# PUPILS' APPROACHES TO DIFFERENT CHARACTERIZATIONS OF VARIABLE IN LOGO

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THESIS SUBMITTED IN FULFILMENT OF THE REQUIREMENT
FOR THE PH. D. DEGREE OF THE UNIVERSITY OF LONDON

UNIVERSITY OF LONDON INSTITUTE OF EDUCATION
DECEMBER, 1993

BIBL. LONDIN. UNIV.

# ABSTRACT

This thesis is based on research investigating 12-13 years old pupils' potential to approach through Logo different characterizations of variable prior to formal algebra teaching. The characterizations of variable considered are variable as general number, variables in a functional relationship and variable as specific unknown. Pupils' potential is defined by their capability to solve a range of specific Logo-based tasks involving one characterization of variable. During the solution process peer collaboration was encouraged and support was available from the researcher.

The experimental work consisted of case studies of 6 pairs of pupils over a period of one year working in three specially designed Logo microworlds each one focussing on one characterization of variable. The observation of these pupils' work was carried out during their normal computer time with the researcher acting as the teacher of the whole group (34 pupils). Special attention was paid to the strategies pupils used to solve the posed tasks, researcher's interventions, peer collaboration and the influence of the Logo setting. Data included researcher's notes taken during the observation of pupils' work, pupils' written records, individual interviews and the responses to a paper and pencil questionnaire given at the beginning and at the end of the study.

Previous studies of pupils' learning of algebra show that they have difficulties with each one of the three characterizations of variable whether they are algebra beginners or more advanced students. Evidence from this study shows that pupils can develop a potential to work with these characterizations of variable prior to formal algebra teaching. Crucial elements provoking this development were: the design of the activities appealing to pupils' prior numeric knowledge; the Logo environment; social interaction between pupils and with the researcher; the use of communicative and egocentric speech.

The results of this research show that notions that are crucial to the use of variable can be developed prior to formal algebra teaching. Pupils' arithmetic can provide a basis for the development of these notions.

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# ACKNOWLEDGEMENTS

First of all I want to thank my supervisors, Prof. Celia Hoyles and Dr. Rosamund Sutherland, for their support, advise and encouragement during my stages at the Institute of Education as well as during their visits to the Department of Mathematical Education, CINVESTAV, in Mexico. I am very grateful to Dr. Teresa Rojano, my supervisor in Mexico, for her valuable comments and steady encouragement during these years. I also wish to thank Marta Gabriela Araujo who assisted me during the data collection.

I am indebted to the group of pupils from the school 'Centro Escolar Hermanos Revueltas' in Mexico City and in particular to the twelve case study pupils who were always ready to explain their way of working. I thank Dr. Eugenio Filloy, head of the school, who offered me all the facilities for developing the experimental work.

I would also like to thank the administrative team at the Department of Mathematical Education, CINVESTAV, in Mexico, in particular Guadalupe Guevara, Enrique Oaxaca and Marina Peral.

I am grateful to the British Council, to the Programa Nacional de Formación y Actualización de Profesores de Matemáticas (PNFAPM) and to the Centro de Investigación y de Estudios Avanzados del I.P.N. (CINVESTAV) for the economic support given to this research.

Finally I could not have made this research without the support of my family. In particular my sons Manolo, Flavio and Fabian were always very supportive and often they had to take care of themselves and help each other when I disappeared into my work.

# CHAPTER 1

# INTRODUCTION

The concept of variable is central to algebra yet it is surprisingly difficult to define. The reason for this difficulty is that a variable has characterizations which vary according to the problem in which it is embedded. For instance, a variable may represent a specific unknown, that is a specific but unknown number that can be calculated considering the given constraints; a general number, that is an indeterminate number involved in a general method; or it might be used to represent a functional relationship between two changing values.

Research investigating pupils' learning of algebra reports that pupils have difficulties with each one of these characterizations of variable whether they are algebra beginners or more advanced students. However, there has not been investigation of pupils' potential to cope with the different characterizations of variable within environments specifically designed to facilitate this learning prior to formal algebra teaching. This research sets out to do this and additionally to investigate how these environments can help pupils construct a multifaceted notion of variable and lead them to the acceptance and use of its symbolization.

The feasibility of approaching the three characterizations of variable prior to formal algebra teaching is suggested by an analysis of the historical construction of the concept of variable. This shows that problems involving the determination of specific unknowns, the idea of variation, and the search for general methods for solving similar problems were approached before the algebraic language was developed. Algebraic language developed in response to a deep necessity to express and handle mathematical ideas. The appearance of symbolism subsuming all this prior experience stimulated the blossoming of new ideas and led to the construction of formal algebra together with the multifaceted concept of variable. The historical development of algebra indicates that algebraic ideas should be introduced to students in order to help them mathematize situations; moreover it also suggests that situations might be devised which are approachable without formal algebra yet, through involvement in them, pupils are stimulated to perceive the need for algebraic language.

Variables are generally used in school texts without providing any introductory experience which might serve as a basis on which the idea of variable in its different meanings could be developed. In fact, pupils are faced with formal problems involving different characterizations of variable (e.g. equations, tautologies, functions) which they are taught to solve using a set of appropriate rules. This approach suggests, albeit implicitly, that with sufficient drill and practice, students will learn to cope with the different characterizations of variable in an appropriate way and moreover finally integrate their experience to form a multifaceted concept of variable. Research indicates that these expectations have few roots in reality.

Research using computer environments suggests new ways that pupils can be helped to approach the concept of variable. In particular, research investigating the development of pupils' algebraic thinking in Logo settings suggests that in these environments pupils have fewer difficulties. In particular they are more able to consider that a variable may represent a range of values as opposed to considering it representing only one arbitrary value, they are able to attach meaning to the symbols in the Logo procedures, they are able to operate on variables and finally they appear comfortable to use 'unclosed' algebraic expressions. Logo-based environments have thus been shown to facilitate pupils in seeing the power of general methods for solving similar problems and in constructing ways to express these methods.

These are promising results but leave considerable scope for further study. For example, no attention to date has been paid to the different characterizations of variable and the design of environments to facilitate their appropriation prior to any formal algebra teaching. This study sets out to fill this gap. It aims to investigate the feasibility to create pupils' potential to approach different characterizations of variable in specially designed Logo microworlds prior to the study of algebraic language. Pupils aged 12-13 years old without prior formal algebra instruction will be studied. The characterizations of variable considered are:

| | |
|---|---|
| variable as specific unknown (VSU) | - the variable represents a specific but unknown number that can be calculated under given constraints |
| variable as general number (VGN) | - the variable represents an indeterminate number involved in a general method |

| | |
|---|---|
| variables in a functional relationship (VFR) | - variables represent numbers whose values move within a range of values linked one to each other by a relationship |

It is considered that a pupil has developed a potential to cope in Logo with one of these characterizations of variable when (s)he is able to solve specially designed Logo-based tasks involving it. The tasks might be solved by the pupil alone or cooperating with a more capable peer or with researcher's guidance. The researcher helps pupils by providing hints, giving nudges, encouraging pupils to reflect on their own approaches, introducing new concepts or showing pupils the way of solving the given problem. Details on what is considered to indicate pupils' potential in relation to each one of the different characterizations of variable will be specified in each section.

It is anticipated that this study will provide insight into pupils' ways of working with these notions and on the kind of support that might be facilitative. This information might suggest new ways to help pupils construct the idea of variable beyond the specificities of Logo environments.

This thesis is organized in 11 chapters. Chapter 2 presents the background to the study. It summarizes the outcomes of research investigating pupils' learning of elementary algebra. Pupils' approaches to variable as general number (VGN), variables in a functional relationship (VFR) and variable as specific unknown (VSU) are analyzed. Implications for teaching are discussed in reference to Vygotsky's idea of *zone of proximal development*. This discussion is followed by an analysis of how the idea of variable is presented in Mexican school texts for primary school and the first grade of secondary school. A brief survey of the historical development of the notions underlying different characterizations of variable is presented.

Chapter 3 surveys the results of research which has investigated pupils' way of working with variable in Logo. Chapter 4 presents the methodology to be adopted in the present study. Chapter 5 describes a paper and pencil questionnaire specifically designed for this investigation to provide a pre-algebraic profile of pupils concerning their interpretation and use of different characterizations of variable. It gives an overview of pupils' responses and the profiles of a sample of 12 pupils chosen for the case studies.

Chapter 6 describes the three Logo microworlds involving the different characterizations of variable. Chapters 7, 8 and 9 analyze the responses of the case study pupils when

working in the microworlds involving the idea of variable as general number (Chapter 7); variables in a functional relationship (Chapter 8); variable as specific unknown (Chapter 9).

Chapter 10 analyzes the case study pupils' responses in the final interview and their answers to the extract of the questionnaire applied at the end of the study. It discusses pupils' progress and their potential to approach different characterizations of variable. The conclusion of the study and suggestions for further research are given in Chapter 11.

There are two appendices presenting respectively the questionnaire applied at the beginning of the study (Appendix 1) and the Logo work sheets used during the study (Appendix 2).

# CHAPTER 2

# BACKGROUND TO THE STUDY

## 2.1 The multifaceted concept of variable.

'What are these things called variables?'. In this way Wagner (1983) titles one of her articles where she stresses the complexity of this concept and the difficulties beginning algebra students have with it. The concept of variable is not easy to define. Variable is used in different contexts with different meanings, and depending on this we treat it in different ways. Usiskin (1988), for example, highlights four different uses of variable and he links them to different conceptions of algebra.

| Conception of algebra | Uses of variable |
|---|---|
| Generalized arithmetic | Pattern generalizers (translate, generalize) |
| Means to solve certain problems | Unknowns, constants (solve, simplify) |
| Study of relationships | Arguments, parameters (relate, graph) |
| Structure | Arbitrary marks on paper (manipulate, justify) |

(Usiskin, 1988, p. 17)

Although only one of these uses of variable may appear within a specific task, it is usual for students to have to cope with problems whose solution requires dealing with more than one of them. For example, to solve the following problem it is necessary to work with different uses of variable:

'Find an equation for the line through (6, 2) with slope 11 '

(Usiskin, 1988, p.14)

When the problem is approached by considering the general relation between the points of a line and its slope: $y = mx + b$, the student has to be able to cope with general numbers. This formula describes a general straight line and any value can be assigned to each one of the variables involved. However, for a particular line, **m** and **b** are not representing general numbers but constants. In the given example the value of the slope is given and it has to be substituted to **m**. **b** is an unknown that has to be determined

using the data. **x** and **y** are two variables linked by a functional relationship, **x** can be considered an argument that can assume any value, and the value of **y** changes depending on **x**. That is, to solve this problem students have to cope with general numbers, with constants, with unknowns, with variables in a functional relationship and they need to be able to shift from one interpretation of variable to another even when these different characterizations of variable are embedded in the same symbolic representation.

A competent user of algebra is able to give different interpretations to the variable depending on the problem faced. (S)he recognizes when (s)he is facing an equation (1) or a tautology (2) even if they look very similar:

$$X + 5 = X + X \quad (1)$$
$$X + 5 = 5 + X \quad (2)$$

(S)he is able to simplify an algebraic expression and to cope with variability when dealing with functional relationships and to discriminate among the actions that have to be performed with variables in each specific case. To a competent user the differences between the variable uses can seem transparent and insignificant, because (s)he has already recognized them, but these are crucial for beginners and frequently they turn out to be the obstacles that block pupils' learning of algebra (Matz, 1982).

Variables in mathematics are usually represented by literal symbols. However, the same symbols are used to denote different characterizations of variable and different symbols are used to represent the same thing. This might blur distinctions that influence the way variable is manipulated in each case and hide the conditions that determine where and how the variable is varying (Matz, 1982). The symbolism used has been often considered conducive to obscuring the different uses of variable and to create confusion for beginning algebra students (Thorndike et al., 1923; Van Engen, 1953; Wagner, 1981; Matz, 1982). Proposals were made (Menger, 1956) to use symbols that would reflect the conceptual variety behind the different uses of variable. However, approaches like this would not free students from the necessity to be able to interpret the same symbolic variable in different ways. As it was already stressed, in order to solve a problem it is often necessary to shift between different interpretations of the same literal symbol.

In order to elucidate how pupils interpret literal symbols representing variables in usual school algebra contexts, Küchemann (1980) carried out a study with more than 3000 pupils aged 13-15 years old. He administered a written questionnaire asking pupils to

interpret and to manipulate algebraic expressions, and to solve problems involving the use of variables represented by literal symbols.

Küchemann (1980) identified six different pupils' interpretations and uses of literal symbols:

*'Letter evaluated'*.The letter is assigned a numerical value;
*'Letter not used'*. The letter is ignored or its existence is acknowledged without giving it a meaning;
*'Letter as object'*. The letter is regarded as a shorthand for an object or as an object in its own right;
*'Letter as specific unknown'*. The letter is regarded as a unique but unknown number and pupils can operate upon it directly;
*'Letter as generalized number'*. The letter is seen as representing, or as being able to take, several values;
*'Letter as variable'*. The letter is seen as representing a range of unspecified values, and a systematic relationship is seen to exist between two such sets of values.

These findings highlight that pupils have different ways of interpreting letters representing variables. That is, algebra beginners consider that literal symbols used as variables can be interpreted in different ways and their meaning might vary with the problem. But the results reported by Küchemann show also that pupils' interpretation is not always the appropriate one and often it is the source of erroneous answers.

From a Piagetian perspective, Küchemann considered that his classification of pupils' interpretation of literal symbols was reflecting an increasing degree of difficulty. He regarded the first three categories indicative of a low level of response and argued that to consider that a child has an understanding of "beginning algebra", it is necessary for him (her) to be able to cope, at least, with simple items requiring the use of 'letter as specific unknown'. He claimed that pupils have fully understood the use of letter in algebra when they can cope with 'letter as variable'. The order introduced by Küchemann suggests that for pupils is easier to work with 'letters as specific unknowns' than to handle 'letters as generalized numbers', and that they cope easier with 'letters as generalized numbers' than with 'letters as variables'.

Although it might be considered that pupils are approaching elementary algebra when their responses belong to the last three categories the hierarchical view reflected by Küchemann's classification might insinuate an order in which different uses of variable have to be taught. In fact, to work with literal symbols used to represent variables implies to work with different uses of variable. Therefore, to consider that pupils cope

easier with one interpretation of letters than with another implies that it is easier for pupils to work with one characterization of variable than with another. However, each one of the uses of variable can be taught at different levels of complexity, therefore a hierarchical organization of them does not seem appropriate for teaching purposes. Furthermore, researchers investigating pupils' work with different characterizations of variable, namely, variable as specific unknown, variable as general number and variables in a functional relationship, have found that beginning algebra pupils have difficulties with each one of them.

## 2.2 Pupils and the different uses of variable

### 2.2.1 Pupils and variable as general number.

The ability to generalize is one of the most important characteristic of intelligence (Krutetskii, 1976). It is a general ability, that is, it is used in different domains and its use can be observed in very young children. However, the ability to generalize in mathematics is a specific ability, as Krutetskii and his team found; it is the specific ability to generalize numerical and spatial relations, that are expressed by numbers or literal symbols. He found a tight link between the degree of ability to generalize mathematical material and the development of ability to study mathematics. He argues that pupils who were able in mathematics easily grasped what was general in externally different situations, whereas less able pupils needed a lot of training and practice in order to be able to generalize.

Krutetskii states that mathematic generalization evolves and moves from the ability to see what is already known and general in the particular to what is unknown and general. It also moves from the necessity to go from many particular examples that are prerequisite for a generalization, to the ability to generalize from one single example. A point stressed by Krutetskii's study is that mathematical ability and in particular the ability to generalize in mathematics, are not innate but they are acquired. These arguments highlight crucial points that should be taken into account when designing teaching environments to help pupils develop the ability to generalize in mathematics.

Variable is a tool used in mathematics to express generalization. When we want to express a pattern, a regularity or a general method mathematically, we use variables to express the general numbers involved.

Research investigating pupils' capability to work with variable as general number in traditional algebra tasks found that the great majority of the pupils tested have difficulties to cope with this characterization of variable. It was found (Küchemann, 1980; Booth, 1984) that when faced with general numbers represented by a literal symbol a variety of interpretations of the literal symbol may occur and these may vary depending on the problem pupils are solving. Pupils might ignore the letter or interpret it as a specific unknown and assign it a specific value or they might interpret it as an object. Similar results were obtained in a study developed with secondary school pupils in Mexico (Avila et al., 1990).

These outcomes show that pupils are disoriented when they have to interpret literal symbols representing general numbers. Booth (1984), for example, hypothesized that pupils might have a "natural" tendency to interpret letters as standing for specific numbers. Both, Küchemann (1980) and Booth (1984) suggested that the difficulty pupils have with letters as generalized number might depend on pupils' cognitive development.

However, the results of investigations developed in computational environments did not support this suggestion showing that in these environments pupils can be helped to develop an understanding of letter as general number. Thomas and Tall (1986), for example, extended Booth's teaching experiment to computational environments. They designed a software aiming to enable pupils 12 years old with no previous experience in algebra to improve their understanding of general algebraic concepts by structured exploration of particular examples. In particular, they focused on the use of letters as generalized number and letters as variable (Küchemann's definition). The results of a post-test and a delayed post-test showed that the experimental group behaved significantly better than the control group on questions requiring an understanding of the use of letters as specific unknowns, letters as general numbers and letters as variables.

In logitudinal studies in which pupils worked in Logo-based environments Noss (1985) and Sutherland (1987) found that pupils aged 8-11 and 12-13 years old respectively could approach in Logo the idea of general number. They were able to consider that the variable name used for the input to a general Logo procedure was representing a range of values. Sutherland (1987) also found that after a three years Logo work in which pupils had the experience of writing and running general procedures assigning different values to the inputs, six of the eight pupils of her study were able to carry to the algebra context the understanding that a variable name might represent a range of numbers.

These findings suggest that approaches different from traditional ones might reduce pupils' difficulties and help them work with the idea of general numbers and its symbolic representation. Implicitly they also suggest that conventional instruction does not offer enough opportunities for pupils to construct the idea of general number and to develop a meaning for the symbol used to represent it. Therefore, the necessity to look for approaches that would help pupils work with general number and its symbolization.

Suggestions like these are not new. Mason (1985), for example, has investigated how to help pupils to deal with algebra as generalized arithmetic. He identified four steps in the generalization process: 'seeing' - mentally grasping a pattern or a regularity; 'saying' - articulating the insight in words; 'recording' - using symbols to formulate the generalization; 'testing' - verifying the validity of the formulation. In consequence he stresses the relevance of offering pupils the opportunity to become aware of patterns or relationships and to stimulate the necessity to express them instead of rushing children into the use of literal symbols.

These arguments suggest the convenience of working with the idea of general number prior to the introduction of formal algebra language. They implicitly indicate that some of the difficulties pupils have to cope with literal symbols representing variables as general number might reflect a lack of understanding of the concept that the symbol is representing. To design appropriate environments might help pupils with arithmetic background shift from the realm of specific numbers to consider them as objects that can be subsumed under the more general concept of general number. In this way, pupils might be led to approach the idea of general number and its symbolization prior to the introduction of formal algebra.

### 2.2.2 Pupils and variables in a functional relationship

Variables may be used to express a relationship between two changing quantities. When used in this way their main characteristic is that they move within a range of values and the change in the value of one of them produces a change in the value of the other. This characteristic is emphasized when functional relationships are viewed as the expression of change. Dirichlet's definition of continuous function, for example, reflects this view:

> 'Let by a and b be understood two fixed values and by x a variable quantity, which gradually assumes all values lying between a and b. Now, if a single finite y corresponds to every x, in such a way that while x continuously passes through

the interval a to b, y=f(x) likewise varies gradually, then y is called a continuous function of x for this interval' (quoted in Hamley, 1934, p. 14).

However, in present curricula functions are very often defined by two sets and a rule of correspondence which assigns to one element of one set exactly one element of the other set. In this 'modern' definition that was influenced by the development of the set theory, the idea of change and the relation between variables' movement is not stressed any more.

When this definition is used to introduce pupils to functions they may have difficulties to cope with the ideas of domain and range of variation either when the function is represented in graphic or in algebraic form (Markovits et al., 1986; Sfard, 1989). They may also have difficulties with constant functions, with functions represented by a disconnected graph and with functions defined piecewise (Markovits et al., 1986).

Markovits and her colleagues (1986) suggest that these difficulties are a direct consequence of the 'modern' definition of function where the idea of variability is downgraded. Without claiming for a return to a more traditional approach these researchers suggest that pupils should be helped to narrow the gap between the old and the new definition of function. This implies that to support pupils' work with functions and the notions underlying them, namely, domain, range, monotonicity, maximum and minimum, they should be helped to perceive variables as entities whose value changes under given constraints, as well as to focus on the rule of correspondence linking them.

To develop these complementary views of variables in a functional relationship might be interpreted as a need for developing 'versatile thinking' (Tall and Thomas, 1991). 'Versatile thinking' refers to the capability to move freely and easily between the sequential/analytic and global/holistic modes of thinking (Tall and Thomas, 1991, p. 3). Concerning variables in a functional relationship this might be interpreted on the one hand as the capability to consider variables in a static, one-to-one correspondence and to be able to calculate the value of one of them in correspondence to the value assigned to the other. On the other hand, to be capable to look at variables as entities moving in a related way within ranges of values and to cope in this way with the idea of change.

Nevertheless, to cope with the idea of change is not easy for pupils. Küchemann (1980), for example, in the mentioned study (see 2.1, Chapter 2) found that only few pupils could interpret 'letter as variable' and consider that the changes in one set of values depended on the changes in another set of values. Situations involving a dynamic process are often perceived by pupils as static ones. Evidence for this is provided by pupils' own

ways of representing change situations (Bednarz and Dufour-Janvier, 1991). When dealing with the dynamic process pupils seem to look only for some essential characteristic that can be described pointwise. Pupils have also difficulties to perceive change when this is represented by means of graphics or other codes used in mathematics to illustrate dynamic concepts. They tend to interpret these representations in a static way. Difficulties to cope with change were observed for young pupils, 1st and 2nd grades (Bednarz and Dufour-Janvier, 1991) and it was also stressed as a characteristic that pupils studying elementary algebra may have (Heid and Kunkle, 1988).

It was found (Heid and Kunkle, 1988) that to use computer-generated tables of variable values over a given range of numbers might help pupils focus on how the values of algebraic expressions change. To analyze tables helped algebra beginners deduce domains where functions were increasing or decreasing and appreciate the effects of change in related variables.

However, tabular notation might not be helpful to work with non-linear functions. Kieran (1992), for example, found that when working with tabular records 9-grade students (15 years old) who already have had some algebra instruction had difficulties to derive an overall view of the behaviour of non-linear functions from a tabular representation as well as to identify maximum and minimum values. In contrast, to work with a graphic representation provided these students with a global perspective. Nevertheless, other studies have found that younger pupils (Heid, 1989) or pupils classified as being of low-level ability (Dreyfus and Einsenberg, 1981) might have difficulties in working with graphics but not with tables. Dreyfus and Einsenberg (1981) suggest that notions underlying a functional relationship (e.g., domain, image) should be introduced in a graphical setting for high-level students and in a table setting for low-level students.

### 2.2.3 Pupils and variable as specific unknown

During primary school, prior to any algebra teaching, pupils often start dealing with simple problems where they are asked to find a specific unknown value. Usually the unknown value is not represented by a literal symbol, but by another signs (e.g. a box, a line, a void space). When pupils begin the study of algebra the sign is substituted by a literal symbol.

Algebra beginners may reason with unknowns and find the correct value when these are involved in simple word problems. Karplus (1981) found that pupils 12-14 years old

were able to solve number puzzle problems of the form 'I am thinking of a number. I add 12 to my number and then multiply by 6. I get 90. What is the number?' (Karplus et al., 1981, p. 148). He observed that the most frequent reasoning pattern used by pupils to solve this kind of word problems was to work backward from the given result, being trial-and-error the second most used strategy. Algebra beginners may also cope with variable as specific unknown when it appears in one-step equations, that is, equations of the form $ax = b$ , $a + x = b$ and $x + a = b$ (Kieran, 1984). To solve this kind of equations pupils use inverse operations or they substitute different numbers to the literal symbol in order to find the value that balance both sides of the equation. These findings show that faced with simple word problems or with one-step equations algebra beginners may conceptualize a specific unknown and determine its value working backward from the given result or using trial and error.

However, when a specific unknown is involved in equations with a more complicated structure (e.g., $a + x = b + c$, $ax + b + cx = d$, $ax + b = cx + d$) pupils have difficulties to cope with it. These equations cannot be solved in one step. To solve them it is necessary to perform first numeric operations and/or to operate with the unknown. This implies to be capable to perceive the equation globally in order to rearrange the terms before trying to calculate the value of the specific unknown (Herscovics and Linchevski, 1991b). When faced with these types of equations algebra beginners use different strategies in order to avoid operating with numbers and with unknowns. For example, it was found that faced with an equation with one appearance of the specific unknown but with more than one numeric term after the equal sign pupils tended to cut the equation after the first numeric term and to ignore the others (e.g. they solved the equation $4+ x-2+5 = 11+3-5$ as it were the equation $4 + x - 2 +5 = 11$) (Kieran, 1984).

To solve equations of the form $ax + b + cx = d$ implies to be able to operate with specific unknown. That is, this type of equations cannot be solved using only arithmetic operations. To operate with the specific unknown presents difficulties to most algebra beginners. It was observed that to solve this type of equations pupils may try to invert the operations before grouping similar terms (Kieran, 1984). That is, they try to extend to these equations a rule they use successfully with one-step equations. They may also determine the specific unknown by systematic substitution (Herscovics and Linchevski, 1991a). These different approaches show that pupils tend not to operate with specific unknown.

Difficulties to operate with the specific unknown were detected as well when this appears on both sides of the equal sign (Kieran, 1984; Filloy and Rojano, 1984, 1989;

Herscovics and Linchevski, 1991a). Filloy and Rojano (1984, 1989) found that the transition from solving equations of the form $x + a = b$ to the solution of equations of the form $ax + b = cx + d$ is neither immediate nor spontaneous. To solve this kind of equations algebra beginners often use trial and error assigning different values to the different appearances of the specific unknown in the same expression ('polysemia' error) (Filloy and Rojano, 1984). The 'polysemia' error can be avoided by telling pupils explicitly that all the appearances of the specific unknown represent the same value (Herscovics and Linchevski, 1991a). The difficulties pupils have with this type of equations led Filloy and Rojano to suggest that a didactical cut occurs when the specific unknown appears on both sides of the equation. To cope with this kind of equations it is necessary to be familiar with algebraic syntax. Although originated from arithmetic, to cope with algebraic syntax demands to break with certain arithmetical notions. Therefore the presence of the suggested didactical cut. Because of the difficulties pupils have to operate with the unknown also when multiple appearances of it occur on the same side of the equal sign Herscovics and Linchevski (1991a) considered the didactical cut in terms of a cognitive obstacle pupils may have to operate on or with the specific unknown.

These results highlight the difficulties pupils have in operating with or on the specific unknown. Traditional teaching that emphasizes manipulative skills seems not to be helpful for pupils to overcome these difficulties. Moreover, some of the approaches to specific unknown used in elementary school might originate incorrect equations solving strategies. For example, pupils are often taught that to solve an equations is equivalent to invert the operations. Kieran (1988) suggests that this approach may lead pupils to conceive the variable 'as the result of a set of inverse operations - with the letter having no existence of its own within the given equation' (Kieran, 1988, p. 95). This interpretation might be an obstacle for making sense of equations that cannot be solved by inverting the operations and for operating with or on specific unknown. It cannot be expected that pupils will spontaneously overcome these difficulties. As it was stressed by Filloy and Rojano (1989), to help pupils overcome the difficulties they have with operating on and with the specific unknown is an educational task.

Approaches were made to help pupils overcome the mentioned difficulties. Filloy and Rojano (1985b, 1989), for example, used concrete models (the geometrical model, the balance model) to help pupils operate with specific unknown. Their findings show that although in these environments pupils could solve equations with more than one appearance of specific unknown, to use models may hide what is intended to be taught. It may lead pupils to stay within the concrete context and progress within it delaying the

construction of algebraic syntax. Teaching intervention was found crucial to help pupils progress from the model to the construction of algebraic notions.

It was also found (Herscovics and Linchevski, 1991b) that individualized teaching specially designed in order to help pupils overcome the inability to operate on or with specific unknown, led pupils to acquire the capability to group similar terms in an equation. However, it was observed that difficulties persisted when pupils had to cope with the unknown in the form of a singleton.

To look in pupils' background for a cognitive basis on which to build the new knowledge might be also crucial as was suggested by Chalouh and Herscovics (1988). A teaching experiment was designed (Chalouh and Herscovics, 1988) to help pupils without any formal algebra instruction create meaning for expressions involving one unknown and one operation, and for expressions with several unknown and multiple operations. The results obtained showed that pupils could pass from using signs representing place holders to letters, to letters representing a hidden quantity, and to letters representing an unknown quantity. They were also able to cope with algebraic expressions with multiple terms. To indicate explicitly the frame of reference and to ask pupils to answer 'in algebra' was found to be essential in order to help them shift from the arithmetic context to the algebraic one. These findings show that for algebra beginners it might be crucial to let them know clearly the mathematical context in which they are expected to work. This suggests that some erroneous behaviour might be influenced by not having clear the context in which they are expected to solve the problem.

These results show that great efforts have been made in order to look for approaches that would help pupils overcome the difficulties they have with specific unknown. However, more work is needed in order to help pupils grasp the idea of variable as specific unknown and be able to cope with it when dealing with problems and solving equations.

### 2.2.4 Implications for teaching

So far the difficulties pupils have with different characterizations of variable when working in algebra have been stressed. In addition, suggestions that might help pupils overcome these difficulties have been highlighted.

All these suggestions arise from research investigating algebra beginners or pupils that have had no formal algebra instruction at all. They claim for the necessity to approach each one of the different characterizations of variable in a way different from the traditional one. They manifest the necessity to devise appropriate environments where pupils might use their prior mathematic knowledge in order to approach these concepts in a meaningful way. Implicitly these arguments suggest that work should be done prior to approaching variables in a formal way. This leads to consider the feasibility to design environments in which pupils might be led to approach different characterizations of variable prior to formal algebra teaching. To work in these environments might help pupils create a potential to cope with different characterizations of variable and awaken in this way developmental processes that might help pupils cope later with a more formal treatment of variable. The idea of helping pupils create a potential to work with new concepts is linked to Vygotsky's idea of *zone of proximal development*.

In recent years the theory of the Soviet psychologist L.S. Vygotsky (1896-1934) has become a focus of interest for many psychologists studying children cognitive development and the formation of concepts. In particular, Vygotsky's notion of *zone of proximal development* is awakening a growing interest among educational researchers.

This notion derived from both, a critique of the psychological tendencies that were developing in the first half of this century, and the search for a Marxist theory of cognitive development. Vygotsky (1978) criticized the relation established between development and learning by the most outstanding contemporary theories. He stressed, for example, that theories as the one developed by Piaget viewed development as a prerequisite for learning. He considered that a second major tendency was elaborated by James (1958) and that it identified development with learning. He discussed also Koffka's approach stressing that in this view development and learning mutually influence each other but development is always a larger set than learning. He considered that the most frequent error of these different theories was to consider only one developmental stage, the one referring to child's actual development, that is, 'the level of development of a child's mental functions that has been established as a result of certain already *completed* developmental cycles' (Vygotsky, 1978, p. 85).

Without denying that a relation between child's cognitive development and his capability to learn certain subjects exists, Vygotsky claimed that in order to establish a relation between development and learning abilities, two developmental levels must be considered, namely, the actual and the potential development. The actual development is determined by the child's capability to solve problems alone. The potential development

is determined by the child's capability to solve problems in collaboration with a more capable peer or with adult guidance. Vygotsky defined the *zone of proximal development* as ' the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers' (Vygotsky, 1978, p. 86) and he considered that 'an essential feature of learning is that it creates the zone of proximal development; that is, learning awakens a variety of internal developmental processes that are able to operate only when the child is interacting with people in his environment and in cooperation with his peers' (Vygotsky, 1978, p. 90).

These ideas reflect Vygotsky's theory of the social origins of individual higher mental functions such as thinking, voluntary attention, logical memory, the formation of concepts. He considered that any function in the child's cultural development appears twice. First, it appears on the social plane, that is, between people as an interpsychological category, and then within the child as an intrapsychological category (Vygotsky, 1981). As pointed out by Wertsch (1984) this does not simply mean that individual's mental processes develop in a social environment but, that the organization of mental processes is the direct reflect of the social life in which they are formed. Therefore, the social environment in which the new concepts are approached acquires in this theory a crucial role. The mastery of a new concept is considered a consequence of interaction with more competent persons and this interaction creates the *zone of proximal development.* That is, the *zone of proximal development* is not in the child waiting to be awakened by a more competent person but, for a particular subject it is created as a consequence of negotiation established between the child and the expert (McLane, 1987).

The notion of *zone of proximal development* was for Vygotsky directly linked to instructional purposes. In contrast with those who considered that instruction can be provided depending on the developmental level reached, Vygotsky considered that 'instruction is good only when it proceeds ahead of development. It then awakens and rouses to life those functions which are in a stage of maturing, which lie in the zone of proximal development' (quoted in Rogoff and Wertsch, 1984, p. 3). At this stage children can cope with these new functions only with help of adults or more capable peers. This contrasts with usual educational practice that is organized considering pupils' level of unaided competence. When instruction focus on the levels pupils can reach with help, they are led to work successfully with topics that were previously considered out of their competence. When these ideas are applied to school instruction the role of the

teacher changes from being a transmitter of a body of knowledge to being an expert who by interacting with pupils helps them develop a potential for solving new problems and constructing new concepts. Peers collaborations acquires as well an outstanding role in contrast with individual and often competitive work. The organization of the social environment in which teaching activities take place also becomes crucial.

Because social interaction is considered by Vygotsky to be the fundamental vehicle of education and the creation of the *zone of proximal development*, speech is necessarily involved. In Vygotsky's theory speech and the evolution of its use have a crucial role in the formation of higher mental functions. He considers that even for young children the primary function of speech is social communication. Child's communicative speech evolves later to egocentric speech, that is, speech-for-oneself. Vygotsky states that this appears 'when the child transfers social, collaborative forms of behaviour to the sphere of inner-personal psychic functions' (Vygotsky, 1989, p. 35). Egocentric speech is then used by the child to direct himself during an activity he develops without the expert help. In this way, speech changes from having an interpersonal to having also an intrapersonal function (Vygotsky, 1978) and it is used as a problem solving tool. In Vygotsky's theory egocentric speech leads to inner speech, that is, to individualized verbal thought. In this context, egocentric speech is considered to be the transition step between social interchange characterized by communication and individualized inner mental functions (Kozulin, 1989).

Although Vygotsky's ideas and the implications for education have been analyzed and discussed by many psychologists (e.g. Griffin and Cole, 1984; Campione, Brown, Ferrara and Bryant, 1984; McLane, 1987; Palacios, 1987), these have not yet influenced substantially the pedagogical practice, in particular, the teaching of mathematics. Learning is still often considered to be an individual and not a social matter. However, interesting attempts to promote, for example, peer collaboration and to analyze the role of teacher intervention were made (see Hoyles and Sutherland, 1989). But a lot of work still has to be done in order to design environments which create pupils' potential to work with mathematical concepts, in particular with different characterizations of variable. This is not the usual school approach, for example, in Mexican schools .

## 2.3 The variable in Mexican school texts

### 2.3.1 Introduction

Because this study will be conducted in Mexico with Mexican pupils, the way different characterizations of variable are introduced in Mexican school texts will be briefly discussed. These texts reflect the official mathematic program used during the last twenty years. This has been analyzed and criticized from its appearance. Some methodological inconsistences and the incoherence between the declared psychological background and its practical realization have been highlighted (Rojano, 1977).

In Mexico the study of algebra starts officially during the second year of secondary school (pupils aged 13-14 years old). However, pupils are usually presented with symbolic variable earlier: during the primary school and during the first grade of secondary school. Therefore, the official texts used in the fourth, fifth and sixth grades of primary school, and some of the widely used mathematical texts for the first grade of the secondary school were analyzed.

### 2.3.2 Variables in a functional relationship

In the mathematical texts for primary school variables in a functional relationship appear principally in geometry. They are represented by literal symbols and they are used mainly to provide pupils with formulae for areas and volumes. Although the geometric formulae represent functional relationships between the variables involved, this aspect is not stressed. Pupils are expected to memorize formulae and to use them in order to obtain a result by substituting given values to the literal symbols involved. In this way literal symbols are emphasized as representing 'place holders', that is, symbols to which specific but arbitrary numbers might be substituted. A similar approach is used in the first year of secondary school where pupils are introduced to formulae for calculating the volume of geometric objects.

Few texts for the first year of secondary school introduce the idea of function. This is introduced, for example, in order to discuss problems of ratio and proportion (e.g. Parra Cabrera and Walls Medina,1970, p. 228). Function is presented as a dependence between two quantities (ibid.) and also as a rule of correspondence between two sets (ibid.). This

approach to functions emphasizes the idea of correspondence between variables but not that of change.

### 2.3.3 Variable as general number

The analyzed texts showed that in primary school the only contact pupils have with variable used as general number is in the fifth grade where pupils are introduced very briefly to the symbolic expression of some general arithmetic rules. The following example illustrates how the general commutative rule is deduced from few particular examples and how it is symbolized by means of letters used as general numbers:

'Observe that:

$2 + 3 = 3 + 2$
$7 + 2 = 2 + 7$
$9 + 6 = 6 + 9$
$3 + 4 = 4 + 3$
$7 + 8 = 8 + 7$
$5 + 3 = 3 + 5$
$0 + 2 = 2 + 0$
$1 + 3 = 3 + 1$
etc..

we can conclude that if a and b represent two whole positive numbers, it will be $a + b = b + a$'.

(Matemáticas 5, p. 94)

No additional explanation concerning the meaning of the literal symbols used is given. After this introduction pupils are invited to solve some exercises in order to foster the just given general commutative rule:

' If $4 + 3 = 3 + m$ then $m =$
If $7 + 23 = x + 7$ then $x =$ '

(Matemáticas 5, p. 94)

In these exercises the literal symbol is used to represent a specific unknown and not a general number. However, when faced with these exercises pupils have not yet had experiences in solving equations where the specific unknown is represented by a letter. Their experience is with literal symbols representing place holders. Therefore, these

exercises might contribute to strengthen the interpretation of literal symbols as place holders even when they represent specific unknowns. In a similar way the general associative and the distributive rules are introduced. There is no additional work with general numbers during the primary school.

The main aim of the mathematics course for the first year of secondary school in Mexico is to introduce pupils to generalized arithmetic. Pupils are usually presented with generalized arithmetic rules involving the use of variables as general numbers represented by a literal symbol. Throughout the books algebraic expressions representing generalized arithmetic rules are used. However, no construction of general methods is promoted and no explanation about the meaning of the literal symbols used is given. Moreover, on occasion the initials of meaning words are used to represent general numbers. This is illustrated by the following example that presents in a formal way what is meant by 'division':

$$'D = dc + r \quad \text{with} \quad r < d'$$

(Habacuc,1980,p. 74)

Here D stays for 'dividendo' (the number that is divided), d for 'divisor' (the number that divides), c for 'cociente' (quotient) and r for 'remanente' (remainder). This notation far from helping pupils to interpret the literal symbols as general numbers is leading to an interpretation of them as labels.

These approaches suggest that it is taken for granted that pupils will be able to conclude, by themselves, that in general rules literal symbols represent general numbers.

### 2.3.4 Variable as specific unknown

In primary school pupils deal with simple equations involving the idea of specific unknown. The specific unknown is always represented by a sign other than a letter (e.g. a box, a line) or a void space. Pupils are asked to find the 'missed number' in problems like

$$6 + __ = 0 \text{ or } 7 + __ = 3$$

or to complete expressions, for example, in

$$2/9 + __/9 = 6/9, \quad 1 - 3/7 = __/__ , \quad 6/11 = 66/__$$

No introduction to the use of literal symbols to represent specific unknowns is given. During the first grade of secondary school pupils are on occasion introduced to equation solving. For example, in Serralde (1982) some numeric examples concerning the subtraction of natural numbers lead to the following general conclusion:

' If a + b = c then c - a = _____ or c - b = _____ '

(Serralde et al., 1982, p. 37)

After this and some numeric exercises dealing with subtraction, pupils are introduced to equations solving in order to apply the previously deduced general rule:

'5 + a = 12
a = 12 - 5 ... converting it to a subtraction
a = 7 .... because 5 + 7 = 12'.

(Serralde et al., 1982, p. 39)

No explanation about the use of literal symbols is given although they are used to represent both, variable as specific unknown and variable as general number. This suggests an implicit assumption that pupils will be able to interpret the literal symbols in a correct way, that is, as general numbers when facing general statements, and as specific unknown when coping with equations.

### 2.3.5 Other uses of literal symbols

The analysis of the selected texts shows that literal symbols are widely used also as labels and as objects. For example, literal symbols are used as labels to name true and false propositions:

'a = the cat is an insect
.....
f = 3 x 9 = 8 x 3
.....
l = x is an odd number
..... '

(Serralde et al., 1982, p. 3)

They are used to name sets, objects of a set and to represent the general element of a set as it is illustrated by the following example:

'A = {x is a letter of the word 'mathematics'}
Which of these sets is a subset of A ?:
B = {a,e,i}
C = {m,a,t,e,o}
D = {x is a letter of the word 'house'}
E = {a,m,o} '

(Escareno et al., 1979)

These different uses of literal symbols appear very often in the same text. This might contribute substantially to create confusion about the meaning literal symbols may have in mathematics.

### 2.3.6 Summary

This review of Mexican school mathematic texts shows that during the primary school pupils are presented with variables in a functional relationship. These are represented by literal symbols. But these are usually emphasized as 'place holders' and not as representing variables in a functional relationship. During the first year of secondary school the concept of function is on occasion briefly introduced in a formal way and the correspondence between the variables involved is stressed. However, the idea of change is not stressed and there are no activities leading to the construction of the idea of variables in a functional relationship as entities moving in a linked way.

In the first year of secondary school pupils are faced with generalized arithmetic expressions involving the use of literal symbols to represent variables as general numbers. However, there is an absence of activities leading to the construction of the idea of variable as general number.

In primary school or during the first year of secondary school pupils are not introduced to the use of literal symbols to represent variables as specific unknowns, although they are used to solve simple equations where the specific unknown is represented by other signs. However, on occasion they may be expected to solve equations where a letter is used to represent a specific unknown.

In the analyzed texts there is no activity leading to the construction of different uses of variable. Pupils are rushed to the symbolic representation of variables. This suggests that they are expected to be able to make sense of the literal symbols and to work with different uses of variable. This approach may reflect an implicit belief that by drill and practice pupils will acquire the capability to cope with each one of the different uses of variable as well as to shift from one to another when required by a specific task. Although it is not expected that pupils will work alone and it is expected that the teacher will play a fundamental role in helping pupils cope with different uses of variable, the teachers usually reinforces the approach reflected by the school texts.

Due to the complexity of the concept of variable it is perhaps too optimistic to expect that an approach like the one reflected by the analyzed texts will help pupils to cope successfully with it. Moreover, the outcomes of research investigating pupils' difficulties with different characterizations of variable suggest the necessity to work with them prior to the introduction of algebraic symbolism. This suggestion is further supported by the historical development of the notions underlying some of the different characterizations of variable, namely, variable as specific unknown, variables in a functional relationship, variable as general number. These notions developed during centuries before the appearance of a symbolic system that subsumed them under a common symbolism and started the development of symbolic algebra.

## 2.4 A survey of the historical development of the notions underlying different characterizations of variable

### 2.4.1 Introduction

This section provides a glimpse of the development of the notions underlying different characterizations of variable prior to the appearance of algebraic symbolism. It is based on a revue of some popular texts on the history of mathematics (Cajori, 1980; Bell, 1945; Rouse Ball, 1960; Klein J., 1968; Bunt et al., 1976; Neugebauer, 1969; Gillings, 1982; Youschkevitch, 1976) and not on original sources.

### 2.4.2 The early presence of specific unknown

The study of Babylonian and Egyptian documents (about 2000 B.C.) as well as the analysis of early Hindu and Chinese (I. to VII. century A.D.) mathematics show that in these civilizations problems were solved where the determination of a specific unknown quantity was required (Neugebauer, 1969; Gillings, 1972). Problems involving what today we would call second and third degree equations and systems of linear equations with two or more unknowns were solved by Babylonians, Hindu and Chines (Neugebauer, 1969; Franci and Rigatelli, 1979; Bunt & al., 1976; Cajori, 1980). Although Egyptian algebra was far less advanced, they were solving numerical equation of first and second degree (Neugebauer, 1969; Gillings, 1972).

These cultures did not have anything approximating modern notation. Babylonian and Egyptian algebra was rhetoric (Neugebauer, 1969), that is, totally expressed in natural language. Chinese algebra was manipulative and linked to direct calculation. However, an attempt of representing the specific unknown can be observed in Chinese algebra. They used a computing board and carried out calculations by means of rods. To represent the specific unknown whose value they were looking for they used a rod that was easily distinguished from the others (Cajori, 1980).

An interesting approach to the symbolization of specific unknown quantities is observed in Hindu mathematics. They identified it by a color name (Rouse Ball, 1960) and when several unknowns occurred they gave them distinct names. They used the initials of the names as a symbol to represent the unknown quantity. Sometimes the word 'color' was used as synonymous for 'unknown quantity' (Rouse Ball, 1960). Although Hindu algebra cannot be considered symbolic, it might be classified as syncopated (Rouse Ball, 1960), that is, a combination of natural language and symbols, due to the use of words' abbreviations.

The most important approach to the use of a symbol to represent a specific unknown occurred in Greek algebra whose development reflected the Mesopotamian tradition (Neugebauer, 1969). Diophantus (about 250 A.D.) in 'Arithmetic', his main work, used a symbol to represent the 'unknown quantity' and its powers in problems that led to equations of first and second degree. However, as remarked by Klein, J. (1968) in his comment on Diophantus:

> 'When the unknown or its sign is introduced in the process of solution it is precisely not indeterminacy in the sense of "potential" determinacy which is intended ... is the unknown to be understood as an "indeterminate multitude" only

from the point of view of the completed solution, namely as "provisionally indeterminate" and as a number *which is about to be exactly determined in its multitude ...'* (Klein J.,1968, p. 140).

However, these findings and comments indicate that in ancient cultures prior to the appearance of symbolic algebra problems involving the idea of specific unknown were very often approached and solved .

### 2.4.3 The early idea of functionality

Some of the ancient mathematics texts discovered in Babylonia and Egypt contained numeric tables (e.g. Babylonian multiplication tables; Egyptian fraction tables) or empirically collected astronomical data (Neugebauer, 1969). These tables testify a first record of dependence between quantities.

Tables were used to determine an unknown quantity. This was found by looking for it in the table or by calculating it by interpolation. Referring to Babylonians, Youschkevitch (1976) remarks that these

> 'empirically tabulated functions thereafter became the mathematical foundation for the whole subsequent development of astronomy' (Youschkevitch,1976, p. 40).

However, this author considers that Babylonians and Egyptian had no general notion of variable quantity and function.

Klein J. (1968) and Youschkevitch (1976) indicated that a view of a general notion of variable quantity is missing also in Diophantus' work. Klein J., for example, signals

> '... it appears to be Diophantus' peculiarity - corresponding to the primitive stage of algebra which he represents - to pose problems belonging to indeterminate analysis ..., but always to transform these problems into determinate equations by means of arbitrary numerical assumptions which permit a univocal (although by no means exclusively integral) solution.' (Klein J.,1968, p. 133)

In a similar way Youschkevitch comments:

> '... it would have been an impermissible modernization to see the idea of a variable quantity in the proper sense in the works of Diophantus, who did use substitutions for calculation of rational roots of indeterminate equations and

> whose method does make it possible in many instances to calculate an infinite number of values of the unknown of the indeterminate problem. At best, it is possible to speak ... about the actual use of a *substitution variable*, but not about the *fully free variable* characteristic of the algebra of Viete' (Youschkevitch 1976, p. 44).

Comparing Hindus' with Diophantus' treatment of indeterminate equations, Cajori (1980) remarks that while the latter was satisfied with a single solution, the former tried to find all the possible solutions. Referring to Aryabhata's solutions in integers of linear equations of the form ax + by = c (modern notation) the same author stresses that probably these kind of equations grew out of astronomical problems and were used to make astronomical predictions. This comment suggests the possibility of an early connection between the idea of 'specific unknown' in an indeterminate equation and the idea of 'variable' in a functional relationship, but a deeper historical investigation would be needed to support this suggestion.

Although the ideas of change and variable quantity were considered by the Greeks, they did not become 'an object of mathematical study' (Cajori, 1980 p. 44). However, the idea of functionality reached an interesting development among the Greeks. The search for quantitative interdependence of various physical quantities is revealed in attempts to determine the simplest laws of acoustics by the early Pythagoreans (about 600-500 B.C.) (Youschkevitch, 1976). It seems also that Greeks did not restrict themselves to the use of tabulated functions, as Babylonians or Egyptians (Youschkevitch, 1976). They developed the notion of "symptom" in relation to the theory of conic sections:

> 'A symptom of some conic section represents, a modern mathematician would say, for each point of the given curve one and the same functional dependence between its semichord y and the segment x ... ' (Youschkevitch, 1976, p. 40).

I want to emphasize the importance of the idea of "symptom", that seems to be a first approach to the idea of 'one and the same functional dependence' relating elements of two sets, an idea developed in western mathematics in the XIV century. However, due to the absence of any algebraic formula or analytical expression, this dependence was always described verbally or by means of geometrical algebra (Youschkevitch, 1976).

However, as Youschkevitch says referring in general to ancient civilizations:

> '... there was no general idea of functionality in ancient times' (Youschkevitch 1976, p. 42).



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Additionally the same author comments:

> '... antique mathematical literature lacks not only words tantamount to the term function but even an allusion to that more abstract and more general idea which unifies separate concrete dependences between quantities or numbers in whichever form (verbal description, graph, table) these dependences happen to be considered. There is a good distance between the instinct for functionality (Bell) and the perception of it...' (Youschkevitch,1976, pp. 42-43)

A more general concept of the notion of function was developed in XIV century when the schools of natural philosophy at Oxford and Paris declared mathematics to be the main instrument for studying natural phenomena. Youschkevitch observes that the theories that developed in XIV century are suggesting

> 'a conscious use of general ideas about independent and dependent variable quantities; though direct definitions of these quantities are lacking, each of them is designated by a special term .... in these theories, a function is defined either by a verbal description of its specific property or directly by graph' (Youschkevitch, 1976, p. 46-47).

These comments show that the idea of one-to-one correspondence between quantities and that of change can be found in early civilizations. A more advanced idea of functional dependence between quantities is observed in the development of the idea of "symptoms" by the early Pythagoreans. A general notion of independent and dependent variables appears until the XIV century, although there is an absence of a symbolism to represent them.

### 2.4.4 The early search for general methods and the appearance of general number

The analysis of ancient mathematics texts reveals an existing interest for deducing general methods for solving problems involving equations and for making astronomical predictions. In Babylonia and Egypt, for instance, similar problems were grouped together and one was solved in detail as an example of how to solve the others (Neugebauer, 1969; Gillings, 1972). The detailed solution of one problem was intended to illustrate general procedures to solve classes of problems (Bunt et al., 1976). Neugebauer (1969) emphasizes that the main focus in Babylonian 'problem tasks' was on the general procedure and not on the numeric examples The numeric examples were used only to illustrate the procedure. The same author comments that in some Babylonian

astronomical tables, data were followed by a detailed example to illustrate the procedure to be used to determine an astronomical magnitude. The example ended by the sentence "such is the procedure" (Nuegebauer 1969, p. 100). In this way the procedure used was emphasized and not the particular example chosen only to illustrate a procedure.

The use of repeated solutions of similar arithmetic problems in order to communicate a general procedure is suggested also by Egyptian arithmetic. Bunt et al. (1976) consider that this testifies to the existence of some generalization, even though no procedure was ever stated explicitly or proved.

From the texts consulted the presence of other attempts to look for general methods in ancient times could not be deduced. Moreover, on occasion the absence of the search for a general method is explicitly signaled. For example, Klein J. (1968) stresses that a crucial weakness in Diophantus' work was his lack of interest in looking for general solutions. He comments that Diophantus always looked for solutions with some kind of given restrictions, for example, for integer solutions or for mutually related solutions.

### 2.4.5 Vieta's work - A turning point

Vieta (1540-1603), as other scientists in the second half of XVI century, was strongly influenced by ancient Greek texts, but the way of reinterpreting them reflects a conception and an understanding of the world that differed from that of the ancient times and which led to the development of completely new ideas (Klein J., 1968). Vieta focused his work on the development of a general method, the "Analytic Art", which was intended to be an instrument for the solution of problems in general (Klein J., 1968).

Vieta's "Analytic Art" was strongly influenced by the method presented by Pappus with reference to geometric theorems and problems, and by the procedures of Diophantus' Arithmetic. He conjectured 'that a generalized procedure which is not confined to figures and numbers lies not only behind the geometric analysis of the ancients but also ... behind the Diophantine Arithmetic ...' (Klein J., 1968, p.158)

However, the great innovation and crucial difference with respect to the ancient "general methods" for solving problems was Vieta's conception of the object handled by the general method. Even if, general methods were developed in ancient times they were always applied to numbers, magnitudes, points, periods of time, etc.. There was no conception of a "general object" on which the general method was applied. Vieta

conceived a general object, "species", and treated it as an object he could operate on by well defined rules. This "general object", while preserving its link with numbers, represented general magnitudes and became the object of a general mathematical discipline, not identifiable either with arithmetic or with geometry. This "general object" represented a potential value, however, it was treated as an actual mathematic object capable to be manipulated and operated on (Klein J., 1968).

Thus, with Vieta, a completely new concept appears: the concept of "general number". However, it must be stressed that, even if Vieta conceived very clearly the generality, dealt with it and introduced a symbolism for it (he denoted unknown quantities by vowels and coefficients by consonants), he did not have yet a completely developed symbolic system to express his theory. In fact, to describe the "Analytic Art", as well as the operative rules to be followed when dealing with "species", he used rhetoric or syncopated algebra.

Although Vieta did not focus on problems involving functionality and he did not apply his new method to further this concept, the creation of symbolic algebra had a great influence on the development of the idea of function and related variable quantities (Youschkevitch, 1976). In the XVII century, Fermat and Descartes, independently of each other, applied the new symbolism to geometry and presented the analytical method of introducing functions. Descartes considered the new algebra as a useful tool to model and think about problems implying indeterminate or unknown quantities and he, for the first time, considered that

> 'an equation in x and y is a means for introducing a dependence between variable quantities in such a way as to enable calculation of the value of one of them corresponding to a given value of the other one' (Youschkevitch, 1976, p. 52).

After this first approaches algebraic language was gradually adopted to express functionality.

### 2.4.6 Summary

This short overview shows that the presence of problems requiring the determination of a specific unknown quantity can be traced back to the earliest mathematical texts. The idea of change and one-to-one correspondence also appeared in different ancient cultures. During thousands of years the mathematics that developed linked to these concepts, searched for methods useful for solving different problems. However, the 'general

methods' proposed were never expressed in a general way either symbolically or rhetorically, but always by means of particular examples. Although a first approach to the idea of 'general number' may be viewed in De Nemore's De Numeris Datis in XIII century (Filloy and Rojano, 1985a), it is with Vieta in XVI century that general methods are first expressed symbolically by means of 'general numbers'.

This indicate that there were centuries of experience in solving problems involving the determination of a specific unknown, in working with related variable quantities and in looking for general methods before the source of a symbolic language that offered a common symbolic system to handle these concepts. The introduction of symbolic variables is a culmination of a long historical process as well as the starting point of a completely new discipline. At the present time the generic term 'variable' subsumes the notions of specific unknown, general number and variables linked by a functional relationship. Therefore, the term 'variable' as it is used today is a multifaceted concept, that contains the historical development of specific unknown, of general number and of inter-related quantities.

It is not my claim that to introduce pupils to the multifaceted concept of variable the historical development of different characterizations of variable should be followed. In fact, there are distinctive characteristics between the historical and the individual development of these concepts. In the course of history these concepts were invented and elaborated. On the contrary, pupils are expected to assimilate them. However, the historical development of these concepts may suggest approaches that might be used to facilitate pupils' work with them. This brief historical overview suggests that the development of these notions in a pre-symbolic stage was crucial for the appearance of algebraic symbolism. The creation of algebraic symbolism gave sense and unified all these different uses of variable. A plausible explanation for pupils' failure when they are introduced directly to symbolic algebra might be a lack of previous experience with all those different uses of variable at a pre-algebraic level. This kind of experience would provide pupils with a richer background upon which algebraic language might be initially introduced as a tool for handling problems involving different characterizations of variable.

# CHAPTER 3

# LOGO, PUPILS AND VARIABLE

## 3.1 Introduction

The programming language Logo was developed in the late 1960s by Papert and his colleagues at the Massachusetts Institute of Technology. Arguing that our daily world is poor in mathematics, Logo was specifically designed to help pupils of different ages develop mathematical thinking through programming. In particular, Papert (1980) suggested that learning Logo could help students learn algebra and geometry. However, Logo is now more than a programming language. It has become the kernel of a philosophy of education in constant development.

Educational research in the last fifteen years has devoted considerably efforts to creating and evaluating mathematic learning in Logo environments and in studying the linkages between Logo and school mathematics. In particular, which algebraic notions can be developed in Logo have been investigated (Sutherland,1987, 1989, 1991, 1992). Special attention has been paid to the notion of variable in the context of turtle graphics (Hillel, 1992). When pupils' approach to the concept of variable in Logo is analyzed, what is usually referred to by the term 'variable' is its characterization as general number. Not much investigation has been made concerning the possibility to approach in Logo other characterizations of variable, namely variables in a functional relationship and variable as specific unknown.

## 3.2 Logo and variable as general number

The idea of variable is usually introduced in Logo through general Logo procedure (GLP). A GLP is a formal way of expressing a generalization in Logo and to do this symbolic variables are used. Therefore, in a GLP variables represent general numbers.

To write a GLP is not straightforward for pupils. It requires several abilities:

- coping with the idea of procedure
- analyzing a problem in order to identify what varies and what is invariant

- naming and declaring the variable
- understanding that variables represent any value
- operating on the variable
- writing 'unclosed' algebraic expressions (e.g. :X + 3).

### 3.2.1 The idea of procedure in Logo

Pupils cope with procedures both in daily life, for example, when following the rules of a game, and in school, for example, when solving a problem. However, they are not used to reflect on them (Papert, 1980). The emphasis is usually on obtaining a result and not on the steps followed to obtain the result.

To write a Logo procedure implies not only to obtain a result but also to make explicit all the steps producing it and to codify them in the programming language Logo. A Logo procedure is a list of commands (Logo primitives or procedures) which has been given a name. Once written it becomes an entity that can be changed and used on its own or within another procedure. Pupils often have difficulties recognizing a Logo procedure as an entity with all these aspects (Papert, 1980) and they tend to consider it only as a way to save a list of instructions. In this way they miss the idea of a procedure as a device to be used in a wider project.

However, evidence was provided that pupils can progress in understanding, for example, the ideas of subprocedure and modularity. Hoyles and Sutherland (1989) found that when pupils are encouraged to use procedures and subprocedures within projects and they are led to reflect on the state of the turtle within their procedures they can progress in the understanding of these notions. Lemerise and Kayler (1986) provided evidence to show that with teacher support pupils aged 9-12 years old could progress to a better understanding of the idea of procedure in Logo.

### 3.2.2 Distinguishing the variable aspects from the invariant ones

In order to write a GLP it is necessary to analyze the given problem to identify what are its general features and to distinguish them from the specific ones. This ability is not always present in pupils. Working with 12-13 years old pupils Sutherland (1989) found that it was problematic for them to analyze what was invariant within a situation. However, Logo provides facilities that may help pupils make this analysis. In order to write a GLP pupils often write first of all a fixed Logo procedure (FLP), solving in this

way the problem for a particular case and after this they use the FLP in order to write the GLP by substituting a variable to the specific numbers (Healy, Hoyles and Sutherland 1990). Although this is a useful method helping pupils to approach gradually the writing of GLPs, it must be considered that when using it some errors may occur. For example, pupils may over generalize. That is they might substitute a variable to all the numbers that appear in the FLP independently of their specific role. This would indicate that pupils are not yet distinguishing between what is variable and what is invariant in the given problem. However, the interaction with the computer and appropriate teacher's support might help pupils re-analyze the problem in order to identify the variant and the invariant elements and gradually construct a correct version of a GLP.

### 3.2.3 The naming of variables

In Logo meaningful words are often used to name variables. This gives a clear referent to the variable and helps pupils to accept it. However, it can also lead students to think that the specific word used has meaning in itself (Sutherland, 1987) and that the role of a variable is determined by its name (Hillel and Samurcay, 1985). Concerning the naming of variables in Logo researchers have different points of view. Hillel (1992), for example, considers that the convenience of using meaningful names may depend on the particular situation and he suggests that 'there might be situations in which the use of X and Y may actually remove some conceptual problems' (Hillel, 1992, p. 19). On the other hand, Sutherland (1987) suggests that pupils need to be encouraged to use different variable names: meaningful names, 'nonsense' names and single letters in order not to link a specific name with the role of a variable.

When writing general Logo procedures with more than one input it is necessary to assign different names to the variables. This might help pupils to accept different variables names and understand that different variable names can represent the same value. Algebra research found that when working with algebraic expressions pupils are not always aware that arbitrary letters can be used to represent a variable and that two variables may have the same value. They often consider that changing a letter implies changing its value (Wagner, 1981). On the contrary, Sutherland (1987) found that within Logo environments four out of the eight pupils she studied understood this idea.

### 3.2.4 Understanding that in a general Logo procedure variables stand for any value

When working in usual algebra environments pupils have difficulties in understanding that a variable stands for a range of values (Küchemann, 1980; Booth, 1984). Research studying how working in Logo may affect pupils' understanding of algebraic concepts, showed that in specially designed Logo environments pupils can understand that a variable represents a range of values (Noss, 1985; Sutherland, 1987). Six of the eight 12-13 years old pupils studies by Sutherland (1987) were also able to transfer this understanding to paper-and-pencil algebra tasks.

Working with younger pupils, 8-11 years old, who had not yet been introduced to any formal algebra, Noss (1985) found that they could use a name to stand for a range of values. For example, one of the studied pupils used :Peter and :Jane as variables' names. These names were used to represent the number of marbles two children had in a given problem. When interviewed this child argued that it was possible to 'Type in how ever size you want it... How ever many they want. How many they want Peter to have, and how many they want Jane to have' (Noss,1985 p. 415). This indicates that she considered that variables names were representing any number of marbles. Noss explains his findings suggesting that to run the procedure typing in one value at a time may have helped pupils to conceptualize a variable standing for a range of values.

With this comment Noss points to an important characteristic of Logo namely, to its interactive character. This Logo feature favours the development of a dialectic process between the symbolization of a general statement and its use by obtaining a particular example of the general problem. This link general $\leftrightarrow$ particular might be a crucial aspect helping pupils construct gradually the characterization of variable as general number. Logo is so not just a formal language in which generality can be expressed, but it is an environment where the formal expression of a generalization becomes a tool to be used. It was found that to use a GLP can help pupils attach meaning to the symbols, to manipulate them and to see the results of doing it (Hoyles and Noss, 1987b). The process of going from a particular case to the expression of a general case and back again to a specific instantiation of the generality is favoured by Logo and it seems to be essential for the development of the understanding of variable as general number. Sutherland (1991) from a Vygotskyan perspective compares the learning of algebraic ideas in Logo with the learning of natural language. She stresses its dialectical character and comments that 'For many pupils the computer-based symbolism is an essential tool in

their negotiation of a generalization, and the way they come to progressively formalize informal methods' (Sutherland, 1991, p. 44).

This aspect contrasts with how general expressions are usually presented at school at the beginning of the teaching of algebra (see for example, 2.3.3, chapter 2). General expressions are given to algebra beginners, or sometimes deduced very quickly by the teacher, and the emphasis is on the teaching of manipulative rules and on the development of manipulative skills. Pupils have to cope with general expressions without knowing what they are representing and what for they can use them. It is only later, when dealing with more sophisticated mathematic knowledge (e.g. analytic geometry) that general mathematic expressions are used as tools and eventually they may acquire meaning for students.

### 3.2.5 Operating on variables and accepting 'unclosed' algebraic expressions

When writing GLPs it is very often necessary to operate on variables and to write expressions that could be considered similar to 'unclosed' algebraic expressions (Fig. 3.1). From now on these Logo expression (e.g. :X + :X/3) will be called 'unclosed' expressions or 'unclosed' algebraic expressions.

```
TO F :X
FD :X
RT 90
FD :X / 3
BK :X / 3
LT 90
FD (:X/3) * 2
RT 90
FD :X + :X/3
END
```

**Fig. 3.1** Operating on variables and writing 'unclosed' algebraic expressions in a general Logo procedure (GLP).

It was observed (Noss and Hoyles,1987; Sutherland,1987) that in Logo pupils may avoid operating on variables by naming additional variables. However, Sutherland (1991) suggests that this pupils' approach seems to provide a structure to their generalization

process. In fact, during the process of defining the procedure pupils may remove additional variables as the relationship between the variables involved in a specific problem is explicitly perceived (Sutherland, 1987). But not always the relationship between variable elements in a problem is perceived by pupils or, if it is perceived, pupils are not always able to express it in a general way. For example, Hillel (1992) found that in order to write a GLP to draw regular polygons pupils used two variables (Fig. 3.2), even if they were aware of the relationship between the number of sides and the angle and they could handle it numerically. However, they were not able to express the relationship in a general way.

```
TO POLY :N :A
REPEAT :N [FD 30 RT :A]
END
```

**Fig. 3.2** Example of pupils' avoidance of operating on variables.

The way in which pupils use variables in Logo, their capability to operate on it and to use them in 'unclosed' algebraic expressions might be linked to the way they have been introduced to them. In order to operate on variables pupils need support otherwise they 'can develop the practice of adding unrelated variable to their Logo procedures without making any relationship between these variables explicit' (Sutherland, 1991, p. 44). Sutherland (1991) reports that after having been introduced to the use of variable as a scaling factor in a teacher-devised task, 12 out of 17 studied pupils aged 12-13 years old could use variables in their own tasks and they could confidently operate on them and use 'unclosed' algebraic expressions in Logo procedures. She also points out that younger children, 10-11 years old, who have been taught Logo by their class teacher, could operate on variable when working on a task that required it. Working with 12-13 years old pupils without formal algebra instruction it was found (Ursini, 1990c) that after a short Logo experience (9 hours), they could use variables and operate on them in order to write 'unclosed' expressions. Given the procedure

```
TO GUESS.WITH :X
PRINT :X + 3
END
```

pupils were asked to run it for different values. After this, they were invited to invent a new expression, to substitute it to :X + 3 , to hide the procedure and challenge a peer to discover the expression by using the procedure as an input-output machine. Pupils were very motivated by this activity. They were able to invent several 'unclosed' expressions

(e.g., 7 * :X -2, :X * 2 / 2 + 1, (42 - :X) / 2) and they modified the given GLP by substituting the new expression to the given one. To deduce the hidden expression they assigned different values to the variable showing that they were considering that any value could be assigned to it. Based on the numeric results obtained they were able to deduce the hidden expression.

## 3.3 Logo and variable as specific unknown

Although the use of variables in Logo is tightly linked to GLPs, to write a GLP is usually not the final aim of a task. It represents a step in a larger process whose final aim is to obtain certain results. A GLP is a tool to be used. In order to run a GLP (written by the child or given to her (him) already developed) it is necessary to assign a specific value to the variable involved. In this process the variable looses its general character and it assumes a specific value. It was found that pupils usually do not have any difficulty to make these assignments (Hillel, 1992). However, pupils very often use Logo procedures only as exploring tools and their activity is characterized by 'planning in action' and by the lack of a specific goal. Sometimes, for example, children assign values without understanding what the input signifies and how it relates to the output obtained (Hillel, 1992). However, Logo tasks might be designed in order to lead pupils to approach the variable as a specific unknown. They might be asked, for example, to determine the specific value that would give a particular result and to assign it to the input. They might be required to solve an equation in order to determine the particular input giving a desired output. Leron and Zazkis (1986) illustrate this by the following problem:

> 'Here is a function called FUN.
>
> ```
> TO FUN :X
> PRINT :X * 3 + 5
> END
> ```
>
> Someone had executed FUN with an unknown value of :X and got 20 as a result. Find the unknown value of the input :X.'
>
> (Leron and Zazkis, 1986, p. 189)

Although a task like this may be sufficient to motivate pupils to solve equations, it can also be embedded in a wider project so that solving the equation become part of a process tending to reach a determinate goal. In such an approach to equation solving,

manipulative skills are not stressed and the searched value might be obtained by systematic substitution. However, pupils might be introduced to manipulative skills later after having worked with the idea of variable as specific unknown in meaningful tasks. Logo seems to offer this possibility. Nevertheless, not much work has been done in Logo involving the characterization of variable as specific unknown.

## 3.4 Logo and variables in a functional relationship

Logo is a functional language. The underlying model of operation is based on the idea of mathematical function. Logo procedures (primitives and programs) are functions. From the first approach to this programming language users deal with functional relationships, specifically with the idea of correspondence and with the idea of change. However, these notions are not transparent, but masked as procedures with inputs and outputs. Pupils approach them in a practical way by using Logo procedures as input-output machines in order to achieve a result. The focus of the Logo activities is usually on the particular output obtained and the ideas of correspondence and change are not stressed. Therefore, although when working in Logo pupils are working with functions and variables linked by a functional relationship, it is not to be expected that they will spontaneously develop function related concepts.

However, the experience acquired in using GLPs might be used as a starting point on which more advanced learning about functions and variables in a functional relationship can be built. For example, Leron and Zazkis (1986) suggest that by appropriate Logo activities and discussion pupils can be introduced to look at functions as input-output procedures; to see equivalent procedures as equivalent functions; to deal with domain of definition; to cope with constant and inverse functions; to compose functions and to deal with functions of several variables. However, not much investigation has been done in this area.

