WINDOWS ON THE INFINITE: CONSTRUCTING MEANINGS IN A LOGO-BASED MICROWORLD

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Thesis submitted in fulfilment of the requirements for the Ph.D. degree of the University of London

Institute of Education, University of London

August, 1997

For Manuel

Abstract

This thesis focuses on how people think about the infinite. A review of both the historical and psychological/educational literature, reveals a complexity which sharpens the research questions and informs the methodology. Furthermore, the areas of mathematics where infinity occurs are those that have traditionally been presented to students mainly from an algebraic/symbolic perspective, which has tended to make it difficult to link formal and intuitive knowledge. The challenge is to create situations in which infinity can become more accessible. My theoretical approach follows the constructionist paradigm, adopting the position that the construction of meanings involves the use of representations; that representations are tools for understanding; and that the learning of a concept is facilitated when there are more opportunities of constructing and interacting with external representations of a concept, which are as diverse as possible.

Based on this premise, I built a computational set of open tools — a *microworld* — which could simultaneously provide its users with insights into a range of infinity-related ideas, and offer the researcher a window into the users' thinking about the infinite. The microworld provided a means for students to construct and explore different types of representations — symbolic, graphical and numerical — of infinite processes via programming activities. The processes studied were infinite sequences and the construction of fractals. The corpus of data is based on case studies of 8 individuals, whose ages ranged from 14 to mid-thirties, interacting as pairs with the microworld. These case studies served as the basis for an analysis of the ways in which the tools of the microworld structured, and were structured by, the activities.

The findings indicate that the environment and its tools shaped students' understandings of the infinite in rich ways, allowing them to discriminate subtle process-oriented features of infinite processes, and permitted the students to deal with the complexity of the infinite by assisting them in coordinating the different epistemological elements present. On a theoretical level, the thesis elaborates and refines the notion of situated abstraction and introduces the idea of "situated proof".

Acknowledgements

I would like to express my deepest gratitude to:

First and foremost, my supervisor, Professor Richard Noss, who not only guided and assisted me through this long journey, but never failed to encourage and support me; very special thanks go to him.

Dr Luis Enrique Moreno A. of the *CINVESTAV-IPN,* who supervised me in Mexico. Both he and Richard were wonderful mentors and friends, and I thank them both for their unwavering faith in me, and all their patience.

Professor Celia Hoyles, who helped in my supervision in the beginning and gave me some very helpful pointers and guidance.

Professor Rosamund Sutherland, for her support and friendship during this period.

All the students who participated in this research and without whose cooperation it would not have been possible. In particular I would like to thank the students who participated in the main study: Consuelo, Verónica, Elvia, Alejandra, Victor, Jesús, Manuel, and Martin.

The *British Council*, and Mexico's *Consejo Nacional de Ciencia y Tecnología* (CONACYT) for their financial support.

In Mexico: the *Centro de Investigación y de Estudios Avanzados del IPN* (CINVESTAV-IPN), in particular the *Departamento de Matemática Educativa* of that institution, and the *Programa Nacional de Formación y Actualización de Profesores de Matemáticas* (SEP/CINVESTAV) for their overall support.

Peter Warburton, for his constant encouragement, his helpful advice and assistance in proof-reading this thesis, and for always providing a friendly ear when I needed to let things out.

My parents and family who have always believed in me and paved the way for me to get here.

Last, but never least, my late husband Manuel, without whose loving encouragement I would have never begun this endeavour, and whose inspiration helped me finish it.

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Chapter 1:

Introduction

"Mathematics is the science of infinity."
— H. Weyl

There is no question that calculus is a difficult area in mathematics education, and has been the focus of study of many educational researchers. The concepts of calculus are mental constructs like all mathematical objects. In the Piagetian view, the intellectual constructs represented by the formal definitions, which should be distinguished from the cognitive structures, cannot be constructed without an intense work of "reflective abstraction"[1] at the cognitive level.

This seems to be true in particular for the concept of infinity, because infinity is not "extractable" from sensory experience, it is a mental construct which often defies common sense. This is why infinity has been recognised as a difficult concept and has historically been the origin of paradoxes and confusions. It has also been argued (see Chapter 3) that the spontaneous conceptions and intuitions that people have of infinite processes and of infinite (mathematical) objects can become obstacles for the adequate construction of formalised versions of these concepts.

