct an understanding of the structure of the decade numeration system. Considering that adults usually make groups of tens when they quantify large collections of ones, Kamii explored whether this strategy was employed by younger children as well.

In order to do so, she asked 100 Genevan children from the first to the fifth grades to count groups of coins, while she recorded their spontaneous counting strategies; that is, whether they counted by 'ones', 'twos' or 'tens'. In a further stage of her experiment, she 'imposed' the counting by tens in order to see children's process of construction of 'tens' on the system of 'ones'. This time, children were given heaps of 'tens' and asked whether they wanted to count the heaps (following the advice of a child from "another school"), *or* mix the tokens up and count them by ones.

Kamii's (1986) results on the spontaneous counting task showed that there is a considerable lag between counting ones and being able to regroup those ones into tens. The majority of the children counted by ones and only 3 (out of 22) 4th graders and 1 (out of 18) 5th graders (i.e. 9 and 10 year-olds) were able to count by tens, spontaneously. Also, only a minority of the 2nd and 3rd graders counted by twos, which Kamii interpreted as a faster way of counting by ones.

The results of the counting by tens task (in heaps), however, were quite different. For a richer description, Kamii (1986) divided children who were able to count by tens into two categories: those who counted by tens only, and those who needed to count several ones up to a ten, and then repeated the procedure for the remaining tens.

Table 4 shows that from the second grade onwards, practically all children assessed had some idea that 10 ones are equal to ten, and that another 10 ones will give a twenty, and so on. In other words, and according to Kamii (1986), the process of constructing larger units seems to start from the first grade but, nevertheless is a very slow one.

TABLE 4
Frequencies of children who counted by tens (Kamii, 1986)

	Grade level				
	1st [21] *	2nd [18]	3rd [19]	4th [22]	5th [18]
counted by tens only	0	7	15	8	14
counted each ten by ones	8	10	4	14	4
failed	13	1	0	0	0

* Numbers in parentheses represent N in each grade

Although Kamii's (1986) results show that there is a relation between counting ones and counting tens, the evidence is not very clear in explaining the nature of this relation. In other words, the data allows inferences about *when* this relation happens, but not *how* it happens. Also, the data presented suggests inconsistency in the use of either "tens only" and "counted tens by ones", where no age trend is shown: nearly 40% of the second graders and almost 80% of the third graders were able to count in tens. However, only 36% of the fourth graders counted in tens.

On the other hand, the number of children needing to count ones to reach a ten diminishes until the third grade, grows on the fourth grade and diminishes again later. This data suggests that the choice of strategy cannot be accounted for in Kamii's task which represents considerable limitation in the assessment of children's understanding of ones and tens.

According to Kamii (1986), initially children can only count ones; then they start counting up to a ten, but lose track of the remaining counts. At a further stage they integrate the counts: 1, 2, 3...8, 9, 10 1, 2, 3 8, 9, 10...20!1, 2, 3 8, 9, 10...30! and so on. At a more proficient level, children count straight by tens in a 10, 20, 30 fashion. Unfortunately, however, no clear explanation for differences in development is offered, other than Piagetian reversibility:

"Children have to create a system of tens, by reflective abstraction, on the system of ones they have already constructed in their heads" Kamii, 1986; p. 84).

Kamii's (1986) study, probably reflecting the Piagetian view that counting has a secondary role in children's number understanding, may have led to the underestimation of children's abilities in counting by describing their development as an all-or-nothing event. Her classification of children's developmental differences in their understanding of the system of tens could have taken into account children's development from the *unbreakable* chain level to the *breakable* chain level, where they become able to count ones from an arbitrary number (Fuson et al., 1982). Indeed, some of the children in her study could count by two's but the implications of this ability were overlooked. Russell and Ginsburg (1984) have also overlooked this development in counting.

The second limitation, refers to what may arguably constitute a problem of many studies following the tradition of research in 'one-component', as is the case in Kamii's study. Its focus on counting alone does not clarify which other maths achievements may be involved in children's grasping of the structure of the decade system. This criticism seems pertinent considering the results of other studies suggesting that children's progression from counting ones to using the numeration system may involve other number components, such as addition (Carragher, 1985; Nunes and Bryant, 1996).

More importantly, from the methodological point of view, Kamii's (1986) use of tokens with face-value one may have not facilitated children's answers; rather it may have lead to confusion resulting in erroneous responses. Desforges and Desforges (1980) reported that a seven year-old girl failed to answer correctly to a task similar to Kamii's. After reviewing his procedure, Desforges concluded that the girl's failure was related to the use of an excessive number of one-tokens, which, in the main, confused her.

3.3.2 1.1 Shopping tasks

To study children's ability to count with different units, some authors have used an alternative known as the shopping task claiming that the context of coins provide a good estimate of children's grasp of informal aspects of additive composition of number, a property of the decade system (e.g. Carraher, 1985; Resnick, 1986; Carraher and Schliemann, 1990; Saxe, 1991; Nunes and Bryant, 1996). Although additive composition is applicable to all general number systems, it is the knowledge that any number is composed of ones, ten, hundreds and so on, that is useful in the assessment of ways in which children construct the system of tens on the system of ones.

The understanding of additive composition in seven year-olds is expected to be rather intuitive, but nevertheless of importance as a

"basis for highly flexible application of well-known concepts, notations, and transformational rules" (Resnick, 1986; pp. 166).

Resnick, who based her argument on a well researched case-study of a seven year-old, defined intuitive knowledge as self-evident to its user, but as not requiring justification in terms of prior premises (see also Vergnaud, 1983; Karmiloff-Smith, 1995).

Relative Values

The ability to count units of different denominations develops during the early years of school. It has as a prerequisite the understanding of the relative size of the different units. Carraher (1985) and Nunes and Bryant (1996) assessed this with the relative values task. In this task, two rows of play-money with the same amount of coins, are put on the table, one said to be the child's and the other one, the experimenter's. For example, the child is given three 1p coins and the experimenter has three 5p coins. The experimenter then asks: "Who do you think will buy more sweets, you or me ?". The child is also invited to justify the answer.

A response like "we both buy the same.... because we have the same amount of coins", has been interpreted as not recognising the meaning of the different denomination in the coins. On the other hand, a response like "you buy more because you have more coins; you have 15p and I only have 3p", has been interpreted as having recognised the meaning of the different denomination in the coins. The experiment has been also done with coins of 1p and 10p.

Counting with different denominations

The counting of units of different denominations task (also known as the shopping task), verifies whether the child is able to understand that any number is composed of units, tens, hundreds that add exactly to it. For example, it assesses whether children know that number $24 = 10 + 10 + 1 + 1 + 1 + 1$). According to these authors, the

shopping task is suggested to assess aspects of additive composition of number, an essential property of the decade numeration system.

In this shopping task, the child plays the role of the customer and the experimenter is the shopkeeper. The shop-situation is considered to make the counting of different denominations more meaningful to children. Play-money is also made available to the child in each item. In a typical item, the child is asked to buy from the shop a toy costing 15p, and is given three 10p and eight 1p coins to pay for the toy. Responses such as giving all the coins to the experimenter have been interpreted as inability to count and combine units of different denominations.

In all items the number of coins given is never enough to respond correctly to the item without taking into account denomination - in this example, the child was given eleven coins in total. Responses where the child separates one coin of 10p and counts other five 1p coins have been interpreted as being able to combine units of different denominations. The experiment includes several other items up to 3 and 4-digit prices.

The use of coins has further advantages compared to the other forms of assessment described above. One of them is the possibility of observing children's handling of units of different denominations. Another, is the possibility of witnessing children's production of quantities that they could not write in the form of numerals. This possibility enables the assessment of children's number understanding before they are able to write multi-digit numbers. These are all important features in a study that observes the simultaneous development of several number components, as in the present case.

3.3.2.2 Children's performance in shopping tasks

Carraher (1985) investigated the understanding of the decade system in a study with 72 Brazilian children, aged 5 to 8 with a relative values task and a shopping task. She reported that sixty percent of the children succeeded in the relative values task, where some of these children could not even count the total amount of money in the arrays (e.g. 4 coins of 10 cruzeiros), but were nevertheless able to recognise that four 10 cruzeiro² tokens buys more sweets than four 1 cruzeiro tokens. Also, a significant proportion of the same children, (i.e. 39%), gave correct answers in the shopping task. Similar results in the same age category were obtained in a study using the same tasks with 5 and 6 year-old British children (Nunes and Bryant, 1996).

The data obtained with the relative values and the shopping tasks shows that the use of coins with different denominations enables a more accurate assessment of young children's ability to count units of different denominations, which justifies its inclusion in the present study.

2 - Brazilian currency of the time.

3.4 ARITHMETICAL OPERATIONS

3.4.1 Addition and Subtraction

Arithmetic word problems constitute an important part of the mathematics programme of primary schools. Initially, they were used to help children to apply the formal mathematical knowledge and skills learned at school to everyday life situations. Later, word-problems were seen as a vehicle for developing students' general problem-solving capacity or for making mathematics lessons more motivating. Presently, word-problems have also been used to help children's early learning of a particular concept or skill, in order to promote a clear understanding of the basic arithmetic operations (Carpenter and Moser, 1984; De Corte and Verschaffel, 1989; Bergeron and Herscovics, 1990; Fuson, 1992; Verschaffel and De Corte, 1998).

Based on this, word problems have been a widely accepted method of assessment of children's understanding of addition and subtraction word-problems (e.g. Carpenter and Moser, 1982; Riley et al., 1983; Nesher, 1982; Fuson, 1992; Verschaffel and De Corte, 1998).

3.4.1.1 Assessment and methodology

Research in addition and subtraction word-problems has taken two main perspectives: one, has been explaining children's understanding of the different levels of difficulty in addition and subtraction problems, according to problem structure. Another, has

been clarifying the development of children's counting strategies. Why some children develop from the more basic counting strategies to the more complex derivation and recall strategies, while others lag behind ?

In their assessments authors have used a common framework which classifies problems according to their structure. Further to this classification, authors have also assessed children's performance in function of the unknown quantity: either the result set ($a+b=x$), the change set ($a+x=c$), or the start set ($x+b=c$). These problems pose particular difficulties as children must rearrange the quantities in order to find the solution.

3.4.1.1.1 The relevance of problem structure

Addition and subtraction problems vary in difficulty according to two main dimensions: *syntactic variables* and *semantic structure*. Syntactic variables refer to the number of words used in the problem and the sequence of information. Semantic structure, on the other hand, relates to the type of action involved in the problems. Although this differentiation has been made, the evidence suggests that the semantic structure of a problem is much more important than syntax in determining the processes that children use in their solutions (e.g. Carpenter, Hiebert and Moser, 1981; Carpenter and Moser, 1982; Verschaffel and De Corte, 1998).

Several authors have agreed to adopt a common framework to characterise different problem structure (Carpenter and Moser, 1982; Riley et al., 1983). This analysis proposes four broad classes of addition and subtraction problems: Change, Combine,

Compare and Equalise where the different kinds of word problems are seen to represent the sum in different ways and in different contexts and,

"Therefore, if one type of problem turns out to be much more difficult than another, the difference between the two should tell us something about the way in which children represent the sum in different ways and in different contexts" (Bryant, 1994; p. 20).

Table 5 displays some examples of word-problems according to their structure. Both types of Change problems involve action: joining or separating. In both sets the action occurs over time, with an initial condition (T1), which is followed by a change occurring at T2, resulting in a final state (T3). In both sets, too, there are three different types of problems depending on which quantity is unknown. In the first type, both the start set (first) and the change set (second) are given and the result set (final) needs to be found. In the second type, the start and result sets are given. In the third type, the start set is missing.

Both Combine and Compare problems involve static relationships, where there is no action. Combine problems involve the relationship among a particular set and its two, disjointed subsets. In one problem type, two subsets are given and the child is asked to find the result of their union. In the other, one of the subsets and the union are given, and the solver must find the remaining subset.

TABLE 5

Some types of addition and subtraction word-problems (from Riley et al., 1983; Carpenter and Moser, 1983; Verschaffel and De Corte, 1998)

Join	Separate
<i>Change</i>	
Joe had 3 marbles. Then Tom gave him 5 more marbles. How many marbles does Joe have altogether ?	Joe had 8 marbles. Then he gave 5 marbles to Tom. How many marbles does Joe have now ?
<i>Combine</i>	
Joe has 3 marbles. Tom has 5 marbles. How many marbles do they have altogether ?	Joe and Tom have 8 marbles altogether. Joe has 3 marbles. How many marbles does Tom have ?
<i>Compare</i>	
Joe has 8 marbles. Tom has 5 marbles. How many marbles does Joe have more than Tom ?	Joe has 8 marbles. Tom has 5 marbles. How many marbles does Tom have less than Joe ?
<i>Equalize</i>	
Joe has 8 marbles. Tom has 5 marbles. How many marbles does Tom have to win to have as many marbles as Joe ?	Joe has 8 marbles. Tom has 5 marbles. How many marbles does Joe have to lose to have as many marbles as Tom ?

Compare problems involve the comparison of two distinct sets, a referent set and a compared set. The third entity in these problems is the difference, i.e. the amount by which the larger exceeds the other. Here, any of the three entities could be the unknown - the difference, the referent set, or the compared set. The larger set can be either the referent set or the compared set, originating six different types of Compare problems. Equalise problems are a combination of Compare and Combine problems. There is similar action to the one found in the Change problems, but it is based on the comparison of two disjointed sets.

3.4.1.1.2 Description of strategies for solving addition and subtraction problems

Brownwell's (1935) and Ilg and Ames' (1951) suggestion that children used different strategies to solve addition problems - most of them involving the use of fingers to count - led to a whole new area of research in addition and subtraction. Their findings supported the idea that children would understand the principles involved in addition not by systematic repetition, but through activities based on concrete materials. This represented a change of focus from the importance attributed to the memorisation of results (Thorndike, 1922).

Lately, several authors have confirmed Brownwell's (1935) initial findings and have clarified the various strategies used by children in addition and subtraction problems. Pioneering work in this area was carried out by Carpenter and Moser (1982) in a three-year longitudinal study that followed more than 100 children from grade 1 through 3. Their results demonstrated convincingly that from an early age have a wide variety of material counting strategies (based on the use of concrete objects) and verbal counting strategies (based on forward and backward counting) for solving addition and subtraction problems. Many of these strategies are never taught explicitly in school.

The data from Carpenter and Moser's (1982) as well as numerous other studies has shown that the younger children begin by using count-all from the first addend (CAF), then progress to count-on from the first addend (COF), eventually begin using count-on from the larger addend (COL; Groen and Resnick, 1977; Carpenter and Moser, 1982; Riley et al., 1983), and finally start using number facts and, sometimes derived facts (Table 6).

A child who uses the counting-on strategy will solve the 3+5 problem by counting-on from 3... 4, 5, 6, 7, 8, instead of using the earlier counting-all strategy (i.e. 1, 2, 3, ... 1, 2, 3, 4, 5 ... 1, 2, 3, 4, 5, 6, 7, 8). Counting-on is, for this reason, considered to be a more sophisticated strategy than counting-all, because children start counting from the given total of one of the addends (Carpenter and Moser, 1982; 1983; Riley et al., 1983). Later, children who count-on from the first addend, will begin to count-on from the larger addend: 5 ... 6, 7, 8. This procedure is also known as the Min strategy.

TABLE 6

Counting strategies used in Addition problems

Count all from the 1st addend (CAF); e.g. $3+5=$ 1, 2, 3 ... 1, 2, 3, 4, 5 ... 1, 2 (...) 7, 8.
Count on from the first addend (COF); e.g. $3+5=$ 3 ... 4, 5, 6, 7, 8 (5).
Count on from the larger addend (COL); e.g. $3+5=$ 5 ... 6, 7, 8 (3).
Number facts (NF); result is retrieved from memory.
Derived facts (DF); e.g. $5+7=$ $5+5=10$ $+2=12$

Finally, children will answer addition word-problems from memory, having practised enough number ties, which they can now recall. One step towards this will be the memorisation of doubles (i.e. $2+2=4$), and later the derivation of responses from these doubles - derived facts strategy. In the case of $5+7$, children may derive that $5+5=10$ ($+2$) =12. The use of the derived facts strategy illustrates children's understanding of the part-whole schema (Resnick, 1983).

In the case of subtraction problems, children begin by counting the starting set using objects, and then remove the amount in the smaller addend. The remaining number of objects is the result. Using a more sophisticated strategy, without objects, children

attempt to count down from the total, until the amount in the smaller set is all counted. This procedure is fairly complicated, considering that children need to count backwards and also keep track of their counts. Another, more sophisticated, alternative involves using an “addition strategy”; i.e. counting up from the addend, to the total.

Carpenter and Moser (1983), described a complex procedure called ‘choice’, which requires deciding between the above described counting-up or counting down, according to which may seem more efficient. In the 9-3 case, for example, it seems more effective to count-down (9..... 8, 7, 6) than to count up (3..... 4, 5, 6, 7, 8, 9.... 6).

3.4.1.1.3 Explaining strategy development

The reason for children’s change of strategies has been attributed to conceptual development (e.g. Resnick, 1983), search for efficiency (Baroody and Ginsburg, 1986), and to individual differences (e.g. Gray, 1991; Gray et al., 1997).

Children’s progression to the COL strategy is quite interesting as it involves changing the order of the two addends according to which is larger. The use of this revolutionary procedure could mean that children have grasped commutativity - i.e. the order in which two numbers are added makes no difference to the result. However, has the use of a particular procedure led to the discovery of a particular concept, or is the development of these two independent ?

In order to explain children's discovery of the COL procedure, Resnick (1983) suggested that children may remember particular pairs' solutions - probably prompted by someone else more experienced - which they assume to be commutative from the start. Another possibility is that their application of the part-whole schema to addition is at the basis of their development of the counting-on strategy. Resnick's (1983) suggestion has been that children initially apply a part-whole schema by assigning addends to slots in the whole, whose parts can be added in either order to discover the value of the whole. The discovery that addend order is irrelevant to the final result, is what allows children to initiate the use of a more competent counting strategy, such as COL.

Baroody (1987), on the other hand, believes that the invention of the COL strategy is not so much conceptually based, but rather the result of the child's attempt to save cognitive effort. This was based on evidence that not all the children who used COL understood commutativity of addition (Baroody and Gannon, 1984).

It is possible that particular mathematical situations may elicit the use of certain procedures over others, because of the part-whole relations that are made evident. There is evidence that the use of COL, as well as the understanding of commutativity, might be facilitated in the context of combine problems (e.g. John has 3 marbles. Tom has 5. How many altogether ?), rather than change problems (Rose has 3 marbles. Tom gave her 5 more. How many does Rose have now ?; Fuson, 1979). In combine problems the order in which the sets are counted is irrelevant, although this is not true in the case of the change problems, which include a temporal dimension.

Finally, Gray et al. (1997) suggest that strategy development is linked to individual differences. Gray et al. gave several addition word problems to first, second and third

graders. He then divided his sample of children into two main groups, high and low achievers, according to mathematical ability. Gray and colleagues found that across all age groups the high-achievers used more sophisticated strategies such as derived facts or recalled facts, whilst the low-achievers used the more primitive strategies, such as counting-all.

Based on this evidence, Gray (1997) concluded that his data did not support the idea that most children develop better strategies on their own. On the contrary, his evidence supports the argument that children see mathematics in idiosyncratic ways, where the choice of the 'wrong' way, may imply a slower development.

So far, no agreement has been reached about the reasons that motivate children to use particular strategies, instead of others. The lack of conclusive evidence is further justified by the fact that children do not always use their most proficient strategy. Whilst some children may be perfectionists (Siegler and Jenkins, 1989) - i.e. they only rely on retrieval when they are sure of the answer - others may simply use earlier strategies to please the experimenter or to give a correct answer. Probably, the reasons that justify strategy development may be a combination of the three main factors outlined above.

Alternatively, Siegler (Siegler and Robinson, 1982; Siegler and Shrager, 1984; Siegler and Jenkins, 1989; Siegler and Shipley, 1995; Siegler and Stern, 1998; Shrager and Siegler, 1998) presented several explanations for the development of children's choice of strategies, through the use of computer models such as ASCM (Siegler and Shipley, 1995) and the SCADS (Shrager and Siegler, 1998).

These models provide alternative explanations for results from both traditions, the chronometric and the observational studies - pioneered by Groen and Parkman (1972) and Ilg and Ames (1951), respectively - by separating the trials according to reported and observed strategies. This data is not, however, conclusive (e.g. Baroody, 1994). It represents a rich account of children's development of strategies and encourages further theorisation, but it also sends research into a completely new avenue of inquiry. According to Cowan, the models

"show how adaptive strategy choice and evolution can result from associative strength rather than conscious decisions" (Cowan, in-press; p.21).

and

"This may seem strange to educators who assume that strategies develop from the deliberate application of principles or that children must know what strategy they are using" (Cowan, in press; p.16).

This, unfortunately, suggests that we are some way from being able to explain the development of strategies in children whilst studying isolated number components. However, further insights may come from observing the relation between several number components, where further research is still needed.

3.4.1.2 Children's performance

Riley, Greeno and Heller (1983) assessed children's ability to respond to addition and subtraction word-problems. The number of the sets used in the problems were under 10, and materials were made available to the children. Some of the results of their study, which included children from kindergarten to third grade, are shown in Table 7.

3.4.1.2.1 Differences in problem structure

The data show that children's performance varies according to problem type and structure (i.e. which set is missing). Combine problems (which include two static measures) are slightly less difficult than Change problems, which involve action and time). However, according to Carpenter and Moser (1982), children seem to treat Change and Combine addition problems as though they were equivalent.

Compare problems, on the other hand, are significantly more difficult for children than Change problems. Compare problems require knowledge of the matching strategy, a strategy unknown to children at this stage of schooling (Carpenter and Moser, 1982; Riley et al., 1983). There are no clear differences between addition and subtraction word-problems within the result unknown problems or within the inverse (i.e. start-set unknown) problems.

TABLE 7

Children's rate of success in different types of addition and subtraction word-problems (Adapted from Riley et al., 1983)

Problem type	Percentage of correct responses			
	Kindergarten	Grade		
		one	two	three
SIMPLE ADDITION				
Combine problem with result unknown	100	100	100	100
Change problem with result unknown ($a+b=?$)	87	100	100	100
Change problem with middle unknown ($a+?=c$)	61	56	100	100
Compare problems with difference unknown (see Table 5)	17	28	85	100
SIMPLE SUBTRACTION				
Change problem with result unknown ($a-b=?$)	100	100	100	100
INVERSE ADDITION				
Change problem with start unknown ($?+b=c$)	9	28	80	95
INVERSE SUBTRACTION				
Change problem with start unknown ($?-b=c$)	22	39	70	80

The position of the missing set at the start of the problem also represents a significant problem for children. This relates to the fact that start missing problems (i.e. $?+b=c$) are virtually impossible to represent with fingers, as the start quantity does not exist. This type of problems demand knowledge of part-whole schema as there is a need in representing the situation mentally (Resnick, 1983). The data in Table 7, further shows that problems where the middle set is missing (i.e. $a+?=c$) are much closer to the change result unknown problems, in level of difficulty.

Regarding the use of strategies to solve addition problems, shown in Table 8, the data show that children's strategies change with time. Younger children start by relying more on the strategies shown on the left of the Table and, with development, older

children use more the strategies on the right. The main developmental jump seems to happen somewhere in between the second and the third grade.

Interestingly, more children seem to use the counting-on from the larger strategy, than the counting-on from the first strategy. Also, more children seem to rely on the recalled facts strategy than on the derived fact strategy.

TABLE 8
Development of strategies in change result unknown word-problems (adapted from Carpenter and Moser, 1983). Results are in percentages.

Grade	Strategy used				
	Count all	Count-on from first	Count-on from larger	Derived fact	Recalled fact
first	46	3	8	2	1
second	41	14	26	6	6
third	11	15	32	9	32

The choice of tasks to be included in the present study included two levels of difficulty in addition and subtraction word-problems, in order to cover a developmental span typical of the three primary years of schooling. To reduce bias in the interpretation of results, the types of problems were reduced to one. Combine problems are too easy for children and Compare and Equalise problems are, on the other hand, too difficult (Carpenter and Moser, 1982). Furthermore, this study takes the view that the conceptual structures attributed to children at various levels can be illustrated by restricting the discussion to change problems (Cobb, 1987). Change problems seem to be the more suitable as they resemble more the typical addition and subtraction situation.

"Change problems describe situations involving action such as giving or taking, whereas combine and compare problems describe static situations" (Cobb, 1987; p. 163).

Having controlled the "type of problem" factor, the level of problem difficulty will be a function of the structure of the problem. As outlined earlier, there are three possibilities: result-set, change-set, or start-set unknown. Based on Riley et al's., (1983) data, change-set problems are of almost similar difficulty to result-set problems. For this reason, change-set problems were discarded. The remaining two possibilities, results-set and start-set unknown problems were used. Finally, a counting-on task was included, where one of the addends was hidden in a box (Hughes, 1986).

3.4.2 Multiplication

The assessment of children's understanding of early multiplication is embedded in a difficult conceptual definition problem (e.g. Piaget, 1952; Vergnaud, 1983; Kouba, 1989; Davydov, 1991; Carpenter et al., 1993; Steffe, 1994; Kieren, 1994; Clark and Kamii, 1996). Do addition and multiplication develop concurrently (Piaget, 1952), or is the development of multiplication inevitably tied to the use of addition strategies (Fischbein et al., 1985; Kouba, 1989), at least in the early stages ?

This debate leaves those interested in the development of children's understanding of multiplication with the difficult task of defining a good criteria for the assessment of multiplication (and not addition). The first implication in seeing multiplication as a result of addition is that its emergence can only be expected later than addition. However, in a recent volume dedicated to this issue, Kieren (1994) concludes that no evidence has yet settled this theoretical divergence.

Meanwhile, an alternative way of looking at the development of multiplication has been suggested by Steffe (1994) who proposed that multiplication is entered, from the child's point of view, from the practice and development of more elaborate counting schemes, such as double-counting, and not from a clear grasp of what the operation of multiplication should be. For similar results in the United Kingdom, see Anghileri (1997).

A child using a double-counting strategy will count the contents of each set, without losing track of the counted sets. For example, a child who is trying to find out how

many wheel three cars may have will count: "1, 2, 3, 4 (4, first car); 1, 2, 3, 4 (8, second car); 1, 2, 3, 4 (12, last car)".

Also, authors have agreed upon a common classification for multiplication problems (Vergnaud, 1983; Schwartz, 1988) where, simultaneously, the development of the child's counting schemes can be assessed. Using this framework, recent reports have confirmed that children's acquisition of mental multiplication possibly begins with the use of counting strategies (Steffe, 1988; Cooney, Swanson and Ladd, 1988; Carpenter et al., 1993).

3.4.2.1 Assessment, methodology and children's performance

Fischbein, Deri, Nelo and Marino (1985) suggested that the concept of multiplication is intuitively attached to a repeated addition model - i.e. 4×5 can be seen and solved as $5+5+5+5$. According to these authors,

"Each fundamental operation of arithmetic generally remains linked to an implicit, unconscious, and primitive intuitive model. Identification of the operation needed to solve a problem with two items of numerical data takes place not directly but as mediated by the model (Fischbein et al., 1985; p.

4)

3.4.2.1.1 Addition and Multiplication as sequential operations

This argument was based on evidence that children had difficulties in solving multiplications that could not be interpreted as "multiplication makes bigger" or "multiplication makes lots of". An example of a multiplication that contradicts this perception is one that uses a multiplier that is a decimal number or a fraction - i.e. $4 \times 0.5 = 2$. Whenever difficulties like these arose, children resorted to a more familiar model to check their responses - i.e. repeated addition.

Fischbein et al. (1985) gave 42 problems (12 multiplication, 14 division and the remaining were additions and subtractions) to 628 Italian children aged 10 to 13. Children were not required to calculate the answer, but simply to indicate which operation would most suitably solve each of the problems. Whilst any of the 12 multiplications could be solved by repeated addition, what varied amongst them was the types of numbers used. In two of the multiplication problems, both the multiplier and the multiplicand were whole numbers. In the remaining problems either the multiplier or the multiplicand were decimals. Fischbein et al's. (1985) prediction was that those problems where the multiplier was not a whole number would be more difficult for children.

Almost all the children succeeded in problems that used integers. Although the presence of decimal numbers in either position affected children's performance, children tended to perform better when the decimal number was the multiplicand. As predicted, those children who could not solve the problems by means of multiplication had to resort to the use of a repeated addition model. Based on this Fischbein et al.

(1985) argued that whenever the numerical data of the problem does not fit the constraints of the model the children may not choose the correct operation and the solution effort may be diverted or blocked.

Several other authors have presented data supporting Fischbein et al's. (1985) hypothesis. Greer (1987) obtained empirical data from children and adults similar to Fischbein's. Data about younger children was also provided by Kouba (1989).

However, the main limitation of Fischbein et al's. (1985) study is that it does not provide data about younger children, aged five, six or seven. For studies with younger children authors have used multiplication word-problems (e.g. Vergnaud, 1983; Schwartz, 1988; Greer, 1992; Verschaffel and De Corte, 1998). Before we turn to those studies we first need to briefly outline a widely accepted classification (Vergnaud, 1983), which is used in this thesis.

3.4.2.1.1.1 Classification of multiplication word-problems

Vergnaud's (1983) classification of multiplication word-problems regards simpler multiplication problems as part of a broader multiplicative conceptual field including more complex notions such as ratio, rational numbers, vector space and so on. His emphasis is on the dimensions (hence dimensional analysis) and unit structures of these problem types. Vergnaud claims that the multiplicative conceptual field takes considerable time to be fully grasped, probably until adulthood.

Vergnaud (1983) defined three main classes of problems: isomorphism of measures, product of measures, and multiple proportions. An *isomorphism of measures* problem such as "each car has four wheels; how many wheels would 3 cars have altogether?" implies a four-place relation since there are two basic dimensions, M1 (cars) and M2 (wheels), with each dimension comprising two numbers, as shown:

Cars (M1)	Wheels (M2)
1	4
3	?

These dimensions do not exist in addition problems. Every time the child adds one car, to keep this relation constant, s/he has to add four new wheels to the ones s/he already has. Therefore, the number of wheels in the basic set represents the ratio. The ratio, which expresses the relation between the two sets (and not the number of objects in neither set) must be maintained constant in multiplication problems. Also, the number of times that a replication is carried out has a particular meaning and corresponds to a scalar increase or decrease. This factor does not bear a relation with the number of objects in the sets, but it indicates the number of times that two sets must be replicated.

Product of measures problems consists of the Cartesian composition of two measure spaces, M1 and M2, are mapped onto a third, M3. For example: "What is the area (M3) of a room whose length (M1) is 4 meters and its width (M2) is 3 meters?". These problems involve 3 measures and the child must deal with double proportions, rather than with a single proportion as in isomorphism of measure problems. Finally, in a multiple proportion problem, M3, is proportional to two different independent

measures M1 and M2. For example, "if one cat eats 100 grams of food per day, how many grams will 4 cats eat in 3 days ?" Multiple proportion problems involve magnitudes that have intrinsic meaning; none of them can be reduced to the product of the others. The next section briefly reviews some studies that have used isomorphism of measures word-problems.