## 3.5 Concluding remarks

The discussion developed so far suggests that Logo can be a good environment to approach different characterizations of variable, namely, variable as general number, variables in a functional relationship, variable as specific unknown. All this aspects might be approached by writing and running GLPs. However, it has to be also considered that

pupils working in Logo tend to avoid the use of variables (Hillel, 1992; Sutherland, 1992). They prefer to work in direct mode and to use the 'planning in action' approach (Hillel, 1992). It cannot be expected that pupils who have had no prior experience in formalizing and generalizing in Logo or algebra, will spontaneously conceive to do it (Sutherland, 1992). Therefore, it is not enough to offer pupils the opportunity to work in Logo in order to help them approach different characterizations of variable. In order to work with variables in Logo pupils need the support of specifically designed Logo activities and of teacher interventions (Sutherland, 1992; Hillel, 1992).

Because working with variables in Logo implies working with GLPs, to approach in Logo different characterizations of variable pupils need to be introduced first of all to the writing of GLPs and to the use of variable as general number. Although pupils' spontaneous approaches might be encouraged, teacher's intervention has shown to be crucial to help pupils write GLPs and cope with variable as general number. Even if not much work has been done in Logo with regard to variable as specific unknown and variables in a functional relationship, Leron and Zaskis (1986) comments suggest the possibility to design activities involving these characterizations of variable. The project developed for this thesis considers these suggestions.

# CHAPTER 4

# RESEARCH METHODOLOGY

## 4.1 Introduction

The study reported in this thesis investigated the feasibility to create pupils' potential to approach through Logo different characterizations of variable prior to formal algebra teaching. Pupils aged 12-13 years old without prior formal algebra instruction were studied. The characterizations of variable considered were:

| | | |
|---|---|---|
| variable as specific unknown (VSU) | - | the variable represents a specific but unknown number that can be calculated under given constraints |
| variable as general number (VGN) | - | the variable represents an indeterminate number involved in a general method |
| variables in a functional relationship (VFR) | - | variables represent numbers whose values move within a range of values linked one to each other by a relationship |

## 4.2 The questionnaire

To select a sample to be case studied a questionnaire was designed (Appendix 1). It provided a pre-algebraic profile concerning pupils' interpretation and use of VSU, VGN and VFR when these appear in simple algebra tasks. The questionnaire was tested with 65 pupils 12-13 years old starting the first year of secondary school in Mexico City and other two Mexican towns. It was applied to 30 pupils during the pilot study and to a group of 34 pupils for the main study. On both occasions the analysis of the responses obtained provided a means of choosing samples to be closely observed during the Logo-based activities.

## 4.3 The Logo microworlds

In Mexico the official curriculum for the primary school does not include computer workshops. Therefore it was considered that the majority of the case study pupils will not have experience with computer and in particular with Logo. For that reason a sequence of introductory activities to Logo was designed. Pupils were introduced to some basic Logo primitives of the turtle graphics environment and to the syntax for writing fixed Logo procedures (FLP).

From the review of the literature (see 2.2, Chapter 2) there was evidence that when working in algebra pupils have difficulties to cope with each one of the different characterizations of variable namely, variable as specific unknown (VSU), variable as general number (VGN) and variables in a functional relationship (VFR). The survey of the historical development of algebraic ideas (see 2.4, Chapter 2) showed that different characterizations of variable were approached before the development of formal algebra. This suggested the possibility to design special environments where pupils might work with different characterizations of variable prior to formal algebra teaching. Additionally, the results of Logo research (see Chapter 3) showed that Logo environments might favour pupils' work with variables but they also stressed that pupils do not use the notion of variable spontaneously.

Therefore, aiming to create pupils' potential to work with these characterizations of variable prior to formal algebra teaching a series of Logo activities were designed. They integrated three microworlds each one focussing on one characterization of variable and the main notions underlying it (see Table 4.1). For each activity of the microworld a work sheet was designed (Appendix 2).

| Microworld | Central Ideas |
|---|---|
| Microworld 1<br>Generalization and general number | - Emphasizing the method to solve similar problems vs. focusing on the results.<br>- A general Logo procedure is the expression of a general method.<br>- The variable in a general Logo procedure represents a general number. |
| Microworld 2<br>Variables in a functional relationship | - The variable's movements can be restricted to an interval.<br>- Variables behave in a linked way. |
| Microworld 3<br>Variable as specific unknown | - The variable represents one particular value.<br>- The value of the variable can be calculated under given constraints. |

**TABLE 4.1** - The Logo microworlds

## 4.4 The pilot study

A pilot study was carried out with a group of 30 pupils 12-13 years old with no prior formal algebra instruction. They were attending the first year of a private secondary school in Mexico City. A sample of 12 pupils to be closely observed was selected by analyzing the responses given to the questionnaire by the whole group. The selected pupils had no prior experience in working with Logo.

The pilot study was carried out during the school normal computer workshop and the researcher was the teacher of the group. There were 8 computers in the school therefore the group was split into two groups each one attending the workshop one hour a week. Pupils worked in pairs and they were encouraged to discuss and cooperate in order to solve the posed tasks. To work in pairs was not a consequence of the reduced number of computers but it was considered that cooperation and discussion will help pupils approach the characterizations of variable embedded in the activities. This consideration was influenced by a Vygotskian perspective. Vygotsky (1981) considers that the formation of concepts appears first at the social level and they are later internalized as a consequence of social interaction, being speech and cooperative activities fundamental aspects of this process. Moreover, prior investigation developed in Logo environments

(Hoyles and Sutherland, 1989) showed that to work in pairs helped pupils overcome challenging goals of a mathematical nature, provoked reflection, helped pupils to learn to articulate their thoughts, to communicate and to negotiate them. However, it was also stressed that pupils need time in order to learn to work together.

The advantages of working in pairs stressed by Hoyles and Sutherland were confirmed during the pilot study. Additionally it was observed that allowing pupils to choose their partner favoured cooperative work. Therefore during the main study pupils worked in pairs and their own choice of the peer was accepted by the researcher.

The researcher had a double role, she was the teacher of the group and the observer of the work of the selected pupils. Additionally an assistant helped her with the data collection. The pilot study showed that it was difficult for the researcher and one assistant to attend the group and to observe more than three of the selected pairs in each session. For that reason during the main study the group was split into two sub-groups each one including three case study pairs.

After the introductory sessions pupils worked successively on the three microworlds, namely: microworld 1 - Generalization and general number; microworld 2 - Variables in a functional relationship; microworld 3 - Variable as specific unknown. This order was suggested by the fact that pupils had no prior experience with variables in Logo. It was considered suitable to introduce pupils first of all to the idea of general Logo procedure (GLP) and to variable as general number (VGN) because working with variables in Logo implies working with GLPs. After the experience of writing GLPs and working with VGN (microworld 1) pupils used already written GLPs as tools to approach the idea of VFR (microworld 2) and VSU (microworld 3). Since this order worked properly it was maintained for the main study.

All the activities designed for the study were piloted. After this some of them were eliminated. For example, the pilot study included activities concerning pattern recognition and its expression in Logo by using the primitive REPEAT. It was hypothesized that to acquire experience in pattern recognition and its expression would help pupils to focus on a method for solving a problem as opposed to focus on the result. It was expected that this experience would favour pupils acceptance and use of GLPs as a way of expressing in Logo general methods for solving similar problems. That is, that recognition of regularity in pattern tasks would help pupils to recognize regularities in a set of similar fixed Logo procedures (FLPs). Although the results obtained when pupils worked on pattern recognition in Logo were very interesting (see Ursini, 1991) the

former hypothesis was not confirmed. For this reason the activities involving pattern recognition were not included in the main study.

One of the original aims of this investigation was to explore if prior to formal algebra teaching pupils could be helped to develop a potential to discriminate between the different characterizations of variable and shift from one to another when required by a Logo-based task. This aim was influenced by the analysis of the Mexican school texts for the first year of secondary school (see 2.3, Chapter 2). In these texts different characterizations of variable are used from the first beginning. This suggests that pupils are expected to be able to discriminate between them and to shift appropriately from one to another when they appear in usual algebra context.

Pupils worked first of all on Logo activities involving each one of the different characterizations of variable and after this they had to solve tasks in which they needed to discriminate and shift between them. In the light of the experience acquired during the pilot study there was evidence that to cope with each one of the different characterizations of variable was not straightforward for pupils, they needed time and substantial support to cope with them. This implied that to develop the study it would take more time than had been originally planned. Additionally these results suggested the importance of creating pupils' potential to approach in Logo each one of the different characterizations of variable prior to helping them create a potential to discriminate between them and shift from one to another. Therefore after this experience the emphasis of the study was to investigate the feasibility to create pupils' potential to approach through Logo different characterizations of variable prior to formal algebra teaching. The activities concerning discrimination were eliminated. The activities designed to help create pupils' potential to work with different characterizations of variable were maintained and expanded.

## 4.5 What is meant by pupils' potential?

The investigation reported in this thesis reflects the influence that some of the Vygostky's ideas had on the researcher. In accordance with Vygotsky (1978) considerations that *zones of proximal development* might be created for particular subjects, the feasibility to create pupils potential to work with different characterizations of variable was investigated.

Vygotsky defined the *zone of proximal development* as "the distance between the actual development level as determined by independent problem-solving and the level of potential development as determined through problem-solving under adult guidance or in collaboration with more capable peers" (Vygotsky, 1978, p.86). In the present study adult guidance and collaboration with more capable peers were considered important. However, these were not assumed to be the only elements contributing to create pupils' potential to cope with different characterizations of variable. Because in order to deal with different characterizations of variable pupils cope with particular problems involving them, it was considered that the particular activity presented to pupils might also be crucial to help creating pupils' potential. Therefore, special attention was paid to the design of the activities.

It was assumed that a pupil has developed the potential to cope in Logo with one of the characterizations of variable when (s)he is able to solve the specially designed Logo-based tasks involving it. The task might be solved by the pupil alone or cooperating with a more capable peer or with researcher's help. All these approaches were considered to indicate pupil's potential to cope in Logo with the characterization of variable involved in the given task.

The capability to solve a given Logo-based task on their own suggests that pupils might solve other similar Logo-based tasks. This was assumed to indicate their potential to cope in Logo with the characterization of variable involved.

To solve the given task in cooperation with a more capable peer or with researcher's help was considered to show pupils' capability to participate actively in the solution of the task. This behaviour was considered to indicate that pupils had the potential to cope in Logo with the characterization of variable involved. Their capability to participate actively was considered to suggest that in the future they might be able to solve the posed tasks on their own.

## 4.6 The main study

### 4.6.1 The classroom setting

The main study was developed during one academic year. A qualitative approach was adopted and 12 pupils were case studied. They were selected by applying the

questionnaire at the beginning of the academic year to a group of 34 pupils 12-13 years old starting the first year of a private secondary school in Mexico City. Pupils attending this school were middle-class children coming from different primary schools. The school was chosen because it offered facility for developing the study: it has 9 computers (2 IBM compatibles and 7 Commodore-64), and additionally to the curriculum subjects pupils had to attend a computer workshop one hour a week during the whole school year. The researcher was the teacher of the workshop and she was allowed to decide its contents. To facilitate the research the 34 pupils were split into two groups each one attending the workshop one hour a week. In each group three pairs to be case studied were included.

When starting the study the pupils were all new to the school and the majority did not know each other. They were invited to work in pairs and initially they were allowed to change their partner. This aimed to help them choose a peer to whom they will be happy to work with and to facilitate in this way their involvement in the activities and cooperation between them. None of the pupils had experience with Logo and only a few of them had been in touch with computers when playing computer games.

During the study all the pupils, observed and non-observed ones, worked in the same computer room and on the same activities. That is, the social environment in which the investigation developed was the normal school class and not a laboratory.

### 4.6.2 The researcher's role

The researcher was the teacher of the group. At the beginning of each session she briefly explained what was the task pupils were expected to solve. Each pair was given the work sheet designed for that activity and they worked at their own pace. Pupils were invited to collaborate in order to solve the task. Discussion between peers and with the researcher was encouraged.

During the sessions the researcher observed the work of all the pairs intervening to motivate them, congratulate them for their advances and encourage them to continue. On occasion she organized a group discussion in order to create an environment propitious for the introduction of a new concept (e.g. the Logo primitives, the syntax for writing fixed and general Logo procedures). Although attending the whole group the researcher paid special attention to the six case study pairs. She took notes of the way they were approaching the task, the difficulties they had and the kind of intervention needed by each pair. Additionally an assistant helped her with the data collection.

### 4.6.3 Data collection

In order to build up a pre-algebraic pupil profile the written questionnaire was applied at the beginning of the study. After analyzing the answers and selecting the sample, the 12 chosen pupils were individually interviewed on the items of the questionnaire in order to refine their profile.

During the Logo sessions pupils were asked to record the work developed with the computer on the paper work sheets. These notes were a fundamental part of the data. In each session the researcher and an assistant took notes of pupils' approach to the task, the difficulties they had and the kind of intervention provided. Because the research was developed during the normal computer workshop that was attended by the whole group the details of pupils' dialogues and of researcher's intervention were not systematically recorded during the Logo-based activities. However, they were observed and notes were written which were considered when analyzing the data.

In particular during the whole study special attention was paid to:

- the strategies pupils used to solve the posed task
- the role of the researcher's intervention
- the role of the Logo setting
- the role of language

The case study pupils were individually interviewed after working in microworlds 1 and 2. These interviews aimed to provide more inside on pupils' potential to work in Logo based environment with the characterization of variable involved in the microworld. At the end of the study, that is, after microworld 3, pupils were individually interviewed once more. The questions of the last interview aimed at verifying pupils' potential to work in Logo with the three characterizations of variable they worked with in the three microworlds. All the interviews were tape recorded.

This study was not designed in order to help pupils use in non-Logo environments their experience of working with different characterizations of variable in Logo. However, in order to observe some possible effects of pupils' Logo experience on the way they were solving traditional paper and pencil algebra tasks, at the end of the study an extract of the questionnaire was applied.

# CHAPTER 5

# THE QUESTIONNAIRE

## 5.1 Introduction

To select a sample to be case studied a questionnaire was specially designed for this research (Appendix 1). Some of the items were borrowed from the Concepts in Secondary Mathematics and Science project (CSMS), the DIME pre-Algebra project (Geoff Giles), the De la Aritmética al Algebra project (Rojano, 1985) and from Mason's (1985) Routes to/Roots of Algebra. The questionnaire provided a pre-algebraic profile regarding pupils' capability to work with variable as specific unknown (VSU), variable as general number (VGN) and variables in a functional relationship (VFR) when these appear in simple algebra tasks.

Concerning VSU pupils were asked to:

- solve simple equations with one VSU;
- solve simple equations with multiple appearances of the VSU (i.e. 3 + A + A = A + 10);
- translate to symbols word sentences involving a VSU.

Concerning VGN pupils were asked to:

- interpret a variables representing a VGN;
- translate to symbols word sentences involving a VGN;
- use an already given VGN in order to construct an algebraic expression;
- express a generalization.

Concerning VFR pupils were posed questions where they had to:

- interpret an algebraic expression involving a VFR;
- translate to symbols a word sentence involving a VFR;
- fill in a table given a functional relationship expressed analytically.

The questionnaire was piloted with a group of 65 pupils aged 12-13 years old starting the first grade of secondary school in Mexico City and two other Mexican towns (Ursini, 1990b). During the pilot study 30 pupils completed the questionnaire. For the main study it was applied to a group of 34 pupils at the beginning of the academic year. In both

occasions the results obtained provided a profile of the pupils concerning their interpretation and use of the different characterizations of variable in usual algebra context and this influenced the selection of a sample to be case studied.

The mathematics teacher of the group offered to apply the questionnaire during the regular time for mathematics classes (50 minutes). Anticipating that it would take more than 50 minutes to answer the questionnaire, it was split into three parts. The first part (Appendix 1, Part 1) included items involving VSU, VGN and one item concerning VFR (item 1.3). This organization reflected the one observed in Mexican school texts where different characterizations of variable are used simultaneously. To get more insight on pupils' approaches to VGN the second part of the questionnaire included problems involving generalization and its expression (Appendix 1, Part 2). Information on pupils' approaches to VFR was provided by the items grouped in the third part of the questionnaire (Appendix 1, Part 3). Concerning VSU it was considered that the items included in Part 1 of the questionnaire were providing enough information.

Pupils spent approximately fifty minutes on answering Part 1 of the questionnaire and fifty minutes on answering Part 2 and Part 3.

After the application of the questionnaire 12 pupils were selected to be case studied (see 5.3, Chapter 5). In order to design a better profile of each one of them they were individually interviewed. The interview clarified the answers they gave to the questionnaire. The results reported below concern the responses given by the whole group and they are occasionally illustrated with fragments of the interviews.

## 5.2 Results of the questionnaire

To analyze the responses given to the questionnaire the items were grouped depending on the characterization of the variable involved:

- items involving VSU (Appendix 1, Part 1, items 1.1, 1.2, 2.2, 2.5, 2.6, 3, 7),
- items involving VGN (Appendix 1, Part 1, items 1.4, 1.5, 2.1, 2.3, 2.4, 4, 5 and 6; Part 2, all the items),
- items involving VFR (Appendix 1, Part 1, items 1.3; Part 3, all the items).

Pupils' answers were analyzed in order to detect if their interpretation and/or use of variable was coherent with the characterizations of variable involved. During the analysis another category appeared:

variable as placeholder (VPH) - the variable is considered to stand for a specific but arbitrary value.

### 5.2.1 Analysis of pupils' responses to items involving variable as specific unknown (VSU)

Items were designed (Fig. 5.1) to explore pupils' capability to calculate the value of a VSU when it appeared in a simple equation represented by a letter (items 2.2 and 3) or by using box notation (item 7) and when it appeared more than once (items 2.5 and 2.6).

2. For each expression write the values you think the letter can have. If you think it can have more than one value, then write some of them.

2.2 $3 + y = 7$ _____

2.5 $X + 5 = X + X$ _____

2.6 $3 + A + A = A + 10$ _____

3. Write for each expression all the values that you think the letter can have.

$Z + 874 = 1093$ _____

$525 + Z = 823$ _____

$4 \div X = 2$ _____

$3 \times M = 15$ _____

7. Solve:

$\square + 512 = 721$ $\square + 2 = 2 + \square$

$437 - \square = 25$ $3 * \square = 15$

$4 \div \square = 2$

**Fig. 5.1** Items involving the calculation of the value of a variable as specific unknown (VSU)

Three kind of responses to these items were identified:

- no answer was given

- an arbitrary value was assigned to the VSU
- value of the VSU was calculated.

Table 5.1 presents the proportion of pupils giving each one of these answers.

| Proportion of pupils | | | |
|---|---|---|---|
| Item | No answer | Arbitrary value | VSU calculated |
| 2.2, 3 | 10/34 | 5/34 | 19/34 |
| 7 | 0/34 | 0/34 | 34/34 |
| 2.5, 2.6 | 24/34 | 10/34* | 0/34 |

* 7 of these pupils tried to balance the equation

**TABLE 5.1** - Pupils' solution of simple equations

Almost a third of the pupils did not answer the items in which the VSU was represented by a literal symbol (i.e. items 2.2, 3). This suggests that they were not familiar with this symbolization. This is illustrated, for example, by the comment written by Etna (Fig. 5.2).

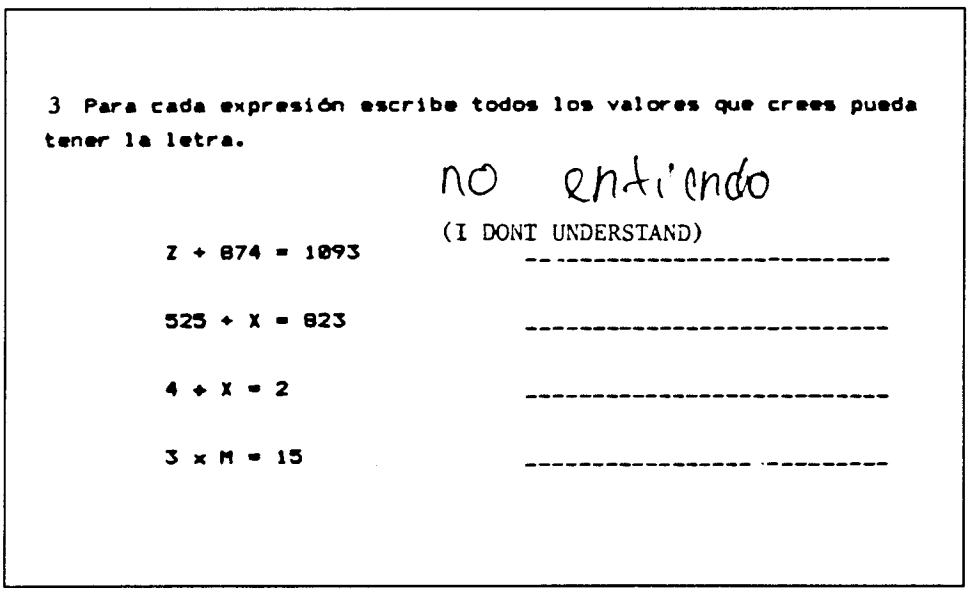



**Fig. 5.2** Etna - Answer to item 3

On the contrary all the pupils were able to calculate the value of VSU when box notation was used (item 7). This notation was already familiar to them from the primary school.

Moreover, pupils' assignment of a randomly chosen value to the literal symbol suggests an interpretation of it as VPH (variable as place holder).

These results indicate on the one hand that the ability to interpret a literal symbol as a VSU could be a result of previous teaching and not a "natural" tendency as suggested by Booth (1984). On the other hand they highlight a tight link between the symbol used to represent a VSU and the capability to calculate its value suggesting a poor conceptualization of VSU. When a letter or a symbol other than a letter was interpreted as a VSU all the pupils knew how to calculate its value considering the constraints of the given equation.

As expected for pupils without algebra instruction nobody could solve equations with multiple appearances of VSU (items 2.5 and 2.6). It is remarkable, however, that seven pupils tried to find a way to determine appropriate values for the different appearances of VSU. They considered the numeric values that appeared in the equation and they assigned values to the different appearances of VSU in order to balance both sides of the equation. This is evidenced, for example, by the solution given by Mariana (Fig. 5.3, items 2.5 and 2.6) and the dialogue the researcher sustained with Sandra during the individual interview (Dialogue 1).

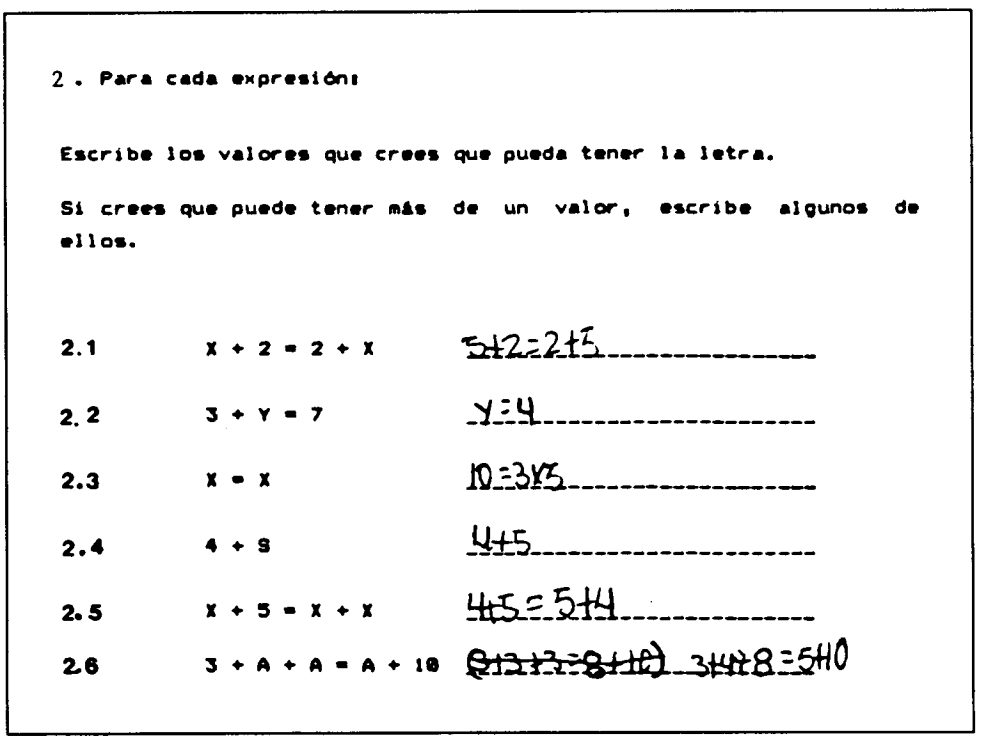



**Fig. 5.3** Mariana - Answers to items 2.5 and 2.6

**Dialogue 1**

*R(esearcher): In the expression X + 5 = X + X what does the letter represent?'*
*S(andra): Well, it could be 7 plus 5 is 5 plus 7 '*
Sandra writes 7 + 5 = 5 + 7
*R.: The same letter can have different values?'*
*S.: 'Well, yes, but here I already know that the value of one of the X must be 5'* and she pointed to the first X after the equal sign.

This approach confirms already reported pupils' tendency to consider that when the same letter appears more than once in an expression it can take on different values ('polysemia' error) (Filloy and Rojano, 1984).

Items 1.1 and 1.2 (Fig. 5.4) aimed to explore if pupils were able to symbolize a VSU when it appeared in an equation presented in words. The word 'formula' was used because during the pilot study it was found to be more familiar for pupils than the word expression.

1. In the following exercises DO NOT calculate the number, write only a formula.
   1.1 Write a formula which means: An unknown number plus 5 is equal to 8.
   1.2 Write a formula which means: An unknown number multiplied by 13 is equal to 127.

**Fig. 5.4** Items requiring to symbolize a variable as specific unknown (VSU)

Four kind of responses were obtained:
- no answer was given,
- the value of the VSU was calculated,
- the VSU was represented by a symbol other than a letter,
- the VSU was represented by a literal symbol.

Table 5.2 presents the proportion of pupils giving these answers.

| Proportion of Pupils | | | | |
|---|---|---|---|---|
| Item | No answer | Numeric answer | Letter used | Sign used |
| 1.1, 1.2 | 5/34 | 16/34 | 11/34 | 2/34 |

**TABLE 5.2** - Pupils' symbolization of a simple equation involving one appearance of variable as specific unknown (VSU)

These results show that a large proportion of pupils could not symbolize a VSU. This confirms pupils' tendency to solve a problem looking for a result (Booth, 1984) even when they are explicitly asked only to symbolize it. It seems that it is difficult for pupils to disregard the action demanded from the posed question and to limit themselves only to reformulate the same question in another language. Perhaps, as Booth points out 'they may not see that it is an appropriate thing to do' (Booth, 1984, p. 38).

However, there were pupils who could symbolize a VSU by using a letter or another sign (□ and * were used). The use of signs different than letters reflects prior teaching where these are often used to represent the 'missed number' in simple equations.

The answers given by some pupils when interviewed after the application of the written questionnaire suggest that in order to symbolize a word problem they tended to calculate first of all the value of the specific unknown involved and, after this, they symbolized the statement by substituting a symbol to this value. This is illustrated by the dialogue sustained with Martha during the interview (Dialogue 2).

**Dialogue 2**

*R(esearcher):* *'Could you write a formula that means "an unknown number multiplied by 13 is 127" ?'*

*M(artha):* *'I could use a letter ... for example, 13 by X or ...'*

*R.:* *'Could you write it, please?'*

Martha writes 13 * X = 127 while talking for herself: *'X must be a number I think about and I have to put the result and I will see what that number is.'*

*R.:* *'Could you now write a formula that means "an unknown number plus 5 is equal to 8" ?'*

*M.:* *'3 plus 5 is 8'* and she writes 3 + 5 = 8.

*R.:* *'Could you now write a formula that means "an unknown number plus 5 is equal to 8" ?'*

*M.:* *'With X, isn't it? Or with another letter.'*

*R.:* *'Can we use any letter?'*

*M.:* *'Yes.'*

*R.:* *'What does it represent?'*

*M.:* *'A number. Any number that gives the result.'* She writes X + 5 = 8.

Summarizing, these results show that all the tested pupils were able to calculate the value of a VSU when it appeared in one-step equations and it was represented by a box. But almost a half of the pupils could not calculate it when it was represented by a letter. Faced with an equation with multiple appearances of VSU represented by a letter pupils could not determine its value. It was also found that less than a half of pupils were able to symbolize a VSU. These results show that for many pupils the capability to work with VSU was linked to the sign used to represent it. This suggests a poor level of conceptualization of VSU.

I would suggest that the different behaviours observed depended essentially on the previous instruction received in primary school where very little teaching involving literal symbols is included and it is up to the particular school or teacher to extend it (see 2.3, Chapter 2).

### 5.2.2 Analysis of pupils' responses to items involving variable as general number (VGN)

Items 2.1, 2.3 and 2.4 (Fig. 5.5) explored pupils' interpretation of VGN when it appeared in a tautology (items 2.1 and 2.3) or in an open expression (item 2.4).

2. For each expression write the values you think the letter can have. If you think it can have more than one value, then write some of them.

2.1 $X + 2 = 2 + X$ _____

2.3 $X = X$ _____

2.4 $4 + S$ _____

**Fig. 5.5** Items requiring to interpret variable as general number (VGN)

Three types of responses were obtained:

- no answer was given
- one arbitrary value was assigned to the VGN
- a mixed approach was used, that is, when the equal sign was present in the expression pupils tried to calculate the value of the VGN; otherwise they assigned it an arbitrary value.

Table 5.3 presents the proportion of pupils giving these answers.

| Proportion of pupils | | | | |
|---|---|---|---|---|
| Item | No answer | One arbitrary value | Mixed approach | Multiple values |
| 2.1, 2.3, 2.4 | 14/34 | 3/34 | 17/34 | 0/34 |

**TABLE 5.3** - Pupils' estimation of the values of variable as general number (VGN)

In general all the pupils who answered these questions assigned only one value to the variable. This confirms Küchemann's (1980) and Booth's (1984) findings reporting

pupils' tendency to regard a literal symbol as standing for a specific value in situations where it represents a general value. This value was arbitrarily assigned or, depending on the item, some pupils tried to calculate it (mixed approach).

It was observed that half of the pupils tried to calculate the value of the VGN when the equal sign appeared in the expression. When there was no equal sign in the expression they assigned an arbitrary value to it or they did not answer the question. This supports Kieran's (1980) finding that faced with algebraic expressions where an equal sign is involved pupils often interpret it as a 'do something' sign.

In their attempts to calculate the value of a VGN pupils used different strategies. For example, they reduced the given expression to an equation. This is illustrated by the dialog sustained with Jonathan when he was interviewed (Dialogue 3).

**Dialogue 3**

*R(esearcher): 'In the expression* $X + 2 = 2 + X$ *, what are the values X can have?'*
*J(onathan): 'The first one is 0, because* $2 + 0 = 2$ *'*
*R.: 'And what about the second one?'*
*J.: 'That one can have any value, because you are adding something to the result'*
*R.: 'Don't they have necessarily the same value?'*
*J.: 'No, they are independent of each other'*

Another approach was highlighted during Oscar's interview (Dialogue 4). This dialog suggests that in order to make sense of the expression and to be able to determine the value of the VGN involved Oscar changed a sign. During the interview he rectified this approach and considered that the VGN was representing any value. However, he still needed to verify his answer looking for a result.

**Dialogue 4**

*R(esearcher): 'In the expression X + 2 = 2 + X what does the letters represent?'*
*O(scar): 'Zero, no ... a number'*
*R.: 'Do you mean a specific number?'*
*O.: 'Yes, isn't it?'*
*R.: 'What number?'*
The researcher has Oscar's questionnaire in front of him and the child gives a quick look to the answer he gave to this question.
*O.: 'I made a mistake here'* (he points to the 0 (zero) he wrote in the questionnaire) *'I thought it was X + 2 = 2 - X.'*
*R.: 'Was this the reason that you answered zero?'*
*O.: 'Yes'*
*R.: 'And now, what do you think, what are the values X can have?'*
*O.: '3, it can be 5 the result'*
*R.: 'How is it?'*
*O.: '3 plus 2 is 2 plus 3'*
*R.: 'Could you write it?'*
Oscar writes 3 + 2 = 2 + 3
*R.: 'Can X have another value?'*
*O.: 'Yes, any value'*
*R.: 'For example ... 100?'*
*O.: 'Yes, 100 plus 2 is 2 plus 100'*
Oscar writes 100 + 2 = 2 + 100.

Table 5.3 shows that three pupils systematically assigned an arbitrary value to the VGN. Although this could indicate that they were interpreting the literal sign as VGN this was not confirmed when they answered the questions posed in Part 2 of the questionnaire. This suggests that they were interpreting the letter as a VPH.

These results show that faced with algebraic expressions involving VGN pupils could not interpret the literal symbol or they interpreted it as VSU and/or VPH.

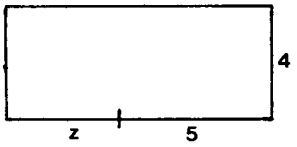
Six items were designed to explore pupils' capability to use a given VGN represented by a literal symbol in order to construct a new algebraic expression (Fig. 5.6).

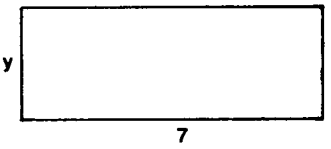
Four types of answers were identified:

- no answer was given
- the given VGN was ignored
- an arbitrary value was assigned to VGN and a numeric answer was given
- the given VGN was used to construct a new algebraic expression.

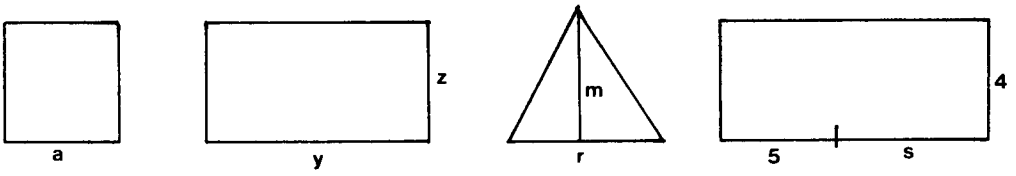
1.5 Write a formula which means: 4 added to n + 5.

4. The **perimeter** of a figure is the addition of its sides. Write a formula to express the **perimeter** of the following figures:

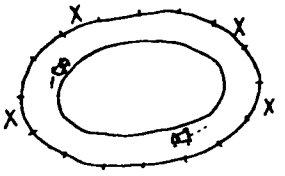



6. Write a formula to calculate the **area** of the following figures:

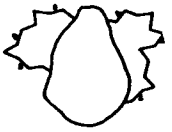



8. A race track is divided into 16 parts of equal length. Each part measures X kilometers. Write a formula to express the total length of the race track.




Write a formula to express how many kilometers a car will move if it travels three times around the track.

9. The next figure is not completely visible. Since we do not know how many sides it has in total, we will say that it has N sides. Each side measures 2 centimeters. Write a formula to calculate the **perimeter** of the figure.



10. A highway which length was X kilometers was made 25 kilometers larger. How would you express the new length of the highway?

**Fig. 5.6** Items requiring to use a given variable as general number (VGN) represented by a literal symbol

Because the proportion of pupils giving answers of these types varied depending on the item, table 5.4 presents this proportion for each item.

| Proportion of pupils | | | | |
|---|---|---|---|---|
| Item | No answer | VGN ignored | Value assigned | VGN used |
| 1.5 | 23/34 | 0/34 | 3/34 | 8/34 |
| 4 | 11/34 | 15/34 | 1/34 | 7/34 |
| 6 | 11/34 | 16/34 | 0/34 | 7/34 |
| 8 | 10/34 | 1/34 | 9/34 | 14/34* |
| 9 | 11/34 | 7/34 | 1/34 | 15/34 |
| 10 | 10/34 | 0/34 | 8/34 | 16/34 |

(*) 7 of these pupils gave incorrect answers

**TABLE 5.4** - Pupils' use of a given variable as general number (VGN)

These results show that item 1.5 ('Write a formula which means: 4 added to n + 5') that implied writing an 'unclosed' algebraic expression was particularly difficult for pupils. The great majority did not answer this question. A reason for this might be that pupils could not assign a meaning to the symbol used in the given expression and it was difficult for them to limit themselves to symbolize the given statement without thinking of a possible result of adding 4 to n + 5. On the contrary the number of pupils who answered the question increased substantially for all the other items where the symbol had an explicit referent and the possibility to obtain a result was implicit (e.g. the area of a geometric shape, item 6; the total length of a race track, item 8). However, approximately half of the pupils ignored the given VGNs and the particular numbers when asked to write formulae to express the perimeter or the area of the given figures (items 4 and 6). They wrote the formulae learned by rote at school. This suggests that when the problem could be referred to a more familiar situation pupils tended to drop down to this. When the problem situation could not be reduced to a familiar one (items 8, 9 and 10) pupils tried to use the given symbol and they constructed an algebraic expression.

Pupils' capability to express a generalization was explored by five items (Fig. 5.7).

1.4 Write a formula which means:
8 multiplies the addition of 3 with an unknown number.

5. Write a formula which means:
An unknown number is bigger than 5.

11. Look at the following pictures:

| | | Number of dots |
|---|---|---|
| 1 | . | 1 x 1 |
| 2 | . .<br>. . | 2 x 2 |
| 3 | . . .<br>. . .<br>. . . | 3 x 3 |
| 4 | . . . .<br>. . . .<br>. . . .<br>. . . . | |

How many dots are there in picture number 4?

Draw pictures number 5 and 6, and write down the respective number of dots.

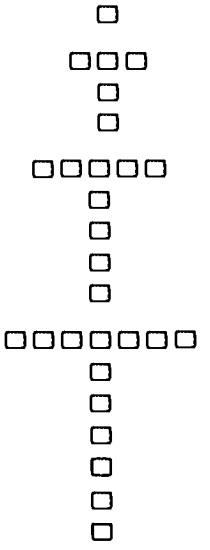
5

6

Imagine that you keep drawing until the picture number m (m represents an unknown number).

How many dots will the picture number m have?

12 Look at the following pictures and the respective numbers:



Number of squares

1

3 + 2 = 5

5 + 4 = 9

7 + 6 = 13

How will the next picture look? Draw it.

In the picture that you drew:
How many squares are there on the horizontal line?

How many of them in the vertical?
How many in total?

Imagine that you keep drawing until you have P squares on the horizontal line. In that figure:
How many squares would be on the horizontal line?
How many of them in the vertical?
How many in total?

13 Look at the following figures and the respective numbers:

| Number of squares | Number of sides |
|---|---|
| 1 | → 4 |
| 2 | → 7 |
| 3 | → 10 |

Complete the numbers:

4 →

5 →

10 →

12 →

Imagine that you keep on drawing until you have Y square
Write a formula to calculate the number of sides when yc
have Y squares.

How many sides will be there if you draw 100 squares?

Fig. 5.7

**Fig. 5.7** Items requiring to express a generalization

Four types of answers were identified:

- no answer was given
- a specific numeric example was given to illustrate the given general statement or the deduced general rule when a succession of shapes was given
- the given succession of shapes was continued by drawing and the correspondent numbers were written
- VGN was used to express a generalization

The proportion of pupils giving these answers varied substantially depending on the item. Therefore Table 5.5 presents this proportion for each item.

| | Proportion of pupils | | | |
|---|---|---|---|---|
| Item | No answer | Numeric example | Drawing continued | VGN used |
| 1.4 | 24/34 | 6/34 | ----- | 4/34 |
| 5 | 7/34 | 18/34 | ----- | 9/34 |
| 11 | 6/34 | 0/34 | 24/34 | 4/34 |
| 12 | 13/34* | 2/34* | 15/34 | 4/34** |
| 13 | 11/34* | 4/34* | 11/34 | 8/34* |

(*) 1 correct; (**) all incorrect.

**TABLE 5.5** - Pupils' use of variable as general number (VGN) to express a generalization

Table 5.5 shows that a large proportion of pupils did not answer item 1.4 ('Write a formula which means: multiply 8 by the addition of 3 and an unknown number'). To solve this item required to symbolize a VGN and use it to write an 'unclosed' algebraic expression. The results obtained might be a consequence of a difficulty of making sense of the given word statement where no calculation was required. This might be a consequence of the tendency to consider that to solve a problem means looking for a result (Booth, 1984). In fact, when an action was implicit in a problem pupils were able to answer it. For example, in item 5 pupils looked for a number bigger than 5. In the items 11, 12 and 13 they continued with the drawing. That is, the great majority expressed a generalization by giving specific examples. However, there were pupils that used a literal symbol to express a generalization. These attempts were not always successful and they might be influenced by pupils' interpretation of the literal symbol as a VPH and not as a VGN. This interpretation is suggested, for example, by Valentin's

solution to item 12 (Fig. 5.8) where he considered that the same literal symbol could be used to represent the squares in the horizontal line, the squares in the vertical line as well as the total number of squares.

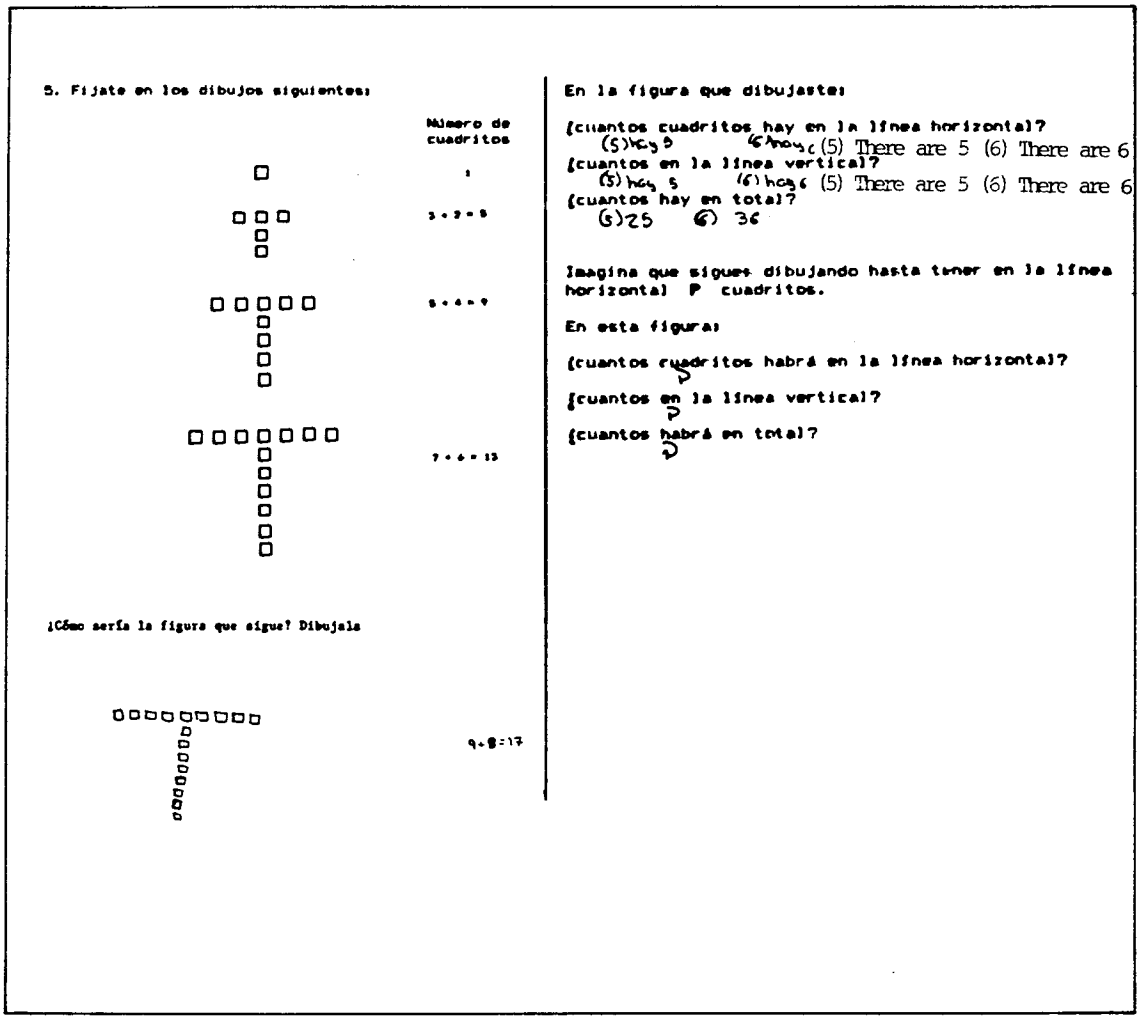



**Fig. 5.8** Valentin - Answer to item 12

A careful analysis of pupils' answers to all the items involving VGN suggests that eight pupils showed an emerging although inconsistent capability to work with the idea of VGN. This is illustrated, for example, by the answers given by Nayeli, one of the pupils that was later chosen to be case studied.

*The case of Nayeli*

Asked to interpret a VGN (items 2.1, 2.3 and 2.4) Nayeli could not interpret it and she wrote 'no me lo enseñaron' (they did not teach this to me). However, to item 1.5 ('Write a formula which means 4 added to n + 5') she answered writing: 4 + n + 5 . To item 5

('Write a formula which means: An unknown number is bigger than 5') she answered writing: $X > 5$ .

Faced with item 4 (Fig. 5.6), in which a perimeter for a rectangle had to be written, she used the given VGN when it was representing the total length of a side but she ignored it when the dimension was given by a number and a VGN.

Nayeli partially answered item 8. She symbolized the length of the race track writing X x 16. But she did not use this expression in order to answer the second part of the question and she wrote 'no me lo enseñaron' (they did not teach this to me). She did not consider the VGN in item 9. She counted the visible lines and multiplied this number by 2. However, to solve item 11 she used a VGN to express the number of dots belonging to the picture number m and she wrote m x m.

These answers show that Nayeli was not able yet to work with VGN in a consistent way. However, they highlight an emerging use of VGN.

Although there were more than eight pupils who were able to symbolize some items using a literal symbol (see Table 5.4 items 8, 9 and 10), the way they used the letter in other items suggests a tendency to look at it as VPH or VSU and not as VGN. This is evidenced, for example, by Sandra's work.

*The case of Sandra*

Asked to interpret a VGN (items 2.1, 2.3 and 2.4) Sandra tried to determine its value. When there was no equal sign in the expression she wrote 'no sabría contestar' (I would not know how to answer).

Asked to write the area of simple geometric figures (Fig. 5.6 items 4 and 6) she ignored the given dimensions and wrote the formula learned in the primary school or she wrote 'no me acuerdo' (I do not remember). Her solution to item 8 (Fig. 5.9) suggests that she was using the literal symbol as a VPH. This was confirmed when she was individually interviewed (Dialogue 5).

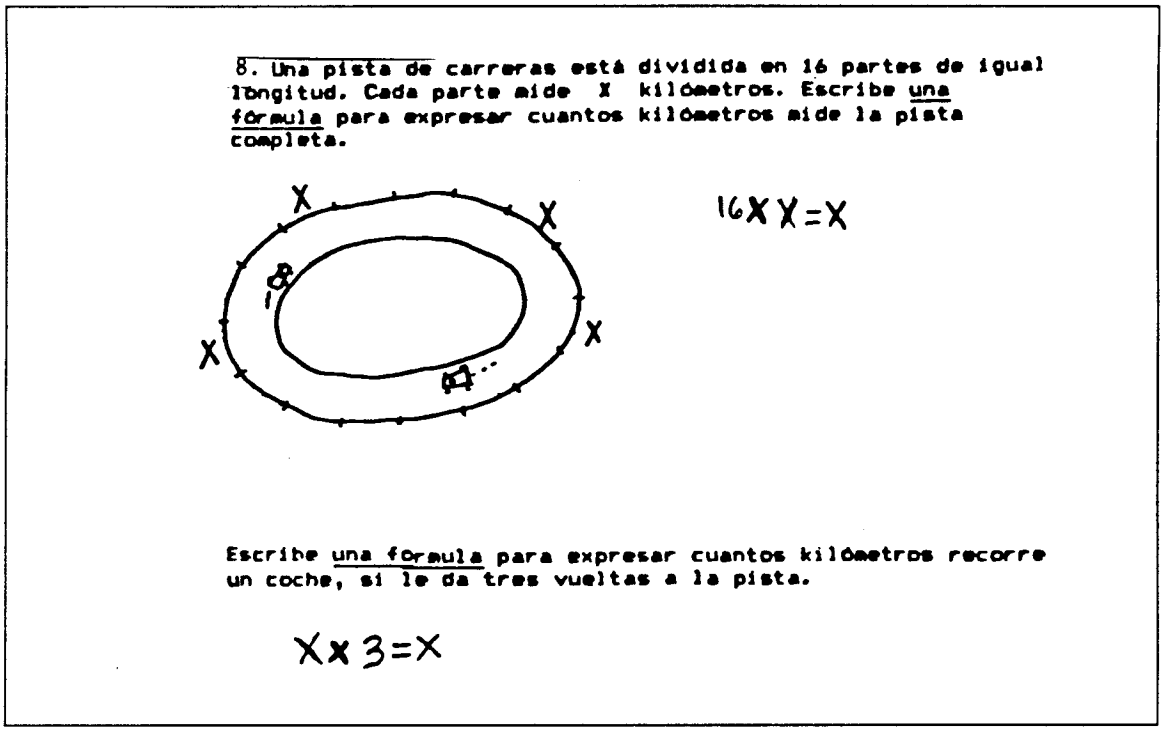



**Fig. 5.9** Sandra - Answer to item 8

**Dialogue 5**

*R(esearcher):* *'Could you tell me what does the X mean in the expression 16 x X = X that you wrote when answering the questionnaire?'*

*S(andra):* (She points to the first appearance of the X in 16 x X = X) *'The amount of kilometers'*

*R:* *'Do you mean the measure of each part of the race track?'*

*S.:* *'Yes'*

*R.:* *'This X* (pointing to the second appearance of the X) *represents the same?'*

*S.:* *'No, it is the result'*

*R.:* *'The measure of the total race track?'*

*S.:* *'Yes'*

*R.:* *'And these X here?'* (pointing to both X in 3 x X = X). *'What does the first X represent?'*

*S.:* *'It is the same as this one'* (pointing to the second X in the expression 16 x X = X).

*R.:* *'A bit confused, isn't it? Is there some way to differentiate here in 16 x X = X, which X refers to the measure of a part of the race track and which X to its total length?'*

*S.:* *'Yes, if we use two different letters'.*

To conclude this section I want to remark that the majority of the tested pupils could not interpret and/or use a VGN. They had difficulties in interpreting and/or writing 'unclosed' expressions. A strong tendency to look for a result even when they were explicitly asked only to symbolize a given statement involving a VGN was observed. However, an

emerging although inconsistent capability to work with VGN was observed in eight pupils.

### 5.2.3 Analysis of pupils' responses to items involving variables in a functional relationship (VFR)

All the items included in part 3 of the questionnaire (Fig. 5.10) aimed to investigate pupils' capability to work with VFR. Questions were posed to see if pupils considered that the variables involved in a functional relationship were related with each other (item 14). They were asked to determine how one variable was varying in relation to another one (items 15 and 16), or how an expression was varying in relation to another one (item 19). Given an algebraic expression representing a functional relationship (item 17) pupils were asked to use it in order to complete a table. They were also asked to determine a range of values that satisfied a given condition (item 18).

14. If X + 3 = Y, what values can X have? What values can Y have?
Is there any relationship between the values of X and Y? (Underline the correct answer).
YES NO I DO NOT KNOW
Explain your answer.

15. If Y = X + 5, the value of Y will be bigger than the value of X. (Underline the correct answer).
ALWAYS NEVER SOMETIMES
Explain your answer.

16. If Y = 7 + X, what happens to the value of Y when the value of X increases?

17. Considering that Y = X + 4, complete the table:

| X | Y |
|---|---|
| 5 | |
| | 12 |
| 3.5 | |
| | 7.5 |

18. Consider Y = 3 + X
If we want the value of Y to be 10, what value must X have?
If we want the value of Y to be 3, what value must X have?
If we want the value of Y to be bigger than 3 but smaller than 10, what are all the values that X can have?
If X takes values between 8 and 15, what values will the value of Y be between?

19. Which is bigger: N + 2 or 2 x N? Explain your answer.

**Fig. 5.10** Items requiring to interpret variables in a functional relationship (VFR)

The answers obtained were classified into three groups:

- no answer was given
- the given functional relationship was not considered
- the given functional relationship was considered

The proportion of pupils giving these answers are presented in Table 5.6.

| Proportion of pupils | | | |
|---|---|---|---|
| Item | No answer | Relationship not considered | Relationship considered |
| 14 | 21/34 | 6/34 | 7/34 |
| 15, 16, 19 | 21/34 | 7/34 | 6/34 |
| 17 | 22/34 | 8/34 | 4/34 |
| 18 | 12/34 | 15/34 | 7/34 |

**TABLE 5.6** - Pupils' answers to items involving variables in a functional relationship (VFR)

As shown in Table 5.6 a large proportion of pupils did not answer these items. I would suggest that a major obstacle was the difficulty they had to link the posed problems with familiar situations. These results show that the great majority of pupils had no experience in working with VFR. Similar proportions of 'no answer' were obtained when pupils worked with other unfamiliar situations as, for example, equations with multiple appearances of VSU (Table 5.1, items 2.5 and 2.6) and 'unclosed' expressions (Table 5.4, items 1.5; Table 5.5 item 1.4).

The proportion of answers increased substantially for item 18. Fifteen pupils answered this item by listing the numbers between 3 and 10 instead of looking for the values of X that gave values of Y between 3 and 10. This indicates a misunderstanding of the question posed. This might be caused by the way the question was formulated. However, the answers given to other items show that pupils had difficulty to perceive that two variable quantities might be linked by a functional relationship. In fact, more than half of the pupils did not consider the given functional relationship when it was necessary in order to answer the item. For example, in item 14 six pupils assigned arbitrary values to both VFR. In item 17 nine pupils filled in the table with arbitrary values or they invented a new rule. Supporting Matz's (1982) findings it was observed that some erroneous answers given to item 15 were a consequence of considering that operators as well as the

value of the literal symbols might change. This is illustrated by the dialogue the researcher sustained with Nayeli when discussing her answer to item 15 (Dialogue 6).

**Dialogue 6**

*Researcher:* *'You answered this item underlying the word 'sometimes'. Could you explain your answer?'* In order to remember why she gave that answer, Nayeli assigns few values to Y and calculates the value of X.

*Nayeli:* *'Yes, it is "SOMETIMES". It will be always greater than X, except if we change the operation. If instead of adding 5 we subtract 5. In that case ...'* (she writes a small '-' sign over the '+' sign and tries out some values for the expression Y = X - 5) *'... it will be always smaller'*

However, there were pupils who although inconsistently considered the given functional relationships in order to solve the problems (Table 5.6). They assigned arbitrary values to one of the VFR and they calculated the value of the other one. Besides the perception of a one-to-one correspondence the answers given by six pupils to items 15, 16 and 19 suggest a perception of change. That is, an emerging capability to consider VFR as entities moving in a linked way was observed. For example, to answer item 16, some pupils wrote 'The value of Y also increases'. This suggests that they were able to assign different values to X in the expression Y = 7 + X and to calculate the corresponding value of Y. After this, they were able to analyze how the value of Y was changing in relation to the variation of the value of X. Additionally the same six pupils used the given relationship when asked to determine a range of values satisfying a given condition in item 18 ('If we want the value of Y to be bigger than 3 but smaller than 10 what are all the values that X can have'). Four of them were able to fill in correctly the table in order to answer item 17.

Item 1.3 (Fig. 5.11), included in Part 1 of the questionnaire, explored pupils' capability to symbolize VFR.

1.3 Write a formula which means:
An unknown number is equal to 6 plus another unknown number.

**Fig. 5.11** Item requiring to symbolize variables in a functional relationship (VFR)

The answers obtained were classified into four groups:

- no answer was given
- specific numeric values satisfying the statement were considered
- the functional relationship was symbolized representing both VFR with the same symbol
- the functional relationship was symbolized representing the VFR with two different symbols

The proportion of pupils giving these answers is presented in Table 5.7.

| Proportion of pupils | | | | |
|---|---|---|---|---|
| Item | No answer | Value assigned | One symbol used | Two symbols used |
| 1.3 | 13/34 | 11/34 | 7/34 | 3/34 |

**TABLE 5.7** - Pupils' symbolization of variables in a functional relationship (VFR)

Table 5.7 shows that the majority did not symbolize the given statement. Eleven pupils tried to interpret it and they gave a numeric answer. They wrote 3+3, 3+3=6, 3x2=6, or a number greater than 6. It might be that to use the terminology 'an unknown number' led pupils to look for a specific number that would satisfy the condition given in the statement.

However, there were pupils who were able to symbolize the given relationship. Seven pupils used the same literal symbol to represent both inter-related variables suggesting in this way the presence of the 'polysemia' error. Three pupils used two different symbols in order to represent the functional relationship linking to variables.

Pupils' responses to items involving VFR show that most of them were not able to work with VFR. They did not perceive the relationship between two variable numbers involved in an expression representing a functional relationship. However, some pupils showed an emerging although inconsistent capability to work with VFR and to consider them as entities moving in a linked way.

### 5.2.4 Summary

The analysis of the responses given to the written questionnaire shows that to cope with different characterizations of variable in usual algebra tasks was not easy for the tested pupils. They were able to calculate the value of a VSU when it appeared once in simple given equations, although a poor level of conceptualization of VSU was suggested by the difficulty some pupils had to interpret a literal symbol as VSU.
A great number of the tested pupils could not interpret and/or use a VGN. They could not interpret and/or write an 'unclosed' expression. But an emerging although inconsistent capability to work with VGN was observed for some pupils.
A large proportion of pupils could not cope with items involving VFR. In general they did not perceive a functional relationship linking two variable numbers. However, there were few pupils who showed an emerging although inconsistent capability to work with VFR.

The great majority of pupils showed a strong tendency to look for a result when they were asked to symbolize a statement given in words. This suggests a lack of experience in focussing on a method to solve a problem and to represent this, as opposed to solving a problem focusing on the results.

## 5.3 The case study pupils

The analysis of pupils' answers to the questionnaire showed that both, the proportion of pupils giving a type of answers and the type of answers given by a particular pupil, varied substantially with the items. Therefore it was not possible to classify pupils in groups regarding their performance with different characterizations of variable and to chose a sample to be case studied based on this classification.

A sample was chosen classifying pupils' answers as correct and incorrect. One point was given to each correct answer. The maximum possible score was 28. Based on the total score obtained by each child pupils were divided into four groups:

| Number of pupils | Bounds of the obtained scores |
|---|---|
| 4 | 15 - 19 |
| 7 | 11 - 14 |
| 10 | 7 - 10 |
| 13 | 4 - 6 |

The sample was formed choosing twelve pupils having different scores:

1 pupil whose score was between 15 and 19

2 pupils whose scores were between 11 and 14

3 pupils whose scores were between 7 and 10

4 pupils whose scores were between 4 and 6

2 pupils whose score was 4 and that had explicitly written in the questionnaire that they had never seen literal symbols used in mathematics before, except for geometric formulae.

These pupils were not completely chosen at random. Researcher's intention was to respect pupils' own choice of the peer they wanted to work with during the study. Therefore, when one pupil was selected for the sample her peer was as well included. The six selected pairs are presented in Table 5.8. Etna and Ernesto were the two pupils who had no prior experience in using literal symbols in mathematics.

| | | |
|---|---|---|
| Valentín (score 15) | and | Ernesto (score 4) |
| Etna (score 4) | and | Itzel (score 4) |
| Martha (score 9) | and | Mariana (score 7) |
| Oscar (score 5) | and | Jonathan (score 5) |
| Claudia (score 11) | and | Sandra (score 11) |
| Nayeli (score 13) | and | Soledad (score 4) |

**TABLE 5.8** - The six selected pairs to be case studied

### 5.3.1 Case study pupils' profile concerning variable as specific unknown (VSU)

The answers given by the selected pupils to the items of the written questionnaire involving VSU are presented in Table 5.9. These answers were confirmed during the individual interviews.

Table 5.9 shows that at the beginning of the study five pupils (Ernesto, Etna, Claudia, Nayeli, Soledad) could not calculate the value of a VSU when it was represented by a literal symbol (items 2.2, 3). On the contrary all the pupils were able to determine it when the box notation was used (item 7). Faced with equations with multiple appearances of VSU (items 2.5, 2.6) only three pupils (Mariana, Oscar, Jonathan) tried to solve them. During the interview Sandra tried to answer one of these items too (item 2.5). Both Mariana (see Fig. 5.3) and Sandra (see Dialogue 1) presented the 'polysemia'

error. Four pupils (Valentin, Claudia, Sandra, Nayeli) were able to symbolize a simple equation given in words (items 1.1, 1.2).

| Pupils' solution of a simple equation | | | |
|---|---|---|---|
| **Item** | **No answer** | **One arbitrary value** | **VSU calculated** |
| 2.2, 3 | Ernesto, Etna, Claudia, Nayeli, Soledad | ---------- | Valentín, Itzel, Martha, Mariana, Oscar, Jonathan, Sandra |
| 7 | ---------- | ---------- | All the pupils |
| 2.5, 2.6 | Valentín, Ernesto, Etna, Itzel, Martha, Claudia, Sandra, Nayeli, Soledad | Mariana*, Oscar, Jonathan | ---------- |

| Pupils' symbolization of a simple equation involving one appearance of VSU | | | | |
|---|---|---|---|---|
| **Item** | **No answer** | **Numeric answer** | **Letter used** | **Sign used** |
| 1.1, 1.2 | Etna, Itzel, Oscar, Soledad | Ernesto, Martha, Mariana, Jonathan | Claudia, Sandra, Nayeli | Valentín |

* Tried to balance the equation

**TABLE 5.9** Pupils' profile concerning variable as specific unknown (VSU)

During the interview it was observed that when a VSU was represented by a literal symbol pupils were in general confused whether it represented an unknown number or any number. This led them to try to calculate its value or to assign it an arbitrary value or to live the question unanswered. This suggests a poor conceptualization of VSU.

### 5.3.2 Case study pupils' profile concerning variable as general number (VGN)

Pupils' answers to items involving VGN are presented in Table 5.10. Table 5.10 shows that when answering these items only five pupils (Valentín, Oscar, Jonathan, Mariana and Sandra) tried to interpret a VGN (items 2.1, 2.3 and 2.4). When it appeared in an expression where the equal sign was not included they assigned an arbitrary value to the VGN. When the equal sign was included they tried to calculate the value of the VGN (see for example 5.2.2., Dialogue 3).

| Pupils' estimation of the values of a VGN | | | | |
|---|---|---|---|---|
| Item | No answer | One value | Mixed approach | Multiple values |
| 2.1, 2.3, 2.4 | Ernesto, Etna, Itzel, Martha, Claudia, Nayeli, Soledad | ---------- | Valentín, Oscar, Jonathan, Mariana, Sandra. | |
| **Pupils' use of a given VGN** | | | | |
| Item | No answer | VGN ignored | Value assigned | VGN used |
| 1.5 | Valentín, Ernesto, Etna, Itzel, Martha,. Oscar, Jonathan, Claudia, Sandra, Soledad | ---------- | ---------- | Mariana, Nayeli |
| 4 | Ernesto, Etna, Itzel, Sandra, Soledad | Martha, Mariana, Oscar, Jonathan, Claudia, Nayeli | ---------- | Valentín |
| 6 | Ernesto, Sandra | Etna, Itzel, Mariana, Oscar, Jonathan, Claudia, Soledad | ---------- | Valentín, Martha, Nayeli |
| 8 | Soledad | Valentín | Ernesto, Etna, Itzel, Martha, Mariana, Oscar, Jonathan | Claudia, Sandra, Nayeli |
| 9 | Ernesto, Etna, Oscar, Jonathan, Soledad | Itzel | Valentín, Martha, Mariana, Nayeli | Claudia, Sandra |
| 10 | Ernesto, Etna, Itzel, Mariana, Oscar, Jonathan, Soledad | ---------- | ---------- | Valentín, Martha, Claudia, Sandra, Nayeli |
| **Pupils' expression of generalization** | | | | |
| Item | No answer | Numeric example | Drawing continued | VGN used |
| 1.4 | Valentín, Ernesto, Etna, Itzel, Mariana, Oscar, Jonathan, Sandra, Nayeli, Soledad | Martha | ---------- | Claudia |
| 5 | Ernesto, Etna, Itzel, Mariana, Oscar, Jonathan, Claudia, Sandra, Soledad | ---------- | ---------- | Valentín, Nayeli |
| 11 | Mariana | ---------- | Ernesto, Etna, Itzel, Martha, Oscar, Jonathan* Claudia, Sandra, Soledad | Valentín, Nayeli |
| 12 | ---------- | Etna* , Mariana | Ernesto, Itzel, Martha, Oscar, Jonathan*, Claudia*,Sandra*, Nayeli*, Soledad | Valentín |
| 13 | Ernesto, Nayeli, Soledad | Etna*, Martha, Mariana*, Oscar* | Valentín, Itzel, Martha,Jonathan*, Claudia, Sandra | ---------- |

*Wrong answer

**TABLE 5.10** - Pupil's profile concerning variable as general number (VGN)

The individual interviews confirmed these results. Additionally it was found that all the pupils, except Etna and Ernesto, were able to consider that in items involving a VGN this might be substituted by an arbitrary number. But, all of them tried to find some criteria that would help them to decide if an arbitrary value was acceptable. If they did not find them they could not answer the question. This indicates that they were interpreting a VGN as a VPH and a VSU but not as a VGN. As shown in Table 5.10, some pupils (Valentin, Claudia, Sandra, Nayeli, and occasionally Martha and Mariana) were able to use although inconsistently a given VGN to symbolize a word given general statement (items 1.5, 4) and/or to express a generalization (items 1.4, 5, 6, 8, 9, 10, 11, 12, 13). However, a careful analysis of the answers given by each pupil to all the items and when individually interviewed suggest that only Valentin and Nayeli might be considered to have an emerging although inconsistent capability to work with VGN. Although the other pupils were able to use a literal symbol to express a generalization, they tended to interpret it as a VPH or a VSU (see 5.2.2, The case of Sandra).

### 5.3.3 Case study pupils' profile concerning variables in a functional relationship (VFR)

Table 5.11 presents pupils' answers to the questionnaire items involving VFR. They show that the great majority of the case study pupils did not answer these items. Some of them tried to answer some items but only Valentin, Mariana and Oscar occasionally considered the given functional relationship. None of the selected pupils could complete a table considering the given functional relationship (item 17). Asked to symbolize a functional relationship (item 1.3) Claudia and Nayeli did it by representing with the same symbol the two inter-related variables suggesting an interpretation of VFR as VPH.

These results were confirmed during the interview. However, it was observed that with researcher's intervention verbalizing the given functional relationship and asking pupils to consider this one in order to fill in the table seven pupils (Valentin, Oscar, Jonathan, Sandra, Claudia, Nayeli and Soledad) were able to fill in a table. This shows that with researcher's help they were able to consider VFR in one-to-one correspondence. In spite of researcher's help the other pupils disregarded the given functional relationship and they filled in the table with arbitrary numbers or they invented a rule.

| Pupils' answers to items with symbolized VFRs | | | |
|---|---|---|---|
| Item | No answer | Relationship not considered | Relationship considered |
| 14 | Ernesto, Etna, Itzel, Martha, Mariana, Sandra, Nayeli, Soledad | Oscar, Jonathan, Claudia | Valentín |
| 15, 16 and 19 | Valentín, Ernesto, Etna, Itzel, Martha, Oscar, Jonathan, Claudia, Nayeli, Soledad | Sandra | Mariana |
| 17 | Valentín, Ernesto, Etna, Itzel, Martha, Mariana, Claudia, Sandra, Nayeli, Soledad | Oscar, Jonathan | ---------- |
| 18 | Ernesto, Sandra, Nayeli, Soledad | Itzel, Etna, Martha, Jonathan, Claudia | Valentín, Mariana, Oscar |

| Pupils' symbolization of VFR | | | | |
|---|---|---|---|---|
| Item | No answer | Value assigned | One symbol used | Two symbols used |
| 1.3 | Etna, Itzel, Martha, Oscar, Jonathan, Sandra, Soledad | Valentín, Ernesto, Mariana | Claudia, Nayeli | ---------- |

**TABLE 5.11** - Pupils' profile concerning variables in a functional relationship (VFR)

### 5.3.4 Summary

Summarizing, the profile of the pupils chosen to be case studied shows that the majority could cope with simple one-step equations involving VSU but they could not cope with equations with multiple appearances of VSU. The majority could not symbolize an equation given in words. The great majority could not interpret a literal symbol representing a VGN. They had difficulties in using a given VGN to construct a new algebraic expression and they were not able to represent a generalization in a formal way. The great majority of the pupils did not answer the items involving VFR. Pupils who tried to answer these items did not consider the relationship linking the variables involved.

These results show that at the beginning of the study the majority of the pupils of the sample had difficulties to work with each one of the different characterizations of

variable considered when they appeared in normal algebra tasks represented by literal symbols. This stresses that the experience acquired in primary school in Mexico is not enough to help pupils work with these characterizations of variable although the review of the Mexican school texts (see 2.3, Chapter 2) suggests that pupils should be able to cope with them. This study aims to investigate if a potential to work with these different characterizations of variable can be created through Logo prior to formal algebra teaching.

# CHAPTER 6

# THE LOGO MICROWORLDS

## 6.1 Introduction

Three Logo microworlds were designed for this study (see Chapter 4, Table 4.1). Each one was constructed around one characterization of variable, namely, variable as general number (VGN), variables in a functional relationship (VFR) and variable as specific unknown (VSU). To work with the activities of the three microworlds pupils needed to have prior experience with some Logo primitives and the writing of fixed Logo procedures (FLPs). Therefore the first five sessions were devoted to introduce pupils to Logo.