Infinity, however, is central to calculus, where, for instance, infinite processes form the basis for the concept of limit, and is also present in other important areas of mathematics. Yet infinity has been neglected as an area of study in school mathematics: it is seldom seen as a main object of study.

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[1] The idea of *reflective abstraction,* introduced by Piaget, refers to the construction of logical-mathematical structures drawn from interiorisation and coordinations of *actions.* Dubinsky (1991, p.99) explains that reflective abstraction is an internal process which "differs from 'empirical abstraction' in that it deals with action as opposed to objects, and it differs from 'pseudo-empirical abstraction' in that it is concerned, not so much with the actions themselves, but with the interrelationships among actions, which Piaget ... called 'general coordinations'."

Furthermore, the areas of mathematics where infinity appears are also those that have traditionally been presented to students mainly from an algebraic/symbolic perspective. Concepts which are already difficult and "abstract", such as that of the limit of an infinite process, are accessed through formal forms of representation, making it difficult to integrate these ideas into the cognitive structures of the learner. These formal mathematical representations often lose their connection with the phenomena which originally gave rise to them (e.g. movement, and infinite processes). For instance, the links of modern calculus with its origins, particularly with physics problems dealing with movement, are no longer apparent in its formal definitions and theorems.

In this work, I focus on the infinite, and, particularly, infinite processes. One challenge is to create situations and ways in which infinity can become more accessible. As is discussed in Chapter 3, I have adopted the position that the construction of knowledge and meanings involves the use of representations; that representations are tools for understanding; and that the learning of a concept is facilitated when there are more opportunities of *constructing* and interacting with as diverse as possible external representations of a concept.

There are several possible types and methods of representation whose implementation is worth researching in an attempt to make the infinite more "concrete". Among these are the integration into the learning environment of (i) the visual — i.e. graphical, geometrical — element, and (ii) the representational systems that can be provided by the computer.

Aims of the research

My general interest is *to investigate the mediating role of computer-based tools in learning and the construction of knowledge.* I postulate that infinity, or at least some of the infinite processes found in mathematics, may become more accessible if studied in an environment that facilitates the construction and articulation of diverse types of representations, including visual ones and the element of *movement.*

Based on this premise, the work presented here involves the design and implementation of a computer based-environment — a *microworld*[2] — intended as an exploratory setting comprising the use and construction of different types of representations (e.g. symbolic; visual; "unfolding", i.e. using movement). This setting and the representational systems it involves were designed to serve as tools for students to explore infinite processes, particularly iterative/recursive[3] processes such as sequences and series.

The major research issue is thus to *investigate how the activities and tools of this microworld shape the understandings about infinity.* The aims are as follows:

1.- To investigate students' conceptions of the infinite as mediated by the different tools and external representations (symbolic, visual, numeric) provided by the microworld.

2.- To probe the ways in which students made use of the environment in order to make sense of the phenomena they observed, and the ways they explored and manipulated ideas in order to make them meaningful.

3.- To look at the ways in which the different forms of representations were coordinated and integrated, in particular through their interaction with the procedural code.

The specific characteristics of the microworld were designed in order that:

a) it allowed for ways to incorporate the visual aspect, which may not have been possible before. For example, it offers the possibility to observe visually the evolution in time of a process: i.e. the process can be perceived as it *unfolds,* thus highlighting its behaviour and eliminating the limitation of only observing the final state (the result of the process); and

b) it could provide representational systems and tools that can be used to create a situation for exploring and expressing ideas.

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[2] The concept of *microworld* is defined in Chapter 3.

[3] Note that although in programming recursion and iteration are different, from a mathematical point of view the two ideas are rather close.

In general, the computer setting provided an opportunity to analyse and discuss in conceptual — and concrete — terms the meaning of a mathematical situation. For example, drawing a geometric figure using the computer, necessitated an analysis of the geometric structure under study and an analysis of the relationship between the visual and analytic representations.

In addition, the microworld can become a "window"[4] for researching its mediating role in the thinking and learning processes of the students, giving a glimpse of the shifting conceptions of students in their interaction with the environment.

Outline of the thesis

Since my research focuses on the concept of infinity, I consider it essential to begin with a review of the main mathematical ideas underlying this concept as they were developed historically and the difficulties that emerged. Thus, in Chapter 2, I present a brief historical and epistemological overview of the development of the concept of infinity in mathematics. The content of this chapter serves as an additional guideline and research tool with which to observe and interpret the work done by the students in the empirical phase of the research.