3.4.2.1.1.2 Isomorphism of measures word-problems

Kouba (1989) observed the development of children's understanding of multiplication, in 43 first, 35 second and 50 third graders. She used a widely accepted classification of multiplication word-problems proposed by Vergnaud (1983), i.e. the dimensional analysis.

In her study, Kouba (1989) used 'grouping' (e.g. You are having soup for lunch. there are ___ bowls. If you put ___ crackers in each bowl, how many crackers do you need altogether ?), and 'matching' problems (e.g. Pretend you are a squirrel. There are ___ trees. If you find ___ nuts under each tree, how many nuts do you find altogether ?)

Kouba (1989) classified children's strategies into (1) direct representation; (2) double counting; (3) transitional counting; (4) additive or subtractive and (5) recalled number fact. Children who used "direct representation" processed the information in a sequential way that reflected or paralleled the structure of the problem. In a typical problem like "there were 6 cups. You put 5 marshmallows in each cup. How many marshmallows did you use altogether ?", these children used 6 containers, placed 5

objects in each container and found the answer by counting the total objects, one by one.

Children who used "double counting" integrated two counting strategies, requiring more abstract processing. Children who used "transitional counting" calculated the answer to the problem by using a counting sequence based on multiple of a factor in the problem (e.g. 4, 8, 12...). According to Kouba,

"Counting by multiples, or skip counting, was labelled transitional counting because it relies on the knowledge of a counting sequence, it is related to multiplication in a more fundamental way than the direct representation strategies are" (Kouba, 1989; p. 153).

Children who used an additive or subtractive strategy clearly identified the use of repeated addition or subtraction to calculate an answer. For example, in calculating four groups of five, the child might have said, 5 plus 5 is 10, 10 and 5 is 15, and 15 plus 5 is 20". Finally, children who used "recalled facts" strategy obtained the answer by remembering the appropriate multiplication combination.

According to her results (shown in Table 9), children are able to solve multiplication problems only in the second grade, the majority of these using and 'additive or subtractive' strategy. From this evidence Kouba (1989) concluded that children's strategies for solving equivalent set multiplication word problems generally fit Fischbein et al's. (1985) intuitive model, in that

"the solution strategies children used in this study reveals that children appear to view multiplication as a different two-step process: make several equivalent sets and put them together" (Kouba, 1989; p. 156).

TABLE 9

Percent of children using each type of solution strategy on multiplication problems (Kouba, 1989)

Grade	n	Type				
		direct representation	double count	transitional count	additive or subtractive	recalled fact
Grouping						
1	43	25	0	2	2	0
2	35	9	0	6	24	3
3	50	8	0	10	8	40
Matching						
1	43	23	0	4	0	0
2	35	18	0	12	18	0
3	50	8	0	4	10	34

There are, however, other studies that support the view that children's understanding of multiplication develops earlier than the second grade. These studies have used both clinical interviews and isomorphism of measures word problems.

3.4.2.1.2 Addition and Multiplication as simultaneous operations

Piaget (1952) has suggested that multiplication is not just a faster way of doing repeated addition but is an operation that requires higher-order multiplicative thinking which children construct out of their ability to think additively. According to Piaget

(1952), young children build their knowledge of one-to-many correspondence (a logical invariant of multiplication) on knowledge of the one-to-one correspondence schema and its use in transitive inferences. Knowledge of both should enable children to realise that if $A=B$ and $C=B$, then $A=C$ (transitivity), and also that if $A=2B$ and $A=C$, then $C=2B$. Later, Piaget (1987) described the differences between addition and multiplication as depending on the number of levels of abstraction and the number of inclusion relationships the child has to make simultaneously.

Piaget (1952) tested his argument that multiplication develops simultaneously with addition by asking children to establish one-to-one correspondence between two sets of objects, and also one-to-many correspondence between another two sets of objects. It must be stressed that these children were not asked to quantify or calculate; they were only required to apply transitive reasoning to the different one-to-many correspondence situations.

3.4.2.1.2.1 One-to-many correspondence and transitivity tasks

In one of the tasks the children were requested to put a blue flower in each of ten vases; then the blue flowers (large) were removed and put in a single bunch. Next, they were asked to put a pink flower (small) in each vase; again, the pink flowers were also joined in a bunch. This way, the children knew that the number of flowers (A) was equal to the number of vases (B) and they also knew that the number of pink flowers (C) was equal to the number of vases.

However, the blue and pink flowers were of different sizes so that the children could not easily compare the number of flowers in the two sets visually. The bunch with the large blue flowers looked different from the bunch with the small pink flowers. Therefore, children would have to understand one-to-one correspondence to conclude that the two sets of flowers were equal in number.

Piaget classified children's responses to this first task in two categories: they either were able to recognise that the number of flowers in the pink and blue bunches was the same and justified this deduction based on the correspondence established between the sets of flowers and the sets of vases, or they did not establish correspondence and answered wrongly, even if given the hint by the experimenter that each flower had its own vase.

In a second task, Piaget verified whether children could reason that if $A=2B$ and $C=A$, then $C=2B$. He inquired what would happen if the pink and blue flowers were put back in the same vases. How many flowers would each vase have now? In case of doubt, children were allowed to go to the flowers in the vases, and verify for themselves that each vase would have now two flowers; $A=2B$.

Next, the flowers were put away, but the vases remained within the child's sight. The children were then asked to pick the right number (C) of tubes (from a box of thin plastic tubes) so that they got one flower per tube. The children knew that there had been two flowers in each vase and only one flower was to be placed in each tube ($C=A$). Piaget was trying to verify whether children would understand that it was necessary to pick twice as many tubes as vases ($C=2B$). Piaget reported three different types of responses.

3.4.2.1.2.2 Children's performance in the Piagetian task

Piaget (1952) reported that a group of children did not anticipate that there would be two flowers in each vase, so they also did not realise that they would have to take two tubes for each vase (on the table) in order to end up with one tube for each flower. This happened even after they had placed the flowers into the vases and had found for themselves that there were two flowers per vase. This group did not use the vases to estimate how many tubes they needed to take out of the box. These children were those who had not realised in the first task that there were as many pink and blue flowers. They failed to make the transitive inference in the one-to-one correspondence in the first place, so they could not understand the one-to-two correspondence between vases and flowers.

A second group of children, was able to establish one-to-one correspondence but could not maintain the lasting equivalence of the corresponding sets. Finally, in the third group all children were able to establish one-to-one correspondence, were able to compose equivalencies and also were capable of understanding the relations of multiple correspondence that were put to them. Based on the evidence provided by the three groups of children, Piaget was able to define three stages of development for multiplication. At the third stage, children are able to grasp the two-to-one relation, which they can then generalise to three, four and five. Gros (5 years; 10 months) provides an example of what has been said:

Gros (5;10) had succeeded in making the inference that the number of blue (X) and pink (Z) flowers was the same. He was then asked how many flowers would be in each vase if they were now all (X+Z) put back into the vases (Y).

G: 1 blue, 1 pink.

Exp: How many is that ?

G: Two.

Exp: And if I added these (a new set of 10), how many would there be in each vase ?

G: Three.

Exp: Why ?

G: I'd put one, one, one.

Exp: And now suppose we wanted to put them in these tubes that will only hold one flower ?

G: (he took 10 + 10 + 10 tubes)

Based on the evidence exemplified here by Gros, Piaget suggested that children as young as 5 or 6 years of age are already capable of understanding some aspects of multiplicative relations. He concluded that:

"the operation of correspondence is revealed in its true light, as being a multiplicative composition. In the various correspondences, one-to-one, two-to-one, three-to-one, etc., the value of each new set is no longer regarded only as an addition, but as multiplication, '1 x n', '2 x n', '3 x n', etc." (1952; p. 219).

The next section briefly outlines a study that has used a Piagetian-inspired task in a more controlled, experimental context.

3.4.2.1.2.3 Recent versions of the Piagetian task

In a recent study, Clark and Kamii (1996) reassessed young children with a simpler version of the Piagetian task. The children were shown three fish of identical width, with lengths of 5, 10 and 15 cm - made of plywood (Figure 2, below). They were also supplied with 100 small chips, to feed the fish. The examiner told the children that fish B ate twice as many as fish A, and that fish C ate three times more than fish A, because of their relative sizes (i.e. B is twice as big as A, and so on). This was demonstrated by placing the smaller fish on top of the others.

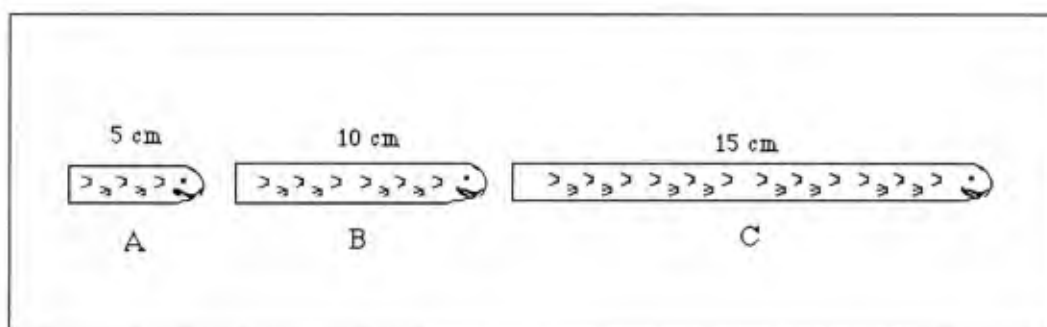


Figure 1. The fish used in the task

The child was then asked "if this fish (A) gets 1 chip of food, how many chips of food would you feed the other two fish ?" The child was also asked about the following variations: (a) when B received 4 chips; (b) when C received 9 chips; (c) when A received 4 chips; (d) when A received 7 chips.

3.4.2.1.2.4 Children's performance on the fish task

Clark and Kamii (1996) tested 336 American students in grades 1-5. In their results, they identified four developmental levels in children's progression from additive to multiplicative structures. Children in level I were only able to think qualitatively in terms of "more" and "less". These children were considered to be not yet numerical or additive. Children in level II displayed additive thinking with numerical sequences of +1 or +2 only. For example, such a child would give 3 to A and 5 to C, after the experimenter gave 4 to B.

Children at level III took into consideration the number of times stipulated by the experimenter (i.e. B is 2 times A and C is 3 times A), but add these numbers. For example, if the experimenter gave 4 to A, they would give 6 (+2) to B and 7 to C (+3). Children at this level use the term "times" but use additive thinking. Children at level IVa display multiplicative thinking but not with immediate success. They succeed only after the experimenters counter suggestion:

Abby (grade 3). For 9 to C she gives 6 to B "...because C eats 3 times, so I take away 3, and 4 to A ... because B eats 2 times, so I take away 2 [4,

6, 9]. After the counter suggestion, Abby thinks [3, 6, 9] is better and explains her reasoning multiplicatively; but got 4 to A, she gives 6 to B and 7 to C [4, 6, 7]. "I just added 2 more and 3 more." Again, she prefers the multiplicative counter suggestion [4, 8, 12]. For 5 to A she gives 2 groups of five to B and 3 groups of five to C [5, 10, 15]. For 7 to A she gives 14 in two groups to B and 21 in three groups to C [7, 14, 21].
(Clark and Kamii, 1996; p. 47)

Finally, children at level IVb display multiplicative thinking with immediate success. As Table 10 shows, their evidence suggests that multiplicative thinking is already possible by 19% of the first graders (result obtained by the sum of the percentages in levels IV a and b).

TABLE 10
Percentage of Children at Each Developmental Level by Grade (Clark and Kamii, 1996)

Level	Grade [N]				
	first [58]	second [65]	third [59]	fourth [78]	fifth [76]
Below additive (I)	14	2	-	-	-
Additive level (II)	53	43	14	15	7
Additive level (III)	14	11	22	3	3
Multiplicative level (IVa)	17	35	42	54	42
Multiplicative level (IVb)	2	9	22	28	49

Solid multiplicative thinking, on the other hand, seems to be displayed by only 9% of the second graders. Based on this, it would be interesting to compare these results with studies using isomorphism of measures word-problems where children's

counting strategies could be analysed more in depth. Such as study is briefly reviewed in the next section.

3.4.2.1.3 Isomorphism of measures problems and counting strategies

Research looking at children's development of the counting scheme in the context of isomorphism of measures word-problems, complements that provoked by the debate outlined in previous sections. This approach is particularly useful to this thesis as it enables an exploration of two different components: knowledge of the number-word list and knowledge of multiplication.

Two independent studies using the same type of problems, Kouba (1989), outlined earlier, and Carpenter et al., (1993) - described in this section - have produced different results on the development of children's understanding of multiplication, and the ages when it emerges. Whereas the results of Kouba's (1989) study suggests that children understand multiplication only on the second grade, Carpenter et al's. (1993) data shows that kindergarten children can solve the same multiplication word-problems. The difference in the results could be justified by the attention paid to children's counting schemes in Carpenter et al's. (1993) study.

3.4.2.1.3.1 Studies with isomorphism of measures problems

Carpenter et al. (1993) compared the results of 70 kindergartener's performances in addition and multiplication problems (e.g. Robin has 3 packages of gum. There are 6 pieces of gum in each package. How many pieces of gum does Robin have

altogether ?). They were interested in clarifying whether the differences in children's performance between addition and multiplication operations can be accounted by the fact that multiplication problems are inherently more difficult to solve than addition problems (e.g. Schwartz, 1988; Kouba, 1989; Greer, 1992; Clark and Kamii, 1996), or whether differences are due, in the most part, to differences in exposure (e.g. Nunes and Bryant, 1996; Carpenter et al., 1993). Whereas children are exposed to addition and subtraction problems from the first year of schooling, the systematic introduction of multiplication problems only happens from their second grade, which could account for differences in performance.

Although Carpenter et al. (1993) used the same type of multiplication word-problems as Kouba (1989), their data does not support Kouba's (1989) or Clark and Kamii's (1996) conclusions that multiplication develops after addition. In Carpenter et al.'s (1993) study special attention was paid to children's counting schemes.

Their strategies were classified into: (1) direct modelling; (2) counting; (3) derived fact; (4) other and (5) uncodable. Children who used direct modelling used counters to model directly the action or relationships described in the problem. Those classified as 'counting' did not use counters or fingers to model directly the problem but counted up or back from a given number or skip counted to give an answer. Children classified as 'derived fact' used recalled number facts to provide an answer. Children who got the correct answers but the interviewer could not reliably code the response on the basis of the child's actions and explanations, were classified as 'uncodable'.

In their study, Carpenter et al. (1993) showed that at least 14 kindergarteners were able to use some form of counting without counters which resembles Kouba's (1989) 'transitional counting' (i.e. 1, 2, 3, 4 (pause) 4, 5, 6, 8 (pause) 9, 10, 11, 12).

Carpenter et al. (1993) also showed that the kindergarten children in their study were more successful in solving multiplication word problems than the first grade students in Kouba's (1989) study, and as successful as the third-grade students - although Kouba's students used more mental strategies in their solutions.

TABLE 11

Number of children correctly solving each problem and the number and kind of valid strategies used (N=70; partially reproduced from Carpenter et al., 1993)

Problem	number correct	valid strategies	Strategy				
			direct modelling	counting	derived fact	other	uncodable
Additive structure*	51	62	54	5	2	0	1
Multiplication	50	60	46	14	0	0	0

* Separate result unknown problems

Carpenter et al's. (1993) data, support the view that children's differences in performances are due to the differences in exposure that they have had to multiplication and that

"children can solve a wide range of problems, including problems involving multiplication and division situations, much earlier than generally has been presumed" (Carpenter et al., 1993; p. 439).

Their data also supports Steffe's (1988) alternative view that children's early grasp of multiplication can be successfully assessed through their use of more sophisticated

counting schemes (such as double-counting) in isomorphism of measures word-problems. Finally, Carpenter et al's. (1993) results suggest that children's understanding of multiplication in the early years of school can be accurately assessed through the use of isomorphism of measures word-problems, where particular attention should be paid to children's counting schemes.

3.4.2.1.3.2 Studies with relative values problems

Alternatively, Nunes and Bryant's (1996) relative values task is also considered to assess children's early understanding of multiplication. In this task, two rows of play-money with the same amount of coins, are put on the table, one said to be the child's and the other one, the experimenter's. For example, the child is given three 1p coins and the experimenter has three 5p coins. The experimenter then asks: "Who do you think will buy more sweets, you or me ?". The child is also invited to justify the answer.

A response like "we both buy the same.... because we have the same amount of coins", has been interpreted as not recognising the meaning of the different denomination in the coins. On the other hand, a response like "you buy more because you have more coins; you have 15p and I only have 3p", has been interpreted as having recognised the meaning of the different denomination in the coins and as having the ability to count them up to find the total. In the 15p example, the child will (double) count 1,2,3,4,5...6,7,8,9,10...11,12,13,14,15, displaying the application of the one-to-many correspondence principle (Piaget, 1952). This approach is also particularly useful to this thesis as it enables an exploration of two different components, i.e. knowledge of the number-word list and multiplication.

Carraher and Schliemann (1990) applied this task to seventy-two 5- to 7-year-old Brazilian pre-school children. Results were classified in three categories: 40% of the children did not take into account the relative value when answering; 15% took into account the relative value but could not justify the answer by counting the totals, and 45% could use the relative values and justified the answers by counting the totals in each array. Nunes and Bryant (1996) also applied the same tasks to 5- and 6-year-old British children and found that the results did not differ significantly from those obtained by the Brazilian study.

The present study used isomorphism of measures word-problems and the relative values task, as these are more suitable for the assessment of multiplication in 5- and 6-year-old children.

3.5. WRITTEN MULTI-DIGIT NUMBERS AND KNOWLEDGE OF PLACE VALUE

Children's understanding of the written Arabic system (i.e. place value) has been assessed by several authors (Luria, 1969; Ginsburg, 1977; Brown, 1981; Miura and Okamoto, 1989; Ross, 1989; Bergeron and Herscovics, 1990; Power and Dal Martello, 1990; Miura et al., 1993; Sinclair et al., 1992; Sinclair and Scheuer, 1993; Seron, Deloche and Noel, 1991; Seron and Fayol, 1994; Nunes and Bryant, 1996).

Within these, one group argues that children learn about place value from experience with written numbers, i.e. by observing the relations between different digits ('numbers-first hypothesis', e.g. Luria, 1969; Bergeron and Herscovics, 1990; Sinclair et al., 1992; Sinclair and Scheuer, 1993). Another group suggests that knowledge of place value can only be developed after children have understood the structure of the numeration system ('numbers-after hypothesis', e.g. Ginsburg, 1977; Nunes and Bryant, 1996).

3.5.1 The numbers-first hypothesis

Reflecting the view of the first group of authors, Sinclair and Scheuer (1993) have hypothesised a developmental order for children's learning of place value, according to which the

"understanding of written numerical notations is a construction process that is necessary to the understanding of our numeration system, and it participates in and directly influences mathematical cognition. The grasp of numerical notation is thus deserving study on its own right, and is not to be approached exclusively as means of representing knowledge acquired in other domains." (Sinclair and Scheuer, 1993; p. 203).

According to this view, the realisation that a 3 with a 2 on its right is read as thirty-two, could give children valuable hints about the notion that different positions mean units of different sizes. Some evidence for this hypothesis comes from studies with brain-lesioned patients (Luria, 1969) and from research with school-children (e.g. Bergeron and Herscovics, 1990; Sinclair, et al., 1992; Sinclair and Scheuer, 1993).

3.5.1.1 Studies with brain-lesioned patients

Luria's (1969) observation of his patients' difficulties in writing multi-digit numbers while doing additions and subtractions, led him to the conclusion that the inability to write numbers would have a devastating effect on their understanding of the numeration system. In Luria's (1969) opinion, evidence of misunderstanding of the numeration system came from the inability to make distinctions between digits in the tens and hundreds, in written form.

However, a limitation of Luria's (1969) study is that it may not be appropriate to generalise findings based on brain-damaged patients to the case of children who are developing their numeracy skills. Although Luria assumes that his subjects had good

calculative capacities before the lesion, nothing is known about their understanding of the structure of the numeration system during the same period. This makes it impossible to establish a causal relation between losing the 'ability to write numbers' and having a poorer 'understanding of the numeration system'.

Possibly, the lesion may have affected both skills negatively, which could also explain Luria's (1969) results. In order to establish what the relation between these skills is, a study is needed where both are measured over a period of time. A further limitation of Luria's (1969) research is that it was specific to clinical case-studies only.

3.5.1.2 Studies with school children

There is, however, further evidence to support the 'numbers-first' hypothesis. Sinclair and Scheuer (1993), argue that children start by attempting to match particular quantities to multi-digit numbers, which they then try to read. At the same time, they also attempt to acquire the conventional knowledge that allows them to do so - that is, place value. While they acquire this conventional knowledge and try to make it their own,

"they are puzzling out what the underlying characteristics of the system are"

(Sinclair and Scheuer, 1993; p. 219).

These authors' position can be interpreted as slightly different from Luria's (1969), as their data suggest an interrelated process, according to which written numbers are seen

as the starting point for children's understanding of both place value and the structure of the numeration system.

Also, Bergeron and Herscovics (1990) have provided evidence that children initially develop an understanding of a positional notation in the decades (i.e. knowing that twelve is always written as 12 and not as 21) which is seen as a pre-requisite for grasping place value. They have shown that children's understanding of positional notation goes through several levels of understanding. Initially,

"concatenated digits acquire a global meaning: '12' is no longer one and two, but twelve. However, children do not as yet perceive the importance of relative position and may very well consider '21' as another way of writing twelve. We call this level of understanding that of juxtaposition" (Bergeron and Herscovics, 1990; p. 194).

Next, children become aware of the importance of relative position, but associate the position with order of writing and not with the left-right direction of reading (e.g. "twelve" may be written 21, the child writing from right to left). Bergeron and Herscovics termed this the chronological level. Finally, when children reach the conventional stage, they can produce bi-digit with digits in their conventional position whatever the direction of writing ("twelve" is always written 12). These levels are not mutually exclusive and

"the same child may be at the chronological level in the teens but still produce juxtaposition errors in the twenties" (Bergeron and Herscovics, 1990; p. 198).

The limitation of studies supporting this view are conceptual, rather than methodological. Here, the written Arabic (e.g. 122) and the written verbal (e.g. one hundred and twenty-two) number systems are seen to develop as a whole, when they can also be seen as separate number components (Ginsburg, 1977; Fuson, 1990; Nunes and Bryant, 1996). Due to this confusion, very little research has been devoted to the simultaneous acquisition of the two main systems used to code quantities (Seron and Fayol, 1994; Nunes and Bryant, 1996).

3.5.2 The numbers-after hypothesis

From an opposite perspective, another group of authors contend that children's understanding of place value is a more complex process than the previous researchers have suggested. Their view is that this development involves two separate conceptual structures, which in the previous approach are seen as inseparable (Ginsburg, 1977; Resnick, 1983; Fuson, 1990; Nunes and Bryant, 1996). One refers to the understanding of the conventions involved in place value, which require the ability to write multi-digit numbers. Another, relates to the conventions involved in the understanding of the structure of the decade numeration system itself. Whilst the first one relates to how digits should be 'put right', the second deals with the understanding that the numeration system implies the combination of units of different sizes: ones, tens, hundreds and so on - where written numbers are not involved.

In fact, this conceptual confusion between structures has been highlighted by Sinclair et al. (1992) - subscribers of the numbers-first view -, who have recently admitted that children's understanding of place value entails more than just cracking an arbitrary written code. Although they see this code as

"indissolubly linked to understanding the number system itself" (Sinclair et al., 1992; p. 193)

they have not forwarded

"any hypotheses (or clear ideas) about how a grasp of the structure of our written numerals comes about" (Sinclair et al., 1992; p. 193).

Nunes and Bryant, (1996) provided important evidence to support the idea that understanding the decade system and knowing about place value are two separate conceptual structures. They interviewed 72 pre-school Brazilian children (aged 6) and 20 number-illiterate adults, and assessed them on their grasp of place value and their understanding of the structure of the numeration system (or number-word sequence).

Nunes and Bryant (1996) reported that some pre-schoolers, as well some number-illiterate adults, showed an understanding of the structure of the numeration system before knowing about written numbers or place value. However, the fact that the

adults had no formal instruction about written numbers does not rule out the possibility that they may have learned place value somewhere else.

The evidence presented supports both the view that children may learn about place value from experience with numbers, and the view that their understanding of place value is significantly bolstered by a previous grasp of the structure of the decade numeration system. A review of the studies supports the view that the assessment of children's understanding of the numeration system and their grasp of place value should be carried out as separate number components, which will contribute to the clarification of the development of their understanding of written multi-digits.

3.5.3 Assessment and methodology

Typically, studies about children's understanding of place value have involved tasks where the child is asked to give evidence of their understanding of the meaning of different roles of digits in the units, tens and sometimes hundreds position, in terms of some numerical correspondence. In one type of tasks, children are presented with a two-digit notation (e.g. 14 or 26) and a corresponding collection of small (usually identical) objects, and are asked to set up a correspondence between the different digits and the objects in the collection (see, for e.g., Kamii, 1986; Ross, 1989).

In another type of tasks, children are asked to construct alternative representations of a given multi-digit number, by using Dienes blocks (see, for e.g., Resnick, 1983; Ross, 1989), or cards on which different numbers of units, tens and hundreds are written, as in 4 ones or 5 tens (see Bednarz and Janvier, 1982). The converse, asking

children to what number a display made up of some other numerical representations (blocks, bundles or sticks) corresponds is also used (see Resnick, 1983; Ross, 1989).

These choices of tasks seem to be a result of assessing children's understanding of place value and their grasp of the numeration system as one number component, instead of two. Here, authors assess the ability to write multi-digit numbers whilst seeking clear evidence that children have understood the system of units of different sizes (ones, tens, etc.). This represents a tacit recognition that two components are being assessed although, in reality, more attention seems to be paid to the child's explanation of the system rather than the ability to write the numbers, which is self explanatory (i.e. the probability of writing a multi-digit number correctly by chance, without knowledge of place value, seems small. In this study, tasks assessing children's understanding of the decade system were used in a separate number component, as explained earlier.

Regarding children's production of Arabic multi-digit numbers *per se*, Nunes and Bryant (1996), assessed five and six year-olds' (Brazilian and English) ability to write and recognise numbers from single-digits, up to 4-digit numbers (the results are shown in table 6, by categories). Power and Dal Martello (1990) also looked at the production of Arabic numbers in 7 year-olds, and Seron, Deloche and Noel (1991) investigated this same ability in 7 to 9 year-olds. In all three studies, children were asked to write down digit numbers that were dictated to them, in Portuguese, English, Italian and French, respectively. As this thesis centres on 4 to 7 year-old children, I focus on the first two studies.

3.5.4 Children's performance

In all these studies, results showed that children produce a significant amount of errors, both lexical and syntactical (Power and Dal Martello, 1990). The lexical errors were not numerous and consisted of making digit or word substitution errors (e.g. writing the number 35 as 53). On the other hand, the syntactical errors found (which were more numerous), consisted of the production of sequences of digits within which individual digits were correct but their combination was wrong. These errors usually provoked a lengthening of the digit sequence such as the number 124 being written as 100204. They were found to be an important indicator of the emergence of children's understanding of place value (Nunes and Bryant, 1996).

Power and Dal Martello (1990) found that in the errors made by their seven year-old Italian children, 87% were purely syntactic, 3% were purely lexical and 11% were mixed. Nunes and Bryant (1996) used a similar error classification, but are less clear in their report: 15% of their sample refused to write numbers over 10, about 21% made lexical errors and the rest made syntactical errors.

Nunes and Bryant (1996) also reported that children find it easier to recognise numbers than to write them down (Table 12). Furthermore, there seems to exist a clear difference in writing two-digit numbers and in writing three and four-digit numbers. Different types of knowledge seems to be involved: whereas nearly half the children could write numbers in the teens and the 2-digits (above 20) categories, numbers dropped dramatically in the three- and four-digit categories.

TABLE 12
Percentage of five and six year-old children writing and recognising numbers (adapted from Nunes and Bryant, 1996)

	Categories						
	under 10	teens	2-digits	3-digits*	100	200	4-digits
written numbers	93	48	47	7	68	35	2
number recognition	97	60	66	24	79	37	2

* under 200

However, as predicted by Nunes and Bryant (1996), the difficulties in writing three-digit numbers is not related to the size of the number being written, but to the underlying difficulties generated by the combination of units of different sizes. As Table 12 shows, a significant proportion of the same children are able to write correctly numbers 100 and 200.

3.6 SUMMARY OF METHODS TO BE USED IN THE STUDY

This chapter has briefly reviewed the methods used for the assessment of each separate number component. The case was made for the choice of certain methods, instead of others. These will now be summarised in order to provide a global picture of the methods used in the present study.

As noted in the introduction, the assessment will be divided into four number components: (1) counting and knowledge of the number-word sequence; (2) the ability to generate verbal number-words and the understanding of the structure of the

numeration system; (3) arithmetical operations, and (4) the ability to read and write numbers and the understanding of the principles underlying place value.

The first number component, i.e. *counting and knowledge of the number-word sequence*, was assessed through three tasks: (a) one-to-one correspondence and stable order of number labelling; (b) knowledge of the number-word list (i.e. counting range), and (c) continuation of counting. For the first task, a counting task similar to the one suggested by Fuson (1988) was used, where children were required to count both straight and scattered rows of tokens. Children's countings were required to make correct use of the one-to-one correspondence and stable-order principles in both situations. Another task, i.e. counting units of the same denomination (see below), assessed children's ability to count in a different context.

In the *counting range* task, children were invited to count as far as they can, or until they make two successive mistakes in their counts. This task is also based on Fuson's (1988) work. The older children were asked how far they think they can count (n), and were invited to count from $n-12$, to avoid fatigue.

Finally, in the continuation of counting task, children were asked to continue counting from numbers 20 and 10. Both numbers have the advantage of relating to the decade numeration system but do not involve numbers in the teens, which are problematic for children (Siegler and Robinson, 1982; Fuson, 1988).

The second component, the *ability to generate verbal number-words and the understanding of the structure of the numeration system*, was assessed with two tasks: (a) counting with units of the same denomination and (b) counting with units of different denominations, also known as the shopping task. Both tasks are based on the

tasks used by Nunes and Bryant (1996). In the former, children were asked to buy items from a shop and to pay for them with units of the same denomination, i.e. ones. In the latter, the child was also invited to pay for items in a shop situation where coins of different denomination (i.e. ones, fives, tens, hundreds, etc.) were made available to make the payments. Children were required to pay for items according to the categories of prices in the teens, two-digit quantities between 20 and 100, three-digit and four-digit amounts. The present study assessed children's understanding of the structure of the decade numeration system separately from their ability to write multi-digit numbers.

The third component, *arithmetical operations*, was assessed with four addition tasks and two multiplication tasks. The four addition tasks included change increase result unknown word problems (i.e. $a+b=?$), as well as change decrease result unknown word problems (i.e. $a-b=?$). Secondly, it also included several change start unknown word problems (increase and decrease; i.e. $?+b=c$ and $?-b=c$). These tasks are based on the work of Carpenter and Moser (1982) and Riley et al., 1983).

Finally, for a clearer assessment of the counting-on strategy of addition, this study included addition with one hidden addend problems (Hughes, 1986). In these problems children are invited to abstract the quantity that is hidden in a box and to count-on from it. Counting-on was also assessed in other four different situations.