## 6.2 Introduction to Logo

A series of introductory activities were devised (Table 6.1). Work sheets I-1, I-2 and I-6 are based on the work sheets of the Logo Maths Project (Hoyles, Sutherland and Evans, 1986).

| Introduction to Logo | | |
|---|---|---|
| **Activity** | **Aim** | **Worksheet** |
| 1 | Experimenting with Logo primitives (FD, BK, RT, LT, HOME, CS, PU, PD, ST, HT, PENERASE, PENCOLOR). | I-1, I-2 |
| 2 | Drawing shapes in direct mode | I-3 to I-5 |
| 3 | Syntax for writing and editing FLPs. Writing FLPs to draw a 'flag' and squares. | I-6 to I-9 |
| 4 | Writing FLPs to draw regular polygons - the use of REPEAT. | I-10 to I-11 |
| 5 | Writing FLPs to draw 'staircases'. | I-13 |

**TABLE 6.1** - Overview of the introductory activities

During the first four activities the 12 case study pupils were not closely observed. The observation started during the fifth activity. This aimed to observe if pupils were able to

use the Logo primitives fluently and to write fixed Logo procedures (FLPs). Table 6.2 presents the results of this observation.

| Pairs | Use primitives | Work in direct mode | Write FLPs |
|---|---|---|---|
| Valentín/Ernesto | Yes | Yes | Yes |
| Etna/Itzel | Yes | Yes | Yes |
| Martha/Mariana | Yes | Yes | Yes |
| Oscar/Jonathan | Yes | Yes | Need help |
| Claudia/Sandra | Yes | Yes | When asked |
| Nayeli/Soledad | Need help | Yes | Need help |

**TABLE 6.2** - Case study pupils' use of Logo primitives and fixed Logo procedures (FLPs) after four introductory sessions

These results show that after four sessions of working with Logo only one pair, Nayeli and Soledad, still needed help in order to use some of the Logo primitives. This pair had difficulties with the direction of the turtle's turns.

There was a general preference to work in direct mode instead of writing procedures. However, three pairs (Valentin/Ernesto, Etna/Itzel, Martha/Mariana) wrote a FLP spontaneously after solving the problem first in direct mode. One pair (Claudia/Sandra) wrote a FLP after being explicitly asked to do it. It was observed that to write a FLP all these pairs used the record of the commands given in direct mode as support. Two pairs, Oscar/Jonathan and Nayeli/Soledad, needed help from the researcher in order to write a FLP. Oscar/Jonathan needed support with the Logo syntax. Nayeli/Soledad tried to write a FLP by the direct observation of the shape. But, lacking the feedback of the turtle's movements they could not decide what command they had to type next. After being suggested by the researcher to keep a record of the commands given in direct mode they were able to write a FLP on their own.

Although these results suggest that some pupils would need to spend more time on introductory activities the reduced amount of time for the experimental work did not allow to extend this phase. However, the Logo experience acquired by pupils was sufficient for the purposes of the study and they were considered able to start working in the microworlds. To help pupils remember the introductory experience and to use it, the work sheets I-1, I-2 and I-6 were displayed on the walls of the classroom.

## 6.3 The components of the Logo microworlds

The Logo microworlds were designed in order to help create pupils' potential to cope with each one of the different characterizations of variable. To design them different aspects were considered. These are related to the four components considered by Hoyles and Noss (1987a) to be crucial in order to construct a microworld, namely, the technical, the pedagogical, the contextual and the pupil components. The technical component is formed essentially by the programming language and already written procedures. The function of the pedagogical component is to 'structure the investigation and exploration of the concepts embodied in the technical component' (Hoyles and Noss, 1987a, p. 588). The contextual component refers to the social environment in which the programming activity takes place. The pupil component takes into account both pupils' cognitive and affective aspects.

### 6.3.1 The technical component

This component of the microworld was formed by the Logo programing language, researcher's written procedures and researcher's designed work sheets. The Logo turtle graphics environment provided the framework for working with different characterizations of variable. Additionally the primitive PRINT and the syntax for using the four basic arithmetic operations were introduced in microworld 1. This aimed to help pupils use in these environments their prior arithmetic experience in order to work with VGN.

Researcher's written procedures were used in microworlds 2 and 3. In microworld 2 they were offered as tools to explore the idea of interval and to investigate the behaviour of two variables linked by a functional relationship. The procedures written for microworld 3 offered to pupils a way to work with the idea of VSU determining its value by trial and error. It was expected that to use the given procedures in the interactive Logo setting would help pupils work with VFR and VSU.

The aim of the work sheets was to structure pupils' work during the Logo sessions. They explained the goal of the activity and occasionally provided some leading questions or suggestions for solving the task (e.g. see Appendix 2 work sheet G-2 and D-5).

### 6.3.2 The pedagogical component

This component of the microworld concerned the design of the activities and the researcher's role during the Logo sessions.

*The design of the activities*

Since the characterizations of variable pupils were expected to approach were clearly defined, the activities might be considered to be guided tasks. The activities of microworld 1, 'Generalization and general number', were designed considering the difficulties pupils have to cope with VGN in usual algebra environments (Küchemann, 1980; Booth, 1984). They aimed to help pupils approach in Logo this characterization of variable by expressing a generalization. Pupils were led to focus on the method to solve a problem in Logo as opposed to focus on the result obtained by solving the problem. They were introduced to general Logo procedures (GLPs) as a formal way to express generalization, and to variables as representing the general numbers involved in a general method. They were given problems that had to be solved by writing GLPs.

The activities of microworld 2, 'Variables in a functional relationship', were designed bearing in mind the reported difficulties pupils have with the idea of function and its underlying notions (e.g. domain, range, constant functions) when they are introduced to them formally (Markovits at al., 1986, 1988; Sfard, 1989). During these activities pupils were not introduced to the idea of function in a formal way. They worked with the idea of interval and of variable moving within it. They used given GLPs to analyze how to change the value of the input was affecting the behaviour of the output. They were led to focus both on the pointwise/static correspondence between two variables and on their global linked behaviour.

To design the activities of microworld 3, 'Variable as specific unknown', it was considered that pupils have difficulties to determine the value of a specific unknown when solving equations or problems involving it (Galvin and Bell, 1977; Küchemann, 1980; Matz, 1980; Kieran, 1984, 1988; Filloy and Rojano, 1989; Herscovics and Linchevski, 1991a, b). It was also considered that these difficulties might stem from a lack of understanding that a specific unknown represents a particular number that can be calculated under given constraints. For that reason pupils were posed problems in Logo where to determine the value of a specific unknown they did not necessarily have to cope with equations written with algebraic syntax. To solve the posed problem it was necessary to conceive that a particular value had to be determined and that in order to

determine it the given constraints had to be considered. No instruction about posing equations and solving them was given.

*The researcher's role*

The researcher intervened in ways that were conducive to create pupils' potential to work in Logo with different characterizations of variable. She helped pupils understand the aim of the task and use their prior knowledge and experience in order to approach the characterizations of variable involved. She encouraged discussion and cooperation between peers. When she observed that the majority of the pupils had the same difficulty she addressed the whole group. On occasion she organized a group discussion to help all the pupils clarify some crucial points. She intervened individually with each pair giving nudges and encouraging pupils to reflect on the way they were solving the task and on its goal. When necessary, she reminded them a piece of information already used in previous activities. When in spite of these interventions pupils could not advance she indicated explicitly some steps to approach the solution and invited pupils to continue on their own. When a pair was not able to solve a task she provided stronger support by negotiating with pupils its solution and, on occasion, showing explicitly how to solve the given problem. To show explicitly how to solve a task aimed to help create pupils' potential to work with the characterization of variable involved. After that pupils were asked to solve similar problems alone. Some of the interventions provided can be related to the general categories of intervention, namely, motivational, leading to reflection and directional, presented by Hoyles and Sutherland (1989).

In general, the researcher did not provide support beyond the point it was needed. Aiming to push pupils to independent and self-regulated solution of the problems she encouraged pupils' own approaches helping them to use these in an appropriate way in order to reach a correct solution.

### 6.3.3 The contextual component

All the Logo activities of the study were developed at school during the regular computer workshop (one session a week). Pupils would receive marks for their work. Therefore the activities were perceived by pupils as school work which they had to fulfill. However, the researcher explained that their work would be the basis of an investigation. She stressed that she was more interested in observing the actions they carried out in order to solve the tasks than the final results obtained. They were invited to experiment,

to ask for help when they needed it, to talk with their partner and to collaborate in this way with the researcher's project. This created a relaxed atmosphere and contributed to pupils' involvement in the activities.

### 6.3.4 The pupil component

To design the activities the pupils' answers given to the questionnaire were taken into account. Since the pupils of the study came from different primary schools it was also considered that not all of them were necessarily used to work in pairs and to cooperate. Although there are primary schools in Mexico that encourage cooperation there are also those encouraging individual competitive work. For that reason during the introductory Logo sessions pupils were also introduced to work in pairs. Initially they were allowed to change their partner. This aimed to help them chose the partner with whom they would like to cooperate during the academic year. Their preference was accepted by the researcher and this influenced also the selection of the 12 pupils to be case studied. That is, when one pupil was chosen to be case studied her (his) partner was also chosen.

# CHAPTER 7

# VARIABLE AS GENERAL NUMBER

## 7.1 Introduction

Algebra research (Küchemann, 1980; Booth, 1984) has shown that when faced with algebraic expressions involving variables as general numbers pupils have difficulties in considering them as representing a range of values and they tend to interpret them as unknown numbers (Booth, 1984). In contrast, Logo research (Noss, 1985; Sutherland, 1987, 1992; Hoyles and Sutherland, 1989) shows that in specifically designed Logo environments pupils can develop an understanding of variables as standing for a range of values. This, however, is not sufficient to conclude that in Logo pupils are perceiving the variable as a general number. A general number is more than a letter standing for a range of values; it represents the general object of a general method. In Logo a general method to solve a problem is expressed by a general procedure and, as in algebra, the general object upon which that procedure applies is represented by a variable. However, in order to conceive a Logo variable as a general number and not only as a place holder to which a range of values may be substituted, it is necessary for the pupil to be able to perceive a general Logo procedure not only as an input-output machine but also as the expression of a general method to solve similar problems. The variables involved in a general Logo procedure may then be conceived as the general numbers of a general method.

In the present study it will be assumed that a variable is used as general number (VGN) when it represents an indeterminate number involved in a general method. It will be considered that pupils have developed a potential to work in Logo with the idea of variable as general number (VGN) when they can:

- shift from solving a particular problem by means of a fixed Logo procedure (FLP) to devising a general method to solve similar problems by means of a general Logo procedure (GLP);

- consider that a variable in a general Logo procedure (GLP) represents an abstract indeterminate number, that is, a general number, upon which Logo primitives are acting.

## 7.2 The activities

The activities of microworld 1 (Table 7.1) were designed in order to help create pupils' potential to approach in a Logo environment the ideas of general method to solve similar problems and of variable as general number (VGN). The work sheets used are included in Appendix 2 (G-1 to G-16u).

| Microworld 1 - Generalization and General Number | | |
|---|---|---|
| **Numeric Environment** | | |
| **Activity** | **Aim** | **Work sheet** |
| Doubling numbers | Direct pupils' attention to the common method used to solve similar problems. | G-1 and G-2 |
| Group discussion | Introduce the idea of general Logo procedures | G-3 and G-4 |
| Write it in Logo | Express a word given sentence in Logo | G-5 to G-7 |
| A multiplication table | Generalize and express a generalization in Logo | G-8 |
| **Turtle Graphics Environment** | | |
| **Activity** | **Aim** | **Work sheet** |
| Drawing squares | Direct pupils' attention to the common method used to solve similar problems as a scaffold to write general Logo procedures. | G-9 and G-10 |
| Regular polygons<br><br>Universe/Village (Part 1) | Generalize and express a generalization in Logo; variable not operated on. | G-11<br><br>G-12 u/v & G-13 u/v |
| Universe / Village (Part 2) | Generalize and express a generalization in Logo; variable operated on. | G-14 u/v to G-16 u |

**TABLE 7.1** - Overview of the activities of microworld 1

Two sets of activities were designed:

- activities embedded in the Logo numeric setting (work sheets G-1 to G-8)
- activities embedded in the Logo turtle graphics environment (work sheets G-9 to G-16u).

The reason for working in these two environments was not to compare pupils' performance in each one of them but rather to facilitate their approach to the idea of general Logo procedures (GLPs). The results of a previous study (Ursini, 1990c) suggested that GLPs (general Logo procedures) can be more quickly approached by 12-13 years old Logo beginners in a numeric Logo subset than in the turtle graphics

environment. The results obtained in that study showed that given a simple GLP that printed the result of an arithmetic calculation, pupils were able to analyse the procedure, understand it and run it for different inputs. They were also able to modify the given algebraic expression and substitute for it other algebraic expressions written on their own. In contrast, when working in the turtle graphics environment the difficulties they had with basic geometric concepts (e.g. angles) became obstacles which delayed pupils' work with GLP.

For these reasons, in the present study, pupils were first introduced to the idea of GLP and they became familiar with it in the numeric environment. After this they engaged in activities embedded in the turtle graphics environment. To avoid possible difficulties with angles these were explicitly marked on their work sheets (see work sheets G-11 to G-16u).

In both environments, the activities aimed to push pupils to construct a general method for solving similar problems, to consider a GLP as an expression of a general method and to look at the variables involved in the procedure as representing general numbers. The design of the activities considered Krutetskii's (1976) arguments stating that:

- the ability to generalize is closely related to the ability to make distinctions;
- the ability to generalize in mathematics evolves gradually, from the analysis of many particular examples in order to identify their known and general aspects, to the ability to go from one particular example to what is unknown and general.

For these reasons at the beginning of each set of activities pupils were asked to write three FLPs for three instantiations of a given problem and analyse them (work sheets G-2 and G-9) in order to identify their general aspects and to distinguish these from the particular ones. This analysis aimed to provide a basis for deducing with the researcher's help a general method for solving similar problems in Logo, and to approach in this way the idea of GLP as an expression of a general method applied to a general number.

After having been introduced to writing GLPs in the numeric environment (work sheets G-3 and G-4), pupils were asked to express in Logo a word given general sentence (work sheets G-5 to G-7). This implied to write simple algebraic expressions. These activities aimed to help pupils accept and become familiar with the idea of GLP as a way of expressing a general method for solving similar problems.

After this experience, pupils worked on activities in which they had to generalize expressing this in Logo. They were asked to do this first in the numeric environment

(work sheet G-8) and after this in the turtle graphics environment (work sheets G-9 to G-16u). When working in the turtle graphics environment pupils were asked to write GLPs to draw shapes of different size maintaining the given proportions. The sketch of a shape appeared on their work sheets. To write a GLP, pupils had to focus on the method used in Logo to draw that particular shape, generalize the method used and express it by a GLP. For some GLPs the variable had not to be operated on (work sheets G-10 to G-13u/v); for others it had to be operated on (work sheets G-14u/v to G-16u).

## 7.3 Researcher's intervention

During the activities of microworld 1 the researcher intervened in ways that aimed to be conducive to creating pupils' potential to work in Logo with VGN. The interventions provided were of the type described in 6.3.2, chapter 6. In particular, her interventions aimed to help the case study pupils:

- focus on the method used to solve a problem as opposed to focusing on the result obtained;
- write GLPs in both the Logo numeric and the turtle graphics environment;
- consider a VGN as a general object of a general method expressed by a GLP;
- operate on variable.

## 7.4 Results

The results obtained in both the numeric and the turtle graphics environments are discussed using the following organizing principles:

- pupils' potential to focus on the method used in Logo to solve a problem rather than on the result obtained;
- pupils' potential to express in Logo a general method to solve a problem;
- pupils' strategies to write GLPs.

### 7.4.1 Pupils' potential to focus on the method

#### 7.4.1.1 Focussing on the method in the Logo numeric environment

During the first activity of microworld 1 pupils were introduced to the use of the primitive PRINT (work sheet G-1) and asked to write three FLPs to double a number (work sheet G-2). Both the additive and the multiplicative strategy were accepted as correct provided that the same strategy was used to write the three procedures. After this pupils were invited to analyse their procedures in order to identify their similarities and their differences. This activity was aimed at observing pupils' capability to focus on the method to solve a problem rather than on the result obtained.

All the case study pupils were able to write three FLPs to double a number (Fig. 7.1 shows the FLPs written by Claudia and Sandra).

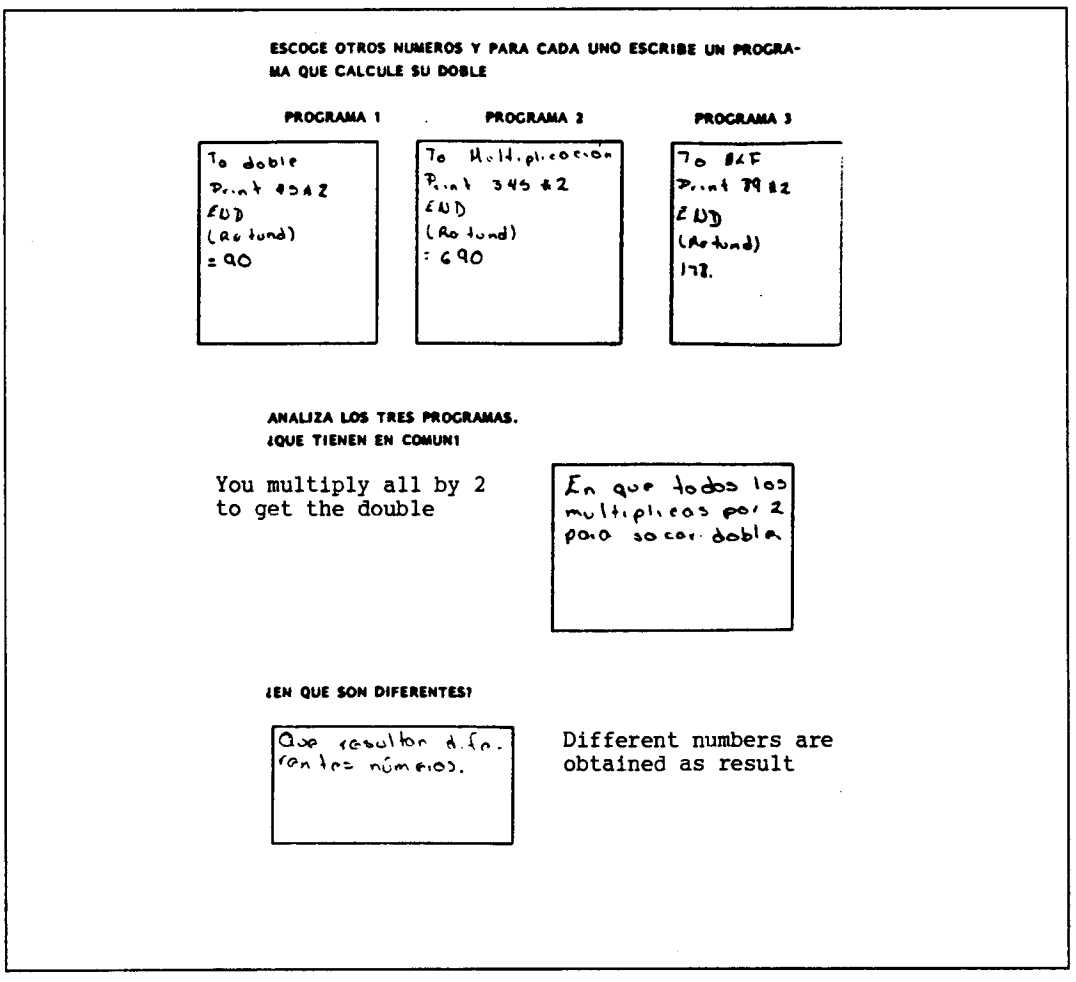



**Fig. 7.1** Claudia and Sandra - Fixed Logo procedures to double a number

This suggests that in the numeric environment when the case study pupils worked with arithmetic calculations they could avoid carrying them out, they were able to focus on the operation acting on particular numbers, and they were able to represent this by a FLP. After this they analysed the three FLPs guided by the questions that appeared on their work sheets (work sheet G-2).

The responses given by the case study pairs to these questions are shown in the two first columns of Table 7.2.

| Pairs | What is similar? | What is different? | | What is similar? | What is different? |
|---|---|---|---|---|---|
| Valentín, Ernesto | The 3 are additions<br>The 3 are procedures | They have different numbers | | All have PRINT, TO, END and + | The numbers |
| Etna, Itzel | The addition | The numbers | I N T E R V E N T I O N | TO<br>PRINT +<br>END | The procedures' names and the numbers; |
| Martha, Mariana | They add twice the same number | The results | | TO<br>PRINT +<br>END | The procedures' names, the numbers and the results |
| Oscar, Jonathan | They are multiplications | The numbers | | TO<br>PRINT<br>*<br>END | The procedure's names and the numbers; |
| Claudia, Sandra | You multiply all by 2 to get the double | Different numbers are obtained as result | | TO<br>PRINT<br>* 2 | The numbers and the results; |
| Nayeli, Soledad | We multiplied and we get the double | The numbers of the operation were different | | TO<br>PRINT<br>END<br>* | The procedures' name, the numbers and the results; |

**TABLE 7.2** - Case study pupils' analysis of three fixed Logo procedures (FLPs) to double a number in order to identify similarities and differences

The answers given by the pupils suggest a strong tendency to look at Logo procedures only as tools to do something: adding, multiplying, doubling a number. The operation was the only characteristic identified as common to the three FLPs. The numbers upon which the operation was acting or the results obtained were pointed out as different characteristics. Pupils' attention was on the arithmetic expression involved in the procedures and on the results obtained. They disregarded the other elements involved in the FLPs, that is, the Logo commands and their structure.

This pupils' behaviour shows their capability to focus on the operation involved in similar arithmetic expressions and to stress this considering that the operation may apply to different numbers. In a traditional school environment this capability would be a good starting point to introduce pupils to the idea of general arithmetic expressions. However, it was considered that this approach was not good enough to introduce them to the idea of GLP as an expression of a general method to solve similar problems. In fact, the method used in Logo to solve the posed problem implies not only the arithmetic expression but also the use of other Logo primitives structured in a determined way.

In order to push pupils to focus on the similar global structure of the three FLPs and help them make explicit the method used in Logo to solve the posed tasks, the researcher asked all the pupils to analyse once more the three FLPs examining them at the same time, line by line. The responses given by pupils after this researcher's intervention are presented in the last two columns of Table 7.2.

Comparing the answers given by pupils before and after the intervention, it is evident that this was crucial for pupils to shift from viewing a procedure as a tool to do something to looking at it also as an object to be analysed. This suggests pupils' potential to look at a Logo procedure as a method to solve a problem and not only as a tool used in order to obtain a result. However, even if pupils were able to identify structure similarities in the three FLPs, they needed additional help to organize these data in order to make explicit a method for solving the posed problem in Logo. Having this in mind a common group discussion was developed. This aimed to push pupils to show the methods they used to solve the given problem and to reach by a common discussion a consensus about them. The identified methods would be used as a starting point to introduce pupils to the idea of GLP and VGN as a means to express a general method for solving a problem.

Pupils communicated their findings to the whole group. In a discussion guided by the researcher, the group identified two methods to solve the posed problem in Logo, as well as the variable elements involved in the FLPs analysed:

| | |
|---|---|
| TO <a name> | TO <a name> |
| PRINT <a number> * 2 | PRINT <a number> + <a number> |
| END | END |

Each one of the two identified methods was emphasized by the researcher as representing a general method for solving in Logo different instances of the posed problem. The idea of variable was then introduced as a means to represent the

indeterminate number handled by the identified general method, that is, as a general number. During the common discussion, it was stressed that the variable was representing all the different numbers used by pupils when writing the three FLPs and that any other number could be substituted for it. At the same time the researcher emphasized that while the value of the variable might change, the method to solve the problem remained fixed.

After this discussion pupils were introduced to the syntax to write a GLP (work sheets G-3 and G-4). It was emphasized that any name, letter or symbol other than a letter could be used to represent a VGN.

**These results show that in this environment pupils' potential to focus on the method to solve a problem was created. They also suggest that pupils' spontaneous tendency is to solve a problem and not to analyse how a problem is solved. To engage in such kind of analysis pupils need to be encouraged by appropriate tasks and researcher's intervention.** I want to stress that to be able to focus on the method to solve similar problems rather than on the result obtained is a crucial step to approach the idea of general method for solving similar problems and, in consequence, to start constructing the idea of general number as a general object and not only as representing a range of values.

#### **7.4.1.2** Focussing on the method in the Logo turtle graphics environment

Pupils' potential to look at a Logo procedure as representing a method to solve a problem was confirmed when they faced the first activity in the turtle graphics environment (work sheet G-9). This suggests that this pupils' capability was not limited to the Logo numeric environment but they were able to extend it to the Logo turtle graphics environment. Pupils were asked to write three FLPs to draw squares of different sizes, to analyse them to identify their similarities and differences and to deduce a general method to draw squares of any size in Logo expressing this by writing a GLP. The responses obtained are presented in Table 7.3.

Table 7.3 shows that three of the case study pairs (Valentin/Ernesto, Martha/Mariana, Claudia/Sandra) needed only a nudge (e.g. *'Remember what you did in the numeric environment'*; *'Identify what parts of the procedure are changing and what are fixed'*) in order to write a GLP after having written and analysed three FLPs to draw squares. This suggests that they were able to focus on the method used to draw squares in Logo, to

deduce by themselves a general method for solving similar problems and to express this in a formal way.

| Pairs | What pupils found similar | What pupils found different | Intervention given to help pupils write a GLP |
|---|---|---|---|
| Valentín Ernesto | They draw squares | Squares' size | Nudge |
| Etna Itzel (absent) | They draw 3 squares | Squares' size | No intervention |
| Martha Mariana | FD, RT | The numbers | Nudge |
| Oscar Jonathan | The 3 are procedures | Squares' size | Strong intervention |
| Claudia Sandra | RT, FD | The numbers | Nudge |
| Nayeli Soledad | FD, RT, TO, END, RT 90 | The numbers after FD | No intervention |

**TABLE 7.3** - Case study pupils' analysis of three fixed Logo procedures (FLPs) to draw squares and the kind of intervention provided to help them write a general Logo procedure (GLP) in the turtle graphics environment

Two pairs (Etna/Itzel-absent and Nayeli/Soledad) did not need researcher's interventions in order to write a GLP to draw squares of any size (Fig. 7.2 shows the GLP written by Nayeli/Soledad).

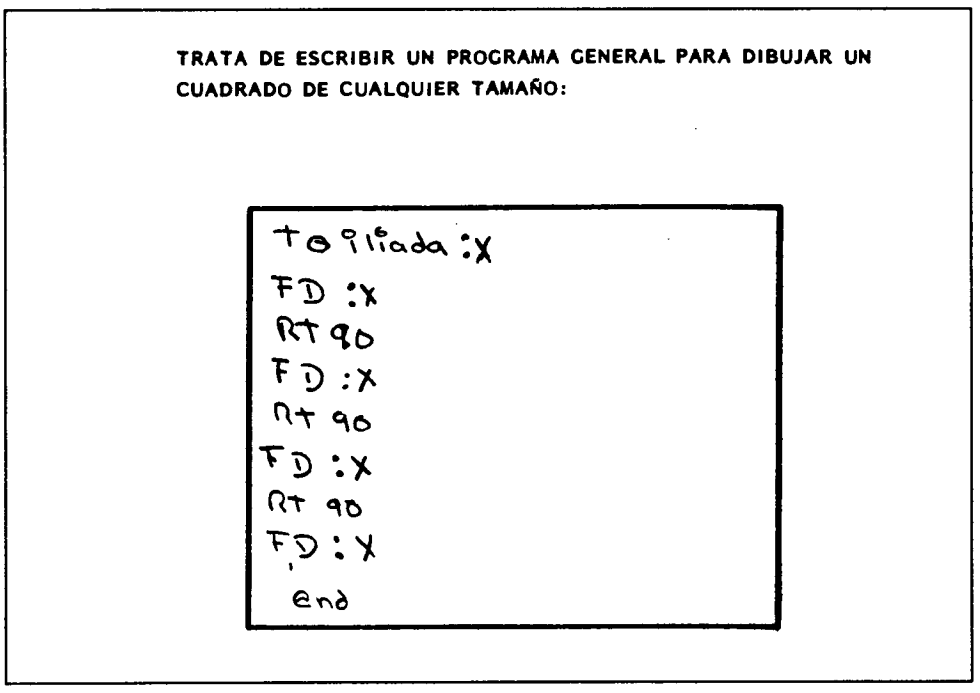



**Fig. 7.2** Nayeli and Soledad - General Logo procedures to draw squares

This indicates that these pairs were able to consider the FLPs written in the turtle graphics environment as objects to be analysed and not only as tools to be used. It was observed that to write a GLP they were combining in a dialectic way the capability to write a GLP acquired during their work in the numeric environment and their approach to FLPs as objects to be analysed. This is evidenced by the way Etna approached this activity. During her work on this task Etna was closely observed and her comments were recorded.

The case of Etna

During the first activity in the turtle graphics environment (work sheet G-9 and G-10) Etna's peer is absent and she works alone.

Etna starts by drawing some squares in direct mode guiding her decisions by self directed speech: *'I have already two sides ... now, I need two more'*. She explains to the researcher that she prefers to work first in direct mode in order to take appropriate decisions about the turtle's turns. After this she writes three FLPs on her worksheet (Fig.7.3).

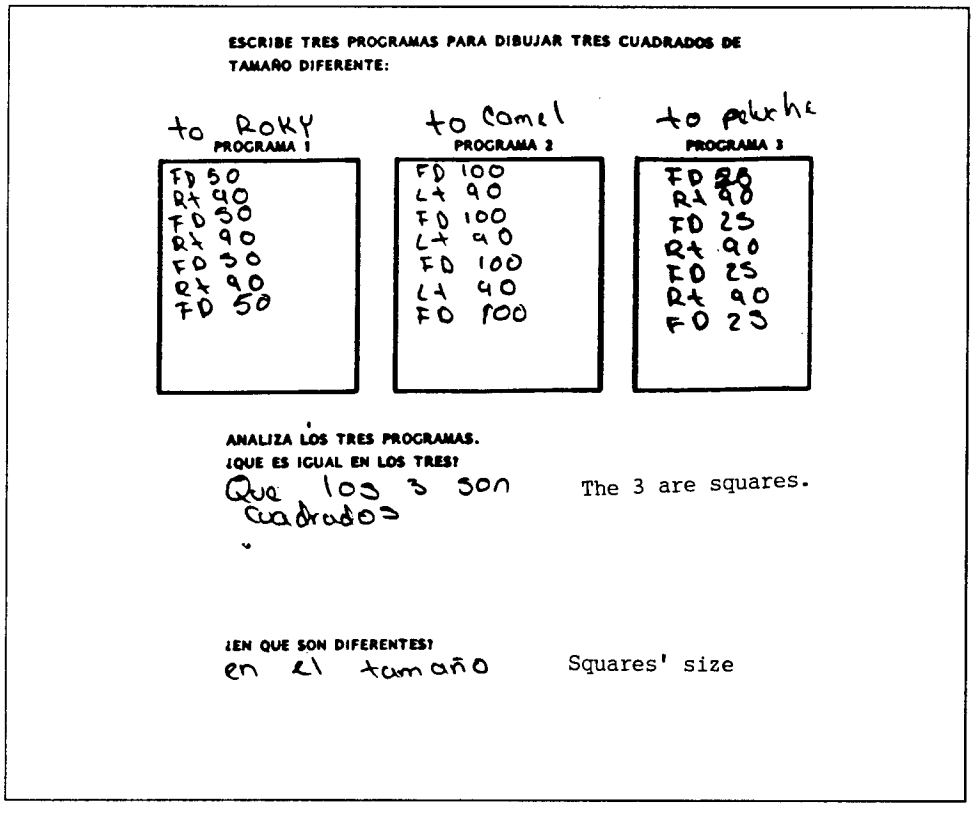



**Fig. 7.3** Etna - Analysis of three fixed Logo procedures to draw squares

She starts analysing them and comments to herself: *'Yes, all are squares and what is varying is the square's side. I will write repeat four and after this FD a number and RT another number'*. This comment shows that Etna is looking at FLPs as objects to be analysed. She perceives a repeating pattern in the three FLPs and she makes a rough classification of the similarities (*'all are squares'*) and of the differences (*'what is varying is the square's side'*). After this she writes:

```
TO ESPECIAL :X
REPEAT 4[FD :X RT :X]
END
```

To write this GLP she is using the experience acquired in the numeric environment. However, the GLP she writes shows that in her attempt to deduce a general method for drawing squares she has not differentiated between the invariant and the variable elements and she has over generalized. On the other hand her verbalization ' .... *I will write repeat four and after this FD a number and RT another number*' suggests that **Etna is not working any more with particular numbers but with general number and she is able to use a symbol to represent the general number**. She runs the procedure realising from the shape that appears on the screen that something is wrong. Without asking for help she analyses her GLP and comments: *'Let me see ... what is wrong? It is repeat four, FD a number, RT a number ... no! ... you are fool, it is RT 90'*. This analysis and the visual feedback obtained, helped Etna to distinguish between the variable and the invariant elements. She modifies the procedure obtaining a GLP to draw squares (Fig. 7.4).

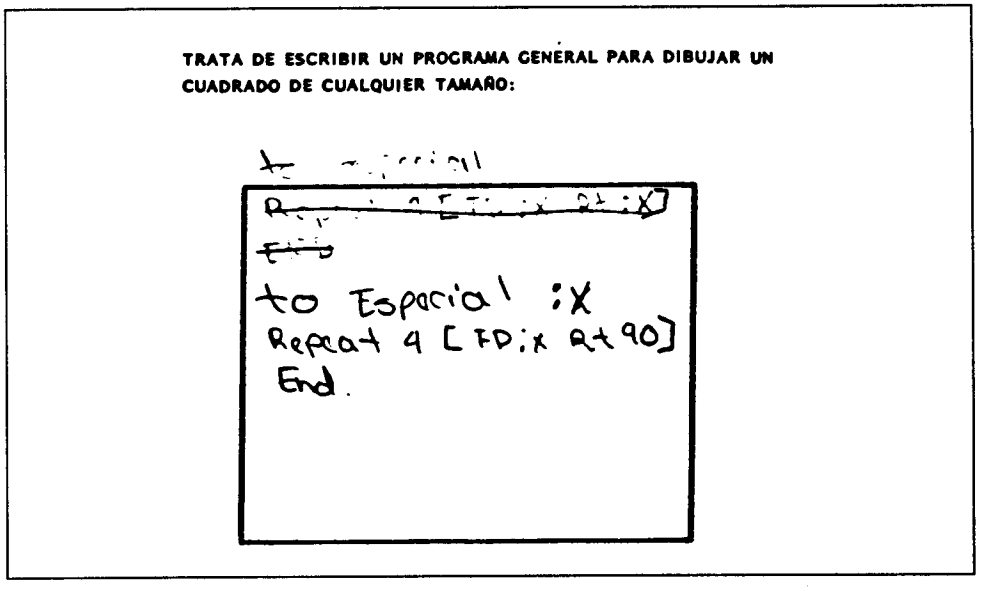



**Fig. 7.4** Etna - A general Logo procedure to draw squares

To do this she did not need the researcher's help. The possibility offered and suggested by the Logo environment to go from the particular to the general and back again to the particular helped Etna during the generalization process as well as to express her generalization in a formal way.

**These results show that Etna's first approach to the task aims to deduce a general method to draw squares in Logo based on the analysis of the three FLPs. During this process she shifts from considering particular numbers to thinking (her thoughts were suggested by her loud comments) with general numbers and with the Logo primitives acting on them. She expresses her hypothesis by a GLP and after this she engages in a dialectical spiral between using and analysing the GLP. The GLP is both a tool to be used and an object to be analysed and Etna shifts between them. This dialectic combination of using and analysing a GLP and the possibility offered by the computational environment to test immediately the correctness of her hypothesis guide her to formulate in Logo a general method for drawing squares.**

As a final comment the role played in this case by the loud self-directed speech has to be stressed. **The loud self-directed speech is an essential element of Etna's analysis of the posed situation. She uses it to organize her thinking and it helps her make a qualitative shift from particular numbers to general number. Faced with an unexpected result she uses it as a tool helping her to direct her analysis and identify the mistake.** Loud self-directed speech was common among the case study pupils although in general it was mixed up with communicative speech directed to the partner. Both kinds of speech were more often observed when pupils were facing new unfamiliar situations. Pupils used them when trying to comprehend a situation and to plan the solution of a problem. It was observed that both kinds of speech tended to disappear when pupils solved the posed task by applying a method already used in a prior activity (e.g. when writing the second or the third GLP where the variable had not to be operated on).

Only one of the case study pairs, Oscar and Jonathan, needed a strong researcher's support in order to write their first GLP in the turtle graphics environment.

The case of Oscar and Jonathan

After writing the three FLPs Oscar and Jonathan need help to analyse them in order to discriminate the similarities from the differences. They do not seem to see any analogy

between the requirement of this task and the analysis done in the numeric environment when they were first introduced to a GLP. The researcher intervenes discussing with them the posed problem in a similar way as it was done in the numeric environment. After this Oscar suggests that it is necessary to use PRINT to write a GLP to draw squares of any size. This suggestion shows that he starts realising that there are similarities between the problems posed in both environments, however, he is completely confused. Instead of getting involved in the problem he tries to remember a possible solution and makes attempts to extrapolate his prior knowledge to the new environment. Jonathan is very distracted during this interchange. It is necessary to help them approach the problem as a completely new task and to show them explicitly how to write a GLP to draw squares.

The difficulties this pair had to write a GLP in the turtle graphics environment suggest that their idea of GLP was tightly linked to the numeric environment. It seems that they had not completely grasped the idea that a GLP can be viewed as representing a general method for solving similar problems independently of the environment where the problems are given.

There may be several reasons explaining Oscar's and Jonathan's difficulty. Very often these pupils did not pay attention to the common discussion as well as to the researcher's explanations. They preferred to play with the computer instead of listening to the researcher when new tasks were given. Specially one of them, Oscar, worked always in a very confused way. He had difficulties to communicate his thoughts clearly. He was very competitive and faced with a task he tried to solve it alone and quickly. He did not listen to his partner's suggestions, except when he was stuck after several unsuccessful trials.

This behaviour led this pair to take turns on the computer, each one assuming the total responsibility of the work, while the partner was watching. In this way they avoided discussions and instead of collaborating each one was given a chance to solve the given problem individually. In the numeric environment, for example, after the initial individual help both of them were successful in writing GLPs, however, **because they tended to work individually they never discussed how they were solving the posed problems. Therefore, they were not encouraged to reflect on their actions by the necessity to communicate it to the partner.**

**The lack of communication probably contributed substantially to blur the idea of GLP as a general method to solve similar problems and of VGN as general number, although in the numeric environment they were able to write GLPs and to**

**use the variable in a correct way.** It might be that in the numeric environment, after having grasped initially the idea of GLP, Oscar and Jonathan got used to write them. Because they were not discussing how to solve the posed tasks they gradually lost their meaning and wrote GLPs in an automatic way. Therefore, in order to grasp once more the idea of GLP and to start writing GLPs in the Logo turtle graphics environment it was necessary for the researcher to solve the tasks with them repeating the tailed analysis of FLPs, deducing a general method and showing how to express this by GLP.

**These results suggest that it is possible for pupils to solve problems in a correct way without having completely grasped the main idea underlying their solution. It seems that after a successful approach they may get used to the method and they do not care about understanding why it works. For this reason when introducing pupils to new ideas it is essential to design activities in which they are very often faced with challenging situations that push them to analyse the solution methods and that avoid a premature automation.**

To conclude this section I want to stress that one of the aims of Logo designers (Papert 1980) was to help pupils to develop an analytical approach to their own way of thinking. This is partially reached, for example, when pupils engage in a debugging process in which they analyse their own procedures that are descriptions of their own way of thinking, in order to localize the errors and to correct them. However, when pupils are involved in such kind of activity their main aim is to make the procedure behave in a pre-determined way. The procedure, even if analysed, is looked essentially as a tool to do something and the child analyses it as a mechanism that needs to be fixed in order to work properly. Once it works it is used and it is, in general, not analysed any more. The way the majority of the case study pupils dealt with the activities discussed in this section shows that **it is possible to encourage pupils to look at procedures that are already working correctly, not only as tools to do something but also as descriptions of methods to solve problems.** Pupils were able to look at procedures as objects to be analysed; they became conscious of the method used to solve a problem and of a formal way of expressing it. The fact that pupils were able to engage in such an analysis and in consequence to write a general procedure shows the potential they have developed to focus on the method for solving a problem and to express this in a formal language. **However, the results obtained suggest also that pupils do not necessarily engage spontaneously in this kind of analysis. They need to be led to it by specially designed tasks with increasing level of complexity and by interventions**

**encouraging them to focus on the method used to solve a problem rather than on the results obtained.**

### 7.4.2 Pupils' potential to express in Logo a general method to solve a problem

The specially designed activities, the researcher's interventions and the common discussion created pupils' potential to focuss on the method to solve a problem in Logo rather than on the results obtained by solving the problem. However, it was not straightforward for all the pupils to accept GLPs and VGN as a means of expressing a general method in Logo.

When asked to write their first GLP in the numeric environment only Claudia and Sandra were able to write it successfully without researcher's help (Fig. 7.5 shows Claudia's and Sandra's record of their GLPs).

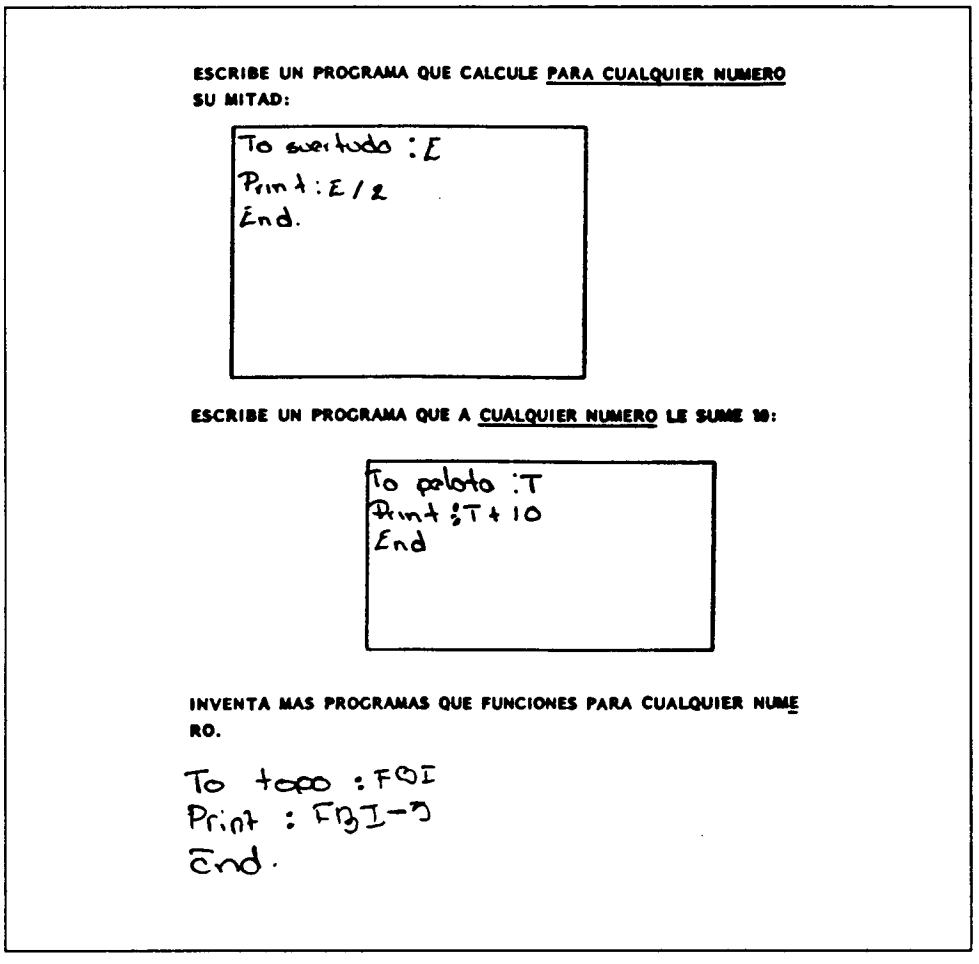



**Fig. 7.5** Claudia and Sandra - Their first general Logo procedures

Observing the work of the other pairs it was noticed that they:

- had difficulty with Logo syntax: they used the symbol * to indicate indistinctly all the operations (Martha/Mariana and Nayeli/Soledad); they forgot to assign a value to the variable (Oscar/Jonathan);
- gave only a particular solution to the posed problem instead of a general solution (Valentin/Ernesto);
- had difficulty to use a symbol to represent a VGN (Etna/Itzel).

Pupils whose difficulties to write a GLP belonged to the first two listed above could overcome them after receiving researcher's individual help. She intervened by helping pupils repeat the analysis done during the common discussion; stressed once more the ideas of GLP and VGN as an expression of a general method for solving a problem and called pupils attention on work sheets G-3 or G-4 where an example of GLP was given. This individual support helped these pupils overcome the initial difficulty and after this they were able to start writing simple GLPs in order to express in Logo word given general sentences. They represented the VGN by letters (Martha/Mariana; Nayeli/Soledad; Claudia/Sandra); by letters or names (Oscar/Jonathan); by symbols other than letters (Valentin/Ernesto). Individually asked what were representing the letters, names or symbols other than letters they were using, all these pupils considered that they represented any number and that when running the procedure any value could be substituted to them.

Etna and Itzel were the only pair who had difficulty in symbolizing a VGN, even if they were able to approach the idea of general method. It was observed that in order to be able to symbolize a VGN, they needed to construct first a meaning for it. This is evidenced by the way this pair approached the activities 'Write it in Logo', work sheets G-5 to G-7.

The case of Etna and Itzel - Constructing a meaning for the symbol

To solve the first task (work sheet G-5) pupils are asked to write a GLP in order to express in Logo a general statement given in words (e.g. 'Write a procedure to calculate the half of any number'). Etna and Itzel verbalize several numeric examples of the given sentence, but they cannot write a GLP. After receiving researcher's individual support stressing the idea of GLP and VGN and calling their attention on work sheets G-3 and G-4, they write a GLP but they are still very insecure about the use of the symbol to represent the variable element.

Asked to write a GLP to multiply any number by itself (work sheet G-6), Itzel timidly proposes writing X by X but, instead of doing it she asks first for the researcher's approval. This suggests Itzel's potential to generalize and express it by using a variable to represent a general number, but to do this she needs the researcher's agreement. Etna seems not to understand Itzel's suggestion and she verbalizes several numeric examples. After this she accepts writing X by X, as proposed by Itzel, but argues that she does not understand very well why it is correct.

The answers given to the questionnaire applied at the beginning of the study (see 5.3, Chapter 5) showed that Etna lacks experience with the use of literal symbols in mathematics. During the present activity, it seems that she is trying to make sense of them. I would suggest that she is trying to use the numeric examples as a scaffold, that is, as a 'hook' which helps her understand the algebraic symbolization proposed by Itzel. However, it seems that she still cannot give a meaning to the symbol and this obstructs its acceptance. This hypothesis is further supported by the way this pair write a procedure to divide any number by 10 and subtract 100 to it (work sheet G-7). Etna and Itzel work on this task without being observed. When the researcher approaches them they have written the following procedure and they ask for help in order to run it:

```
TO DIVIDIR
PRINT   /10 - 100
END
```

Etna explains that the procedure divides any number by 10 and than subtracts 100 and that any number can be substituted to the empty space in the expression <empty space>/10 -100. This solution to the posed problem shows their spontaneous way of symbolizing in Logo the general statement given in words. This suggests that **they have understood the general sentence and are capable of translating it into a formal language, but they do not use a symbol to represent the general number.** On the one hand, this solution testifies Itzel's insecurity about using symbolic variables. In fact, during the prior task, she was already suggesting that the way to solve the posed problem was by using a symbolic variable, but in the present task, perhaps because she lacked the researcher's approval, she drops to a safer level where the variable is not represented by a symbol. On the other hand, this solution shows that Etna has not yet assimilated the idea of using a symbol to represent the idea of 'any number'. This leads these two girls to leave an empty space in their procedure to be filled by any value instead of representing the general number by a symbol. However, although not completely correct this solution represents a progress in the acceptance of the idea of

GLP as an expression of a general method. In fact, Etna has gone beyond the particular examples; they both focus on the operation that will be applied to arbitrary numbers; and they spontaneously represent this by means of an almost correct GLP. Etna's explanation of how the procedure works suggests that she is convinced about the meaning of the symbolization they are using.

The researcher intervenes explaining that Logo cannot interpret an empty space and that it is necessary to give a name to the variable element. This consideration is enough for Etna and Itzel to accept to use a symbol in order to represent the general number in Logo. After this they write GLPs in the numeric environment without needing additional interventions. In general, they use a letter to represent the variable. Asked what is representing the letter they are using both girls say '*It represents any value*'.

**These results suggest that even if the idea of general number and the necessity to operate on it may be conceived, it is not straightforward for pupils to use a symbol to represent it.** I would suggest that in order to use a symbol this pair needed first to construct a meaning for it. Etna tried first to use numeric examples as a scaffold for giving a meaning to the symbol and accepting it. But it seems that even if the numeric examples helped her to approach the expression of a general statement in Logo, she could not yet use a symbol to represent the general number and she left an empty space to be filled by any number. This notation was probably influenced by experiences acquired in primary school in which an unknown is often represented by a box or an empty space. However, Etna's explanation (*'any number can be substituted to the empty space'*) suggests that this pair has not a particular number in mind but they are able to consider a general number. **At this point the researcher's indication to use a symbol instead of the empty space is enough to help this pair accept and use it. It becomes meaningful for them to use a symbol: it is representing the empty space; and the empty space is their own way of representing a general number. After this Etna and Itzel used fluently literal symbols to represent the variable element in their GLPs.**

**The case of Etna and Itzel points to a crucial element for creating pupils' potential to cope with variables, namely to help them construct a meaning for the symbols used to represent them. As shown in section 2.3, chapter 2, pupils are often faced with symbolic variables without being led to construct a meaning for them. This might be a cause of misconception and difficulties pupils have with variables.**

The results obtained during these activities (work sheets G-5 to G-7) show that all the case study pupils developed the ability to write GLPs symbolizing in this way word given general sentences representing general arithmetic expressions. This suggests pupils' acceptance of the idea of GLP as a means to represent a general method to solve a problem and not only as an input-output mechanism. To write a GLP pupils used symbolic variables to represent the non defined numbers and they were able to operate on them with numbers (see for example Fig. 7.6).

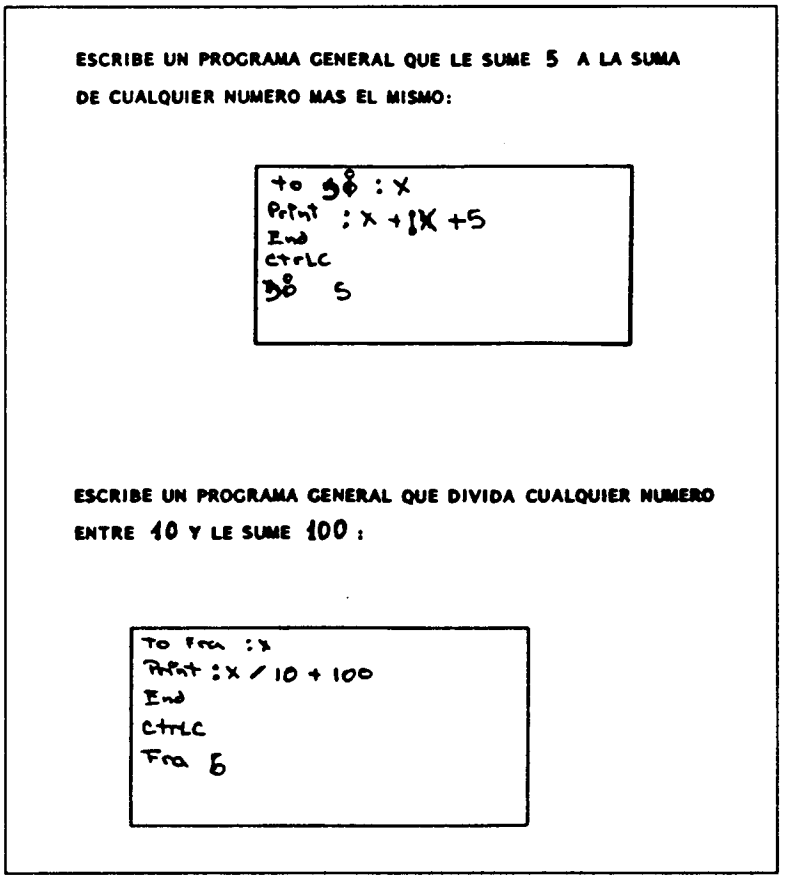



**Fig. 7.6** Nayeli and Soledad - Use of symbolic variables to write general Logo procedures

**This suggests that pupils' potential to approach in this numeric Logo environment the idea of variable as general number (VGN) was created. However, pupils were still tightly anchored to particular numbers and it was not easy for them to relax this link.** This is evidenced, for example, by the way the case study pupils approached the task where they were asked to write a GLP to multiply any number by itself (work sheet G-6). To solve this task all the pupils except Etna/Itzel wrote a GLP multiplying a variable by a particular number. They considered this a correct Logo representation of the posed problem. The researcher's help was crucial to push pupils to relax the link with

particular numbers and to approach a correct symbolization of the given general sentence. This is exemplified by the case of Valentin and Ernesto.

The case of Valentin and Ernesto - Relaxing the link with particular numbers

Valentin and Ernesto face the task of writing a GLP to multiply any number by itself after having written three GLPs by themselves when solving work sheet G-5 and the first problem of work sheet G-6.

When the researcher approaches them they have already written a GLP:

```
TO MESA  : ♥
PRINT  : ♥  * 5
END
```

Questioned if their procedure is multiplying any number by itself, Valentin and Ernesto confirm it. To prove this to the researcher they run the procedure with input 5. Although incorrect, their solution to the problem highlights their way of approaching the idea of GLP and VGN in this environment and stresses their link to the particular. Their GLP suggests that they have approached the task by thinking of a specific example of the general given sentence and they have tried to translate it to a GLP by substituting the arbitrary particular number by a symbol. However, they have not substituted both appearances of the arbitrary particular number maintaining in this way a link with the particular example they had in mind. Several reasons may underlay this behaviour: they may consider that running their GLP with 5 will provide a particular example of the general statement and that giving a particular example is a way of expressing a generalization; they have no experience with a mathematical expression that does not involve numbers at all; they may consider that an arbitrary number can be represented by a symbol but an operation must involve a particular number; they may consider that the same symbol does not necessary represent the same particular value.

Their GLP shows that even if Valentin and Ernesto seem to have only one particular example in mind, the requirement of the task to write a GLP pushes them to substitute a symbol to one of the two appearances of the particular number. I would suggest that in this way they try to communicate that the particular number they have in mind is arbitrarily chosen and any other number can be used instead of it. However, they do not substitute the symbol for both appearances of the arbitrary number. This suggests that to do this a qualitative shift relaxing the link with particular numbers and starting to think with general numbers is necessary. Aiming to help pupils make this shift the researcher

intervenes suggesting to try more numeric examples before writing a GLP. This suggestion considers pupils tendency to rely on numbers, and it apparently encourages pupils to remain in the realm of particular examples. Nevertheless, it aims to help pupils use the particular examples as a trampoline to jump to a qualitatively different way of thinking. This strategy was successful and pushed pupils to relax the link with particular numbers and even if timidly and doubtfully after verbalizing several numeric examples, they alone suggested to write :♥ * :♥. **These results show that for Valentin and Ernesto a potential to relax the link with particular numbers and to shift to a qualitatively different level where they started thinking with general numbers was created.**

Approaches similar to the one used by Valentin and Ernesto were observed for the other case study pupils. This suggests that although pupils might be able to use VGN and operate on it with numbers, a qualitative shift is needed in order to be able to operate on variables with variables. This implies **to relax completely the link with particular numbers and to start working directly with VGN. The results obtained show that to make this shift pupils need substantial support**.

When writing GLPs in the numeric Logo environment all the pupils were able to write 'unclosed' expressions (see for example Fig. 7.6). When solving the written questionnaire applied at the beginning of the study (Chapter 5, Table 5.10, questions 1.5, 8, 9, 10, 11, 12, 13), only some of the case study pupils were able to write sporadically 'unclosed' expressions to answer some of the questions requiring them. I would suggest that to write 'unclosed' expressions in the Logo numeric environment was meaningful for pupils. On the one hand, it was a consequence of having analysed particular examples first and having deduced the general method involved going from the particular to the general. On the other hand, it was part of the Logo syntax, that is, it was something necessary in order to communicate to the computer the general method they had identified. Finally, to write an 'unclosed' expression was not the final step of a process but an intermediate step in the process of writing a GLP that would be run giving a particular result. This suggests that **to help pupils accept and write 'unclosed' algebraic expressions might be crucial to offer them initially the opportunity to work in environments where the link with particular numeric instantiations can be naturally maintained in both directions: in order to go from the particular to the general and in order to go from the general back to the particular. This kind of experience might help pupils to give meaning to 'unclosed' expressions and help them relax gradually the link with particular instantiations of general expressions.**

### 7.4.3 Pupils' strategies to write general Logo procedures (GLPs)

To write GLPs both in the numeric and in the turtle graphics environments, pupils used different strategies. These were identified by analysing pupils' written work, the researcher's and the observer's notes of each one of the activities of microworld 1.

#### 7.4.3.1 Writing general Logo procedures (GLPs) in the numeric environment

Four different strategies to write GLPs in the numeric environment were observed (Table 7.4). These were classified depending on the number of steps required to achieve the goal and the type of activity developed in each step.

<table>
<tr><th>Strategy</th><th>Number of steps</th><th>Step 1</th><th>Step 2</th><th>Step3</th></tr>
<tr><td>N1</td><td>1</td><td>----------</td><td>----------</td><td rowspan="4">GLP written</td></tr>
<tr><td>N2a</td><td>2</td><td>One particular example verbalized</td><td>----------</td></tr>
<tr><td>N2b</td><td>2</td><td rowspan="2">Several particular examples verbalized</td><td>----------</td></tr>
<tr><td>N3</td><td>3</td><td>FLP written</td></tr>
</table>

**TABLE 7.4** - Four strategies to write general Logo procedures (GLPs) in the numeric Logo environment

As shown in Table 7.4 strategies N2a, N2b and N3 share a common characteristic: pupils using them approached the task by giving first of all verbal particular examples of the general statement given in words. This suggests that pupils were trying to make sense of the given generalization by making explicit some possible particular examples which could have originated it.

When the strategy N3 was used, pupils represented in Logo one or more of the verbally given particular examples by writing FLPs. In this way by expressing an arithmetic problem in Logo language they were transferring the problem to the Logo environment. Subsequently they generalized within the new environment expressing this by a GLP. Strategy N3 shows similarities with the method devised by the researcher to introduce pupils to the idea of GLP, where pupils were asked to write first three FLPs, record them

on their work sheet, analyse them in order to infer a general method for solving similar problems and than represent this by a GLP (Work sheets G-1 and G-2). Pupils were probably remembering this approach and they were trying to repeat the same steps in order to write a GLP although they wrote only one FLP and based on this they wrote a GLP.

Strategies N2a and N2b were 2-steps strategies. When using them pupils did not write FLPs after the verbal particular examples. They did not need to represent the arithmetic verbal examples in Logo first, but they were able to shift directly from these to expressing the given generalization in Logo. The use of strategies N2a and N2b suggests confidence in writing FLPs to express arithmetic examples and, therefore, no need to do this explicitly. The difference between these two strategies was the number of particular examples given. When using strategy N2b pupils gave several particular verbal examples in order to deduce the given generalization: they were generalizing from many particulars. When strategy N2a was used pupils generalized from one single particular example. Considering this difference in the generalization process involved, strategy N2a may be considered more advanced than strategy N2b.

Strategy N1 was a 1-step strategy. Given a word general statement involving an algebraic expression, this was expressed in Logo by a GLP without any intermediate step. That is, pupils using this strategy did not need external supports (e.g. verbal particular examples, FLP) to write a GLP.

For each pair the strategies used to write GLPs in this environment were observed (Table 7.5).

| Pair | Strategies used |
|---|---|
| Valentín/ Ernesto | N3, N2a, N2b, N1 |
| Etna/Itzel | N2a, N2b, N1 |
| Martha/Mariana | N2a, N2b, N1 |
| Oscar/Jonathan | N1 |
| Claudia/Sandra | N3, N1 |
| Nayeli/Soledad | N2b, N1 |

**TABLE 7.5** - Case study pupils' strategies to write general Logo procedures (GLPs) in the numeric Logo environment

As shown in Table 7.5 five of the six case study pairs did not use all the time the same strategy to write GLPs. It was observed that except the pair Oscar/Jonathan all the case study pairs approached their first GLP by using a more-than-one-step strategy. However, **a spontaneous tendency to evolve to strategies with less steps was observed. At the end of the activities embedded in the numeric Logo environment all the case study pupils were using strategy N1 to write GLPs. This means that in order to write a GLP to represent a general statement given in words they did not need any more the explicit support of particular examples.**

**This evolution may be interpreted as a gradual internalization of explicit actions initially involved in writing GLPs. This gradual process finally led to a qualitative shift that was observed when pupils started working directly with general numbers instead of working with particular numbers first.** This is evidenced, for example, by the evolution of Valentin's and Ernesto's strategy to write GLPs. This pair was closely observed during the activities reported below.

The case of Valentin and Ernesto - Evolution of a strategy.

Asked to write their first GLP ('Write a procedure to calculate the half any number', work sheet G-5) Valentin and Ernesto approach the task using strategy N3. That is, they start by verbalizing several particular examples, they write after this a FLP and finally they write a GLP.

After this experience Valentin and Ernesto write a GLP to add 10 to any number (work sheet G-5). To do this they use strategy N2b. That is, they give first some verbal numeric examples and in consequence, without writing a FLP, they immediately write a GLP. This suggests that due to the prior experience they are confident that they can represent the particular examples by FLPs. They are probably handling this part of the process within their mind without needing to write it explicitly. By adopting a 2-step instead of a 3-step strategy they have reduced the number of actions that provided them an explicit external support to write a GLP.

The number of actions explicitly performed is reduced further when Valentin and Ernesto write a GLP to subtract 10 from any number (work sheet G-6). To do this they use strategy N2a instead of strategy N2b. That is, they give only one particular verbal example and after this they write a GLP. This suggests that one particular examples is enough for this pair in order to make sense of the word given algebraic expression, differentiate the variable and the invariant elements and generalize representing this by a GLP.

Finally, when solving work sheet G-7 ('Write a general procedure to add 5 to the addition of any number with itself'; 'Write a general procedure to divide any number by 10 an to add 100 to it') Valentin and Ernesto use strategy N1, that is, they write directly a GLP. They are not giving verbal particular examples any more. After reading loudly the posed problem one of them (they take turns on the computer) writes a GLP. This suggests that **they do not need any more the support of explicit verbal examples in order to make sense of the problem and to write a GLP. A qualitative shift has occurred and Valentin and Ernesto are able to think (their thought was indicated by their capability to write a GLP immediately after reading the statement and without talking any more) with general numbers operating on them.** They can represent, for example, the addition of 'any number' with itself or divide 'any number' by 10 (see Fig. 7.7) without the necesity of giving first a particular value to the so called 'any number'.

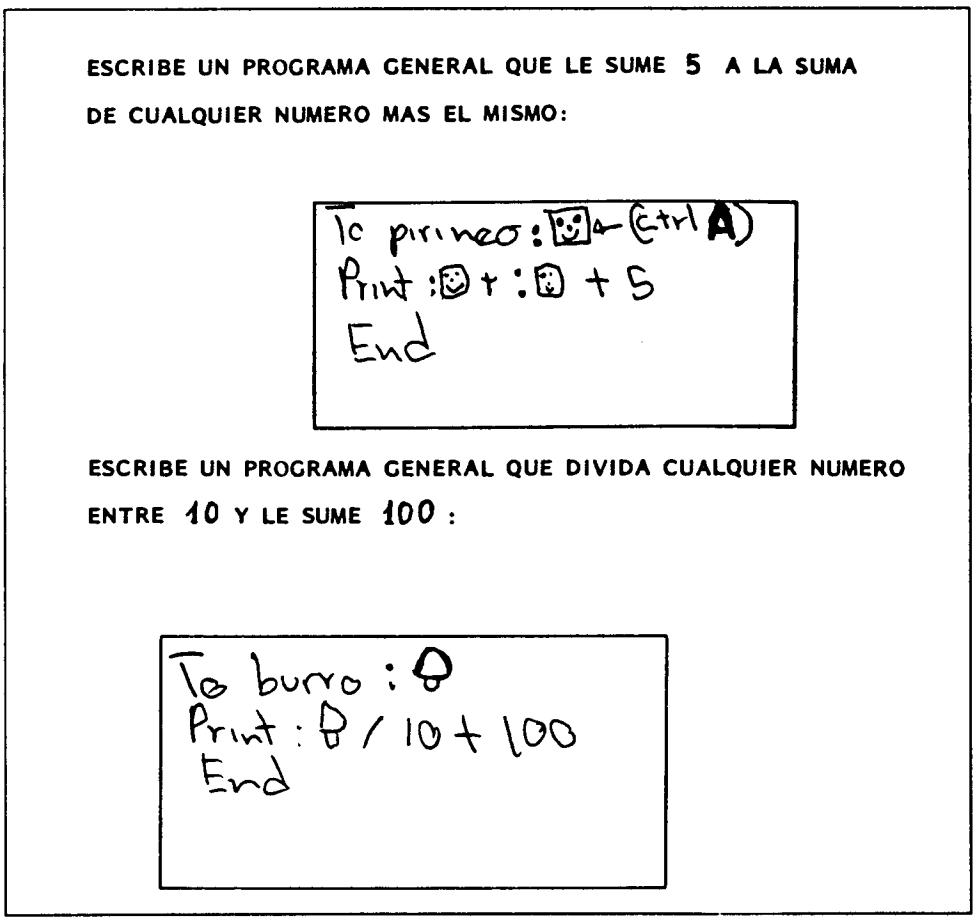



Fig. 7.7 Valentin and Ernesto - General Logo procedures written by using a 1-step strategy (N1)

**These results show a clear evolution of the strategy used by Valentin and Ernesto suggesting that a gradual internalization of the chain of actions that were initially**

**explicitly performed, finally led to a qualitative shift in their way of approaching the problem.**

When individually interviewed at the end of microworld 1 these pupils used strategy N1 to write a GLP to add 3 to any number (Table 7.9, problem 3). However, explicitly asked by the researcher to explain what their GLP represented they answered by giving particular arithmetic examples and emphasized that any value might be substituted to the variable involved. This indicates that faced with the requirement of justifying their answer they were able to return to the detailed actions that were initially explicitly performed (e.g. giving verbal numeric examples of the posed problem). This confirms the hypothesis of internalized actions and a consequent qualitative shift to work directly with general number.

It was observed that all the case study pairs spontaneously tended to use strategy N1 in order to write GLPs in this environment, however, not all of them followed the evolutionary path observed in Valentin and Ernesto. There were pairs, for example Claudia and Sandra, who after having used strategy N3 to write three GLPs, they started using strategy N1. They were not observed using explicitly strategies N2a and N2b.

There were also pairs who did not use strategy N3 at all and they started writing GLPs by using strategy N2a (Etna/Itzel, Martha/Mariana) or strategy N2b (Nayeli/Soledad). After this Etna/Itzel and Martha/Mariana evolved to strategies N2b and N1. Nayeli/Soledad evolved to strategy N1.

Only one pair, Oscar and Jonathan, used strategy N1 from the first beginning. Because Oscar and Jonathan preferred to work individually taking turns on the computer while the partner was watching (see 7.4.1.2, The case of Oscar and Jonathan) they did not discuss between them and each one worked in silence. For this reason the possible steps followed by these pupils in order to write a GLP were not transparent for the observer. However, it must be stressed that Oscar and Jonathan did not need any explicit external support in order to write a GLP in this Logo numeric environment.

These results suggest that even if in this environment all the case study pupils could shift from working with particular numbers to working with VGN and they were able to express a generalization in a formal language, not all the pupils needed the same external support for this. However, a common characteristic to all the case study pupils was a tendency to rely on their arithmetic knowledge and to use numeric examples as a scaffold in order to generalize and write GLPs. **In this Logo environment, pupils were**

**encouraged to rely on the arithmetic knowledge and approach the tasks working with arithmetic examples first. This showed to be an important scaffold that helped pupils approach the idea of VGN.** In this environment, pupils used their arithmetic background and based on this they generalized first from several verbal numeric examples. They shifted then to generalizing on the basis of an analysis of just one verbal numeric example. They were able to make then a qualitative shift to think directly with general numbers. When writing GLPs they showed their capability to express an arithmetic generalization by a formal language.

#### 7.4.3.2 Writing general Logo procedures (GLPs) in the turtle graphics environment

Five different strategies to write GLPs in the turtle graphics environment were observed (Table 7.6). They are classified depending on the number of steps required to achieve the goal and of the actions performed in each step.

| Strategy | Number of steps | Step 1 | Step 2 | Step 3 |
|---|---|---|---|---|
| T1 | 1 | ---------- | ---------- | GLP written |
| T2a | 2 | Direct mode | ---------- | |
| T2b | 2 | ---------- | FLP written | |
| T2c | 2 | Direct mode and written record | ---------- | |
| T3 | 3 | Direct mode | FLP written | |

**TABLE 7.6** - Five strategies to write general Logo procedures (GLPs) in the Logo turtle graphics environment

It was observed that independently of the strategy used pupils usually discussed between them the command to be typed. When strategy T3 was used the given shape was drawn in direct mode without recording the commands. After this a FLP was written and recorded on the work sheet. Based on this record a GLP was written.