In Chapter 3, I present the main theoretical considerations which served as the basis of my study through a literature review: I review research related to visualisation and representation theory, computer-based microworlds, and previous research in mathematics education related to the concept of infinity and the concept of the limit of an infinite process.

The remainder of the dissertation centres on the empirical research and its results. In Chapter 4, I present the methodology used in the study, and how it evolved from a series of preliminary studies. Then in Chapter 5, I describe the design of the microworld, its principles, and its content.

Chapters 6 and 7 deal with the results of the empirical research. In Chapter 6, I present an account of the microworld in practice, and illustrate the way in which the

[4] The idea of "window" for the study of students' thinking processes is elaborated in Chapter 3.

microworld functioned; I delineate the structure of common activities, highlight the role of the exploratory environment, and describe how its tools served as structuring elements for the processes of discovery and construction of meanings to take place.

The main ideas or key issues that emerge from the empirical research are illustrated and analysed in Chapter 7: this chapter looks at the ways in which meanings for the observed phenomena are created through the (re-)construction of connections between the different representations (e.g. the visual and the symbolic). It focuses on how the microworld and its tools were used to construct meanings for the notions of infinity and infinite processes, how the microworld acted as a "domain of abstraction"[5], and how it shaped the processes of discovery and "proof" about properties of the infinite.

Finally, in Chapter 8, I present the overall conclusions and implications of this research.

[5] The idea of "domain of abstraction" is explained in Chapter 3.

Chapter 2:

A Historical Perspective of the Concept of Infinity

"We would be wise to take note of the lessons of history."
— David Tall (1986; p.51)

In this chapter, I present an overview of the historical evolution of the concept of infinity. The purpose of this is to highlight the ways in which infinite processes have appeared and have been conceived in history and mathematics. This is important for several reasons. First, it explains some of the difficulties that had to be confronted in the development of the concept of infinity as attempts were made to incorporate it into mathematics and define it as a field of study. Throughout history examples can be found in which infinity has been a source of difficulties and controversy, often due to the lack of an adequate theoretical framework and operatory field for this concept, and where it was necessary to overcome certain epistemological obstacles. There is evidence (as found by some of the researchers reviewed in Chapter 3, such as Sierpinska, 1987; Waldegg, 1988; Cornu, 1991) that some of these obstacles may have similar manifestations as didactic obstacles for mathematics students.

This evidence confirms Piaget & Garcia's (1989) claim that some of the cognitive processes found in the historical development of science are similarly present in the development of individuals. However, this does not imply that ontogenesis recapitulates phylogenesis since the contextual influences and epistemological *forces* (see diSessa, 1995) which reflect the social situation are different. A fundamental idea in Piaget and Garcia's work is that in every historical period there is a prevailing epistemological framework resulting from social and epistemological paradigms or forces. Thus, the scientific knowledge which is produced is conditioned by this framework; the resulting ideology acts as an epistemological obstacle disallowing the development of knowledge outside the accepted framework. Piaget and Garcia explain that when a moment of crisis (a scientific revolution) occurs there is a break with the existing ideology and a new epistemological framework emerges. The mathematics of infinity are no exception to this, even more so if we consider the highly abstract or mental nature of this concept. In this chapter, I also present an overview of some of the main mathematical ideas related to the concept of infinity. The review shows the central role this area holds in mathematics, as well as its richness and complexity.

I. The concept of infinity: its first manifestations.

A. Intuitive preconceptions of infinity.

Everybody has some personal notion or intuitive ideas of infinity. We can consider these pre-mathematical conceptions, since they are, in fact, cognitive structures which help answer personal inquiries. These notions emerge as answers to the question of what the limits of perception are. That is, in its origin the notion of infinity is related to that which is beyond the perceptual and the material. In fact, the word infinite means "not finished" or "without end" (i.e. endless); thus, it is an idea that is related to that which is boundless, limitless or endless. :

> "Endlessness is, after all, a principal component of one's concept of infinity. Other notions associated with infinity are indefiniteness and inconceivability."
>
> (Rucker, 1982; p.2)

As Rucker points out, infinity is also related to the inconceivable or indefinite[1]. Thus, an infinite process is often said to be that which continues indefinitely. Aristotle (384-322 BC) wrote: "...being infinite is a privation, not a perfection but the absence of a limit...."[2]. The primary notion, relating infinity to that which is beyond perceptual experiences, is also illustrated by the term used by the Greeks to describe it: *to apeiron*

[1] Another notion which is intuitively related to infinity is that of God or the Absolute. Again this is because these are ideas which are beyond a perceptual framework.