The multiplication tasks included in the study were isomorphism of measures problems (Vergnaud, 1983) and relative values problems (Nunes and Bryant, 1996). In both instances, children are invited to count each group of objects (e.g. the wheels in a car) and also to keep track of the total number of objects (three cars). In the

relative values task children are asked to judge and justify comparisons between rows of coins with different denominations, which requires double-counting (Steffe, 1994).

The fourth number component, *the ability to read and write numbers and the understanding of the principles underlying place value*, was assessed separately from the ability to understand the structure of the numeration system which can be done without knowledge of written numbers (Ginsburg, 1977). In this task, children were asked to write and recognise (in different sessions) single-digits, as well as numbers in the teens, two-digits (between 20 and 100), three and four digit numbers.

4

RELATING NUMBER COMPONENTS: QUESTIONS UNDER INVESTIGATION

A fundamental issue that has interested a growing number of researchers is whether the school makes use of the wide range of skills brought by the children into the classroom (e.g. Ginsburg, 1977; Resnick, 1987; Aubrey, 1993; Nunes and Bryant, 1996; Suggate, Aubrey and Pettitt, 1997). However, the majority of studies have investigated separate skills, reflecting the one-component tradition, as discussed in earlier sections. As such, little data relating the performance of the same sample of children in different number components has been made available.

There remains an urgent need to assess the development of several number components in the same children, throughout the first years of primary school. Such study would provide a global picture about children's abilities, and the way each specific type of knowledge may relate to the development of other number understandings. In this particular case, a longitudinal approach seems crucial in order to map the development of these interrelationships in the same children, along time.

4.1 Re-examining children's understanding of the decade system

The development of children's understanding of the decade numeration system. requires further clarification. Arguably, whilst the investigation of number components such as counting and knowledge of the numeration system continue to be carried out separately, their relation shall remain unclear.

Meanwhile, the lack of data on the performance of the same children in these two components has led to conflicting views, where counting is seen both as crucial (e.g. Gelman and Gallistel, 1978; Kamii, 1986), or as secondary (Piaget, 1952; Resnick, 1986; Nunes and Bryant, 1996), and even as an obstacle (Fuson, 1990; Miller and Stigler, 1987; Miura and Okamoto, 1989; Miura et al., 1993) to children's understanding of the decade numeration system.

Without precise data relating the performance of the same children, it can only be speculated that the higher children count up their number word-list, the more probable it is they may be able to tease out the principles underlying the structure of the decade numeration system - the fact that the numeration system is made of units of different denominations; i.e. ones, tens, hundreds, and so on. However, much clarification is needed regarding the mechanisms of this development. It is possible that extending the number-list may not be what teaches children about the structure of ones, tens and hundreds.

It is worth recalling the two dominant views about children's understanding of the decade numeration system. The first, argues that children's numerical reasoning originates in counting and,

"the child's arithmetic system is strongly shaped by the mental entities with which it deals, namely, the representation of numerosity that may be obtained by counting" (Gelman and Gallistel, 1978; p. 185).

According to this perspective, without counting knowledge children would be unable to manipulate mental entities when they reason numerically. From this point of view, children construct the structure of tens on the structure of ones, where counting is granted a fundamental role (Gelman and Gallistel, 1978; Kamii, 1986). The previous chapter briefly reviewed work from Kamii (1986), presenting evidence to support this view. However, it is also recognised that early principles aid initial but not later conceptions of number (Gelman and Meck, 1992).

An alternative view, however, suggests that counting units of the same denomination (ones) plays a minor role in children's grasp of units of different denominations. Children's counting is seen as a necessary condition for further mathematical developments, but it is not, in itself, a sufficient condition for the development of the numeration system. Authors sharing this perspective - of which Piaget (1952) is a famous example - tend to seek evidence that the understanding of the decade numeration system requires, from the child's point of view, something beyond the development of counting,

"a break with simpler concepts of the past, and a reconceptualization of number itself" (Hiebert and Behr, 1988; p. 9).

This argument has been based mostly on evidence showing lack of significant correlations between children's results in counting tasks and results in tasks assessing their understanding of the structure of the decade numeration system (Carragher, 1985; Miller and Stigler, 1987; Fuson, 1990; Nunes and Bryant, 1996). These authors further argue that although Gelman and Gallistel's (1978) work on counting has provided insightful hypotheses to account for the development of children's counting skills, they have not explained the influence of this knowledge on later number acquisitions. In fact, this relationship is not investigated either in their book "The child's understanding of number" (1978) or in later studies (e.g. Gelman and Meck, 1983; 1992).

Likewise, although Fuson (1988) has provided an explanation for the development of children's number-word list from age two to seven, the relationship between this progress and the child's ability to use the decade numeration system, has not been clarified - although it has been generally assumed that the latter is an extension of the former. The next section presents recent work by Nunes and Bryant (1996) supporting the alternative view that counting plays a secondary role in children's understanding of the decade numeration system.

4.1.1 The 'addition hypothesis'

Nunes and Bryant (1996) recently provided evidence that counting, on its own, is not a sufficient condition to enable children to understand the decade system. According to

their argument, evidence that counting relates to knowledge of the decade system, should be provided by data showing that the child can generalise his/her knowledge of counting (i.e. adding one more to the set already counted) to more complex additive compositions involved in the decade system, simply by frequent practising. Positive evidence, they argue, should show a significant correlation between children's ability to count ones and their ability to count ones, tens and hundreds (i.e. units of different denominations).

Carraher (1985) investigated this question in a study with 72 Brazilian children, aged 5 to 8. Children's knowledge of counting was assessed with a task where they had to buy items from a shop and pay with units of single denomination; i.e. ones. Children's understanding of the numeration system was assessed with a relative values task, where children were required to decide which of two arrays of tokens had more money, and a shopping task, where children had to buy items from a shop and pay with coins of different denominations (i.e. ones, tens, hundreds, and so on). The shopping task was thought to make the counting of single and different denominations more meaningful to children.

In her results Carraher (1985) reported that counting units of the same size was not problematic even for the 5 year-olds. Most children obtained a ceiling-effect on these items. More importantly, she also reported that the results in the counting ones task did not correlate significantly with the results of the relative values nor the counting units of different denominations tasks. Based on this data, Carraher (1985) suggested that practice in counting units of single denomination alone, does not teach children to count units of different denominations, because

"Children who know how to count may still not be able to understand the relative values of units and compose totals with different-value units in the context of dealing with money" (Nunes and Bryant, 1996; p. 52).

Nunes and Bryant (1996) also report on a study which replicated Carraher's (1985) results in a similar study with 5 and 6 year-old British children. This argument has been further supported by data from Miller and Stigler (1987), who assessed the relation between counting and children's understanding of the base-10 structure underlying the numeration system, in 96 American and Taiwanese children aged 4 to 6. The systems used in both cultures differ in the sense that Chinese counting continually repeats the order of the first nine numbers, throughout the system - one, two,eight, nine, ten ... ten-one, ten-two, ten-three ten-nine, two-tens, two-tens-one, and so on. This gives the child important clue about the organisation of the system, and

"it should be much easier for Chinese children to induce the difference between primitive and compound numbers, and such is the case (Miller and Stigler, 1987; p. 301).

The western system is less helpful in that sense, and instruction does not help children to grasp the structure of the system from their initial countings to ten (Miller and Stigler, 1987; see also Miura and Okamoto, 1989; Miura et al., 1993). Their results showed that the Chinese children were more proficient in the use of the decade system,

in comparison with the American children. This suggests that a distinction between object-tagging and number-naming systems in counting should be made.

"These systems appear to be largely independent or modular, being affected differently by separate sources of added difficulty" (Miller and Stigler, 1987; p. 301).

This leads to an important point. The argument that some number components (in this case counting) may interfere in the child's acquisition of other, more complex components, has been put forward by other authors. According to Fuson (1990),

"The use of unitary conceptual structures becomes highly automatized in the U.S. first and second graders and interferes with their construction and use of multiunits of ten" (Fuson, 1990; p.360).

According to Nunes and Bryant (1996) children build their understanding of the decade system on previous knowledge of addition, hence the reconceptualisation referred to by Hiebert and Behr (1988; see above). In their words,

"Children's encounters with addition might be the necessary experience for understanding the additive composition that underlies the decade system. There is a well-established change in young children's approach to adding

which, on the face of it, could be the spur for understanding the base-ten system. This is the transition from counting-all to counting-on" (Nunes and Bryant, 1996; p. 52).

4.1.2 Development in addition strategies and children's understanding of the decade numeration system

The argument that counting-on may be related to children's understanding of the numeration system, is based on the idea that the use of this counting strategy may expose children to an early grasp of units of different denominations (Carraher, 1985; Resnick, 1983; 1986; Fuson, 1990; Nunes and Bryant, 1996).

In order to perform counting-on the child must join separate units and collect them into a single multiunit, which gain a different value. In the "4+5" example, the child who uses Counting-on simply unites "1, 2, 3, 4" into a *collected multiunit* of a higher value, i.e. "4" (Fuson, 1990). By Counting-on from 4, the child is judging that number as a unit of a different size. According to Nunes and Bryant (1996),

"This developmental change could well be relevant to the understanding of the decade structure. The child who sees that she does not laboriously have to re-count the larger set may have realised that this set can be treated as a larger unit which can be combined with a smaller one. This child might therefore be in a better position to understand that one can form the number

23 by combining two units of one denomination (two tens) with three of another (three ones)" (Nunes and Bryant, 1996; p. 53).

Nunes and Bryant (1996) investigated the relation between addition and knowledge of the decade system in five and six year-old British children. The addition problems used were simple (i.e. Mary had 8 sweets and her Granny gave her 5 sweets. How many does she have now ?) and enough tokens to represent both addends were provided. Besides a pass/fail score in the addition problems, Nunes and Bryant (1996) also recorded the type of strategy used to solve the problem: either count-all, count-on or recalled-facts.

Nunes and Bryant (1996) reported that only 10% of the five year-olds used count-on (or recalled-facts), whereas a significantly higher number (57%) of the six year-olds used the same strategies. Interestingly, although the pass/fail score of the addition task did not correlate significantly with knowledge of additive composition, the second score (counting strategy used) showed a significant correlation between the use of the count-on strategy and the results of the addition composition tasks.

These results support the argument that children who understand addition but still have not developed more sophisticated counting strategies such as counting-on, are not able to grasp additive composition. Based on the correlations found, they further suggested that children's understanding of additive composition of number may develop from counting-on.

4.1.3 Further inferences from the hidden addend studies

In a study reported by Nunes and Bryant (1996), Kornilaki (1994) attempted to further clarify the relation between the use of the counting-on strategy and knowledge of additive composition, in 5 to 6 year-old Greek children. The question being asked to further explore the addition hypothesis, was: What is it that the 'count-on children' are capable of doing that the 'count-all children' cannot do ?

To increase the probability of children's use of the counting-on strategy, she hid the first addend in a wallet, so that the children could not count it. For example, the children were told that a girl had 8 drachmas (Greek currency) in her wallet, and that she had been given another 7 drachmas, which were placed in front of the child. She also assessed children's ability to count units of the same (i.e. ones) and different denominations (additive composition task).

Her results were consistent with Nunes and Bryant's (1996) suggestion that counting and knowledge of the numeration system are different issues: Children who only counted-all could not pass the additive composition task (shopping task). She also reported that the use of Counting-on (in an addition with hidden addend task) was a necessary but not sufficient condition to pass the shopping task: all the children who passed the Additive Composition task also passed the Count-on task (Table 13, below).

TABLE 13

Cross-tabulation of Kornilaki's (1994) results

Addition with invisible addend	<i>additive composition</i>		Total
	fail	pass	
fail	17	0	17
pass	21	12	33
Total	38	12	50

Based on this, Kornilaki (1994) concluded that her data was consistent with the argument that counting-on may be an embryonic form of children's grasp of additive composition of number and, therefore, a basis for their understanding of the structure of the numeration system.

This conceptual link looked so promising that Kornilaki went on to observe the counting strategies used in the invisible addend addition task in greater detail, expecting to detect further connections between the use of the counting-on strategy and children's understanding of additive composition.

She formed two groups: Children who had failed the count-on task (count-all) and children who passed it (count-on). Those classified as 'count-all' children did not conceive the hidden value represented by the first addend of the problem. They either counted the coins that represented the second addend, or counted the wallet as one object. No significant correlation was found between the count-all children and additive composition, which suggested that knowledge of one-to-one correspondence is insufficient for the child to understand the cardinal as a sufficient representation of the set and to be able to add on to it, in order to grasp additive composition.

On the other hand, the 33 children classified as 'count-on', used several strategies to pass the task (Table 14), which Kornilaki (1994) categorised according to Steffe et al's. (1982) counting types. Twelve of these children used their fingers to count-all the first addend and then continued by counting the available coins (figural). A further 8 children used either head movements or finger tappings to count-all from one up to eight (first addend) and then continued counting the available coins (motor). These two strategies involved an attempt to represent the invisible drachmas in some way, and then the counting of the second addend, to find the final result.

TABLE 14

Strategies and success in the additive composition task (N=33; Kornilaki, 1994)

	Successful counting strategies			
	counted with fingers (Figural)	made head movements (Motor)	counted quickly from one (verbal)	used cardinal number for first addend (abstract)
Pass additive composition	12	7	2	0
Fail additive composition	0	1	5	6

Kornilaki also found that another group of 7 children counted quickly from one, up to the value of the first addend, and then continued counting the visible coins, and 6 children (out of 33) said the cardinal for the first addend and continued counting-on the visible coins. Only children in this last group relied on the cardinal number as a sufficient representation of the hidden set of drachmas.

Kornilaki (1994) reported that with one exception, no children classified as figural or motor counters were able to pass the additive composition task. Furthermore, those 5 out of 7 verbal counters who passed the additive composition task, had counted the first addend quickly from one. Kornilaki (1994) and Nunes and Bryant (1996) argued

that these quick counters could be considered abstract counters, only that they used an 'overt' strategy.

Based on this data, Kornilaki (1994) and Nunes and Bryant (1996) suggested that the use of the count-on strategy may represent children's first experience of mixing and counting units of different sizes. Also, they proposed that count-on could be an embryonic form of additive composition of number. This link, they claim, may be an opportunity for children to start learning about the numeration system, although they admit that further research is needed to settle the question.

4.1.4 The relevance of continuation of counting

The limitation of these studies representing both views is that they have not considered developmental differences between counting level (Fuson et al., 1982; Fuson, 1988) and have overlooked the relevance of children's continuation of counting. This development, as briefly discussed earlier, reflects the child's new understanding about number and is, simultaneously, a precursor of counting-on (Secada et al., 1983).

Table 15 compares the levels used by both types of studies, highlighting the levels of counting not contemplated in Kamii's (1986) and Nunes and Bryant's (1996) studies. Those children classified as 'counting ones' by Kamii (1986) and Nunes and Bryant (1996) on the right column, can be classified in three different groups, according to Fuson (1988).

TABLE 15

Comparison of levels of counting used

Fuson's (1988) model	Kamii (1986) Nunes and Bryant (1996) and other studies
string level (1)	counting ones (1)
unbreakable chain level (2)	counting ones (1)
breakable chain level (3)	counting ones (1)
numerable chain level (4)	counting-on (2)
bidirectional chain level (5)	counting-on (2)

Based on Fuson et al's. (1982) data, it seems appropriate to hypothesise that children who have developed different levels of counting ability may perform differently in tasks assessing their understanding of units of different denominations. No longitudinal data has yet been produced about *the effects of specific counting skills on children's grasp of additive composition of number, a property of the numeration system.*

4.1.5 Part-whole and children's understanding of the decade system

Resnick (1983) suggested an alternative to the shopping task to estimate children's knowledge of additive composition of number. Her assumption is that children's ability to interpret word-problems in terms of part-whole is good evidence that they have informally understood additive composition.

When children are given a typical start-set-unknown problem like 'Paul has some; Charles gave him 5; now he has 8. How many did he have to start with?', they have difficulties in representing an undefined start-set with their fingers or the number-line. Alternatively to the use of the number-line, which only allows children to relate numbers as larger or smaller, they may revert to a part-whole schema to map the problem (Riley and Greeno, 1988). By mapping the problem, they may interchange the different sets in order to find a solution. To rearrange the quantities is considered to depend on an understanding of additive composition (Resnick, 1983).

No data has yet been presented *about estimates of additive composition of number, or about the relationship between the counting-on strategy of addition and knowledge of part-whole problems.*

4.1.6 Arithmetical operations and children's understanding of the decade system

Evidence supporting the argument that children learn about the numeration system from knowledge of addition, rather than from practice with counting (Piaget, 1952; Nunes and Bryant, 1996), also raises a new possibility. As was briefly explained earlier, the numeration system has multiplier words like 'hundred', 'thousand' and 'million' which, according to their position in a word sequence, enter in sum or product relations with the basic numbers (Ross, 1989; Fuson, 1990). For example, 'four-hundred' corresponds to a product relation, whereas 'hundred and four' corresponds to a sum relation.

Based on this, it seems worth exploring the effects of children's knowledge of addition and multiplication on their understanding of the decade numeration system's sum and product relations.

This relationship, which has not yet been explored longitudinally, could develop much sooner than previously thought, having important consequences for children's grasp of the decade system.

4.2 Exploring the further relevance of continuation of counting

Apart from the works of Fuson et al. (1982) and Siegler and Robinson (1982) which have described the development of children's counting strategies, only two studies have investigated the importance of continuation of counting in children's number development.

The first one, argues that continuation of counting is one of three subskills involved in counting-on (Secada et al., 1983). The second one, hypothesises that continuation of counting represents a fundamental development in children's formation of the concept of number, a process that is seen as occurring when addition is learned as a mental operation (Davydov, 1969). Basically, Davydov (1969) argues that when children count-on in addition problems, they begin to understand that a numeral implies quantities, although they still need to count the second addend to obtain the result of the sum.

"Could it be that the curtailment of counting begins when the children learn to count farther from any number?" (Davydov, 1969; p. 43).

From the outset - wrote Davydov (1969) -, the curtailment of the counting procedure looks like adding-on. However, he argued, this development is far more important than learning a counting strategy, as it characterises the use of number in concept form. If this is the case, Davydov (1969) proposed,

" When a quite definite quantity is implied in the numeral itself, which is known in advance, there is the possibility of skipping the middle elements of the series being counted. This circumstance eliminates the necessity of counting the first addend and thereby teaches the child how to use a number as a whole. It is the latter that characterises use of number in concept form (Davydov, 1969; p. 43)."

Unfortunately, Davydov (1969) did not provide data to support his hypothesis, at least in western scientific journals. The present study assesses these two arguments and proposes a third one, based on Davydov (1969), postulating that continuation of counting may be also related to children's understanding of subtraction and multiplication. Evidence of the latter will support the idea that children's knowledge of number may be interrelated to their understanding of the operations; in other words, that children need to understand them in order to grasp number as a whole (Piaget, 1952). This evidence suggests that the relevance of continuation of counting should be further investigated, especially from a longitudinal perspective.

4.3 Children's understanding of written numbers and place value

There is evidence to support both views that (1) children's understanding of the convention of place value is based on their prior knowledge of the structure of the numeration system (Ginsburg, 1977; Fuson, 1990; Nunes and Bryant, 1996), and (2) children base their understanding of place value on practice with written numbers (Sinclair, 1991; Sinclair et al., 1992; Sinclair and Scheuer, 1993)

Also, evidence shows that there is a developmental lag between being able to understand the structure of the numeration system and grasping place value. Whereas the former is understood by five year-olds (Carragher, 1985; Nunes and Bryant, 1996) the latter is grasped by seven year olds (Sinclair, 1991; Sinclair et al., 1992; Sinclair and Scheuer, 1993). No longitudinal data has been produced about *the effects of 'understanding the structure of the numeration system' and their 'knowledge of written numbers' on use of place-value*, as separate number components.

5

METHODOLOGY AND RESULTS

5.1 RATIONALE OF THE PRESENT STUDY

The proposal of this study is to explore longitudinally the development of several number components in the same children, throughout their first three years of learning in school mathematics. The components to be explored are:

- (1) counting and knowledge of the number-word sequence;
- (2) generating verbal number-words and the understanding of the structure of the numeration system;
- (3) arithmetical operations;

(4) the ability to read and write numbers and the understanding of the principles underlying place value.

5.2 RESEARCH QUESTIONS

This study will also explore some relationships between the various number components through a series of examinations of relevant data. The following will be examined: (1) the emergence of additive composition of number and children's understanding of the decade numeration system - the effects of continuation of counting; (2) the effects of children's use of continuation of counting on their understanding of the arithmetical operations; (3) the effects of knowledge of addition and multiplication on children's understanding of the decade numeration system; (4) the effects of children's 'understanding the structure of the numeration system' and their 'knowledge of written numbers' on use of place-value.

5.3 PARTICIPANTS

The participants in this study were 152 primary school children recruited equally from three schools in North London. The Reception, Year 1 and Year 2 groups included fifty-three children, forty-one and fifty-eight children, respectively. The children's age ranges are shown in Table 16. The participants in each class were selected by their teacher who was asked to provide children from three different levels of mathematical achievement: top, average and bottom.

TABLE 16

Age ranges and medians by assessment (in years-months)

	Assessments		
	first	second	third
Reception group	4-8 to 5-2 (4-11) ^a	5-0 to 5-7 (5-3)	5-4 to 5-10 (5-7)
Year 1 group	5-2 to 6-4 (5-11)	5-6 to 6-8 (6-2)	5-10 to 7-0 (6-7)
Year 2 group	6-2 to 7-4 (6-11)	6-6 to 7-6 (7-1)	6-10 to 7-10 (7-5)

^(a) Numbers in parentheses are Medians

The experimenter was blind to the teacher's evaluation of the participants, which was based on either number class work (Reception group), Nuffield maths worksheets or similar (Year 1), and level of achievement in the numerical components of the Standard Assessment Tasks (SEAC, 1992) in the Year 2 group. These tasks are given to all children within the UK at the end of Key Stage 1 (7+). These tests identify levels of competence normally expected of the 'average' seven year-old and may also be used to identify children at both extremes of the spectrum of achievement (e.g. Gray et al., 1997). The participants were all fluent English speakers.

5.4 OVERALL PROCEDURE

Each cohort of participants was assessed three times during the school year — in the autumn, the winter and spring terms — following a mixed cross-sectional and longitudinal design. Before the series of interviews in each school, the experimenter was introduced to the class by the teacher, who explained that the experimenter had come to play some games with everyone.

The games were then described to the children in a general form. All children were interviewed by the same experimenter in a quiet room away from their classroom, for two or three sessions according to their speed. Each child spent 20-30 minutes with the experimenter, in each session.

The children responded to several maths tasks grouped according to the four number components described in the introduction (Table 17). All tasks were based on previous research (authors are shown in parentheses) but were nevertheless applied in a previous pilot-test and adjusted for the main study. The experimenter used a semi-structured interview and tried to ascertain the meaning of each answer by asking the child: "why ?" or "can you show me how you did it ?".

The child was permitted to manipulate objects and there were no time pressures. All of the children's responses were recorded in a scoring sheet. All children started their "games" with a counting task, which was used as a "warm-up" task. The order of presentation of the remaining tasks was randomised across groups.

TABLE 17
Tasks used in the study

(1) Counting And Knowledge Of The Number-Word List

One-to-one correspondence and fixed order of number labelling (*Gelman and Gallistel, 1978*)
Counting range (*Fuson, 1988; Miller and Stigler, 1989*)
Continuation of counting (*Fuson, 1988*)

(2) Generation of Verbal Number Words And Decade Numeration System

Counting units with single denomination (shopping task; *Nunes and Bryant, 1996*)
Counting units with different denominations (shopping task; *Nunes and Bryant, 1996*)

(3) Arithmetical Operations

3.1 Addition and Subtraction

Change increase result-unknown word-problems (addition; *Carpenter & Moser, 1982*)
Change decrease result-unknown word-problems (subtraction; *idem*)
Change start-set unknown word-problems (inversion; *idem*)
Addition with one hidden addend (addition with a box; *Hughes, 1986*)

3.2 Multiplication

Isomorphism of measures word-problems (*Vergnaud, 1983*)
Relative Values (*Nunes and Bryant, 1996*)

(4) Written Numbers And Knowledge Of Place Value

Production of written numbers (*Power and Dal Martello, 1990; Nunes and Bryant, 1996*)
Recognition of written numbers (*Nunes and Bryant, 1996*)

5.5 TASKS USED IN THE STUDY

The range of tasks assessed reflected a combination of two criteria, whenever possible. On the one hand, the present study assessed children's knowledge of several number components, reflecting the range of numerical competencies referred to in the literature. On the other hand, and in order to give ecological validity to the situation in which children would find themselves (*Aubrey, 1993*), this study used assessment

tasks, the majority of which were compatible with key areas of the National Curriculum for mathematics.

5.5.1 COUNTING & KNOWLEDGE OF THE NUMBER-WORD SEQUENCE

(component 1)

5.5.1.1 One-to-one correspondence & fixed order of number labelling

Materials

For the one-to-one correspondence and fixed order of number labelling task, ten red and yellow plastic coins were used (diameter of 2.5 cm).

Rationale

This task was set as a 'warming-up' activity. This task verified whether children were able to establish a one-to-one correspondence between objects and counting sequence and whether they were able to label the numbers correctly (Gelman and Gallistel, 1978).

Procedure

Twenty coloured plastic coins were displayed on the table, scattered. The child was asked: "*can you count these coins ? Try to count them*". The child's procedure was recorded. Children who respected the one-to-one counting and the stable order principles within the first ten objects were classified as pass. The remaining children were given a second trial, where they were required to count the coins in a straight line. Children who made one error at least were classified as 'fail'.

5.5.1.2 Continuation of counting

Rationale

This task verified whether children were able to continue counting from an arbitrary number in the counting-list (Fuson, 1988).

Procedure

After having counted 20 coins (previous warm-up task), children in the first assessment of the Reception group were asked to continue counting from number twenty (i.e. "*do you know what numbers come after twenty ?*"). Those children who failed to continue counting were asked to continue counting from number ten (i.e. "*do you know what numbers come after ten ?*"). Children were invited to continue counting until they reached their own limit. (Counting range task. see below). Children's answers were recorded.

During the second and third assessments of the Reception group, and the three assessments of the Year 1 group, the children were simply asked: "*What is the highest number you can count to ?*" The child was then invited to continue counting from the number s/he said, minus 15. This was done to verify whether the child could pass from the previous decade to the next. For example, a child who said 100 would be asked to continue counting from 85; a child who said '80', would be asked to continue counting from 65 until a second error was made.

If decade errors were made the child was then asked: "*do you know what number comes after 20 ?*". Those who failed to answer to the 'after 20' question were asked:

"do you know what number comes after 10 ?" Children were invited to continue counting until they reached their own limit (Counting range task, see below). Children's answers were recorded. To pass this task children had to continue counting from at least one of the numbers they were asked about.

5.5.1.3 Counting Range

Rationale

This task was based on the work of Fuson (1988) and Stigler and Miller (1989) and assessed children's counting limit, i.e. how far they were able to say the counting words in the conventional order, before making serious mistakes.

Procedure

The younger children were invited to count until they reached their own limit. The older children were asked how far they thought they could count. They were then invited to count from that number minus 12. Similarly to Miller and Stigler (1987), the counting was stopped after two consecutive omissions. For example: 23, 24, (), () 27 (where two consecutive numbers were omitted). In counts like 23, (), 25, (), 27, the errors are not immediately consecutive. Children's answers were recorded.

5.5.2 GENERATING VERBAL NUMBER WORDS AND UNDERSTANDING THE
DECADE NUMERATION SYSTEM (component 2)

5.5.2.1 Counting units with single denomination (shopping task)

Materials

This task used five toys (to be sold in the shop) and 18 yellow plastic coins worth 1 pence each, (2.5 cm in diameter), to be used as money.

Rationale

This task was adapted from Carraher (1985) and Nunes and Bryant (1996) and verified whether children were able to count units of the same denomination, i.e. ones. Each child was asked to buy and pay for 5 objects in a make-believe shop, while the experimenter played the role of the shopkeeper. The items to be purchased in the shop included two toys priced under 10p (7p and 8p), and three toys priced between 10 and 20p (12, 15 and 16p).

Procedure

The experimenter gave the child 18 yellow one-pence coins and explained how much each coin was worth (i.e. one pence). The children were told: *"this is a shopping game. I would like you to buy some toys from my shop. I will be the shopkeeper and you the customer. Use this money (point) to buy the toys. Ready ?I would like you to buy this toy (point) first. It costs 7p (trial 1)... How much money do you have to pay me ?"* The five trials were applied in a fixed order. Children's answers were recorded

and the money reshuffled after each trial. Each correct answer scored one point. Maximum score: five points.

5.5.2.2 *Counting units with different denomination (shopping task)*

Materials

The *counting units with different denominations* task used several toys to be bought from the shop (e.g. little teddy-bears, colouring pencils etc.). Also, play-money was used: nine 1p coins (yellow), four 5p coins (red), six 10p coins (green), five 100p coins (blue) and three 1000p coins (black). The coins had different colours for better identification and were tagged with the amount they represented.

Rationale

This task verified whether children were able to count and combine units of different denominations (adapted from Carraher, 1985 and Nunes and Bryant, 1996). Each child was asked to buy and pay for 12 objects in a make-believe shop, while the experimenter played the role of the shopkeeper.

Procedure

The items to be purchased in the shop included sets priced under 10p (6p, 7p and 8p), under 20p (12p, 15p and 16p), under 100p (26p, 53p), under 1000p (124p and 347p) and above 1000 (1052p and 2340p). The children were told: "*this is the same shopping game but I would like you to buy some more toys from my shop. Now, you can use this other money (point) to buy the toys. Ready ?*" Children were always given money in combinations that did not allow them to buy an item with only one denomination. For example, in the 7p item, children were given two 5p coins and four

1p coins (total: 6 coins). The objects were sold in the above fixed order up to the 16p item. After that, the game was interrupted at the second failure.

Before moving on to a new set of items, the child was shown the coins used in that set and asked to recognise their value. In those cases where the child did not recognise the value of some of the coins, the experimenter explained it to the child. Children's responses were classified as to whether they continued counting from the number in the higher denomination - or not. Each correct answer scored one point. Maximum score: twelve points.

5.5.3 ARITHMETICAL OPERATIONS (component 3)

5.5.3.1 Increase and decrease change result-set unknown word-problems (addition and subtraction)

Materials

Fourteen small wooden bricks were used as materials.

Rationale

This task verified whether children could add and subtract (adapted from Carpenter and Moser, 1982; Riley et al., 1983).

Procedure

Each child had to find the sum (total) of six (3 increase and 3 decrease) problems. The child was allowed to manipulate objects. *"For our next game I'm going to let you use*

these bricks, in case you need them to play the game. (...) I'm going to read you some easy problems, and I would like you to find the result. Ready ?"

The problems read were of the type "George had 3 marbles and John gave him another 5 marbles. How many marbles does George have now ?" Whenever necessary, the problem was re-read to the child up to two times, after saying: "*I'm going to repeat the problem so that you can find out what happened, pay attention*".

The bricks were always mixed after being used. The three increase trials were 3+5, 2+6 and 4+7. The three decrease trials were 6-4, 7-3 and 9-5. The numbers were kept under 5 to facilitate calculation. Addition trials had their second addends slightly bigger, so that if addend order disregard was to occur, it could be more easily noted. Both sets of problems were presented in a fixed order.