Three 2-steps strategies were observed: T2c, T2b and T2a. Pupils using strategy T2c wrote a GLP based on the record of the commands given in direct mode. When strategy T2b was used a GLP was written based on the written record of a FLP. When using strategy T2a pupils worked first in direct mode without taking any record and after this they wrote a GLP from the direct observation of the shape.

Strategies T2a, T2b, T2c and T3 have all a common characteristic: when using them pupils described first of all in Logo the given shape maintaining the given particular dimensions. This description was obtained by drawing the shape in direct mode and/or by a FLP. This suggests a tendency to transfer first of all the given problem to the Logo environment maintaining the level of specificity in which it was posed. The tendency to describe first a particular example in Logo was observed also when the pupils worked in the Logo numeric environment, although in that environment a general sentence was given and not a numeric example.

While transferring the problem to Logo a first analysis was done (e.g. they verified the turtle's turns). It was observed that after this first analysis pupils using strategies T2b, T2c and T3, considered the written record of the FLP or of the commands given in direct mode as an object to be analysed and based on this they wrote a GLP.

When using strategy T1, pupils wrote a GLP based on the direct observation of the given shape without working first in direct mode or writing a FLP. It was observed that on occasion while writing a GLP they were verbalizing their actions. This was the most advanced strategy observed in this environment.

Table 7.7 presents the strategies used by each pair to write GLPs when it was not necessary to operate on the variable and when it was necessary to operate on it.

| Pair | Strategies used | |
|---|---|---|
| | Variable not operated on | Variable operated on |
| Valentín/Ernesto | T2b, T1 | T2b, T1 |
| Etna/Itzel | T3, T1 | T2b |
| Martha/Mariana | T2c | T2c, T1(*) |
| Oscar/Jonathan | T2b, T1(*) | T1(*), T2b |
| Claudia/Sandra | T2c | T1(*), T2b |
| Nayeli/Soledad | T2a | T2b, T1(*) |

(*) non successful

**TABLE 7.7** - Case study pupils' strategies to write general Logo procedures (GLPs) in the turtle graphics environment

Table 7.7 shows that Etna and Itzel, were the only pair using strategy T3. They explained to the researcher that working in direct mode first helped them verify that they

could actually draw the given shape. This suggests that these girls were still not very confident about their skills to draw in Logo and they needed the immediate feedback helping them decide the next turtle's movement.

Only one pair, Nayeli/Soledad, used strategy T2a. Questioned about the purpose of working first in direct mode without recording the commands these girls explained that they preferred to verify the directions of turtle's turns before writing the GLP. This suggests that they were still not confident about the directions of the turtle's movements and they tried to sort out this problem before writing a GLP. It was observed that to write a GLP these girls were verbalizing each one of the commands previously typed in direct mode, they were typing them in the procedure and when the command FORWARD was used they verbalized the number after the FD command but they typed the variable's name instead of the particular numeric value. This suggests that in order to write a GLP Nayeli and Soledad used the direct support of a particular example substituting a variable to the number that had to be changed in order to get a shape of different size. This shows that to write a GLP they were thinking with particular values and not with general numbers.

Table 7.7 shows that **all the case study pairs tried to use strategy T1 to write a GLP in the turtle graphics environment.** They tried to use this strategy when variable had not to be operated on and when it had to be operated on. This tendency to evolve to strategy T1 was spontaneous for all the observed pairs. **However, only one pair, Valentin and Ernesto, could successfully use strategy T1.** It was observed that Valentin and Ernesto evolved to strategy T1 after having used strategy T2b to solve similar problems. When faced with an unfamiliar situation they dropped back to strategy T2b and after that they evolved once more to strategy T1 when solving similar problems. This is evidenced by the following detailed description of the work of Valentin and Ernesto.

<u>The case of Valentin and Ernesto - Evolution of a strategy.</u>

To write a GLP where the variable has not to be operated on (work sheet G-11), Valentin and Ernesto use initially strategy T2b. To write a GLP to draw an hexagon they write a FLP and based on this they write a GLP (see Fig. 7.8). After recording the FLP they use this record and the direct observation of the shape in order to write a GLP. By discussing between themselves they identify the variable and the invariant elements and they substitute a symbolic variable to the particular numbers representing the variable elements. In this way they obtain a GLP.

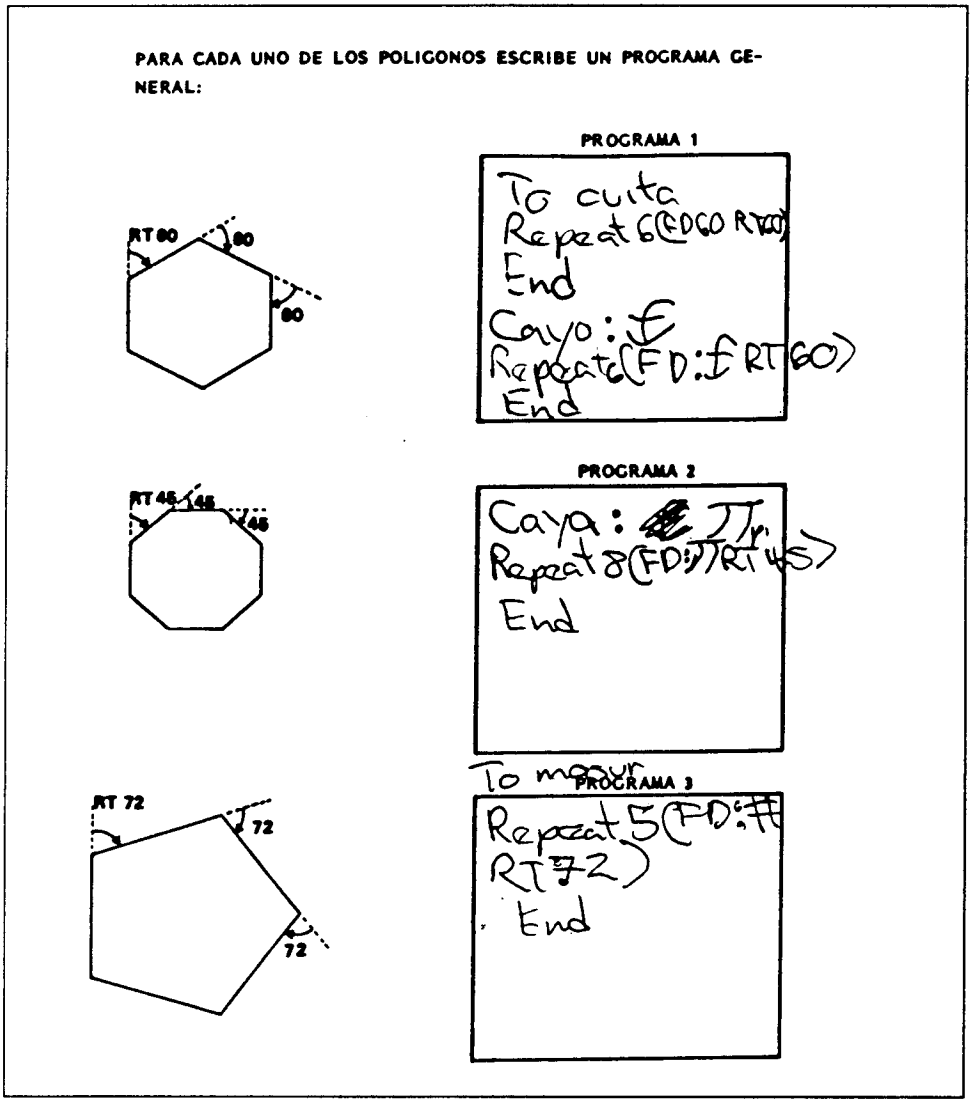



Fig. 7.8 Valentin and Ernesto - Evolution of the strategy used to write general Logo procedures

After this experience in order to write a GLP to draw octagons and pentagons Valentin and Ernesto spontaneously evolve to strategy T1. That is, based on the direct observation of the shape they write a GLP (see Fig. 7.8). To do this Valentin and Ernesto substitute the explicit writing of a FLP by its verbal description. While verbalizing the Logo commands they are analysing the shape and they identify the variable elements. Based on this analysis they write a GLP. I would suggest that to substitute the writing of a FLP by its verbal description indicates that Valentin and Ernesto are starting to gradually internalize the process they used to write a GLP. It was not observed if they were able to evolve to write a GLP without verbalizing first a FLP. However, the researcher observed that this pair consistently used strategy T1 to write GLPs when the variable had not to be operated on.

After this Valentin and Ernesto work on a problem where to write a GLP the variable has to be operated on (work sheet G-14u, see also Fig. 7.9). After observing the given sketch of the shape (work sheet G-14u, upper shape) they approach the task by using strategy T2b, that is, they write first a FLP to draw the shape and then they try to use this as support to write a GLP. This behaviour suggests that they cannot assimilate the new shape to the prior ones (Fig. 7.9), therefore, they do not approach it directly by strategy T1 but they drop back to strategy T2b. In spite of using a 2-step strategy they cannot write a GLP and after a while they ask for help. They explain to the researcher that in contrast with the shapes they drew before (Fig. 7.8), in the shape they are working with there are two distances of different length, one measuring 20 and the other 40. In order to write a GLP they need to tell the computer that one distance will be always the double of the other. But they do not know how to communicate this to the computer.

This Valentin's and Ernesto's explanation suggests that **by writing the FLP and observing the sketch of the figure Valentin and Ernesto have found the general relationship linking the dimensions of the shape (they verbalize this saying '*one is always the double of the other*'), but they cannot express this general relationship in Logo.**

In order to push Valentin and Ernesto to symbolize the general relationship they are verbalizing, the researcher explicitly suggests to operate on the variable. However, they seem not to understand this suggestion. They propose to use two variables or to maintain one variable fixed using the editor to vary it. **It is necessary for the researcher to show them explicitly how to write the expression :♥ * 2.**

This suggests that even if Valentin and Ernesto can perceive the relationship between the numbers involved and generalize it they cannot represent this in a formal way. The expression :♥ * 2 is not new to them. In fact, it was used when they were first introduced to GLPs in the Logo numeric subset (work sheet G-3). However, when faced with it they are surprised and they argue that they did not know that it was possible to multiply a symbol by a number. This suggests that **Valentin and Ernesto cannot use in the turtle graphics environment their capability to operate on the variable shown in the Logo numeric subset. It seems that for these pupils this capability is bounded to the numeric context.**

In contrast with the Logo numeric environment where the numbers had no additional referent, in the turtle graphics environment the numbers have a clear referent: they are

representing lengths or angles. In this environment it seems difficult for Valentin and Ernesto to detach the numbers from their specific referent and this is an obstacle for applying their capability to operate on the variable. The link with the referent does not prevent them from representing the value of a length by a symbolic variable but they cannot make an additional shift and conceive the possibility to operate on a symbolic variable, that is representing the value of a length. To do this they would need to relax the link with the numbers' referents. A similar difficulty to operate on the variable in this environment was observed for all the other pairs. All needed researcher's intervention indicating how to operate on the variable and how to symbolize this.

The researcher's intervention showing explicitly how to operate on the symbolic variable pushes Valentin and Ernesto to break completely the link with the variable's referent and leads them to an over-generalization. In fact, after the intervention they write a GLP by substituting all the numbers in their FLP by the variable, inclusively the numbers representing turtle's turns. When substituting the numbers by a symbolic variable they correctly operate on it, however, they are completely ignoring the given shape. A further intervention suggesting to consider the given shape in order to discriminate between the variable and the invariant elements helps them write finally a correct GLP operating on the variable when necessary (see Fig. 7.9 upper shape, Valentin and Ernesto did not record their FLP on their work sheet).

After this, Valentin and Ernesto need no further researcher's intervention to write GLPs operating on variable. Initially they use successfully the strategy T2b. That is, they write first a FLP and they use this as support to write a GLP by substituting a symbolic variable to the appropriate numbers and operating on it when necessary (see Fig. 7.9, lower shape). After this experience they shift spontaneously to using strategy T1, that is, they write a GLP directly from the observation of the shape (Fig. 7.10) and they consistently use this strategy without dropping back to a more-than-one step strategy.

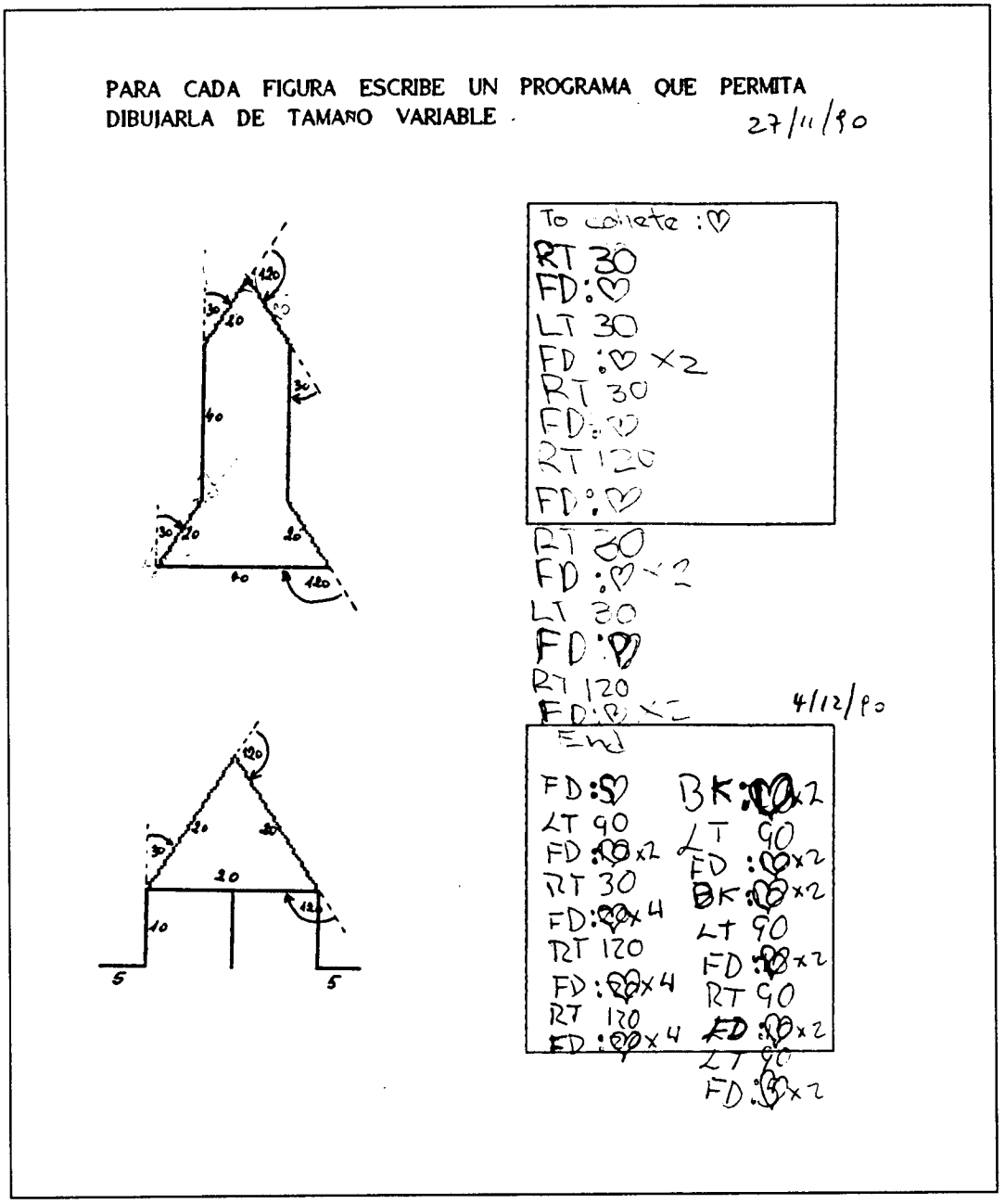



**Fig. 7.9** Valentin and Ernesto - Use of a 2-step strategy (T2b) to write a general Logo procedure

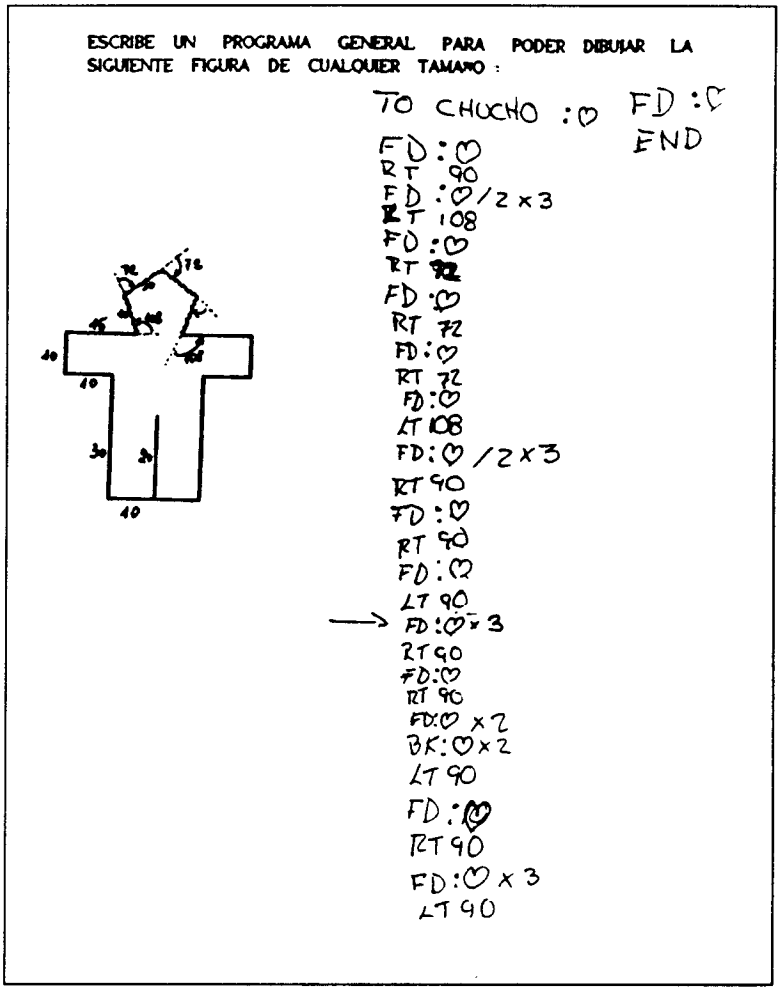



**Fig. 7.10** Valentin and Ernesto - Use of a 1-step strategy (T1) to write a general Logo procedure

During the observation of the work of this pair the researcher realised that Logo passed gradually from being only a written language to communicate with the computer, to being a language that Valentin and Ernesto used, mixed with the natural language, to discuss the task they were facing. This hybrid spoken language (Logo mixed with natural language) was the tool they used to analyse the given shapes. It was also observed that Valentin and Ernesto used the given particular dimensions of a shape as support to operate on the variable. For example, when writing the GLP to draw the shape shown in Figure 7.11 they started by writing a FLP but they immediately shifted to writing directly a GLP using language as support. The following verbalization was recorded, '*Seven is semicolon heart; fourteen is twice seven; so it will be twice semicolon hart*'. While talking Valentin was writing FD :♥ and FD :♥ *2.

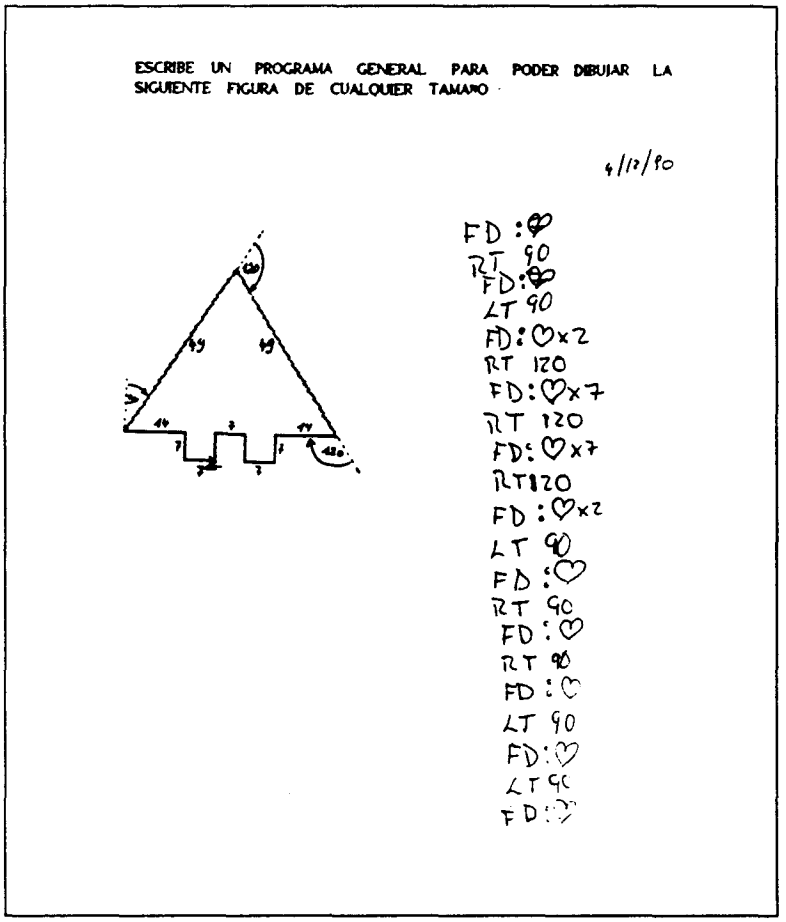



**Fig. 7.11** Valentin and Ernesto - Record of a general Logo procedure written by using the hybrid spoken language (Logo mixed with natural language) as support

These results suggest, on the one hand, a spontaneous tendency to evolve to a 1-step strategy in order to write GLPs after having used successfully a 2-step strategy. **Valentin's and Ernesto's approach to the strategy T1 by reproducing verbally the steps previously explicitly performed, confirms the tendency to gradually internalize the actions previously explicitly performed in order to write a GLP. This stresses as well the crucial role of verbalizations in helping these pupils handle the different elements intervening in the writing of a GLP in this environment.** On the other hand, these results suggest Valentin's and Ernesto's potential to approach the idea of VGN, although in this environment they still need the explicit support of particular numbers in order to operate on variable. The link with particular numbers was a support helping Valentin and Ernesto to approach the idea of VGN. This link was very strong and Valentin/Ernesto could not break it in order to shift to think directly with general numbers in order to write GLPs in this environment. The feasibility to solve these tasks without making this shift might be considered a limitation of their design.

**In contrast with the results obtained in the Logo numeric environment where all the pupils evolved to strategy N1 and they were able to use it successfully, in the turtle graphics environment although a spontaneous tendency to use a 1-step strategy was observed, the great majority was not successful in using it. This suggests that in this environment the great majority of the case study pupils needed the explicit support of a particular example expressed in Logo (for example, the written record of a FLP or of the commands given in direct mode) in order to write a GLP.** It was observed that when trying to use strategy T1 in this environment, they had difficulty in handling all the elements intervening in the writing of a GLP, for example, a long chain of Logo commands and the direction of turtle's turns. This was the main obstacle for their successful use of a 1-step strategy in the turtle graphics environment.

It was observed that when using strategies T2b and T2c the written record of the FLP or of the Logo commands given in direct mode were the basic support used by pupils to write a GLP. Based on these records and without observing the shape any more, they transformed them to a GLP by substituting the numbers following each FD command for a symbolic variable. This is evidenced, for example, by Claudia's and Sandra's record of the GLPs they wrote to draw 'stars' (see Fig. 7.12).

During this activity Claudia and Sandra were not closely observed. But their written record where the particular numbers after the commands FD are crossed out and substituted by a symbolic variable (Fig. 7.12, lower shape), shows that after writing a FLP to draw the shape they are using this as support to write a GLP by substituting a symbolic variable to the numbers.

To use this strategy helped pupils to handle separately the different elements intervening in the writing of a GLP in this environment. In fact, when writing the FLP they solved all the aspects of the problem related to the directions of the turtle's turns and the correct use of the Logo primitives. After this, using the record of the FLP or of the commands given in direct mode, they focussed only on the variable elements without caring any more about the correctness of turtle's turns and of the sequence of the Logo commands. Using this approach, all the case study pupils successfully wrote GLPs in the turtle graphics environment and this perhaps contributed to obstruct any further evolution to the use of a 1-step strategy.

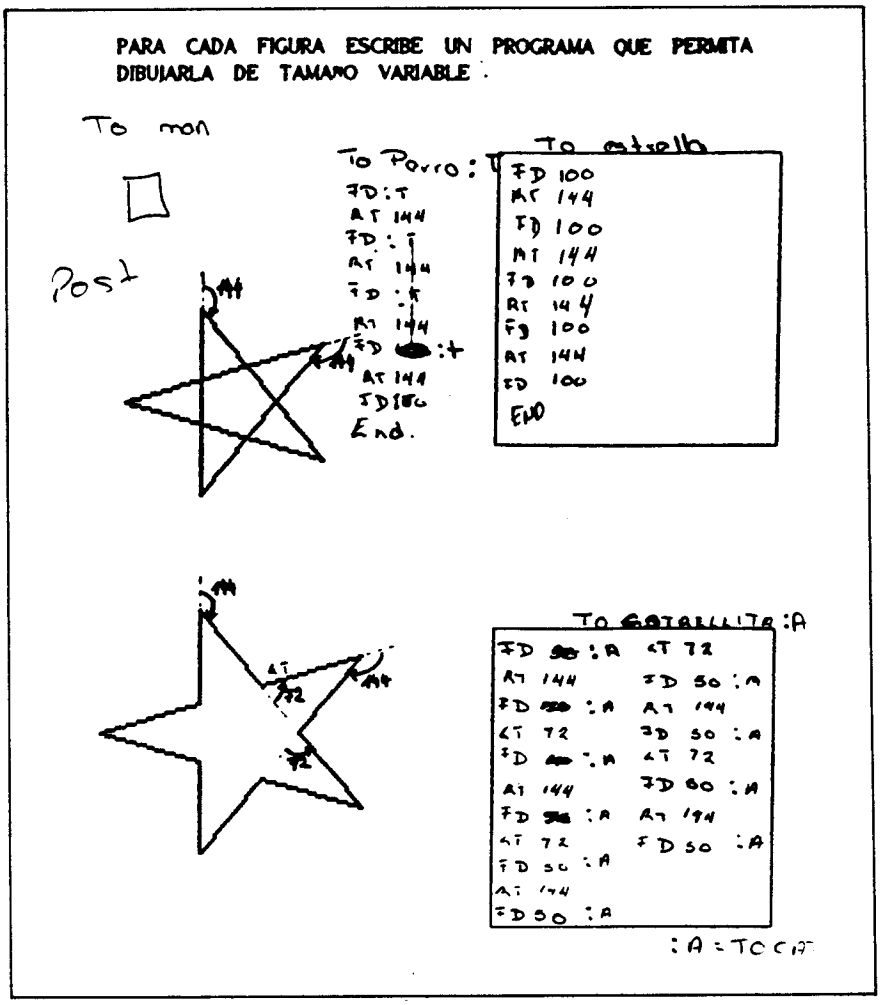



**Fig. 7.12** Claudia and Sandra - Use of a 2-step strategy (T2b) to write a general Logo procedure

## 7.5 Interview

At the end of microworld 1 all the case study pupils were individually interviewed. This aimed to verify for each pupil if a potential to work with the idea of VGN in Logo was created during microworld 1.

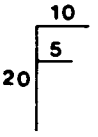
### 7.5.1 The questions

The interview questions dealt with problems embedded in the Logo numeric and in the Logo turtle graphics environment. Additionally two questions were included (questions 1 and 2) which aimed to explore if the experience acquired during microworld 1 influenced pupils' use and interpretation of VGN in a non-computational environment. Figure 7.12

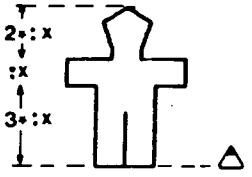
presents the interview questions. The same questions were posed to all the case study pupils.

**Interview after microworld 1**

1. Write a general formula to add 3 to any number.
2. In the expression 5 + X = X + 5
   - What does X represent?
   - What values can X have?
   - For what values do you think the expression is true?
   - Once a value is assigned to X, have all the X of the expression the same value?
   - If the value 3 is assigned to X, how does the expression look?
3. Write a general Logo procedure to print the result of adding 3 to any number.
4. Write a general Logo procedure to draw a letter F of any size but maintaining the given proportions.




5. There is a general procedure which draws the astronaut and puts the turtle besides him. What must be written after FD in order to draw the total height of the astronaut?




6. I have a procedure which calculates and writes a result when it is given a number. How would you write the general formula to perform such calculation?

```
3 -------> 5          TO CALCULO :X
4 -------> 6          PRINT
9 -------> 10         END
15 -----> 17
```

**Fig. 7.13** Interview questions after microworld 1

The first question required to write an 'unclosed' algebraic expression. It was observed if pupils used numeric examples as support to write it. Question 2 tested pupils' interpretation of a VGN involved in an algebraic expression and whether the 'polysemia' error occurred or not. Questions 3 and 4 aimed to give insight on the strategies used by pupils to write GLPs in the numeric Logo environment (question 3) and in the turtle graphics environment (question 4).

Although the activities of microworld 1 were not designed in order to help creating pupils' potential to group similar algebraic terms, question 5 explored pupils' approach to this kind of problems. Question 6 aimed to observe pupils' potential to deduce a general expression based on several particular examples and express this in a formal way.

When pupils had difficulties in understanding a question this was repeated (QR) or formulated again (QRF). When pupils needed additional help the researcher intervened in order to provoke reflection (R), reminding a piece of information (I) and/or negotiating with pupils the solution of the problem (SN), that is, discussing their approach in order to lead them to a correct solution. The type of interventions are presented in Table 7.8.

| Symbol used | Type of intervention |
|---|---|
| QR | Original question repeated several times. |
| QRF | Original question repeated and formulated again. |
| R | Provoking reflection. |
| I | Reminding a piece of information. |
| SN | Negotiating the solution of the problem. |

**TABLE 7.8** - Type of intervention provided during the interview on variable as general number (VGN)

### 7.5.2 Results

Table 7.9 shows the kind of intervention each pupil received in order to solve the interview problems. The symbol '+' indicates that the pupil was able to solve the posed problem without researcher's intervention. The symbol '-' indicates that in spite of researcher's guide the pupil could not solve the posed problem. For questions 3 and 4 the strategy used to write a GLP is also indicated (for a detailed description of the strategies see Table 7.4 and Table 7.6).

Questions 1 was not related to Logo. This might be a reason why some pupils were initially confused and the question had to be repeated or formulated again (e.g. '*Write a formula such that looking at it I will say: this is 3 plus any number*'). After this intervention they were able to write an 'unclosed' algebraic expression to answer the question.

| Pupil's name | Problems | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | | | 4 | | | 5 | 6 | | |
| Valentín | + | + | + | | N1 | + | | T1 | + | + | | |
| Ernesto | QR | QR | + | | N1 | SN | | T2b | R | + | | |
| Etna | + | + | + | | N1 | R | | T1 | - | + | | |
| Itzel | QRF | QRF | I | | N1 | SN | | T1 | + | + | | |
| Martha | QR | + | + | | N1 | + | | T2b | SN | + | | |
| Mariana | + | + | + | | N1 | I | | T1 | SN | + | | |
| Oscar | + | + | I | | N1 | I | | T1 | QR | - | | |
| Jonathan | QR | + | + | | N1 | + | | T1 | QR | + | | |
| Claudia | QR | + | I | | N1 | - | | T2b | - | QRF | | |
| Sandra | + | + | QR, I | | N1 | + | | T1 | + | + | | |
| Nayeli | QRF | + | QR, I | | N1 | + | | T1 | + | + | | |
| Soledad | + | + | + | | N1 | SN | | T1 | - | QRF | | |

**TABLE 7.9** - Type of intervention received by each pupil during the interview on variable as general number (VGN)

This pupils' capability to write an 'unclosed' algebraic expression contrasts with the results obtained at the beginning of the study. When answering the initial questionnaire only few pupils on occasion wrote an 'unclosed' expression (Chapter 5, Table 5.10, items 1.4, 1.5, 8, 9, 10, 11, 12, 13). These different behaviours suggest that **the experience acquired in Logo was influencing pupils' approach to the given algebraic questions and it was helping them to cope with the idea of VGN inclusively in non-Logo environments.** This influence is highlighted by the way some pupils symbolized VGN. For example, Valentin and Ernesto wrote 3 + :♥ , and Oscar wrote 3 + :/. That is, they used a symbol other than a letter preceded by semicolon. When writing GLPs in microworld 1, these pupils used symbols other than letters to represent variables. Asked for the meaning of the symbols used to represent the variable they explained that these were representing any number. They added that it was more amusing to use signs instead of letters, but that the meaning was the same. This shows that being clear the concept they wanted to represent, the type of symbol used was not relevant to them. This suggests that **the difficulties pupils usually have with symbols representing variables in algebra is due to a lack of clearness about the concepts underlying them.**

The influence of the Logo experience was reflected also by pupils' answers to question 2. In fact, all the pupils, inclusively Etna and Ernesto who at the beginning of the study

argued that they had never faced literal symbols in mathematics before, were able to consider that in the expression 5 + X = X + 5, the X was representing any number and that any value could be assigned to it. Except Itzel, who presented the 'polysemia' error, all the pupils considered that the same value should be substituted to both appearances of X. When faced with a question like this at the beginning of the study the majority of the case study pupils gave no answer. Some of them tried to calculate the value of the variable (Chapter 5, Table 5.10, items 2.1, 2.3, 2.4). The 'polysemia' error was observed when pupils tried to answer this item (see 5.2.2, Dialogue 3). **In microworld 1 pupils were writing and running GLPs where the same variable appeared several times in the procedure. This could have influenced the assumption that in the given algebraic expression any value could be assigned to the variable and that the same value had to be assigned to all the appearances of it.** Moreover, a reason for Itzel's 'polysemia' error might be that due to illness she missed several sessions and therefore, her Logo experience was not as complete as those of other pupils.

Pupils' approach to question 3 confirms that in the Logo numeric environment the case study pupils could think directly with general numbers and express an arithmetic generalization by a formal language. In fact, all the pupils used a 1-step strategy to write a GLP. That is, they did not need the explicit support of particular examples. Three pupils (Itzel, Oscar and Claudia) needed to be reminded how to use the primitive PRINT. Two pupils (Sandra and Nayeli) initially did not understand the question. After repeating it several times and reminding them the use of the primitive PRINT they were able to solve the given problem.

The strategies pupils used to write a GLP in order to answer question 4 confirm the tendency to write a GLP by the direct observation of the given shape. In fact, the grat majority used strategy T1. Only three pupils, Ernesto, Martha and Claudia, used a 2-step strategy, T2b, that is they drew the shape in direct mode, recorded the given commands and based on this record they wrote a GLP.

To answer this question (question 4) five pupils (Ernesto, Itzel, Mariana, Oscar and Soledad) needed the researcher's support indicating the directions of the turtle's turns. Three of them (Ernesto, Itzel and Soledad) needed also help in order to maintain the link between the variable and the length it was representing. This confirms the hypothesis that when working in the turtle graphics environment pupils might have difficulties in handling simultaneously all the intervening elements. This might have been a source of difficulty in using strategy T1 when working in microworld 1.

The interview showed that four pupils (Etna, Jonathan, Sandra and Nayeli), who when working with their peers in microworld 1 were not successful in using strategy T1, were able to shift to use it without needing researcher's help. These pupils were not yet self-confident enough to use a 1-step strategy, but the emotional support received by the presence of the researcher helped them adventure in using this strategy during the interview. When working in microworld 1 they did not find this support in their partners, therefore, they adopted better a strategy they felt confident about. In fact, when interviewed their partners wrote a GLP by using strategy T2b (Oscar and Claudia) or by using strategy T1 with researcher's guidance (Soledad). This shows that **Etna, Jonathan, Sandra and Nayeli had the potential to generalize and express this in a formal way by a GLP without the explicit support of a particular example, but they still could not do it completely alone. The researcher's presence was crucial for them to use strategy T1.**

This, however, was not the case of Valentin. When interviewed he used strategy T1 to write the GLP without needing researcher's support. His partner, Ernesto, used strategy T2b when interviewed. In spite of this difference they were the only pair who during microworld 1 used successfully strategy T1. This indicates that **Valentin was confident enough about using this strategy and he did not need his partner's support for this. As well he could help Ernesto understand this strategy and cooperate actively in using it to solve the problem, although Ernesto could not yet use this strategy by his own.**

The interview gave evidence about the way pupils used the support of particular numbers in order to operate on the variable. This is illustrated by Martha's and Valentin's scripts (Fig. 7.14) and comments.

Martha's and Valentin's approach to question 4

After the question is posed, Martha works first in direct mode recording the given commands on paper (Fig. 7.14).

To write a GLP she observes her record commenting:

M(artha): '*First of all I have to define a value for X. It can be 5. I have to write it down ... the value of X now is 5*'.
She writes on the work sheet X = 5 (see Fig. 7.14). After this she looks to the first line of her FLP.

M.: '*Here will be FD X multiplied by 4*'.

She writes FD :X * 4 besides FD 20 and in a similar way she substitutes the numbers following all the commands FD by the variable operated on when necessary.

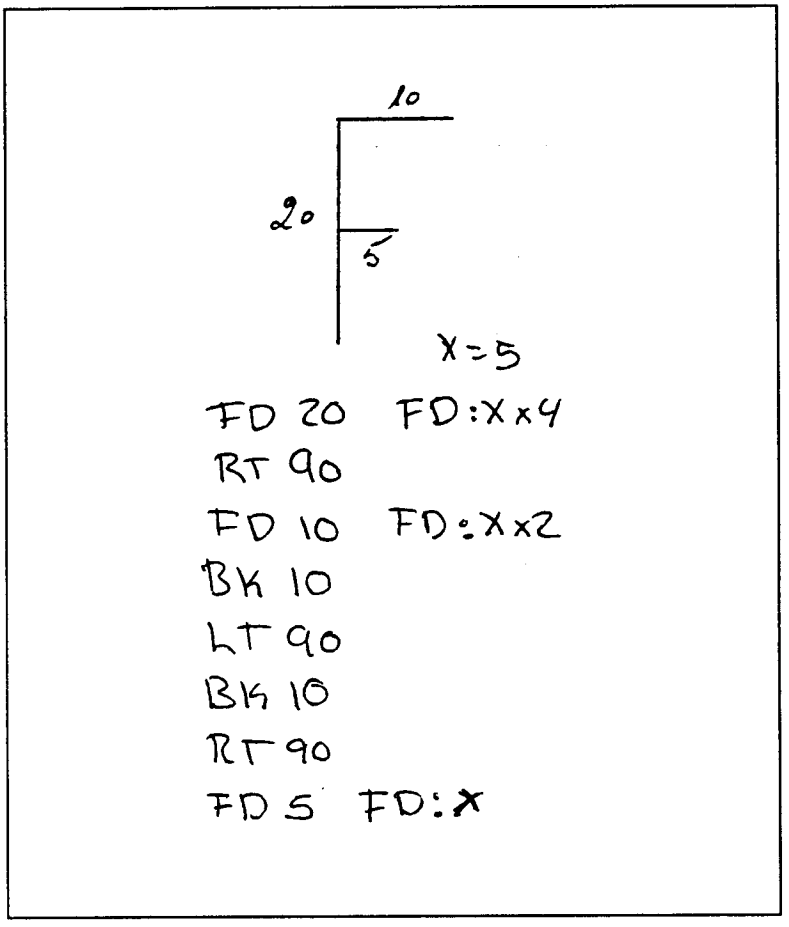



**Fig. 7.14** Martha - Use of a 2-step strategy (T2b) to write a general Logo procedure

The majority of the case study pupils applied an approach similar to the one used by Martha independently of the strategy they used. This is evidenced, for example, by the answer given by Valentin. To write the GLP Valentin was using strategy T1 and he was operating fluently on the variable without talking (see Fig. 7.15).

Asked what was representing the sign he was using, Valentin said that it stood instead of 5. This suggests that in order to operate on the variable he had a particular value in mind. He operated that particular number in order to obtain the other given numbers (e.g. 10 as 5 x 2; 20 as 5 x 4). After that he substituted the symbol to the chosen value for the variable and wrote it down (e.g. :✸ x 2; :✸ x 4). This suggests that in spite of using a 1-step strategy Valentin was working with particular values and not with general numbers.

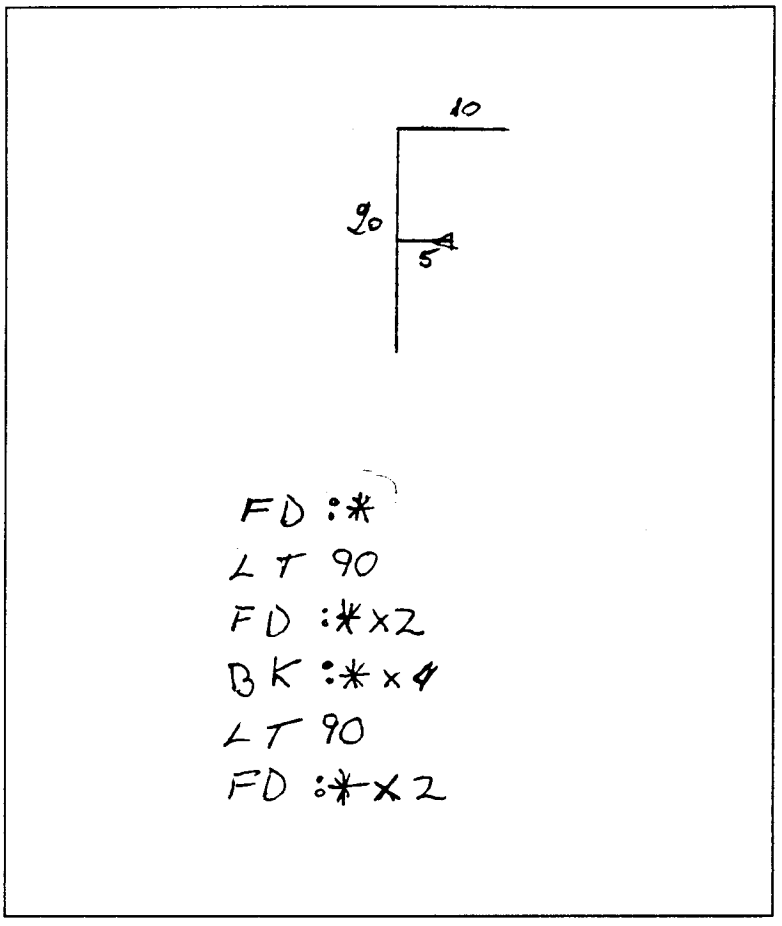



**Fig. 7.15** Valentin - Use of a 1-step strategy (T1) to write a general Logo procedure

However, the way Etna answered question 4 shows that it is possible for pupils to shift to working with general numbers.

Etna's approach to question 4

Etna spontaneously tries to use strategy T1 to write the asked GLP. She writes:

```
TO F :X
FD :X
RT 90
```

To understand what she is doing the researcher intervenes asking her what part of the shape is she drawing.

E(tna): '*I am here*' (she points to the left upper extreme of the shape ↗F) '... *but it would be better to go first here*' (she points to the middle point ↦ F) '... *and so it will be divided by 2 because it is a half of it*'.

She modifies her procedure dividing the X after the command FD by 2.

She continues: '... *then it is RT 90 and FD :X, but if this one was 10* (she refers to the segment just drawn), *this one also will be divided by 2*' (she refers to the segment of length 5).

Her procedure now is:

TO F :X
FD :X/2
RT 90
FD :X/2

This shows that initially Etna considers that X is representing 20. To draw the lengths corresponding to 10 she divides X by 2. After this, she seems to forget this initial assignment and she considers that X represents 10. Therefore, to draw the lengths corresponding to 5 she divides X by 2.

Aiming to clarify what she is doing and to help her reflect on her method the researcher intervenes.

R(esearcher): '*Well, this first part is 10 and you represented it by :X/2, but now this part measures 5. How will you represent it?*'

E.: '*Yes ... look, if for example this part*' (she indicates (F) '*would be 20, the total height would be 40. For that reason to get this part* (she indicates $F_K$) *I would divide 20 by 2*'.

Etna's answer shows that she has grasped what is the relationship between the lengths of the different segments and she can express this by giving a particular example different from the given one. However, her formal expression is not coherent with this explanation.

The researcher intervenes once more trying to push her to rely on a particular example.

R.: '*Well, and in your procedure what is the value you are thinking about?*'
E.: '*It is the total height*'
R. '*In that case your X is representing 20? And X/2 is 10?*'

E.: *'Forget it, I think it is better in this way: I will go forward a number than I will come back half the number; I will turn right 90; then FD a half of that number ...'*

Etna continues in this way and simultaneously she writes her GLP (see Fig. 7.16) where the first four lines belong to her first attempt to write the GLP and the others to her second approach.

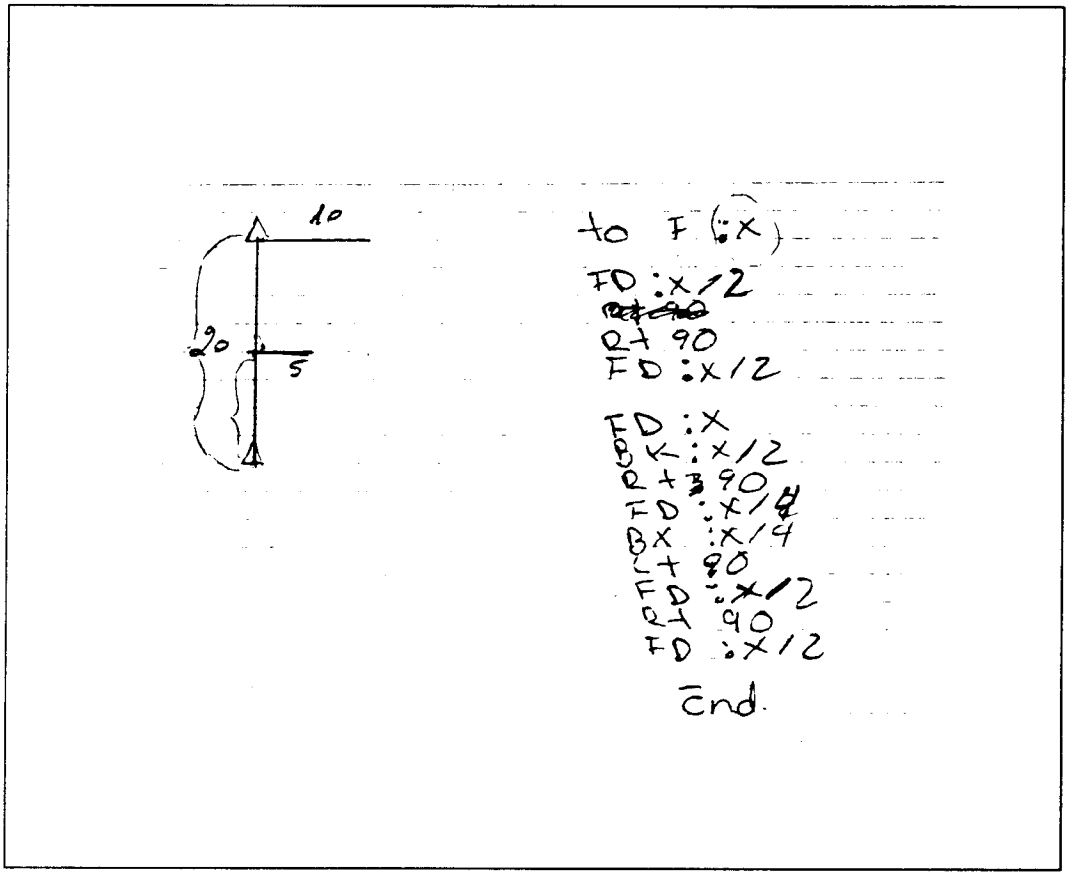



**Fig. 7.16** Etna - Record of the commands applied to a general number

The way Etna approaches this problem shows that in order to operate on the variable, **she is not thinking any more with particular values but she has shifted to working with general numbers (*'I will go forward a number'*). The given particular numbers have helped her realise that they are in proportion and she has generalized this. To express this generalization in a formal way she is not substituting a variable to a particular number but she has shifted to working directly with a general number and she is operating on this.** Even if this is an isolated case it testifies that **it is possible for pupils before any formal algebra teaching to work with general**

**numbers when they are involved in simple problems and approach in this way the idea of VGN.**

Although during microworld 1 pupils were not introduced to grouping similar algebraic expressions this was required to answer question 5.

Table 7.9 shows that four of the case study pupils (Valentin, Itzel, Sandra and Nayeli) correctly grouped the algebraic terms without needing researcher's support. For one pupil (Ernesto) it was enough to ask him to reflect on the problem and without additional help he grouped the terms. Two pupils (Oscar and Jonathan) were able to do it after the researcher repeated the question. Asked to explain their answers Oscar and Nayeli who had symbolized it by writing 6X told *'It is 6X because there are 6 of them'*. Itzel, Jonathan and Sandra who wrote FD :X * 6 explained their answer by saying *'It is FD :X * 6 because it is necessary to add all of them'*. They were referring to the total number of X, however, they were not looking at the variable as an object. In fact, asked for the meaning of X all these pupils said that it was representing any value.

The answers given by this seven pupils suggest that **to group the given terms they were looking at the variable as general number.** This is clearly evidenced by the dialogue the researcher sustained with Ernesto.

After examining the problem, Ernesto writes FD :X * 5. The researcher intervenes:

R(esearcher): *'Why by 5?'*

E(rnesto): *'Because this part measures X, and this one measures 2 times X ... It will be 6, yes, by 6. Because 2 times X and 3 times X and it will arrive approximately here because I did not include this part'* (he indicates the part marked with an X without a coefficient).

This explanation shows that Ernesto is not working with a particular value but he is considering a general number.

R.: *'Well, go on'*.

E.: *'So, it will be FD : X *6'*.

This shows that Ernesto has grouped all the terms and he is able to symbolize the result in a formal way.

Trying to get more insight into his approach the researcher continues:

R.: '*Well, and how did you calculate it?*'

E: '*Well, X represents any number. If it is 10 in order to get the total height I will consider the given measures and I must give the value to X and multiply it .... yes ... isn't it?*'

R.: '*Yes. If for example, as you said, X is 10, how much is the total height?*'

E: '*It depends on the value you give to X*'.

R.: '*Yes, suppose it is 10*'.

E: '*It will be 60*'.

R: '*Can you explain me why it is 60?*'

E: '*Well, because if X is 10, here we have to multiply X by 3 and with this other X we get 40 and here it is by 2 that is 20, so all together 60; this is 6 times 10. It is 6 times the value ... 60 if it is 10*'.

The particular example used by Ernesto to validate his answer shows his capability to shift between the general and the particular. A similar explanation where a particular value was assigned to X, was given also by Oscar and Valentin when asked to explain their answer.

**These pupils' answers suggest that a generalization from a particular example has occurred at an internal level. They all spent a very short time in analysing the posed problem in silence and after this they wrote the expression. Their answers also point to pupils' capability to return to a fully detailed reasoning that was probably not conscious when they were solving the problem. This also indicates that pupils were relying on their arithmetic background in order to approach the grouping of similar algebraic terms.**

Two pupils (Martha and Mariana) could group the terms with the researcher's guidance, but they were very insecure of the result obtained. However, their capability to actively participate in the solution of the task suggests that a potential to group similar terms was created.

Three pupils (Etna, Claudia and Soledad) could not solve the problem. Etna invented rules, Claudia told that without knowing the value of X she could not add the terms and Soledad added only the coefficients ignoring the variable. In spite of researcher's help they could not modify their answers. This suggests that it was still premature for these girls to approach this kind of problems. I would suggest that they needed more

familiarity in handling problems where a VGN had to be used and operated on, before facing tasks where they had to group terms involving a VGN.

The answers given to question 6 show that all the pupils except one, Oscar, were able to deduce a general rule from several particular examples and express this in a formal way. Two pupils (Claudia and Soledad) needed researcher's intervention. The researcher verbalized the given particular examples emphasizing that they were expected to complete the GLP by writing the general rule linking the input and the output. After this intervention they wrote :X + 2. In spite of repeating and formulating the question again one pupils (Oscar) could not understand what he was expected to do. He was completely confused and could not answer this question. I would suggest two reasons for this difficulty. On the one hand this was the last question and he might be tired. On the other hand to be given an almost complete GLP might have confused him. Perhaps to have asked him to write his own GLP he could have done it. In fact, he was able to write a similar GLP to answer question 3.

## 7.6 Discussion and conclusions

Microworld 1 aimed to help create pupils' potential to approach in Logo the idea of variable as general number (VGN). These activities were designed in order to encourage pupils to focus on the method used to solve a problem in Logo rather than on the result obtained and to deduce a general method to solve similar problems. In order to express a general method in a formal way pupils were introduced to general Logo procedures (GLPs) and to VGN. It was considered that a pupil had developed a potential to work in Logo with the idea of VGN when (s)he was able to:

- shift from solving a particular problem by writing a fixed Logo procedure (FLP) to devising a general method to solve similar problems by writing a GLP;
- consider that a variable in a GLP represents an abstract indeterminate number, that is, a general number upon which Logo primitives are acting.

The results obtained during microworld 1 and during the individual interview show that a potential to focus on the method for solving a problem in Logo, to deduce a general method to solve similar problems and express this in a formal language has been created for the case study pupils. They were able to look at FLPs as objects to be analysed and not only as a tool to be used or as a record of instructions to be saved as has been noted

to be the way pupils may conceptualize procedures in Logo (Hillel,1992). However, the results obtained show also that pupils do not necessarily engage spontaneously in analysing the method used to solve a problem. They need to be led to it by specially designed tasks with increasing level of complexity and by interventions encouraging them to focus on the method used to solve a problem rather than on the results obtained.

The way pupils coped with VGN varied depending on the environment in which they were working, namely, the Logo numeric environment and the turtle graphics environment. In the Logo numeric environment all the case study pupils were able to write GLPs operating on and with the variable and to write 'unclosed' algebraic expressions. Moreover, they were able to shift from using particular examples (FLPs and/or verbal examples) as support for operating on a variable to think directly with general numbers and operate on them (see 7.4.3). That is, in this environment a potential to think directly with VGN was created.

When pupils worked in the turtle graphics environment a tendency to avoid operating on variables was observed. Pupils suggestion was to use the 'adding-on' strategy (Hoyles 1986), that is to add on a new variable although they saw a relationship linking the old and the new variable, or to edit the GLP in order to introduce the required variable value (see 7.4.3.2 The case of Valentin and Ernesto). Similar observations were made by Noss and Hoyles (1987), Sutherland (1987) and Hillel (1992). However, after researcher's intervention indicating how to operate on variable and how to symbolize this, all the case study pupils were able to write GLPs operating on variable by their own. This shows that a potential to operate on variable in the turtle graphics environment was created. But in this environment a potential to think directly with VGN was not created for the great majority of the case study pupils. They could not shift to think with general numbers. In order to work with VGN, they always used the support of particular examples.

Pupils used different strategies in order to generalize and express this by a GLP. In both environments a spontaneous tendency to evolve to a 1-step strategy, that is, to generalize from one particular example was observed (see 7.4.3.1 and 7.4.3.2). In the Logo numeric environment this evolution culminated with a qualitative shift to thinking with general numbers. In the turtle graphics environment not all the pupils were successful in using a 1-step strategy and only one pupil could finally make a qualitative shift and work directly with general numbers (see 7.5.2, Etna's approach to question 4).

During the individual interview it was observed that to work in microworld 1 created for the majority a potential to operate with VGN. Pupils were able to group similar terms involving VGN (see Table 7.9, problem 5).

Crucial elements for creating pupils' potential to work in Logo with VGN were: the design of the activities; the researcher's interventions; the Logo setting; the role of language.

*The design of the activities.*

The activities of microworld 1 (see 7.2) were designed aiming to encourage pupils to use their actual mathematic knowledge (essentially arithmetic) and their short Logo experience (see 6.3, Chapter 6) as scaffolds, that is, 'hooks' that assist pupils in overcoming the obstacles to the solution of the task (Hoyles, Healy and Sutherland, 1991), to approach gradually the idea of VGN. Challenging tasks of increasing difficulty and different levels of abstraction were designed in order to encourage pupils:

- to shift from solving an arithmetic problem to focussing on the method used to solve it;
- to deduce a general method to solve similar arithmetic problems expressing this in a formal way by a GLP;
- to generalize first from many instantiations of a problem and then from only one particular example;
- to use a symbolic variable to represent the indeterminate number upon which a general method was applied and to operate on it;
- to extend the experience of working with VGN from the Logo numeric environment, where the particular numbers used had no additional reference, to the turtle graphics environment, where numbers represented lengths.

The results obtained during microworld 1 suggest that to engage in activities where pupils were encouraged to rely on their prior arithmetic knowledge and on explicit particular examples in order to generalize, and where the complexity of the tasks increased gradually, helped create pupils' potential to work with the idea of VGN in Logo.

In the Logo numeric environment this approach helped pupils to shift to a higher level of abstraction (not planed by the researcher) enabling all the case study pupils to think directly with general numbers in order to represent simple word given general sentences by a GLP. When individually interviewed they were able to apply this capability also to

solve simple tasks posed in the usual paper and pencil environment (see 7.4). A possible reason for this qualitative shift not to occur in the turtle graphics environment for the great majority of pupils, might have been the greater complexity of the tasks if compared with those given in the Logo numeric environment. In order to solve the tasks in the turtle graphics environment pupils had to handle simultaneously several elements (e.g. the turtles' turns, the invariant and variable elements); they had to operate on the variable in different ways within the same procedure and they had to cope with numbers that had referent. However, Etna's case (7.5.2) although isolated testifies the possibility to make this qualitative shift also in the turtle graphics environment. It might be that spending more time on tasks similar to the ones given in this environment would have helped finally the majority of the case study pupils to make this qualitative shift.

Although, as Griffin and Cole (1984) remark referring to educational activities, the structure of an activity cannot be claimed to be the only determinant element of mental development it may be a necessary condition, even if not sufficient, to create pupils' potential for it.

*The researcher's intervention*

During the activities of microworld 1 the researcher intervened in ways that aimed to be conducive to create pupils' potential to work in Logo with VGN. In general the researcher did not provide support beyond the point it was needed. However, on occasion support might have been avoided or postponed if the researcher had been more patient (see, for example, 7.4.3.2, The case of Valentin and Ernesto).

A tendency to look at procedures only as tools to be used was observed. Researcher's intervention was crucial to help pupils shift from focusing on the results obtained to focus on the method used to solve a problem in Logo (see 7.4.1.1). By organizing and directing a group discussion she created appropriate conditions to help pupils approach the ideas of GLP as the expression of a general method and VGN as an indeterminate number involved in a general method (see 7.4.1.1). The individual support provided when pupils were stuck in a situation helped them build gradual control and understanding of the problem and approach in this way the ideas of GLP and VGN (see 7.4.2 and 7.4.3.2). However, due to the organization of the work with the whole group she could not always provide to all the pupils the required support. This might have influenced the different achievements reached by the pupils.

It was observed that pupils may conceive a VGN and the necessity to operate on it although they are not yet able to use a symbol to represent it (see 7.4.2 The case of Etna and Itzel). To introduce a symbol to represent a VGN in these circumstances helped pupils make sense of it and in consequence they were able to use it on their own and to operate on it when necessary.

Researcher's intervention encouraging pupils' own approaches and helping them use these in an appropriate way in order to reach a correct solution (see 7.4.2, The case of Valentin and Ernesto) helped pupils relax gradually the link with numeric examples and in the Logo numeric environment they shifted to a qualitatively different level that enabled them to think directly with VGN. To indicate how to operate on VGN in the turtle graphics environment was crucial to create pupils potential for this (see 7.4.3.2 The case of Valentin and Ernesto).

It was observed that during the individual interview the sole researcher's silent presence pushed some pupils to gain self confidence and they ventured to use methods to write a GLP in the turtle graphics environment about which they were not yet completely sure (see 7.5.2). On the contrary when they were working with their peer during microworld 1 they were more cautious. They seemed not to want to take risks and they tended to adopt more familiar strategies.

These results stress the crucial role of interventions in order to create pupils' potential to work in Logo with VGN. They also suggest that to provide occasional individual attention may help pupils to create a potential to shift to more abstract levels supporting in this way their approach to the idea of VGN as an abstract indeterminate number involved in a general method.

*The Logo setting*

To work in Logo was very motivating for pupils. In general all the pupils engaged enthusiastically in the given tasks. It was observed that to work in Logo provided them with the possibility to regulate their own activity in order to approach the ideas of GLP and VGN at their own pace. The design of the tasks and the researcher's interventions initially pushed pupils to approach the writing of a GLP and the idea of VGN by using intermediate steps. That is, they were encouraged to rely on their prior arithmetic knowledge and on particular examples. The feasibility to solve the posed Logo tasks using their prior knowledge and the feedback obtained by the computer when running the

GLPs, as well as the possibility to drop back to a several-steps strategy in order to write a GLP, led pupils to gradually internalize some of the actions initially explicitly performed. In this way pupils were regulating their approach to the given tasks and each pupil could approach gradually the idea of VGN relaxing slowly the link with particular numbers. After this experience an eventual qualitative shift to thinking directly with general numbers was made (see 7.5.2 Etna's approach to question 4). This qualitative shift might have been influenced by the dialectic process

| | | | | |
|---|---|---|---|---|
| particular | $\mapsto$ | general | $\mapsto$ | particular |
| (writing a FLP) | | (writing a GLP) | | (running a GLP) |

favoured by Logo environments in which pupils engaged to write a GLP (see 7.4.1.2 The case of Etna). This dialectic process led pupils not only to the idea that a variable represents a range of values (Noss, 1985; Sutherland, 1987) but it pushed them to start working directly with general numbers without needing any more the explicit support of particular examples to which if necessary they could drop back. This suggests that pupils were looking at the VGN as an indeterminate number involved in a general method. This evolution was observed for all the pupils in the numeric Logo environment and for one pupil in the turtle graphics environment.

It was observed that the necessity to communicate with the computer in a precise formal language, Logo, helped pupils accept the use of symbolic variables in order to express their generalizations (see 7.4.2 Constructing a meaning for the symbol - the case of Etna and Itzel). After the experience of writing GLPs pupils who could not use symbolic variable at the beginning of the study (Etna and Ernesto) used them inclusively to solve simple problems posed in usual paper and pencil environment (see 7.4). This suggests that a transference of the use of symbolic variables from Logo to paper and pencil environment occurred. This confirms Noss' suggestion that under appropriate conditions pupils may use the algebra learned in Logo 'in order to construct algebraic meaning in a non-computational context' (Noss, 1986, p. 354). I suggest that to work in Logo environments helped pupils construct a meaning for symbolic variables. Therefore, they could use them inclusively outside Logo.

In the Logo numeric environment all the case study pupils were able to operate on the variable by a particular number, however all of them needed help in order to operate on the variable in the turtle graphics environment. This suggests that initially it may be easier for pupils to operate on a variable when the numbers it represents have no additional referent than to operate on it when the numbers it represents have a referent. For that

reason in order to help pupils approach the idea of operating on variables in Logo it might be a good start to work first in Logo numeric subsets helping then pupils to extend this experience to the turtle graphics environment.

*The role of language*

Spoken language was an element pupils used very often during the solution of the tasks of microworld 1. Pupils were encouraged to talk between them and with the researcher in order to solve the posed tasks. That is, pupils were encouraged to use communicative speech, that is speech-for-others. However, it was observed that during these activities both, communicative and egocentric speech, that is, speech-for-oneself (Vygotsky's definition, see 2.2.4, Chapter 2) where used very often. Egocentric speech usually appeared after communicative speech. Pupils used it after discussing the task between them or with the researcher. This use of speech appeared also when a pupil worked alone, that is, when her (his) peer was absent. It had always a self-regulative function (see 7.4.1.2, The case of Etna, 7.5.2, Martha's and Valentin's approach to question 4). Shifts between communicative and egocentric speech were observed during pupils' work. It was observed that in the numeric Logo environment, both types of speech disappeared and pupils wrote GLPs and used VGN without verbal support. In accordance with Vygotsky (1986) this use of speech may suggests a possible evolution from communicative to egocentric speech and to its final internalization. However, there were not records of this evolution.

The results obtained suggest that both, communicative and egocentric speech had a crucial role in the evolution of pupils' strategies to write GLPs and use VGN (see 7.4.3.1 and 7.4.3.2, The case of Valentin and Ernesto). It was observed that when, in the turtle graphics environment, discussion was avoided from the beginning and not as a consequence of pupils' evolution in the solution of the tasks, pupils needed stronger researcher's support in order to write GLP and cope with VGN (see 7.4.1.2 The case of Oscar and Jonathan).

The relation observed between pupils' gradual approach to the idea of VGN and the way in which language was used seems to be in accordance with Vygotsky's (1978) claim that higher mental functions appear first at the social (interpsychological) level and after this at the individual (intrapsychological) level (see 2.2.4, chapter 2). In this view communicative speech is a crucial element allowing the transition from social to individual regulation of the activity leading to the solution of a task. Egocentric speech, that is speech-for-oneself, is considered the transitional form between social

communicative speech and internal speech; and it is viewed as a tool used by the pupil to organize and control her (his) own behaviour (Vygotsky, 1978, 1989).

The frequent use of egocentric speech (Vygotsky's definition) observed during microworld 1 suggests that Logo based environment may favour its appearance helping in this way pupils to start internalizing the idea of VGN. However, to internalize this concept is not straightforward and it was observed that faced with unfamiliar situations pupils who were already writing GLPs using VGN without needing any more verbal support, dropped back to use communicative and egocentric speech. The appearance of an hybrid language where Logo and natural language were mixed up was observed. Pupils used this hybrid language to analyse the posed problem and to direct their actions (see 7.4.3.2, The case of Valentin and Ernesto). A similar use of Logo and natural language was observed by Hoyles, Healy and Sutherland (1990).

The frequent appearance of egocentric speech (Vygotsky's definition) in these Logo environments points to Logo as an environment where pupils' way of approaching the idea of VGN might be externalized by egocentric speech (Vygotsky's definition) favouring in this way researcher's observation of pupils' way of approaching this concept in Logo.

Some elements that may be possible obstacles for pupils' approach to the idea of VGN in Logo-based environments were also observed. These were:

- the lack of a meaning for the symbolic variable
- a premature automation for writing GLPs.

*A meaning for the symbol*

A resistance to use a symbol to represent a VGN was observed. There were pupils who initially preferred to represent the VGN by a void space that was a notation already familiar to them (see 7.4.2 Constructing a meaning for the symbol - The case of Etna and Itzel). This behaviour suggests a difficulty in giving a meaning to the symbol used by the researcher to represent a VGN but not in working with this idea.

When the idea of VGN was introduced the researcher empathized that any letter, name, or non-sense word could be used to represent it. This aimed to help pupils understand that the name of the variable does not affect its role (Sutherland, 1987). However, to present pupils with all these different possible ways to symbolize a VGN might have

created confusion for some pupils. Although they grasped the idea of VGN and they were able to work with it, they were not able to use a symbol to represent it. Researcher's intervention indicating explicitly that to represent a VGN in Logo a symbol have to be used instead of a void space helped these pupils to establish a connection between the concept they were already working with and its symbolization. In this way, the symbol acquired a meaning. After this they were able to symbolize a VGN using different letters for it and to operate on it when necessary. This suggests that difficulties pupils have with literal symbols in usual algebra environments (Küchmann, 1980; Booth, 1984) might originate in a lack of clearness about the concept underlying them.

The conventional teaching of elementary algebra does not offer enough opportunities for pupils to construct a meaning for the symbols used to represent general numbers. There seems to be an implicit assumption that pupils by their own will make sense of the meaning of the symbols by using them. However, as James and Mason (1982) remark 'Behind the formal symbols of mathematics there lies a wealth of experience which provides meaning for those symbols. Attempts to rush students into symbols impoverished the background experience and leads to trouble later' (James and Mason, 1982, p. 249). Referring to elementary school mathematics Hiebert (1989) considers that a critical instructional problem is to help pupils build a meaning for symbols. I would extend this comment suggesting that it might be critical to assist pupils in building a meaning for the symbols representing general numbers in order to help them approach elementary algebra. It seems that in Logo a meaning for the symbolic variable used as VGN can be built very quickly and this is supported by the dialectic interaction particular↔general always present in Logo. However, even in Logo not always enough attention is paid to the construction of the meaning of the symbols used. This might be a cause for pupils confusion about how they should name a variable and the belief that the name affects its function (Hillel, 1992). The results obtained in the present study show that in Logo-based environments pupils may be led to construct a meaning for the symbol used to represent VGN.

*Premature automation*

It was observed that pupils may automate the actions required to write GLPs in order to solve similar tasks without having grasped the essence of the concept involved (see 7.4.1.2 The case of Oscar and Jonathan). This may lead pupils to try to apply the same actions in situations that even if involving the same concept need another approach to be solved. This was the case when Oscar suggested to use the primitive PRINT in order to

write a GLP in the turtle graphics environment. A similar tendency to automation was observed by Filloy and Rojano (1989) when studying pupils with short algebra experience that used concrete models to solve algebraic equations. When coping with a new concept a premature automation may carry with it the risk of hiding it. For that reason when aiming to help pupils approach a new concept by reflecting on it in order to gradually master it, a premature automation might be an obstacle. It may be convenient to offer pupils non-routine tasks where the new concept is approached by different methods in different settings. To approach the idea of VGN in the Logo numeric environment and in the turtle graphics environment helped pupils to grasp the idea of general method and of VGN. However, the tasks posed in each one of the two settings were similar. This may have pushed some pupils to a premature automation delaying in this way the creation of a potential for approaching the idea of VGN.

# CHAPTER 8

# VARIABLES IN A FUNCTIONAL RELATIONSHIP

## 8.1 Introduction

During microworld 2 pupils worked on activities involving variables in a functional relationship (VFR). To work with VFR implies to cope with functional relationships and with some non trivial mathematic notions underlying them, for example:

- domain and range of variation
- monotonicity
- maximum and minimum.