[2] As quoted by Rucker (1982), p.3.

which literally means the unlimited or unbounded "but can also mean infinite, indefinite or undefined," (Rucker, op.cit.; p.3).

Originally, the Greeks perceived mathematics as an empirical abstraction (i.e. related to "reality"[3]). Perhaps because of this their number system was finite (they only had numbers for which they had a use). Thus, the Greeks refused to accept the apeiron, and in fact feared infinity (the *Horror Infinity),* not only because it is something that cannot be extracted from sensory experience, but also because it was beyond their conceptual framework: not surprisingly they lacked the proper notation to be able to incorporate it into their mathematical system (see Maor, 1987; p.3).

B. The first encounters with infinity: the infinitely large (outward infinity) and the infinitely small (inward infinity).

In the nineteenth century, Cantor (1945-1918) distinguished between the Absolute infinity (the Absolute), the physical infinities and the mathematical infinities. The physical infinities are described by Rucker (op.cit., p.10) as follows:

> "There are three ways in which our world appears to be unbounded, and thus, perhaps, infinite. It seems that time cannot end. It seems that space cannot end. And it seems that any interval of space and time can be divided and subdivided endlessly."

Thus, the infinite is first related to lengths of time or spatial magnitudes. In first instance this happens when thinking of the "very large". I call this an "outward" notion of infinity: notions of endlessness or limitlessness are first related to issues such as the eternity of time or the immensity of the Universe. Children often relate the infinite to "the very large": large sets, such as the number of grains of sand in a desert or the number of stars in the sky, are said to be infinite. These are the largest *concrete* sets that people can think of, and thus constitute their first idea of an infinite number or, at least, the closest to infinity that *exists.*

In fact, Archimedes (287-212 BC) used the idea of all the grains of sand in the world as a basis for his discussion of infinity in his *Sand-Reckoner,* where he proved

[3] In fact, much of their progress in mathematics resulted from their search for solutions to problems from the "real" world.

that in fact, this large number *can* be counted and thus is finite[4]. The notion of *counting* is, in fact, an underlying idea to (the potential) infinity: the *potential* action of adding objects to a large set. But the relevance of this discussion is that it constitutes an epistemological change: Archimedes redefined the Greek number system[5] by explaining a way in which numbers of any size as large as wanted could be constructed. He showed that although there is no concrete correspondent to the infinite in the perceptual world, infinity exists (as a mental construct) since the numerical list can be continued indefinitely. In this, he was following Aristotle who made a distinction between the *potentially infinite* and the *actually infinite,* and accepted only the *potentially* infinite: time goes on forever, the natural numbers are potentially infinite. This example also illustrates the importance and limitations of numerical representations: as Waldegg (1988) explains, it relates the conceptual problem of infinity with the indefinition of large quantities.

A second way in which the notion of infinity emerges is when considering the idea of repeated subdivisions. I call this an "inward" notion of infinity. In relation to this, Rucker points out the following:

> "The question of the existence of an infinity in the small becomes the question of whether or not the space-scale continuum extends *downward* indefinitely."
>
> (Ibid., p.35).

In ancient Greece, there were two schools of thought: one (e.g. Democritus, c. 460-370 BC; the Pythagorean Atomists), considered that repeated subdivision of the line would lead to a primary indivisible unit, and another which believed in infinite subdivision and denied the existence of those final units.

In any case, as Lévy (1987) points out — and as was recognised by Aristotle — number and measure are both related to infinity: the sequence of natural numbers is endless, and a magnitude can be divided into other magnitudes, regardless of whether one believes in infinite divisibility or in the existence of an infinite amount of

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[4] He estimated the number of grains of sand in the world to be less than 10^{63} (using modern notation). — It is also interesting to note that in relation to the other large set I mentioned, the number of visible stars to the naked eye from one point is estimated to be only about 2800, according to Maor (1987, p.16).