Children who correctly answered one trial at least were classified as 'pass'. Responses resulting from miscounts by one were not treated as errors. Children's strategies were classified into "no counting strategy", used "counting-all" and used "counting-on".

5.5.3.2 Change start-set unknown word-problems (inversion)

Materials

Fourteen small wooden bricks were used as materials.

Rationale

Each child was given four (two increase and two decrease) inversion problems to verify whether they could apply the part-whole schema (adapted from Carpenter and Moser, 1983; Riley et al., 1983).

Procedure

The child was allowed to use objects. *"For our next game I'm going to let you use these bricks, in case you need them to play the game. (...) I'm going to read you some easy problems, and I would like you to find the result. Ready ?"* The problems read were of the type "Libby bought some oranges in the morning. Her Mum gave her another 5 in the afternoon. Now she has 8 oranges. How many oranges did Libby buy in the morning ?"

Whenever necessary, the problem was re-read to the child up to two times, after saying: *"I'm going to repeat the problem so that you can find out what happened, pay attention"*. The bricks were always mixed after being used. The 2 increase trials were $?+5=8$, $?+6=10$. The two decrease trials were $?-4=6$, $?-7=3$. Both sets of problems were presented in a fixed order. The child's answers and procedures were recorded; a justification, either procedural or verbal was asked for.

Children who correctly answered one trial at least were classified as 'pass'. Responses resulting from miscounts by one were not treated as errors. Children's strategies were classified into "no counting strategy", used "counting-all" and used "counting-on".

5.5.3.3 *Addition with a hidden addend (addition with a box)*

Materials

Fifteen small coloured round chips (sweets) and a small plastic box were used.

Rationale

This task represented a specific assessment of the counting-on strategy. Each child had to find the sum (total) of bricks inside and outside a box (adapted from Hughes, 1986 and Nunes and Bryant, 1996).

Procedure

The experimenter hid 3 bricks inside the box and put 4 bricks outside and said: *"Inside the box there are 3 bricks and these (point) are outside. How many bricks are there altogether ?"*. Whenever necessary, the number of bricks inside the box was repeated. Children in the Reception group were given the trials 3in+4out, 4in+6out and 5in+3out. Children in the Year 1 and Year 2 groups were given the trials: 3in+4out, 9in+3out and 11in+4out. Both sets of problems were presented in a fixed order.

The child's answers and procedures were recorded. Responses resulting from miscounts by one were not treated as errors. Children who correctly answered one trial at least were classified as 'pass'. Particular attention was paid to the ways children counted the items inside the box. Children's strategies were classified into "no counting strategy", used "counting-all" and used "counting-on".

5.5.3.4 *Isomorphism of measures word-problems (multiplication)*

Materials

A cardboard-made puppet, a small toy-car, several paper-made gloves, bracelets and plastic wheels were used as materials.

Rationale

This task verified whether children could operate with composite units (adapted from Piaget, 1952 and Steffe, 1988). Each child had to find the result of three multiplication items similar to those used by Aubrey (1993), corresponding to Vergnaud's (1983) multiplication word-problems' classification.

Procedure

The child was shown a model (puppet) with a pair of gloves put on and asked (demo): *"This is a thinking game. I'll need your help to buy some clothes for Mary (showed puppet) and her friends. Ok ? I bought two gloves for Mary (show puppet with a pair of gloves on). How many gloves do I have to buy for 3 puppets like Mary ?"* If the child failed to give a correct answer the experimenter corrected it, explaining why.

The procedure was repeated with the remaining trials in a fixed order: *How many wheels does a car have ? (show a car). How many wheels do you think 3 cars have ?"; "I bought 3 bracelets for Judy (show puppet with 3 bracelets on). How many bracelets will I have to buy for 3 puppets like Judy ?"*

Those children who correctly answered one trial at least were classified as 'pass'. Responses resulting from miscounts by one were not treated as errors. Children's

strategies were classified into "no counting strategy", used "counting-all" and used "counting-on".

5.5.3.5 *Relative Values*

Materials

Eighteen yellow plastic coins of 1p, three red plastic coins of 5p, two blue plastic coins of 10p. All plastic coins were 2.5 cm in diameter and had the amounts they represented marked on them.

Rationale

This task verified whether children could respect and operate with units of different denominations (Carraher, 1985; Nunes and Bryant, 1996).

Procedure

In each trial, the experimenter and the child were given different amounts of money to buy sweets (Table 18). Two arrays of coins were put in front of each participant in the 'game' and the child was told: "Let's pretend we go to a store to buy some sweets. *Who buys more sweets; you or me ?*". After each trial the child is asked: "*Why ?*"

TABLE 18

Items used in the relative values task

Trial 1	one coin of 1 (Exp) — one coin of 5 (Child)
Trial 2	three coins of 1 (Exp) — one coin of 5 (Child)
Trial 3	three coins of 5 (Exp) — three coins of 1 (Child)
Trial 4	nine coins of 1 (Exp) — one coin of 10 (Child)
Trial 5	eighteen coins of 1 (Exp) — two coins of 10 (Child)

The answers were classified into judgements (who buys more ?) and justifications (why ?). Each correct judgement scored one point; maximum score for the first assessment of the Reception and Year 1 groups: 3 points. The maximum score of the remaining assessments was 5 points. All justifications were recorded for later categorisation.

5.5.4 WRITTEN NUMBERS AND KNOWLEDGE OF PLACE VALUE (component 4)

5.5.4.1 Production and recognition of written numbers

Materials

Paper and pencil were used for the production of written numbers task. The recognition of written numbers task used paper and pencil and fifteen 5x12cm cards with numbers printed on them.

Rationale

This tasks assessed whether children were able to write and recognise multi-digit numbers, up to four digits (e.g. Power and Dal Martello, 1990; Nunes and Bryant, 1996). The written numbers and the number recognition tasks were presented in different sessions, days apart.

Procedure

The children were asked to write down and recognise single-digits (2, 3, 4 , 7, 9; warm-up' items), numbers in the teens (12, 15), two-digit numbers (37, 40, 79), three-digit numbers (124, 200, 347), and four-digit numbers (1052 and 2340). The trials were presented in a fixed order and the children were read (or were asked to recognise) all numbers in the list. Each correct answer scored one point.

5.6 RESULTS BY NUMBER COMPONENT

This section reports on the results obtained by the same children in three different cohorts on the following number components: (1) counting and knowledge of the number-word sequence; (2) generating verbal number-words and the understanding of the structure of the numeration system; (3) arithmetical operations; and (4) the ability to read and write numbers and the understanding of the principles underlying place value.

No gender differences were found in the vast majority of the results obtained so this type of analysis was discontinued.

5.6.1 COUNTING AND KNOWLEDGE OF THE NUMBER-WORD SEQUENCE

5.6.1.1 One-to-one correspondence and fixed order of number labelling

The vast majority of the Reception children in the first assessment were able to respond correctly to the *one-to-one correspondence and fixed order of number labelling tasks*. The results were seventy-four (i.e. 39 out of 53) and seventy-nine percent (i.e. 42 out of 53), respectively.

Given the high results obtained, the assessment of these tasks was not repeated. Children's ability to count units was hence forth assessed by the counting units with single denomination tasks (small). According to McNemar tests, there were no

significant differences between these two tasks in the first assessment (see Table 20 in the pass single denomination results, Reception group, assessment one).

5.6.1.2 Continuation of counting

Children's results in the continuation of counting task were classified into pass/fail (Table 20). Twenty children (38%) in the first assessment of the reception group (i.e. at the start of school), were able to *continue counting from an arbitrary number* in the count-list. These numbers increased in the second (32 out of 53) and third assessments (43 out of 53), where close to ceiling-level results were reached. In the first assessment of the Year 1 group, 32 (out of 41) could continue counting, and by the second assessment of this year group, over 90% of the children were able to pass this task.

TABLE 20
Frequencies (and percentages) of success in the continuation of counting and counting range tasks - component 1

assessments	Reception (N=53)			Year 1 (N=41)			Year 2 (N=58)		
	one	two	three	one	two	three	one	two	three
CC*	20 / 38	32 / 60	43 / 81	32 / 78	38 / 93	39 / 95	56 / 97	57 / 98	56 / 97
Counting range	24a (3-100)b	50 (10-150)	65 (13-200)	78 (12-197)	89 (15-199)	107 (18-200)	-	-	-

* Continuation of counting

a mean

b minimum-maximum

Evidence that 38% of the reception children can already continue counting, at the start of schooling, suggests that a significant proportion of the children entering school already display flexible and abstract strategies to deal with the number-line. Furthermore, continuation of counting and the ability to count ones seem to represent different abilities where not all the children who count ones will necessarily know how to continue counting.

5.6.1.3 Counting range

Table 20 also shows the results of the counting range task. Some children in the first assessment of the Reception group were able to count up to 100. The averages for the three assessments of this groups were 24, 50 and 65. Some children in the second assessment reached 150 and others reached 200 in the third assessment. The averages for the three assessments of the Year 1 group were 79, 89 and 107. Throughout the various assessments, however, some children still could not count further than the teens.

5.6.2 VERBAL NUMBER-WORDS AND UNDERSTANDING THE STRUCTURE OF THE NUMERATION SYSTEM

The results of the "counting units of single denomination" and "counting units of different denominations" were categorised as "pass small" (items 6, 7 and 8p) and "pass large" (items 12, 15 and 18p). In order to pass, children were required to respond correctly to all the items in each category. Results are shown in Table 21.

5.6.2.1 Counting units with single denomination

The data show that children had no difficulties in passing the single denominations task, especially the items under 10 (i.e. small). Over ninety percent of newcomers to school were able to pass this task. The results on this task were at ceiling level throughout the nine assessments.

Children found it slightly more difficult to pass the large items (i.e. between 10 and 20), although over 50% of the children in the first assessment of the Reception group were able to succeed in these items as well. Results reached ceiling-level by the first assessment of the Year 1 group.

No children passed the larger items without having passed the smaller items. According to McNemar tests, the differences between passing the small and the large items were significant in the three assessments of the Reception group only (binomial tests; $p > 0.001$, in the three assessments). On the whole, these data support the argument that counting ones is not a difficult task for a great majority of children in primary school.

5.6.2.2 Counting units with different denomination

Children's achievement on the counting money with different denominations task is also shown on Table 21. The data show that the size of the denominations used has a

significant influence on children's counting. Results in the different denominations task (large) were significantly lower, in comparison with the single denominations task (large). McNemar tests showed significant differences in all assessments: binomial test, $p < 0.001$, for the first two assessments and binomial test, $p < 0.05$ for the third assessment of Reception. Binomial test, $p < 0.001$ for the first two assessments and binomial test, $p < 0.01$ for the third assessment of the Year 1 group. Binomial test, $p < 0.01$ for the first assessment of the Year 2 group.

There were no significant differences between the large and the small categories of the different denominations task in all the assessments, according to McNemar tests, so only the results for the pass large category are presented. Only six children passed this task in the first assessment of the Reception group. However, by the third assessment of the Reception group over a third of the children in the same group had passed the task.

TABLE 21
Frequencies (and percentages) of success in the generating verbal number-words and understanding of the structure of the numeration system tasks

ASSESS.	Reception (N=53)			Year 1 (N=41)			Year 2 (N=58)		
	ONE	two	three	ONE	two	three	ONE	two	three
<i>pass single denomination</i>									
small	47 / 92	44 / 86	50 / 94	41 / 100	41 / 100	40 / 100	58 / 100	58 / 100	58 / 100
large	27 / 53	30 / 59	29 / 55	38 / 93	40 / 98	38 / 93	55 / 95	56 / 97	55 / 95
<i>pass different denomination**</i>									
large	6 / 11	13 / 25	20 / 38	18 / 44	23 / 56	28 / 68	45 / 78	51 / 88	51 / 88

** No differences were found between the pass small and the pass large categories in all the assessments

Although there is a difference between counting ones and counting units of any other size (fives or tens), the data further suggest that the ability to count units of *different*

denominations does not depend on the size of the units being counted; i.e. there is no significant difference between counting with fives and ones or tens and ones, according to McNemar tests. This data supports the idea that the ability to combine units of different denominations is not a function of the size of the units counted.

Overall, results increased steadily throughout the remaining assessments and approached ceiling level by the first assessment of the Year 2 group. Further McNemar tests showed that there were significant differences between the results in the different denomination (large) task and the addition (binomial tests, $p < 0.01$ for the first eight assessments) and multiplication tasks (binomial tests, $p < 0.01$ for the first seven assessments).

5.6.3 ARITHMETICAL OPERATIONS

5.6.3.1 Addition and Subtraction

The results of the arithmetical operations tasks are shown in Tables 22 and 23. To pass any of these tasks, children were required to pass one item, at least. McNemar tests showed that there were no significant differences between addition and subtraction results in all assessments, so the subtraction results are not shown.

The data shows that a significant proportion of the Reception children (25%) were able to solve addition and subtraction word-problems at school entry. These word-problems were of the result-unknown type (either increase or decrease). Later, in the

second assessment, nearly half the children passed the addition and subtraction tasks respectively (i.e. 44 and 56%), and 64% of children passed both tasks in the final assessment of this age group.

TABLE 22

Frequencies (and percentages) of success in the arithmetical operations tasks - component 3

assessments	Reception (N=53)			Year 1 (N=41)			Year 2 (N=58)		
	one	two	three	one	two	three	one	two	three
Addition*	13 / 25	23 / 44	34 / 64	29 / 71	34 / 83	35 / 85	51 / 88	56 / 97	54 / 93
Inversion**	3 / 6	3 / 6	13 / 25	10 / 24	9 / 22	19 / 46	32 / 55	42 / 72	42 / 72

* change increase result unknown word-problems

** change start-set result unknown word-problems

Over two-thirds of the children in the first assessment of the Year 1 group passed the addition (71%) and subtraction (78%) problems. After that, results in these two tasks approached ceiling-level: 83% passed addition and 78% passed subtraction. In the final assessment, 85% and 93% of the children passed these tasks. In the Year 2 group, 88% and 90% of the children passed the tasks, showing that the results had clearly reached ceiling-level at this point.

5.6.3.2 Inversion

The results of the inversion task (change start-unknown word-problems) are also shown on Table 22. To pass this task, children had to respond correctly to one item, at least. No significant differences were found between the increase and decrease items of

this task, according to McNemar tests. For the subsequent analyses data from these two subtypes of problems were combined.

Only 3 out of 53 Reception group children passed this task in the first and second assessments. However, one fourth of the children succeeded in this task by the last assessment of the group. In the Year 1 group (N=41), 10, 9 and 19 passed this task in the first, second and third assessments respectively. By the first assessment of the Year 2 group (N=58) more than half the children passed this task (32 children). Later, in the second and third assessments, over two thirds of the children passed this task. Children had obtained the equivalent results in the addition items by the first assessment of the Year one group, almost two years earlier, in developmental terms.

McNemar tests confirmed that children found these problems significantly more difficult than the addition problems (binomial test, $p < 0.01$ for the first assessment of the Reception group; binomial test, $p < 0.001$ for all remaining assessments) as the former imply recurring to a part-whole schema for its mapping (Resnick, 1983; Riley and Greeno, 1988), whereas addition problems can be solved by relying on the number-line alone.

McNemar tests also confirmed that there were significant differences between the inversion problems and the isomorphism of measures problems of multiplication in all the assessments (binomial test, $p < 0.001$ for the first assessments; binomial test, $p < 0.01$; $p < 0.05$ and $p < 0.01$ for the first, second and third assessments of the Year 2 group, respectively). The results of the relative values problems of multiplication and inversion differed significantly only in the second assessment of the Year 1 group (binomial test, $p < 0.01$).

5.6.3.3 Addition with one hidden addend

The results of the *addition with a hidden addend task* are shown in Table 23. To pass, children had to respond correctly to one item at least. The scores were significantly higher than those obtained by the same children in the addition word-problems, as reported by Hughes (1986). Differences were significant in the first four assessments, according to McNemar tests (i.e. from the first assessment of Reception until the first assessment of the Year 1 group: binomial test, $p < 0.001$; binomial test, $p < 0.001$; binomial test, $p < 0.05$; binomial test, $p < 0.05$, respectively).

Nearly two-thirds of the children passed the hidden addend task at school entry. Thirty-six and 42 children passed the same task on the second and third assessments, respectively. The results approached ceiling-level in the first assessment of the Year 1 group, where 36 children passed the task. Later, in the second and third assessments of the same year group, almost all the children passed the hidden addend task.

The data show that the hidden addend does not represent a major obstacle for children's adding, as they easily revert to a counting-all strategy to count the items in the box. The great majority of children simply asked the experimenter "how many inside?" and tapped that number on top of the box. More importantly, the results support the argument that the situation plays an important role in the teaching of children's understanding of specific mathematical concepts, such as addition (Vergnaud, 1982; Hughes, 1986; Nunes and Bryant, 1996).

5.6.3.4 Multiplication

The results of the multiplication tasks (isomorphism of measures and relative values) are shown in Table 23. In order to pass these tasks, children had to respond correctly to one item at least. McNemar tests showed significant differences between the results of these two tasks in eight out of nine assessments; i.e. the second and third assessment of the Reception group (binomial test; $p < 0.01$; binomial test; $p < 0.001$), in all the assessments of the Year 1 group (binomial test; $p < 0.01$), and the three assessment of the Year 2 group (binomial test; $p < 0.001$; binomial test; $p < 0.01$; binomial test; $p < 0.05$). This suggests that these tasks assess different aspects of multiplication.

Isomorphism of measures problems

Results in this task were similar to those obtained by the same children in the addition and subtraction tasks. McNemar tests showed no significant differences between the results on addition and multiplication tasks in eight out of nine assessments. The exception was the third assessment of the Year 1 group (binomial test; $p < 0.05$).

About one third of the children passed this task at school entry, and 20 and 28 children also passed it in the second and third assessments of the Reception group. Over one-half the children sampled passed the multiplication problems by the end of the Reception Year.

TABLE 23
Frequencies (and percentages) of success in the arithmetical operations tasks - component 3

<i>assessments</i>	<i>Reception (N=53)</i>			<i>Year 1 (N=41)</i>			<i>Year 2 (N=58)</i>		
	<i>one</i>	<i>two</i>	<i>three</i>	<i>one</i>	<i>two</i>	<i>three</i>	<i>one</i>	<i>two</i>	<i>three</i>
<i>Addition (box)*</i>	30 / 58 <i>a</i>	36 / 69 <i>a</i>	42 / 81 <i>a</i>	36 / 88	35 / 90 <i>b</i>	38 / 95 <i>a</i>	55 / 95	53 / 96 <i>c</i>	56 / 98 <i>a</i>
<i>Multiplication†</i>	18 / 34	20 / 39 <i>b</i>	28 / 55 <i>b</i>	29 / 71	32 / 82 <i>b</i>	30 / 86 <i>e</i>	43 / 75 <i>a</i>	47 / 89 <i>d</i>	50 / 89 <i>b</i>
<i>Relative values</i>	10 / 20 <i>c</i>	9 / 17	14 / 27 <i>a</i>	17 / 41	22 / 54	25 / 63 <i>a</i>	33 / 58 <i>a</i>	46 / 79	46 / 81 <i>a</i>

a one subject missing
b two subjects missing
c three subjects missing
d five subjects missing
e six subjects missing

* addition in box with one hidden addend
† isomorphism of measures

About two thirds of the children passed this task by the first assessment of the Year 1 group and nearly 90% of these children also passed the task by the third assessment of the Year 1 group. Results reached ceiling level by the second assessment of the Year 2 group. The data support the view that addition and multiplication are synchronous operations, as suggested by several authors (Piaget, 1952; Carpenter et al., 1993; Nunes and Bryant, 1996).

Relative values problems

Twenty percent of the children at school entry pass this task. These numbers increase slightly to nearly thirty percent, by the third assessment of this group. About forty percent of the children in the first assessment of the Year 1 group pass this task, and these numbers increased to about two-thirds, by the final assessment of this year group.

5.6.3.5 Use of the counting-on strategy

The use of the counting-on strategy was assessed in five different word-problem situations, including addition, subtraction, hidden addend, inversion and multiplication problems. The results (seen in Table 24) show that the development of counting-on is slower, compared with the same children's attainment in the other tasks.

TABLE 24

Frequencies (and percentages) of use of the counting-on strategy - component 3

assessments	Reception (N=53)			Year 1 (N=41)			Year 2 (N=58)		
	one	two	three	one	two	three	one	two	three
Any of five*	6 / 11	15 / 28	16 / 30	14 / 34	12 / 29	21 / 51	30 / 52	44 / 76	46 / 79
Addition box†	3 / 6	7 / 14	13 / 25	9 / 22	10 / 24	18 / 44	28 / 48	41 / 71	43 / 74
Addition**	1 / 2	5 / 9	7 / 13	7 / 17	3 / 7	7 / 17	12 / 21	28 / 48	32 / 55
Subtraction‡	3 / 6	7 / 13	5 / 9	0 / 0	1 / 2	6 / 15	1 / 2	23 / 40	26 / 45
Multiplication††	2 / 4	4 / 8	5 / 9	2 / 5	3 / 7	3 / 7	11 / 19	20 / 34	24 / 41
Inversion***	0 / 0	0 / 0	3 / 6	1 / 2	1 / 2	3 / 7	8 / 14	16 / 28	12 / 21

a one subject missing
b two subjects missing

* In any of the five word problems assessed
† addition in a box with one hidden addend
** change increase result-unknown word problems (e.g. $5+3=?$)
‡ change decrease result-unknown word problems (e.g. $5-3=?$)
†† Isomorphism of measures
*** increase and decrease start-set unknown word problems ($?+3=8$)

Only 6 children used counting-on at school entry. Display of this specific strategy rose to 15 and 16 children in the second and third assessments of the Reception group (N=61). Fourteen children in the Year 1 group (N=41) used this strategy in the first assessment, 12 did the same in the second assessment, and 21 did so in the third assessment.

Thirty children (about one-half) in the Year 2 group (N=58) counted-on in the first assessment. By the second assessment of this year group, the majority of these children used counting-on in word-problems (i.e. 44 and 46 in the second and third assessments, respectively).

The data show that children use less counting-on in any of the word-problem tasks than in the addition with one hidden addend task.

5.6.4 WRITTEN NUMBERS AND KNOWLEDGE OF PLACE VALUE

5.6.4.1 Production and recognition of single-digit numbers

The results of children's single numbers (production and recognition) are shown in Table 25. By the third assessment of the Year 2 group, still some children could not write all the digits 2, 3, 4, 7 and 9. Whereas in the Reception group children fail for not knowing how to write some of the digits assessed, by the Year 2 children err by inverting the numbers (i.e. writing "S" for a "2"). On the other hand, 86% of the children by the third assessment of Reception group seemed to recognise all single-digits assessed.

About half the children could write all the single digits by the beginning of Year 1. Results show an effect of number size. Single digits are comparatively easier than 2-digit numbers across all assessments in the three year groups. McNemar tests show that differences are significant in seven out of nine assessments (binomial $p < 0.01$ for the first seven assessments).

TABLE 25

Means (and standard deviations) of success in the written number and number recognition task (single-digits)

assessments	Reception (N=53)			Year 1 (N=41)			Year 2 (N=58)		
	one	two	three	one	two	three	one	two	three
written numb.	1.9 / 1.7	2.6 / 1.5	2.9 / 1.6	3.8 / 1.4	3.9 / 1.4	3.9 / 1.3 ^b	4.3 / 1.2	4.4 / 1.3	4.4 / 1.1
recognition	—	4.7 / 0.8 ^a	4.8 / 0.4 ^c	—	5.0 / 0.0 ^a	5.0 / 0.0 ^f	—	4.9 / 0.1	4.8 / 0.9 ^e
	^a one subject missing						^e seven subjects missing		
	^b three subjects missing						^f fifteen subjects missing		
	^c four subjects missing								

Children performed better at recognising numbers, than in writing them. Results between the written numbers and the number recognition tasks differed significantly in all the assessments, according to Wilcoxon matched-pairs signed-ranks tests ($Z=-5.8$, $p<0.001$ and $Z=-5.2$, $p<0.001$ in the second and third assessment of the Reception group; $Z=-4.0$, $p<0.001$ and $Z=-3.4$, $p<0.001$ in the second and third assessments of the Year 1 group; $Z=-3.5$, $p<0.001$; $Z=-2.3$, $p<0.05$ for the second and third assessments of the Year 2 group.

5.6.4.2 Production and recognition of multi-digit numbers

The results of the *production of written multi-digit numbers* task are shown on Table 26. In order to pass children had to respond correctly to one out of two items in each category. The data shows that there is a size effect between numbers in the teens and 2-digit categories. McNemar tests showed significant differences between these

categories in all assessments of the Reception and Year 1 groups (binomial; $p < 0.01$ for all). The same significant differences were found between the 2 and 3-digits categories (binomial; $p < 0.01$ in seven out of nine assessments), and between the 3 and 4-digit categories in the last four assessments (binomial; $p < 0.01$); the differences among the first five assessments were not significant).

Twelve children were able to write numbers in the teens by school entry (first assessment of Reception group). Numbers increased slightly by the final assessment of this group, where nearly half of the children passed this task. More children passed the teens task compared with the 2-digit task in the majority of the assessments.

TABLE 26

Frequencies (and percentages) of success in the written numbers task (multi-digits)

assessments	Reception (N=53)			Year 1 (N=41)			Year 2 (N=58)		
	one	two	three	one	two	three	one	two	three
Teens	12 / 23	14 / 26	25 / 47	25 / 61	30 / 73	30 / 79 ^a	46 / 79	50 / 86	51 / 88
2-digits	0 / 0	3 / 6	7 / 13	18 / 44	22 / 54	24 / 63 ^a	44 / 76	45 / 78	45 / 78
3-digits	0 / 0	0 / 0	1 / 2	0 / 0	2 / 5	4 / 11 ^a	12 / 21	18 / 31	22 / 38
number 200	0 / 0	0 / 0	4 / 8	1 / 2	19 / 46	21 / 55 ^a	42 / 72	39 / 67	42 / 72
4-digits	0 / 0	0 / 0	0 / 0	0 / 0	0 / 0	0 / 0 ^a	8 / 14	10 / 17	15 / 26

^a one subject missing

Children were able to write 3-digit numbers only from the second assessment of the Year 1 group onwards. Only 5% of the children succeeded at this stage. Only children from the Year 2 group could write 4-digit numbers. Children found it much easier to write the control number (200): results were similar to those obtained in the 2-digit task. These data support the view that children's difficulties in writing larger numbers are not only a function of size, but mostly a function of the complexity of the units involved in that number.

TABLE 26A

Frequencies (and percentages) of success in the number recognition task (multi-digits)

assessments	Reception (N=53)			Year 1 (N=41)			Year 2 (N=58)		
	one	two	three	one	two	three	one	two	three
Teens	—	31 / 60 <i>a</i>	33 / 67 <i>c</i>	—	35 / 88 <i>a</i>	23 / 88 <i>e</i>	—	54 / 95 <i>a</i>	46 / 90 <i>d</i>
2-digits	—	15 / 29 <i>a</i>	16 / 32 <i>b</i>	—	29 / 73 <i>a</i>	19 / 73 <i>e</i>	—	49 / 86 <i>a</i>	44 / 86 <i>d</i>
3-digits	—	0 / 0 <i>a</i>	5 / 10 <i>b</i>	—	9 / 23 <i>a</i>	7 / 27 <i>e</i>	—	32 / 56 <i>a</i>	35 / 70 <i>d</i>
number 200	—	6 / 12 <i>a</i>	12 / 24 <i>b</i>	—	25 / 63 <i>a</i>	19 / 73 <i>e</i>	—	47 / 82 <i>a</i>	45 / 88 <i>d</i>
4-digits	—	0 / 0	1 / 2	—	1 / 3 <i>a</i>	0 / 0 <i>e</i>	—	13 / 23	16 / 32

- a* one subject missing
- b* three subjects missing
- c* four subjects missing
- d* seven subjects missing
- e* fifteen subjects missing

Regarding the recognition of numbers, shown on Table 26A, children found it much easier to recognise multi-digits. Results were consistently higher in the recognition tasks in all assessments of all categories. McNemar tests showed that differences were significant in the vast majority of cases (binomial; $p < 0.01$).

5.7 THE EMERGENCE OF ADDITIVE COMPOSITION OF NUMBER AND CHILDREN'S UNDERSTANDING OF THE DECADE NUMERATION SYSTEM - THE EFFECT OF CONTINUATION OF COUNTING ³

5.7.1 - Introduction

Children's understanding of additive composition of number implies knowledge that numbers are compositions of other smaller numbers and that any number can be composed by ones, tens, hundreds and so on. This notion, which is thought to form a conceptual base for the development of children's elementary arithmetic and their understanding of the decade numeration system (Resnick, 1983; 1986; Carraher, 1985; Nunes and Bryant, 1996), is nevertheless difficult to assess. The main obstacle has been inferring how widespread and consistent this knowledge might be. In the case of younger children, researchers have opted for the assessment of informal versions of additive composition before formal schooling - i.e. those that can be used but not explained.

In this context, two different types of tasks have been proposed to estimate children's knowledge of additive composition: their ability to solve start-set-unknown word-problems (Resnick, 1983) and their ability to combine coins of different denominations (Carraher, 1985; Carraher and Schliemann, 1990; Nunes and Bryant, 1996). There is, however, little data relating estimates of when children show some understanding of additive composition (Resnick, 1983; 1986; Carraher, 1985; Nunes and Bryant, 1996).

³ - The study reported in this section has been published in Educational Psychology (The Emergence of Additive Composition of Number, Martins-Mourao and Cowan, 1998).

Resnick's (1983) assessment assumes that children's ability to interpret word-problems in terms of part-whole is good evidence that they have informally understood additive composition. When children are given a typical start-set-unknown problem like 'Paul has some; Charles gave him 5; now he has 8. How many did he have to start with?', they have difficulties in representing an undefined start-set with their fingers or the number-line.

Alternatively to the use of the number-line, which only allows children to relate numbers as larger or smaller, they may revert to a part-whole schema to map the problem (Riley and Greeno, 1988). By mapping the problem, they may interchange the different sets in order to find a solution. To rearrange the quantities is considered to depend on an understanding of additive composition (Resnick, 1983).

From a different perspective, Carraher (1985) and Nunes and Bryant (1996) proposed to estimate children's understanding of additive composition through their ability to combine coins of different denominations, i.e. ones and tens. In a typical item of this task - presented as a shopping situation -, the child is given three 10p coins and eight 1p coins to pay for an item costing 16p. To pass this task requires *decomposing* the total amount to be paid into one unit of 10 and several ones, and *composing* the quantity from units of different denominations. Carraher (1985) and Nunes and Bryant (1996) reported that some five and six year-olds succeed in this task.

Finally, Nunes and Bryant (1996) also suggested that

"children's encounters with addition might be the necessary experience for understanding the additive composition that underlies the decade system" (Nunes and Bryant, 1996; p. 52).

According to them, children's use of the counting-on strategy - a particular procedure used in addition word-problems - could be the crucial step for their success in combining coins of different denominations and, therefore, an earlier measure of their informal understanding of additive composition.