Prior research (e.g. Markovits, 1986, 1988; Sfard, 1989; Breidenbach et al., 1992; Tall and Bakar, 1990) has shown that when the idea of function is introduced in a formal way defined by a rule of correspondence between two sets even students with algebra experience have difficulties to cope with it and its underlying notions. It was also found that algebra beginners have difficulties to cope with the idea of change (Heid and Kunkle, 1988) and that in order to favor pupils understanding of functions, they might be helped to cope with the idea of change (Markovits et al., 1986). Therefore, it was decided not to introduce the idea of function in a formal way but to design activities where pupils might work with it and its underlying notions based on the visual perception of change and using their arithmetic background, mainly their experience with integers and their order.

The notion of 'functional relationship' assumed was the dynamic one: a quantity that depends on another changing quantity. This conception of a functional relationship was present from the ancient times, before any analytic expression representing a functional relationship was introduced (see Youschkevitch, 1976, for a detailed analysis of the development of the concept of function). It was considered that at least two aspects characterize VFR:

- a static aspect (SA): when the pointwise/static correspondence between two quantities is considered;
- a dynamic aspect (DA): when two quantities are viewed as moving entities and their global linked behaviour is perceived.

It was assumed that variables are in a functional relationship (VFR) when they represent numbers whose values move within a range of values linked one to each other by a relationship. It was considered that pupils had developed a potential to work in Logo with the idea of VFR when they were able to cope with both aspects, namely SA and DA, and shift from one to the other when required by a Logo-based problem.

## 8.2 The activities

The activities of microworld 2 (Table 8.1) were designed in order to help create pupils' potential to cope in Logo with both, the static aspect (SA) and the dynamic aspect (DA) of VFR. The work sheets used are included in Appendix 2 (D-1 to D-5).

| Microworld 2- Variables in a functional relationship | | |
|---|---|---|
| **Activity** | **Aim** | **Worksheet** |
| An interval for each shape | Direct pupils' attention to SA and to the movement of a variable within an interval. | D - 1 |
| Shape ↔ Interval | | D-2 and D-3 |
| What happens to the shape? | Direct pupils' attention to SA and DA | D-4 and D-5 |

**TABLE 8.1** - Overview of the activities of microworld 2

The two first activities were designed in order to create pupils' potential to approach SA, that is, to consider a pointwise/static correpondence between two elements (e.g. a number and a shape) and to approach DA, that is to consider a variable as an entity moving within a numeric interval. For the first activity the computer was loaded with a GLP that drew six different shapes in correspondence to six numeric intervals. The GLP was not visible to pupils and the six shapes were not shown to them in advance. They were asked to run the GLP in order to discover the six figures and to determine for each one *all* the possible inputs that produced it (work sheet D-1). To determine *all* the inputs implied to determine a numeric interval in correspondence to each shape. This required, on the one hand, to consider that inputs and outputs were in pointwise correspondence. That is, it was necessary to approach SA. On the other hand, to determine an interval implied to consider that to obtain a determinate shape the values of the input could vary within certain limits. Moreover, it was necessary to fix these limits

by determining the bounds of the interval where the values of the input might move. That is, it was necessary to approach DA.

During the second activity pupils were asked to make predictions based on the written record of the intervals determined during the first activity. They were asked not to use computers but to rely only on their written records. A list of arbitrary numbers was given and pupils were asked to predict for each number the corresponding shape (work sheet D-2). After this, they were given a work sheet with the drawing of the six shapes (work sheet D-3) and asked to determine at least three numbers in correspondence to each one of them. To solve these tasks it was necessary to identify for each shape the corresponding interval and to consider that any value of a determined interval was producing the same shape. This implied considering the value of the input as an entity moving within an interval. That is, they had to approach DA. It was also necessary to consider the correspondence number $\leftrightarrow$ shape. That is, they had to approach SA.

The third activity aimed to observe pupils' potential to cope with SA and DA and shift from one to the other. Given a GLP that drew six shapes of changing size pupils were asked to discover for each shape the linked behaviour of the input and the corresponding size of the shape and to describe it qualitatively (work sheet D-4 and D-5). Pupils were explained that it was necessary to know how the values of the input were related to the size of the shape in order to be able to draw a 'jungle' on the screen. In order to draw a 'jungle' they were expected to combine all the six shapes each one having a previously planned size.

During microworld 2 pupils were not asked to write GLPs, but to use procedures written by the researcher. The reason for this was to avoid pupils being distracted by writing GLPs and using variables as general numbers (activities already developed during microworld 1), and to push them to focus essentially on the static and the dynamic characteristics of VFR. It was considered that to include more than one characterization of variable at the same time could create confusion for pupils who were just starting to work with them. Thus it was decided not to fully exploit Logo to explore this environment. Due to the experience of writing and running GLPs acquired in microworld 1, it was considered that to run a given GLP, although not written by them, would make sense to pupils. This consideration was justified by a prior experience (Ursini, 1990c) where after only a short time spent in writing and running GLPs, pupils were able to use fluently a researcher's written GLP as an exploring tool.

## 8.3 Researcher's intervention

During the activities of microworld 2, the researcher's interventions aimed to help create pupils' potential to work with VFR in Logo-based environment. She intervened giving nudges stressing the goal of the activity, posing precise questions. Her interventions aimed to help pupils:

- approach SA
- cope with SA
- approach DA
- cope with DA
- shift from DA to SA.

## 8.4 Results

The results obtained are discussed considering:

- pupils' potential to approach SA and DA
- pupils' potential to cope with SA and DA and to shift from one to the other.

### 8.4.1 Pupils' potential to approach the static aspect (SA) and the dynamic aspect (DA) of variables in a functional relationship (VFR)

For the first activity (work sheets D-1) the computer was loaded with a GLP named DIBUJA (DRAW), that drew six different shapes, each one in correspondence with a specific numeric interval (Fig. 8.1).

When the input to the procedure belonged to one of the intervals, the corresponding shape appeared on the screen, otherwise nothing happened. In order not to include too many distractors in the problem, while a value was varying within the same interval the size of the shape remained constant. Pupils were asked to run the given procedure and to find out *all* the possible values that provided a specific shape. During this activity pupils' strategies to determine a numeric interval and the ways in which they expressed a numeric interval were observed.

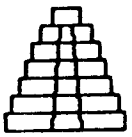
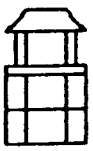
| Shape | Domain of definition |
|---|---|
| BIRD | (-10, 199.9) |
| FLOWER | (199.9, 379.9) |
| PYRAMID  | (379.9, 499.9) |
| PALM | (499.9, 749.9) |
| TOWER  | (749.9, 999.9) |
| MONKEY | (999.9, 1 500) |

**Fig. 8.1** - Domain of definition of the six shapes

**8.4.1.1** Pupils' approach to the task

All the pupils enthusiastically approached the task. They were very competitive trying to identify the six shapes. They tried numbers at random without recording them. In correspondence to each shape they recorded only one number. In this way all the case study pupils established spontaneously, that is, without intervention, only a one-to-one correspondence between one shape and one number. None of the pupils looked for intervals. After having identified the six shapes, they considered that they had finished the

task. Their behaviour indicate that they have grasped the static aspect of VFR but they were not considering the value of the input as a moving entity. So they could approach SA but they were not approaching DA.

In order to identify each shape all the pairs assigned several values to the input. While doing this they realised that there were different values corresponding to the same shape, however they completely disregarded them. Evidence for this was provided by the answer given by pupils when individually asked if there were other values that gave the same shape. Valentin, for example, said: *'Yes, there are a lot of values, but it doesn't matter because you will get always the same six shapes'*. Other pupils gave similar answers. This suggests that even if pupils were assigning different values to the input, their aim was to pinpoint only one value in correspondence to each shape. This approach was perhaps favoured by the design of the task. In fact, even if the pupils were told that the procedure was drawing six different figures, they did not know them in advance and they were very impatient to discover them. Another reason for this approach might be due to pupils' interpretation of the variables involved in the problem. I would suggest that a reasonable interpretation of this behaviour is that pupils were looking at the independent variable, that is at the input, as both a place holder (they assigned arbitrary values to it) and as a specific unknown (they aimed to identify one particular input in correspondence to each shape), but they were not interpreting it as a dynamic entity moving within a range of values. They were reducing the relationship between the inputs and the shapes to a one-to-one static correspondence. A similar behaviour was observed when some of them tried to answer Part 3 of the questionnaire (Appendix 1) given at the beginning of the study. Faced with an algebraic expression representing a functional relationship they fixed one of the variables and calculated the other (see 5.2.3, Chapter 5). They were reducing in this way a functional relationship to one (or a collection of) equation(s) and, in this way, they were considering only a pointwise/static correspondence between variables.

#### **8.4.1.2** Intervention

After this spontaneous approach to the problem the researcher intervened emphasizing that *all* the numbers that were in correspondence with a shape had to be find out and not only one single number. She wanted to observe if this explicit requirement was enough to push pupils to shift from considering only a one-to-one static correspondence between

one particular value and a shape, to consider the value of the input as a moving entity. That is, she aimed to help create pupils' potential to approach DA.

#### **8.4.1.3** Pupils' approach to the task after intervention

After the intervention all the observed pairs could identify a numeric interval in correspondence to each shape. It was observed that to determine an interval they considered as a suitable starting point the already recorded single value. They increased (decreased) it systematically or at random and guided by the visual feedback they identified a neighborhood of numbers where two different shapes appeared. After that they approximated the limits of the interval systematically increasing (decreasing) the input.

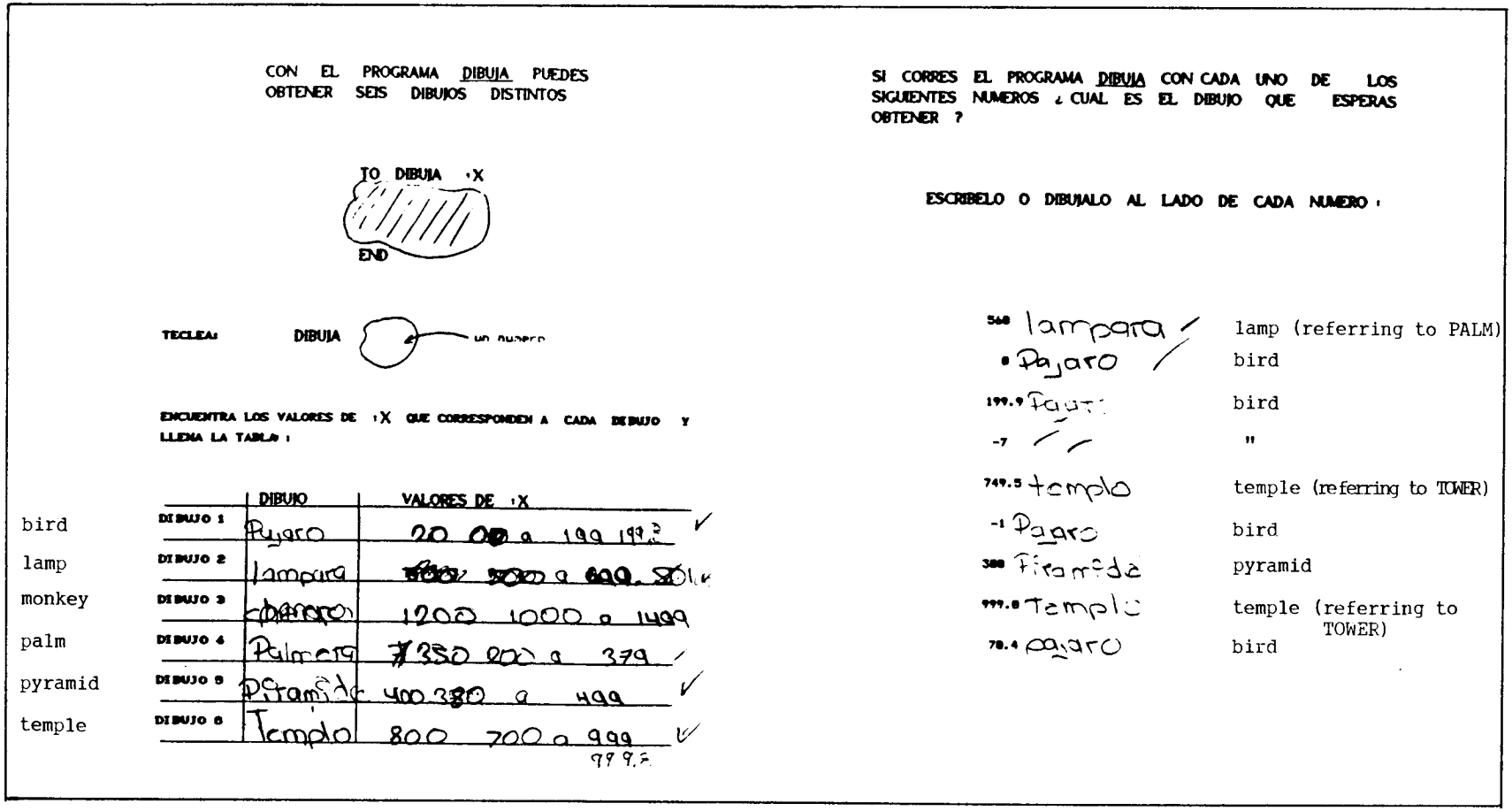
Moreover, the notation pupils used to record the intervals suggests that they were considering the intervals as entities and not only as a collection of elements. Two very similar notations were spontaneously used by pupils:

(a number) to (a number) (e.g. 1000 to 1499, 200 to 379),

(a number) - (a number) (e.g. 1000 - 1499, 200 - 379).

It may be that pupils were considering the relationship between the shape and the corresponding interval only as a collection of pointwise/static relationships between each element of the interval and the shape but, establishing a correspondence between shapes and intervals-as-entities suggests that pupils were considering that any value of the interval will produce the same shape. That is **they were able to consider the value of the input freely moving within a range of values. This indicates that pupils' potential to approach DA was created.** This consideration is supported by the way pupils worked during the second activity.

During the second activity pupils were asked to make predictions (work sheet D-2). Without using computers and based only on the written record of the intervals all the pupils correctly predicted the shape corresponding to a given number. They made correct interpolations for decimal numbers and extrapolations for negative numbers. Figure 8.2 illustrates this showing the record of the intervals and the predictions made by Oscar and Jonathan. All the case study pupils were also able to choose three arbitrary values belonging to the appropriate interval in correspondence to each one of the six shapes (work sheet D-3).



CON EL PROGRAMA DIBUJA PUEDES OBTENER SEIS DIBUJOS DISTINTOS

TO DIBUJA :X
END

TECLEA: DIBUJA (un número)

ENCUENTRA LOS VALORES DE :X QUE CORRESPONDEN A CADA DIBUJO Y LLENA LA TABLA:

| | DIBUJO | VALORES DE :X |
|---|---|---|
| bird | DIBUJO 1 Pajaro | 20 [illegible] a 199 199.2 ✓ |
| lamp | DIBUJO 2 lampara | [illegible] a 699. [illegible] ✓ |
| monkey | DIBUJO 3 [illegible] | 1200 1000 a 1499 |
| palm | DIBUJO 4 Palmera | [illegible] 350 200 a 379 ✓ |
| pyramid | DIBUJO 5 Piramide | 400 380 a 499 ✓ |
| temple | DIBUJO 6 Templo | 800 700 a 999 ✓ 999.2 |

SI CORRES EL PROGRAMA DIBUJA CON CADA UNO DE LOS SIGUIENTES NUMEROS ¿ CUAL ES EL DIBUJO QUE ESPERAS OBTENER ?

ESCRIBELO O DIBUJALO AL LADO DE CADA NUMERO :

| | |
|---|---|
| 568 lampara ✓ | lamp (referring to PALM) |
| 8 Pajaro ✓ | bird |
| 199.9 Pajaro | bird |
| -7 ✓ ✓ | " |
| 749.5 templo | temple (referring to TOWER) |
| -1 Pajaro | bird |
| 388 Piramide | pyramid |
| 999.8 Templo | temple (referring to TOWER) |
| 78.4 pajaro | bird |

Fig. 8.2 Oscar and Jonathan - Record of the intervals and predictions

These results show that all the case study pupils were able to consider that each interval was in correspondence with a determined shape and that any value belonging to a specific interval gave the same shape. This confirms that in this environment pupils were able to approach SA and DA. This is further supported by the answer given, for example, by Sandra when asked by the researcher if any value other than the three values she recorded in correspondence to the PALM gave the same shape. She explained: *'If you want to get a PALM you type DIBUJA and any value between 200 and 379'*. This answer indicate that Sandra considered that there was a correspondence between a value and a specific shape but, she also considered that to obtain a determined shape the value of the input could move freely within the corresponding interval.

Moreover, these results show that pupils were able to consider that when the value of the input moved within an interval the outcome might remain constant. But when the value of the input moved from one interval to another the outcome might change. This kind of experience could constitute a first approach to the notion of function defined piecewise. **This indicates that environments might be devised to create pupils' potential to cope with piecewise functions prior to a formal introduction to this notion.**

### 8.4.2 Pupils' potential to cope with the static aspect (SA) and the dynamic aspect (DA) of variables in a functional relationship (VFR) and to shift from one to the other - Pupils' analysis of six shapes

The third activity aimed to help create pupils' potential to cope with SA and DA and to shift from one to the other. For this activity the procedure DIBUJA was modified in order to draw shapes of different size depending on the value given to the input. Different functional relationships linking the value of the input and the size of the shapes were included (Fig. 8.3).

While the input was increasing, the size of the shapes was:

- monotonically increasing (FLOWER),
- monotonically decreasing (TOWER),
- constant (BIRD and MONKEY),
- increasing, reaching a maximum and then decreasing (PYRAMID),
- decreasing, reaching a minimum and then increasing (PALM).

After running the GLP a shape appeared on the screen. Its size depended on the given input. Additionally a written message appeared in which the size of the shape was given

by a number. The visual and the numeric representations of the size aimed to help pupils look both, at the pointwise/static correspondence between the value of the input and the size of the shape, and at their linked movement.

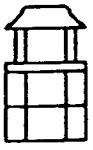
| Shape | Description of the functional relationship |
|---|---|
| BIRD | Constant<br>Range of variation: (-10,199.9)<br>Size = 30 |
| FLOWER | Monotonically increasing<br>Range of variation: (199.9, 379.9) |
| PYRAMID | Function with a maximum<br>Range of variation: (379.9, 499.9)<br>Maximum value for X = 420 |
| PALM | Function with a minimum<br>Range of variation: (499.9, 749.9)<br>Minimum value for X = 680 |
| TOWER  | Monotonically decreasing<br>Range of variation: (749.9, 999.9) |
| MONKEY | Constant<br>Range of variation: (999.9, 1 500).<br>Size = 20 |

**Fig. 8.3** - Description of the functional relationship linking the input and the size of the shape

### 8.4.2.1 Pupils' perception of variables' movement

To test pupils' capability to cope with DA in these environments a first task was posed. Pupils were asked to run the GLP DIBUJA for two given inputs that drew the FLOWER; to try more values to get the same shape; and to describe the linked movement of the inputs and the size of the shape (work sheet D-4). The functional relationship linking the value of the input and the size of the shape was monotonic increasing (see Fig. 8.3).

During this activity it was observed:

- whether or not pupils run the GLP with inputs belonging to the appropriate interval;
- how pupils described the global linked movement of the VFR involved .

The results obtained are shown in Table 8.2.

| Pair | Inputs belonging to the interval | Pupils' description of the relation |
|---|---|---|
| Valentín / Ernesto | Yes | When the number is bigger, the shape is bigger. |
| Etna / Itzel | Yes | To each number corresponds a size. |
| Martha / Mariana | Yes | If we half the value the size changes and it gets smaller. |
| Oscar / Jonathan | Yes | To each number corresponds a size |
| Claudia / Sandra | Yes | Each shape has its size. |
| Nayeli / Soledad | Yes | Smaller the number, smaller the shape. |

**TABLE 8.2** - Case study pupils' descriptions of a functional relationship

Table 8.2 shows that all the case study pupils assigned to the input different values belonging to the appropriate interval. That is, all the case study pupils considered that to obtain the given shape the value of the input could move within the corresponding interval.

Pupils' qualitative descriptions of the relationship linking the inputs and the size of the shape, show that only two pairs (Valentin/Ernesto, Nayeli/Soledad) were able to shift from considering a pointwise/static correspondence between the values of the input and the value of the size and to give a qualitative description of the global behaviour of the variables linked by a monotonic function. This indicate that only two pairs were able to cope with DA for this monotonic functional relationship.

To describe the global behaviour of the variables linked by a monotonic function Valentin/Ernesto and Nayeli/Soledad used the so called '*the... the...*' pattern (e.g. *the* smaller the number *the* smaller the shape). The answers given by these pairs suggest that they were considering the values of the independent variable as an ordered set of numbers. The lack of a record of the given inputs and its corresponding size as well as the reference to the shape and not to its size when describing the relationship, suggest that the dependent variable considered was the size of the shape visually perceived. However, it seems that they were able to establish an order in the visually perceived size and link its variation with the values assigned to the input.

The descriptions given by the other pairs show that they were considering only a pointwise/static correspondence between the input and the size of the shape. This shows on the one hand their capability to cope with SA and to approach DA. On the other hand, this shows that they did not cope with DA spontaneously.

#### **8.4.2.2** Providing scaffolds

Bearing in mind the results described above and aiming to create pupils' potential to cope with DA, some intermediate tasks were introduced before asking pupils to give a qualitative description of the behaviour of VFR. These were meant to be used as scaffolds to shift to DA.

Pupils were asked to take the following steps. They were suggested first of all to construct a table recording on their work sheets all the inputs they were trying out and the corresponding size of the analysed shape. It was hypothesized that to construct a table by themselves would push pupils to perceive the linked movement of variables. This activity would allow pupils to use their spontaneous tendency to see VFR in a pointwise/static correspondence. Additionally the experience of increasing/decreasing the input and to record this and the corresponding value for the size, were expected to push pupils to shift to looking at the global linked movement of VFR. In this way, it was expected that pupils' capability to cope with SA could be used to help them shift to DA.

After this pupils were required to determine the values of the input where the shape had respectively its minimum and its maximum size. This aimed to push pupils to focus on the two extreme values of the size of the shape. This was expected to help pupils focus on the variation of the size of a shape.

Finally, they were asked to answer the question, 'What happens with the size of the shape between the smallest and the biggest value of X?'. This question aimed to help pupils organize the values of the independent and the dependent variable in an order structure, focus on the global linked movement of VFR and give a qualitative description of their linked movement. That is, it aimed to help pupils cope with DA..

To analyse each one of the six shapes pupils were suggested to follow the above mentioned steps (work sheet D-5). The following observations were made:

- whether or not pupils followed the suggested steps in the given order
- whether or not pupils used for each shape inputs belonging to the appropriate interval
- pupils' strategies to fill a table
- whether or not pupils located maximum and minimum values
- pupils' way of describing the movement of VFR

The results obtained are displayed in Table 8.3. The last five columns of the table describe pupils' perceptions of the movement of VFR. It was considered that pupils were perceiving a global linked movement of variables when the answers were of the type 'when the value of X increases the size increases'. In the table this is indicated by 'global linked'. When 'intervals specified' is added this means that intervals in which non-monotonic functions were increasing or decreasing were explicitly indicated by the pupils. When a non-monotonic function was viewed as a monotonic one, this is specified by 'treated as monotonic decreasing (increasing)'. Answers of the type 'small - small', 'big - big' were considered to reflect only a pointwise/static perception of the behaviour of VFR. In the table this is indicated by 'pointwise static'. Answers of the type 'the size does not change' or 'in the middle the size is the smallest' showed that pupils focussed only on the behaviour of the dependent variable. In the table this is indicated by 'dependent variable only'.

The results reported in Table 8.3 show that all the pupils tried out the GLP using several inputs belonging to the interval corresponding to the chosen shape. This confirms that the case study pupils were able to consider the value of the input moving within a limited range of values.

| Pair | Suggested steps followed | Inputs beloging to the interval | Strategy used to fill tables | Maximum located | Minimum located | Description of the movement of the variables linked by: | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | Monotonic increasing functions | Monotonic decreasing functions | Functions with maximum | Function with minimum | Constant function |
| Valentin Ernesto | No (1st shape analyse) Yes (all other shapes) | Yes | Not ISI/not R (1st shape) ISI/R (all other shapes) | Yes | Yes | Pointwise static | Global linked | Global linked intervals specified | Global linked interval specified | Dependent variable only |
| Etna Iztel | No | Yes | Not ISI/NotR | No | No | Pointwise static | Pointwise static | Pointwise static | Pointwise static | Dependent variable only |
| Martha Mariana | No | Yes | Not ISI/R | Yes | Yes | Global linked | Global linked | Treated as monotonic decreasing | Movement perceived but not linked | Dependent variable only |
| Oscar Jonathan | Yes | Yes | ISI/R | Yes | Yes | Not analysed | Global linked | Global linked intervals specified | Global linked intervals specified | Global linked |
| Claudia Sandra | No | Yes | Not ISI/R | Only for monotonic functions | Only for monotonic functions | Global linked | Global linked | Dependent variable only | Not analyzed | Dependent variable only |
| Nayeli Soledad | Yes | Yes | ISI/R | Yes | Yes | Global linked | Global linked | Global linked intervals specified | Dependent variable only | Dependent variable only |

**TABLE 8.3 - Case Study Pupils' analysis of six shapes**

It was observed that to fill tables pupils used three different strategies:

- the input was systematically increased and the majority of the tested values were recorded (ISI/R),
- the input was not systematically increased and the majority of the tested values were recorded (not ISI/R)
- the input was not systematically increased and only few of the tested values were recorded (not ISI/not R).

The results displayed in Table 8.3 suggest a link between pupils' perception of the movement of VFR and the strategy they used to construct the numeric tables. Moreover, it was observed that pupils who followed the suggested steps in the given order were filling in the table by increasing systematically the input. It was observed that three pairs (Valentin/Ernesto, Oscar/Jonathan, Nayeli/Soledad) followed the suggested steps in the given order, filled in the tables increasing systematically the inputs and they recorded the majority of the tested values. Two pairs (Martha/Mariana, Claudia/Sandra) did not followed the suggested steps in the given order, they did not fill in the tables increasing systematically the inputs but they recorded the majority of the tested values. One pair (Etna/Itzel) did not follow the suggested steps in the given order, they did not fill in the tables increasing systematically the inputs and they recorded only few of the tested values. The way in which these different approaches influenced pupils' perception of DA is illustrated by a detailed description of the work of three case study pairs.

#### **8.4.2.3** Aspects influencing pupils' perception of the movement of variables in a functional relationship (VFR)

*Suggested steps followed in order + Input systematically increased + Majority of the tested values recorded*

It was observed that pupils using this approach started by filling in the table. To do this, they increased systematically the input recording the majority of the tested values. After this they analysed the table looking for maximum and minimum values of the dependent variable and in correspondence they found the values for the independent one. Finally, they analysed the table once more in order to give a qualitative description of the linked movement of VFR. This approach suggests that these pupils were first of all considering the input moving within a limited range of values and being in one-to-one correspondence with the size of the shape. After that they shifted to looking at the input as a specific unknown: they located the minimum and maximum values of the size and

they identified the corresponding values for the input. While doing this they were coping with SA and they were approaching DA. Finally they shifted back to looking at the input as moving in an interval but they linked its movement to the variation of the size of the shape. They were coping with DA.

This suggests that **to follows the given steps in order, to increase systematically the input and to record the majority of the tested values helped pupils cope with DA.** Moreover, the qualitative descriptions of the linked movement of VFR given by these pupils suggest that they were relying more on the analysis of the numeric table than on the visual perception of the image. Evidence for this are provided by the way Nayeli and Soledad analysed the PYRAMID.

The case of Nayeli and Soledad

Nayeli and Soledad are analysing the PYRAMID. The input and the size of this shape are linked by a functional relationship with a maximum (see Fig. 8.3 for more details). This is the second shape they analyse. The interval they have located in correspondence to this shape is [380, 499]. Nayeli is typing and Soledad is recording the values of the inputs and the corresponding values of the size on the work sheet (Fig. 8.4).

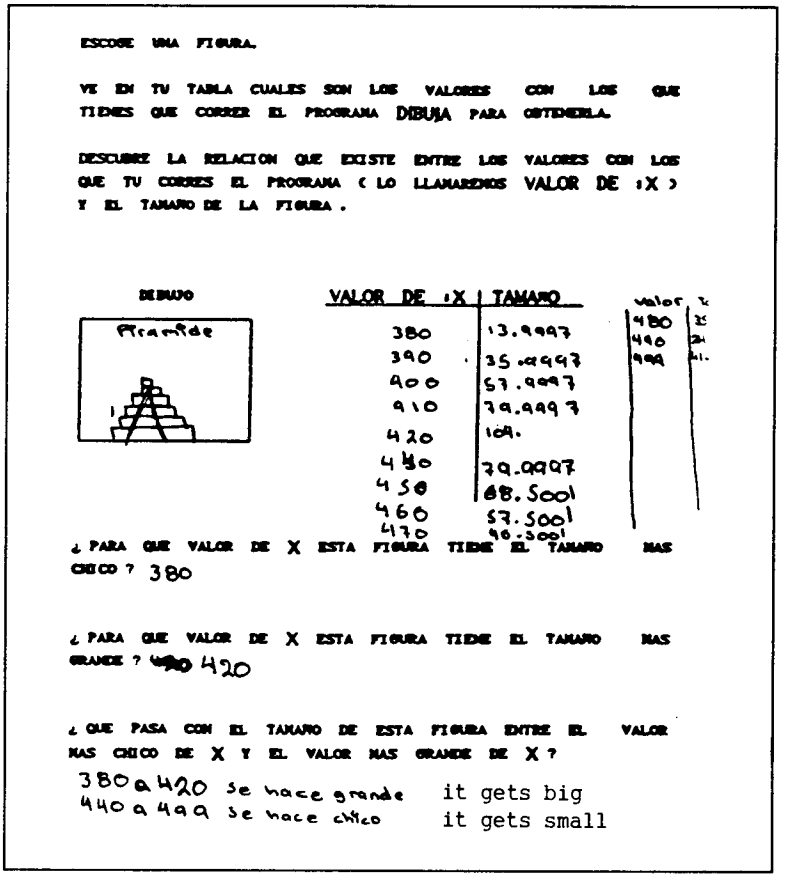



**Fig. 8.4** Nayeli and Soledad - Analysis of the PYRAMID

The first input they use is 380. This value corresponds to the lower limit of their interval. After this they increase the input systematically by 10. The last value they use is 499, that is the value corresponding to the upper limit of their interval. This suggests that **Nayeli and Soledad are considering that the input moves within a limited range of values and that these values are in one-to-one correspondence with the values of the dependent variable, that is, with the size of the shape. So they can cope with SA and they are approaching DA.**

After filling the table they do not try out any more inputs. To answer the questions that appear on the work sheet they analyse only the data of the table and identify the values of the input corresponding to the minimum/maximum size of the shape. This suggests that **Nayeli and Soledad have shifted from considering the variable moving within the interval and they are analysing the table looking for the particular value of the variable that satisfies the condition to give the minimum/maximum size. They are working with SA.**

Once identified these two particular values Nayeli and Soledad answer the third question by writing on the work sheet:

'380 to 420 it gets big
440 to 499 it gets small'

This answer suggests that **after determining the minimum/maximum values Nayeli and Soledad shift back to considering the input moving within the interval, they relate this movement to the corresponding movement of the size of the shape, and based on the analysis of the table they describe this linked movement in a qualitative way. They are coping with DA.**

These results indicate that in this environment this pair has the potential to look at VFR both, in a pointwise/static correspondence and as entities moving in a linked way. So, **to work in this environment has created for this pair the potential to cope with SA and DA and they are able to shift between them.**

On the other hand, the way Nayeli and Soledad answer the third question shows that their analysis depends essentially on the analysis of the tabulated data. In fact, based on this information they extrapolate the global movement of VFR indicating that the value of the size of the shape is increasing when the input moves from 380 to 420 and that it decreases for values moving from 440 to 499. They probably realise that there is a

change in the behaviour of the dependent variable between 420 and 440 but, they have no recorded information about what happens between those two values, therefore, they ignore this subinterval.

These results suggest that **to analyse a table obtained by increasing systematically the input might be a scaffold for coping with DA. Moreover, these outcomes indicate that pupils' capability to cope with SA might be used as a scaffold to cope with DA.**

*Suggested steps not followed in order + Input not systematically increased + Majority of the tested values recorded*

It was observed that pupils who were not following the suggested steps in order (Martha/Mariana, Claudia/Sandra) were filling the table guided essentially by the two first posed questions: 'For what value of X the shape has its biggest size?', 'For what value of X the shape has its smallest size?'. They were not increasing systematically the inputs but they started by running the procedure DIBUJA with the two extreme values of the corresponding interval, or with values near to them. After that they tried some more intermediate values without any order.

This suggests that these pupils were trying to get a global perception of the behaviour of the size of the chosen shape in order to direct their search of the minimum and maximum values of the size. It was observed that the choice of the inputs was guided by the combination of the perception of the variation of the visual image and the already recorded values.

After only few trials they considered located the minimum and maximum values of the size of the shape and in correspondence they identified the values of the input. This suggests that these pupils tended from the first start to consider the variable as a specific unknown. Even if they were able to assign different values to the input, showing in this way that they were able to consider the independent variable moving within an interval, they were using this property as a means to find a particular value. They were coping with SA.

It was observed that in order to answer the third question, 'What happens with the size of the shape between the smallest and the biggest value of X?', these pupils relied both, on the visual experience acquired when looking for the minimum and maximum values of

the size by running the GLP and on the table filled in a non systematic way. Based on these elements they were able to give a global qualitative description of the movement of VFR when the functional relationship involved was monotonic. That is, when a monotonic function was considered these pupils were able to shift to DA.

However, they could not shift to DA when the variables were linked by a non-monotonic functional relationship. Although they perceived that both variables were varying, they could not organize the information obtained when running the GLP. The non systematic way of varying the value of the input in order to explore the behaviour of the size of the shape obstructed the shift to DA. This is evidenced, for example, by the way Martha and Mariana analysed the PALM. The input and the size of this shape were linked by a non-monotonic functional relationship (see Fig. 8.3 for details).

The case of Martha and Mariana

The interval Martha and Mariana have recorded in correspondence to this shape is [500, 749.8]. They start by running the procedure for 500 (Fig. 8.5), the lower bound of the interval, getting a big PALM on the screen.

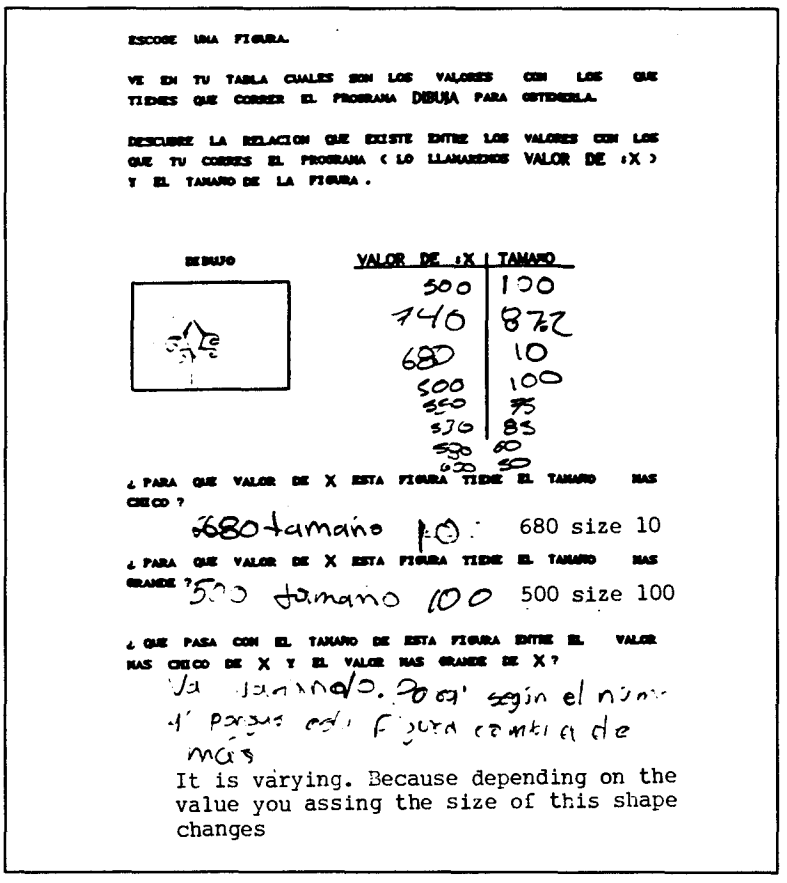



**Fig. 8.5** Martha and Mariana - Analysis of the PALM

The size of the PALM that appears on the screen leads them to forget for a moment the interval they are using and they try a smaller value for the input. They are confused when instead of a smaller PALM they obtain a PYRAMID. They probably expect that a decrease in the input will produce a decrease in the size of the PALM, leading them in this way to find its minimum size. Realising that they have used an input that does not belong to the appropriate interval, they do not record this value. This shows that **Martha and Mariana can cope with SA and approach DA.**

After this they immediately run the procedure with input 740, a value near the upper bound of their interval. They are probably considering that the size of the shape decreases in the interval and they try to confirm this. Because they do not get a substantially smaller PALM they try one intermediate value, 680. Realising that the size obtained is smaller, they verify once more the size corresponding to the lower bound of the interval running the GLP with 500.

This suggests, on the one hand, that they are looking for the minimum size. On the other hand this indicates a reticence to accept a non-monotonic behaviour. After this they run the procedure for four more intermediate values without any systematic order. They correctly locate on their table the values of the input corresponding to the minimum and the maximum value of the size. But after this they cannot give a description of the global movement of the two linked variables. They write 'It is varying. Because depending on the value you assign the size of this shape changes' .

**Their table filled by increasing the input in a non-systematic way is not enough support to help Martha and Mariana analyse the behaviour of VFR when the function involved is non-monotonic. They cannot shift to DA.** However, when monotonic functional relationships were involved they were able to shift to DA.

These results suggest that **to work with non systematically organized data is not an obstacle to shift to DA when monotonic functional relationships are analysed. The visual perception seems to be enough to help pupils make this shift. But, to have the information systematically organized seems to be crucial in order to shift to DA when non- monotonic functions are analysed.**

*Suggested steps not followed in order + Input not systematically increased + Few values recorded*

In general pupils recorded the values they were giving to the input and the corresponding values for the size of the shape. However, there was one pair (Etna/Itzel) who recorded only few values although they run the GLP for several values. They were the only pair who instead of locating maximum and minimum values defined loosely intervals where the size was considered 'big' or 'small'. Instead of giving a global description of the movement of the two variables this pair established a pointwise correspondence between loose defined intervals.

The case of Etna and Itzel

Etna and Itzel used the same method to analyse all the given shapes. The way they analysed the PALM illustrates the approach they used.

The PALM (see Fig. 8.3 for details) is the second figure Etna and Itzel analyse. At the beginning of the session the researcher observes that they run the procedure DIBUJA for several random values all belonging to the appropriate interval. This shows that they are considering that a variable can move within a limited range of values. **That is, Etna and Itzel can approach DA.** Additionally the unsystematic way of assigning values to the input suggests that they are looking for information that would help them answer the three questions that appear on the work sheet. They record only some of the tested values (Fig. 8.6) and the corresponding value for the size. **This shows that Etna and Itzel can cope with SA.**

Instead of locating two particular values for X corresponding respectively to the minimum and maximum value of the size of the shape Etna and Itzel write respectively 'We must assign a number between 500 and 749' and 'We must assign a number belonging to the beginning or/and one belonging to the end' (Fig. 8.6). These answers suggest that the posed questions, although asking for particular values, do not push Etna and Itzel to look at the variable as a specific unknown whose value has to be determined. The idea of variable moving within an interval seems to dominate. The researcher intervenes asking them to answer the posed questions in a more precise way locating the particular values for the inputs corresponding to the maximum/minimum values of the size. This intervention aims to help Etna and Itzel shift to SA. Etna and Itzel react to this by defining in a loose way two subintervals of the original one, where the input can move and produce respectively 'small' and 'big' PALMS. They complete their table by recording these qualifiers in a third column.

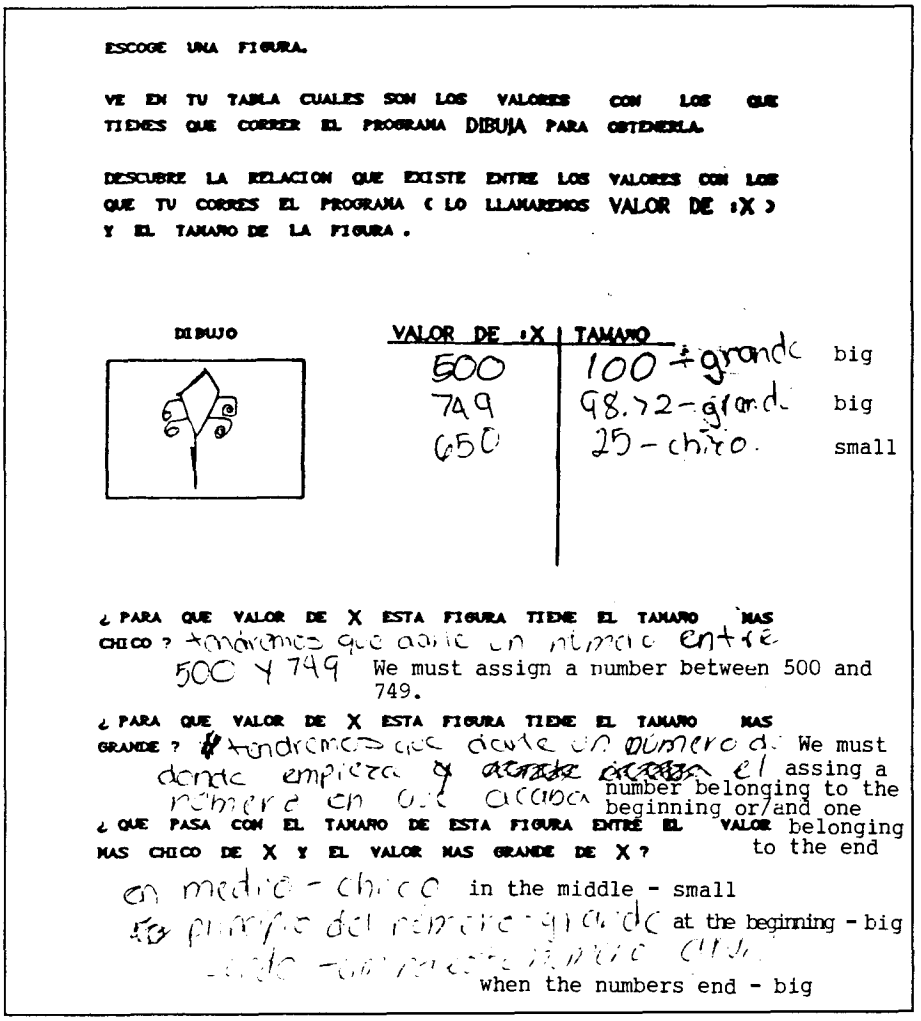



Fig. 8.6 Etna and Itzel - Analysis of the PALM

**These results suggest that although initially Etna and Itzel coped with SA, to run the procedure with different inputs let them to shift to perceiving the variable as a moving entity and they cannot shift back to SA.** The idea of variable as an indeterminate moving entity is dominant for Etna and Itzel. Therefore, they consider enough to locate intervals where the variable can move in order to produce shapes of 'big' and 'small' size, respectively. They cannot look at the variable as a specific unknown. Therefore, in spite of the researcher's requirements they cannot determine the values of the input corresponding to the maximum and minimum size of the shape.

To the third question, 'What happens to this shape between the smallest and the biggest value of X?', Etna and Itzel answer:

in the middle - small
at the beginning - big
when the number ends - big

In this way they do not give a global description of the relationship linking the movement of the input and the size of the shape. They establish only a pointwise correspondence between loose defined intervals. To handle them they use the qualifiers: small and big. This tendency to make a rough distinction between 'small' and 'big' values, seems to contribute to blur the idea of a "continuous" variation of the size in correspondence to the input, and Etna and Itzel do not shift to DA.

To perceive the variable as an entity moving within an interval seems to lead Etna and Itzel to record only few of the values assigned to the input. They are probably considering them as representative of all the possible values the input can take. **Not to have a comprehensive enough tabular record of inputs-outputs seems to be an obstacle for this pair to cope with both, SA and DA and to shift between them.**

*The tabular notation as scaffold to shift to the dynamic aspect (DA) of variables in a functional relationship (VFR)*

The results discussed so far suggest that to construct a comprehensive enough tabular representation of a functional relationship was an important scaffold that helped pupils approach the perception of the global linked behaviour of VFR. On the contrary, this was not easily achieved when pupils had not the support of having constructed a comprehensive enough tabular representation. This is further evidenced by the way Valentin and Ernesto analysed two different shapes: one after recording only few of the tested values; and the other after having constructed a table by increasing systematically the value of the input and recording the majority of the tested values.

The case of Valentin and Ernesto

The first shape Valentin and Ernesto analyse is the PALM (see Fig. 8.3 for details). When the researcher approaches them they have already run the procedure for several values but they have recorded only three inputs and, in correspondence, instead of the value of the size they have written three qualifying adjectives: 'medium', 'small' and 'big' (Fig. 8.7).

Valentin explains to the researcher that the analysed shape is first big then it gets small and after this it gets big again. This explanation suggests that Valentin and Ernesto have perceived how the size of the shape is varying. But they are not relating the variation of the size with the variation of the input. They are analysing only the behaviour of the dependent variable.

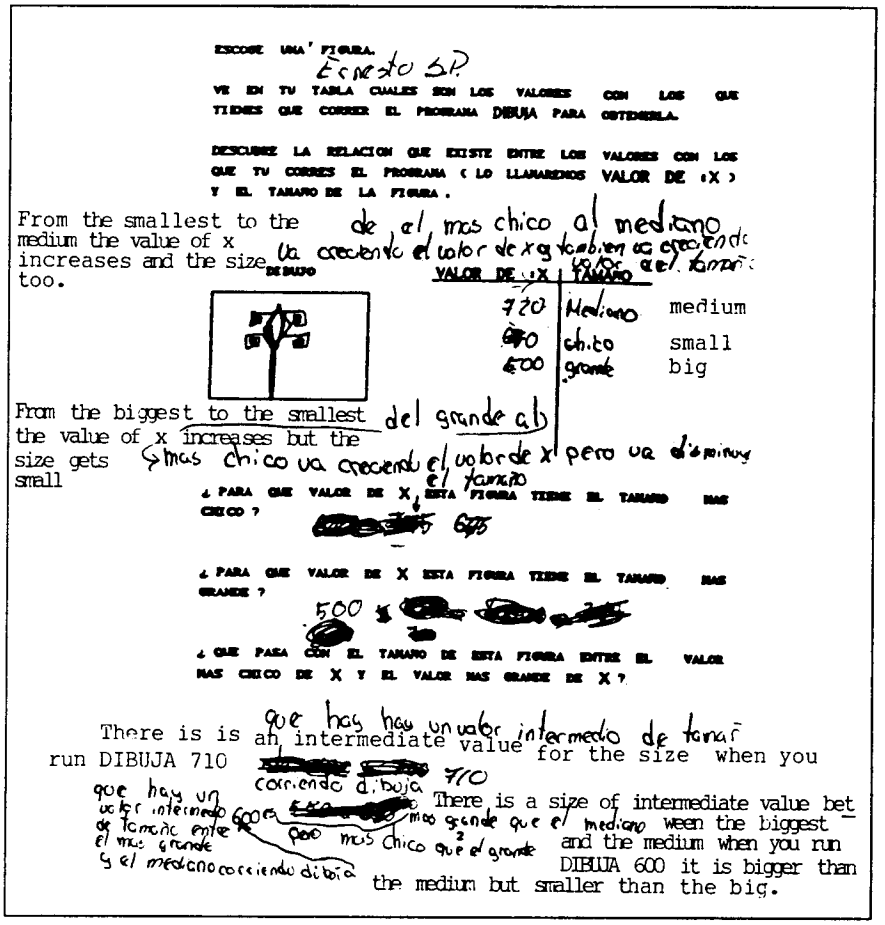



**Fig. 8.7** Valentin and Ernesto - Analysis of the PALM

In order to help them link the behaviour of both the independent and the dependent variable the researcher asks them to locate the values of the input where the size is respectively maximum and minimum. This leads them to establish a first feeble but explicit link between independent and dependent variables. In fact, they say '*at the beginning it is big and in the middle it is small and at the end it has a medium size*'. However, instead of locating the maximum and minimum values they establish a one-to-one correspondence between three zones of the interval loosely defined ('beginning', 'middle', 'end') and three qualifiers that aim to describe the average size of the shape ('big', 'small', 'medium').

A further researcher's intervention aims to push Valentin and Ernesto to establish a pointwise correspondence between particular values. She asks them to determine one particular value for the maximum and one for the minimum. In order to determine these two values it is necessary for Valentin and Ernesto to shift from viewing the variable as moving within an interval, or a subinterval, to viewing it as a specific unknown. They are

able to make this shift and after trying out several inputs they identify the particular values of the input corresponding respectively to the maximum and minimum value of the size. These two values are expected to be a scaffold that might help pupils to perceive the global linked movement of the variables. They are expected to realise that when the value of the imput increases the size of the shape first decreases, it reaches a minimum and then increases. However, when asked what happens to the size of the shape between the smallest and the biggest value of X, the two pupils drop back to their first approach. They describe only the behaviour of the dependent variable.

The researcher intervenes asking them to explain what happens to the size of the shape when the value of X increases. Reacting to this explicit question Valentin says: '*From 500 to 675 X increases but the size gets smaller. From 675 to 720 X increases and the size gets bigger*'.

This very clear explanation shows that Valentin was able to shift to DA and to express verbally the linked behaviour of the input and the size of the shape. Nevertheless, asked to write the explanation on the work sheet Valentin and Ernesto write very confused sentences (Fig. 8.7). They try to use once more the qualifiers they introduced to describe the behaviour of the dependent variable and this seems to obstruct their written explanation.

These results show that **Valentin and Ernesto have the potential to cope with DA. However, to shift to DA is still difficult for them and they need researcher's support guiding them by asking explicit questions.**

After this experience, Valentin and Ernesto spontaneously, that is without researcher's intervention, change their strategy in order to analyse the behaviour of the other shapes. They run the GLP DIBUJA by increasing systematically the values of the inputs and they record the majority of them and the corresponding outputs. After this they analyse the table, locate the values of the input where the size of the analysed shape is maximum/minimum respectively and after this they give a description of the global linked movement of VFR. That is, **they are following the steps in the order suggested by the researcher. The qualitative description they give of the global linked movement of variables shows that they are now able to shift to DA.** This is evidenced by the way this pair approach the analysis of the PYRAMID (for the behaviour of this shape see Fig. 8.3).

To analyse the PYRAMID, Valentin and Ernesto construct first of all a table by increasing systematically the input (Fig. 8.8). After this they locate the minimum value, 380, but they fail to locate the maximum, 420. They consider that the maximum value of the input, 499, corresponds to the maximum value of the size. This mistake is probably due to a vertical analysis of only one side of the table, the column representing the values of the input. Moreover, this suggests that in order to determine these values Valentin and Ernesto are analysing only the numeric table disregarding the visual image.

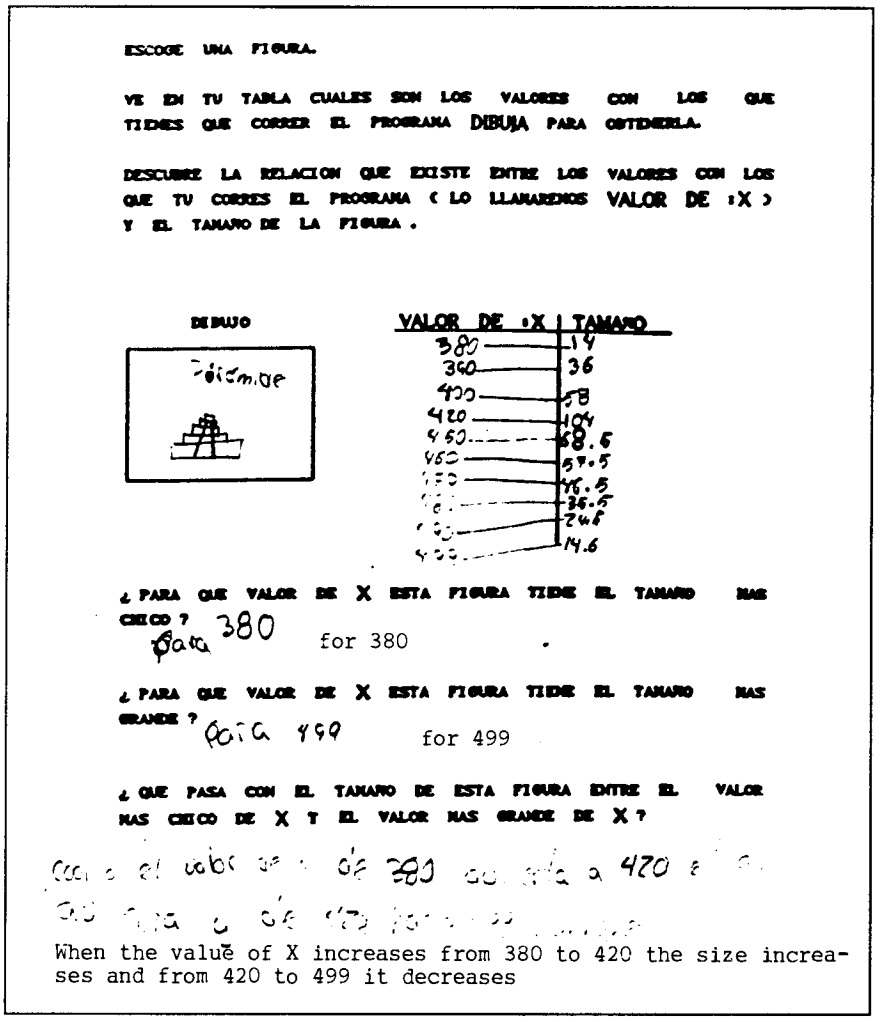



**Fig. 8.8** Valentin and Ernesto - Analysis of the PYRAMID

However, when describing the global relationship linking the movement of the size of the shape to the value of the input they consider the correct value for the minimum. In fact, they write on their work sheet: 'When the value of X increases from 380 to 420 the size increases and from 420 to 499 it decreases'. This suggests that **to describe the global relationship they are combining both, the analysis of the numeric table and the visually perceived behaviour of the shape.**

This answer shows that Valentin and Ernesto have shifted to DA. Additionally it shows that they are able to locate the correct value of the input, 420, corresponding to the maximum size. That is, they can cope with SA.

To analyse the behaviour of this shape Valentin and Ernesto do not need the researcher's intervention. **The visual image, the numeric table and the precise questions that appear on the work sheet seem to provide enough support leading Valentin and Ernesto to perceive and describe precisely the linked movement of VFR**. If compared to the way they analysed the figure prior to this one, it is clear that the new element they have introduced to their analysis is a comprehensive enough numeric table. It seems that **the numeric table they construct by themselves is a crucial element that helps them perceive and describe the connection between the movement of two variables without needing any more the researcher's support. In this context the numeric table systematically constructed by using SA is a scaffold that helps them shift to DA.**

However, although crucial the numeric table was only one element that contributed to create Valentin's and Ernesto's potential to shift to DA. The other intervening elements were: the varying visual image; the precise questions they answered in the given order; and the experience acquired during the prior activity when they received the researcher's support. I would suggest that the dialogue with the researcher helped this pair grasp what they were expected to do. This understanding pushed them to change their strategy and to make a comprehensive enough tabular record. This helped them focuss on the linked behaviour of VFR. All these elements contributed to Valentin's and Ernesto's shift to DA.

#### **8.4.2.4** Variables linked by a constant functional relationship

Pupils' approach to variables linked by a constant functional relationship deserves special attention. The results reported in Table 8.3 show that when analysing a shape whose input and size were linked by a constant functional relationship (see Fig. 8.3 the BIRD and the MONKEY), five pairs could not link the behaviour of the dependent and the independent variable. All these pupils filled in a table and they realised that the size was not changing. This is evidenced by the way they answered the questions asking for the maximum and minimum values. They wrote 'the size doesn't change' (Valentin/Ernesto, Etna/Itzel, Martha/Mariana), 'it does not increase' (Sandra/Claudia), 'nothing happens'

(Nayeli/Soledad). These answers show that pupils were focussing only on the behavior of the dependent variable. Perhaps the fact that there was no change in the dependent variable obstructed their capability to link its behaviour to the movement of the independent one. This indicate that **when working with constant functions the great majority of the case study pupils could not shift to DA although they were able to cope with SA, to approach DA and to make a comprehensive enough tabular record of the inputs-outputs.**

Only one pair, Oscar and Jonathan, were able to shift to DA when working with constant functions.

The case of Oscar and Jonathan

After analysing four shapes of variable size Oscar and Jonathan are analysing the BIRD whose size is constant in the whole interval where it is defined (see Fig. 8.3). The interval these pupils have recorded in correspondence to this shape is [0, 199.8]. To analyse all the previous shapes they have filled the table by increasing the input systematically. They start analysing this shape with the same strategy. They run the procedure for 10, a value near the lower bound of their interval and they systematically increase it by 10 (Fig. 8.9).

After trying out few values they ask the researcher if they can jump directly to try values near the upper bound of the interval because, they say, *'the size seems to be always the same'*. The researcher does not confirm or deny their hypothesis and asks them if they are completely sure about it. This question pushes them to explore more and they continue increasing systematically the input trying some more values.

This shows that based on few data Oscar and Jonathan tend to extrapolate the global behaviour of the dependent variable. But the experience they acquired when analysing the other shapes pushes them to be cautious and they decide to explore more before giving a conclusion. Finally they write that there is no minimum or maximum value. To describe the behaviour of VFR they write: 'The value is increasing and the size remains the same'.

This shows that to solve this task Oscar and Jonathan focus first of all on the pointwise/static correspondence between the values of the input and the values of the size of the shape and they fill in the table with these data. After this their attention goes to the behaviour of the dependent variable, the size of the shape, and they comment *'the size seems to be always the same'*. After recording more values there is another shift of

attention to the behaviour of VFR and they can describe the global linked movement. This different shifts of attention were observed during Oscar's and Jonathan's analysis of all the other shapes from the first start. For all the four shapes they analysed prior to the constant shape, they were able to describe the global linked behaviour of VFR. They were coping with DA. This might suggest that **they got used to coping with DA and, therefore, they were able to use it also for analysing a constant functional relationship.**

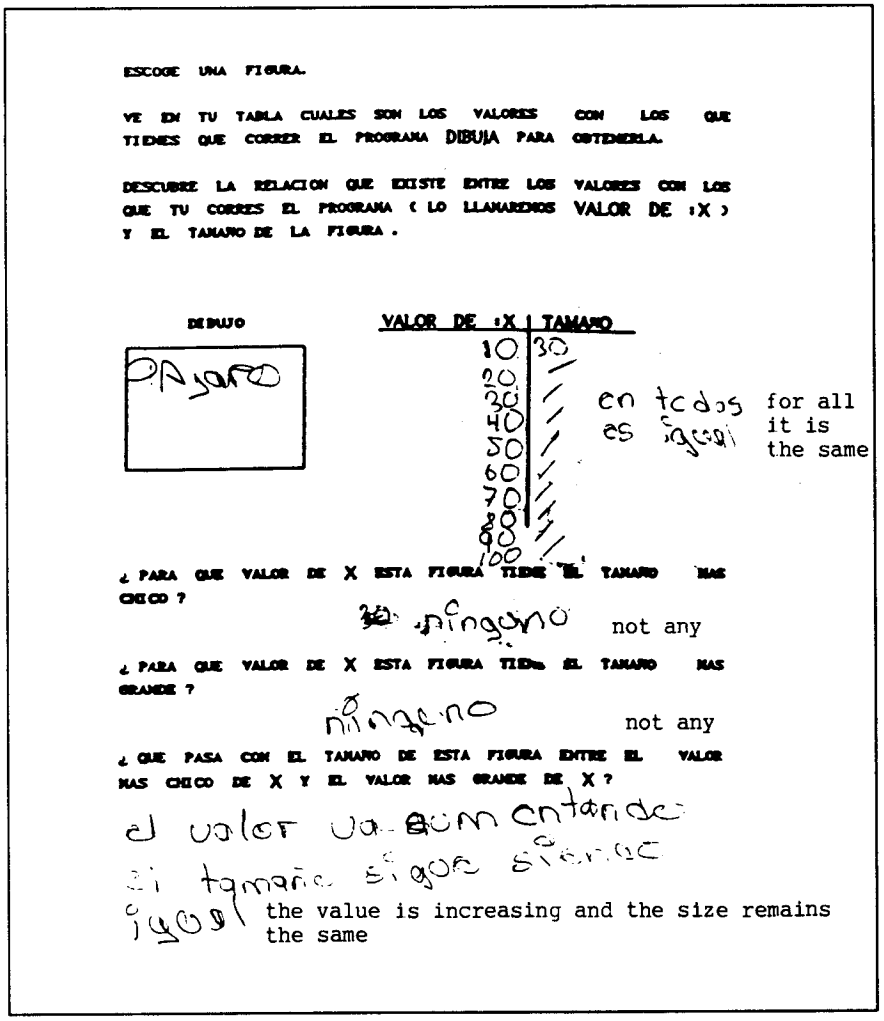
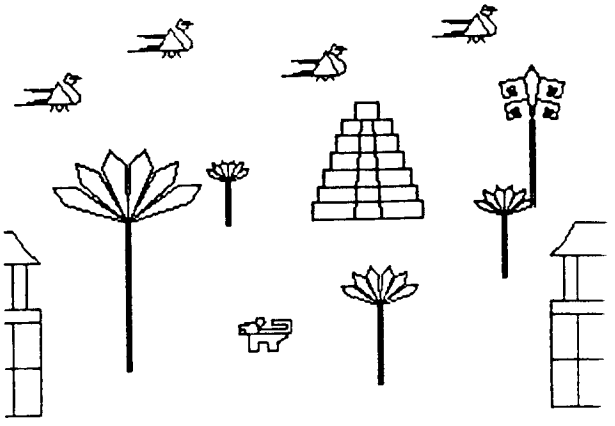
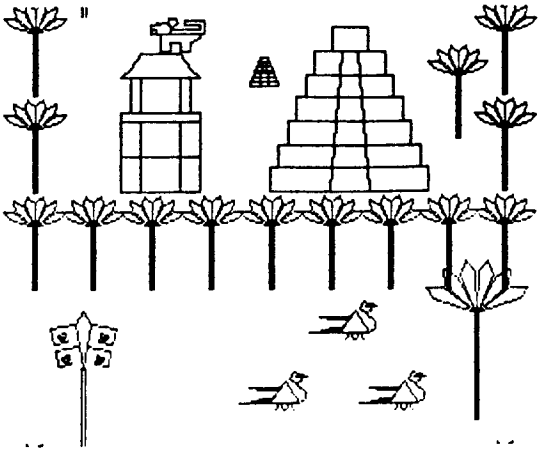



**Fig. 8.9** Oscar and Jonathan - Analysis of the BIRD

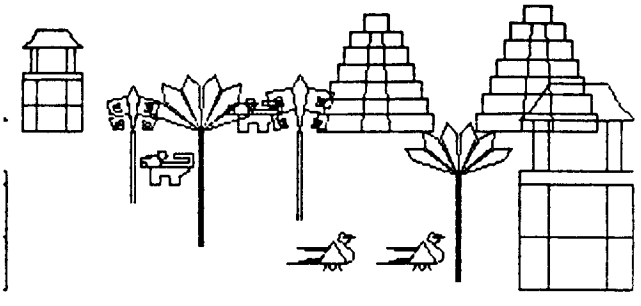
This was not the case for the other pairs. Because each pair could freely chose the order in which they will analyse the six shapes, there were pupils who worked with constant functional relationships at the very beginning of this activity. Therefore, when analysing these functional relationships their experience with DA was still very short. This was, for example, the case for Valentin and Ernesto.



Martha and Mariana



Oscar and Jonathan



Etna and Itzel

**Fig. 8.10** The drawings of the 'jungle'

Although for Etna and Itzel the constant functional relationship was the fourth they analysed, this pair had no experience with DA. In fact, for none of the analysed shaped they were able to make this shift.

The other pairs (Martha/Mariana, Sandra/Claudia, Nayeli/Soledad) were not consistent in the description of the behaviour of VFR. For example, when analysing a non-monotonic functional relationship they tended to focus only on the behaviour of the dependent variable and not on the linked movement. But when working with monotonic functional relationships they could cope with DA.

**The case of Oscar and Jonathan suggests that an appropriate sequence of activities might help create pupils' potential to cope with DA even when a constant functional relationship is analysed.**

After completing the analysis of the six shapes the case study pupils draw the 'jungle'. Figure 8.10 shows the 'jungle' drawn by Etna and Itzel, Martha and Mariana, and Oscar and Jonathan. These drawings show that pupils were able to use the information obtained during the analysis of each shape in order to combine figures of different sizes. To do this they have to cope with intervals, with SA and DA and to shift between them. The different shapes that appear in their drawings suggests that they could work with intervals and cope with SA. However, there are few appearances of the same figure having different sizes. This suggests a difficulty to shift between SA and DA.

## 8.5 The interview

At the end of microworld 2 all the case study pupils were individually interviewed. This aimed to verify for each pupil if a potential to work in Logo with VFR, that is to cope with SA and DA and to shift between them, was created during microworld 2.

### 8.5.1 The questions

Figure 8.10 presents the problems posed during the individual interview. All the case study pupils were posed the same questions.

**Interview after microworld 2**

1. If we run the procedure DIBUJA with values from 200 to 379.9, we get a FLOWER. Can you tell me:
   a) between what values will be the size of the FLOWER when X takes values between 250 and 300?
   b) between what values X can take its values if we want the size of the FLOWER to take values between 55 and 90?
2. Given the procedure:

```
TO FUNCION :X
PRINT :X + 7
END
```

   Use it to answer the following questions:
   a) If we name Y the result obtained, could you fill in the table?

| X | Y |
|---|---|

   b) If the value of X varies between 10 and 20, between what values will Y vary?
   c) If the value of X increases, what happens with the value of Y?
   d) If the value of X decreases what happens with the value of Y?
   e) For what values of X will be Y greater than 23 but smaller than 50?
   f) If X = 49, what is the value of Y?
   g) What must be the value of X if we want Y to be 80?
3. This problem is similar to the previous one, but you will not use the computer to answer the questions.

   If we know that X + 3 = Y,

   a) What values can X have?
   b) If the value of X varies between 10 and 20, between what values will Y vary?
   c) If X increases what happens with Y?
   d) If X decreases what happens with Y?
   e) For what values of X will be Y greater than 15 but smaller than 25?
   f) If X = 26, what is the value of Y?
   g) What must be the value of X if we want Y to be 78?

**Fig. 8.11** Interview questions after microworld 2

For the first problem the computer was loaded with the GLP named DIBUJA. This was the same GLP pupils used during microworld 2. The shape considered was the FLOWER. The functional relationship linking the values of the input and the size of this shape was monotonic increasing (see Fig. 8.3 for details). To answer the questions implied to consider the linked movement of VFR and to stop their movement in order to determine the limits of the interval in correspondence to the limits of the given interval. That is pupils had to cope with SA and DA and shift between them in order to determine the required interval.

In the second problem pupils were asked to type into the computer a given GLP named FUNCION and to use it as an exploring tool in order to answer the posed questions. Question **a** implied to consider that any value could be assigned to the input, and to make a tabular record of some values of the input and the corresponding outputs. That is, pupils had to cope with SA and to approach DA. Question **b** and **e** required to limit the movement of one variables to a given interval and determine in correspondence the interval in which the inter-related variable will vary. This implied coping with both SA

and DA and to shift between them. Questions **c** and **d** implied focussing on the global linked movement of VFR, that is to cope with DA. To answer questions **f** and **g** it was necessary to consider the pointwise/static correspondence between the values of VFR, that is to cope with SA.

Although the activities of microworld 2 were not designed in order to help pupils use in traditional algebra environments the experience acquired in Logo-based environments, a third problem explored if the Logo experience influenced pupils' potential to work with VFR when they appeared in a functional relationship expressed algebraically. The questions in problem 3 were similar to the ones in problem 2. They required to cope with both SA and DA and to shift between them (questions **b** and **e**), to cope with DA (questions **c** and **d**) and to cope with SA (questions **f** and **g**). Question **a** aimed to explore if pupils considered that any value could be assigned to X in the expression $X + 3 = Y$.

During the individual interview the researcher intervened repeating the original question (QR) or formulating it again (QRF). When pupils needed stronger support the researcher intervened by negotiating with them the solution of the problem (SN), that is by discussing their approach in order to lead them to a correct solution. The type of intervention provided are presented in Table 8.4.

| Symbol used | Type of intervention |
|---|---|
| QR | Original question repeated |
| QRF | Original question repeated and formulated again |
| SN | Negotiating the solution of the problem |

**TABLE 8.4** - Type of interventions provided during the interview on variables in a functional relationship (VFR)

### 8.5.2 Results

Table 8.5 shows the type of intervention pupils received in order to answer the interview questions. Symbol "+" indicates that the question was answered without researcher's help. Symbol '-' indicates that in spite of researcher's help the pupil could not answer the question.

| Pupils' names | Problems | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | | 2 | | | | | | | 3 | | | | | | |
| | a | b | a | b | c | d | e | f | g | a | b | c | d | e | f | g |
| Valentín | QR | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + |
| Ernesto | QRF | + | + | + | + | + | QR | + | + | + | QR | + | + | + | + | + |
| Etna | SN | QRF | QRF | QR | + | + | + | + | QR | + | + | + | + | + | + | + |
| Itzel | QRF | QR | QRF | + | + | + | + | + | QR | + | + | + | + | QRF | + | QR |
| Martha | + | + | QRF | QR | + | + | + | + | + | + | + | + | + | + | + | + |
| Mariana | SN | + | + | + | + | + | SN | + | QR | + | SN | + | + | SN | + | + |
| Oscar | QR | + | + | QRF | + | + | QRF | + | + | + | + | + | + | + | + | + |
| Jonathan | QR | + | + | QR | + | + | + | + | + | + | + | + | + | + | + | + |
| Claudia | - | + | + | SN | + | + | QR | + | + | + | QRF | + | + | - | + | - |
| Sandra | QRF | QR | + | QRF | + | + | QRF | + | + | + | QRF | + | + | + | + | + |
| Nayeli | QRF | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + |
| Soledad | QR | QR | + | QR | + | + | QR | + | QR | + | + | + | + | + | + | + |

**TABLE 8.5** - Type of intervention received by each pupil during the interview on variables in a functional relationship (VFR)

The results of the individual interview reported in Table 8.5 show that Martha was the only pupil who was able to answer question **a**, problem 1, without researcher's help. She could cope with SA and DA and shift between them.