[5] Until then, the Greek number system could not express values larger than 100, 000, 000.

indivisibles. And in both notions of infinity — the 'outward' one of the infinitely big (the infinite by addition) reflected in the indefinite sequence of natural numbers, and the 'inward' one of repeated subdivisions — infinity is reflected through the idea of repeating a process indefinitely; that is, the idea of *iteration* is central to this concept. Aristotle in his *Physics* (Book III, Chapter 6) expressed this idea in the following way:

> "For generally the infinite has this mode of existence: one thing is always taken after another, and each thing that is taken is always finite, but ever other and other."

As Moore (1991) explains, Aristotle's description of the infinite as that which goes on forever (the potentially infinite) is highly significant, since it was among the first characterisations of the *mathematically* infinite. Until then, most of the early Greeks had managed to "circumvent" the mathematical infinity.

II. Some key historical events leading to the incursion of the infinite in mathematics.

Infinity in mathematics has repeatedly been the source of conflict and paradox, from the problem of continuity to the paradoxes emerging from the theory of infinite sets, which led to extensive studies of the logical foundations of mathematics in this century. Following are some historical episodes that I consider interesting and relevant for my work.

The problem of the continuum and the definition of the real number line.

The problem of the correspondence between number and space (arithmetic and geometry) is fundamental to the ideas of infinity and of the continuum. Aristotle gave lengthy discourses on the nature of the infinite and on the difference between the discrete and the continuous: for him, number is a discrete quantity, while magnitudes are continuous because they can be divided *ad infinitum*. Furthermore, as Lévy (op.cit., p.29) explains, in Aristotle's conception it is due to *movement* that the infinite exists, and that the infinite only exists in nature in that it expresses quantity. In the Book III of his *Physics,* Aristotle says:

> "... motion is supposed to belong to the class of things which are *continuous;* and the *infinite* presents itself first in the continuous — that is how it comes about that 'infinite' is often used in definitions of the continuous ('what is infinitely divisible is continuous')."[6]

Zeno's paradoxes (paradoxes of the infinitely small):

With the focus on movement, the problem of the relationship between the discrete and the continuous was highlighted by Zeno of Elea (c. 450 BC). Zeno challenged the belief that, in Struik's (1967, p.43) words, "the sum of an infinite number of quantities can be made as large as we like, even if each quantity is extremely small ($\infty \times \varepsilon = \infty$), and also that the sum of a finite or infinite number of quantities of dimension zero is zero". His arguments highlighted the difficulty of saying that the line is formed by points. If continuous magnitudes (space and time) are infinitely divisible, then movement cannot exist (paradoxes of *The Dichotomy* — or *The Runner*; and *Achilles and the Turtle*). But if they are *not* infinitely divisible, due to the existence of elementary indivisible parts, then again movement is not possible (paradoxes of *The Arrow* and *The Stadium*).

For instance, Zeno's Dichotomy paradox from the first category is translated and explained by Rucker (op.cit.; p.125) as follows: If you are at 1 on the number line, there are two ways to get to 2: moving one unit all at once, or using the infinite procedure of moving 1/2 unit, then 1/4, then 1/8, etc. This fact is usually represented by the equation $1+1 = 1 + 1/2 + 1/4 + 1/8 + \ldots$ As Rucker explains, Zeno viewed this as paradoxical, because he assumed *a priori* that no actual infinity could exist, so that no infinite process could be regarded as completed; therefore the equivalence between a finite quantity and an infinite process seemed impossible.

Aristotle believed that one of the problems leading to the first paradoxes is that Zeno is mixing two types of infinity — the infinite divisibility of space, and the infinite extension of time — when he suggests that the infinite division of space requires an infinite amount of time to be completed. Moore (1991) explains that for Aristotle, Zeno's paradoxes revealed the incoherence of anything being divided into infinitely many parts (which implies the construction of an actual infinity). Thus, for the second two paradoxes, Aristotle argued that it was false to assume that the

[6] Aristotle, *Physics,* Book III, Chapter 1, p.278.

continuum was formed of indivisible elements. Jones (1987) adds that Zeno's paradoxes emerged from mixing the discrete with the continuous — by applying a number to a magnitude — and Aristotle's solution was to separate the discrete from the continuous. It was much later, in the present century, that Russell (1872-1970) explained that the points in a segment cannot be counted since the set of points in the real continuum are non-denumerable (an idea discussed further below). Thus continuous measures cannot be formed by putting together punctual particles. As expressed by Bruyère (1989, p. 29), intuition often fools us when it pushes us to mistake the infinity of the natural numbers with that of the real numbers.