Children who can count-on will solve the problem "5+3" by using the total number of the first set as a starting point; i.e. 5... 6, 7, 8. This is an advance over counting-all, i.e. 1, 2, 3, 4, 5 (1st) ...1, 2, 3 (2nd)... 1, 2, 3, 4, 5, 6, 7, 8 (both). This developmental change could be relevant to the understanding of units of different denominations: the child who sees no need to re-count the first addend

"may have realised that this set can be treated as a larger unit which can be combined with a smaller one" (Nunes and Bryant, 1996; p. 53).

This child, they argue, by transposing knowledge from one situation to another, may be in a better position to understand that the number 16 is composed by the combination of a unit of a larger denomination (i.e. one ten) with six units of a single denomination (i.e. six ones).

No data have been produced about the relation between performances in start-set-unknown problems and the shopping task, in the same children - nor about the ability to count-on and solving start-set-unknown problems. In fact, the only exception to the sparse data made available on the relation between these three tasks is Nunes and Bryant's (1996) suggestion that children need to have discovered counting-on in addition problems before they are able to combine units of different denominations.

These authors reported data from an unpublished study by Kornilaki (1994; cited in Nunes and Bryant, 1996), showing that all the 5 and 6 year-olds who passed the shopping task also passed a hidden addend task. This task used a wallet, where the first addend of the problem was hidden. While some may pass this task by counting-on others succeeded by counting-all. Thus passing the hidden addend task is not by itself clear evidence that counting-on is a necessary condition to pass the shopping task.

Evidence that children may be able to use counting-on but may fail to display it (e.g. Carpenter and Moser, 1983; Siegler and Jenkins, 1989) requires modifications in the way this strategy has been assessed. The problem is that counting-on in the context of word problems can only be evaluated indirectly: what is being assessed is the child's ability to solve a problem and not the choice of strategy, which is free.

Nevertheless, more confident results about the relation between counting-on and the ability to combine units of different denominations should be obtained by expanding the assessment of counting-on in the same children to several situations simultaneously, i.e. addition, subtraction, inversion and multiplication word-problems besides the hidden addend task. Failing to use counting-on in any of these tasks will be a stringent criteria to define display of this strategy.

Another possibility, however, is that children may not depend on counting-on to combine coins of different denominations, depending instead on an earlier skill known as continuation of counting (Martins-Mourão and Cowan, 1997). Continuation of counting is a counting skill that enables children to count up from an arbitrary number in the count list, and is displayed by 3 and 4 year-olds who are not yet proficient in addition.

According to Secada et al. (1983), continuation of counting is a subskill of counting-on. When children are asked to buy an item costing 16p in the shopping task and pass by counting: 10 ... 11, 12, 13, 14, 15, 16, it is not clear whether they are counting-on or simply continuing counting. However, evidence that children fail to display counting-on but pass the shopping task would support the view that children rely on continuation of counting to pass this task.

There are other reasons to believe that counting-on may not play a relevant role in children's ability to combine units of different denominations: one, is the view that the use of counting-on may be restricted to solving word-problems (Davydov, 1969). The other, is that it foremost serves to save the child cognitive effort but does not reflect, in itself, a conceptual development (Baroody and Gannon, 1984).

In view of the evidence that some children may be able to use the counting-on strategy but may not always display it (e.g. Carpenter and Moser, 1983) the present study investigated whether different situations have an impact on the child's use of this strategy and whether this influence remains constant over time. Can some situations elicit more use of counting-on than others ?

Also, by assessing the performances of the same children in start-unknown problems and the shopping task, along one school year, this study attempted to compare the results of these two different estimates of additive composition. Are they equally difficult ?

It finally explored Nunes and Bryant's (1996) suggestion that knowledge of addition, and specifically the use of counting-on is a necessary condition for the understanding of additive composition of number. Can children who count-on benefit from this knowledge to grasp additive composition in a different context ? Or, can the understanding of additive composition be developed from the use of continuation of counting, an earlier skill developed before knowledge of addition ?

5.7.2 - Display of counting-on in different word-problem situations

No significant gender differences were found in any of the tasks assessed. So for the subsequent analyses data from boys and girls were combined. Children's strategies used in the addition with one hidden addend and the word-problem tasks [i.e. increase change result-unknown problems (addition) and decrease change result-unknown problems (subtraction), change start-set unknown problems (inversion) and multiplication] were classified into 'does not use count-on', or 'uses count-on'.

No differences were found between the increase and decrease items on the change result-unknown problems or the change start-set unknown problems, so these results were combined. The frequencies of use of counting-on in word-problem situations are shown in Table 27. Numerous children passed the hidden addend task in particular,

by using counting-all only, which shows that use of counting-on cannot be assumed from success in this task alone. Patrick's protocol shows an example of counting-all use in the hidden addend task.

Exp: Inside the box there are 5 bricks and these [3] are outside. How many bricks are there altogether ?

Patrick (4, 10): How many here ? (pointing at box).

Exp: Five.

Patrick: 1, 2, 3, 4, 5 (tapping each finger on top of the box). I'm going to leave my hand here (on top of the box) ... 1, 2, 3 (counting the bricks outside) ... looks at hand on the box and counts) 1, 2, 3, 4, 5 ... 6, 7, 8. Eight!

Katie's protocol shows an example of counting-on use.

Exp: Inside the box there are 9 bricks and these [3] are outside. How many bricks are there altogether ?

Katie (6, 11): nine (taps on the box quickly), 10, 11, 12. Twelve!

In the first assessment, forty children used counting-on, but a further ninety-two children passed the counting-on task by using counting-all only (these numbers are not shown in Table 1). In the second assessment, fifty-eight children used counting-on and another seventy-six used counting-all. In the final assessment, seventy-four children used counting-on and sixty-one used counting-all.

To compare the frequency of counting-on use in the four tasks (i.e. hidden addend, change result unknown, multiplication and change start unknown), a *Cochran Q* test was conducted for each assessment. There were overall significant differences in the first ($Q= 61.9$ $df=4$, $p<.001$); second ($Q= 60.6$, $df=4$, $p<.001$); and third assessments ($Q= 96.6$, $df=4$, $p<.001$).

As Table 27 shows, children were most likely to use counting-on in the hidden addend task, although some of them used counting-on at least once in any of the other tasks, except hidden addend. McNemar tests confirmed that children were reliably more prone to use counting-on in the hidden addend task than in the change result-unknown problems: $X^2 = 9.6$, $df=1$, $p<.01$; $X^2 = 5.6$, $df=1$, $p=.02$; $X^2 = 10.6$, $df=1$, $p<.01$; in the first, second and third assessments respectively.

Children were less likely to use counting-on in start-unknown problems. McNemar tests show significant differences in start-unknown and multiplication problems in the second (binomial test, $p<.05$) and third assessment (binomial test, $p<.01$).

TABLE 27

Frequencies of counting-on use in each task by assessment (N=152)

<i>Assessment</i>	<i>Word problems</i>				At least once
	Hidden addend	Change result unknown	Multipli- cation	Change start unknown*	
First	40	22	15	9	50
Second	58	43	27	17	71
Third	74	54	32	18	83

* *Part-whole word problems*

While there was a general increase in the use of counting-on in all the word-problem tasks, it was more marked in some situations. McNemar tests show that the number of children using the counting-on strategy in the hidden addend task grew significantly from the first to the second ($X^2 = 10.5$, $df=1$, $p<.01$), and from the second to the third assessments ($X^2 = 7.5$, $df=1$, $p<.01$).

The same pattern was found in the change result-unknown problems from the first to the second assessment ($X^2 = 5.9$, $df=1$, $p<.05$) and from the second to the third assessment (*binomial test*, $p<.05$). In the multiplication task, there was a significant increase from the first to the second assessment only ($X^2 = 4.7$, $df=1$, $p<.05$). No significant increase was found in the development of children's counting-on use in start-unknown problems.

5.7.3 - Performance in the additive composition tasks

The frequencies of success in the start-set unknown problems and the shopping task are shown in Table 28. Within the start-set unknown problems, McNemar tests

showed no significant differences between the increase and decrease versions, so these were combined. In the shopping task, no differences were also found between success on the 5&1 and the 10&1 items; only the latter are displayed in Table 28. Children were more likely to pass the shopping task in each assessment; first ($X^2 = 16.5$, $df=1$, $p<.001$), second ($X^2 = 23.8$, $df=1$, $p<.001$) and third ($X^2 = 17.5$, $df=1$, $p<.001$).

TABLE 28

Frequencies of success in the additive composition tasks and its precursors (N=152)

<i>Assessment</i>	<i>Additive Composition tasks</i>		<i>Precursors</i>	
	Shopping task	Start-unknown problems	Continuation of counting	Counting On
First	69	45	108	50
Second	87	54	127	71
Third	99	74	139	83

There was a general increase in the success on both tasks, although it was more marked in the shopping task. Results in the shopping task differed significantly from the first to the second ($X^2 = 11.1$, $df=1$, $p<.001$), and from the second to the third assessment (*binomial*, $p=.02$). However, changes in success in the start-unknown problems were only marked later, from the second to the third assessment ($X^2 = 10.2$, $df=1$, $p<.01$).

5.7.4 - *The relationship between counting-on, continuation of counting and additive composition*

To assess the status of counting-on and continuation of counting as precursors to additive composition I examined the relation between use of the counting-on strategy on any of the word-problem tasks, and success in continuation of counting and success on the two measures of additive composition. The frequencies of success in these tasks are shown in Table 28. Results show that more children were able to succeed in the shopping task than to display counting-on. According to McNemar tests, the differences were significant in the first ($X^2 = 9.3, df=1, p<.01$), second ($X^2 = 5.9, df=1, p=.02$), and third assessments ($X^2 = 6.6, df=1, p=.01$).

The data show that children used two different strategies to pass the shopping task. One group counted all the units in the coin of higher denomination and then continued counting the remaining units - i.e. in the 10p + 6p item, children would count: 1, 2, 3 ... 8, 9, 10 (10) ... 11, 12, 13, 14, 15, 16 (16). Another group of children counted 10, 11, 12, 13, 14, 15, 16, for instance. However, evidence that eleven out of sixty-nine children (in the first assessment), fourteen out of eighty-seven (in the second assessment) and three out of ninety-nine (in the third assessment) passed the shopping task by counting-all, supports the argument (1) that counting-on is not a necessary condition for success in the shopping task and that, for this reason, (2) the other successful strategy used by children must be continuation of counting - rather than counting-on.

Other results further support the view that it is possible to pass additive composition tasks, without knowledge of counting-on. Children were divided into two groups: those who displayed counting-on, and those who never displayed it. The same

children were further divided into those who used continuation of counting, and those who did not. The crosstabulations of these four groups with the results in the additive composition tasks are shown in Table 29.

The data show that a considerable number of children were able to pass either the start-unknown problems or shopping task, having failed to display counting-on in any of the five situations assessed in the previous section. According to Table 29, fourteen, ten and fifteen children passed the start-unknown problems having failed to count-on (in the first, second and third assessments respectively). Also, twenty-seven (in the first and second assessments) and twenty-five children (in the third assessment), passed the shopping task having failed to count-on. Furthermore, seven, nine and one child who failed to count-on in any of the word-problem situations, passed the shopping task by using the counting-all strategy.

TABLE 29

Relation between counting-on and continuation of counting and additive composition tasks. Results are in frequencies (N= 152) (from left to right, numbers correspond to assessment one through three)

		<i>Additive composition tasks</i>											
		Start-unknown problems						shopping task					
		fail			pass			fail		pass			
counting ON	no	88	71	54	14	10	15	75	54	44	27	27	25
	yes	19	27	24	31	44	59	8	11	9	42	60	74
continuation of counting	no	44	25	14	0	0	0	44	25	14	0	0	0
	yes	63	73	64	45	54	74	70	57	61	38	70	77

On the other hand, the data presented in Table 29 show that no children were able to pass any of the additive composition tasks without also passing the continuation of counting task. This evidence supports the idea that children's grasp of additive composition of number - in any of the two forms assessed - presupposes their ability to continue counting from an arbitrary number in the list, but not their ability to count-on.

Regarding the relation between continuation of counting and counting-on, no children who failed to continue counting were able to use counting-on (with one exception in the first and second assessments). This data supports Secada et al's. (1983) model suggesting that the former is a subskill of the latter.

TABLE 30
Significant relationships between counting-on and continuation of counting and additive composition tasks

	<i>Additive composition tasks</i>	
	Start-unknown problems	Shopping task
<i>Counting-on</i>		
Assessment 1	n.s.	X ² =9.3; df=1; p>.01
Assessment 2	X ² =6.9; df=1; p>.01	X ² =5.9; df=1; p>.05
Assessment 3	n.s.	X ² =6.6; df=1; p=.01
<i>Continuation of counting</i>		
Assessment 1	X ² =61.01; df=1; p>.001	X ² =33.6; df=1; p>.001
Assessment 2	X ² =71.01; df=1; p>.001	X ² =36.2; df=1; p>.001
Assessment 3	X ² =62.01; df=1; p>.001	X ² =35.2; df=1; p>.001

Table 30 compares the significance of the relationship between both counting-on and continuation of counting, and the additive composition tasks. According to McNemar tests, the relationship between continuation of counting and start unknown problems is

significant in three out of three assessments, whereas the relation between counting-on and start-unknown problems is significant in only the second assessment.

The relations between counting-on and continuation of counting and the shopping task are both significant, although they are more significant between continuation of counting and the shopping task, according to the values of X^2 as well as the p values -shown on the right-hand column of Table 30.

5.7.5 - Predictive effects of continuation of counting

To explore the effect of continuation of counting (CC) on the additive composition tasks, high crosslag correlations (Spearman) were performed. Results show significant correlations between continuation of counting in the first assessment and the shopping task ($r=0.6$, $p<0.001$) and start-unknown problems ($r=0.5$, $p<0.001$) in the third assessment. This suggests that those children who understand continuation of counting earlier are in a better position to grasp additive composition of number, later.

Table 31 crosstabulates the results of continuation of counting in previous assessments, with the results of the additive composition tasks, in later assessments, i.e. from the first to the second assessment, from the first to the third, and from the second to the third assessments.

Table 31 shows that, with rare exceptions, children who failed continuation of counting in the first assessment could not pass the additive composition tasks in the

second and third assessments. The few children that failed to continue counting in the first assessment but passed the start-unknown problems or the shopping task on the third assessment, all learned to continue counting sometime in between these assessments. The same pattern occurs between the second and the third assessment. These data support the argument that continuation of counting is a necessary condition for children's understanding of additive composition of number.

TABLE 31
Relation between counting-on, continuation of counting and additive composition tasks across different assessments. Results are in frequencies (N= 152)

		Additive composition tasks							
		<i>Start-unknown problems</i>				<i>shopping task</i>			
		ass 2		ass 3		ass 2		ass 3	
		fail	pass	fail	pass	fail	pass	fail	pass
counting-on (ass 1)	no	82	20	68	34	60	42	51	51
	yes	16	34	10	40	5	45	2	48
counting-on (ass 2)	no			58	23			49	32
	yes			20	51			4	67
continuation of counting (ass 1)	no	42	2*	41	3**	40	4†	36	8††
	yes	56	52	37	71	25	83	17	91
continuation of counting (ass 2)	no			24	1			23	1
	yes			54	73			30	98

* these children displayed continuation of counting (CC) in the second assessment

** two of these children displayed CC in ass 2; the remaining child displayed CC in ass 3

† three displayed CC in ass 2

†† seven displayed CC in ass 2. One displayed CC in ass 3

On the other hand, numerous children who failed to count-on in previous assessments, pass the additive composition tasks in later assessments. One such example is the relation between counting-on in the first assessment and success in the shopping task

in the third assessment: of the one hundred and two children who failed to count-on (first assessment), half (fifty-one) failed the shopping task and the other half passed.

5.8 THE EFFECTS OF CHILDREN'S USE OF CONTINUATION OF COUNTING ON THEIR UNDERSTANDING OF THE ARITHMETICAL OPERATIONS

5.8.1 Introduction

There is evidence that the early counting strategy that enables children to count from an arbitrary number (i.e. continuation of counting) is more important than it has been formerly realised in previous research. It seems, therefore, worthwhile to investigate the relation between continuation of counting and counting strategies used in the word-problems.

If, as suggested by Secada et al's (1983) model, continuation of counting is a subskill of counting-on, then, all children who display counting-on must also display continuation of counting. On the other hand, data showing that children who fail to continue counting are nevertheless able to use counting-on will be inconsistent with Secada et al.'s model, and will suggest that continuation of counting is not a necessary condition for the use of counting-on. Data showing that children either fail both tasks, or pass continuation but fail to count-on, or pass both tasks, will be consistent with Secada et al's. prediction.

Data showing that children who fail to continue counting can nevertheless pass the operations tasks, will be consistent with the argument that continuation of counting is not a necessary condition for the understanding of the operations. However, data showing that children who fail continuation of counting do not pass any of the 3 tasks, and all children who pass the 3 tasks also pass the continuation task, support the idea that children's use of continuation of counting is presupposed in their understanding of addition, subtraction and multiplication.

5.8.2 The Relation Between Continuation of Counting and Counting Strategies in used in the word-problems

Children's results on the continuation of counting, addition, subtraction and multiplication tasks were classified as 'fail' or 'pass'. To pass, children had to respond correctly to one item, at least. Children's counting strategies in the addition, subtraction problems were categorised as in the previous section: 'No counting strategies', 'count-all only' and 'count-on'. Children were allowed miscounts of -1 or +1.

The crosstabulation of the continuation of counting task and both the accuracy and counting strategies used in Addition, Subtraction, Addition with one hidden addend, Inversion and Multiplication problems is shown in Table 32.

Regarding the use of strategies, the data shows that across assessments, children of all groups who fail to continue counting do not display counting-on in any of the five tasks, with one exception. This can be seen by crossing the "fail CC" lines with the

"pass C-on" column where only zeros can be found; with the above mentioned exception in the second assessment (subtraction and addition with one hidden addend problems).

TABLE 32
Relation between counting pattern and the accuracy and strategies used in addition, subtraction, addition with hidden addend, Inversion and Multiplication problems (Results are in frequencies)

CC*	assessment one						assessment two						assessment three					
	NCS†		ct-all		ct-ON		NCS†		ct-all		ct-ON		NCS†		ct-all		ct-ON	
	f**	p	f	p	f	p	f	p	f	p	f	p	f	p	f	p	f	p
Addition																		
<i>Change result unknown word-problems (N=152, 152, 152)</i>																		
fail	33	0	5	6a	0	0	21	0	2	1b	0	0	9	0	3	2c	0	0
pass	15	0	6	67	0	20	10	4	5	72	0	36	12	4	4	72	1	45
Subtraction																		
<i>Change result unknown word-problems (N=150, 150, 151)</i>																		
fail	32	3	4	4d	0	0	18	3	0	2c	0	1b	10	0	2	2c	0	0
pass	14	4	4	81	0	4	13	9	6	68	0	30	10	9	4	78	0	36
Addition with one hidden addend box problems (N=145, 151, 148)																		
fail	16	0	9	18e	0	0	9	0	6	8f	0	1	3	0	2	8	0	0
pass	2	1	3	62	0	40	5	2	2	57	0	55	5	6	2	48	1	73
Inversion																		
<i>Change start unknown word-problems (N=152, 152, 152)</i>																		
fail	44	0	0	0	0	0	25	0	0	0	0	0	14	0	0	0	0	0
pass	53	0	10	36	0	9	72	2	1	35	0	17	35	0	29	56	0	18
Multiplication																		
<i>isomorphism of measures word-problems (N=151, 143, 142)</i>																		
fail	33	3	1	6a	1	0	9	0	13	2c	0	0	6	1	5	1b	0	0
pass	14	5	12	62	0	14	5	1	17	70	0	26	12	5	11	69	0	32
* CC- continuation of counting			a four percent of all subjects						e twelve percent of all subjects									
† NCS- No counting strategy			b under one percent of all subjects						f five percent of all subjects									
* f, p - fail, pass			c one percent of all subjects						g two percent of all subjects									
						d three percent of all subjects												

On the other hand, of those children who pass the continuation of counting task, all succeed in the use of the counting-on strategy, with one exception in the addition and

addition with one hidden addend problems (one participant out of 152 represents less than one percent of the sample observed).

The use of continuation of counting also seems to have a significant effect on children's correct use of counting-all to succeed in any of the five tasks, i.e. in the increase of their accuracy in the word-problems. A significantly greater proportion of children who continue counting use counting-all to succeed (these results can be found in the crossing of the "pass CC" lines with the "pass counting-all" column). Only a minority of children who fail to continue counting are able to pass addition by using counting-all. Only 4, 0.6 and 1.3 percent of all the children succeed in doing this in the first, second and third assessments, respectively.

5.8.3 The effect of Continuation of Counting on Children's Knowledge of Addition and Subtraction

The children who passed the word problems without using counting strategies are displayed in the 'No counting strategies' (NCS, p) column. These children used number-facts, either knowingly (see the crossing with the CCpass line) or randomly (see the crossing with the CCfail line). The number of children in the latter situation is quite small and represents under 2% of all children and occur only in the first and second assessment of the subtraction word problems (see Table 32). The number of children passing the word-problems can be found by adding the number of children who pass by counting-all (ct-all) and counting-on (ct-on).

Table 32, which crosstabulates the results of the continuation of counting and addition, subtraction and inversion word-problems (i.e. additive structures), shows that the use of continuation of counting has a significant effect on children's accuracy in any of the above mentioned tasks. In all assessments, the vast majority of children who fail to continue counting are not able to pass any of the addition, subtraction and inversion word-problems. Exceptions can be found but these are never higher than 4% of all children (i.e. six out of 152 children), as it is the case of the six children that pass addition having failed to continue counting.

The data show that there is a significant difference in performance in addition and subtraction in function of the counting pattern displayed (i.e. pass/fail continuation of counting) in all assessments. In the first assessment, only 6 children (under 4%) pass addition having failed to continue counting. In the second and third assessments these percentages drop to one percent on the observed sample of 152 children.

Of the 108 children passing the continuation of counting task in the first assessment, 87 (i.e. 81%) also pass addition. Of the 127 who pass continuation of counting in the second assessment, 112 (i.e. 88%) succeed in addition. Finally, in the third assessment, 121 out of the 138 (i.e. 80%) who succeed in continuation of counting also pass addition. Differences are significant in all assessments (Table 33).

The effect of continuation of counting over performance in subtraction word-problems is also significant in all assessments, according to chi-square tests. Only 4 (3%), 3 (2%) and 2 (1%) children pass subtraction having failed to continue counting in the first, second and third assessments, respectively.

TABLE 33

Crosstabulation of Continuation of counting and the addition, subtraction and multiplication tasks by assessment. Results are presented in frequencies

		assessments					
		first		second		third	
Addition							
<i>Change result unknown word-problems</i>							
CC*	fail	38	6	23	1	12	2
	pass	21	87	15	112	17	121
		59	93	38	113	29	123
		44	108	24	127	14	138
		$X^2=58.9; df=1; p<0.001$		$X^2=75.7; df=1; p<0.001$		$X^2=44.3; df=1; p<0.001$ Fisher's exact test, $p<0.001$	
Subtraction							
<i>Change result unknown word-problems</i>							
CC*	fail	36	7	18	6	12	2
	pass	18	89	19	107	14	123
		54	96	37	113	26	125
		43	107	24	126	14	137
		$X^2=59.6; df=1; p<0.001$		$X^2=38.9; df=1; p<0.001$		$X^2=50.8; df=1; p<0.001$	
Multiplication							
<i>Isomorphism of measures word-problems</i>							
CC*	fail	35	9	22	2	11	2
	pass	26	81	22	97	23	106
		61	90	44	99	34	108
		44	107	24	119	13	129
		$X^2=39.5; df=1; p<0.001$		$X^2=50.2; df=1; p<0.001$		$X^2=28.9; df=1; p<0.001$ Fisher's exact test, $p<0.001$	
Multiplication							
<i>Relative values</i>							
CC*	fail	34	7	23	2	12	1
	pass	54	53	52	75	52	84
		88	60	75	77	64	85
		41	107	25	127	13	136
		$X^2=12.9; df=1; p<0.001$		$X^2=21.8; df=1; p<0.001$		$X^2=14.2; df=1; p<0.001$	

* Continuation of counting

Furthermore, of the total of children who pass the continuation of counting task (108, 126 and 137), 90 (i.e. 83%), 106 (i.e. 84%) and 123 (i.e. 90%) also succeed in the subtraction tasks. These results suggest that there is a relation between continuation of counting and children's performance in addition and subtraction tasks.

A similar pattern can be seen in the relation between continuation of counting and the inversion tasks (change start-unknown word-problems). Children who fail to continue counting do not succeed in the inversion tasks in all assessments. Accuracy also increases for those who pass continuation of counting: all of these children use counting-on. Also, at least three-fourths of those children passing the continuation of counting task will succeed by using a counting-all task.

5.8.4 The effect of Continuation of Counting in Children's Knowledge of Multiplication

The results of the crosstabulation of continuation and multiplication (Table 33) show that across assessments, and with few exceptions, children who fail to continue counting also fail both multiplication tasks. In the first assessment of the isomorphism of measures task (all year groups) only nine children (i.e. 6%) who failed to continue counting, passed the multiplication task. In the second assessment, only two children (1%) passed multiplication and in the third assessment only one child did the same.

Also, of those children passing the continuation of counting task (see row pass CC), 81 (out of 107; i.e. 76%), 97 (out of 119; i.e. 82%) and 106 (out of 129; i.e. 82%) also pass the multiplication task in assessments one, two and three respectively. The data suggest that children's use of continuation of counting is related to their knowledge of multiplication.

The crosstabulation of the results in continuation of counting and the relative values tasks, also shows a similar pattern to the one discussed above. Only a maximum of 7

children (i.e. 5%) fail continuation of counting and pass the relative values task, in the first assessment. These numbers decrease to two and one children, in the remaining assessments.

5.8.5 Summary

This study was set up to explore whether children's use of continuation of counting was related to (1) their use of the counting-on; and (2) their understanding of addition, subtraction and multiplication problems. It was found that continuation of counting is related to all the above mentioned: with rare exceptions, children who fail to continue counting do not use the counting-on strategy nor pass any of the addition, subtraction and multiplication problems.

This data, which is consistent with both Secada et al's. (1983) model and Davydov's (1969) hypothesis, supports the argument that children's specific early number competencies (i.e. their ability to continue counting from an arbitrary number) are related to their knowledge of the operations, as assessed in this study.

As mentioned earlier, correlational studies are limited as they cannot establish causality between variables. However, and considering other data suggesting that there is a developmental gap between continuation of counting and the use of counting-on and knowledge of addition, subtraction and multiplication (Fuson et al., 1982; Siegler and Robinson, 1982; Secada et al., 1983), the evidence presented supports the argument that continuation of counting is a necessary but not sufficient condition for the understanding of the operations.

5.9 THE EFFECTS OF KNOWLEDGE OF THE ARITHMETICAL OPERATIONS ON CHILDREN'S UNDERSTANDING OF THE DECADE NUMERATION SYSTEM

5.9.1 - Introduction

Several authors have suggested that both addition and multiplication are involved in children's understanding of the numeration system which, in turn, involve addition and product relations (e.g. Piaget, 1952; Skemp, 1971; Ross, 1989; Fuson, 1990). Together, they enable faster ways of counting making it quicker and more precise to count by tens or hundreds, than by ones. A child who grasps the meaning of the number 345 will have to understand product relations such as 3×100 (and 4×10), and additive relations such as $40 + 5$. However, the relation between addition, multiplication and children's understanding of the decade numeration system require further clarification.

Are Addition and Multiplication consecutive operations ? It is argued that children's understanding of the numeration system may develop from previous understanding of operations such as addition and multiplication, which are inevitably linked to the structure of the numeration system as the only means to transform number (e.g. Piaget, 1952; Skemp, 1971).

Evidence of a relation between children's knowledge of addition (and multiplication) and their understanding of additive composition should be provided by data showing that children who understand addition and multiplication have a significantly improved

performance in units of different denominations task (shopping task), in comparison with those who do not understand these operations.

The relation between these two tasks (i.e. addition and shopping task; multiplication and shopping task) entails four possibilities: (1) children may fail both tasks; (2) they may pass the addition (or multiplication) task, but fail the additive composition task; (3) they may pass both tasks; and (4) they may fail the addition task and pass the additive composition task. Only two of these possibilities clarifies whether there is a relation between these two variables.

Evidence that children fail the addition task (or multiplication) and pass the additive composition task will show that addition is not related to additive composition. Evidence that only children who pass addition (or multiplication) also pass additive composition will show that there is a relation between these two variables.

Evidence that children pass the addition (or multiplication) task but fail the additive composition task is expected, since it has been argued that knowledge of the former helps children's understanding of the latter but does not determine it. The same rationale applies to the exploration of the relation between multiplication and additive composition task.

5.9.2 - Are Addition and Multiplication consecutive operations ?

The results of the addition and multiplication tasks were classified as pass/fail. To pass, children had to respond correctly to one of the items in the tasks. Children's

responses to the shopping task were also classified into pass/fail. To pass, children had to respond correctly to one of the items in any of the categories of this tasks.

Table 34 presents the frequencies and percentages of correct responses in the addition and multiplication tasks. The data shows that a similar number of children pass the addition and multiplication tasks. McNemar tests confirmed that there were no significant differences between the results in both tasks in eight out of nine assessments. Significant differences were found in the first assessment of the Year 2 group only (binomial test; $p < 0.05$).

TABLE 34
Frequencies (and percentages) of success in the addition and multiplication tasks by assessment of each year group

	Reception (N=53)			Year 1 (N=41)			Year 2 (N=58)		
	<i>ass1</i>	<i>ass2</i>	<i>ass3</i>	<i>ass1</i>	<i>ass2</i>	<i>ass3</i>	<i>ass1</i>	<i>ass2</i>	<i>ass3</i>
Pass addition	13(25)	23(44) ^a	34(64)	29(71)	34(83)	34(85) ^a	51(88)	56(97)	54(93)
<i>significant differences</i>	<i>n.s.</i>	<i>n.s.</i>	<i>n.s.</i>	<i>n.s.</i>	<i>n.s.</i>	<i>n.s.</i>	$p < 0.05^*$	<i>n.s.</i>	<i>n.s.</i>
Pass multiplication	18(34)	20(39)	28(55) ^a	29(71)	32(82) ^b	30(86) ^d	43(75) ^a	47(89) ^c	50(89) ^b

- a* one subject missing
- b* two subjects missing
- c* five subjects missing
- d* six subjects missing
- * binomial test

For a clearer understanding about the developmental relation between addition and multiplication, Table 35 presents a crosstabulation of the results obtained in both tasks. The results show that it is possible to pass multiplication problems having failed the addition items, and *vice versa*. Up to nearly one fifth of children passed addition and failed multiplication in the first assessment. Eleven and seven percent of the children did the same in the second and third assessment, respectively. Likewise, about one fifth of the children were able to pass addition having failed the multiplication tasks, in

the first assessment. Seventeen and thirteen percent of the children were in the same situation in the second and third assessments, respectively.