Claudia was the only pupil who in spite of researcher's intervention could not answer question **a**, problem 1. She run the GLP for several values, commented that to each input corresponded a different size and that when the value of the input was increasing the size was increasing. However, she could not determine the interval in which the size of the shape was varying for inputs between 250 and 300. Claudia could cope with SA and DA but in spite of researcher's intervention a potential to shift from one to the other could not be created.

To answer question **a**, problem 1, Etna and Mariana needed researcher's guide. Their comment *'for input 250 the size is 35, for 240 it is 40"* shows that they saw a one-to-one correspondence between the input and the size of the shape. That is, they could cope with SA. But to focus on the global linked movement of the input and the size, and to determine the interval in which the size was varying they needed researcher's help. That is, they needed help for coping with DA and for shifting between SA and DA. This help was provided by negotiating with them the way to approach a correct solution. All the

other pupils were able to answer question **a,** problem 1, after it was repeated or formulated again (e.g. *'We can vary the value of the input from 250 to 300. Find out between which values will vary, in correspondence, the size of the shape'*).

Two spontaneous approaches to this question were observed. There were pupils (Valentin, Ernesto, Itzel, Oscar, Sandra, Soledad) who first of all deduced a general quantitative rule linking the input and the size of the shape. Valentin, for example, was typing and talking: *'For 250 it is 35, for 260 it is 40, for 270 it is 45, for 280 it is 50, for 290 it is 55.'* Suddenly he stopped typing and said: *'For 300 it is 60. This means that when the input increases 10 the size increases 5'.* To give a quantitative description of the functional relationship was a spontaneous pupils' tendency not promoted by the researcher. Moreover, during microworld 2 pupils were not asked to deduce quantitative rules linking VFR. This pupils' behaviour might be considered a spontaneous approach to experiencing linear functional relationships. These answers show that they were able to cope with SA and DA. They were able to describe DA qualitatively and also quantitatively.

In order to help these pupils to determine the asked interval the researcher intervened by repeating or formulating the original question again. This pushed pupils to give a qualitative description of the linked behaviour of VFR and after this to integrate both, the quantitative and the qualitative description in order to determine the limits of the interval. This is illustrated by the following fragment of the dialogue between Valentin and the researcher. In his first approach Valentin has run the procedure DIBUJA for 250, 260, 270, 280 and 290 and he has deduced that when the input increases 10 the size increases 5.

R(esearcher): *'Can you tell me between what values will be the size of the flower when X takes values between 250 and 300?'*

V(alentin): *'It is changing, when the value of the input increases the size increases, between 250 and 300'.*

R.: *'Between 250 and 300 varies the value of the input. And the size?'*

V.: *'Between 35 and 60'.*

R.: *'But how do you know that the size in 300 will be 60? You did not try input 300. Do you need to try it?'*

V.: *'No, I am calculating it'*

A second approach to answer question **a**, problem 1, was observed. There were pupils (Jonathan, Nayeli) who answered the original question by giving a qualitative description

of the linked movement of the variables (e.g. *'when the value increases the size increases'*). After the researcher's intervention and based on the data obtained by running the GLP and that were still visible on the screen, they were able to determine the required interval approximating its bounds by running the GLP. These pupils did not give a quantitative description of the functional relationship.

**These results indicate that all the pupils could cope with SA. They had the potential to cope with DA and to shift between SA and DA. However, to shift between these two aspects characterizing VFR was not easy for the great majority of the case study pupils. To make these shifts the researcher's help was still crucial.**

After the experience of answering question **a,** problem 1, the great majority answered question **b** without researcher's intervention. For three pupils (Itzel, Sandra, Soledad) the question has to be repeated. For one pupil (Etna) it was formulated again (*'We want to know all the values of the input to obtain the size of the FLOWER between 55 and 90'*). That is, pupils had the potential to cope with SA, DA and to shift between them.

Table 8.5 shows that all the pupils were able to fill in a table recording the inputs to the given procedure and the corresponding outputs (question **a**, problem 2). For three pupils (Etna, Itzel, Martha) it was necessary to formulate the question again (e.g. *'Run the procedure FUNCION to fill in this table. Use X to represent the inputs and Y to represent the outputs'*). This results confirm pupils' capability to consider that any value can be assigned to an input and their capability to cope with the one-to-one correspondence between inputs and outputs. That is, pupils could cope with SA and approach DA.

When shifts between SA and DA were required (questions **b** and **e**) the majority needed the researcher's intervention. To intervene by repeating or formulating the questions again (e.g. *'If I assign to X any value between 10 and 20, for example, 11, 11.5, 12, and so on, but always values greater than 10 and smaller than 20, between what values will Y vary?'*) was crucial for helping the great majority understand the questions and answer them correctly. **This confirms pupils' potential to cope with SA and DA and to shift between SA and DA although they still needed researcher's help in order to understand what they were expected to do.**

The answers given to questions **c** and **d**, problem 2, show that **all the pupils could cope with DA for a monotonic functional relationship.** To answer these questions they did not need researcher's intervention. Pupils had also a potential to cope with SA when the

value of X (or Y) was given and the value of Y ( or X) has to be calculated (questions **f** and **g**, problem 2). When the value of X was given and they were asked to calculate the corresponding value of Y, none of the pupils needed researcher's intervention (question **f**). But when the value of Y was given and the value of X has to be calculated (question **g**), some pupils (Etna, Itzel, Mariana, Soledad) had difficulties to understand the question. This has to be repeated. These pupils tended to assign the given value to X and to calculate Y as in question **f.** Researcher's intervention helped them understand the question and assign the given value to Y in order to calculate the corresponding value of X.

To answer questions **f** and **g**, pupils did not run the procedure FUNCION. They considered the given expression :X + 7 (see Table 8.11, problem 2) and calculated the value of Y (or X) mentally after assigning to X (or Y) the given value. This is shown, for example, by the dialog sustained by the researcher with Soledad. To answer these questions Soledad is not running the procedure. She is only looking at the procedure FUNCION displayed on the screen.

R(esearcher): *"If X is 49 what is the value of Y?*
S(oledad): *"If X is 49, Y is ... 49 plus 7, that is, 56".*
*R:* *"And for Y to be 80 what must be the value of X?"*
*S:* *"87".*
*R:* *"Y is 80".*
*S:* *"Ah, sorry".*
*R:* *"What is its value?"*
*S:* *"... If this is Y, the result is ... if Y is 80, ... what is the value of X? ... it is, 73".*

The answers given to problem 3, show that **the majority of the case study pupils were able to work with VFR when they appeared in a functional relationship expressed algebraically.** Table 8.5 shows that all the pupils considered that any value could be assigned to X in the expression X + 3 = Y (question **a**). After assigning a value to X, they were able to calculate the corresponding value of Y. The great majority were able to limit the variation of one variable and to define in correspondence the interval in which the other variable will vary (questions **b** and **e**). To answer these questions some pupils needed the researcher's intervention who repeated or formulated the question again (e.g. '*X can assume any value between 10 and 20. That is, it cannot be neither smaller than 10 nor greater than 20. If X assumes these values, between what values will Y vary?*'). Mariana was the only pupils who needed stronger researcher's intervention in order to

answer these questions. For example, to answer question **b**, instead of considering that X was assuming values between 10 and 20 she assigned these values to Y and she calculated the corresponding values of X. She had difficulties to understand the question. After the researcher indicated her that the given values had not to be assigned to Y, she rectified assigning them to X. After this, she was not able to shift spontaneously from considering the pointwise/static correspondence between X and Y to viewing their global linked movement. That is, she could not shift from SA to DA. After researcher's intervention asking her to assign different values to X she was able to define the corresponding interval for the values of Y. She was also able to consider that the value of Y will move within the interval. This shows that Mariana had difficulties to shift from SA to DA but with help she could do it.

Table 8.5 shows that all the pupils were able to analyse the algebraic expression to deduce the linked behaviour of the VFR involved (questions **c** and **d**). That is, they could cope with DA.

**These results show that faced with a simple algebraic expression involving VFR the case study pupils had the potential to cope with SA and DA and to shift between them. In order to shift between SA and DA (question b and e) some pupils needed researcher's intervention.**

These outcomes contrast with the results obtained at the beginning of the study when the written questionnaire was applied. Table 5.11 (Chapter 5) shows that the great majority of the case study pupils gave no answer to items involving algebraic expressions with VFR. This shows that they could not cope with SA and DA. Only Valentin, Mariana and Oscar on occasion considered that VFR were linked by a one-to-one correspondence. With researcher's help received when they were individually interviewed after answering the written questionnaire also Jonathan, Sandra, Claudia, Nayeli and Soledad were able to consider VFR in one-to-one correspondence.

This contrast in the capability pupils had to cope with VFR involved in a simple algebraic expression before and after having worked in microworld 2, suggests that **the experience acquired during their work in Logo-based environments created a potential to work with VFR. Moreover, pupils were able to use this potential to work with VFR also outside Logo-based environments.** It was not the aim of this study to investigate how to work in Logo-based environment was influencing pupils work with traditional algebraic tasks. However, these results highlight these changes in pupils' behaviour and they point to the necessity for further investigation focussing on

how the work in Logo-based environment might be linked to traditional algebra tasks and help in this way to improve pupils' work with different characterizations of variable.

## 8.6 Discussion and conclusions

Microworld 2 aimed to help create pupils' potential to cope in Logo with variables linked by a functional relationship (VFR). It was considered that a VFR is characterized by a static aspect (SA), that is when the pointwise static correspondence between two quantities is considered, and a dynamic aspect (DA), that is when two quantities are viewed as moving entities and their global linked behaviour is perceived.

A pupil was considered to have developed a potential to work with VFR in Logo when s(he) was able to cope with both the static aspect (SA) and the dynamic aspect (DA) characterizing VFR and to shift from one to the other when required by a Logo-based problem.

Because the case study pupils had no prior formal algebra instruction, they were not faced with algebraic expressions representing functional relationships, but special Logo based activities involving the idea of VFR were proposed. The design of the activities aimed to use only pupils' arithmetic background and visual perception of varying shapes as scaffold to help them approach the idea of VFR.

The results obtained during the activities of microworld 2 and when the pupils were individually interviewed show that a potential to determine numeric intervals and to approach DA, that is to consider a variable moving within a numeric interval was created. Moreover, pupils could also accept that when the value of the input moves within an interval the size of the shape may be constant and when the value of the input moves from one interval to another the outcome may change. This may be considered a first non formal approach to piecewise functions and it suggests that activities might be designed to help develop pupils potential to cope with this kind of functions prior to formal introduction to them.

All the case study pupils were able to consider VFR in one-to-one correspondence, that is they could cope with SA. This was observed when pupils worked with both monotonic and non-monotonic functional relationships. When dealing with monotonic functional relationships they showed also a potential to cope with DA and shift between

SA and DA. However, when non monotonic functional relationship were analysed the majority had difficultly to shift from SA to DA, although with help they could make this shift.

A link between pupils' potential to cope with DA and the strategy used to fill in numeric tables was observed. It was found that pupils that filled in a table by increasing systematically the input coped better with DA than pupils who were assigning non systematically ordered values to the input. That is, systematically organized data helped pupils to shift to DA. On the contrary non-systematically organized data obstructed this shift. Pupils who filled in tables with only few values perceived SA but they could not shift to DA.

The great majority of the case study pupils could coupe with SA when working with constant functional relationships but they had difficulty to shift from SA to DA. However, the successful approach of one pair suggests that an appropriate sequence of activities might help pupils shift to DA even for constant functional relationships.

These results show that in these specially designed Logo-based environments a potential to cope with VFR prior to any formal algebra instruction was created. To work with VFR pupils coped with some very sophisticated mathematic notions, namely, the idea of interval, maximum, minimum, monotonicity. To approach these notions they used their prior arithmetic knowledge and the visual perception of variation.

Four elements were crucial in helping pupils succeed with VFR:

- the design of the activities
- the researcher's interventions
- the Logo environment
- the role of language

*The design of the activities*

When designing the activities it was considered that to work with VFR implied coping with the idea of functional relationships and with some non-trivial mathematic notions underlying them, for example:

- variables moving in numeric intervals
- monotonicity
- maximum and minimum

These are complex notions and in the usual school curriculum their introduction is postponed to upper grades. They are usually introduced in a formal and abstract way in

pre-calculus courses, after pupils have received formal algebra instruction. Pupils have then to cope with these notions when dealing with functions presented analytically, by graph, in tabular notation or by arrow diagrams. It was found (see 2.2.2, Chapter 2) that students have difficulties to make sense of these concepts and to handle them.

In order to work with VFR the case study pupils were not introduced to usual function representations (e.g. analytic expression, graphs, arrow diagrams) but they had to work with Logo-based situations involving variable elements linked by a functional relationship. A number of notions underlying the idea of functional relationship were embedded in these situations. Pupils were not introduced to them prior to face the activities. It was expected that the design of the tasks and the researcher's interventions would push pupils to approach these notions using as scaffolds their arithmetic background and the visual perception of the varying size of a shape.

To define intervals and consider variables moving within them, to fill in tables, to determine maximum and minimum values were not emphasized as important actions per se, but they were presented as intermediate activities that would help pupils perceive the global relation linking the values of the input and the size of the shape. This was emphasized as a useful information that would help them draw the 'jungle'. That is, to cope with these notions was a necessary step in the achievement of a goal. During these activities pupils coped with these notions without having to deal with their formal definition. The results obtained show that in these circumstances the potential to approach these non-trivial mathematic concepts was created. Pupils were able to handle each one of these notions separately, for example, when they were asked to determine an interval or to locate a maximum/minimum value of the dependent variable and the corresponding values of the independent one. They were also able to handle them all together when they were analysing the global behaviour of VFR. However, it was also observed that when some of these notions were approached as a scaffold for analysing the global behaviour of a shape, there were pupils who handled them correctly, while when they were approached per se, sometimes they failed (see 8.4.2.3, The case of Valentin and Ernesto). This shows that to approach these notions as part of a wider project might help create a potential for algebra beginners to work with them at an informal level.

This approach contrasts with other approaches (e.g. Markovits et al. 1988) that try to determine possible causes obstructing pupils' understanding of the concept of function and its underlying notions presented in a formal way and try to offer instructional procedures to correct students' misconceptions. In the present study the aim was to help young pupils use their arithmetic knowledge and the visual perception to approach some

complex notions prior to any formal introduction to them. The results obtained show, on the one hand, the feasibility of designing situations where young pupils prior to any formal algebra instruction can approach some of the concepts underlying a functional relationship and that are usually regarded as difficult to handle. On the other hand, these results suggest the possibility to create pupils' potential to cope with both the static and the dynamic aspects of VFR and to shift from one to the other. This potential might be a basis upon which, later on, a more formal concept of functional relationship and its underlying notions might be constructed and formal definitions might be easier accepted and handled by pupils.

*The Logo environment*

The basic element underlying the activities of microworld 2 was a GLP written by the researcher (a GLP named DIBUJA). This GLP was loaded in the computer and it was not visible for pupils. They had to use it as an input-output device and to discover the relationship between the variation of the values of the input and the corresponding outputs. It could be argued that because the GLP was hidden to the user it could have been written in any other programming language, that combines graphic and text modes, and not necessarily in Logo. This is true, considering that the only required conditions were the possibility to assign an input and to obtain in correspondence both, the image of a shape whose size might vary depending on the value of the input, and a message where the size of the shape was given by a number. Nevertheless, it was considered that to assume that they were working in Logo and running a GLP, instead of considering that they were using a computer program whose language and structure they could hardly imagine, might help pupils use their experience with VGN in order to approach VFR.

The results obtained suggest that this assumption was correct. In fact, as a consequence of the experience acquired during microworld 1, where they were writing and running GLPs, pupils considered that any value could be assigned to the input. Therefore, during microworld 2 they did not need help in order to use the GLP DIBUJA as an exploring tool assigning different arbitrary values to the input. After this, the researcher's intervention and the design of the activities pushed pupils to develop strategies for limiting the values of the input to those values that would produce the desired effects (see 8.4.1.3). In this way although viewing the input as capable to take any value they were able to limit these to an interval. While doing this pupils were constructing a more complex idea of variable: it represents any value but on occasion its value might be restricted to an interval. In this way they were using their experience of coping with VGN to approach VFR.

Further on, in order to analyse the link between the values they were assigning to the input and the outputs obtained, pupils applied this new view of variable. In fact, they were able to restrict their exploration to each one of the located intervals by running the GLP for values belonging only to the chosen interval. In that way they were using the given GLP as a tool to explore how some of its own components, the inputs and outputs, were behaving. To do this, pupils had to shift to a more abstract level and consider that the variable elements involved in the problem could be considered as objects whose behaviour might be studied.

This shift was favored by the use of the given GLP as an input-output device. In fact, this use emphasized the inputs and the outputs hiding the process that was transforming the input into the output and that could have distracted pupils' attention from focussing on the linked behaviour of inputs and outputs. Additionally, this use of the GLP and the visual feedback of a shape of changing size in conjunction with the questions that guided pupils' exploration helped create a potential to focuss on both SA and DA and to shift between them.

*The researcher's interventions*

Another crucial element that contributed to create pupils' potential to work with VFR was the researcher's intervention. For example, to emphasize that *all* the inputs giving the same shape had to be find out (see 8.4.1.2) was crucial to help pupils determine intervals and approach DA, that is, to consider the value of a variable moving within an interval. To pose precise questions during the activity pushed pupils to organize the information and to look at the problem from a different perspective. This helped them to shift from SA to DA (see 8.4.2.3, The case of Valentin and Ernesto). To ask pupils to reflect on their approach helped a pair analyse carefully a constant functional relationship and shift to DA in order to explain the linked behaviour of VFR (see 8.3.2.4 The case of Oscar and Jonathan).

It was observed that when, due to the organization of the work with the whole group, the researcher did not intervene opportunely as, for example, in the cases of Etna/Itzel and Martha/Mariana (see 8.4.2.3), pupils had difficulties to cope with DA. On the contrary, when helped by the researcher as, for example, during the individual interview the great majority were able to cope with both SA and DA (see 8.5.2 Table 8.5).

Researcher's intervention was also crucial to help pupils shift between SA and DA (see 8.4.2.3 The case of Valentin and Ernesto, 8.5.2). Even if the great majority of the case study pupils were able to cope with each aspect separately, they needed help to shift

from one to the other. This might be considered as a point of difficulty for pupils working with this characterization of variable. Using the framework proposed by Tall and Thomas (1991) this might be interpreted as a difficulty to shift between the sequential/analytic and global/holistic modes of thinking. However, the results obtained show that with help the majority of the case study pupils were able to shift between SA and DA.

The results obtained show that to create a potential for pupils to cope with VFR the researchers intervention was crucial. The intervention was particularly significant to create pupils' potential to cope with DA and to shift between SA and DA.

*The role of language*

During microworld 2 the majority of the case study pairs engaged in discussions negotiating how to collect the data and how to interpret them. On occasion their talk was a self-directed loud speech helping them to reflect on the problem. Due to the conditions of the research these dialogues were not recorded; however, they were observed by the researcher and this influenced her analysis of the results. These dialogues testify pupils' engagement with the activities and confirm the potential of Logo based environments to provoke mathematic discussions and reflections.

The researcher observed that during these discussions and in order to answer the posed questions, pupils were using the information provided by the computer (e.g. the visual image, the display of numeric values representing the size of the shape), their knowledge about integers and their order, their conception of variable as representing any value and their capability to cope with SA. The interaction of all these elements in their discussions contributed to create pupils' potential to work with VFR.

To conclude this chapter it has to be stressed that the activities of microworld 2 were designed to help create pupils' potential to work with VFR in Logo-based environments. It was found that this potential was created. Additionally it was found that pupils' potential to work with VFR was not restricted to Logo-based environments. In fact, when individually interviewed after working in microworld 2 the majority of the case study pupils were able to solve also a non-Logo-based problem involving VFR (Fig. 8.10, problem 3). To answer the questions of this problem the case study pupils showed to have a potential to cope with SA and with DA and shift between them. These results suggest that to work in microworld 2 created for the great majority of the case study pupils a potential to work with VFR that could be extended also to a non-Logo setting.

# CHAPTER 9

# VARIABLE AS SPECIFIC UNKNOWN

## 9.1 Introduction

Prior studies have shown that algebra beginners can conceptualize a specific unknown when asked to solve very simple verbal problems in which it is explicitly asked to determine its value (Karplus et al., 1981). They are also able to solve one step equations (Kieran, 1984). However, it was found that pupils have great difficulties with the manipulation of the specific unknown in equations involving various combinations of operations, numbers and literal terms, that is, when dealing with equations that cannot be solved in one step (Davis, 1975; Galvin and Bell, 1977; Küchemann, 1980; Matz, 1980; Kieran, 1984, 1988; Filloy and Rojano, 1989; Herscovics and Linchevski, 1991). There are researchers (e.g. Kieran, 1988) who suggest that some of these difficulties might be a consequence of the way pupils are led to interpret a specific unknown when first introduced to this idea in elementary school. For instance, to emphasize the solution of an equation by referring to the corresponding inverse operation may lead pupils to conceive a literal symbol in an equation as 'the result of a set of inverse operations' (Kieran, 1988, p. 95) instead of viewing it as representing a number within a sequence of operations. These results suggest that some of the difficulties algebra beginners have with equation solving may have their origin in the traditional teaching that emphasizes manipulative skills but does not stress the idea of specific unknown as representing a particular number that can be determined considering the given problem constraints.

In the present study it was assumed that the idea of VSU has the following basic characteristics:

- the VSU represents a specific non-arbitrary value
- the value of a VSU can be calculated
- to calculate the value of a VSU the constraints posed by the data must be considered

Pupils were considered to have developed a potential to work in Logo with the idea of VSU when:

- pupils were able to conceive that to solve the given problem it was necessary to determine a value that was not given
- pupils' work reflected the assumptions of the basic characteristics underlying the idea of VSU

## 9.2 The activities

The activities of microworld 3 (Table 9.1) were designed in order to help create pupils' potential to approach in Logo the idea of VSU. They did not aim to teach pupils how to operate on or with VSU. The work sheets used are included in Appendix 2 (work sheets SU-1 and SU-2). The abbreviation VSU is used to indicate both a VSU involved in a Logo-based problem and a VSU that appears in an equation.

| Microworld 3.- The variable as a specific unknown | | |
|---|---|---|
| **Activity** | **Aim** | **Work sheet** |
| Drawing a 5-steps staircase | Determine the value of two VSUs in order to solve a Logo problem. | SU - 1 |
| Fitting the astronaut into the rocket's door. | Direct pupils' attention to solving an equation with more than one appearance of VSU. | SU - 2 |

**TABLE 9.1** - Overview of the activities of microworld 3

During the first activity (work sheet SU-1) pupils were asked to draw a 5-steps staircase in order to complete the drawing of a 'rocket' displayed on the screen of the computer (Fig. 9.1). It was emphasized that exactly 5 steps had to be drawn, that all the steps should have the same dimensions, and that the 5 steps had to fit exactly the given dimensions for the total height and width of the staircase. The total width and height of the staircase, respectively 60 and 70, were displayed on the screen. The VSUs of the problem and the relationship linking them with the data were not explicitly indicated. Pupils were not asked or suggested to write equations.

In order to solve the posed problem pupils needed:
- to identify the two specific unknowns
- to relate them to the given data
- to calculate their values
- to use the calculated values to draw the 5-steps 'staircase'

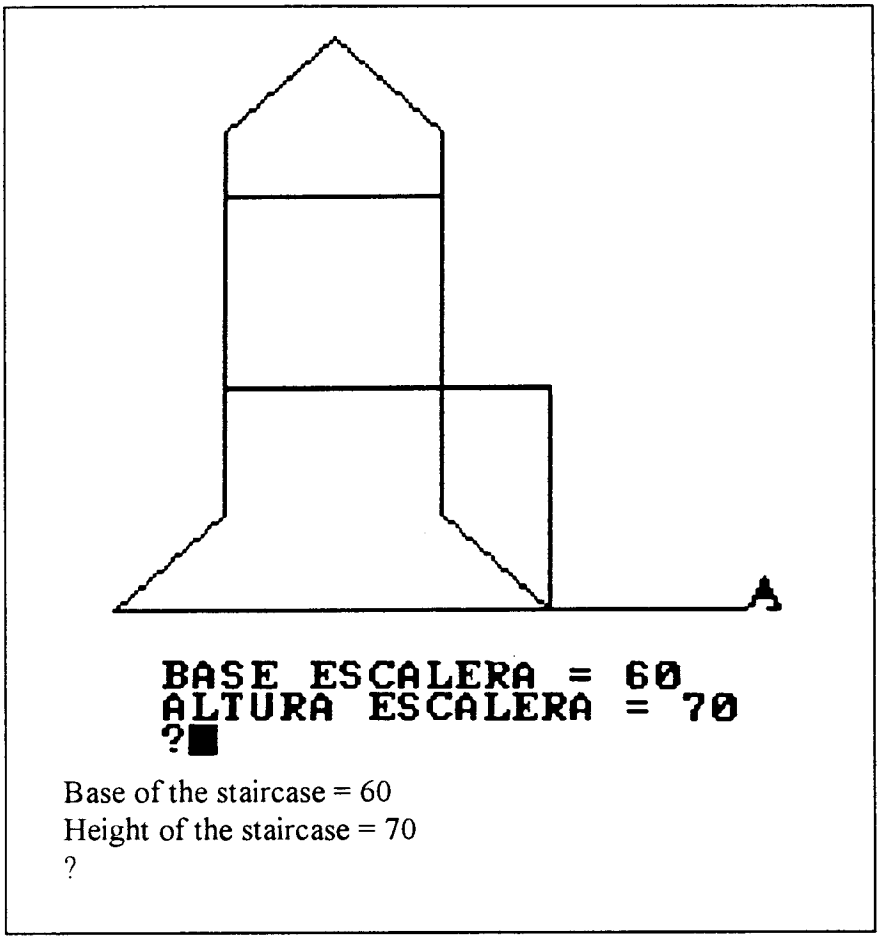



**Fig. 9.1** Screen display corresponding to the first activity involving variable as specific unknown (VSU).

In the second activity (Appendix 2, work sheet SU-2) the drawing of a 'rocket' with a 'door' and the height of the 'door' were displayed on the screen (Fig. 9.2). Pupils were given a GLP named ASTRONAUTA that drew an 'astronaut'. They were asked to draw an 'astronaut' that fit exactly the 'rockets door'. They had to determine the precise value of the input to the procedures that drew an 'astronaut'. Pupils were told that this value could be determined by trial and refinement but this would be very time consuming. The alternative approach suggested by the researcher was to try to solve the equation that appeared on their work sheet (Fig. 9.3). The researcher explained that in the given equation each term containing the VSU represented a part of the total height of the 'astronaut' and the addition of all the terms represented its total height. The value of the VSU obtained by solving the given equation, had to be used as input to the procedure ASTRONAUTA in order to obtain an 'astronaut' of the required size.

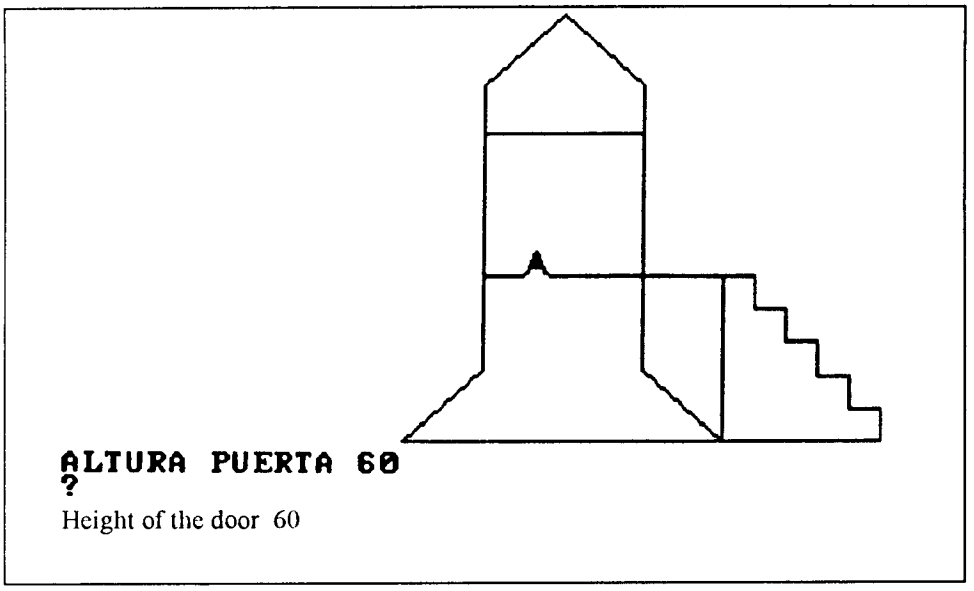



Fig. 9.2 Screen display corresponding to the second activity involving variable as specific unknown (VSU)

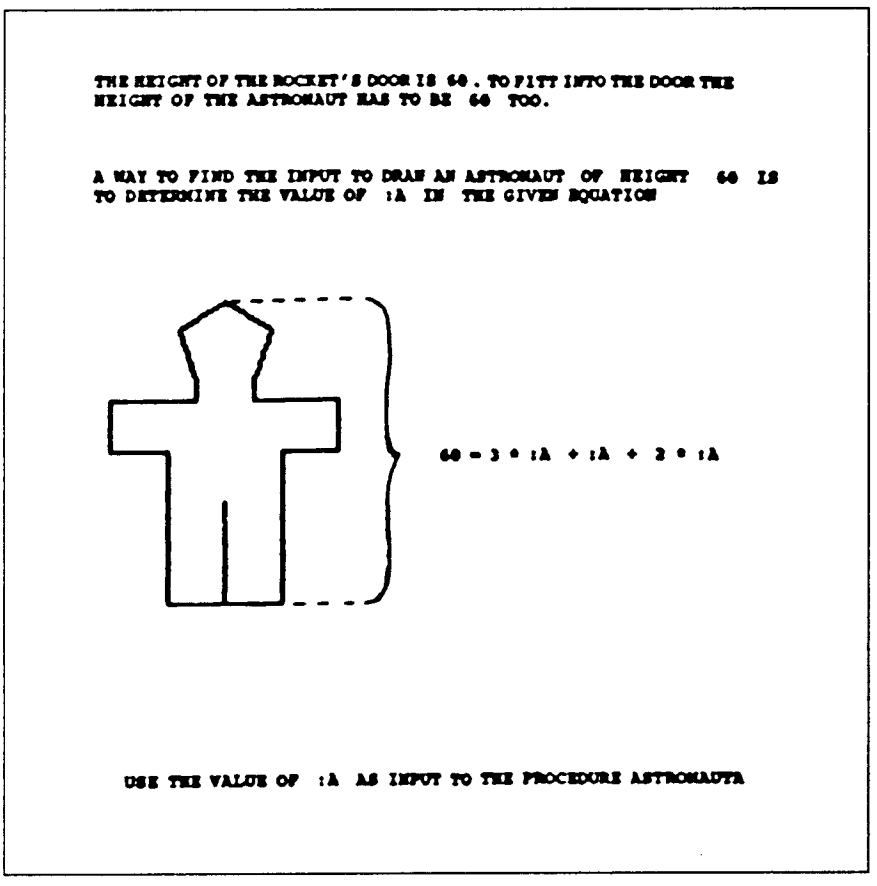



Fig. 9.3 Work sheet SU-2

In contrast with the first activity, in this task the VSU was explicitly indicated and its relationship with the data of the problem were explicitly given by the equation. To solve

the given equation required grouping similar terms, that is to operate with VSU. It was not expected that pupils could do this. However, this task was expected to create pupils' potential to conceptualize a VSU that appeared in an equation with multiple appearances of it and to determine its value in a situation in which to use arithmetic resources was not enough.

For both activities pupils had the option to clear the screen and to go back to the original situation by running an already loaded GLP. This GLP was not completely new to pupils. In fact, it was a slightly modified version of a GLP written during Microworld 1 by the pupils involved in the project 'The universe': the GLP to draw a 'rocket'. To work with a procedure already familiar to pupils, even if not written by them, was expected to encourage pupils to use it in order to re-establish the original situation on the screen whenever they needed it. This aimed to facilitate pupils' involvement in the task encouraging the testing of their hypotheses.

## 9.3 Researcher's intervention

During the activities of microworld 3, the researcher intervened in order to help create pupils' potential to work in Logo with VSU. In particular, her interventions aimed to help pupils:

- conceptualize a VSU
- focus on the constraints of the problem in order to take them into account to calculate the value of a VSU
- use the value of a VSU in order to solve the given problem

## 9.4 Results

The results obtained are discussed considering:

- pupils' potential to conceptualize a VSU, that is to realise that in order to solve each one of the given problems a specific numeric value has to be calculated
- pupils' potential to calculate the value of a VSU taking into account the constraints of the problem
- pupils' potential to use the obtained value for a VSU in order to solve the given problem

### 9.4.1 Conceptualizing a variable as specific unknown (VSU)

During the first activity pupils had to determine the value of two VSUs but no equation was explicitly given (see Fig. 9.1). The great majority of the case study pupils approached the task by dividing by 5 the given total height and width of the 'staircase' in order to obtain the height and width of the individual 'step'.

This indicates that the great majority of the case study pairs had grasped the structure of the posed problem. They were able to analyse it and to identify the unknown values that needed to be determined. **This suggests that in this particular environment the great majority of the case study pupils were able to conceptualize a VSU.** The only pair who needed individual researcher's help were Oscar and Jonathan.

The case of Oscar and Jonathan - Failure to conceptualize a variable as specific unknown (VSU)

Oscar and Jonathan approached the first activity in a very competitive way. Aiming to be the first of the group to get the 'staircase' they did not analyse the problem, they did not consider the data. They tried to draw quickly the 5 steps by trial and refinement guided exclusively by the visual feedback. They did not care if the steps had all the same width and height but they tried only to draw a total of 5 steps to enter the 'rocket'.

They were not successful and realising that Valentin and Ernesto had already solved the problem they copied from them the command

REPEAT 5 [FD 14 LT 90 FD 12 RT 90]

Asked by the researcher if they could explain why the values used for the width and the height of each step were working, they could not answer. This suggests that Oscar and Jonathan were not conscious that there were two specific unknowns in the problem and that these could be calculated. The researcher intervened stressing that the activity was not meant to be competitive. She suggested to them that they take their time in order to analyse carefully the posed problem. In order to help them approach the idea of VSU she asked Oscar and Jonathan to make explicit verbally which were the values they needed to determine. She also asked them if the values could be calculated. Without additional help Oscar and Jonathan identified the two specific unknowns. They decided which were the calculations they needed to do and calculated the values of the two specific unknowns considering the data of the problem. After this they drew their own 'staircase'. These

results indicate that with help Oscar and Jonathan were able to analyse the given problem, to identify the two VSUs and to calculate their values.

The results of the questionnaire applied at the beginning of the study showed that Oscar and Jonathan were able to solve simple given equations (see Table 5.9, items 2.2 and 3, Chapter 5). However, this ability seems not to have been helpful to approach the first activity. This suggests that when solving equations they were probably applying automatically rules learned in primary school without necessarily approaching the idea of VSU while doing it. This suggests that **to be able to solve simple equations is no guarantee that the characteristics of the idea of VSU has been grasped.** It is expected that an experienced student will be able to work with VSU without needing to make explicit the ideas that characterize it. However, in order to approach that capability **it is necessary not only to help pupils acquire manipulative skills but also to offer them initially experience encouraging reflection on the basic ideas characterizing a VSU.**

During the second activity (see Fig. 9.2) pupils were asked to determine the value of the input to the GLP ASTRONAUTA that would produce an 'astronaut' of size 60. This value was also represented by the VSU involved in the equation

$$60 = 3 * :A + :A + 2 * :A$$

that appeared on their work sheet (Fig. 9.3).

It was observed that all the pairs approached this task trying to determine the value of the VSU, that is the input to ASTRONAUTA that would draw an 'astronaut' of size 60. This indicates that in this Logo environment in which the VSU was explicitly indicated, all the case study pairs were able to conceptualize it. **Although they considered that any value could be used as input, they were able to consider that there was only one particular value that could solve the posed problem.**

### 9.4.2 Calculating the value of a variable as specific unknown (VSU)

When working on the first activity pupils had to determine the value of two VSUs. It was observed that none of the case study pupils wrote an equation. After having identified the VSUs, namely, the height and the width of the individual step, all the pupils

calculated them by dividing by 5 the given values for the total height and width of the staircase. This is evidenced by the record of the calculations made by Nayeli and Soledad (Fig. 9.4).

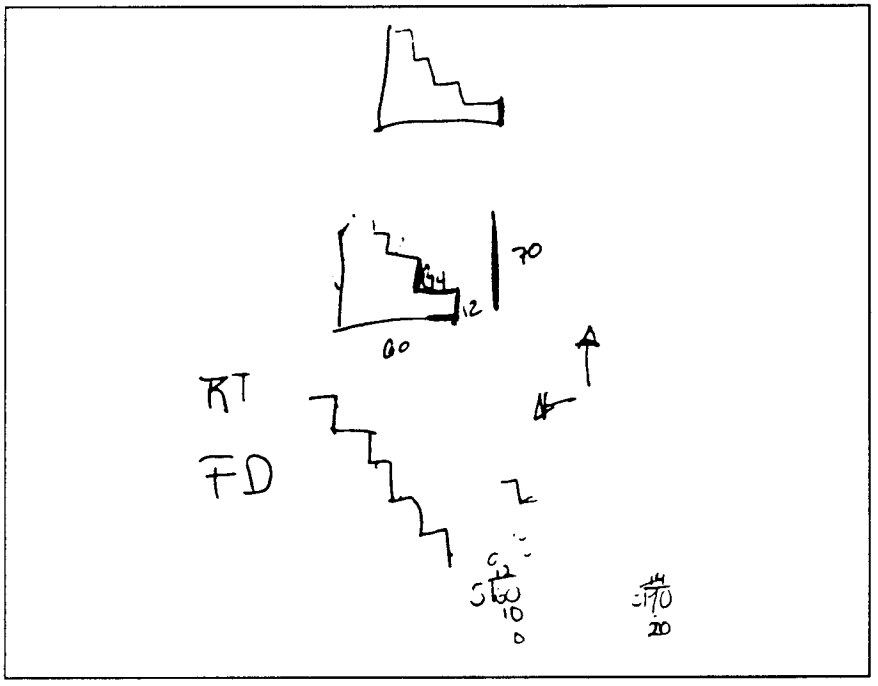



**Fig. 9.4** Nayeli and Soledad - Calculations made to determine the values of two variables as specific unknown (VSUs)

This approach suggests that the case study pupils had the potential to analyse the given problem situation. They were able to decide which were the operations needed to determine the VSUs. They were able to carry them out considering the constraints of the problem, that is, that exactly 5 steps had to be drawn and that these had to fit exactly the given dimensions for the total height and width of the 'staircase'.

When individually asked by the researcher to explain how they solved the posed task, two pairs, Etna/Itzel and Martha/Mariana, answered by giving a general method for solving problems similar to the posed one. Etna, for example, explained: *'You have to divide both the total height and the total width by 5'*. The use of the terms 'total height' and 'total width' instead of the given particular values suggests that **this task created for these pupils a potential to go beyond the solution of the particular example and to devise a solution of the general problem subsuming to this the particular given case.** That is, these pupils were able not only to identify the specific unknown of the particular problem they were facing but, they could also generalize this to all similar cases. They were able to consider that independently of the particular size of the

'staircase' there were always two specific unknown values that had to be calculated in order to solve the problem. Moreover, their answer was giving a general method for determining the values of the two VSUs that could be applied to any instantiation of the problem.

Besides the verbalization of the deduced general method for solving the general problem, **Martha and Mariana spontaneously tried to symbolize this in Logo. This suggests the presence of a spontaneous shift from an arithmetic to an algebraic approach to VSU.** It is considered that the VSU is perceived arithmetically when it is viewed as a specific value that can be calculated by operating with numbers. The VSU is perceived algebraically when it is viewed as a specific value that can be represented by operating on general numbers.

The case of Martha and Mariana - Shifting from an arithmetic to an algebraic approach to variable as specific unknown (VSU)

After explaining to the researcher that to solve the posed task they divided by 5 the width and the height of the 'staircase', Martha and Mariana spontaneously, that is without researcher's intervention, tried to write a GLP in order to draw a 5-steps 'staircase' for any 'rocket'. They wrote:

```
TO ESCALERAS :X
REPEAT 5[FD ALTURA ESCALERA/5 LT 90
                (Stair's height)
          FD BASE ESCALERA/5 RT 90]
                (Stair's base)
END
```

Although syntactically incorrect, this symbolization of the general solution of the problem shows that Martha and Mariana were trying to represent the general height and width of the 'staircase' and they were able to conceptualize two VSUs whose specific values had to be calculated by dividing by 5 the general height and width.

They asked for help in order to run their GLP. They explained to the researcher that they were trying to write a GLP to draw any 5-steps staircase but, they did not know how to assign the value to :X in order to give it both, the values of the stair's height and width. The case study pupils had no experience in writing GLPs with two inputs and the design of this study did not consider to introduce pupils to GLPs with more than one input.

Therefore, after congratulating Martha and Mariana for their good, although not successful, attempt the researcher invited them to approach the second activity.

The case of Martha and Mariana suggests that their prior Logo experience, where they wrote GLPs and they approached the idea of general numbers, have created a potential:

- to generalize the posed problem
- to shift from using arithmetic to using algebraic ideas to determine the value of a VSU
- to try to represent formally the general method devised to solve the general problem

Although Etna and Itzel did not try to express their generalization symbolically, their verbal description (*'You have to divide both the total height and the total width by 5'*) suggests a similar tendency. **These results indicate that this environment created for these pairs a potential for shifting from an arithmetic to an algebraic approach to VSU.**

This shift was not observed in the other four pairs. Although this was not originally planned, the researcher intervened in order to create pupils' potential to deduce a general approach to the problem and to shift from an arithmetic to an algebraic approach to VSU. Pupils were explicitly asked to find a method to draw a 5-steps 'staircase' for any value of the total height and width of the 'staircase'. All the four pairs approached this requirement by giving several correct particular examples, but they did not verbalize a general method. They run the GLP named COHETE that draw the 'rocket', the base and the height of the 'staircase' (Appendix 1, work sheet SU-1) for different inputs. In correspondence to each input they obtained two specific values for the height and width of the 'staircase'. To draw the '5-steps staircase' they divided these values by 5 and they used the results to draw the 5 steps. They explained that they would use the same approach independently of the particular value. To give an example, Sandra and Claudia run COHETE with a randomly chosen value, 111. They obtained 55.5 for the width and 64.75 for the height of the 'staircase'. They divided these values by 5 obtaining 11.1 and 12.95 and they wrote REPEAT 5 [FD 12.95 LT 90 FD 11.1 RT 90] in order to draw the 5 steps. They explained that this method could be used for any particular value assigned to the input.

This pupils' approach suggests that to give particular examples was considered a way of illustrating the general method devised. In fact, they considered that the VSUs as well as the solution method did not vary for different instantiation of the problem. This indicates pupils' capability to conceptualize a VSU and to devise general methods for determining

it, even if they were not able yet to shift from an arithmetic to an algebraic approach to VSU. This way of solving the problem might be related to the one used in ancient time by Egyptians and Babylonians where particular examples were used to illustrate a general method for solving similar problems (see 2.4.4, Chapter 2). In ancient time there was not a symbolic language appropriate to express general methods. Similarly, the case study pupils did not handle yet algebraic language and their use of the formal Logo language was still very incipient. However, the spontaneous approach to symbolization observed in Martha and Mariana indicates that **to work in Logo-based environments might create pupils' potential to look for a symbolic way of expressing their general methods and create in this way pupils' potential to shift from an arithmetic to an algebraic approach to VSU.**

During the second activity (Appendix 2, work sheet SU-2) pupils used different strategies in order to determine the value of the VSU involved in the posed problem. Three approaches were observed:

- the given equation was ignored and the given GLP ASTRONAUTA was used as an exploring tool that helped pupils determine the value of the input by trial and refinement
- the equation was considered and the VSU was determined by trial and refinement or it was calculated
- after determining the value of the VSU using the GLP ASTRONAUTA as an exploring tool the given equation was used as verifier

Table 9.2 presents the strategy used by each pair. It shows that to solve the posed problem three pairs, Valentin/Ernesto, Martha/Mariana and Claudia/Sandra, completely ignored the given equation. They determined the value of the VSU, that is the value of the input by running the given GLP named ASTRONAUTA for different inputs and, guided by the image, they approximated it by trial and refinement. Asked by the researcher if they could solve the equation, these pairs said that they were not understanding the given expression very well and that they preferred to use a trial and refinement approach running the given GLP.

This suggests that although these pupils were able to conceptualize a VSU for the given problem and they worked in order to determine its value, they had difficulty to approach an equation involving several appearances of VSU. A reason for this might be that they did not know how to handle the three appearances of the VSU in the given equation.

This hypothesis is supported by the way Valentin and Ernesto approached the solution of the given equation when explicitly asked by the researcher to try it.

| Pairs | Equation ignored | Equation considered | | |
|---|---|---|---|---|
| | Input determined by trial and refinement | VSU determined by trial and refinement | VSU calculated | Equation used as verifier |
| Valentín, Ernesto | Yes | No | No | No |
| Etna, Itzel | No | Yes | No | No |
| Martha, Mariana | Yes | No | No | No |
| Oscar, Jonathan | Yes | No | No | Yes |
| Claudia, Sandra | Yes | No | No | No |
| Nayeli, Soledad | No | No | Yes | No |

**TABLE 9.2** - Pupils' strategies to determine the specific values of an input when it could be determined by solving an equation with multiple appearances of the VSU.

The case of Valentin and Ernesto - Ignoring the given equation

To determine the value of the VSU Valentin and Ernesto ignore the given equation. They use the GLP as an exploring tool and, by trial and refinement, they find that to draw an 'astronaut' of height 60 the value of the input to the procedure ASTRONAUTA must be 10. Although they have determined the precise value for the input and they have solved the posed task, the researcher asks them to try to solve also the given equation:

$$60 = 3 * :A + :A + 2 * :A$$

After observing it Valentin and Ernesto divide 60 by 3. This might indicate that to solve the posed equation they are cutting it after the first term. In this way they obtain an equation of a more familiar type and, therefore, a one they can handle. But their

approach might also indicate a spontaneous attempt to consider the three appearances of the VSU in the given equation although disregarding their numeric coefficients. The equation chosen for this task did not give the opportunity to the researcher to deduce what was originating the approach used by Valentin and Ernesto. Asked to explain it, they argue that their approach must be wrong because they obtain 20 instead of 10. In fact, 10 is the value they have determined by trial and refinement and which correctness they have verified by running the given GLP. These contradictory results lead Valentin and Ernesto to doubt the way they tried to solve the equation. Arguing that the value they have to obtain is 10 they conclude that after dividing 60 by 3, the result has to be divided also by 2. This pushes them to look at all the terms of the equation, but they cannot find a way to justify these operations. They argue that they have no explanation for their method but that it works.

These results show that Valentin and Ernesto can accept that the value of the VSU of the given problem can be determined by solving the equation and that the VSU involved in the equation is the same that can be determined by trial and refinement using the GLP. However, their lack of experience in working with equations with multiple appearances of the VSU leads them to ignore the equation and to approach the solution of the problem by using the more primitive method of trial and refinement.

In this environment Valentin and Ernesto could not operate with the VSU. However, their attempt to consider all the terms intervening in the equation although not successful suggests that **to offer pupils the possibility to determine the value of a VSU by a method they already handle (e.g. trial and refinement) and to face them after this with the requirement to determine the same value by solving an equation, might be a good starting point to introduce them to the necessity of operating with VSU.** To introduce pupils to operating with VSU was not the aim of microworld 3, therefore, this activity was not further developed and pupils were not introduced to the operation with VSU. Valentin and Ernesto were told that their conclusion to divide 60 by 6 was correct, even if, they were not able yet to justify it.

This was not the first time the case study pupils were asked to operate with algebraic expressions during this study. In fact, when individually interviewed after microworld 1 pupils were asked to add three terms similar to those that appeared in the equation given in this activity. These were

3 * :X  :X  2 * :X

and pupils had to group them in order to determine the total height of the 'astronaut' (see 7.5.1, Fig. 7.13, question 5, Chapter 7). They were not faced with an equation but they were asked to write an algebraic expression equivalent to the addition of the three terms. On that occasion Valentin and Ernesto were two of the seven pupils who correctly grouped the terms without researcher's support.

A reason for these different behaviours might be that when interviewed after microworld 1, the recent experience of having worked with VGN as well as the questions made pushed pupils to consider the variable as a VGN. On that occasion the majority of the case study pupils were able to operate with VGN. However, during the present activity pupils were led to consider the variable as a VSU. **To consider that a particular value had to be determined for the VSU seems to have inhibited them to operate with it. To consider a variable as a VSU pushed pupils to look immediately for its value and they were not able to group first similar terms.**

In order to group the terms they would have to shift from considering the variable as a VSU to consider it as a VGN. This shift was not made and pupils could not group the terms. This suggests that **to help pupils approach the operation with VSU it might be helpful on the one hand to offer them experience in working with VGN and operating on and with it. On the other hand, to offer them also experience in working with VSU in simple situations where its value can be determined without operating with it. Finally, to put them into situations in which it is necessary to shift from VGN to VSU in order to operate with VSU before calculating its value.**

Table 9.2 shows that three of the case study pairs (Etna/Itzel, Oscar/Jonathan, Nayeli/Soledad) considered the given equation in order to determine the value of the VSU involved in the problem. Each one of them worked with the equation in a different way.

Etna and Itzel approached the problem by solving the equation by trial and refinement. They substituted the same value to all the appearances of the VSU. This shows that they were considering that the different appearances of the literal symbol had all the same value and that all of them were referring to the same VSU whose value they were trying to determine. This indicates that **they were not presenting the 'polysemia' error.** To consider that all the appearances of VSU had to have the same value might be influenced by the experience of having written GLPs with several appearances of the same variable

representing the same value. Another reason might be that they were told that there was only one particular value that gave the correct solution of the problem.

To make the calculations indicated in the equation Etna and Itzel considered the precedence of the operations. This is evidenced by the parenthesis they added to the expression when trying to determine the value of the VSU (Fig. 9.5).

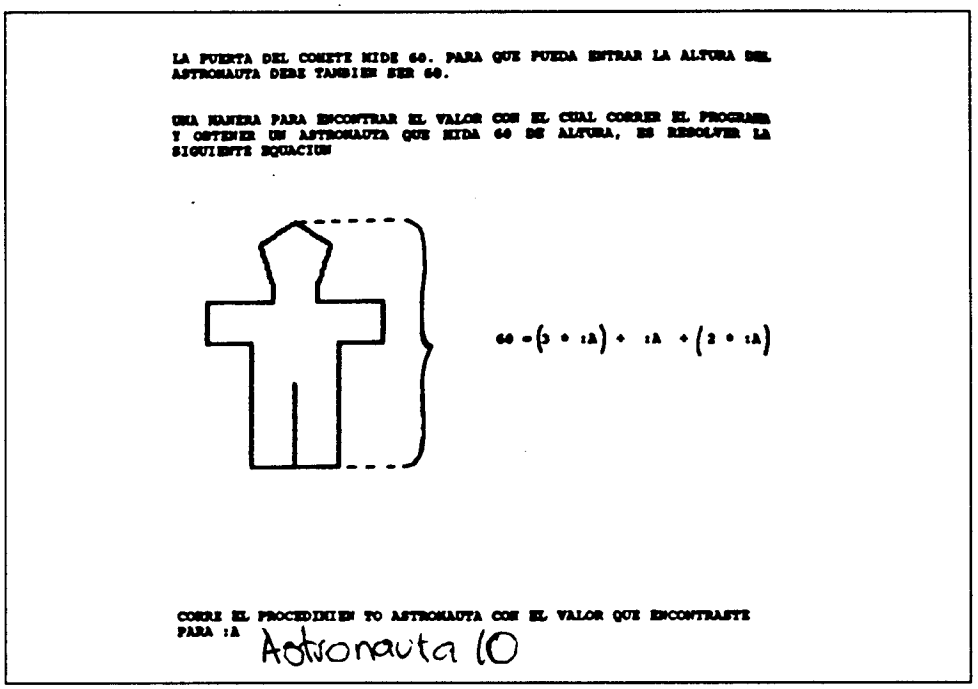



**Fig. 9.5** Etna and Itzel - Approach to an equation with multiple appearances of variable as specific unknown (VSU)

Nayeli and Soledad were also a pair who approached the problem by considering the equation. **They cut the equation after the first term arguing very convinced that it was not necessary to consider the other terms because they were not essential.** In order to help them rectify the methods they were using the researcher intervened. She invited them to verify the correctness of the calculated value by using it as input to the GLP. They were surprised and confused by the unexpected size of the 'astronaut'. Due to the time restrictions of the study and to absences of these girls to some sessions after this one this pair could not continue with this activity.

However, this experience suggests that **Logo-based environments might be designed in order to help pupils who have already developed erroneous methods for solving equations, to become concious that these are not appropriate.** This might be a

starting point to introduce them to different methods leading to a correct solution of equations.

Oscar and Jonathan developed a different and very interesting way of using the given equation in order to solve the posed task.

The case of Oscar and Jonathan - The equation as verifier

After looking at the equation Oscar and Jonathan turn to the given GLP. For example, similarly to Valentin and Ernesto they use the GLP to determine the searched value for the input by trial and refinement. However, after this they do not consider the task finished and they substitute the value determined by trial and refinement to all the appearances of the VSU in the equation. They make the corresponding calculations and they obtain the given correct height for the 'astronaut'. After this they consider the task completed.

Asked by the researcher why they are substituting in the equation the value determined by trial and refinement using the GLP, they explain that it is in order to assure that they have found the correct value. This shows that **Oscar and Jonathan are using the equation in order to verify the correctness of the value determined by trial and refinement using the GLP.** This suggests that they consider that in order to solve the posed task it is necessary to determine one precise value for the VSU involved. They seem to assume that the visual feedback could have led them to accept as correct an approximately correct value and, on the contrary, only one precise value can be the solution of the equation. For that reason they consider necessary to verify the value determined by trial and refinement using the GLP, by substituting this value to the multiple appearances of the VSU in the equation.

The results obtained during these activities show that **all the case study pupils were able to determine the value of the VSU involved in the problems. This shows that these Logo-based environment created pupils' potential to considered that a VSU has a specific value that can be calculated.**

### 9.4.3 Using the calculated value of a variable as specific unknown (VSU)

During the first activity (Appendix 2, work sheet SU-1) it was observed that after calculating the values for the two VSUs four pairs (Etna/Itzel, Oscar/Jonathan, Martha/Mariana, Nayeli/Soledad) spontaneously used these values in order to draw the 5-steps 'staircase'. This suggests that to calculate the values of the two specific VSUs was a meaningful activity for these pairs and they knew how to use the obtained values in order to solve the posed problem.

On the contrary there were two pairs (Valentin/Ernesto, Claudia/Sandra) who, after having calculated the two values, used only one of them to draw both, the width and the height of the steps. The visual feedback obtained did not push them to review their approach or to verify the values obtained for the two VSUs in order to identify why they were not obtaining the expected 'staircase'. Valentin and Ernesto tried to correct the drawing by trial and refinement. Claudia and Sandra considered that even if the 'staircase' obtained was not fitting exactly into the drawing it was solving the posed problem in an acceptable way.

This suggests that these two pairs were not linking the activity of determining the values of the two VSUs with the solution of the problem. A reason for this might be a difficulty to shift from paper and pencil calculations to the Logo environment and to use in Logo the results obtained in paper and pencil. Another reason for the behaviour of these two pairs might be a difficulty to manage simultaneously all the intervening elements: the turtle's turns; the perception of the repeating pattern; the values of the two VSUs. However, the researcher's intervention asking them why they were not considering the calculated values was enough to push these pairs to review their approach and to use the values of the two VSUs in order to draw the stairs.

These results show that for this Logo-based activity **a potential to use the calculated value of a VSU for solving a given problem was created.**

During the second activity only two pairs (Etna/Itzel, Nayeli/Soledad) determined the value of the VSU by solving the given equation. After this, only one pair (Etna/Itzel) spontaneously used the determined value as input to the procedure ASTRONAUTA. This shows their capability to use the calculated value in order to solve the given problem. Nayeli and Soledad used the determined value only after being asked by the researcher to do this. This shows that this pair needed help in order to use the value of the VSU to solve the given problem. All the other pairs determined the value of the

VSU and solved the problem at the same time. In fact, they were approximating the solution of the problem by trial and refinement and in this way they determined the correct value for the inputs to the GLP, that is,the value of the VSU. Therefore, the two steps of determining and using the value of the VSU could not be distinguished.

## 9.5 Discussion and conclusions

The activities of microworld 3 aimed to help create pupils' potential to conceptualize a VSU involved in a Logo-based problem and to approach the basic characteristics underlying the idea of VSU, namely:

- the VSU has a specific non-arbitrary value
- the value of a VSU can be calculated
- to calculate the value of a VSU the constraints posed by the problem must be considered

It was assumed that a pupil had developed a potential to approach in Logo the idea of VSU when:

- (s)he was able to conceive that to solve the given problem it was necessary to determine a value that was not given
- her (his) work reflected the assumptions of the basic characteristics underlying the idea of VSU

The results obtained show that in these Logo-based environments all the case study pupils developed the potential to conceptualize a VSU, that is, to consider that in order to solve the given problem a specific value had to be calculated. This supports Karplus's (1981) findings reporting pupils' capability to reason with unknowns when working with very simple word problems.

To solve a simple Logo-based problem in which the value of a VSU could be determined in one step (e.g. by dividing to particular numbers) none of the case study pupils wrote an equation. Only one pair used trial and refinement. The great majority analysed the problem situation, considered the given constraints and carried out the necessary arithmetic operations in order to determine the value of the VSU.

Given the options to determine a VSU by trial and refinement using a given GLP as an input-output machine or by solving a given equation with several appearances of the

VSU, the great majority of pupils avoided the equation. However, there were pupils who tried to solve it or they used the equation to verify the value of the VSU already determined by trial and refinement using the GLP (see 9.4.2). These observations suggest that to present an equation with multiple appearances of VSU as an alternative to the trial and refinement method to solve a problem might help pupils become familiar with this type of equations. This kind of experience might also help them link the idea of VSU that appears in a problem with that of VSU that appears in a equation. Moreover, this might lead pupils to try to solve this type of equations. This environment did not help create pupils' potential to operate with VSU. This was not the aim of microworld 3. However, the results obtained suggest that environments similar to this one might be designed in Logo to help create pupils' potential to operate with VSU. Further research is needed in order to investigate this possibility.

The results of the present study suggest that pupils' capability to operate on VSU was influenced by the characterization of variable emphasized. In fact, it was observed that to consider a variable as a VGN facilitated operating with it, on the contrary to view it as a VSU obstructed the operation with it (see 9.4.2, The case of Valetin and Ernesto - Ignoring the given equation). This finding should be considered when activities aiming to help pupils operate with VSU are designed.

Pupils who worked with an equation with multiple appearances of VSU in this Logo environment did not present the so called 'polysemia' error observed by Filloy and Rojano (1989) for pupils of similar ages and older when working in usual paper and pencil environments. The case study pupils of the present study considered that the same value had to be assigned to all the appearances of VSU. This might be a consequence of their prior experience in writing GLPs in which the same variable was used several times representing always the same value.

In these Logo-based environments, the case study pupils developed the potential to use the calculated value of a VSU in order to solve a given problem. To use a calculated value to solve a problem might lead to engage in a verification process. However, it was observed that pupils did not engage spontaneously in this process (see 9.4.3). This confirms prior results obtained in usual algebra environments that show that pupils do not tend to verify their answers (Galvin and Bell, 1977). However, it was found that Logo-based environments might help create pupils' potential to verify their answers (see 9.4.2).

The activities of microworld 3 were not designed in order to create pupils' potential to shift from an arithmetic to an algebraic approach to the VSU, that is from considering a VSU as a specific value that can be calculated by operating on numbers to viewing it as a specific value that can be represented by operating on general numbers. However, it was found that these Logo-based environments created for some pupils a potential to make this shift. In fact, there were pupils who without researcher's intervention generalized a given particular situation devising a general method for determining the VSU in order to solve similar Logo-based problems. They expressed the devised general method verbally. Moreover, one pair tried spontaneously to symbolize the general method by writing a GLP (see 9.4.2). This suggests that this environment created for these pupils a potential to shift from an arithmetic to an algebraic approach to VSU. It was observed that asked by the researcher to generalize a given particular Logo-based situation devising a general method for determining the VSU the majority of the pupils were able to do this. However, they expressed this by giving several particular examples. This suggests that although they were able to deduce a general method the majority of the case study pupils was still tightly anchored to arithmetic and they could not shift yet to and algebraic approach to VSU.

Some of the elements intervening during these activities and that were crucial in helping create pupils' potential to work with VSU, were:

- the design of the activities
- the researcher's interventions
- the Logo setting
- the role of language

*The design of the activities*

The activities of microworld 3 were designed in order that pupils could work on them by using their prior arithmetic and Logo knowledge. The results obtained suggest that the possibility offered to pupils to work with VSU without having to cope with algebraic symbolism, favoured the creation of pupils' potential to conceptualize a VSU and to calculate its value using the given constraints of the problem.

Additionally, the second activity included an equation with multiple appearances of VSU. The equation was a formal representation of the relationship between the VSU and the data of the problem. To present the equation aimed to help pupils link the idea of VSU in a Logo problem situation with that of VSU in an equation. To solve the equation was an

alternative to using the given GLP as an input-output device and to solve in this way the posed problem by trial and refinement.

Although this design of the activity was favouring the use of trial and refinement method the results obtained show that the pupils tended to avoid this method when the possibility of an alternative approach was perceived. This suggests that pupils use trial and refinement when there are not other methods available or when they are not secure to be successful in using them. Moreover, it was observed that to present an equation with multiple appearances of VSU as an alternative to trial and refinement method for solving a problem may lead pupils to its acceptance as well as to attempts to solve it although they are not familiar with this type of equations.

It was also observed that some of the already reported errors like cutting the equation after the first term (e.g. Kieran, 1984) appeared when pupils tried to solve it (see 9.4.2). However, the design of the activity allowed pupils to know the value of the VSU prior to solving the given equation. To known in advance the value they had to obtain pushed them to question their own method for solving the equation and to look for another strategy (see 9.4.2, The case of Valentin and Ernesto - Ignoring the given equation). Activities like these might be appropriate for introducing pupils to equation solving strategies. They might be also useful to help pupils who had already developed wrong equations solving strategies to question them.

A spontaneous approach to use a given equation with multiple appearances of VSU as a verification device was also observed (see 9.4.2 The case of Oscar and Jonathan - The equation as verifier). Although isolated this approach indicates that to present equations as a means to solve problems might favour the verification process.

These findings show that Logo environments can be designed where with appropriate guidance pupils can test their own hypothesis concerning how to solve equations with multiple appearances of VSU and they can question their own methods. To work in these or similar environments can help pupils avoid the development of wrong approaches and the installation of misconceptions concerning the solution of equations with multiple appearances of VSU. This kind of experience might be used to give meaning to the later teaching of equation solving strategies.

*The researcher's interventions*

During microworld 3 the researcher intervened in order to create pupils' potential to conceptualize a VSU in a problem situation, to calculate its value and to use the calculated value in order to solve the given problem.

The most of the case study pairs were able to conceptualize a VSU without researcher's help. However, it was observed that not all the pupils were able to conceptualize a VSU when it was not explicitly indicated (see 9.4.1). The researcher's intervention was crucial to help these pupils realise that there was a specific unknown involved in the problem. After this they were able to calculate its value without needing additional help. The lack of a conceptualization of VSU was obstructing the calculation of its value. Although it seems obvious that the value of a VSU cannot be calculated if it is not perceived, the teaching school programs usually do not pay enough attention to this aspect when the idea of VSU is introduced and pupils are rushed to learn equation solving strategies where the VSU is explicitly given. This may lead pupils to learn to solve equations applying rules in an automatic way without being necessarily able to conceptualize a VSU and to understand why for they are applying the learned strategies. These findings suggest that more attention should be paid to the conceptualization of VSU prior to the teaching of manipulative skills.

The researcher's requirement to explain the method used to solve a simple Logo problem where a VSU had to be determined pushed some pupils to shift from an arithmetic to an algebraic view of VSU. It was considered that the VSU is perceived arithmetically when it is viewed as a value that can be calculated by operating with numbers. The VSU is perceived algebraically when it is viewed as a specific value that can be represented by operating on general numbers (see 9.4.2). Moreover, it was observed that this researcher's intervention pushed a pair first to generalize the method and then to try to symbolize it in Logo. These results indicate that in specially devised settings and with appropriate intervention pupils may be stimulated to perceive the need for algebraic language. These experiences might be a basis upon which new algebraic concepts might be later constructed in a meaningful way with teacher's support.

The researcher's intervention was also crucial for helping some pupils use the value of a VSU after having calculated it (see 9.4.3). This resistance to use the calculated value might be related to the already reported tendency pupils have not to verify the correctness of the value of a VSU (Galvin and Bell, 1977) and to consider the task finished once the value has been obtained. Another reason might have been the number

of elements pupils had to handle simultaneously, when coping with the first activity. These were: conceptualize the VSUs of the problem, calculate their values, decide the turtles turns in order to draw the steps, use the calculated values for the VSUs in order to move the turtle an appropriate distance. However, researcher's intervention asking them why they were not considering the calculated values was enough for these pupils to overcome the initial difficulty and to use the calculated values for the VSUs.

The results obtained show that the researcher's intervention was a crucial element to help create pupils' potential to cope with the basic characteristics underlying the idea of VSU.

*The Logo setting*

The turtle graphics environment and the researcher's written GLPs provided the framework in which pupils were expected to work with VSU.

It was found that given a Logo-based problem the great majority of the case study pupils were able to conceptualize a VSU, calculate its value and use the calculated value. In this environment it was purposeful for pupils to go through all these steps. In fact, to do it represented the possibility to complete a drawing on the computer screen.

The results obtained suggest that to work with the idea of VSU in Logo after having worked in microworlds where the characterizations of VGN and VFR were approached, pushed some pupils to generalize the method used to determine a VSU in a Logo-based problem and this created a potential to shift from an arithmetic to an algebraic approach to VSU. Moreover, the prior experience in writing GLPs pushed one pair to write a GLP in order to symbolize the devised general method for determining the value of VSU for problems similar to the posed one.

Although this was not a general tendency these outcomes suggest that Logo might be an appropriate environment in which to design activities to help pupils shift from an arithmetic to an algebraic perception of VSU and to express this in a symbolic language. That is, to work in Logo might stimulate the development of a more abstract way of thinking as well as the necessity for a formal language as a means to express their thoughts.

To work in Logo favoured the use of trial and refinement methods to determine the value of a VSU. On occasion this helped the great majority of the case study pupils solve

the given problem (see Table 9.2). It was found that the solution obtained by trial and refinement using a GLP might be used by pupils as a scaffold to approach the solution of an equation with multiple appearances of VSU (see 9.4.2). This suggests that the trial and refinement method might be used initially to help pupils approach an unfamiliar situation involving VSU. Once obtained the solution of the problem by trial and refinement, algebraic manipulative techniques might be introduced as a more efficient way to determine the value of VSU.

To work with VSU in Logo led pupils to consider that in an equation all the appearances of VSU have the same value. In this way to work in Logo-based environments helped pupils to avoid the 'polysemia' error observed when pupils worked on paper and pencil tasks (Filloy and Rojano, 1989). This behaviour was probably influenced by the experience of having written and run GLPs with several appearances of the same variable representing the same value.

These results suggest that Logo might be an appropriate setting to design activities aiming to create pupils' potential to approach some of the characteristics underlying the idea of VSU. The results obtained show as well that Logo microworlds can be designed to help pupils make sense of equations with multiple appearances of VSU and of the necessity to operate with VSU.

*The role of language*

Another element present during these activities was the use of natural language. Compared with the way this element was used in microworlds 1 and 2 it was observed that during microworld 3 pupils talked less. This might be related to their perception of the tasks. In fact, when explained that they were expected to determine precise values in order to obtain specific drawings they commented that this would be very easy to do. They considered these kind of activities more familiar than the activities of the prior microworlds. This assumption led pupils to try to solve the posed tasks without engaging in discussions. A common behaviour observed was that after thinking (their thought was manifested by their silence while they were observing the work sheet) on the posed problem one pupil proposed some concrete actions to be taken (e.g. to divide 60 and 70 by 5; or to use the given GLP as input-output machine). Without further discussion they started working in silence.

Because of these short verbal interchanges between pupils and due to the fact that the researcher and the assistant were simultaneously attending the group and observing the case study pupils, it was difficult to be present when the dialogs occurred and take detailed notes of them. For that reason although language intervened in pupils' approach to VSU, the specific role it had cannot be discussed in details. In any way a reduced use of language during these activities was observed. This suggests that the role of language diminishes when pupils work on tasks they consider related to a more familiar type and that are therefore considered easy to solve. This might be interpreted from a Vygotskian (1978) perspective where it is considered that more complex are the actions demanded by a situation the greater is the importance played by speech in accomplishing them. However, more investigation is needed to study the relationship between pupils' use of language and their capability to cope with VSU. Further research might highlight whether pupils' work with VSU might be improved by encouraging the use of spoken language.