The discovery of the incommensurables and the problem of the continuum:

The problem of relating (natural) numbers (which are discrete) and magnitudes (which are continuous) was first brought to the foreground much earlier (around the 5th century BC) by the discovery of the incommensurable ratios by the Pythagoreans, which would lead to the definition of the irrational and real numbers. The Pythagoreans had, until that point, believed that everything could be understood in terms (i.e. ratios) of natural numbers, and they identified number with geometry until their discovery shattered that identification (see Kline, p.33). Rucker (op.cit.; p.62) explains how this highlights the way in which the conception of continuity evolved:

> "For the Pythagoreans the 'line' was a rational number line which they considered as continuous, but the discovery of the irrationals led to 'holes' in that line. The conception of 'continuity' for the Pythagoreans is of considering a dense set as continuous. Modern mathematics has established the distinction between density and continuity, but it is not immediate for the uninitiated".

The development of the real number system:

The one to one correspondence between magnitude and number would not be possible until the sixteenth century with the works of mathematicians such as Simon Stevin who recognised irrational numbers as numbers and advocated in *La Disme* (1585) the use of a decimal notation.

With the emergence of the decimal notation, all numbers — integers, rational and irrational — could uniformly be classified as real numbers. In the nineteenth century, Cantor defined a real number as an infinite sequence of digits; in his approach

a real number is the *limit* of the infinite series expressed by: $a+\frac{a_1}{10}+\frac{a_2}{100}+\frac{a_3}{1000}+\ldots$
The importance of this is that he disassociated numbers from magnitudes, by treating them as arbitrary infinite sequence: $\pm a.a_1a_2a_3a_4\ldots$ As Rucker (op.cit.; p.115) explains: "the ordinary representation of a real number as an endless decimal expansion can be viewed as a description of an *infinite* procedure for locating a particular point (or infinitesimal neighbourhood) on the marked line."

Cantor also described the essential difference between the set of discrete natural numbers and the set of all real numbers when he proved that the two sets could not be put into a one to one correspondence. He was thus relating non-denumerability with the continuum.

The method of the indivisibles and the method of exhaustion.

The problem of the measure of spatial magnitudes led to the development among the ancient Greeks of two methods respectively connected to the two conceptions of repeated subdivision. It is interesting that these methods would later form the basis of our modern infinitesimal calculus.

Democritus conceived solids as constituted of an infinite number of parallel planes: indefinitely thin and indefinitely close plates. Democritus' approach is an atomist *infinitesimal* one: i.e. these layers are infinitesimally thin and indivisible. Thus, the *method of the indivisibles* separates a solid, area or line into an infinity of indivisibles. Later, Archimedes also used this method — known as "The Method" — for exploration and discovery purposes, but he did not consider it rigorous enough to be used as proof, for which he instead used the *method of exhaustion* described below. Unlike Democritus, Archimedes did not consider infinitesimal differences as nullifiable[7], only small enough to prevent irregularities and be valid in a heuristic approach. However, what I find interesting about this method is that it involves an actual conception of infinity: the solid or area is considered to be constituted by an infinite set of parts. Struik (1967; p.48) adds in reference to this method:

[7] Democritus considered that space was continuous — an infinite number of elementary layers in contact with each other. This is a different conception of continuity from Aristotle's where magnitudes

"...our modern limit conceptions have made it possible to build this 'atom' theory into a theory as rigorous as the exhaustion method."

It is interesting to relate this to modern mathematics where a continuous region of *mathematical* space is considered to be constituted by an infinite number of *mathematical* points. A finite number of points, which have no length cannot constitute a line segment, which *does* have length. Thus every line segment (or continuous plane segment or region of space) must consist of an infinite number of points.