TABLE 35
Crosstabulation of results of all year groups in the addition and multiplication tasks by assessment.
Results are presented in percentages (N=152)

	assessment 1 (N=151)		assessment 2 (N=143)		assessment 3 (N=142)	
	<i>Multiplication</i>					
		fail	pass		fail	pass
<i>Addition</i>	fail	40	19	fail	27	11
	pass	21	71	pass	17	88
<i>r (Spearman)</i>	0.45; $p < 0.001$		0.53; $p < 0.001$		0.59; $p < 0.001$	
<i>chi-square</i>	30.2; $df=1; p < 0.001$		39.4; $df=1; p < 0.001$		49.9; $df=1; p < 0.001$	

Chi-square tests confirm that children's understanding of addition has a significant influence in their understanding of multiplication (Table 35). Furthermore, the data shows that the results between the two tasks are highly correlated (and significant) in all three assessments.

On the whole, the data does not support Fischbein et al.'s (1985) argument that addition and multiplication are consecutive operations. The data suggests, on the contrary, that children's knowledge of addition and multiplication develop simultaneously, i.e. as synchronous operations as suggested by Piaget (1952) and lately confirmed by Carpenter et. al. (1993).

5.9.3- The Relation between Addition, Multiplication & Additive Composition of Number

Table 36 presents a comparison of frequencies between the addition, multiplication and additive composition of number tasks. The data shows that the additive composition task was more difficult for children. McNemar tests confirmed that differences were significant between addition and additive composition (in five out of nine assessments) and between multiplication and additive composition (in six out of nine assessments).

TABLE 36
Frequencies (and percentages) of success in the Addition, Multiplication and additive composition tasks. Asterisks show significant differences between results, according to McNemar tests

	Assessments								
	Reception (N=53)			Year 1 (N=41)			Year 2 (N=58)		
	one	two	three	one	two	three	one	two	three
Addition	13(25) n.s.	23(44) ***	34(64) *	29(71) **	34(83) **	34(85)a ***	51(88) n.s.	56(97) n.s.	54(93) n.s.
Additive comp.	6(11) **	13(25) ***	20(38) **	18(44) **	23(56) **	28(68) ***	45(78) n.s.	51(88) n.s.	51(88) n.s.
Multiplication	18(34)	20(39)b	28(55)b	29(71)	32(82)b	30(86)d	43(75)a	47(89)c	50(89)b

a one subject missing
b two subjects missing
c five subjects missing
d six subjects missing
 * p<0.001 (binomial test)
 ** p<0.01 (binomial test)
 *** p<0.05 (binomial test)

To further explore the combined relationship of knowledge of addition and multiplication and children's understanding of additive composition of number, the results of these two tasks were combined and classified as "failed all" (i.e. failed both addition and multiplication), "pass addition only", "pass multiplication only" and "pass

both". The results of the additive composition task were categorised as "fail", "pass items in the teens, or under 20", "pass items under 100" and "pass items over 100" (Table 38).

TABLE 38
Crosstabulation of results between addition and multiplication tasks and the Additive composition task by assessment. Results are presented in frequencies.

	<i>Additive composition of number (N=151, 143, 142)</i>											
	assessment one				assessment two				assessment three			
	pass				pass				pass			
	fail	<20	<100	>100	fail	<20	<100	>100	fail	<20	<100	>100
fail all	37	2	1		26	1			21			
pass addition only	13	7		1	13	3	1		8	3	2	
pass multiplication only	13	5	1		6	4		1	6			1
pass both	19	17	8	27	17	25	18	28	17	20	13	51

The results shows that, with some exceptions, children need to have grasped both addition and multiplication in order to succeed in the additive composition. Only 3 children (out of 151; 2%) pass items of the additive composition task in the first assessment. One child also succeeds in the second assessment. Also, partial understanding of addition without multiplication (or vice versa) means limited success in the additive composition task: remarkably, children who succeed in addition or multiplication (but not in both) can only pass items in the "under 20" category. On the other hand, the majority of the children who pass both tasks succeed in the additive composition task: seventy-three, eighty-one and eighty-six percent of these children also succeed in the additive composition task.

5.9.4 Summary

This study was set up to investigate whether children's knowledge of addition and multiplication were related to their understanding of the structure of the decade numeration system, assessed through an additive composition task. It was found that in both cases, the vast majority of children who displayed knowledge of additive composition of number had also passed the addition and multiplication tasks (and, with few exceptions, children who failed any of the two operations could not pass the additive composition task). This evidence supports the argument that children's knowledge of both operations is related to their grasp of the structure of the numeration system. The suggestion that addition and multiplication develop simultaneously, is supported by Piaget's (1952) hypothesis of these as discontinuous operations.

The results suggesting that children's knowledge of addition is related to their understanding of additive composition are not consistent with Kornilaki's (1994) and Nunes and Bryant's (1996) predictions. According to these authors' data, children's knowledge of addition did not relate to additive composition, although their use of counting-on (in particular) did. Their prediction then was that only children who were proficient addition solvers (i.e. used the counting-on strategy) would be in a better position to grasp additive composition.

The present data, which assessed children younger than those observed by Kornilaki (1994), suggests a different picture: count-all children can pass the additive composition task, and therefore, children who pass the addition task pass the additive composition task. Furthermore, and as the data shown in Study 5.7 supports, children's earliest experience with multiunits may be continuation of counting. Finally,

the results of both study 5.8 and study 5.9 suggest that children's understanding of the numeration system relies, at least partially, on the interrelation between specific number competencies and knowledge of the operations.

5.10 THE EFFECTS OF CHILDREN'S UNDERSTANDING THE STRUCTURE OF THE NUMERATION SYSTEM' AND THEIR 'KNOWLEDGE OF WRITTEN NUMBERS' ON USE OF PLACE-VALUE

5.10.1 Introduction

There are two views about children's understanding of place value: one, contends that written numbers and place value are a prerequisite for children's understanding of the structure of the numeration system (Luria, 1969; Kamii, 1986; Bergeron and Herscovics, 1990; Sinclair, et al., 1992; Sinclair and Scheuer, 1993). According to this view, knowledge of place value is a question of conventional representation.

Another view, conversely, suggests that children's understanding of place value does not depend on knowledge of written numbers, but is related to their prior grasp of the structure of the decade numeration system. In order to understand the decade system, children do not need to know about written numbers (e.g. Ginsburg, 1977; Resnick, 1983; Carraher, 1985; Carraher and Schliemann, 1990; Fuson, 1990; Nunes and Bryant, 1996). According to this view, knowledge of place value is a question of conventional abstraction.

The argument that children may base their knowledge of place value (PV) in prior understanding of the structure of numeration system (USNM) implies that only those children who display knowledge of the latter will be able to write larger numbers (where knowledge of place value is required). However, it must be noted that not all children who have grasped the structure of the system will necessarily know the convention of place value. Furthermore, those children without insights about the structure of numeration system are expected to make errors while writing numbers, such as assembling numbers into a large string of digits, as they hear them. For instance, number 'one hundred and twenty five' should be written as 100205.

On the other hand, if children learn about the structure of the numeration system at the same time as they learn to write larger numbers and begin understanding place value, then those children who display knowledge of the structure of the numeration system should have no advantage in their ability to write larger numbers, compared to other children who have not yet grasped this structure. Meanwhile, it is important to clarify whether children are nevertheless able to write large multiunits *per se*, independently of their understanding of both place value and the structure of the Numeration System. Examples of numbers that can assess this type of knowledge are 40 (in the decades) and 200 (in the hundreds). If children are able to write 200 correctly but still make errors such as 100204 this suggests that they have learnt how to write some numbers without understanding the system.

In summary, the relation between the understanding of the structure of the numeration system (USNS) and use of place value (PV) implies two possibilities. On the one hand, if children acquire USNS from experience with PV, then children who know about PV should present significantly better results in the USNS task.

If children learn PV from USNS, then children who have understood USNS should present significantly better results in the PV task, compared with those who have failed the USNS task. It should be noted, however, that not all children who pass the USNS task will necessarily pass the PV task (Carragher, 1985; Nunes and Bryant, 1996).

5.10.2 Types of Responses

The written numbers and number recognition tasks included one control item for the 3 and 4-digit items (i.e. 200). It is worth recalling that although 200 is a "big number" (from the child's point of view), it does not require a grasp of units of different size in written form. Children's success in writing this number correctly will support the argument that children's difficulties are not related to the number of digits, but to the complexity of the units involved in its writing (Nunes and Bryant, 1996).

The numbers used in both number tasks were in most cases equal to those used in the additive composition task. The only exceptions were the numbers used in the 2-digits category. The additive composition task used numbers 26 and 53 and the written numbers and number recognition tasks used numbers 37 and 79.

Children' results were categorised into "teens" (12 and 15), "2-digits" (37 and 79), "3-digits" (124 and 347) and "4-digits" (1052 and 2340). To pass the categories children had to write correctly one of the items, at least. Furthermore, children's answers to the written numbers task were classified into: **(1) no answer** (does not know; scribbles; tallies); **(2) incorrect** (used correct single-digits placed incorrectly, e.g. wrote 73 for 37); **(3) assembles** (units are assembled side-by-side disregarding place-value; e.g. wrote 124 as 100204), and **(4) correct response** (wrote the correct

number taking place-value into account). The responses to the number recognition task were classified into 'pass' or 'fail'.

Table 38 shows that there is an effect of number size on children's refusals (no answer) across assessments; i.e. children tend to refuse more often the larger numbers than the smaller ones. The exception to this pattern are the responses given to the control items, which are easier for children to respond to.

The gap between the control item '200' and the rest of the categories widens along the several assessments. This data supports Nunes and Bryant (1996) prediction that numbers which do not combine units of different sizes are easier for children to write. The percentage of incorrect answers was generally low with the exception of the 'teens' category, which suggests that children tend not to take risks in writing numbers, unless they have some idea of what the correct result might be.

On the other hand, the data show that in all assessments children never (or practically never) attempt to assemble (e.g. write 204 for number 24) two-digit numbers. Finally, children seem to respect place-value in 2-digit numbers well before they master it in 3 and 4-digit numbers. This supports the idea that children may use different strategies to write two-digit numbers and numbers with three or more digits, as predicted by Bergeron and Herscovics (1990).

Regarding the use of place-value, the data suggest that children seem to have a 'head start' on the 'teens' and '2-digits' categories, but significantly more difficulties with the 3 and 4-digits. However, across assessments, the pattern of development seem to be similar amongst all categories. The data shown also suggest that the majority of the children of all age groups assessed go through a stage of "assembling" before being

able to use place value. The data show that children make this mistake more than any other one.

TABLE 39
Percentages of types of responses in the written numbers task by category, year group and assessment

	<i>Assessments</i>														
	<i>first</i>					<i>second</i>					<i>third</i>				
	teens	two digits	three digits	four digits	200	teens	two digits	three digits	four digits	200	teens	two digits	three digits	four digits	200
Reception (N=53)															
No answer	43	100	100	100	100	26	59	100	100	100	13	43	74	100	61
Incorrect	32	0	0	0	0	38	19	0	0	0	34	19	8	0	11
Assembles	0	0	0	0	0	0	0	0	0	0	0	0	17	0	17
Correct	25	0	0	0	0	36	23	0	0	0	53	38	2	0	11
<i>Correct answers (number recognition)*</i>						60	29	0	0	12	67	32	10	2	24
Year 1 (N=41)															
No answer	12	37	59	100	68	2	20	39	93	29	2	15	22	68	16
Incorrect	22	5	0	0	29	22	10	0	5	7	10	7	2	0	11
Assembles	0	0	42	0	0	0	0	54	2	15	0	0	66	32	13
Correct	66	59	0	0	2	76	71	7	0	49	88	78	10	0	61
<i>Correct answers (number recognition)*</i>						88	73	23	3	63	89	73	27	0	73
Year 2 (N=58)															
No answer	2	10	22	72	22	0	3	14	64	10	2	7	9	41	12
Incorrect	16	3	2	2	0	9	5	0	2	3	9	5	2	2	3
Assembles	0	0	55	12	5	0	0	52	17	10	0	0	48	29	5
Correct	83	86	21	14	72	91	91	35	17	76	90	88	41	28	79
<i>Correct answers (number recognition)*</i>						95	86	56	23	83	90	86	70	32	88

* *second and third assessments only*

The results in the control item (200) suggest that the fundamental difficulty in writing numbers is knowledge of place-value. In those numbers where this knowledge is not necessary, children perform at a much better level. It must be noted, however, that a significant number of children did write 200 in an assembled form (i.e. 2100), although it might seem easier to write 2 and '2 zeros'.

The data also show a fixed pattern according to which children show better results in the number recognition task, in comparison to the written number task. The only exception to this pattern is the 2-digits category.

5.10.3 Understanding the numeration system and using place value (written numbers)

For this analysis, the results of the shopping task were initially categorised into 'fail', "passes the teens category" (i.e. 12 and 15p), "passes the 2-digit category" (i.e. 26 and 53p), "passes the 3-digit category" (i.e. 124, 347) and "passes the 4-digit category" (i.e. 1052 and 2340p)". In order to pass, children were required to respond correctly to one item of the relevant category.

However, and to clarify the relationship between the additive composition task and the written numbers task, the results of the former were re-categorised into three categories: "fail", "passes items under 10, teens and 2-digits" (i.e. under 100) and "passes 3 and 4-digits" (i.e. over 100). McNemar tests had shown no significant differences between passing the small and large categories in the shopping tasks. Children's responses to the written numbers tasks were collapsed into 'no answer/incorrect' (NA/Inc), 'assembles' (assemb) and 'correct' (correct).

In order to ascertain whether children might gain some insights about place value from experience with written multi-digit numbers, children's responses to the written numbers task were categorised into "writes single-digits only" and "writes numbers above 10". The crosstabulation of these results with the results obtained in the additive composition task (pass/fail) revealed that 8, 11 and 9% of all the children (in the first,

second and third assessment) passed additive composition and failed to write numbers equal to ten, or above (multi-digit task). This data supports the view that it is possible to show understanding about the structure of the decade numeration system without being able to write down multi-digit numbers (e.g. Ginsburg, 1977; Nunes and Bryant, 1996).

Data showing that children who fail the additive composition task also fail the written numbers task will support the argument that children base their knowledge of place-value on knowledge of the structure of the decade numeration system. Conversely, evidence that children pass the writing number task and fail the additive composition task will support the idea that children do not learn about place value from prior understanding of the structure of the numeration system. The results of the crosstabulation of additive composition and written numbers is shown on Table 40.

The results show that knowledge of additive composition is related to children's success in the writing multi-digit numbers in all categories. Results referring to the teens and 2-digit categories (i.e. under 100) were analysed separately.

In all assessments, a greater proportion of children who pass the additive composition task, also pass the written number task, compared with those who fail the additive composition task. Differences are significant in all assessments of each category: $X^2=27.8$; $df=2$; $p<0.001$, $X^2=29.6$; $df=2$; $p<0.001$ and $X^2=29.5$ $df=2$; $p<0.001$ in the first, second and third assessments of the teens category. $X^2=58.5$; $df=2$; $p<0.001$, $X^2=56.4$; $df=2$; $p<0.001$ and $X^2=37.1$ $df=2$; $p<0.001$ in the first, second and third assessments of the two-digits category. $X^2=132.7$; $df=4$; $p<0.001$, $X^2=83.0$; $df=4$; $p<0.001$ and $X^2=114.4$ $df=4$; $p<0.001$ in the first, second and third assessments of the three-digits category. $X^2=94.7$; $df=4$; $p<0.001$, $X^2=69.2$; $df=4$;

$p < 0.001$ and $X^2 = 87.2$ $df = 4$; $p < 0.001$, in the first, second and third assessments of the four-digits category.

TABLE 40
Crosstabulation of additive composition with written numbers results. Results are presented in percentages

Additive composition	Written Numbers (N=152)								
	Assessments								
	<i>first</i>			<i>second</i>			<i>third</i>		
	NA/inc*	Assem	Corr	NA/inc	Assem	Corr	NA/inc	Assem	Corr
<i>Teens category</i>									
fail	39	0	31	26	0	25	18	0	26
pass <100	3	0	15	6	0	24	5	0	24
pass >100	0	0	12	0	0	18	0	0	26
<i>2-digits category</i>									
fail	50	0	20	34	0	17	26	0	19
pass <100	1	0	17	4	0	26	5	0	24
pass >100	0	0	12	0	0	18	1	0	24
<i>3-digits category</i>									
fail	55	15	0	40	11	1	32	13	0
pass <100	5	14	0	11	16	3	6	22	2
pass >100	0	4	8	1	7	11	1	8	17
<i>Control 200</i>									
fail	62	1	7	40	3	9	31	5	9
pass <100	7	1	11	11	4	16	6	5	17
pass >100	1	0	11	0	1	18	1	1	24
<i>4-digits category</i>									
fail	70	0	0	51	1	0	42	3	0
pass <100	17	1	0	28	2	0	24	6	0
pass >100	3	3	5	7	5	7	4	11	11

* NA/inc (No answer/incorrect); Assem (Assembles the number); Corr (correct)

Another feature of the data is that no children attempted to 'assemble' 2-digit numbers. The data suggest that children may rely on other sources to succeed in the 2-digits tasks (teens and 2-digits). Between one fifth and one fourth of the children do so (see

crossing between "fail" and "corr" lines on Table 40). One possibility is that children may write 2-digit numbers from memory or aided by language cues.

Turning now to the 3 and 4-digit categories, the data in Table 40 show that children found it less difficult to write number 200 compared with other 3-digit numbers. This suggests that writing numbers *per se* is not difficult for many children, although some children write 200 an 2100. The difficulty seems to be related to making sense of the units of different size in written form. Furthermore, children who failed the additive composition task were not able to use place value correctly in digits over 100, in all the assessments (with one exception in the second assessment of the Year 1 group).

As predicted, not all children who pass additive composition, pass the written numbers items. However, it is clearly shown by the data that only those children who pass the additive composition task also display place value.

5.10.4 Understanding the Numeration System and using of Place Value (number recognition)

Table 41 crosstabulates the results of the additive composition task and the responses in the number recognition task. The data show a similar pattern between this crosstabulation and the one between additive composition and written numbers. Children who pass the additive composition task are more successful in the number recognition task. Differences were significant in all items crosstabulated: $X^2=24.9$; $df=2$; $p<0.001$ and $X^2=19.8$; $df=2$; $p<0.001$ in the second and third assessments of the teens category. $X^2=52.2$; $df=2$; $p<0.001$ and $X^2=45.6$; $df=2$; $p<0.001$ in the second and third assessments of the 2-digits category.

TABLE 41

Crosstabulation of additive composition with number recognition results. Results are presented in percentages

Additive composition	Number recognition					
	Assessments					
	first		second (N=149)		third (N=127)	
	fail	pass	fail	pass	fail	pass
<i>Teens category</i>						
fail	—	—	18	34	18	33
pass teens & 2-digit categories	—	—	1	28	1	23
pass 3 & 4-digit categories	—	—	0	19	1	25
<i>2-digits category</i>						
fail	—	—	34	18	34	17
pass teens & 2-digit categories	—	—	4	26	2	21
pass 3 & 4-digit categories	—	—	0	19	2	24
<i>3-digits category</i>						
fail	—	—	49	3	48	4
pass teens & 2-digit categories	—	—	22	8	13	11
pass 3 & 4-digit categories	—	—	2	17	2	22
<i>Control 200</i>						
fail	—	—	46	24	39	32
pass teens & 2-digit categories	—	—	1	16	1	15
pass 3 & 4-digit categories	—	—	0	12	0	13
<i>4-digits category</i>						
fail	—	—	52	0	51	0
pass teens & 2-digit categories	—	—	29	1	24	1
pass 3 & 4-digit categories	—	—	10	9	12	12

* NA/inc (No answer/incorrect); Assem (Assembles the number); Corr (correct)

This pattern is clearer in the case of the 3 and 4-digit numbers: with few exceptions, children who fail the additive composition task are not able to interpret written numbers correctly. Differences were also significant: $X^2=72.8$; $df=2$; $p<0.001$ and $X^2=62.8$; $df=2$; $p<0.001$ in the second and third assessments of the 3-digits category. $X^2=56.2$; $df=2$; $p<0.001$ and $X^2=52.9$; $df=2$; $p<0.001$ in the second and third assessments of the 4-digits category.

5.10.5 Predictive effects of additive composition

To explore the effect of additive composition on the written numbers tasks, high cross-lag correlations (Spearman) were performed. Results (Table 42) show significant correlations between additive composition in the first assessment and written numbers in the second and third assessments. Correlations are higher for the 3 and 4 digit categories. This supports the argument that those children who understand the structure of the decade numeration system are in a better position to grasp place value.

TABLE 42
Correlations between additive composition and written numbers and number recognition tasks across different assessments.

	Assessments							
	<i>first</i>				<i>second</i>			
	teens	2 digits	3 digits	4 digits	teens	2 digits	3 digits	4 digits
	Written numbers							
additive composition (A1)	0.38*	0.44*	0.67*	0.68*	0.29*	0.36*	0.67*	0.75*
additive composition (A2)					0.43*	0.49*	0.71*	0.71*
	Number recognition							
additive composition (A1)	0.32*	0.47*	0.69*	0.50*	0.27**	0.46*	0.60*	0.52*
additive composition (A2)					0.38*	0.54*	0.67*	0.45*

* p<0.001

**p<0.01

Table 43 crosstabulates the results in the additive composition task (pass/fail only) and success in the written and number recognition tasks. In all assessments the influence of knowledge of additive composition is significant in children's written number ability. It is worth noting that children who failed additive composition in the first assessment

but pass in the three-digit category in the third assessment (for example) will have passed the additive composition task sometime between the first and third assessment.

TABLE 43
Crosstabulation and significant relationships between additive composition and written numbers tasks across different assessments. Results are in percentages

		Additive Composition							
		<i>teens</i>		<i>two digits</i>		<i>three digits</i>		<i>four digits</i>	
		fail	pass	fail	pass	fail	pass	fail	pass
Assessment three									
additive (A1) composition	fail	6	64	21	49	35	35	62	8
	pass	0	30	1	30	0	30	7	23
		[1]		[2]		[3]		[4]	
additive (A2) composition	fail	5	46	20	32	32	19	49	3
	pass	1	48	2	47	3	46	20	28
		[5]		[6]		[7]		[8]	
Assessment two									
additive (A1) composition	fail	10	60	27	43	47	22	67	3
	pass	0	30	0	30	3	27	17	13
		[9]		[10]		[11]		[12]	
		[1] X ² =4.2; df=1; p<0.05		[5] X ² =5.4; df=1; p=0.02		[9] X ² =7.2; df=1; p<0.01			
		[2] X ² =14.8; df=1; p<0.001		[6] X ² =26.4; df=1; p<0.001		[10] X ² =24.4; df=1; p<0.001			
		[3] X ² =35.3; df=1; p<0.001		[7] X ² =55.1; df=1; p<0.001		[11] X ² =41.8; df=1; p<0.001			
		[4] X ² =62.9; df=1; p<0.001		[8] X ² =49.9; df=1; p<0.001		[12] X ² =38.0; df=1; p<0.001			

Across the different assessments, of those children who pass the additive composition task, the vast majority pass the written numbers task. Differences are significant in all cases, according to chi-square tests.

Table 44 crosstabulates the results in the additive composition task (pass/fail only) and success in the number recognition tasks. Similar to the pattern shown on Table 43, the influence of knowledge of additive composition is significant in children's number recognition ability. In all the assessments, of those children who pass the additive

composition task, the vast majority passes the number recognition task. Differences are significant in all cases, according to chi-square tests.

TABLE 44
Crosstabulation and significant relationships between additive composition and number recognition tasks across different assessments. Results are in percentages

		Additive Composition								
		<i>teens</i>		<i>two digits</i>		<i>three digits</i>		<i>four digits</i>		
		fail	pass	fail	pass	fail	pass	fail	pass	
Assessment three										
additive (A1) composition	fail	18	52	37	34	58	13	71	1	
	pass	1	29	1	28	5	24	16	12	
		[1]		[2]		[3]		[4]		
additive (A2) composition	fail	18	39	35	23	52	5	74	0	
	pass	1	42	3	39	10	32	30	13	
		[5]		[6]		[7]		[8]		
Assessment two										
additive (A1) composition	fail	19	51	37	33	65	5	71	0	
	pass	0	30	1	29	7	22	20	9	
		[9]		[10]		[11]		[12]		
[1]	X ² =9.1; df=1; p<0.01			[5]	X ² =18.1; df=1; p=0.001			[9]	X ² =15.1; df=1; p<0.001	
[2]	X ² =27.3; df=1; p<0.001			[6]	X ² =36.9; df=1; p<0.001			[10]	X ² =33.2; df=1; p<0.001	
[3]	X ² =45.7; df=1; p<0.001			[7]	X ² =55.9; df=1; p<0.001			[11]	X ² =70.6; df=1; p<0.001	
[4]	X ² =35.4; df=1; p<0.001			[8]	X ² =26.3; df=1; p<0.001			[12]	X ² =37.2; df=1; p<0.001	

5.10.6 Summary

This study was set up to explore whether (1) children's understanding of the conventional knowledge of place value is based on previous understanding of the structure of the numeration system; or (2) whether children's understanding of the structure of the numeration system is based on previous understanding of place value

The results showed that it was possible to display knowledge about the structure of the numeration system without being able to write numbers above 10. The data presented support the argument that a greater proportion of children who understand the structure of the numeration system will succeed in reading and writing 2-digit numbers. Furthermore, these children are more likely to use knowledge of place value in 3 and 4-digit numbers.

However, children's success with 2-digit numbers cannot be solely explained by their previous grasp of the structure of the numeration system alone. It seems to be the case that it is easier for children to write 2-digits correctly, compared with writing 3- and 4-digits; it is possible that they learn to write 2-digits without interpreting them and also by relying on verbal cues. The effectiveness of this process seems to be quite high if the fact that children do not attempt to assemble the 2-digit numbers is taken into account. Instead, they either fail to write them or they succeed.

With one exception, no child showed knowledge of place value in 3 and 4-digits numbers without also displaying understanding of the structure of the numeration system. Taken together, the evidence is consistent with the argument that those children who have already understood the structure of the numeration system will be in a significantly better position to grasp place value.

6

DISCUSSION

6.1 INTRODUCTION

The purpose of the present study was to explore the developmental relations amongst four basic number components, in the same children, throughout their initial primary school years. The main aim was to attempt to provide some preliminary evidence towards a unifying view clarifying the development of children's number competence, where number events are investigated as an interactive and meaningful whole.

For that purpose, each number component has been examined separately, in the same children, and their developmental relations observed over time. As a result, a preliminary proposal about children's number development, from early counting, at age four, to knowledge of place value, at age seven, is put forward.

In order to explore the relations between number components, the present study found its base on fundamental previous research which has produced crucial evidence about the separate development of children's understanding of the most important number

components examined here, such as counting and knowledge of the number-word sequence (e.g. Gelman and Gallistel, Fuson, 1988); the ability to generate verbal number-words and the understanding of the structure of the numeration system (Bednarz and Janvier, 1982; Kamii, 1986; Nunes and Bryant, 1996); the understanding of arithmetical operations (e.g. Carpenter and Moser, 1982; Vergnaud, 1983; Siegler and Shrager, 1984; Riley et al., 1983; Nesher, 1982; Baroody, 1987, 1989; Kouba, 1989; Gray, 1991; Fuson, 1992; Carpenter et al., 1993; Steffe, 1994; Siegler and Shipley, 1995; Verschaffel and De Corte, 1998); and the ability to read and write numbers and the understanding of the principles underlying place value (e.g. Luria, 1969; Ginsburg, 1977; Brown, 1981; Sinclair and Scheuer, 1993; Power and Dal Martello, 1990; Nunes and Bryant, 1996).

Although these previous studies had consistently suggested that each of these number components emerge and develop at different times, and that the grasp of some complex components require knowledge of the simpler ones, no unifying view describing the development of children's number competence had yet been proposed. Such a unifying conceptual framework seemed interesting to pursue, as it will be helpful to teachers in their everyday classroom activities and to children in their learning endeavour.

The above mentioned research was crucial to the present exploration, by enabling the refinement of specific assessment procedures in each individual component. This, in turn, has led to a detailed description of children's typical achievements in nearly all the mentioned number-components, in function of their ages. Mostly, these are the same tasks that were used in the present exploration of the developing relationships between the main number components used by the same children. The present thesis also benefited from previous studies relating some number components, usually two, and where some relevant hypotheses acted as a starting point for this investigation.

The choice of place value as a learning target relates to its recognition as the most important instructional task in mathematics in the primary school years (e.g. Resnick, 1986), as referred to in the introduction chapter. Place value is a fundamental milestone in children's number development as it forms the basis for the understanding of written multi-digit numbers and represents a fundamental developmental step in the child's ability to compute written algorithmical calculations correctly. Also, the choice of place value as a learning target, considers the needs of educationalists by being curriculum-related. It has, however, the recognised limitation of not covering all aspects of the number curriculum for the primary school years.

The choice of methodology used was the investigation of three different cohorts, which allowed both the observation of the same children over a period of time and, where necessary, cross-sectional comparisons. It was assumed that a cohort study would be uniquely able to identify typical patterns of development and to reveal factors operating on those samples which elude other research designs. These types of studies allow for the examination of individual variations in characteristics and the production of individual growth curves. They are particularly appropriate for the examination of causal relationships, as this task involves identifying changes in certain characteristics that result in changes in others (Baltes and Nesselroade, 1979; Plewis, 1985; Cohen and Manion, 1998).

The current chapter is the final part of this research. First, it briefly describes the results of the assessment of the different number components examined. Secondly, it reviews the findings, emphasising the relevant relations between number components that help children reconceptualise number in progressively more sophisticated ways, from early counting to the understanding of place value. Based on these measures, five

levels of development of children's number understanding were distinguished, from the use of early counting skills, at age four, until their understanding and correct use of place value, as early as age seven. Interpretations, limitations, and suggestions for further research and educational implications are considered.

6.2 RELATIONS BETWEEN NUMBER COMPONENTS

The results of the separate assessment of all number components support the data presented in the literature, by confirming previous ideas about children's attainment in these tasks, so the next sections will be dedicated entirely to the relations between number components. In order to explore the relationships between the several number components above mentioned, several perspectives were investigated. The first one was the relation between continuation of counting (component one) and:

- (1) children's understanding of additive composition (component two);
- (2) their use of counting-on in several different situations (component three); and
- (3) their knowledge of the arithmetical operations (component three).

The second perspective was the relation between children's understanding of the arithmetical operations (component three) and their knowledge of the decade numeration system (component two). The final perspective was the relation between children's understanding of the decade numeration system (component two) and their correct use of place value (component four). The next sections summarise the findings obtained in the present study.

6.2.1 The importance of continuation of counting: summary of findings

This section reports on the relation between continuation of counting, counting-on (and its use in different situations) and additive composition of number. Some preliminary data about the use of the counting-on strategy in several different situations will be discussed.

The results of this study show that children's use of the counting-on strategy is situation-dependent: children were more likely to use counting-on in the hidden addend task than anywhere else. However, the fact that a substantial number of children could pass the hidden addend task simply by counting-all, confirms that a pass alone is not proof that these children are capable of counting-on, as assumed in previous studies (e.g. Nunes and Bryant, 1996). A careful analysis of the strategies used in the hidden addend task still needs to be done, before any classification is applied.

6.2.1.1 The relation between continuation of counting and counting-on

To examine the relevance of continuation of counting to other number components, a study was set up to further investigate its relation to the development of the counting-on strategy, to which the former had been suggested to be developmentally related (Davydov, 1969; Secada et al., 1983).