# CHAPTER 10

# DISCUSSION OF PUPILS' POTENTIAL

## 10.1 Introduction

After having worked in the three Logo microworlds all the case study pupils were individually interviewed. This was the final interview and it was made at the end of the study. It aimed to verify if working in Logo-based environments had created pupils' potential to work in Logo with different characterizations of variable, namely, variable as general number (VGN), variables in functional relationship (VFR) and variable as specific unknown (VSU).

The Logo microworlds were not designed in order to help pupils use in non-Logo environments their experience of working with different characterizations of variable. However, the experience acquired in the Logo-based environments might have influenced pupils way of solving traditional paper-and-pencil algebra task. To explore this possibility at the end of the study pupils were given an extract of the questionnaire they answered at the beginning of the research (Appendix 1).

## 10.2 The final interview

During the final interview the same problems were posed to all the case study pupils (Fig. 10.1). To solve these problems pupils used a computer.

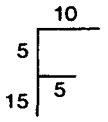
The first problem required to cope with VGN. Pupils worked on a problem similar to this one when individually interviewed immediately after microworld 1 (see Fig. 7.13, question 4, Chapter 7). To solve it pupils had to write a GLP in which the VGN had to be operated on.

To answer the first question of the second problem, question **a**, pupils had to work with VFR. They were required to determine an interval. This implied to cope with both, the static aspect (SA) and the dynamic aspect (DA) characterizing VFR and to shift between them.

To answer the second question of the second problem, question **b**, pupils had to cope with VSU. They had to be able to conceptualize a VSU and to determine its specific value taking into account the constraints of the problem.

**FINAL INTERVIEW**

1. Write a procedure to draw the letter F of any size but maintaining the given proportion.




2. Analyse and use the following procedure in order to answer the questions:

```
TO NO. SE :Y
LT 90
FD :Y
RT 90
FD 3* :Y
PU
HOME
PD
END
```

a) If the drawing starts at HOME, what are all the values :Y can have so that the shape remains within the screen window?

b) What values must :Y have so that the height of the drawing is equal to 75?

**Fig. 10.1** The questions of the final interview

### 10.2.1 Results

To answer these problems pupils needed different types of interventions. These are presented in Table 10.1.

| Symbol used | Type of intervention |
|---|---|
| QR | Original question repeated |
| QRF | Original question repeated and formulated again |
| R | Provoking reflection |
| SN | Negotiating the solution of the problem |

**TABLE 10.1** - Type of intervention provided during the final interview

The type of intervention required by each pupil are presented in Table 10.2. The symbol '+' is used to indicate that the pupil did not need the researcher's intervention to solve the problem. For problem 1 the type of intervention provided is followed by the indication of the strategy pupils used to write a GLP (see Table 7.6, Chapter 7, for a detailed description of the strategies).

| Pupils' name | Problem 1 | | Problem 2 a | Problem 2 b |
|---|---|---|---|---|
| Valentín | + | T2b | + | + |
| Ernesto | SN | T1 | QR | QR , R |
| Etna | + | T1 | + | + |
| Itzel | SN | T1 | + | + |
| Martha | SN | T1 | + | + |
| Mariana | SN | T1 | + | + |
| Oscar | + | T2b | + | + |
| Jonathan | + | T1 | + | + |
| Claudia | SN | T1 | + | + |
| Sandra | + | T2b | + | + |
| Nayeli | + | T2b | + | + |
| Soledad | SN | T1 | QR | QRF , R |

**TABLE 10.2** - Type of intervention received by each pupil during the final interview

#### 10.2.1.1 Pupils' potential to work in Logo with variable as general number (VGN)

Table 10.2 shows that all the case study pupils were able to write a GLP to answer problem 1. Six pupils (Valentin, Etna, Oscar, Jonathan, Sandra, Nayeli) were able to write it on their own. Six pupils (Ernesto, Itzel, Martha, Mariana, Claudia, Soledad) needed researcher's intervention. These pupils were able to write a GLP on their own but they were not operating on the VGN. By emphasizing the given numeric example and on occasion by giving more numeric examples the researcher directed pupils' attention to the necessity to operate on the VGN. After this, they were able to operate on it. This is illustrated by the dialogue between the researcher and Ernesto after he wrote a GLP without operating on the VGN.

R(esearcher): *'Do all the lines have the same size?'*

The researcher indicates the letter F drawn on the work sheet.

E(rnesto): *'No ... yes ...'*

His eyes go from the work sheet to the screen. His 'no' refers to the figure on the work sheet. His 'yes' refers to the drawing that appears on the screen of the computer after running his GLP. Ernesto is confused.

R.: *'Consider the given values. We have to maintain the given proportions for any size of the F.*

E.: *'My procedure is wrong'.*

Ernesto clears the screen and he starts writing a new procedure. He looks at the given numbers, but he does not write a FLP. He wants to write directly a GLP.

E.: *'It's difficult'.*

R.: *'But you have already worked on similar tasks'.*

E.: *'Yes, but I don't remember. I should remember, but I don't ... With two variables ...'*

R.: *'What will your variable represent?'*

E.: *'The first one will be 15 ... and then it is 5'.*

R.: *'Could you use only one variable?'*

E.: *'Ah, yes, I know, it's easy. It is divided by 3'.*

Ernesto writes:

```
TO F :♥
FD :♥
RT 90
FD :♥/3
BK :♥/3
LT 90
FD :♥/3
RT 90
FD :♥/1.5
END
```

To write the GLP the majority spontaneously used a 1-step strategy, T1. That is, they wrote the GLP and they operated on the VGN by the direct observation of the given figure. The tendency to use a 1-step strategy to write GLPs was already observed during pupils' work in microworld 1 (see 7.4.3.2, Chapter 7). Table 10.2 shows that two pupils (Etna, Jonathan) used successfully this strategy without needing researcher's help. Six pupils (Ernesto, Itzel, Martha, Mariana, Claudia, Soledad) needed researcher's help in order to use it.

Four pupils (Valentin, Oscar, Sandra, Nayeli) used a 2-step strategy, T2b and they did not need researcher's help. That is, they wrote first a FLP and they used this as support to write the GLP and to operate on the VGN.

To symbolize the input to the GLP the great majority used a letter. Only Valentin and Ernesto used a symbol other than a letter. As they usually did during microworld 1, they used a heart (♥) to symbolize the VGN. During the interview each pupil was asked for the meaning of the symbol (s)he was using. All the pupils, included Valentin and Ernesto, considered that the symbol was representing any value. They also commented that the operations that were indicated in the GLP will be carried out when a specific value will be assigned to the input.

These results show that all the case study pupils had developed a potential to work in Logo with VGN. However, to operate on it the great majority still needed support. This was provided by the researcher or by particular examples expressed in Logo.

#### 10.2.1.2 Pupils' potential to work in Logo with variables in a functional relationship (VFR)

To answer question **a**, problem 2, an interval had to be identified. All the pupils spontaneously determined it. The great majority did not need the researcher's help. For two pupils (Ernesto, Soledad) it was necessary to repeat the question. After that they did not need additional support.

To determine the interval pupils used different strategies. The great majority of pupils (Etna, Itzel, Mariana, Martha, Jonathan, Sandra, Claudia, Nayeli, Soledad) tried a random input and by trial and refinement they approximated the value that gave the biggest shape within the screen window. They answered the posed question saying that all the values between 1 and 40 were drawing the shape of the required size. Their approach to this question shows that these pupils considered that the input was representing any number and that any value might be assigned to it. But, in order to fulfil the given condition they were able to limit the variation of the input to an interval.

There were pupils (Valentin, Ernesto, Oscar) who calculated the distance between HOME and the top of the screen. They increased systematically the input for the primitive FORWARD and after this they added all the assigned values. In consequence, they divided the calculated distance by 3 in order to obtain the input to the GLP.

This method shows that these pupils realised that the height of the shape was obtained by multiplying the input by 3 (see Fig. 10.1, problem 2, command FD 3*:Y). Based on this relationship they calculated the value for the input corresponding to the largest shape. This suggests that they were looking at the input as a VSU for which a precise value had to be determined. However, asked for the values the input might have, these pupils said any value might be substituted for it. They also clarified that in order to obtain the shape within the screen window any input between 1 and 40 might be used.

In order to determine the interval pupils had to consider the functional relationship linking the input and the size of the shape. This was independent of the strategy used. That is, to solve this task it was necessary to cope with both, the static aspect (SA) and the dynamic aspect (DA) of the given functional relationship and shift between them. The results obtained show that pupils had developed a potential to cope in Logo with SA and DA and to shift between them. That is, they had developed a potential to work in Logo with VFR and to do this they did not need any more researcher's help.

#### **10.2.1.3** Pupils' potential to work in Logo with variable as specific unknown (VSU)

Pupils' answers to question **b**, problem 2, show that the great majority of the case study pupils were able to work in Logo with VSU without researcher's help. To answer this question pupils had to realise that the height of the figure was represented in the given GLP by the expression FD 3*:Y. They had to consider that 3*:Y had to be equal to 75, to conceive the variable involved as a VSU which value could be calculated, and to calculate it. That is, they had to cope with the basic characteristics underlying the idea of VSU.

To answer this question only two pupils (Ernesto, Soledad) needed researcher's intervention. They multiplied 75 by 3 in order to obtain the total height. To repeat the question or to formulate it in a different way (e.g. 'I want the drawing to have height 75. How would you determine the input to obtain exactly that height?') and to ask them which was the command in the GLP that draw the line representing the height of the figure, was enough for helping them calculate the precise value for the input. All the other pupils were able to calculate it without researcher's intervention.

These results show that all the pupils were able to conceptualize a VSU and to determine its value considering the constraints of the problem. As already observed during

microworld 3 (see 9.4.2, Chapter 9) none of the pupils posed an equation. They analysed the problem and solved it by making the correct arithmetic operation in order to obtain the searched value. This confirms pupils' potential to cope in Logo with the basic characteristics underlying the idea of VSU.

### 10.2.2 Discussion of pupils' potential

The results obtained during the final interview show that pupils were able to use the experience acquired during their work in the three Logo microworlds to solve similar Logo-based problems involving VGN, VFR and VSU. This indicates that the activities of the three microworlds were conducive to create a potential for pupils to work in Logo with simple problems requiring to cope with these characterizations of variable.

The support required by pupils to write a GLP and operate on the VGN in order to solve the first problem, suggests that to work in Logo with VGN was not straightforward for the great majority of the case study pupils although a potential to do this was created. In order to cope in Logo with VGN it is necessary to shift from the particular to the general and to express this in a formal language. That is, although particular examples might be used as scaffold to solve a problem involving VGN, the problem cannot be solved within the numeric domain. This might be a reason why the case study pupils still needed help for coping in Logo with VGN.

This, was not the case for the problems involving VFR and VSU. In fact, to work with these characterizations of variable pupils were given an already generalized problem, expressed by a GLP. They had to run the given GLP with particular numbers in order to generate particular examples of the given general problem. That is, they had to shift from the general back to particular examples. To determine intervals, to look for the global linked behaviour of the VFR involved, and to determine a VSU, pupils had to cope with numbers. That is, the posed problems involving VFR and VSU were solvable within the realm of numbers. This might explain why to solve these tasks the great majority did not need help.

These outcomes indicate that the pupils were still tightly linked to the particular and to shift to a more abstract level and cope with the general was not straightforward. However, to work in specially designed Logo environments created a potential for this. Moreover, the obtained results show that several non trivial mathematic notions (e.g. the notion of interval, monotonicity) might be experienced in Logo by working with numbers

produced by running GLPs. These kind of experience might be offered to pupils prior to introducing them to formal algebra in which these notion are approached in a formal way.

Pupils' work with questions **a** and **b**, problem 2, suggests as well that a potential to shift from one characterization of variable to another was created during pupils' work in Logo-based environments. In fact, pupils considered that the variable involved in the given GLP was representing any value. That is, they were looking at it as a VGN. But, in order to satisfy the given condition and to obtain the shape within the screen window, they were able to look at the functional link between the values of the input and the size of the shape. They were able to restrict the possible values for the input to an interval. This indicates that they were able to shift from VGN to VFR within the same task. Finally pupils were able to consider that in order to satisfy a given condition a specific value had to be assigned to the input and to calculate this value. In this way pupils shifted to VSU. That is, in the same task pupils were able to shift from VGN to VFR to VSU. This study did not aim to explore the possibility to create pupils' potential to shift between different characterizations of variable. However, these results suggest the possibility to develop activities in Logo that might help pupils work with different characterizations of variable within the same task. This might be an early experience in which pupils might cope with the multifaceted concept of variable prior to formal algebra teaching.

## 10.3 Re-application of the questionnaire

At the end of the study pupils were given a written questionnaire. This was an extract of the questionnaire they answered at the beginning of the study (Appendix 1). Items that provided redundant information were not given (items 2.6, 6, 7, 15). Items 11, 12 and 13 were excluded because they implied pattern recognition and this was not an explicit focus in the microworlds. Item 19 was slightly modified (Fig. 10.2) by introducing a table frame, however, pupils were not asked to fill in a table.

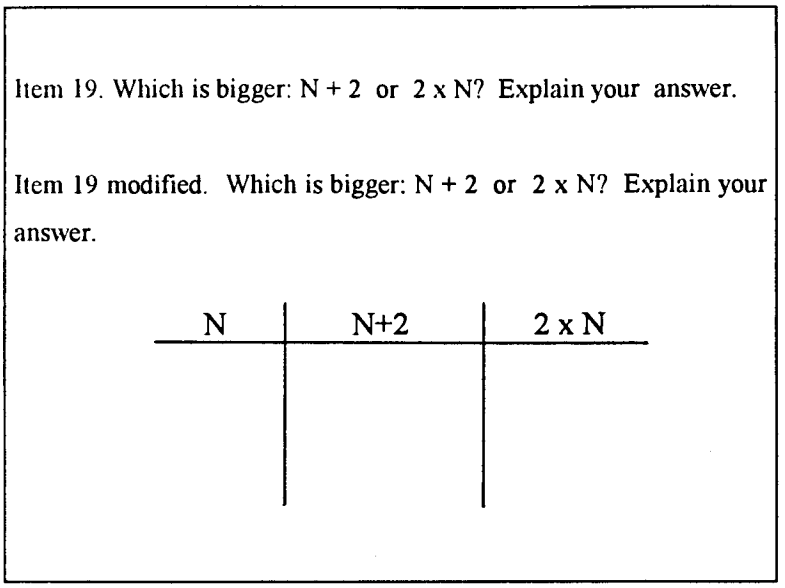



**Fig. 10.2** Item 19

In the written questionnaire pupils had not to cope with Logo-based problems but with traditional paper-and-pencil algebra tasks involving variable as general number (VGN), variables in a functional relationship (VFR) and variable as specific unknown (VSU). These different characterizations of variable appeared in algebraic expressions and they were represented by literal symbols. This aimed to explore whether the pupils' work in Logo-based environments was influencing the way they were solving traditional algebra tasks. The activities of this investigation were not designed in order to help pupils use in non-Logo environments the experience acquired in the Logo-base microworlds.

### 10.3.1 Results

The ways pupils answered the extract of the written questionnaire are presented on the right side of tables 10.3, 10.4 and 10.5. To facilitate a comparison with the ways pupils answered the same items at the beginning of the study, these are presented on the left side of the same tables.

#### **10.3.1.1** Pupils' solutions to items involving variable as specific unknown (VSU)

Table 10.3 shows how pupils solved simple equations in which the VSU was represented by a literal symbol (items 2.2, 2.5, 3); and how pupils symbolized a written statement representing an equation (items 1.1, 1.2).

| Pupils' solution of a simple equation | | | | | | | |
|---|---|---|---|---|---|---|---|
| Beginning of the study | | | | End of the study | | | |
| Item | No answer | One arbitrary value | VSU calculated | No answer | One arbitrary value | VSU calculated | Several arbitrary values |
| 2.2 , 3 | Ernesto, Etna, Claudia, Nayeli, Soledad. | - ---------- | Valentín, Itzel, Martha, Mariana, Oscar, Jonathan, Sandra. | Claudia, Soledad. | ---------- | Valentín, Itzel, Martha, Mariana, Oscar, Ernesto, Etna, Jonathan, Sandra, Nayeli. | ---------- |
| 2.5 | Valentín, Ernesto, Etna, Itzel, Martha, Claudia, Sandra, Nayeli, Soledad. | Mariana*, Oscar, Jonathan | ---------- | Soledad | Oscar, Etna, Claudia | Valentín, Ernesto, Itzel, Martha, Jonathan, Nayeli. | Mariana, Sandra. |
| **Pupils' symbolization of a simple equation involving one appearance of VSU** | | | | | | | |
| **Item** | **No answer** | **Numeric answer** | **Letter used** | **Sign used** | **No answer** | **Numeric answer** | **Letter used** | **Sign used** |
| 1.1 , 1.2 | Etna, Itzel, Oscar, Soledad | Ernesto, Martha, Mariana, Jonathan | Claudia, Sandra, Nayeli. | Valentín | ---------- | ---------- | Valentín, Ernesto, Etna**, Itzel, Mariana, Martha, Oscar, Jonathan, Claudia, Sandra, Nayeli, Soledad | ---------- |

(*) Tried to balance the equation.
(**) Wrote a GLP.

**TABLE 10.3 - Comparison between pupils' answers to items involving VSU, given at the beginning and at the end of the study.**

The results indicate that at the beginning of the study the majority was able to solve simple equations when the VSU was represented by a letter (items 2.2, 3). The great majority could not determine the value of the VSU for an equation with multiple appearances of it (items 2.5). The majority could not symbolize an equation given in words (items 1.1, 1.2). Three pupils (Claudia, Sandra, Nayeli) symbolized it by using a letter. One pupil (Valentin) used a symbol other than a letter.

A progression in the way pupils answered these items at the end of the study was observed. The great majority were able to solve simple equations with one appearance of VSU when this was represented by a letter (items 2.2, 3). Half of the pupils were able to determine the correct value of a VSU when it appeared in an equation with multiple appearances of it (item 2.5). This capability represents a progression.

To answer item 2.5 pupils had to solve an equation with multiple appearances of VSU ($X + 5 = X + X$). In the final test, the great majority answered this item by writing one value. This shows that the great majority did not present the 'polysemia' error (Filloy and Rojano, 1989), that is, they were not considering that different values might be assigned to different appearances of the same literal symbol within the same algebraic expression. There were two pupils (Mariana, Sandra) who answered this item by writing a list of numbers (Fig. 10.3 shows the answer given by Mariana). The numbers chosen suggest that these pupils were regarding the given equation as a general statement and they were interpreting the VSU as a VGN. Therefore, the list of numbers might indicate that any value may be assigned to the variable and not that different values may be assigned to the different appearances of VSU.

The absence of the 'polysemia' error might have been a consequence of the experience acquired in Logo when writing and running GLPs. Pupils were familiar with multiple appearances of the same variable representing the same value within a GLP. This experience might have helped pupils to consider that in an algebraic expression the same literal symbol is used to represent the same value. Moreover, during the second activity of microworld 3 pupils were asked to determine one specific value for the input of a given GLP that could be calculated by solving an equation with multiple appearances of VSU (section 9.2, Chapter 9). The input was represented by a literal symbol and the same literal symbol was used to represent the VSU in the equation. When writing GLPs the great majority of the pupils used a letter to represent the VGN. Therefore, they were used to consider that any value could be assigned to a literal symbol representing the input. However, the activities of microworld 3 required the pupils to consider that when a specific result had to be obtained, the literal symbol was representing only one

particular value. This experience might have influenced pupils capability to consider that in an equation with multiple appearances of VSU the literal symbol represented the same specific value.

All the pupils were able to symbolize an equation given in words (e.g. 'Write a formula which means: An unknown number plus 5 is equal to 8'). To symbolize the VSU all the pupils used a literal symbol (items 1.1, 1.2). This capability also represents a progression and it might be a consequence of the experience acquired in Logo where literal symbols were used to represent different characterizations of variable.

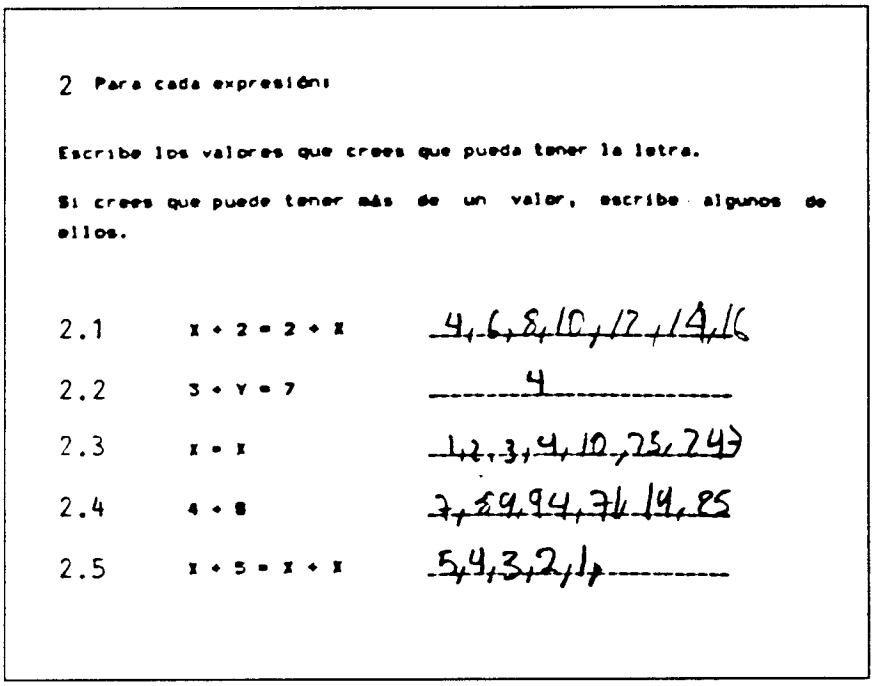



**Fig. 10.3** Mariana - Answer to item 2.5

**10.3.1.2** Pupils' solutions to items involving variable as general number (VGN)

Table 10.4 presents pupils' answers to items requiring to interpret a VGN in an algebraic expression (items 2.1, 2.3, 2.4), to use a given VGN in order to construct a new algebraic expression (items 1.5, 4, 8, 9, 10), to use a VGN to express a generalization (items 1.4, 5).

| Pupils' estimation of the values of a VGN | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Beginning of the study | | | | | End of the study | | | |
| Items | No answer | One arbitrary value | Mixed approach | Multiple values | No answer | One arbitrary value | Mixed approach | Multiple values |
| 2.1, 2.3, 2.4 | Ernesto, Etna, Itzel, Martha, Claudia, Nayeli, Soledad. | ---------- | Valentín, Oscar, Jonathan, Mariana, Sandra. | ---------- | ---------- | Ernesto, Etna, Itzel, Oscar, Jonathan, Claudia. | ---------- | Valentín, Mariana, Martha, Sandra, Nayeli, Soledad. |
| Pupils' use of a given VGN | | | | | | | | |
| Item | No answer | VGN ignored | Value assigned | VGN used | No answer | VGn ignored | Value assigned | VGN used |
| 1.5 | Valentín, Ernesto Etna, Itzel, Martha, Oscar, Jonathan, Claudia, Sandra, Soledad. | ---------- | ---------- | Mariana, Nayeli | ---------- | ---------- | ---------- | Valentín, Ernesto, Etna, Itzel, Nayeli, Martha, Oscar, Jonathan, Claudia, Sandra, Nayeli, Soledad* |
| 4 | Ernesto, Etna, Itzel, Sandra, Soleded. | Martha, Mariana, Oscar, Jonathan, Claudia, Soledad. | ---------- | Valentín | ---------- | Etna | Itzel, Oscar, Jonathan, Claudia. | Ernesto, Mariana, Valentín, Martha, Sandra, Nayeli, Soleded. |
| 8 | Soledad. | Valentín | Ernesto, Etna, Itzel, Martha, Mariana, Oscar, Jonathan | Claudia, Sandra, Nayeli | Claudia, Soledad. | ---------- | Itzel, Oscar | Valentín, Ernesto, Etna (GLP), Mariana, Martha, Jonathan, Sandra, Nayeli. |
| 9 | Ernesto, Etna, Oscar, Jonathan, Soledad. | Itzel | Valentín, Martha, Mariana, Nayeli | Claudia, Sandra | Claudia | ---------- | Martha, Etna, Itzel, Oscar | Valentín, Ernesto, Mariana, Jonathan*. |
| 10 | Ernesto, Etna, Itzel, Mariana, OScar, Jonathan, Soleded. | ---------- | ----- | Valentín, Martha, Claudia, Sandra, Nayeli | Claudia, Soledad | ---------- | ---------- | Valentín, Ernesto, Etna, Martha, Mariana, Jonathan, Sandra, Nayeli. |

| Pupils' exspression of generalization | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Beginning of the study | | | | End of the study | | | | |
| Item | No answer | Numeric example | VGN used | No answer | Numeric example | VSU used | VGN used | VFR |
| 1.4 | Valentín, Ernesto, Etna, Itzel, Mariana, Oscar, Jonathan, Sandra, Nayeli, Soledad. | Martha | Claudia | Itzel, Jonathan, Sandra | Etna* | Mariana, Oscar | Valentín, Ernesto*, Martha*, Claudia', Nayeli, Soledad. | ---------- |
| 5 | Ernesto, Etna, Itzel, Martha, Mariana, Oscar, Jonathan, Claudia, Sandra, Soledad. | ---------- | Valentín, Nayeli | Itzel, Claudia | ---------- | Valentin | Ernesto, Mariana, Nayeli, Soledad*. | Etna, Oscar, Jonathan, Sandra, Martha |

(*) Wrong answer

**TABLE 10.4 Comparison between pupils' answers to items involving VGN, given at the beginning and at the end of the study.**

As can be seen from Table 10.4 at the beginning of the study the majority of pupils did not answer items 2.1, 2.3, 2.4. At the end of the study all the pupils answered these items. This indicates that all the pupils were able to interpret the literal symbol in the given algebraic expression. Six pupils (Valentin, Martha, Mariana, Sandra, Nayeli, Soledad) wrote a list of arbitrary numbers (see, for example, Fig. 10.3, items 2.1, 2.3, 2.4) or they wrote 'any value can be assigned'. To give a list of numbers might be a way to indicate that any value could be assigned to the VGN. Six pupils (Ernesto, Etna, Itzel, Oscar, Jonathan, Claudia) answered these items giving only one arbitrary number. This might indicate that they were considering that any value could be assigned, and they were giving one specific example.

To answer item 1.5 an 'unclosed' algebraic expression had to be written. At the beginning of the study the great majority did not answer this item. This suggests that the case study pupils could not write an 'unclosed' algebraic expression. At the end of the study all the case study pupils were able to use a given VGN and to write an 'unclosed' algebraic expression.

The way pupils answered items 4, 8, 9, 10 at the beginning of the study shows that the great majority did not use the given VGN to construct a new algebraic expression. In contrast, at the end of the study the great majority was able to use the given VGN and to construct an algebraic expression.

To answer item 1.4 required to symbolize the general word sentence 'Write a formula which means: multiply 8 by the addition of 3 and an unknown number'. At the beginning of the study the great majority did not answer this item. At the end of the study only three pupils (Itzel, Jonathan, Sandra) did not answer it. One pupil (Etna) tried to give a numeric example. She wrote 3 * 2 + 2, showing a wrong interpretation of the given sentence. In fact, she calculated the value that would give 8 when added to its multiplication by 3. Two pupils (Mariana, Oscar) wrote an equation (e.g. 3 + z = 8). Three pupils (Ernesto, Martha, Claudia) wrote an 'unclosed' expression (e.g. 8 x 3V) but this was not representing the given sentence. Only three pupils (Valentin, Nayeli, Soledad) could interpret the given sentence and symbolize it correctly (for example, Nayeli wrote 8 x (3 + X) ).

These results shows that the great majority had difficulty to interpret the given sentence and to symbolize it correctly. However, also incorrect these attempts show a great progression if compared with the large number of 'no answer' given at the beginning of the study. This suggests that at the end of the study pupils were much more confident in

dealing with problems involving VGN. They tried to interpret and to symbolize them. When working in the Logo numeric environments all the pupils were able to interpret and to write a GLP to represent similar general sentences (e.g. 'Write a general procedure to divide any number by 10 and to add 100 to it', work sheet G-7). This experience probably helped the majority of pupils gain confidence in working with generalizations and its symbolic expression.

The number of pupils who answered item 5 ('Write a formula which means: An unknown number is bigger than 5') increased in a remarkable way at the end of the study. Only two pupils (Itzel, Martha, Claudia) did not answer this item. One pupil (Valentin) wrote the equation $8 + X = 15$, whose solution gave a number greater than 5. Five pupils (Etna, Oscar, Jonathan, Sandra, Soledad) symbolized the given sentence by writing an expression with two inter-related variables (e.g. $X + 5 = Y$) . This suggests that these pupils were considering that any positive number could be assigned to X and the result represented by Y will be always greater than 5. They were probably not used to the symbol '>', therefore they found their own way to symbolize the given sentence. This solution was probably influenced by the experience acquired in microworld 2 where they worked with variables linked by a functional relationship. Three pupils (Ernesto, Mariana, Nayeli) symbolized the given sentence by writing $X > 5$.

These result suggest that the experience acquired in Logo have influenced pupils' work with algebra items involving VGN. Their capability to express a generalization by writing GLPs, to use a literal symbol to represent the input to a GLP and to consider that it was representing any value (see 10.2.1.1, Chapter 10) might have influenced pupils capability to consider that a letter in an algebraic expression might represent any value. Their experience in writing GLPs in which they had to operate on the variable might have helped them manipulate given VGNs to write algebraic expressions. The experience of working with VFR helped pupils to interpret and solve problems involving VGN that they could not solve at the beginning of the study.

#### 10.3.1.3 Pupils' solutions to items involving variables in a functional relationship (VFR)

The most dramatic progression was observed in pupils' answers to items involving VFR (Table 10.5, items 1.3, 14, 16, 17, 18, 19). VFR appeared in algebraic expressions representing functional relationships.

| Pupils' answers to items with symbolized VFRs | | | | | | |
|---|---|---|---|---|---|---|
| | Beginning of the study | | | End of the study | | |
| Item | No answer | Relationship not considered | Relationship considered | No answer | Relationship not considered | Relationship considered |
| 14 | Ernesto, Etna, Itzel, Martha, Mariana, Sandra, Nayeli, Soledad. | Oscar, Jonathan, Claudia | Valentín | | Itzel, Martha, Soledad. | Valentín, Ernesto, Etna, Martha, Oscar, Jonathan, Claudia, Sandra, Nayeli. |
| 16 and 19 | Velentín, Ernesto, Etna, Itzel, Martha, Oscar, Jonathan, Claudia, Nayeli, Soledad | Sandra | Mariana | | Martha, Soledad. | Valentín, ERnesto, Etna, Itzel, Mariana, OScar, Claudia, Sandra, Nayeli. |
| 17 | Valentín, Ernesto, Etna, Iztel, Martha, Mariana, Claudia, Sandra, Nayeli, Soledad | Oscar, Jonathan | | Itzel | Ernesto, Martha | Valentín, Etna, Martha, Oscar, Jonathan, Claudia, Sandra, Nayeli, Soledad. |
| 18 | Ernesto Sandra, Nayeli, Soledad. | Etna, Itzel, Martha, Jonathan, Claudia | Valentín, Mariana, Oscar. | Ernesto, Sandra, Nayeli, Soledad. | Itzel, Etna, Martha, Jonathan, Claudia | Valentín, Mariana, Oscar |

| Pupils' symbolization of VFR | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Item | No answer | Value assigned | One symbol used | Two symbols used | No answer | Value assigned | One symbol used | Two symbols used |
| 1.3 | Etna, Itzel, Martha, Oscar, Jonathan, Sandra, Soledad. | Valentín, Ernesto, Mariana | Claudia, Nayeli | ---------- | Martha | ---------- | Ernesto, Etna* | Valentín, Itzel, Martha, Oscar, Jonathan, Claudia, Sandra, Nayeli, Soledad |

(*) Wrote a GLP.

**TABLE 10.5 - Comparison between pupils' answers to items involving VFR, given at the beginning and at the end of the study.**

Table 10.5 shows that at the beginning of the study the great majority of the case study pupils did not answer the items involving VFR. On the contrary, at the end of the study the great majority answered them correctly. This indicates pupils' capability to interpret literal symbols in an algebraic expression as VFR and to use VFR to express a functional relationship.

To answer these items it was necessary to take into account the relationship linking VFR. The great majority of pupils were able to consider the given relationships. They were also able to complete a table (item 17) and to cope with monotonicity (item 16). The great majority was also able to use literal symbols to symbolize a functional relationship expressed by a word sentence ('Write a formula which means: An unknown number is equal to 6 plus another unknown number') (item 1.3).

These pupils' capabilities are illustrated, for example, by the way they answered item 19 ('Which is bigger: N + 2 or 2 x N?'). At the beginning of the present study the great majority did not answer this item. Mariana was the only one of the case study pupils who tried to consider the given relationships. She answered this item writing '2 x N is bigger because multiplication outputs bigger than addition'. From this sentence it cannot be deduced if she assigned different values to N and compared the obtained values for the given expressions, or if she just applied an incorrect rule.

On the contrary, at the end of the study the great majority answered this item. They assigned different values to the independent variable, they calculated the values of N + 2 and of 2 x N, they filled in a table and they answered the posed question (Fig. 10.4 shows the answer given by Valentin).

Although none of the pupils gave a completely correct answer (they did not consider what happened when the value of N was smaller than 2) the way they approached this item shows that the great majority was able to cope with both the static aspect (SA) and the dynamic aspect (DA) underlying VFR. To fill in a table they considered the value of N and the values of N + 2 and 2 x N in one-to-one correspondence. That is, they were coping with SA. All these pupils assigned to N only values bigger than 3 and all of them considered that 2 x N was always bigger than N + 2. This suggests that to deduce which expression was bigger they analysed their table considering the linked movement of the values of N + 2 and 2 x N in relation with the variation of the value of N. That is, they were coping with DA.

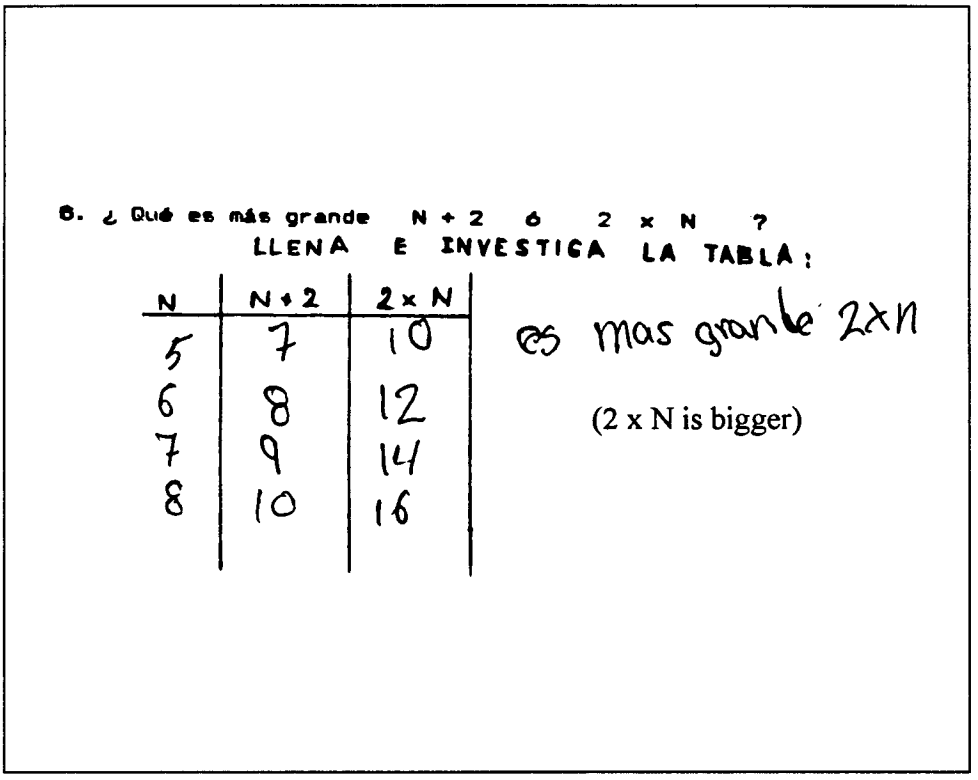



**Fig. 10.4** Valentin - Answer to item 19

This item was the same used by Küchemann when testing pupils in the Concepts in Secondary Mathematics and Science project (CSMS). Küchemann used this item in order

> "to see whether children would recognize that the relative size of two expressions (2n and n+2) was dependent on the value of n"
>
> (Küchemann,1981, p. 111).

He found that only 6/100 pupils 14 years old were able to answer this item correctly. The way the case study pupils of the present study answered this item shows that the great majority recognized that the relative value of the two expressions depended on the value of N. This capability was probably influenced by the experience acquired in Logo. When working in microworld 3 pupils filled in and analysed tables in order to describe the global linked behaviour of the values of the input and the output. That is, in Logo-based environment pupils coped with both, SA and DA.

No progression was observed in pupils' answers to item 18. To answer this item it was necessary to cope with SA, DA and to shift between them. The great majority could cope with SA and they answered the first two questions ('Consider Y = 3 + X. If we want the value of Y to be 10, what value must X have? If we want the value of Y to be

3, what value must X have?'). But they had difficulty to shift between SA and DA in order to answer the questions ('If we want the value of Y to be bigger than 3 but smaller than 10, what are all the values that X can have? If X takes values between 8 and 15, between which values will be the value of Y?'). Similar difficulties to shift between SA and DA were observed when pupils worked in Logo-based environments (see 8.4.2, Chapter 8). However, it was found that in Logo a potential to shift between SA and DA was created (see Table 10.2, problem 2, question a).

#### **10.3.1.4** Concluding comment

The results presented in Tables 10.3, 10.4 and 10.5 show a general progression in pupils' capability to work with different characterizations of variable in traditional algebra task. This progression might have been influenced by the regular mathematic course pupils attended during the academic year. But, when consulted the mathematic teacher commented that during her classes there was no work with different characterizations of variable. Therefore, it is sensible to consider that the observed progression was due mainly to the experience pupils acquired during their work in the Logo-based microworlds. Further studies should be developed to support this hypothesis. For example, environments could be designed in which pupils had to work with both Logo-based situations and paper and pencil algebra tasks involving different characterizations of variable. It might be investigated if to work in these environments helps pupils cope better with different characterizations of variable in algebra.

# CHAPTER 11

# CONCLUSION

This study has investigated the feasibility of creating pupils' potential to approach through Logo different characterizations of variable prior to formal algebra teaching. Pupils aged 12-13 years old, without prior formal algebra instruction, were studied. The characterizations of variable considered were variable as general number, that is, when a variable represents an indeterminate number involved in a general method, variables in a functional relationship, that is, when variables represent numbers whose values move within a range of values linked one to each other by a relationship, variable as specific unknown, that is, when a variable represents a specific but unknown number that can be calculated under given constraints. Aiming to help create pupils' potential a series of Logo-based activities were designed, which integrated three microworlds each one focussing on one characterization of variable.

It was considered that a pupil has developed a potential to cope in Logo with one of these characterizations of variable when (s)he was able to solve the Logo-based tasks related to this characterization. The tasks might be solved by the pupil alone or cooperating with a more capable peer or with the help of the researcher. The researcher intervened to help pupils understand the aims of the task, encouraging them to use their own approaches, provoking reflection, giving nudges and hints, and introducing new pieces of information. On occasion she indicated some steps to approach the solution or she showed explicitly how to solve a given task. Her interventions were aimed at creating pupil potential.

## 11.1 Summary and interpretation of findings

Research investigating pupils' learning of algebraic concepts reports that both, algebra beginners and more advanced students have difficulties in coping with variable as general number (Küchemann, 1980; Booth, 1984), variables in a functional relationship (Dreyfus and Einsenberg, 1981; Markovits et al., 1986; Sfard, 1989;) and variable as specific unknown (Kieran, 1984; Filloy and Rojano, 1984, 1989; Herscovics and Linchevski,

1991a) when they appear in normal school algebra tasks. The results of the present study show that pupils with an arithmetic background working in Logo-based environments can develop a potential to cope with each one of these characterizations of variable prior to formal algebra teaching. The creation of the potential was observed during pupils' work in specially designed Logo microworlds and it was verified by interviewing each pupil individually.

Moreover, the Logo-based experience seems to help pupils develop a potential to work with these characterizations of variable also in non-Logo environments. A comparison between pupils' capability to solve normal school algebra paper and pencil tasks involving these characterizations of variable at the beginning and at the end of the study showed a general progression, even if the progress of a particular pupil may depend on her (his) initial stage. The activities were not designed in order to provoke this progression. These findings suggest that to work in Logo-based environments provides pupils with a framework that helps them give meaning to symbols representing different characterizations of variable and to work with them. This supports Noss's (1986) findings showing that Logo experience may help pupils construct meaning for algebraic concepts. However, more investigation is needed to study how the Logo-based experience might be used in order to help pupils improve their work with normal school algebra tasks involving different characterizations of variable.

### 11.1.1 Pupils' potential to work with variable as general number

Previous studies have found that when working on algebra tasks involving variable as general number pupils have difficulties in interpreting it in an appropiate way. They might consider it as a specific unknown or as an object or they might ignore it (Küchemann, 1980; Booth, 1984). To write and run general Logo procedures was found to help pupils understand that a variable may represent a range of values (Noss, 1985; Sutherland, 1987). In the present study it was considered that a general number represents more than a range of values. It represents an indeterminate number involved in a general method. The results obtained show that to work in specially designed Logo-based environments helped pupils develop a potential to cope with this characterization of variable. This potential was shown by both, pupils' capability to deduce general methods to solve similar problems expressing these by general Logo procedures, and by the evolution of the strategies they used for these general methods.

To help create this potential pupils were led to focus on the method used to solve a particular problem in Logo rather than on the result obtained, and to deduce a general method based on the analysis of several particular examples. General Logo procedures were then introduced as tools to express general methods in Logo. This approach contrasts with the way in which pupils are usually introduced to general Logo procedures by substituting a variable for the elements that are expected to vary in a fixed procedure. Although this helps pupils understand that any value may be substituted for the variable (Noss, 1985; Sutherland, 1987, 1992), it does not necessarily help pupils to conceive it as an indeterminate number with which they can work. In contrast the approach used in this study led pupils to consider a variable as an indeterminate number involved in a general method and also to conceive that any value may be assigned to it. Pupils' capability to focus on the method and to deduce a general one implies that they were able to consider that the particular number involved was not essential. This suggests the existence of a potential to work with indeterminate numbers.

There was evidence that pupils may develop this idea of general number before being able to use a symbol to represent it. To introduce a symbol in these circumstances was found to provide pupils with a way of recording something they were already familiar with. The symbol acquired a meaning and working with it was relevant to pupils. This facilitated its further use as general number. This supports Mason's (1985) suggestion that to help pupils deal with symbols to express a generalization, conditions should be created in which symbolism is introduced as a necessity for pupils to express themselves. Moreover, the results obtained suggest that the way in which a symbol is introduced creates its meaning. This determines its use and its use produces new forms of behaviour.

However, to develop the understanding that a variable might be used to represent an indeterminate number involved in a general method, does not imply that operating on a variable is straightforward for pupils. The already reported tendency to avoid operating on symbols representing general numbers in Logo (Hoyles, 1986; Noss and Hoyles, 1987; Sutherland, 1987; Hillel, 1992) was observed. But to show pupils how to operate on general numbers and to encourage the use of particular examples as support helped create a potential to operate on them within Logo environments and also in non-Logo environments.

To encourage the analysis of particular cases in order to deduce a general method favoured a spontaneous gradual evolution of pupils' strategies to write general Logo procedures and this did not anchor pupils to the particular. Pupils evolved from writing and analysing several fixed procedures, to writing and analysing only one fixed

procedure, to writing a general procedure by the direct analysis of one particular case not expressed in Logo. These observations support Krutetskii's (1976) affirmation that the generalization process evolves from generalizing from many particulars to the ability to generalize from one single example. Moreover, these results show that working in Logo favours the development and the formal expression of intermediate steps used by pupils as support to generalize and express a generalization in a formal way. The observed evolution shows both, that to introduce general Logo procedures as the expression of a pupil-deduced general method is meaningful to them, and that Logo may be used by pupils as an instrument that helps them deduce general methods and express them formally.

Furthermore, the observed evolution culminated with a spontaneous shift to thinking directly with general numbers. Pupils shifted to a level of abstraction in which they used and operated on general numbers without needing the explicit support of particular numbers any more. To engage in the dialectical spiral particular $\leftrightarrow$ general, favoured by the interactive nature of Logo, allowed pupils to break gradually the link with particular examples and to make this shift. To work in the Logo numeric environment favoured this shift. In contrast, in the turtle graphics environment the great majority of pupils could not make it. A cause for this seems to have been a difficulty in handling simultaneously several elements (e.g. turtle's turns, the invariant and variable elements). These results suggest that prior to formal algebra teaching pupils may develop a potential to think directly with general numbers but, this potential may be still context bounded. Depending on the environment pupils may need special support in order to develop this capability. These findings suggest that special conditions should be created to help pupils who start the study of algebra to work with general numbers independently of the context in which they are embedded.

Moreover, it was found that to use and to operate on a general number (e.g. to multiply it by a particular value) in order to write general Logo procedures may create pupils potential to operate with general numbers (e.g. to group similar terms involving general numbers). Without needing the explicit support of particular numbers the great majority of pupils of this study were able to solve a task that required grouping together of similar terms involving general numbers when they were explicitly required to do this.

### 11.1.2 Pupils' potential to work with variables in a functional relationship

To work with variables in a functional relationship pupils had to cope with the idea of correspondence between two quantities (static aspect) and with the idea of variation (dynamic aspect). The literature review shows that to cope with the idea of variation is not straightforward for pupils (Bednarz and Dufour-Janvier, 1991; Heid and Kunkle, 1988). Pupils tend to perceive dynamic processes in a static way. Pupils' lack of work with variation is suggested also as a possible cause for the difficulties they have later with notions as, for example, domain, maximum and minimum, constant functions, when introduced to the idea of function in a formal way. The results of the present study show that pupils aged 12-13 years old with an arithmetic background can develop a potential to perceive both, the static and the dynamic aspect characterizing variables in a functional relationship and to shift between them when required by a task.

This potential was shown by pupils' capability to determine numeric intervals, to fill in numeric tables, to locate values corresponding to the maximum and the minimum size of a shape, and to give a qualitative description of the global behaviour of two inter-related variables. To cope with these notions required pupils to deal with the static or with the dynamic aspect or with both of them.

Determining an interval of numbers led pupils to conceive of a variable as an entity moving within a limited range of values. Moreover, pupils' capability to consider that only one shape may correspond to any value of an interval, but different shapes may correspond to different intervals, suggests that a potential to cope with piecewise functions at an informal level was created. There is evidence that older pupils have difficulties in coping with these functions when they are presented algebraically or by graphs (Markovits et al. 1986). The results of the present study show the possibility of creating pupils' potential to work with this type of functions earlier, at an informal level. This kind of experience might be used later as a basis upon which a more formal approach to piecewise functions may be constructed. I do not wish to suggest that to shift from an informal to a formal level is an easy step, but that to have an informal experience may be a basis upon which a more formal approach to this type of functional relationships might be built.

Heid's and Kunkle's (1988) suggestion that to work with tabular records may enhance pupils' perception of the dynamic nature of variables was confirmed. To analyse the numeric table constructed on their own led pupils to shift from using variables to considering them as objects whose behaviour can be analysed. The conjunction of the

visual perception of a shape of changing size and the construction and analysis of the tabular record helped the great majority of pupils to cope with both the static and the dynamic aspects and shift between them when monotonic functional relationships were analysed. Pupils were able to locate the values corresponding to the maximum and minimum size of a shape, and to describe qualitatively the global behavior of two inter-related variables. This was not the case when non-monotonic functional relationships were analysed. In spite of the tabular records and of the variation of the visual image the majority were not able to locate maximum and minimum and to derive an overall view of the behaviour of the inter-related variables. Pupils could cope with the static and the dynamic aspects but they had difficulty shifting between them. However, it was observed that when the data of a comprehensive table were generated by pupils in a systematic way (e.g. by increasing or decreasing systematically the value of the independent variable) they had less difficulty in locating maximum and minimum values and in perceiving the global linked behaviour of variables. This suggests that encouraging pupils to construct tables by increasing or decreasing systematically the value of the independent variable may help them create a potential to cope with both, the static and the dynamic aspect and shift between them also when non-monotonic functional relationships are analysed.

Pupils' capability to cope in these environments with the ideas of interval, maximum and minimum values contrasts with the reported difficulties older pupils have with these notions when these are introduced in a formal way (Markovits et al., 1986). A characteristic of these Logo-based environments was not to present these notions formally but to approach them through the determination and use of numeric intervals and the construction and analysis of numeric tables. Another characteristic was not to present these as important notions per se but as useful steps in order to achieve a goal that was attractive to pupils. This helped pupils see the applicability of these concepts. This motivated them to engage in the activities and to work with ideas that were still unfamiliar to them. The results obtained suggest a dialectical link between the idea of interval, maximum and minimum values and the idea of variation: to work with intervals and with maximum and minimum values, helped pupils develop a potential to work with variation, to cope with variation led pupils to develop a potential to work with these notions.

To analyse constant functional relationships was difficult for the great majority of pupils. Although pupils coped with the correspondence between variables and they could look at the independent variable as an entity moving within a range of values, they were not able to describe the linked behaviour of variables. To perceive that the value of one of the

variable was not varying seems to have inhibited their capability to consider the linked behaviour of inter-related variables. This tendency to focus only on the behaviour of the dependent variable might be a cause for the already observed difficulties (Markovits et al., 1986; Sfard, 1989) which pupils have later when introduced to constant functions in a formal way. But, it was found that the experience of analysing first monotonic and non-monotonic functions may help create pupils' potential to cope with constant functional relationships.

These results show that notions (e.g. interval, maximum, minimum, monotonic, non-monotonic, piecewise and constant functions) that are usually introduced at upper school levels may be approached at informal levels by young pupils prior to the study of elementary algebra. Experiences like this might provide a basis upon which a formal approach to these concepts may be later constructed.

### 11.1.3 Pupils' potential to work with variable as specific unknown

To solve Logo-based tasks involving variable as specific unknown pupils were not expected to solve equations, but to conceptualize a specific unknown, to determine its value using their own approaches, and to use the calculated value in order to solve the problem.

The results show that the great majority were able to conceptualize the specific unknown when both, this was explicitly indicated or it was not. This finding is in accordance with Karplus's (1981) observation that pupils have the capability to conceptualize an unknown when working with simple word problems. However, it was observed that pupils may try to solve a Logo-based problem using a trial and refinement approach without conceptualizing the specific unknown involved. To use trial and refinement does not necessarily imply that the specific unknown has not been conceptualized but, this method can be used without conceptualizing it. This suggests that although to conceptualize a specific unknown is crucial in order to work with it, this is not always straightforward for pupils. This suggests that activities promoting the conceptualization of variable as specific unknown should be offered to pupils. Algebra curricula often offer few opportunities to identify the specific unknown of a problem. Instead of this, pupils are usually rushed to acquire manipulative techniques to determine the value of a specific unknown involved in an equation and, therefore, already explicitly given.

To determine the value of a specific unknowns involved in Logo-based problems pupils used arithmetic methods, that is, they operated with the data of the problem without writing equations. When they could not apply these methods the great majority turned to trial and refinement. This suggests that pupils use trial and refinement only when there are not other methods available to them. Evidence is provided that to offer an alternative method to trial and refinement (e.g. to solve an equation with multiple appearances of specific unknown in order to determine the unknown of a problem) may push them to try to use it even when they are not familiar with it.

Spontaneous attempts to solve a given equation with multiple appearances of specific unknown were observed. All the pupils avoided the grouping of similar terms. Consistent with prior findings (Filloy and Rojano, 1984, 1989; Herscovics and Linchewski, 1991a) this suggests that pupils have difficulty to operate with variable as a specific unknown. However, this result contrasts with the capability these pupils had to group the same similar terms immediately after having worked with variable as general number. On that occasion the majority were able to group them. This suggests that pupils' capability to group similar algebraic terms may depend on the characterization of variable they are perceiving in the symbol. To consider that it represents a general number was found to favour the grouping of similar terms. On the contrary, to consider that it represents a specific unknown inhibited grouping them. Therefore, in order to operate with variable as specific unknown it seems to be crucial to develop the capability to shift between viewing it as a specific unknown and considering it as representing a general number. Further investigation is needed to study if activities leading pupils to develop the capability to shift between different characterizations of variable may help them operate with variable as specific unknown.

Pupils' attempts to solve the given equation with multiple appearances of specific unknown (e.g cutting the equation after the first term following the equal sign) show that they tried to construct a suitable approach based on their prior knowledge. The method used was not always appropriate. The design of the task leading pupils to engage in a verification process (they used the calculated value as input to a given general Logo procedure) pushed them to question their own methods for solving the equation. This suggests that Logo might be used to lead pupils to test their own solution methods to solve equations with multiple appearances of specific unknown, and to question them when they are incorrect. With appropriate guidance experiences like these might be used to avoid the development of misconceptions and the installation of erroneous methods. This, however, was not the aim of this study and pupils were not led to develop methods to solve equations with multiple appearances of specific unknown.

Another interesting finding was that the experience of deducing and expressing general methods helps create pupils' potential to shift in Logo from an arithmetic to an algebraic approach to variable as specific unknown, that is, from considering it as a specific value that can be calculated operating on numbers to viewing it as a specific value that can be represented by operating on general numbers. A spontaneous tendency to use variables to express through Logo a general method to determine the value of a specific unknown for similar problems was observed. This confirms other observations showing that after intervention pupils are able to use variables in Logo (Sutherland, 1993). Nevertheless, using variables was not the general tendency. Although all the pupils were able to deduce a general method they tended to exemplify it by giving a specific example. This approach might be related to the one used in ancient times by Babylonians and Egyptians who lacked an appropriate symbolic system to express a general method. The pupils of the study lacked experience with algebraic symbolism and their experience with Logo language was very short. In comparing pupils' behaviour with that observed in ancient cultures I do not wish to suggest that ontogenesis repeats phylogenesis. Although the behaviours may be compared they have not the same social origins and necessities. Children assimilate already developed knowledge and in the course of history adults develop new knowledge. The observed capability to deduce a general method to determine the value of a specific unknown suggests that Logo is an appropriate environment in which this potential can be created. Working in Logo was shown to stimulate the development of a more abstract way of thinking and the necessity for a formal language to express these thoughts.

## 11.2 Discussion of results from a Vygotskian perspective

Four main elements common to the three Logo microworlds were crucial to help create pupils' potential to work with these characterizations of variable. These were the design of the activities, the researcher's interventions, working with Logo and the use of spoken language. All these components were interwoven during pupils' work. It is tempting to discuss this observation in relation to Vygotsky's (1978) idea of zone of proximal development.

Vygotsky (1978) claims that whether this zone is created and specific cognitive processes develop depends on the social interaction between the pupil and a more capable peer or an adult. This study suggests that social interaction is not the only critical element that intervenes in oder to create pupils' potential to work with mathematical

concepts (e.g. different characterizations of variable). A fundamental role is played by the particular task chosen to lead pupils to work with a specific concept and by the setting in which the task is developed.

In the present study collaborative work and discussion with peers were encouraged. The researcher intervened to help pupils understand the goal of a task, analyse and reflect on it, and on occasion showing how to solve it. That is, the social interaction encouraged pupils to formulate and try out their own hypothesis, to discuss and to rectify them in order to solve a given task. It also pushed pupils to break gradually the link with numbers and to shift to working with general numbers and to cope with variation.

However, no less important was the role played by the activities in which pupils engaged. Because the concepts pupils were expected to approach were clearly defined from the beginning of the study, the task were designed by the researcher and they were not left to pupils' inventiveness. Two main general aspects characterized the design of the activities. These were helping pupils use their prior knowledge as support to start working with variables and to give pupils clearly defined goals. The mathematical background of the pupils of the study was mainly arithmetical. Therefore, the design of the tasks encouraged them to work first with numbers (e.g. to write fixed Logo procedures, to construct numeric tables, to use arithmetic approaches to calculate the value of a specific unknown). The numeric examples were then used as support to lead pupils to shift to a more abstract level in which these examples were analysed and properties about them were deduced and symbolized (e.g. write a general Logo procedure based on the analysis of a fixed Logo procedure) or qualitatively described (e.g. the global linked behaviour of two variables in a functional relationship). The strategies to solve the given problems that were implicit in the design of the tasks provided the foundation that led pupils to develop new strategies. To have clearly defined goals that were meaningful to them (e.g. to draw a jungle, to put the astronaut into the rocket) helped pupils focus on concepts that were still unfamiliar to them. They gradually acquired the ability to handle these notions (e.g. express a generalization in a formal way; determine intervals; locate maximum and minimum; perceive variation; conceptualize, calculate and use the value of a specific unknown) by solving different but similar tasks. It was not left to chance that perhaps they will come to understand the purpose of the tasks. Their objective was clearly explained. This awareness led to engagement and to the work with complex mathematical notions.

Another crucial element helping to develop pupils' potential was working with Logo. The dialectical spiral particular $\leftrightarrow$ general in which pupils engaged when writing general

Logo procedures based on the analysis of fixed procedures led them to deduce general methods, to verify them and to shift gradually from the particular to the general. To have to communicate with the computer helped pupils accept the use of symbolic variables and led them to the necessity of expressing themselves in a formal way. The feasibility to work first with particular examples and use the same basic structure in order to express a general case helped pupils give meaning to the use of and the operation on symbolic variables. This led to a final shift to working directly with general numbers. The visual feedback helped pupils reflect, question and modify their methods to solve a problem. To combine the visual feedback and the numeric outcomes helped them shift from a static to a dynamic perception of variables linked by a functional relationship. The experience of expressing a generalization in a formal way led some pupils to an algebraic view of specific unknown.

Vygotsky (1978) claims that learning is not development but it provides a basis for subsequent development. It awakens a variety of developmental processes that result in the zone of proximal development. 'The initial mastery of, for example, the four arithmetic operations provides the basis for the subsequent development of a variety of highly complex internal processes in children's thinking' (Vygotsky 1978, p. 90). The results of this study support these claims. During pupils' work in each microworlds spontaneous developments of pupils' capability to work with different characterizations of variable were observed. For example, after having been introduced to write general Logo procedures pupils' strategies to write them spontaneously evolved. The researcher did not teach pupils how to group similar terms involving general numbers, however, the majority of pupils could do this. Pupils were not shown how to determine a numeric interval, they were only asked to find out *all* the value that gave the same shape. This requirement and working in Logo led them to develop the capability to determine intervals and to use them. These and similar observations suggest that to work in these environments awakened processes that led pupils to develop capabilities to cope with variables. Moreover, after working in the Logo-based environments it was observed that pupils coped better with simple usual algebra tasks involving different characterizations of variable. The Logo-based experience seems to have created this pupils' capability. The difficulties pupils have to cope with algebraic variables already reported by prior studies suggest that usual algebra tasks and usual school environments may not be conducive to creating this potential.

These observations provide arguments which show that to help create pupils' potential to work with specific mathematical notions the particular activities and setting chosen are

crucial. A value of this study is to have shown that a zone of proximal development to work with different characterizations of variable can be created. This shows that a potential to learn these concepts can be created before the teaching of formal algebra. However, the extension of this zone depends on the individual pupil. A limitation of this study is not to have explored deeper the zone of proximal development created by the Logo-based activities for each pupil.

Vygotsky (1978) asserts that all higher psychological functions are first social and they are gradually internalized by means of mediating factors as, for example, language. 'This transformation is the result of a long series of developmental events. The process being transformed continues to exist and to change as an external form of activity for a long time before definitively turning inward' (Vygotsky, 1978, p. 57). During this study transformations leading to the internalization of the concepts pupils were working with were observed. These transformations were suggested by pupils' use of speech. Both, communicative (speech-for-others) and egocentric speech (speech-for-oneself) were often used by pupils. Shifts from communicative to egocentric speech were observed and on occasion both types of speech disappeared and pupils worked in silence. This suggests transformations leading to a gradual internalization of the notion pupils were dealing with. When faced with a complex task both, communicative and egocentric speech reappeared. This suggests that the internalization process develops in spiral returning to a more primitive approach when the complexity of the task involving the concept faced increases.

## 11.3 Implications of the study

Some implications for research methodology and for teaching can be made on the basis of the results obtained.

In contrast with other studies that focused on pupils' work with only one specific aspect of variable (e.g. pupils' capability to write 'unclosed' expressions; pupils' difficulties when solving equation with a particular structure), this study has investigated pupils' work with the multifaceted concept of variable. That is, the same pupils worked during a short period of time with different characterizations of variable. This study shows the feasibility to design environments that favour this kind of investigation.

This study was carried out in a school environment working with a whole group. The results obtained show the feasibility of investigating the development of pupils' potential in their natural school setting and not shaped by a laboratory environment. Moreover, this strengthens the results obtained showing that pupils' potential can be developed in a normal school environment.

The researcher was the teacher of the group. This allowed her to organize and to lead the work of the whole group in order to favour the purposes implicit in the design of the activities. She could control the kind of support and the moment in which this was provided in order to help create pupils' potential. Although the general type of interventions were planned from the beginning of the study, the details of the specific help required by each pair could not be planned. To be clear about the purposes to be reached helped her to decide how to intervene. The results obtained suggest that although it was not always easy to cope with, having the double role teacher/researcher may have great benefits when the development of pupils' potential for learning mathematical concepts is being investigated.

Pupils' work with variables usually starts when they begin the study of algebra. The results of this study suggest that pupils' experience with different characterizations of variable can be substantially extended prior to formal algebra teaching by helping create pupils' potential to work with them. To create this potential pupils' capability to work with numbers was used. This shows that pupils' arithmetic background can be used as support to work with complex algebraic notions at an informal level (e.g. generalization and its expression, operation on/with general numbers, intervals, maximum, minimum, monotonicity, constant functional relationships). The results obtained suggest that elementary teaching should emphasize the idea of number and extend pupils' work with numbers in order to provide a basis upon which pupils' potential to work with different characterizations of variable might be afterward developed.

The results obtained also show that pupils could develop a potential to work in Logo-based environments during the same period of time with different characterizations of variable. Moreover, the interview at the end of the study showed that this experience helped pupils develop a potential to shift from one characterization of variable to another when explicitly asked to do this in order to solve a Logo-based task. Although this capability was investigated only in Logo-based environments the results obtained suggest that these different characterizations of variable cannot be ordered depending on their level of complexity. This implies that the teaching of them does not necessarily need to be hierarchically organized. The level of complexity at which each one of them can be

approached might be controlled and gradually increased. This observation leads to suggest a spiral teaching model in which different characterization of variable are approached at the same time and at different levels of complexity. A spiral approach to variables might help pupils develop very early the capability to work with all of them and to shift from one characterization to another. This capability might help pupils cope later with algebraic expressions. This contrasts with the way in which pupils work with variables at school, for example, in Mexico. Usually variable as specific unknown is first approached at different levels of complexity and after a long period variables in a functional relationship are introduced. There is very little work to help pupils cope with variables as general numbers although they often appear in the school texts even before the formal teaching of algebra begins.

Evidence was provided that algebraic symbolism is not a prerequisite for helping create pupils' potential to work with different characterizations of variable in Logo-based environments. The results obtained show that symbols to represent variables may be introduced as a means of helping pupils express something they are already able to work with. By using symbols pupils can express and manipulate their thought in a concise and efficient way. An approach like this reflects the historical development that led to the appearance of symbolic algebra. This appeared as a consequence of the necessity to have an appropriate system of symbols to allow the handling of general problems that were until then solved rhetorically or numerically. However, very often the way in which elementary mathematics is taught does not create this necessity and algebraic symbols are introduced when pupils have nothing to express with them in this approach. The symbols precede the necessity for them and in order to make sense of them pupils have to guess their possible meaning. This comment does not wish to suggest that the historical development of mathematics has to be followed in order to help pupils cope with variables. In fact, there are basic distinct characteristics between the historical development of a concept and its learning. In the course of history concepts are invented and elaborated. In school pupils are expected to assimilate them. However, a sensible explanation for the difficulties algebra beginners have in coping with variables might be a lack of experience with different characterizations of variable prior to being introduced to algebraic symbolism to handle them. This study shows that an anti-historical approach often used in school is an option and not a necessity. It is possible to create environments in which pupils develop a necessity for symbolic variables to express their ideas. This, however, does not imply that the researcher considers symbols as tools by which already elaborated ideas are expressed. Results of this study show, for example, that when a symbol to represent a general number is introduced in Logo a dialectical link between its use and the development of new concepts making use of the symbols begins. This

suggests that symbols can be introduced as a necessity to express an idea although the original idea might be modify as a consequence of using a symbol to express it.

During all the activities it was observed that when faced with new tasks pupils used their prior knowledge to solve them. That is, they tried to find common aspects in different tasks and, in consequence, they tried to solve the new task by using a method that have been successful for solving prior ones. Evidence for this was given when pupils wrote general Logo procedures, operated on and with general numbers, defined intervals, filled in tables, analysed the behaviour of inter-related variables. This suggests that pupils used analogy to make sense of the problems and to solve them. Analogy is a way of reasoning that is used in common life but also by great mathematicians. It often concurs in solving mathematical problems as well as in discoveries. Polya (1954) considers that perhaps no discovery in elementary or in advanced mathematics could be made without analogy. In the present study this kind of reasoning was favoured by the design of the tasks. The results obtained suggest that encouraging pupils' use of analogy may facilitate the learning of new mathematical concepts as, for example, different characterizations of variable. This suggests that this type of reasoning should be encouraged in school and pupils should be helped to develop it and apply it correctly.

## 11.4 Suggestions for further research

In spite of the extremely interesting ideas offered by the theory of Vygotsky, it is surprizing that this theory has received so little attention from researchers in mathematic education, at least from the point of view of practical experimentation in the classroom. This study has shown that working with ideas akin to that of *zone of proximal development*, the kind and amount of help pupils need in order to cope with mathematical concepts can be assessed. Information can be gained concerning pupils' capability to work with the same concept in different contexts as well as to shift from one concept to another within the same task. More investigation is needed in mathematics education from a Vygotskian perspective. For example, to investigate if the idea of zone of proximal development can be used to structure pupils' work in domains already known to be difficult to deal with (e.g. functions and graphs).

To learn mathematics is still often considered to be an individual matter. Computers are frequently used to promote individual work. In this study computer environments were

designed to favour cooperation between pupils and with researcher. More investigation is needed to see if working within environments in which a collective rather than an individual approach to mathematical concepts is emphasized helps pupils cope better with these concepts.

This study suggests that the use of natural language is a critical element in helping pupils cope with different characterizations of variable. A limitation of the study was that pupils' dialogues and researcher's interventions were not recorded during the work in the Logo microworlds. More work is needed to investigate how the use of speech influences pupils' work with mathematical concepts. In particular there is a need to investigate if encouraging the use of communicative speech (e.g. group discussion, cooperative work) and egocentric speech (e.g. speaking aloud during individual work) facilitates pupils' work with different characterizations of variable in non-Logo environments. How pupils' understanding of algebraic concepts is affected by communicating these concepts has not yet been investigated.

Several different elements were found to be crucial in helping create pupils' potential to cope with different characterizations of variable prior to formal algebra teaching. But being beyond the scope of this study, I did not investigate how pupils' own development affected the creation of this potential. More research needs to be carried out to clarify this. For example, it would be worth to explore if pupils younger than the pupils of this study can be helped to develop a potential to cope in Logo with these notions. That is, investigation is needed to see if a potential to cope with algebraic concepts can be developed during primary schooling.

In this study Logo was not fully exploited when functional relationships and specific unknown were involved. Further research could deepen understanding of the role of Logo in creating pupils' potential to work with variables. Very often more than one characterization of variable is involved in a problem. A limitation of this study was to focus on the creation of pupils' potential to work with one characterization of variable at a time. Further studies are needed to investigate if working in Logo-based environments helps create pupils' potential to discriminate between different

characterizations of variable and to shift between them when they appear within the same task represented by the same symbol.

A disparity between Logo numeric and turtle graphics environments in helping create pupils' potential to work with general numbers was found. It was conjectured that the difficulties pupils found in the turtle graphics environment seemed to be due to the existence of a concrete referent external to the variable itself. More investigation is needed to prove this conjecture.

The results of the questionnaire applied at the end of the study suggest that after the Logo experience pupils improved their work with normal algebra tasks. More investigation is needed to find out whether teaching normal algebra after working with Logo helps pupils work better with it.

During this study the researcher was the teacher of the computer workshop. She had no influence on the activities being carried out in the mathematics class. More research needs to be carried out in order to explore whether working with different characterizations of variable in Logo-based environments in parallel with working on these characterizations in the usual mathematics class helps improve pupils' work with them in both environments.

This study was carried out in a normal school environment. The results obtained have validity for classroom practice. The author believes that in order to suggest ways to improve the teaching and learning of mathematics more research should be developed within school settings.

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# APPENDIX 1

## The questionnaire

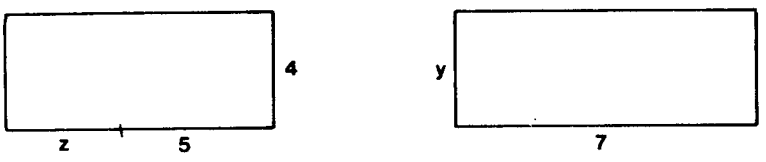
## Part 1

1. In the following exercises DO NOT calculate the number, write only a formula.
   1.1 Write a formula which means: An unknown number plus 5 is equal to 8.
   1.2 Write a formula which means: An unknown number multiplied by 13 is equal to 127.
   1.3 Write a formula which means: An unknown number is equal to 6 plus another unknown number.
   1.4 Write a formula which means: multiply 8 by the addition of 3 and an unknown number.
   1.5 Write a formula which means: 4 added to n + 5.