Of all the children examined, in all assessments of all age groups, none who failed to continue counting was able to use the counting-on strategy in either addition,

subtraction, addition with one hidden addend (box), inversion or multiplication. The evidence presented therefore shows that continuation of counting is presupposed in counting-on. This is consistent with Secada et al's (1983) previous findings. According to Secada et al's. (1983) subskill model, based on the cross-sectional evidence obtained with 73 first-graders (aged 6.3 to 7.6), Counting-on involves 3 subskills: (1) the ability to continue counting from an arbitrary point; (2) the ability to make a transition from the cardinal number of the first addend to the counting meaning; and (3) the ability to shift from regarding the objects in each addend set separately to regarding them as objects within the count of the combined set of objects (sum set). It was also found that except for some rare cases, children who did not continue counting, could not display counting-on.

The data also show that the use of continuation of counting has a significant effect on children's accuracy in any of the above mentioned tasks. In all assessments, the vast majority of children who failed to continue counting could not pass any of the addition, subtraction, inversion and multiplication word-problems.

6.2.1.2 Performance in the additive composition tasks

Regarding the relation between start-unknown problems and the shopping task, the data show that these tasks are two different estimates of children's understanding of additive composition, and that start-unknown problems are more difficult than the shopping task. It seems possible to succeed in any of the two, having failed the other, although only a few children passed the start-unknown problems having failed the shopping task.

One possible explanation for this is that the start-unknown problems assess a more conceptual understanding involved in additive composition - where the use of materials is less helpful - whereas the shopping task is directed to a more intuitive understanding of this mathematical principle.

6.2.1.3 The relation between continuation of counting, counting-on and additive composition

Regarding the relation between counting-on and estimates of additive composition, the results show that despite being more likely to pass the additive composition tasks if they used counting-on, there was a substantial number of children who passed them but did not show any evidence of counting-on. This does not support the argument that counting-on is a necessary but insufficient condition for the grasp of additive composition (Nunes and Bryant, 1996).

The data rather supports the view that counting-on is a consequence of having already understood additive composition (Resnick, 1983), and most probably a consequence of being a proficient addition solver (Carpenter and Moser, 1982; Riley et al., 1983). It is still possible that children may know how to use counting-on without displaying it (e.g. Siegler and Jenkins, 1989), which could have biased the results. However, that possibility seems unlikely considering that this strategy was assessed in five different situations simultaneously, and still no evidence of children's use of counting-on was found.

On the contrary, the evidence presented further strengthens the argument that the skills involved in continuation of counting are more important than has been formerly realised. No child at any point passed the additive composition tasks without also being able to continue counting. This supports the view that continuation of counting is implied in informal understandings of additive composition. While both counting-on and the ability to combine coins of different denominations develop from continuation of counting, the data suggest that there is no necessary link between them. Davydov (1969) had already underlined the importance of continuation of counting, suggesting its relation to children's understanding of addition. Unfortunately, he did not provide data to illustrate his argument, which has now been done.

Based on this, it seems possible to suggest that the argument that counting-on may represent children's first experience with units of different denominations, and therefore "the spur for understanding the base-ten system" (Nunes and Bryant, 1996; p. 52), rather applies to continuation of counting. In a similar way to the child who counts-on, the child who continues counting from, say 10, must also judge this number as a unit of different size, composed of ten ones.

There is evidence to support the argument that continuation of counting allows the child to establish this relationship much earlier, at age 3 or 4 (Fuson, et al., 1982). Ginsburg (1977) suggested that children need to form their own theories about number, before they can understand more complex number conventions. Karmiloff-Smith (1995) argued that children redescribe their knowledge at increasingly more sophisticated levels. This provides grounds to believe that children's first insights about the decade system, a complex number convention, may come from their ability to break and manipulate the number-line. This, however, is not sufficient for their grasp of additive composition.

The present evidence that continuation of counting is a necessary but insufficient condition for children's understanding of additive composition of number, needs to be confirmed with data from intervention studies. Also, further research is needed to clarify which other conceptual structures help children in their grasp of additive composition.

Previous research provides important clues for this clarification. Nunes and Bryant's (1996) argument that addition plays an important role in this development deserves further investigation, although in a different direction. This direction is provided by Resnick's (1983) suggestion that children's initial understanding of part-whole problems can be assessed with Change Result Unknown problems - the start-unknown problems used in this study were difficult part-whole problems. Considering that Change Result Unknown problems are addition problems, and that addition is an ability presupposed in children's ability to combine units of different denominations, it seems therefore possible that additive composition of number may emerge from the interrelated development of continuation of counting and addition. This relation will be discussed in the next section.

6.2.1.4 The relation between continuation of counting and children's understanding of the arithmetical operations

To further explore the relation of continuation of counting to other number components, a study was set up to examine its developmental relation with children's understanding of the arithmetical operations. The results showed that there was a

significant difference in performance in the arithmetical operations, in function of the counting pattern displayed (i.e. either uses or does not use continuation of counting) in all the assessments of all age groups . Only a small minority of children (i.e. always under 5%) who failed to continue counting also passed any of the arithmetical operations word-problems.

The data therefore show that continuation of counting is also relevant to the development of addition, as suggested by Davydov (1969), as well as subtraction, inversion and multiplication word-problems, as hypothesised in the present study. Differences were significant in all assessments. These data provide further evidence to support Davydov's (1969) proposal that continuation of counting is an important development in children's understanding of number, by teaching the child how to use number as a whole and to see (and use) number in concept form.

This, again, supports the view that the use of continuation of counting helps the child to see, at a very early age, that units of different sizes can be related in a meaningful whole and can be part of the same system. This important development was also highlighted by Fuson (1988) and Steffe (1992), from which it was possible to hypothesise and suggest, based on the present evidence that children's level of counting ability (string level, unbreakable chain level, breakable chain level and so on) would relate to their understanding of other number components.

At another level, the importance of continuation of counting in children's understanding of the arithmetical operations also supports the Piagetian (1952) view that children's knowledge of number is interrelated with their understanding of the operations, as grasp of the latter seem to be necessary for the child's understanding of

the former. Evidence to support this view was provided by three cohorts assessed longitudinally.

6.2.2 The importance of knowledge of the arithmetical operations on children's understanding of the decade numeration system: summary of findings

To examine the relationship between children's knowledge of the arithmetical operations and their understanding of the decade numeration system, a study was set up to investigate the relation between several tasks with arithmetical operations' word-problems and a shopping task. This study intended to further clarify previous suggestions that children's grasp of the decade system, which entails sum and product relations, was related to their previous understanding of addition (Resnick, 1983) their use of the counting-on strategy of addition (Nunes and Bryant, 1996), and their knowledge of multiplication.

6.2.2.1 The developmental relation between addition and multiplication

The exploration of the effects of knowledge of the arithmetical operations on children's understanding of the decade numeration system required preliminary clarification about the developmental relation between addition and multiplication. The data obtained in this study showed no significant differences between the results of tasks assessing addition and multiplication.

No evidence was found to support Fischbein et al's. (1985) argument that the concept of multiplication is intuitively attached to the model of repeated addition. Furthermore, the data obtained supports the argument that addition and multiplication develop simultaneously, as empirically verified by Piaget (1952) and more recently by Carpenter et al. (1993).

Alternatively to Fischbein et al's (1985) proposal, Piaget (1952) had suggested that multiplication is an operation that requires higher-order multiplicative thinking which children construct out of their ability to think additively, rather than just a faster way of doing repeated addition. According to this author, young children build their knowledge of one-to-many correspondence (a logical invariant of multiplication) on knowledge of the one-to-one correspondence schema and its use in transitive inferences. Knowledge of both should enable children to realise that if $A=B$ and $C=B$, then $A=C$ (transitivity), and also that if $A=2B$ and $A=C$, then $C=2B$.

Similarly, the evidence presented also supports Steffe's (1994) view suggesting that the realm of multiplication might be entered, from the child's point of view, through the practice and development of more elaborate counting schemes, such as double-counting, and not from a clear grasp of what the operation of multiplication should be. The counting strategies have been granted a similar role in children's understanding of addition and subtraction (e.g. Carpenter and Moser, 1982; 1983; 1984).

Although the secondary role Piaget (1952) attributed to counting is well known, it is also clear that Steffe (1994) is referring to "more elaborate counting schemes", as is also Fuson (1988), when they highlight the relevance of children's counting in their development of number. It seems, therefore, that Piaget (1952) and Steffe (1994)

might be referring to the same thing when they mention one-to-many correspondence and double-counting.

The relevance attributed to double-counting, which is related to the ability to count-on, reinforces the relevance of continuation of counting in children's development of multiplication, which is now supported by the exploratory data presented in this study.

6.2.2.2 The relation between the arithmetical operations and additive composition

Regarding the relation between arithmetical operations and additive composition of number, the data show that children's understanding of the sum and product relations involved in the numeration system presupposes their previous understanding of addition and multiplication in its earlier forms. This relation seems to be phased in such a way that knowledge of addition helps children grasp the decade structure in numbers under 20 (where sum relations are prevalent). Moreover, knowledge of multiplication helps children to understand the decade structure in numbers above 20, where product relations are prevalent [i.e. $53 = (5 \times 10) + 3$]. Only children who pass both addition and multiplication can pass additive composition items above 100.

This data is not consistent with Nunes and Bryant's (1996) prediction that children's understanding of additive composition develops from the use of counting-on, rather from addition. Several children who never use counting-on were able to pass the additive composition task (shopping task).

The results presented rather support the view suggested by Resnick (1983) - in an earlier section - that additive composition develops from earlier understandings of addition, assessed by change result-unknown word-problems. According to this author, change result-unknown word-problems are simpler part-whole problems. But does use of continuation of counting expose children to even simpler part-whole situations ?

It follows that the evidence presented here supports the argument that additive composition develops from the interrelation between continuation of counting (as shown earlier), addition and multiplication.

6.2.3 The relation between understanding the structure of the numeration system and children's correct use of place value: summary of findings

To examine the relationship between children's understanding of the structure of the decade system and their use of place value, a study was set up to investigate the relation between a shopping task and a written multi-digit task. This study intended to clarify whether children learned about place value from experience with written multi-digit numbers (e.g. Luria, 1969; Bergeron and Herscovics, 1990; Sinclair and Scheuer, 1993), or from a previous understanding about the structure of the numeration system (e.g. Ginsburg, 1977; Fuson, 1990; Nunes and Bryant, 1996).

Results showed that it is possible for some children, throughout all assessments, to display understanding about the structure of the decade numeration system without being able to write numbers equal or higher than 10. This evidence does not support the view that children learn about place value from experience with written multi-digit

numbers, as suggested by some authors (e.g. Luria, 1969; Sinclair and Scheuer, 1993). It does support, however, the opposite view that mathematical notation is an integral part of number development (Bialystok, 1992; Hughes, 1986; Sinclair and Scheuer, 1993), although not essential to the emergence of certain ideas about number (e.g. Ginsburg, 1977; Fuson, 1990; Nunes and Bryant, 1996).

This data also support evidence from other sources, such as cross-cultural studies, that have demonstrated that number notation is not a necessary condition for the development of arithmetic principles. Cultures without systems of number notation nonetheless use number computations that obey formal arithmetic principles. Such cases are the Dioulan cloth merchants and tailors of the Ivory Coast (Petitto, 1978; cited by Karmiloff-Smith, 1995) and other African cultures that use practical base-6 mathematics with groupings of cowry shells despite an apparent absence of written symbols (Zaslavsky, 1973) or the Brazilian street vendors who perform partition and iterative addition without making use of externalised notations (Carragher et al., 1985). The present study's evidence lends support to the idea that

"... an external number notation system is not universal, but counting, additive arithmetic operations, and conservation seem to be" (Karmiloff-Smith, 1995; p. 107).

The discrepancy in the results between writing the number 200 (which does not require the combination of units of different sizes) and writing numbers such as 124 and 347 (which require the afore mentioned combination), also suggests that children's difficulties are not related to the size of the number being written, but to the

combination of units it entails. It could be argued that children could never write "200" wrongly anyway. However, a qualitative analysis has shown that up to nearly a fifth of the children "assembled" this number (i.e. wrote it as 2100).

Regarding the relation between children's understanding of the structure of the numeration system and their use of place value, the results show that knowledge of additive composition has an effect on children's success in the writing multi-digit numbers in all categories. However, this effect seems to be different in the case of numbers under and over 100.

In the case of the 2-digit numbers, although it was possible for some children who failed the additive composition task to use place value correctly, a significantly greater proportion of children who pass the additive composition task also pass use place value correctly (compared with those who fail the additive composition task).

Regarding to numbers above 100, no child who failed to display knowledge of the structure of the numeration system was able to use place value correctly, in all the assessments. These data support the view that the understanding of the structure of the numeration system and the ability to write multi-digit numbers are two separate conceptual structures, as suggested by several authors (Ginsburg, 1977; Fuson, 1990; Nunes and Bryant, 1996).

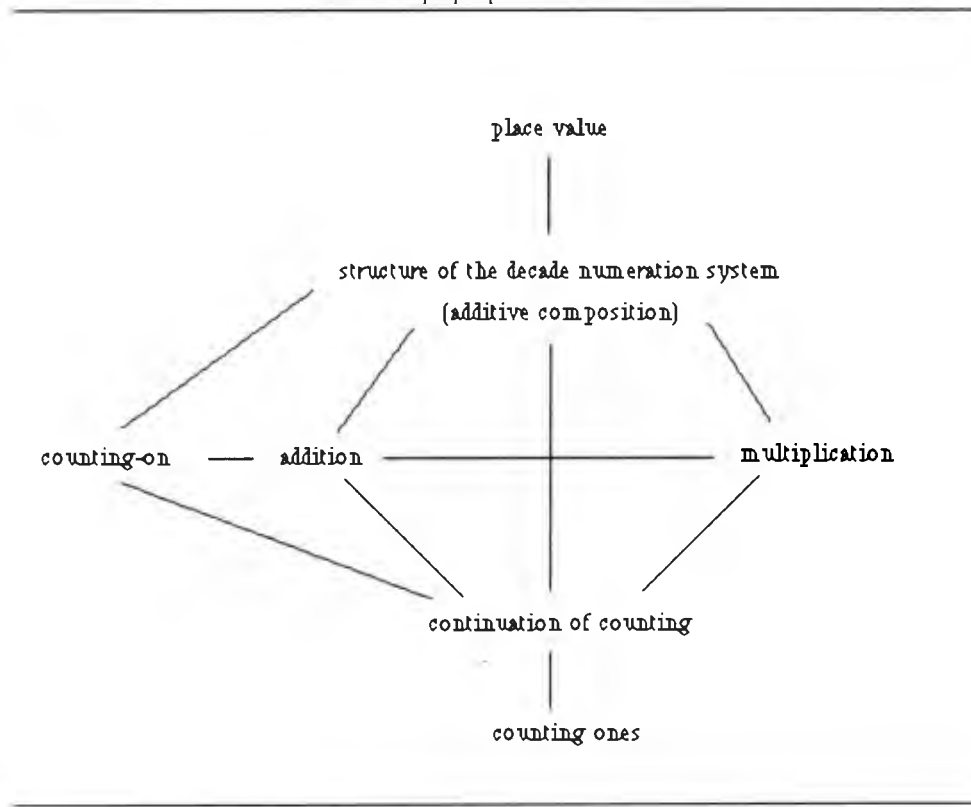
Furthermore, the data suggest that children base their knowledge of place value on previous understanding about the structure of the decade numeration system (Ginsburg, 1977; Fuson, 1990; Nunes and Bryant, 1996), and not the other way around (e.g. Luria, 1969; Bergeron and Herscovics, 1990; Sinclair and Scheuer, 1993).

6.3 PRELIMINARY MODEL OF CHILDREN'S UNDERSTANDING OF NUMBER, FROM EARLY COUNTING TO THE UNDERSTANDING OF PLACE VALUE

After the examination of the most relevant relationships between number components it is now possible to propose an outline of a model of children's number understanding from early counting to knowledge of place value (Table 44a).

TABLE 44a

Relationships proposed in the model



According to this model, children develop through a sequence of qualitatively distinct stages, or levels, as they reconceptualise their understanding of the structure of the numeration system until they finally begin using place value correctly. The proposal is hypothetical, although the suggestions are based on the results of the present study.

6.3.1 Level 1

Although counting ones is, undoubtedly, a necessary condition for further understandings about number, being able to count ones and make correct use of the counting principles (Gelman and Gallistel, 1978) alone, does not necessarily mark the beginning of an understanding about number, as defended by Piaget (1952). Level 1 is characterised by an ability to count ones and an inability to pass any of the remaining tasks assessed in this study. Children in this level, when requested to count from an arbitrary number, will try to count from one up to the number questioned, and then continue their counts.

6.3.2 Level 2

The evidence presented has shown that the display of continuation of counting represents a major developmental step that can be displayed by infants as young as 3 or 4 years of age, before being able to solve addition word-problems (Fuson, 1988). There is also evidence to suggest that the development of continuation of counting may occur almost concurrently with the correct use of the counting principles, and may

develop either from children's counting by groups or from subitising (Gelman and Gallistel, 1978).

Clearly, however, the present data have also shown that children who count by ones *and* use continuation of counting may develop faster than those children who count ones *but do not* display continuation of counting. Only on rare occasions, children who counted ones but failed to continue counting were able to pass the addition or multiplication word-problems.

According to the evidence presented, this developmental period along which children do not display continuation of counting may stretch up to the beginning of Year 1, when a significant number of children still do not display continuation of counting in a very easy task, such as the one used in this study. By the middle of this school year, however, almost all children will have displayed continuation of counting.

Level 2 is, therefore, characterised by an ability to count units of the same size (i.e. ones) and the display of continuation of counting from an arbitrary number. However, these children will fail all the remaining tasks assessed in this study.

6.3.3 Level 3

Level 3 is marked by the ability to display continuation of counting and knowledge of both the arithmetical operations, addition and multiplication. The majority of the children passed both arithmetical operations tasks anyway, so the very few that passed only one of them were categorised as "unclassifiable". Some of these children may

also be able to use the counting-on strategy, although its use is not a necessary condition for their future understanding of additive composition, in the next level.

6.3.4 Level 4

Level 4 is marked by the ability to pass the continuation of counting task, and both arithmetical operations and the additive composition tasks. Children in this level understand the structure of the decade numeration system, both in their sum and product relations. They expand their understanding of addition and multiplication to other domains such as the numeration system. However, they are not yet able of making correct use of place value, although some of them already possess an idea of how the system works, conceptually.

A small proportion of the children in this level displayed knowledge of continuation of counting and an understanding of additive composition, but passed only one of the arithmetical operations - addition. Nine, six and four percent of the children were in this situation in the first, second and third assessments, respectively. These children were able to pass the initial trials (i.e. under 20 category) of the additive composition task, which did not involve product relations (e.g. 12 and 16).

6.3.5 Level 5

Level 5 is characterised by the mastery of all the tasks involved in the model: continuation of counting, the arithmetical operations, additive composition of number

and place value. At this level, children will need to have a grasp about addition and multiplication in the context of the decade numeration system which will, in turn, place them in a better position to understand the theory behind (Ginsburg, 1977; Fuson, 1990) the system of written multi-digit numbers.

6.3.6 Some statistical evidence

Table 45 shows the frequencies of children in each level, by assessment. If the model is correct, then it should be possible to categorise the great majority of the children observed (N=152) in one of the five levels according to their performance in the tasks assessed in the study. Also, and assuming that children progress from one stage to the next over time, they should be expected to move upward in the level scale (or stay in the same one) but not to move downward. The longitudinal data allows us to test this prediction by plotting each individual child's position in the level sequence, from one session to the next.

TABLE 45

Frequencies of children by level of attainment. Results are in percentages.

	Assessments		
	one (N=152)	two (N=152)	three (N=152)
level 1 - fail all (except counting ones)	26	15	8
level 2 - pass continuation of counting only	15	15	15
level 3 - pass continuation of counting and operations	11	11	11
level 4 - pass continuation of counting, operations and additive composition	36	40	41
level 5 - pass all of the former and place value	7	13	18
<i>Missing</i>	<i>1</i>	<i>5</i>	<i>6</i>
<i>Unclassifiable</i>	<i>4</i>	<i>1</i>	<i>1</i>

The data presented on Table 45 confirm the first premise. Ninety-seven, ninety-nine and ninety-nine percent of the children were classified successfully in one of the levels defined in the model, in the first, second and third assessments of the study. Regarding the small minority categorised as "unclassifiable" in the first assessment, Julia (4,8) passed the multiplication and additive composition tasks only. Innes (6,0) and Charlie (6,1) passed both operations but failed to continue counting. Kitty (5,2) passed addition and multiplication only and Thomas (6,5) passed all tasks except multiplication.

In the second assessment, Crystle (5,2) passed the additive composition only and Hayley (6,3) passed all tasks except additive composition. In the third assessment, Sarah (5,6) passed the addition and multiplication tasks only. These situations may be possibly due to the children being distracted while being assessed. None of these children were categorised as "unclassifiable" again.

Table 46 shows the crosstabulation of levels (L1, L2, etc.) from one assessment to the next. The data support the second premise, as the vast majority of the children were found to either stay in the same level or move upwards, from a given assessment to the next.

Ninety-five percent of the children did so from the first to the second assessment, 97% followed the same direction from the first to the third assessment, and 92% of the children did the same from the second to the third assessment. Regarding the small percentage of children that are seen to be moving downward in the level scale, it should be noted that these cases appear to be spread along the various levels.

TABLE 46

Crosstabulation of stages between different assessments (N=152). Results are in percentages

		Assessment two					Assessment three								
		L1	L2	L3	L4	L5			L1	L2	L3	L4	L5		
Ass 1	L1	15	10	2	1		Ass 1	L1	9	13	3	3			
	L2	1	4	4	6			L2	1	4	4	8			
	L3		1	3	7	1		L3		1	2	8			
	L4			2	29	6		L4			1	1	23	13	
	L5			[5%]	1	7		L5				[3%]	1	7	
		L1	L2	L3	L4	L5			L1	L2	L3	L4	L5		
Ass 2	L1	7	8	1	1		Ass 2	L1	7	8	1	1			
	L2	1	8	3	5			L2	1	8	3	5			
	L3		1	5	5	1		L3		1	5	5	1		
	L4			1	4	30		6	L4			1	4	30	6
	L5				[8%]	1		12	L5				[8%]	1	12

6.4. LIMITATIONS

Difficult decisions had to be made concerning the design of the study. These imply the acceptance of limitations. The broader the scope of a study, as it is the present case, the more difficult it becomes to control for extraneous variables. An equilibrium had to be found between the resolve to understand an emerging reality in its wide complexity, and the temptation to control the outcome for a more precise explanation, all in one study. As it was thought that a first approach to the issue would benefit from a wider understanding, a choice was made in favour of a more exploratory design. A

correlational design was, therefore, deemed to cover in a more suitable way, the aims set.

This study has had three main limitations. On the one hand, and considering that the conclusions that can be drawn from the present correlational data are limited, further investigation will be required to test the hypotheses raised here. The fact that all children who pass task B also pass task A does not signify that knowledge of B depends solely on knowledge of A. However, it still seems possible to conclude that those children who pass A seem to be in a better position to understand B. It was not intended, with this work, to claim that this is the only way children learn about place value. Far from that, the idea is mainly to promote debate and encourage research in a new direction where several number components are investigated collectively.

It would have been interesting to have introduced some intervention studies, as relations do not suggest causality. The examination of causality would have implied the introduction and application of an intervention. Rigorous manipulation of the significant interrelations found would be expected to better clarify the degree of change in children's achievement.

A second limitation relates to the type of data used, mainly categorical, that did not allow the utilisation of more sophisticated statistics. The use of parametric statistic would have contributed to the more accurate measurement of knowledge and, therefore, change in time.

Finally, there are issues about scoring that were a constant cause of worry throughout. It was assumed that passing one item of each task was good evidence of some understanding of the task. The choice was based on previous methodology practices,

such as the case of Gelman and Gallistel (1978). Moreover, it was thought to be important that this criteria should be applied evenly across all tasks used in the study. The application of different criteria to different tasks (less stringent in some tasks, more stringent in others) would introduce uncontrollable bias in the results. On the other hand, the choice of using a stringent criteria overall (i.e. children should pass all the items presented in the task (usually three), would make the tasks extremely difficult to pass. As a consequence, patterns in the data would be hidden, especially in the case of the smaller children who were just starting school.

6.5 EDUCATIONAL IMPLICATIONS

The data presented in this study will hopefully represent a step towards a theoretical framework that will enable the understanding of children's number development, as a meaningful whole. Although this "whole" is obviously limited to the four number components assessed in this study, some educational implications can nevertheless be discussed in the way of adjustments to the strategies used by teachers.

Meanwhile, it is clear that the order and speed of development of the number components included in the study may be influenced by the content and methods of school instruction. However, the impact of these methodologies on specific research projects is extremely difficult to measure and was considered to be beyond the present scope. Studies to evaluate teaching methodologies will be necessary, for instance, to account for the advantages of the recent National Numeracy Strategy (1999), which suggests the introduction of written methods later, at age 8. This will most probably influence children's ideas about number in ways that would be interesting to clarify.

In this study, the focus was not on whether children learn at school, but rather what could be done to improve that learning even further. For this reason, the study takes a Piagetian (1952) perspective in suggesting what relations between number components may be implied in children's development of new ideas about number, and whether children have displayed an understanding about these relations. This, independently of what helped them (i.e. what specific methodology) in their development.

The data presented confirm the idea that (1) children bring into the school considerable amounts of mathematical knowledge; and (2) children develop informal mathematical knowledge on their own, before it is taught at school. Of the fifty-two children assessed at school entry, nearly 20% (i.e. 10) were already at Level 2, according to the model suggested in this study. More surprisingly, five of these children (i.e. 10%) were placed in Level 3, and another five children were classified in Level 4.

From another perspective, 32% (i.e. 17 out of 56) of the children in the second assessment of the Year 2 group — i.e. 6- and 7-year-olds — were already able to use place value correctly. These numbers increased to 43% by the third assessment of the same year group. How can this be possible if place value is usually taught later in schools ?

The response seems to be in the evidence that children draw new knowledge from previous mathematical experiences, wherever these may have taken place. However, an interesting way to clarify these developmental relations (between experiences), is through the assessment of several number components, which will offer new insights on how children relate them. This will clarify the specific role of each number component in the whole of number development. Some of the most important implications of this interrelated approach are as follows.

The first implication relates to the renewed importance of counting in children's number development. Based on the data here presented, it seems worth suggesting that children would benefit if counting was treated as a way to progressively reconceptualise the units - from counting units of the same size to the ability to combine units of different sizes - rather than just a means to count objects (e.g. Steffe, 1992). From this perspective, counting could be seen as the main trunk, and a purposeful activity, to which other number components will become linked, enabling progressively faster and more complex ways of counting.

Although counting objects will not, on its own, help the child to grasp more complex understanding about number, it is a necessary step to enable the child to continue counting. As it was shown, it seems worth developing children's ability to continue counting as a way to enable the manipulation of the number-line in progressively more abstract ways. The evidence showed that the ability to continue counting effectively predicts further mathematical abilities, in some cases, years later.

The second implication is that it seems more beneficial for children if they were taught to use counting as a ladder into the realms of addition *and* multiplication, simultaneously. The data suggests that it is easier to help a child to use double-counts - and have him realise later that that is known as multiplication - than starting from teaching the child the concept of multiplication, as repeated-addition. Children as young as five are able to pass simple multiplication tasks, as they also pass addition tasks.

Further, the data supports the view that it helps the child to see the more complex forms of counting (i.e. counting and combining ones, tens, hundreds and so on) as a

natural continuation of the earlier forms of counting, rather than as a separate conceptual entity - known as the "numeration system". Such a link exists in languages like Japanese and Chinese, with significantly improved results in these children's performance in maths (e.g. Miller and Stigler, 1987). In Chinese, "twelve" is reproduced as "ten-two", which helps these children to see counting and the numeration system as a continuum. A way to reinforce this connection in the English Language could be through the use of coins which help making the decade numeration system "visible" to the child.

A third implication that stems from the data presented is that it is also useful to base children's understanding of place value in their knowledge about the structure of the numeration system (Fuson, 1990; Nunes and Bryant, 1996). Again, the handling of coins in the form of shopping tasks provides an important step towards the possibility of representing multi-digit numbers mentally, and a way towards writing them correctly. As shown in this study, no matter how much insistence is placed upon exposing children to different combinations and orders of assembled numbers (in written form) this, on its own, will not necessarily help their mastery of place value. It is true that some children may be able to write two-digit numbers without previous understanding of the decade system. However, the data has also shown that this will not help them to learn to write three and four digit numbers correctly.

From this, stems that success in a learning target such as place value seems to be dependent on many previous acquisitions, on which is worth investing since the beginning of school. A possible beginning is teaching children about continuation of counting.

A fourth implication comes in the form of a warning. As suggested by the data, not all counting strategies have significant conceptual relevance. Whereas continuation of counting exposes the child to the mental manipulation of the number-line at a very early age, counting-on seems to be less relevant in children's number development as it emerges as a consequence of being proficient with addition problems. Counting-on, therefore, seems to be a more economical way of solving word-problems, rather than a different way of conceptualising number.

Finally, helping children to relate number components will help them see the new number component acquisitions as enhancements of the previous ones, and as new possibilities of handling number. These connections are more difficult to make if the child has to follow the strict requirements of the maths curriculum which is organised according to attributed order of difficulty (e.g. multiplication is harder than addition and should, therefore, be taught later). Investment in the teaching of the interrelations proposed in this study may help children see mathematics as a conceptual tool that may be used both inside and outside the school.

It is also hoped that a clearer idea about children's level of number understanding, as proposed in this study, will enable teachers to offer a more appropriate intervention, at crucial moments, adapted to the specificity of each child. The use of different pedagogical techniques may, in this way, be guided by the relevant assessment of what the child can do, as well as by good knowledge of the developmental process that controls number acquisition.

Although researchers are tempted to isolate each number component for a better understanding of its acquisition, it seems important to remember that for the child it may be harder, not easier, to understand something broken down into all the precise

little rules than to grasp it as a meaningful whole. Perhaps it would be more useful if children were offered a view of the forest, as so many children still get lost. Teaching children about the relations between number components will hopefully give them the autonomy and self-confidence they need to find their way forward.

REFERENCES

- Anghileri, J. (1997). Uses of counting in multiplication and division. In: I. Thompson (Ed.), *Teaching and learning early number*. Open University Press.
- Aubrey, C. (1993). An investigation of the mathematical knowledge and competencies which young children bring into school. *British Educational Research Journal*, 19(1), 27-41.
- Ashcraft, M. H. (1992). Cognitive arithmetic: A review of data and theory. *Cognition*, 44, 75-106.
- Baltes, P. B. and Nesselroade, J. R. (1979). History and Rationale of Longitudinal Research. In *Longitudinal Research in the Study of Behaviour and Development*. New York: Academic Press.
- Baroody, A. J. (1985). Mastery of basic number combinations: internalization or relationships or facts ? *Journal for Research in Mathematics Education*, 16, 83-98.
- Baroody, A. J. (1989). Kindergarteners' mental addition with single-digit combinations. *Journal for Research in Mathematics Education*, 20, 159-172.
- Baroody, A. J. (1987). The development of counting strategies for single-digit addition. *Journal for Research in Mathematics Education*, 18, 141-157.
- Baroody, A. J. (1994). An evaluation of evidence supporting fact-retrieval models. *Learning and Individual Differences*, 6, 1-36.
- Baroody, A. J. and Gannon, K. E. (1984). The development of the commutativity principle and economical addition strategies. *Cognition and Instruction*, 1(3), 321-339.

- Baroody, A. J. ; Ginsburg, H. P. and Waxman, B. (1983). Children's use of mathematical structure. *Journal for Research in Mathematics Education*, 14, 156-168.
- Baroody, A. J. and Ginsburg, H. P. (1986). The relationship between initial meaningful and mechanical knowledge of arithmetic. In J. Hiebert (Ed.), *Conceptual and Procedural Knowledge: the Case of Mathematics*, NJ: Erlbaum.
- Bednarz, N. and Janvier, B. (1982). The understanding of numeration in primary school. *Educational Studies in Mathematics*, 13, 33-57.
- Bergeron, J. C. and Herscovics, N. (1990). The kindergarteners' knowledge of numerals. In G. Booker, P. Cobb, and T. N. de Mendicuti (Eds), *Proceedings of the 14th PME Conference*, Mexico.
- Bergeron, J. C. ; Herscovics, N. et. al. (1990). Psychological aspects of learning early arithmetic. In P. Neshier and J. Kilpatrick (Eds.), *Mathematics and cognition: A research synthesis by the international group for the psychology of mathematics education*, pp. 31-52 Cambridge University Press: Cambridge.
- Bialystok, E. (1992). Symbolic representation of letters and numbers. *Cognitive Development*, 7(3), 301-316.
- Brainerd, C. J. (1979). *The origins of number concept*. New York: Praeger.
- Briars, D. and Siegler, R. (1984). A featural analysis of preschoolers' counting knowledge. *Developmental Psychology*, 20. 607-618.
- Briars, D. J. and Larkin, J. H. (1984). An integrated model of skill in solving elementary word problems. *Cognition and Instruction*, 1, 245-296.
- Brown, M. (1981). Number operations. In K. Hart (ed.): *Children's Understanding of Mathematics: 11-16*. Windsor: NFER-Nelson.

- Brownwell, W. A. (1935). Psychological considerations in the learning and teaching of arithmetic. In W. D. Reeve (Ed.), *The teaching of arithmetic: Tenth Yearbook of the National Council of Teachers of Mathematics* (pp. 1-31). New York: Teachers College, Columbia University.
- Bryant, P. (1994). Children and Arithmetic. *Journal of Child Psychology and Psychiatry*, Vol. 36(1), pp. 3-32.
- Butterworth, B. (1999). *The Mathematical Brain*. London: Macmillan.
- Carpenter, T. M. Hiebert, J. and Moser, J. M. (1981). Problem structure and first-grade children's initial solution processes for simple addition and subtraction problems. *Journal for Research in Mathematics Education*, 12, 27-39.
- Carpenter, T. P. and Moser J. M. (1982). The development of addition and subtraction problem-solving skills. In T. Carpenter; J. Moser and T. Romberg (Eds.), *Addition and Subtraction: a Cognitive Perspective* (pp. 9 - 24). Lawrence Erlbaum Associate Publishers.
- Carpenter, T. P. and Moser, J. M. (1983). The acquisition of addition and subtraction concepts. In R. Lesh and M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 7- 44). New York: Academic Press.
- Carpenter, T. P. and Moser J. M. (1984). The acquisition of addition and subtraction concepts in grades one through three. *Journal for Research in Mathematics Education*, 15(3), 179-202.
- Carpenter, T. P., Ansell, E., Franke, M. L., Fennema, E., and Weisbeck, L. (1993). Models of problem solving: A study of kindergarten children's problem-solving processes. *Journal for Research in Mathematics Education*, 24(5), 428-441.
- Carraher, T. N. (1985). The decimal system. Understanding and notation. In L. Streefland (Ed.), *Proceedings of the Ninth International Conference for the Psychology of Mathematical Education* (Vol. 1, pp. 288-303). Utrecht,

Holland: University of Utrecht, Research Group on Mathematics Education and Computer Center.

Carraher, T. N. and Schliemann, A. D. (1990). Knowledge of the numeration system among pre-schoolers. In L. Steffe and T. Wood (Eds.), *Transforming Children's Mathematics Education - international perspectives* (pp. 135 - 141). Lawrence Erlbaum Associates, Hillsdale, NJ.

Carraher, T. N.; Carraher, D. W. and Schliemann, A. D. (1985). Mathematics in the streets and in schools. *British Journal of Developmental Psychology*, 3, 21-29.

Campbell, J. I. D. and Clark, J. M. (1988). An encoding-complex view of cognitive number processing: Comment on McCloskey, Sokol and Goodman (1986). *Journal of Experimental Psychology: General*, 117, 204-214.

Clark, J. M. and Campbell, J. I. D. (1991). Integrated versus modular theories of number skills and acalculia. *Brain and Cognition*, 17, 204-239.

Clark, F. B. and Kamii, C. (1996). Identification of multiplicative thinking in children in grades 1-5. *Journal for Research in Mathematics Education*, 24(1), 41-51.

Cobb, P. (1987). An analysis of three models of early number development. *Journal for Research in Mathematics Education*, 18(3), 163-179.

Cobb, P. and Wheatley, G. (1988). Children's initial understanding of ten. *Focus on Learning Problems in Mathematics*, 10(3), 1-28.

Cohen, L. and Manion, L. (1998). *Research Methods in Education*. Routledge: London and New York.

Cowan, R. (1987). When do Children trust counting as a basis for relative number judgements ? *Journal of Experimental Child Psychology*, 43, 328-345.

Cowan, R.; Foster, C. and Al-Zubaidi, A. (1993). Encouraging children to count. *British Journal of Developmental Psychology*, 11, 411-420.

- Cowan, R. (in-press). Does it all add up ? Changes in children's knowledge of addition facts, strategies, and principles. Chapter in preparation.
- Cooney, J. B., Swanson, H. L. and Ladd, S. F. (1988). Acquisition of mental multiplication skill: Evidence for the transition counting and retrieval strategies. *Cognition and Instruction*, 5(4), 323-345.
- Davydov, V. V. (1969). On the formation of an elementary concept of number by the child. In J. Kilpatrick and I. Wirszup (Eds.), *Soviet studies in the psychology of learning and teaching mathematics. Vol. XIII: The learning of mathematical concepts* (pp. 39-44). Chicago: University of Chicago Press.
- Davydov, V. V. (1991). A psychological analysis of the operation of multiplication. In Leslie Steffe (Ed.), *Psychological Abilities of Primary School Children in Learning mathematics* (Volume 6, pp. 9-85). Reston, VA: National Council of Teachers of Mathematics.
- De Corte, E. and Verschaffel, L. (1989). Teaching word problems in the primary school. What research has to say to the teacher. In B. Greer And G. Mulhern (Eds.), *New Developments in Teaching Mathematics* (pp. 85-106) London: Routledge.
- Dehaene, S. (1992). Varieties of numerical abilities. *Cognition*, 44, 1-42.
- Dehaene, S. and Cohen, L. (1995). Towards an anatomical and functional model of number processing. *Mathematical Cognition*, 1(1), 83-120.
- Dehaene, S., Spelke, E., Pinel, P., Stanescu, R. and Tsivkin, S. (1999). Sources of mathematical thinking: Behavioural and brain imaging evidence. *Science*, 284, 970-984.
- Deloche, G. and Seron, X. (1982). From one to 1: An analysis of a transcoding process by means of neuropsychological data. *Cognition*, 12, 119-149.

- Deloche, G. and Seron, X. (1987). Numerical transcoding: A general production model. In G. Deloche and X. Seron (Eds.), *Mathematical disabilities: A cognitive neuropsychological perspective*. Hillsdale, N. J.: Lawrence Erlbaum Associates, Inc.
- Deloche, G., Seron, X., Larroque, C., Magnien, C. Metz-Lotz, M. N. et al. (1994). Calculation and number processing assessment battery: The role of demographic factors. *Journal of Clinical and Experimental Neuropsychology*, 16, 915-208.
- Denvir, B. and Brown, M. (1986a). Understanding of number concepts in low-attaining 7-9 year olds: Part I: Development of descriptive framework and diagnostic instrument. *Educational Studies in Mathematics*, 17, 15-36.
- Denvir, B. and Brown, M. (1986b). Understanding of number concepts in low-attaining 7-9 year olds: Part II: The teaching studies. *Educational Studies in Mathematics*, 17, 143-164.
- Desforges, A. and Desforges, G. (1980). Number-based strategies of sharing in young children, *Educational Studies*, 6, 97-109.
- Dowker, A. (1998). Individual differences in normal arithmetical development. In C. Donlan (Ed.): *The development of Mathematical Skills*. Hove: Psychology Press.
- English, L. D. and Halford, G. S. (1995). *Mathematics Education: Models and processes*. Hove: Lawrence Erlbaum Associates.
- Fennell, F., Reyes, B., Reyes, R. and Webb, A. (1991). *Mathematics unlimited*. Orlando, FL: Harcourt, Brace Jovanovich.
- Fischbein, E., Deri, M., Nello, M. S., and Marino, M. S. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education*, 16(1), 3-17.
- Fuson, K. (1979). Counting solution procedures in addition and subtraction. Paper presented at the *Wingspread Conference on Addition and Subtraction*, Racine, Wisconsin, 1979.

- Fuson, K. C. (1982). An analysis of the counting-on solution procedure in addition. In T. P. Carpenter; J. M. Moser and T. Romberg (Eds.), *Addition and Subtraction: a cognitive perspective*. Hillsdale, NJ: Erlbaum.
- Fuson, K. C. and Hall, J. W. (1983). The acquisition of early number word meanings: a conceptual analysis and review. In H. P. Ginsburg (Ed.), *The development of mathematical thinking*; pp. 49- 107. New York: Academic Press.
- Fuson, K. C. ; Pergament, G. G.; Lyons, B. G. and Hall, J. W. (1985). Children's conformity to the cardinality rule as a function of set size and counting accuracy. *Child Development*, 56, 1429-1436.
- Fuson, K. C.; Richards, J. and Briars, D. J. (1982). The acquisition and elaboration of the number word sequence. In C. J. Brainerd (Ed.), *Children's logical and mathematical cognition* (pp. 33-92). New York: Springer-Verlag.
- Fuson, K. C. (1990). Conceptual structures for multiunit numbers: Implications for learning and teaching multidigit addition, subtraction and place value. *Cognition and Instruction*, 7(4), 343-403.
- Fuson, K. C. and Kwon, Y. (1992). Effects on children's addition and subtraction of the system of number words and other cultural tools. In J. Bideaud, C. Meljac, & J.-P. Fischer (Eds.), *Pathways to number* (pp. 283-306). Hillsdale, NJ: Lawrence Erlbaum Associates Inc.
- Fuson, K. C. (1988). *Children's counting and concepts of number*. New York: Springer-Verlag.
- Fuson, K. C. (1992). Research on whole number addition and subtraction. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 243-275). New York: Macmillan; Reston, VA: National Council of Teachers of Mathematics.

- Frye, D.; Braisby, N.; Lowe, J.; Maroudas, C.; Nicholls, J. (1989). Young children's understanding of counting and cardinality. *Child Development*, 60, 1158-1171.
- Gal'perin, P. Ya. and Georgiev. L. S. (1969). The formation of elementary mathematical notions. In J. Kilpatrick and I. Wirszup (Eds.), *Soviet studies in the psychology of learning and teaching mathematics. Vol. I: The learning of mathematical concepts* (pp. 189-216). Chicago: University of Chicago Press.
- Geary, D. C. and Widaman, K. F. (1992). Numerical cognition: On the convergence of componential and psychometric models. *Intelligence*, 16, 47-80.
- Gelman, R. (1982). Basic numerical abilities. In R. Sternberg (Ed.), *Advances in the psychology of human intelligence*, 181-205. Hillsdale, NJ: Lawrence Erlbaum Associates.
- Gelman, R. and Gallistel, C. R. (1978). *The child's understanding of number*. Cambridge, Mass.: Harvard University Press.
- Gelman, R. and Meck, E. (1983). Preschoolers' counting: principles before skill. *Cognition*, 13, 343-359.
- Gelman, R. and Meck, E. (1986). The notion of principle: The case of counting. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics*. Hillsdale, NJ: Lawrence Erlbaum Associates.
- Gelman, R. and Meck, B. (1992). Earlier principles aid initial but not later conceptions of number. In J. Bideaud, C. Meljac, & J.-P. Fischer (Eds.), *Pathways to number* (pp. 171-189). Hillsdale, NJ: Lawrence Erlbaum Associates Inc.
- Ginsburg, H. (1977). *Children's arithmetic: the learning process*. New York: D. Van Nostrand Co.
- Ginsburg, H. P.; Posner, J. K. and Russell, R. L. (1981). The development of mental addition as a function of schooling and culture. *Journal of Cross-Cultural Psychology*, 12(2) June, 163-178.

- Gray, E. M. (1991). An analysis of diverging approaches to simple arithmetic: Preferences and its consequences. *Educational Studies in Mathematics*, 22, 551-574.
- Gray, E., Pitta, D. and Tall, D. (1997). The Nature of the Object as an Integral Component of Numerical Processes. In Erkki Pehkonen (Ed.), *Proceedings of the 21st conference of the International Group for the Psychology of Mathematics Education* (Vol 1., pp. 115-130). Lahti, Finland: University of Helsinki, Lahti Research and Training Centre.
- Greeno, J. G., Riley, M. S. and Gelman, R. (1984). Conceptual competence and children's counting. *Cognitive Psychology*, 16, 94-143.
- Greeno, J. G. (1991). Number sense as situated knowing in a conceptual domain. *Journal for Research in Mathematics Education*, 22(3), 170-218.
- Greer, B. (1987). Nonconservation of multiplication and division involving decimals. *Journal for Research in Mathematics Education*, 18, 37-45.
- Groen, G. and Resnick, L. B. (1977). Can Preschool Children Invent Addition Algorithms ? *Journal of Educational Psychology*, 69(6), 645-652.
- Groen, G. J. and Parkman, J. M. (1972). A chronometric analysis of simple addition. *Psychological Review*, 79, 239-343.
- Hart, K. (Ed.).(1981). *Children's understanding of mathematics: 11-16*. London: John Murray.
- Haylock, D. and Cockburn, A. (1997). *Understanding mathematics in the lower primary grades*. London: Paul Chapman Publishing Ltd.
- Hedderston, J. and Fisher, M. (1993). *SPSS made simple*. Belmont, California Wadsworth Publishing Company.

- Hiebert, J. and Wearne, D. (1986). Procedures over concepts: the acquisition of decimal number knowledge In J. Hiebert (Ed.) *Conceptual and Procedural Knowledge: the case of mathematics*. LEA, London.
- Hiebert, J. and Behr, M. (1988). Research agenda for mathematics education. In J. Hiebert and M. Behr (Eds.), *Number concepts and operations in the middle grades*, Vol. 2, 1-32. Hillsdale, NJ: Lawrence Erlbaum Associates; Reston, VA: National Council of Teachers of Mathematics.
- Hittmair-Delazer, M., Semenza, L. and Denes, G. (1994). Concepts and facts in calculation. *Brain*, 117, 715-728.
- Hittmair-Delazer, M., Sailer, V. and Berke, T. (1995). Impaired arithmetical facts but intact conceptual knowledge: a single case study of dyscalculia. *Cortex*, 31, 139-147.
- Hoffer, A. R., Johnson, M. L., Leinwand, S. J., Lodholz, R. D., Musser, G. L., and Thoburn, T. (1991). *Mathematics in action*. New York: Macmillan/McGraw-Hill School Division.
- Houlihan, D. M. and Ginsburg, H. P. (1981). The addition methods of first and second grade children. *Journal for Research in Mathematics Education*, 12, 95-106.
- Hughes, M. (1986). *Children and Number*. Oxford: Blackwell.
- Ilg, F. and Ames, L. B. (1951). Developmental trends in arithmetic. *Journal of Genetic Psychology*, 79, 3-28.
- Kamii, M. (1980). Place value: Children's efforts to find a correspondence between digit and number of objects. *Paper presented at the tenth annual symposium of the Jean Piaget society*, Philadelphia, PA, May.
- Kamii, C. K. (1986). Place value: An explanation of its difficulties and educational implications for the primary grades. *Journal of Research in Childhood Education*, 1(2), 75-85.

- Kamii, C., with Livingston, S. (1994). *Young children continue to reinvent arithmetic, 3rd grade*. New York: Teachers College Press.
- Karmiloff-Smith, A. (1995). *Beyond Modularity*. Cambridge, MA: MIT Press.
- Kieren, T. E. (1994). Multiple views of multiplicative structures. In G. Harel and J. Confrey (Eds.), *The Development of Multiplicative Reasoning in the Learning of Mathematics*, pp. 389-400. Albany, NY: State University of New York Press.
- Kornilaki, E. (1994). Children's understanding of the numeration system. *Unpublished Masters Dissertation*. Institute of Education, University of London.
- Kouba, V. L. (1989). Children's solution strategies for equivalent set multiplication and division word problems. *Journal for research in mathematics education*, 20(2), 147-158.
- Lave, J. (1988). *Cognition in practice. Mind, mathematics and culture in everyday life*. New York: Cambridge University Press.
- Luria, A. R. (1969). On the pathology of computational operations. In J. Kilpatrick and I. Wirszup (Eds.), *Soviet studies in the psychology of learning and teaching mathematics. Vol. I: The learning of mathematical concepts* (pp. 37-74). Chicago: University of Chicago Press.
- Luria, A. R. (1976). *Cognitive Development: Its Cultural and Social Foundations*. Harvard University Press.
- Martins-Mourão, A. and Cowan, R. (1997). Precursors of additive composition of number. In Erkki Pehkonen (Ed.), *Proceedings of the 21st conference of the International Group for the Psychology of Mathematics Education* (Vol. 1., p. 246). Lahti, Finland: University of Helsinki, Lahti Research and Training Centre.

- McCloskey, M., Caramazza, A., And Basili, A. (1985). Cognitive mechanisms in number processing and calculation: Evidence from dyscalculia. *Brain and Cognition*, 4, 171-196.
- McCloskey, M. (1992). Cognitive mechanisms in numerical processing: Evidence from acquired acalculia. *Cognition*, 44, 107-157.
- Meck, W. H., and Church, R. M., (1983). A mode control model of counting and timing processes. *Journal of Experimental Psychology: Animal Behaviour Processes*, 9, 320-334.
- Menninger, K. (1969). *Number words and number symbols: A cultural history of numbers*. Cambridge, MA: The MIT Press.
- Michie, S. (1984). Why preschoolers are reluctant to count spontaneously. *British Journal of Developmental Psychology*, 2, 347-358.
- Miller, K. F. and Stigler, J. W. (1987). Counting in Chinese. Cultural variation in a basic cognitive skill. *Cognitive Development*, 2, 279-305.
- Miller, K.; Perlmutter, M. and Keating, D. (1984). Cognitive arithmetic: comparison of operations. *Journal of Experimental Psychology: Learning, Memory and Cognition*; 10 (1); 46-60.
- Miura, I. T. and Okamoto, Y. (1989). Comparison of American and Japanese first graders' cognitive representation of number and understanding of place value. *Journal of Educational Psychology*, 81, 109-113.
- Miura, I. T., Okamoto, Y., Kim, C. C., Steere, M., and Fayol, M. (1993). First graders' cognitive representation of number and understanding of place value: Cross-national comparisons: France, Japan, Korea, Sweden and the United States. *Journal of Educational Psychology*, 85(1), 24-30.
- Nesher, P. (1982). Levels of description in the analysis of addition and subtraction. In T. P. Carpenter, J. M. Moser and T. Romberg (Eds.), *Addition and*

- Subtraction: A cognitive perspective.* (pp. 25-38). Hillsdale NJ: Lawrence Erlbaum Associates. Inc.
- Newmark, J. (1991). *Math for elementary school teachers.* New York: Addison-Wesley.
- Nichols, E. and Behr, M. (1982). *Elementary school math and how to teach it.* New York: Holt, Rinehart & Winston.
- Noel, M. P. and Seron, X. (1993). Arabic number reading deficit: A single case study. *Cognitive Neuropsychology*, 10, 317-339.
- Nunes, T. and Bryant, P. (1996). *Children Doing Mathematics.* Oxford: Blackwell.
- O'Brien, T. and Casey, S. (1983). Children learning multiplication. *School Science and Mathematics*, 83, 246-251.
- Piaget, J. (1952). *The child's conception of number.* New York: Humanities Press.
- Post, T. R., Cramer, K. A., Behr, M., Lesh, R. and Harel, G. (1993). Curriculum implications of research on the learning, teaching, and assessing of rational number concepts. In T. Carpenter, E. Fennema & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 327-362). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Power, R. D. and Dal Martello, M. F. (1990). The dictation of Italian numerals. *Language and Cognitive Processes*, 5, 237-254.
- Power, R. D. and Dal Martello, M. F. (1997). From 834 to eighty thirty four: The reading of arabic numerals by seven-year-old children. *Mathematical Cognition*, 3(1), 63-85.
- Plewis, I. (1985). *Analysing change: measurement and explanation using longitudinal data.* Wiley.

- Resnick, L. B. (1989). Developing mathematical knowledge. *American Psychologist*, 44(2), 162-169.
- Resnick, L. B. (1987). Learning in School and Out. *Educational Researcher*, 16(9), pp. 13-20.
- Resnick, L. B. (1986). The development of mathematical intuition. In M. Perlumutter (Ed), *Perspectives on intellectual development: Minnesota Symposia on Child Psychology*, vol. 19, 159-194. Hillsdale, NJ: Lawrence Erlbaum Associates.
- Resnick, L. B. (1983). A developmental theory of number understanding. In H. P. Ginsburg (Ed.), *The development of mathematical thinking*; pp. 109-151. Academic Press: New York.
- Resnick, L. B. (1980). The role of invention in the development of mathematical competence. In R. H. Kluwe and H. Spada (Eds.), *Developmental models of thinking*. New York: Academic Press.
- Resnick, L. B. and Ford, W. (1981). *The psychology of mathematics instruction*. Hillsdale, N. J.: Erlbaum.
- Riley, M. S.; Greeno, J. G. and Heller, J. I. (1983). Development of children's problem-solving ability in arithmetic. In H. P. Ginsburg (Ed.), *The development of mathematical thinking*; pp. 153-196. Academic Press: New York.
- Riley, M. S. and Greeno, J. G. (1988). Developmental analysis of understanding language about quantities and of solving problems. *Cognition and Instruction*, 5, 49-101.
- Rittle-Johnson, B. and Siegler, R. S. (1998). The relation between conceptual and procedural knowledge in learning mathematics: A review. In C. Donlan (Ed.): *The development of Mathematical Skills*. Hove: Psychology Press.

- Ross, S. H. (1989). Parts, wholes and place value: a developmental view. *Arithmetic Teacher*, 36(6).
- Russell, R. L. and Ginsburg, H. P. (1984). Cognitive analysis of children's mathematics difficulties. *Cognition and Instruction*, 1(2), 217-244.
- Saxe, G. B. (1988). Candy selling and mathematics learning. *Educational Researcher*, 17(6), 14-21.
- Saxe, G. B. (1982). Developing forms of arithmetical thought among the Oksapmin of Papua New Guinea. *Developmental Psychology*, 18(4), 583-594.
- Saxe, G. B. (1981). Body Parts as Numerals: a developmental analysis of numeration among the Oksapmin of Papua New Guinea. *Child Development*, 52, 306-316.
- Saxe, G. B. and Posner, J. (1983). The development of numerical cognition: cross-cultural perspectives. In H. P. Ginsburg (Ed.), *The development of mathematical thinking* (pp. 291-317). New York: Academic Press.
- Saxe, G. B. (1991). *Culture and Cognitive Development: Studies in Mathematical Understanding*. Hillsdale, New Jersey: Lawrence Erlbaum Associates.
- Schools Examination and Assessment Council, (1992). *Standard Assessment Tasks, Pupil Sheet Booklet, Key Stage 1*. Lincoln: NFER-Nelson.
- Schwartz, J. L. (1988). Intensive quantity and referent transforming arithmetic operations. In J. Hiebert and M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 41-52). Hillsdale, NJ: Lawrence Erlbaum Associates; Reston, VA: National Council of Teachers of Mathematics.
- Secada, W. G.; Fuson, K. C. and Hall. (1983). The transition from counting-all to counting-on in addition. *Journal for Research in Mathematics Education*, 14, 47-57.

- Seron, X. and Fayol, M. (1994). Number transcoding in children: a functional analysis. *British Journal of Developmental Psychology*, 12, pp. 281-300.
- Seron, X., Deloche, G. and Noel, M. P. (1991). In transcodage des nombres chez l'enfant: la production des chiffres sus dictée. In J. Bideaud, Cl. Meljac and J. P. Fischer (Eds.), *Les Chemins du Nombre*, pp. 245-264. Lille: Presses Universitaires de Lille.
- Seron, X. and Noel, M. (1995). Transcoding numbers from the arabic code to the verbal one or vice versa: How many routes ? *Mathematical Cognition*, 1, 215-243.
- Shrager, J. and Siegler, R. S. (1998). SCADS: A model of children's strategy choices and strategy discoveries. *Psychological Science*, 9, 405-410.
- Siegel, S. (1956). *Nonparametric Statistics for the Behavioural Sciences*. New York: McGraw-Hill Book Company, Inc.
- Siegler, R. S. and Robinson, M. (1982). The development of numerical understandings. In H. Reese and L. P. Lipsitt (Eds.) *Advances in Child Development and Behaviour* (16), pp. 241-312. New York: Academic Press.
- Siegler, R. S. and Shrager, J. (1984). Strategy choices in addition and subtraction: how do children know what to do ? In C. Sophian (Ed.), *The Origins of Cognitive Skills*. Hillsdale NJ: Erlbaum.
- Siegler, R. S. and Jenkins, E. (1989). *How children discover new strategies*. Hillsdale, NJ: Erlbaum.
- Siegler, R. S. and Shipley, C. (1995). Variation, selection, and cognitive change. In T. Simon and G. Halford (Eds.), *Developing cognitive competence: New approaches to process modelling* (pp. 31-76). Hillsdale, NJ: Lawrence Erlbaum Associates

- Siegler, R. S. and Stern, E. (1998). Conscious and unconscious strategy discoveries: A microgenetic analysis. *Journal of Experimental Psychology: General*, 127, 377-397.
- Sinclair, A. (1991). Children's production and comprehension of written numerical representations. In K. Durkin and B. Shire (Eds), *Language in Mathematical Education*. Open University Press. Milton Keynes.
- Sinclair, A., Siegrist, F., and Sinclair, H. (1983). Young children's idea about the written system. In D. Rogers & J. Sloboda (Eds.), *The acquisition of symbolic skills*. New York: Plenum.
- Sinclair, A. and Scheuer, N. (1993). Understanding the written number system: 6 year-olds in Argentina and Switzerland. *Educational Studies in Mathematics*, 24, 199-221.
- Sinclair, A.; Garin, A.; Tieche-Christinat, C. (1992). Constructing and understanding of place value in numerical notation. *European Journal of Psychology of Education*, Vol. VII (3), 191-207.
- Skemp, R. R. (1971). *The Psychology of Learning Mathematics*. Harmondsworth: Penguin.
- Song, M. J. and Ginsburg, H. P. (1988). The effect of the Korean number system on young children's counting: A natural experiment in numerical binlignualism. *International Journal of Psychology*, 23, 319-332.
- Sophian, C. (1987). Early developments in children's use of counting to solve quantitative problems. *Cognition and Instruction*, 4, 61-90.
- Starkey, P. and Gelman, R. (1982). The development of addition and subtraction abilities prior to formal schooling in arithmetic. In T. Carpenter, J. Moser and T. Romberg (Eds.), *Addition and Subtraction: a Cognitive Perspective* (99 - 116). Hillsdale, NJ: Lawrence Erlbaum Associates.

- Steffe, L. P.; von Glasersfeld, E.; Richards, J. and Cobb, P. (1983). *Children's counting types — philosophy, theory and application*. Praeger special studies - Praeger scientific. N. Y.
- Steffe, P. S.; Thompson, P. W. and Richards, J. (1982). Children's counting in arithmetical problem solving. In T. Carpenter; J. Moser and T. Romberg (Eds.), *Addition and Subtraction: a Cognitive Perspective* (pp. 83-97). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Steffe, L. P. (1988). Children's construction of number sequences and multiplying schemes. In J. Hiebert and M. Behr (Eds.), *Number concepts and operations in the middle grades* (Vol. 2, 119-140. Hillsdale, NJ: Lawrence Erlbaum Associates; Reston, VA: National Council of Teachers of Mathematics.
- Steffe, L. P. (1992). Learning Stages in the Construction of the number Sequence. In J. Bideaud, C. Meljac and J. P. Fischer (Eds.), *Pathways to Number*, pp. 83-98. Hillsdale, NJ: Lawrence Erlbaum Associates.
- Steffe, L. (1994). Children's multiplying schemes. In G. Harel and J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 3-40). Albany, New York: State University of New York Press.
- Steinberg, R. M. (1985). Instruction on derived fact strategies in addition and subtraction. *Journal for Research in Mathematics Education*, 16, 337-355.
- Suggate, J.; Aubrey, C. and Pettitt, D. (1997). The Number Knowledge of Four to Five Year Olds at School Entry and at the Entry of their First Year. *European Early Childhood Education Research Journal*, Vol. 5, No. 2, 85-101.
- The National Numeracy Strategy. Framework for teaching mathematics from Reception to Year 6. Department for Education and Employment. March 1999.
- Thorndike, E. L. (1922). *The Psychology of Arithmetic*. New York: Macmillan.

- VanLehn, K. (1990). *Mind bugs: The origins of procedural misconceptions*. Cambridge, MA: MIT Press.
- Vergnaud, G. (1982). A classification of cognitive tasks and operations of thought involved in addition and subtraction problems. In T. Carpenter; J. Moser and T. Romberg (Eds.), *Addition and Subtraction: a Cognitive Perspective* (39 - 59). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Vergnaud, G. (1983). Multiplicative Structures. In R. Lesh and M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 127- 174). New York: Academic Press.
- Verschaffel, L. and De Corte, E. (1998). Word problems: A vehicle for promoting authentic mathematical understanding and problem solving in the primary school ?, In T. Nunes and P. Bryant (Eds.) *Learning and teaching mathematics : an international perspective*. Hove: Psychology Press.
- Vygotsky, L. S. (1978). *Mind in Society. The development of higher psychological processes*. Cambridge, MA: Harvard University Press.
- Warrington, E. K. (1982). The fractionation of arithmetical skills: A single-case study. *Quarterly Journal of Experimental Psychology*, 34A, 31-51.
- Wynn, K. (1990). Children's understanding of counting. *Cognition*, 36, 155-193.
- Wynn, K. (1992). Addition and subtraction by human infants. *Nature*, 358, 749-750.
- Yang, M. T. and Cobb, P. (1995). A cross-cultural investigation into the development of place value concepts of children in Taiwan and the United States. *Educational Studies in Mathematics*, 28, 1-33.
- Zaslavsky, C. (1973). *Africa Counts*. Boston, Mass: Prindle, Weber and Schmidt.