2. For each expression write the values you think the letter can have. If you think it can have more than one value, then write some of them.
   2.1 $X + 2 = 2 + X$ _____
   2.2 $3 + y = 7$ _____
   2.3 $X = X$ _____
   2.4 $4 + S$ _____
   2.5 $X + 5 = X + X$ _____
   2.6 $3 + A + A = A + 10$ _____

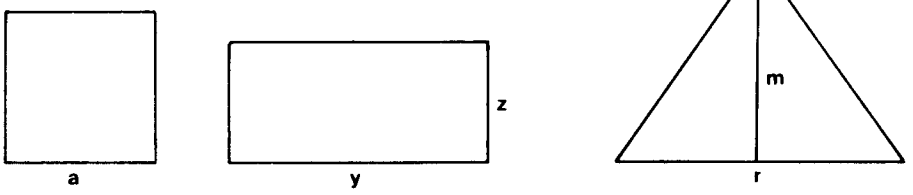
3. Write for each expresion all the values that you think the letter can have.
   $Z + 874 = 1\ 093$ _____
   $525 + X = 823$ _____
   $4 \div X = 2$ _____
   $3 \times M = 15$ _____

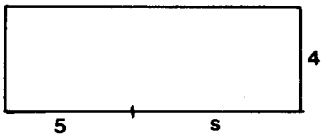
4. The **perimeter** of a figure is the addition of its sides. Write a formula to express the **perimeter** of the following figures:




5. Write a formula which means: An unknown number is bigger than 5.

6. Write a formula to calculate the **area** of the following figures:




7. Solve:

□ + 512 = 721

□ + 2 = 2 + □

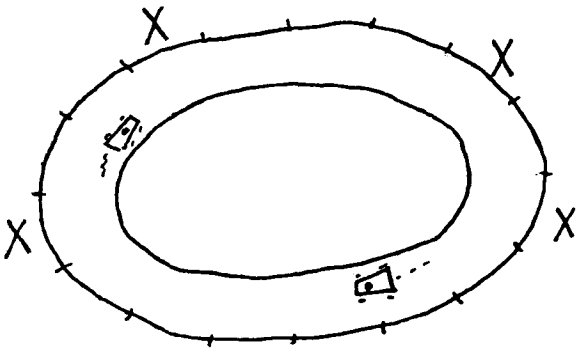
437 - □ = 25

3 * □ = 15

4 ÷ □ = 2

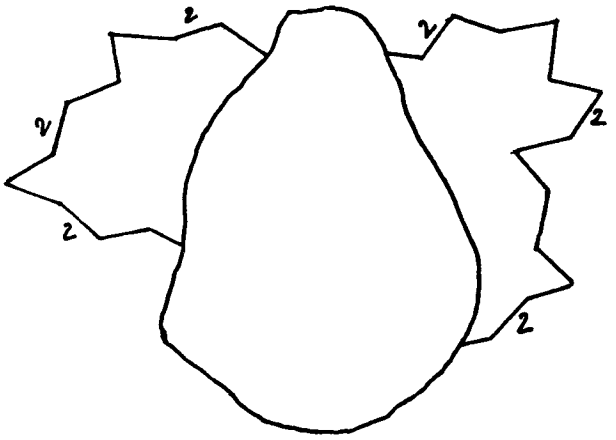
**Part 2**

8. A race track is divided into 16 parts of equal length. Each part measures X kilometers. Write a formula to express the total lenghts of the race track.



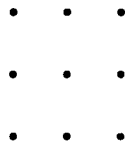
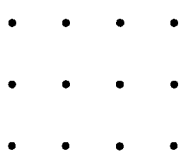

Write a formula to express how many kilometers a car will move if it travels three times around the track.

9. The next figure is not completely visible. Since we do not know how many sides it has in total, we will say that it has N sides. Each side measures 2 centimeters. Write a formula to calculate the perimeter of the figure.




10. A highway which length was X kilometers was made 25 kilometers larger. How would you express the new lengths of the highway?

11. Look at the following pictures:

| | | Number of dots |
|---|---|---|
| 1 |  | 1 x 1 |
| 2 | | 2 x 2 |
| 3 |  | 3 x 3 |
| 4 |  | |

How many dots are ther in the picture number 4?

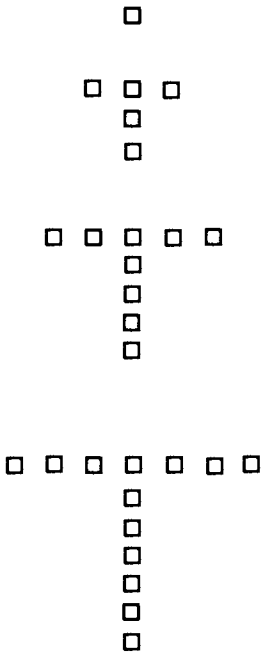
Draw pictures number 5 and 6, and write down the respective number of dots.

5

6

Imagine that yuou keep drawing until the picture number m (m represents an unknown numer). How many dots will the picture number m have?

12. Look at the following pictures and the respective numbes:



Number of squares

1

3 + 2 = 5

5 + 4 = 9

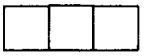
7 + 6 = 13

How will the next picture look? Draw it.

In the picture that you drew, how many squares are on the horizontal line? How many of them are in the vertical? How many in total?

Image that you keep drawing until you have P squares on the horizontal line. In the figure: how many squares would be on the horizontal line? How many of them are in the vertical? How many in total?

13. Look at the following figures and the respective numbers.

| | Numbers of squares | Numbers of sides |
|---|---|---|
| | 1 | 4 |
| | 2 | 7 |
|  | 3 | 10 |

Complete the numbers:

4
5
10
12

Imagine that you keep on drawing until you have Y squares. Write a formula to calculate the number of sides when you have Y squares. How many sides will be there if you draw 100 squares?

## Part 3

14. If X + 3 = Y, what value can X have? What values can Y have?
    Is there any relationship between the values of X and Y? (Underline the correct answer).
    YES NO I DO NOT KNOW
    Explain your answer.

15. If Y = X + 5 , the value of Y will be bigger than the value of X. (Underline the correct answer).
    ALWAYS NEVER SOMETIMES
    Explain your answer.

16. If Y = 7 + X , what happens to the value of Y when the value of X increases?

17. Considering that Y = X + 4, complete the table:

| X | Y |
|---|---|
| 5 | |
| | 12 |
| 3.5 | |
| | 7.5 |

18. Consider Y = 3 + X.
    If we want the value of Y to be 10, what value must X have?

    If we want the value of Y to be 3, what value must X have?

    If we want the value of Y to be bigger than 3 but smaller than 10, what are all the values that X can have?

    If X takes values between 8 and 15, between what values will be the value of Y?

19. Which is bigger: N + 2 or 2 x N? Explain your answer.

| N | N + 2 | 2 x N |
|---|---|---|
| | | |

# APPENDIX 2

## The LOGO work sheets

# Introduction to Logo

**FD** A NUMBER

**BK** A NUMBER

SOME OF THE WORDS
I UNDERSTAND

**RT** A NUMBER

**LT** A NUMBER

# Introduction to Logo

**HOME**
**CS**

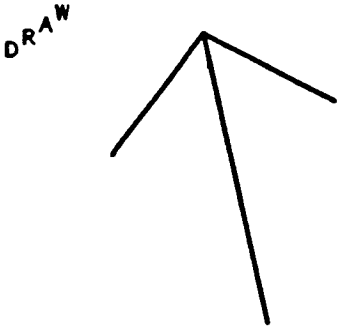
I UNDERSTAND THESE WORDS TOO

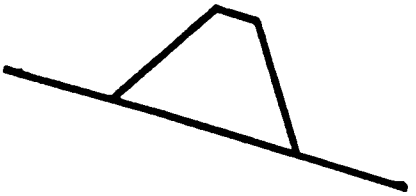
**PU**
**PD**

**ST**
**HT**

**PENERASE**
**PENCOLOR 1**

DRAW





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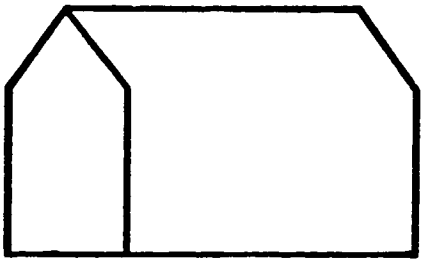
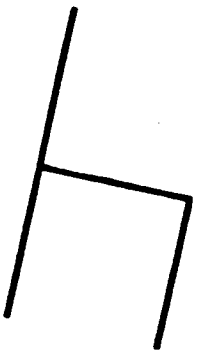


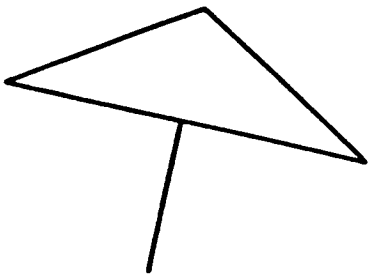


DRAW

# Introduction to Logo

DRAW

DRAW



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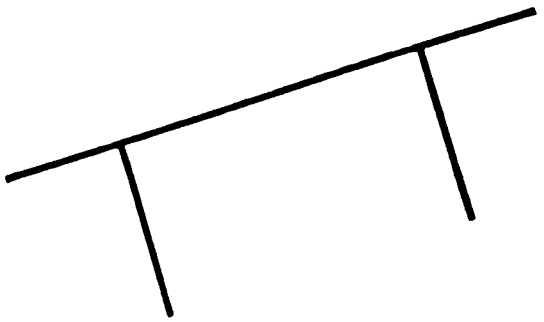

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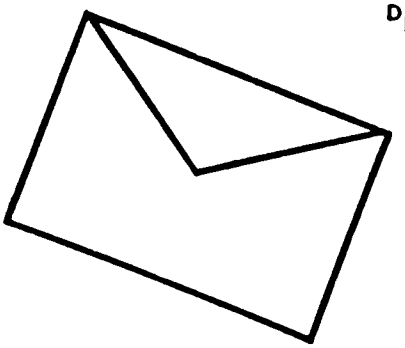

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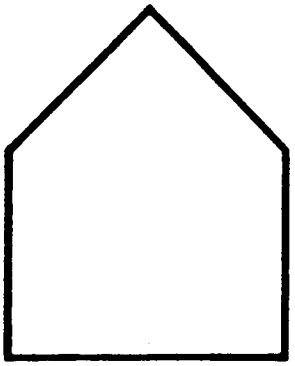

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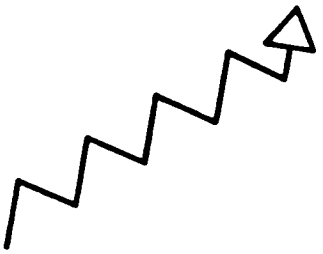

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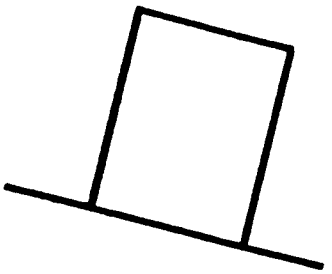

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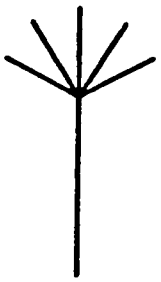

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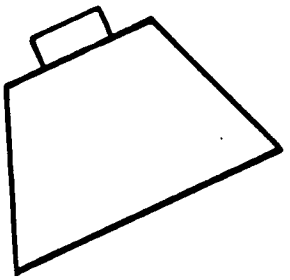
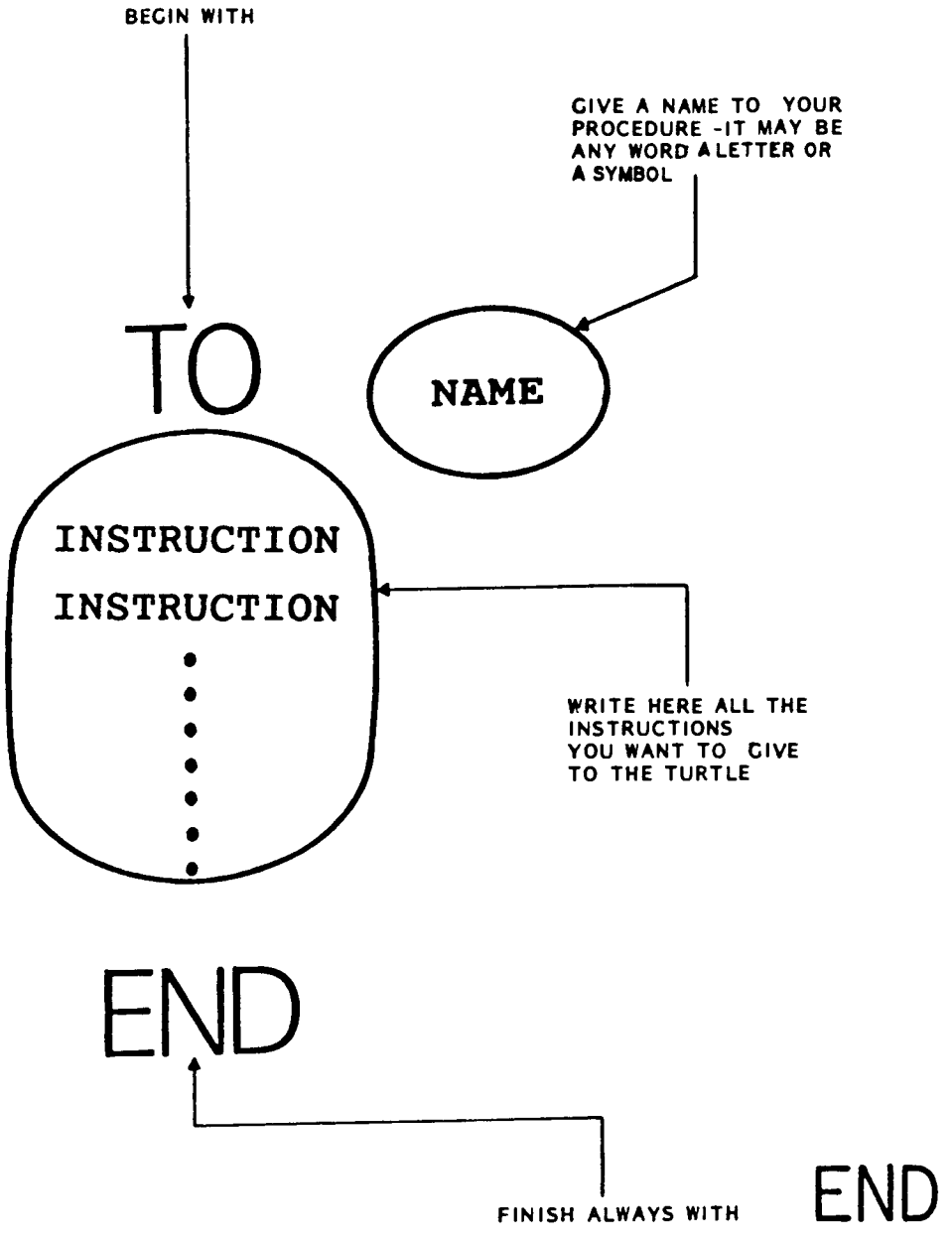


DRAW

# How to write fixed Logo procedures



# How to edit a procedure

TO CORRECT A PROCEDURE YOU MUST EDIT IT.

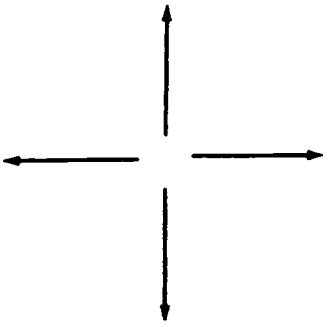
TYPE:

EDIT "NAME (PC)

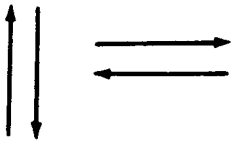
EDIT NAME (COMMODORE)

(WRITE HERE THE NAME OF YOUR PROCEDURE)

USE THE ARROWS OF THE KEYBOARD TO MOVE THE CURSOR:



(PC)



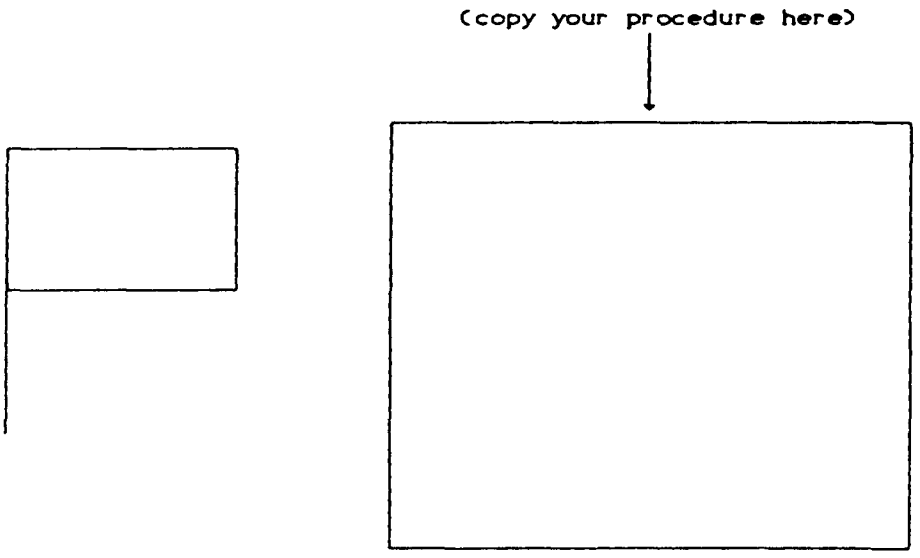
(COMMODORE)

WHEN YOU FINISH YOUR CORRECTIONS KEYS:

ESC (PC)

CTRL AND AT THE SAME TIME C (COMMODORE)

# Write a procedure to draw a flag




TYPE NOW:

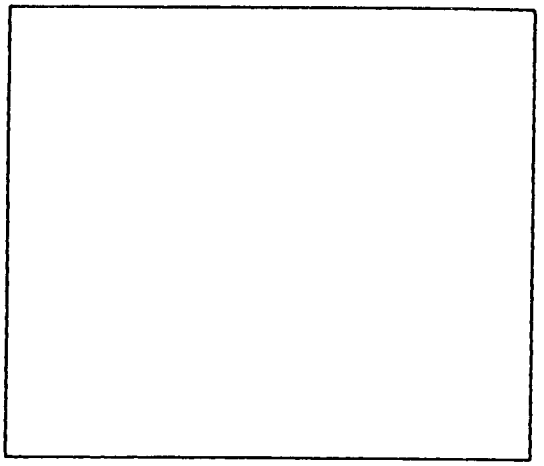
```
NAME      (THE NAME OF YOUR PROCEDURE)
PU
FD  30
PD
NAME      (THE NAME OF YOUR PROCEDURE)
```

USE YOUR PROCEDURE TO FILL THE SCREEN OF FLAGS.

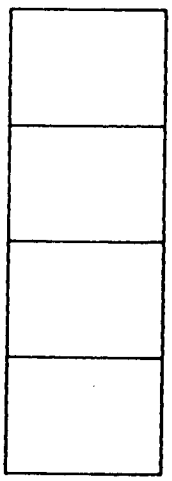
WORK SHEET I-9

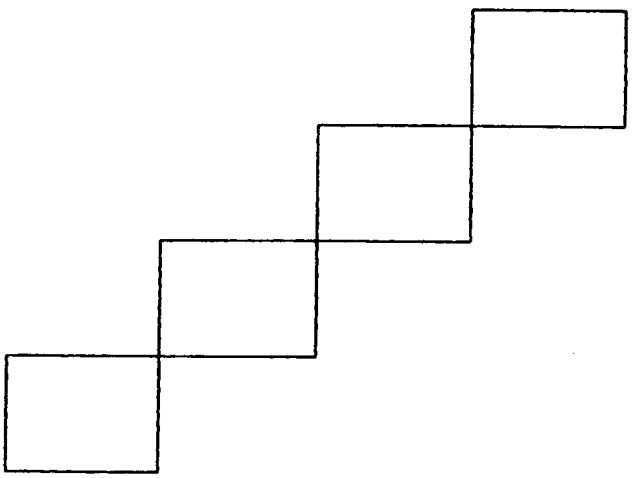
# Write a procedure to draw a square

(copy your procedure here)



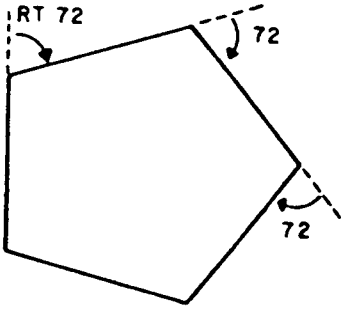
USE YOUR PROCEDURE TO DO THESE DRAWINGS:



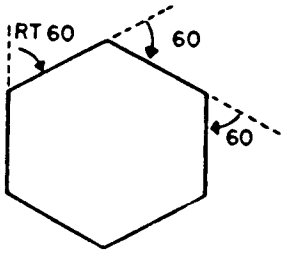


## Write a procedure to draw each one of these polygons

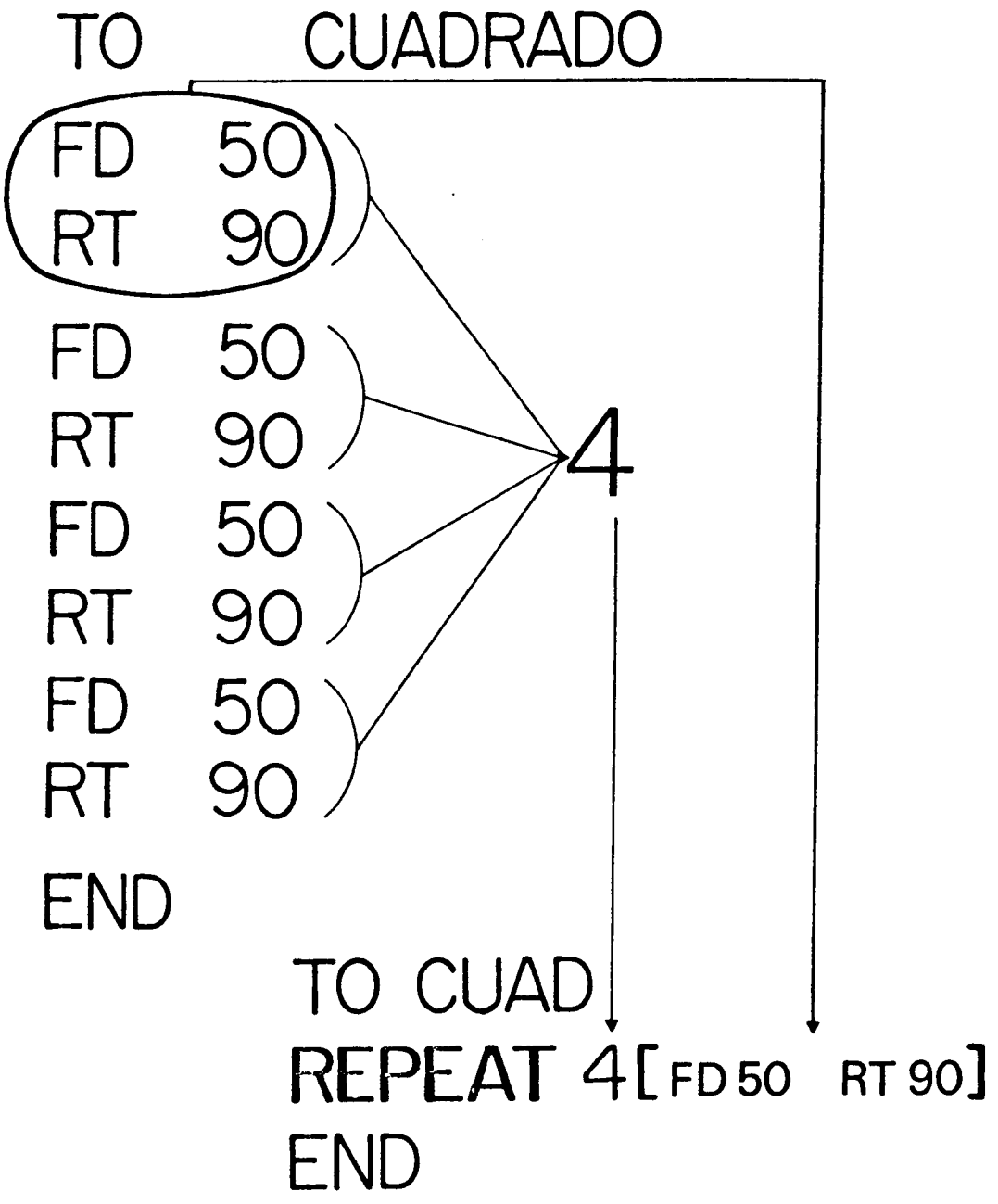
(copy your procedure here)




(copy your procedure here)

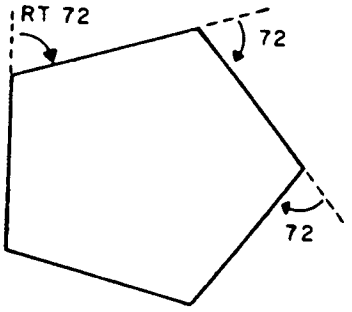


## The use of REPEAT

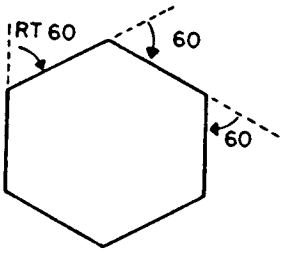


# Rewrite your procedures using REPEAT

(copy your procedure here)



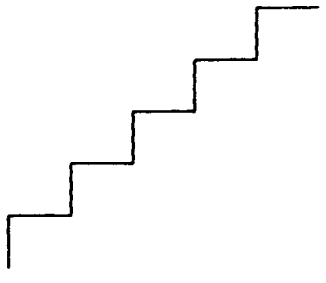

(copy your procedure here)

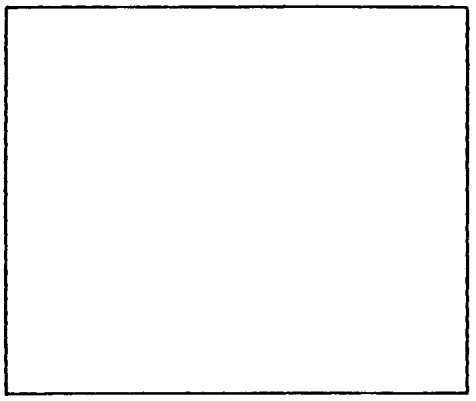


# Can you draw stairs?

WRITE A PROCEDURE TO DRAW
A STAIR WITH 5 STEPS

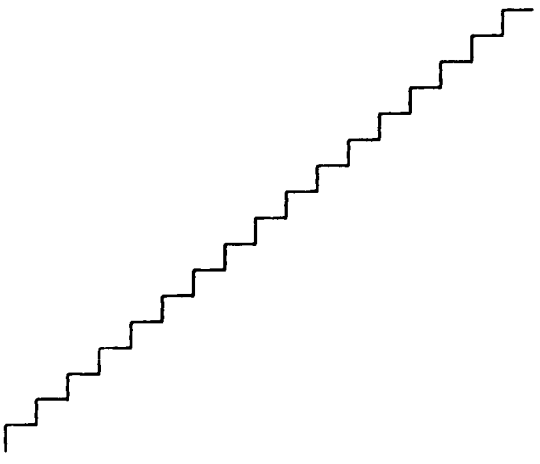
(procedure)

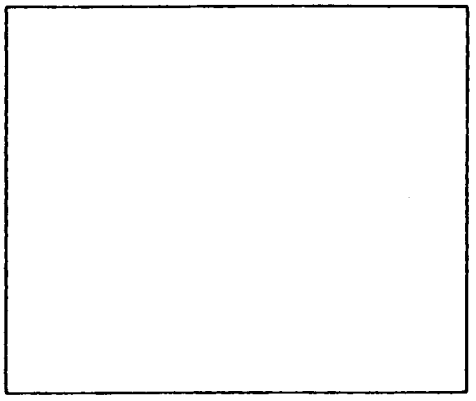




WRITE NOW ANOTHER PROCEDURE TO DRAW
A STAIR WITH 100 STEPS

(procedure)





WORK SHEET G-1

## With Logo we can make calculations

T Y P E :

```
TO SUMA
PRINT 15 + 13
END
```

```
TO MULTI
  PRINT 9 * 8
END
```

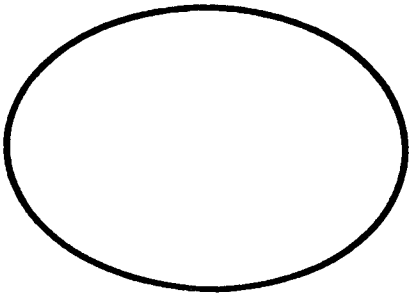
CHOOSE A NUMBER AND WRITE A PROCEDURE TO CALCULATE THE DOUBLE OF THAT NUMBER:

TO

PRINT

END

NAME



# Doubling numbers

CHOOSE OTHER NUMBERS AND WRITE A PROCEDURE TO CALCULATE THE DOUBLE OF EACH ONE.

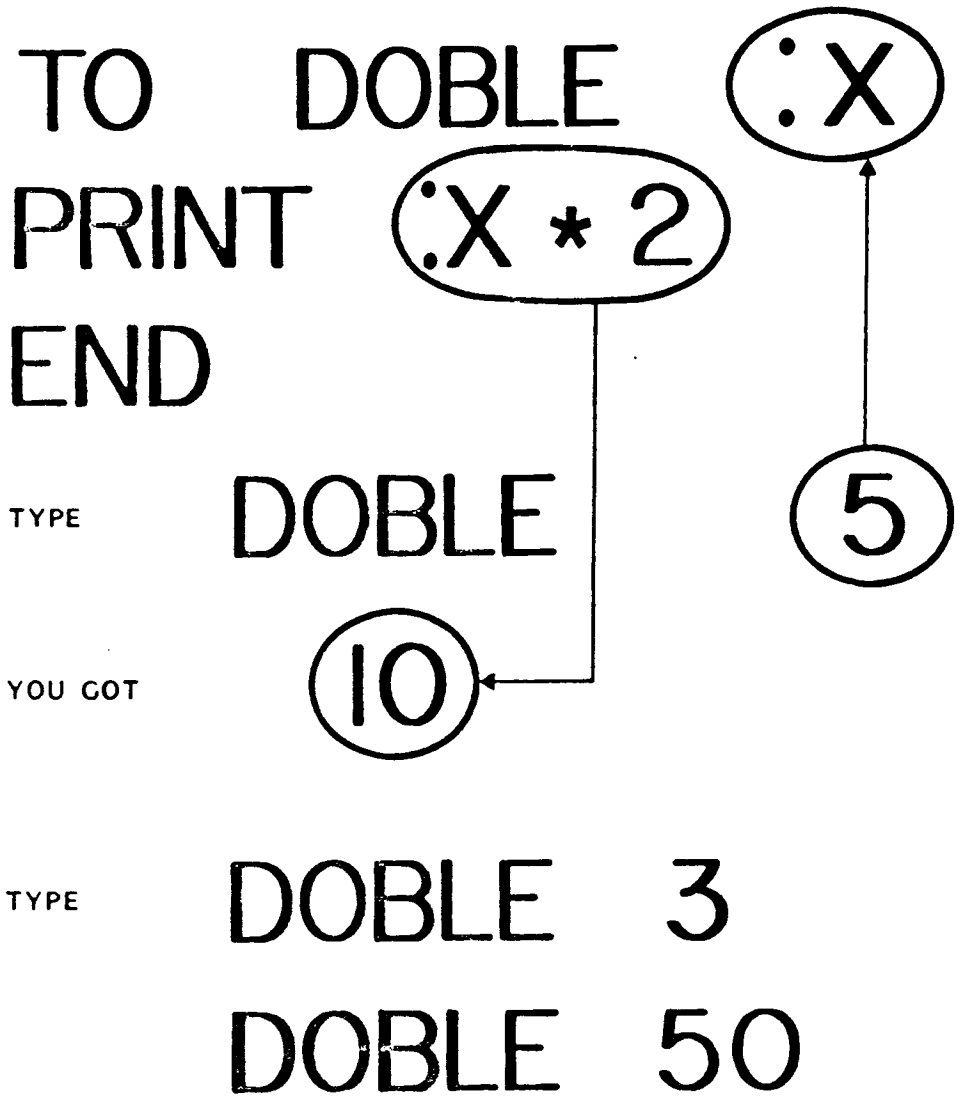
PROCEDURE 1 | PROCEDURE 2 | PROCEDURE 3

ANALIZE THE THREE PROCEDURES.
HOW ARE THEY SIMILAR ?

HOW DO THEY DIFFER?

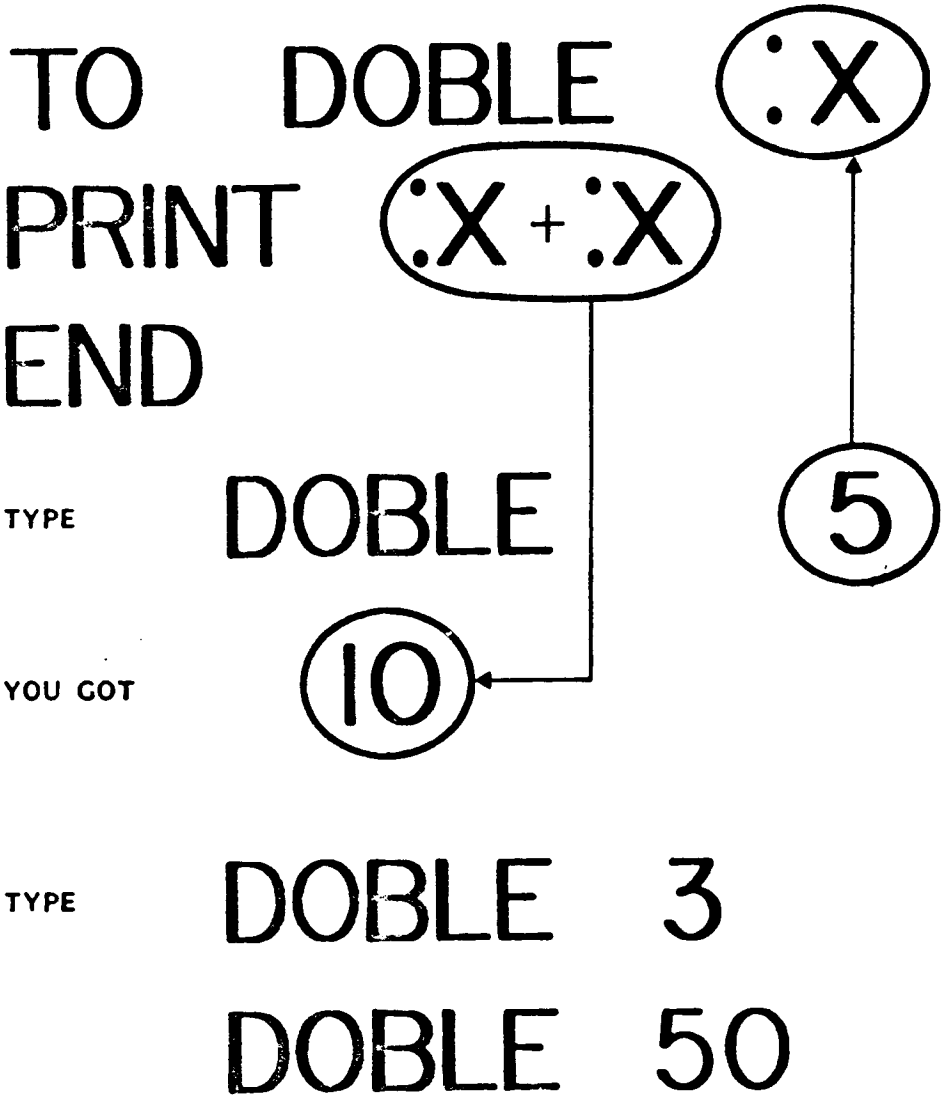
# How to write general Logo procedures

THIS PROCEDURE CALCULATES THE DOUBLE OF ANY NUMBER X:

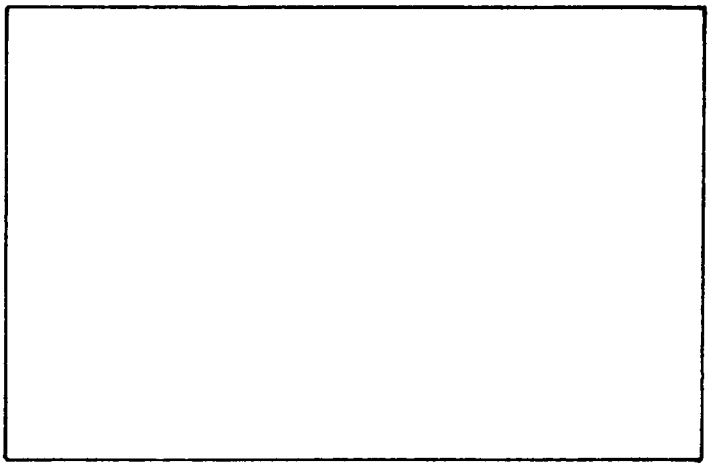


# How to write general Logo procedures

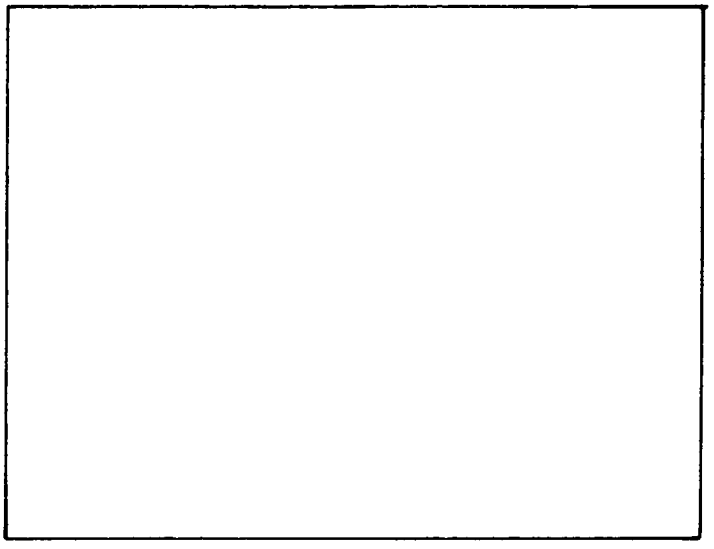
THIS PROCEDURE CALCULATES THE DOUBLE OF ANY NUMBER X:



WRITE A PROCEDURE TO CALCULATE HALF OF ANY NUMBER:

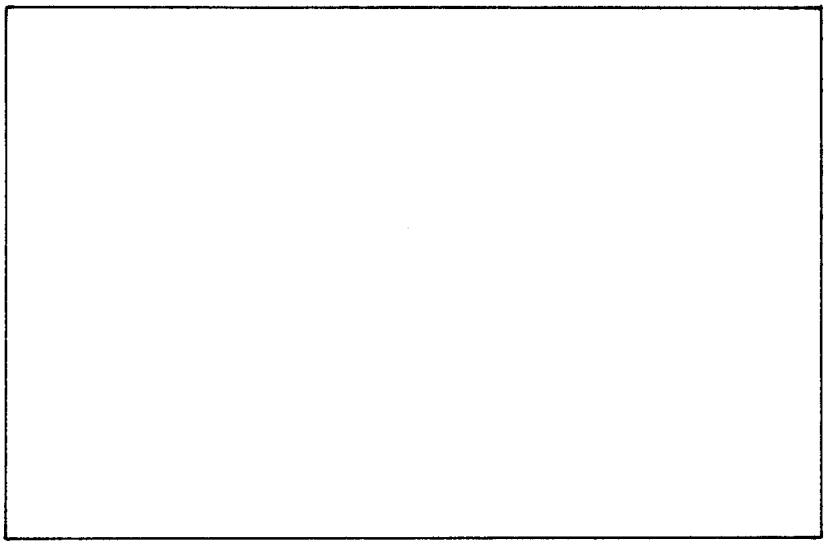


WRITE A PROCEDURE TO ADD 10 TO ANY NUMBER:

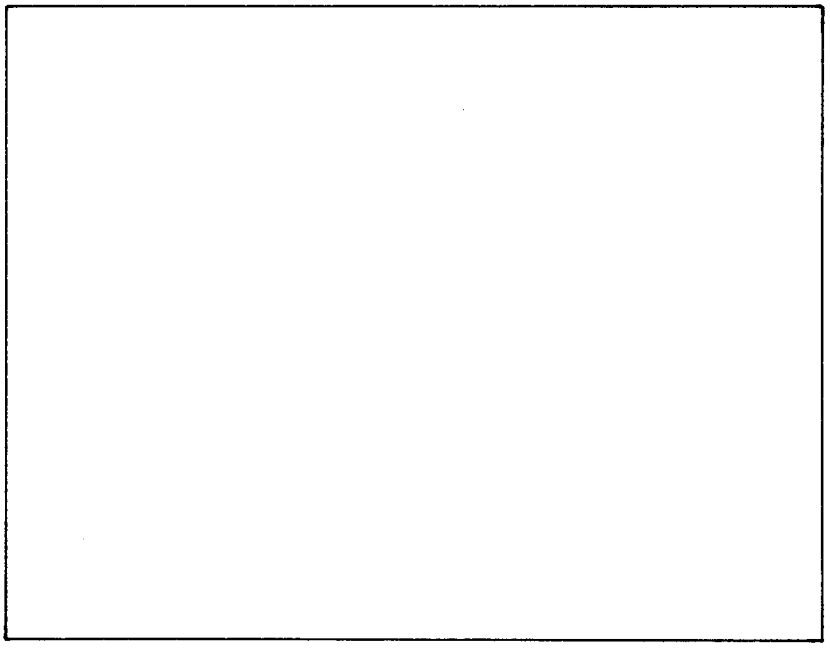


**WORK SHEET G-6**

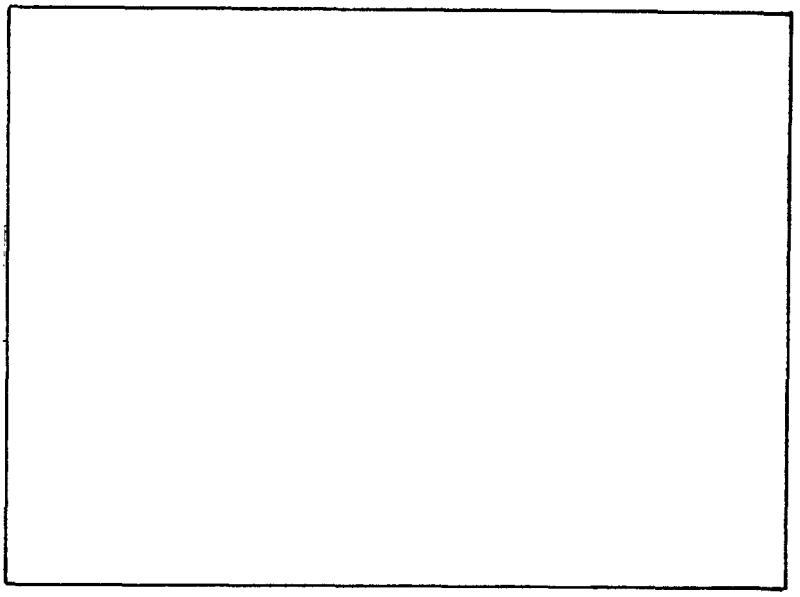
WRITE A GENERAL PROCEDURE TO SUBTRACT 10 FROM ANY NUMBER:



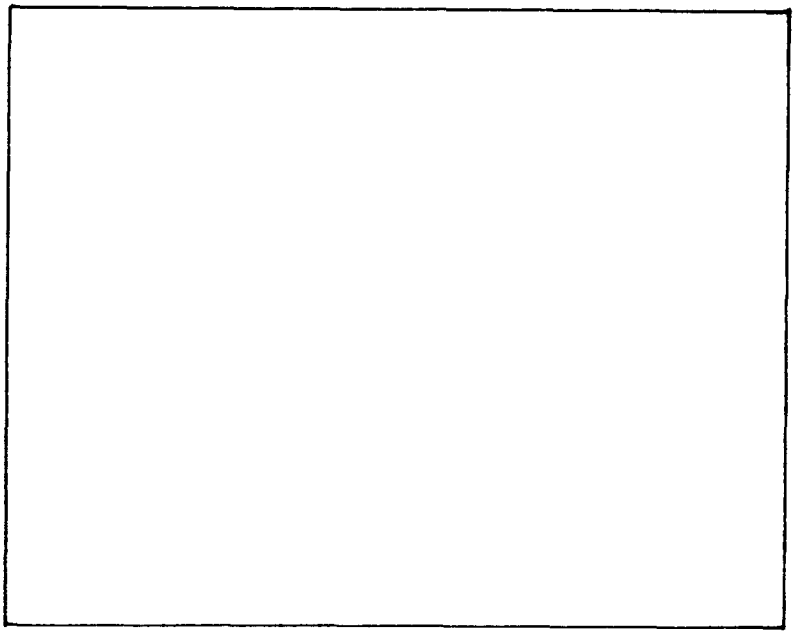
WRITE A GENERAL PROCEDURE TO MULTIPLY ANY NUMBER BY ITSELF:



WRITE A GENERAL PROCEDURE TO ADD 5 TO THE ADDITION OF ANY NUMBER BY ITSELF:



WRITE A GENERAL PROCEDURE TO DIVIDE ANY NUMBER BY 10 AND TO ADD 100 TO IT:



WORK SHEET G-8

**Choose a number and write a procedure that helps you fill in its multiplication table:**

**TABLE**

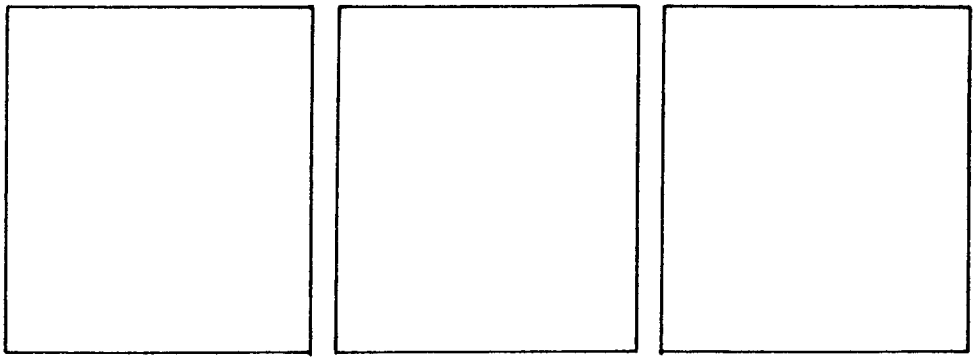
**PROCEDURE**

WORK SHEET G-9

# Drawing squares

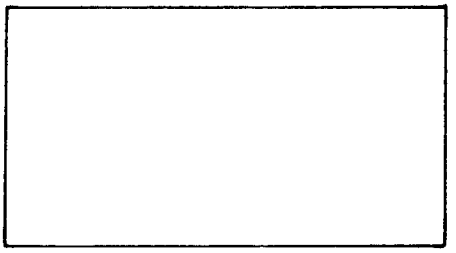
WRITE THREE PROCEDURES TO DRAW THREE SQUARES OF DIFFERENT SIZE:

PROCEDURE 1 PROCEDURE 2 PROCEDURE 3

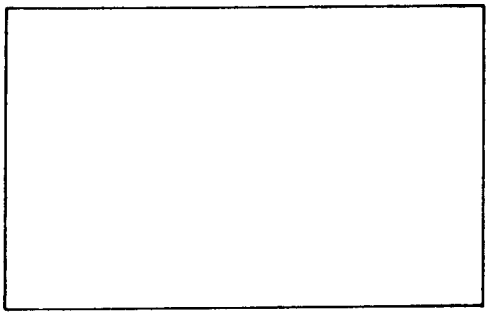


ANALIZE THE THREE PROCEDURES.

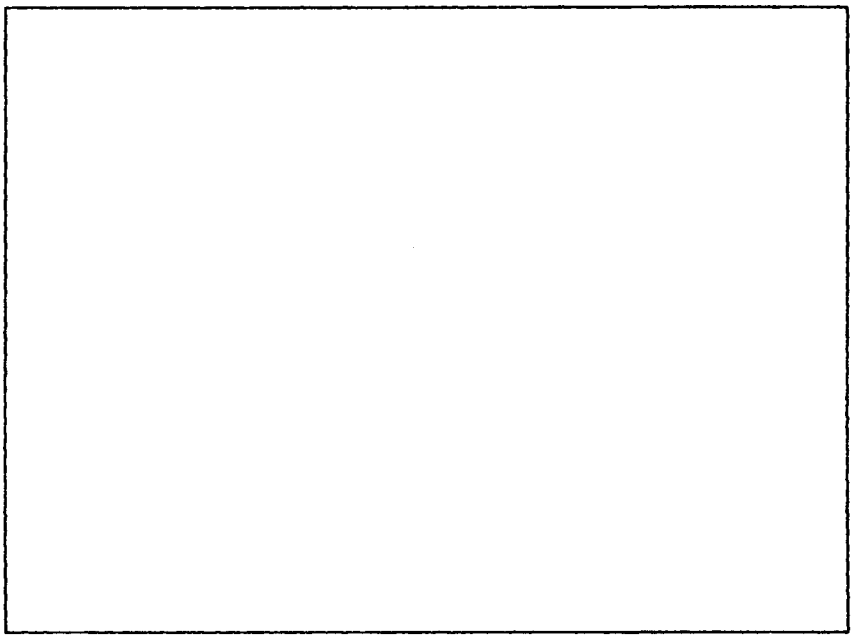
HOW ARE THEY SIMILAR ?



HOW DO THEY DIFFER ?

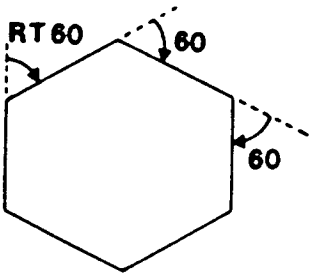


**Write a general procedure to draw a square of any size**

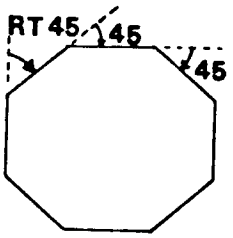


WRITE A GENERAL PROCEDURE FOR EACH POLYGON:

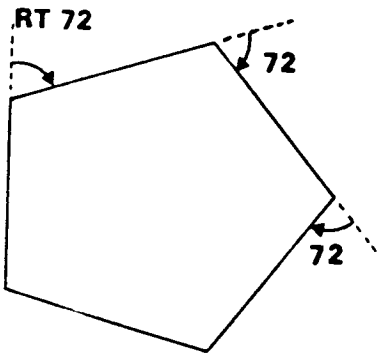
PROCEDURE 1




PROCEDURE 2



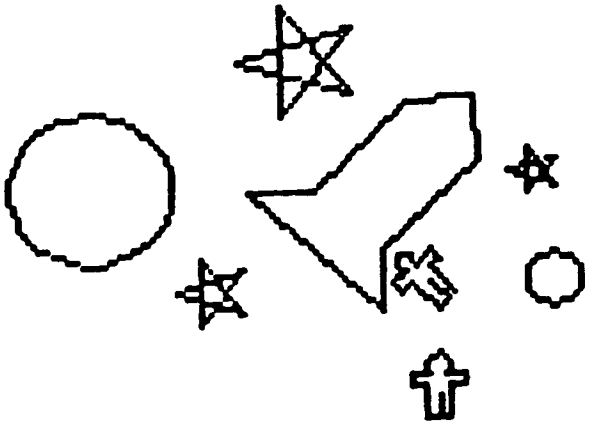

PROCEDURE 3



WORK SHEET G-12U

## Choose your project:

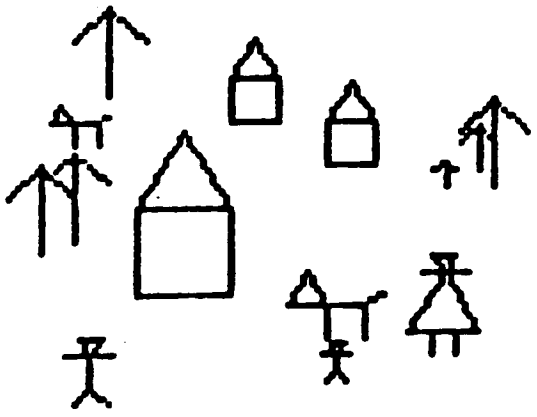
UNIVERSE



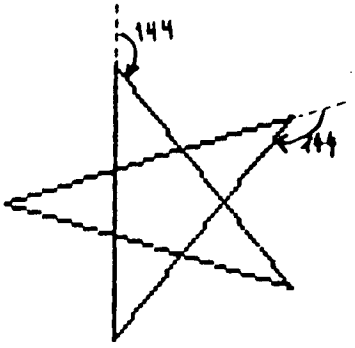
WORK SHEET G-12V

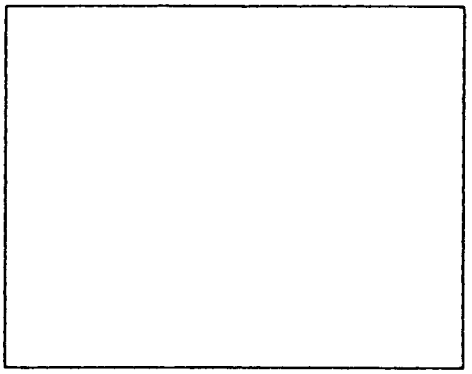
## Choose your project:

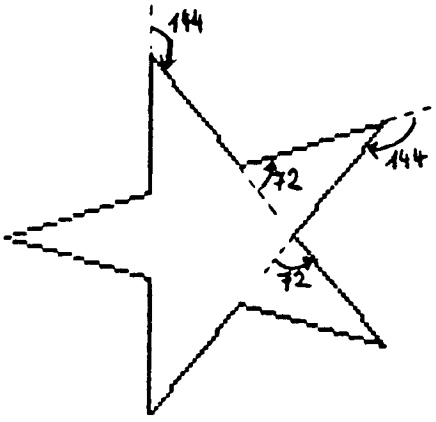
VILLAGE

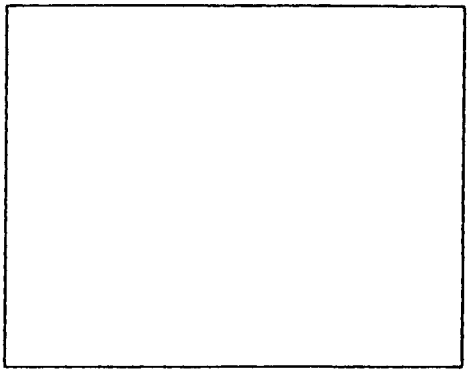


**Write a procedure to draw each figure in a variable size:**

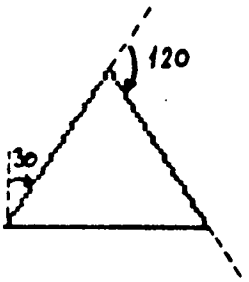


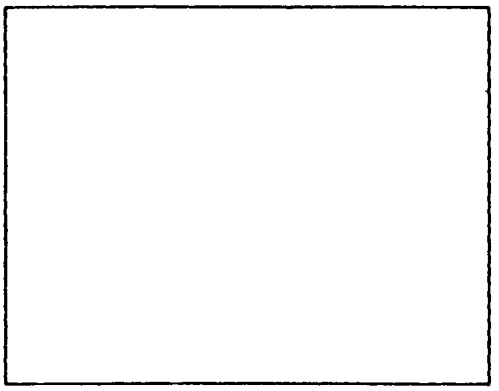



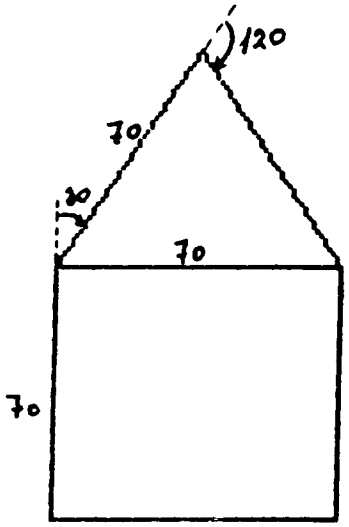


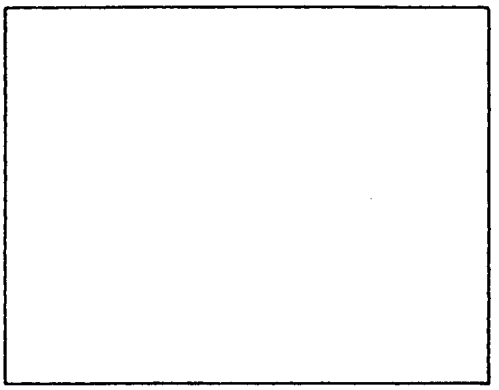



**Write a procedure to draw each figure in a variable size:**

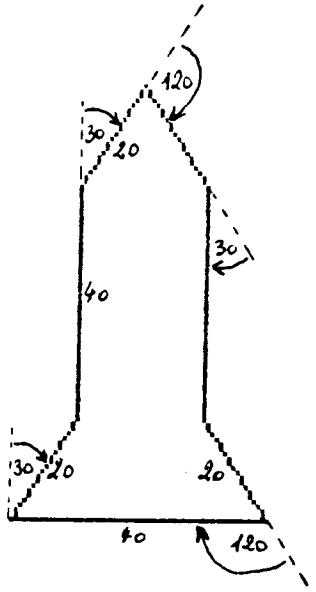


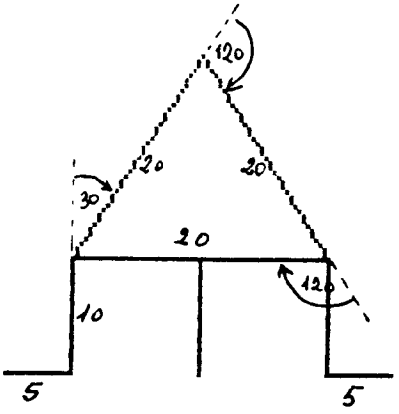





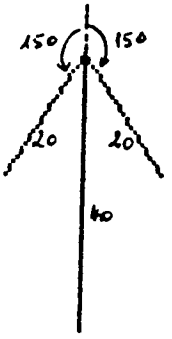



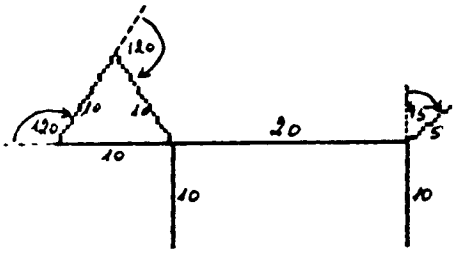
**Write a procedure to draw each figure in a variable size:**



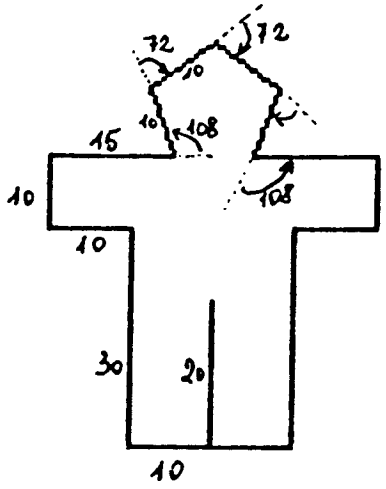



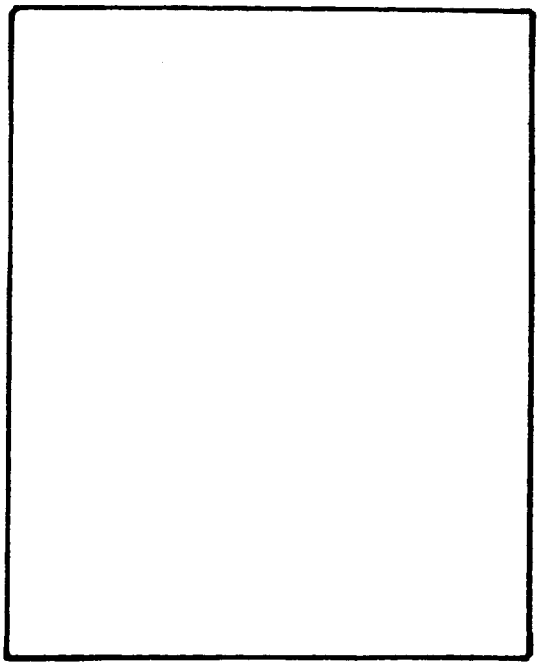
## Write a procedure to draw each figure in a variable size



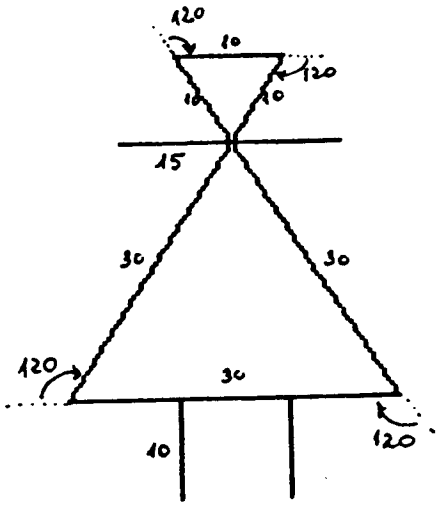



**Write a general procedure to draw the figure in a variable size:**

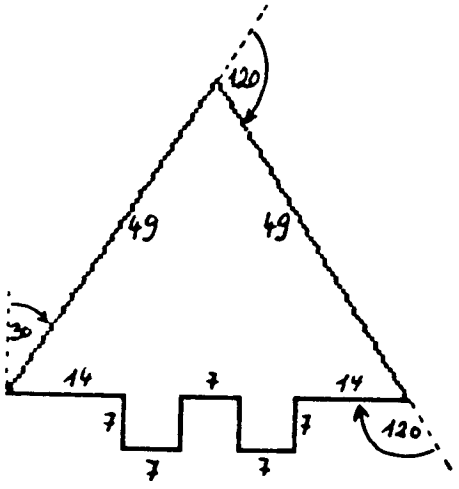


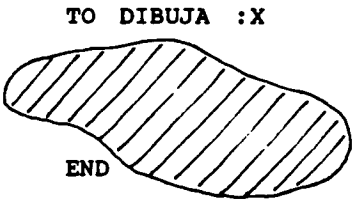
**Write a general procedure to draw the figure in a variable size:**



**Write a general procedure to draw the figure in a variable size:**



# With the procedure DIBUJA you can obtain six different drawings




TYPE:

DIBUJA a number

FIND OUT ALL THE VALUES OF :X THAT CORRESPOND TO EACH SHAPE
AND FILL IN THE TABLE:

| SHAPE 1 | DRAWING | VALUES OF :X |
|---|---|---|
| SHAPE 2 | | |
| SHAPE 3 | | |
| SHAPE 4 | | |
| SHAPE 5 | | |
| SHAPE 6 | | |

WORK SHEET D-2

# Making predictions

IF YOU RUN THE PROCEDURE DIBUJA WITH EACH ONE OF THE FOLLOWING NUMBERS, WHICH IS THE SHAPE YOU EXPECT TO OBTAIN ?

DESCRIBE IT OR DRAW IT BESIDES EACH NUMBER:

560

0

199.90

-7

749.5

-1

380

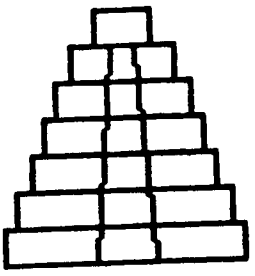
999.8

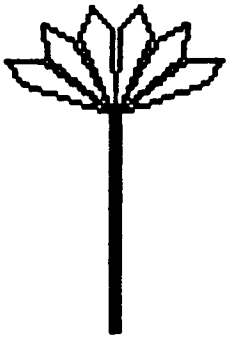
70.4

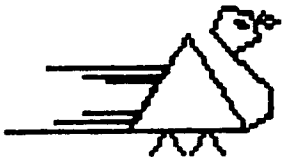
WORK SHEET D-3

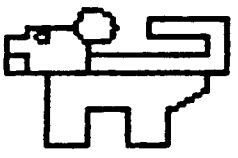
# Making predictions

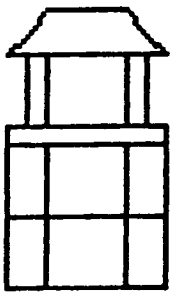
FOR EACH SHAPE GIVE AT LEAST THREE NUMBERS THAT WHEN USED AS INPUTS TO DIBUJA WILL PRODUCE THE SAME SHAPE :











# Analyzing the behaviour of a shape

TYPE: DIBUJA 250

DIBUJA 320

TRY OTHER INPUTS TO DRAW THE SAME SHAPE (USE YOUR TABLE TO FIND THE APPROPRIATE INPUTS)
OTHER NUMBERS THAT YOU KNOW THEY GIVE YOU THE SAME FIGURE.

WHAT HAPPENS TO THIS SHAPE WHEN WE RUN THE PROCEDURE WITH DIFFERENT INPUTS ?

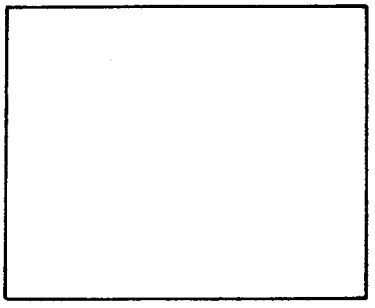
EXPLAIN THE RELATIONSHIP BETWEEN THE VALUES OF THE INPUT AND THE SIZE OF THE SHAPE

# Analyzing the behaviour of a shape

CHOOSE ONE OF THE SHAPES

LOOK IN YOUR TABLE WHAT ARE THE APPRPRIATE VALUES THE INPUT YOU CAN USE TO DRAW THE CHOSEN SHAPE

FIND OUT THE RELATIONSHIP BETWEEN THE VALUES OF THE INPUT (VALUE OF :X ) AND THE SIZE OF THE SHAPE

| SHAPE | VALUE OF :X | SIZE |
|---|---|---|
|  | | |

FOR WHICH VALUE OF X HAS THIS SHAPE THE SMALLEST SIZE?

FOR WHICH VALUE OF X HAS THIS SHAPE THE BIGGEST SIZE?

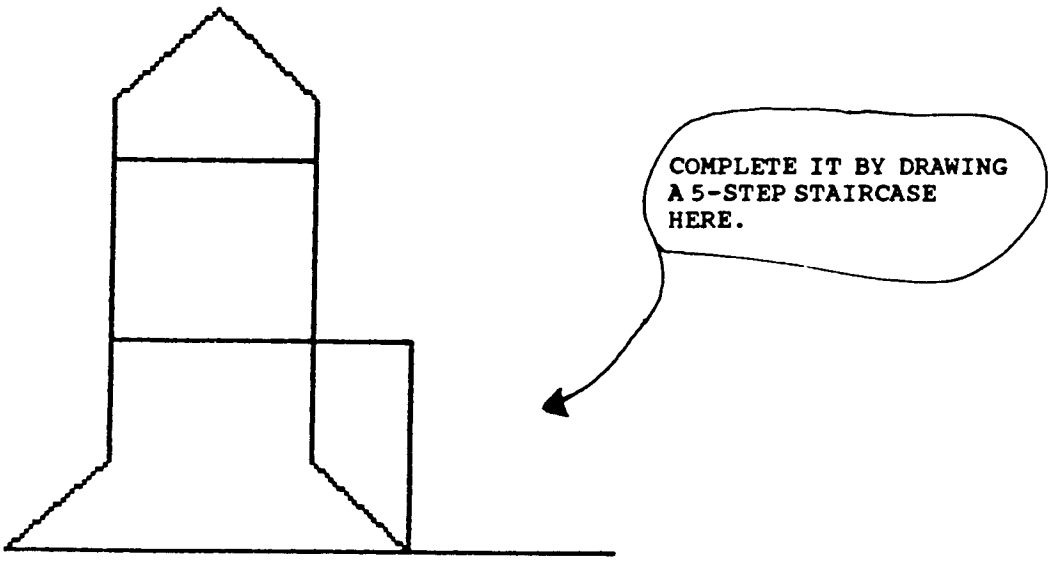
WHAT HAPPENS TO THE SIZE OF THIS SHAPE, BETWEEN THE SMALLEST AND THE BIGGEST VALUE OF X ?

WORK SHEET SU-1

# Can you draw a 5-steps stair?

```
TYPE   COHETE  120
```

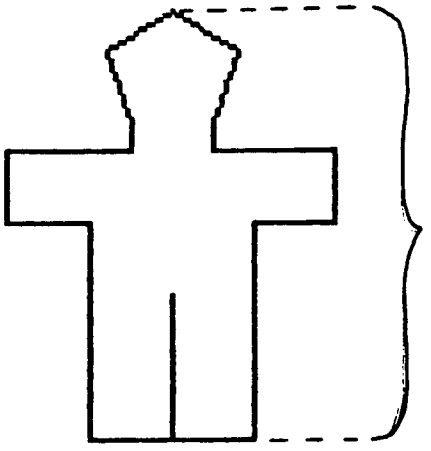
THE FOLLOWING DRAWING WILL APPEAR ON YOUR SCREEN



# Looking for a particular input

THE HEIGHT OF THE ROCKET'S DOOR IS 60 . TO FITT INTO THE DOOR
HEIGHT OF THE ASTRONAUT HAS TO BE 60 TOO.

A WAY TO FIND THE INPUT TO DRAW AN ASTRONAUT OF HEIGHT 6
TO DETERMINE THE VALUE OF :A IN THE GIVEN EQUATION



```
60 = 3 * :A + :A + 2 * :A
```

USE THE VALUE OF :A AS INPUT TO THE PROCEDURE ASTRONAUTA

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