The method of exhaustion, which was perfected by Archimedes, can be described using the example of the measurement of the circle. Using this method he was able to approximate (from below and above) the circumference and area of a circle simply by inscribing and circumscribing polygons with more and more sides upon the circle (see Figure 2.1). This is the technique behind integral calculus. It was based on an axiom by Eudoxus (c. 408-355 BC) where, given two unequal magnitudes, one of the magnitudes can be repeated enough times to exceed the other. In principle, there is no limit to the accuracy obtainable by this method although infinite precision cannot, of course, be obtained in a finite amount of time. In the nineteenth century this method would be explicitly expressed in Dedekind's (1831-1916) *axiom of continuity* which asserts that there is a single right magnitude that exists as the limit of any such process. The difference between Eudoxus and Dedekind was that the latter accepted the *actuality* of what he defined as the infinite set of real numbers.

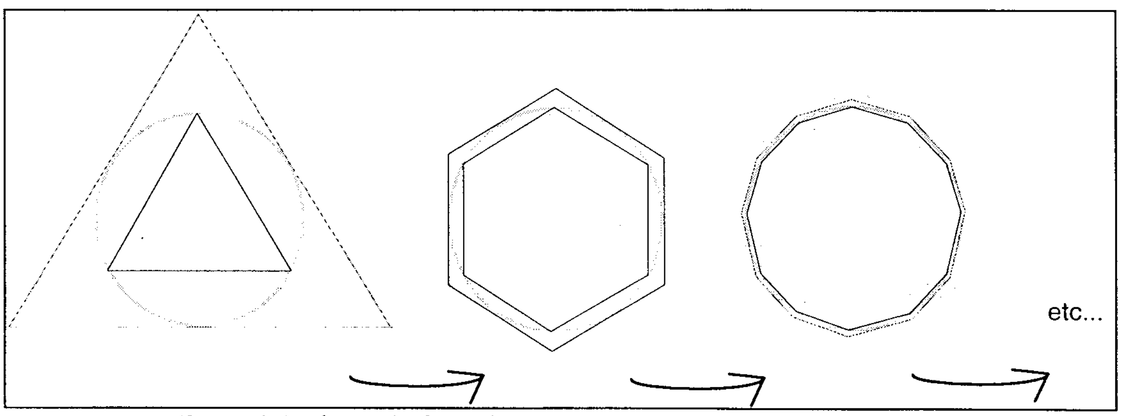


Figure 2.1. The method of exhaustion for the measurement of the circle.

can be subdivided indefinitely. According to Moreno (1995), Democritus' view is one of the first *analytic* conceptions of the continuum.

Struik (1967) explains that the method of exhaustion — as opposed to that of the indivisibles — "avoided the pitfalls of infinitesimals by simply discarding them, by reducing problems which might lead to infinitesimals to problems involving formal logic only," (op.cit., p.46). This method relies on simple iteration and is not concerned with the problem of the existence of the infinitely small (or big).

The method of the indivisibles was reintroduced in the 17th century by mathematicians such as Kepler (1571-1630), Galileo (1564-1642) and Cavalieri (1598-1647), although they were unaware of the work of Archimedes. What is interesting from this is the "Principle of Cavalieri", which appeared in Cavalieri's *Geometria Indivisibilus Continuorum* (1635), where the comparison between, for instance, two areas was reduced to the comparison of all the lines which formed those areas. Cavalieri explains that the totality of indivisibles of a body can be compared with the totality of indivisibles of another body, and that their magnitudes have a definite relationship between them. As explained by Gardies (1984) this method implies putting two infinite sets into a one to one correspondence. Gardies claims that this is where Dedekind would later base his definition that an infinite set can be put into a one to one correspondence with one of its proper subsets.

Infinite series.

The development of mathematical ideas related to infinity was stalled until the 16th century when for the first time infinite processes were explicitly expressed, such as in formulas for π: For example, $2/\pi$ was written in terms of an infinite product by Vieta (1540-1603) and later in a different form by Wallis (1616-1703). (Wallis was also responsible for introducing the symbol '∞' for infinity; an interesting symbol which reflects the recursive nature of infinity in its endless loop.) In 1671, James Gregory (1638-1675) found an expression for π as an infinite series:

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \pm \dots$$

Thus, it is through infinite expressions such as these that infinity is first explicitly expressed in mathematics[8]. But it is also in the domain of infinite series that we find examples representative of the problems which emerge when dealing with the infinite by using finite schemes: that is, before convergent and divergent series were differentiated and formally defined, many mathematicians did not recognise that infinite series could not be manipulated in the same way as finite sums. It was thus that for instance the Grandi series (1703)

1 - 1 + 1 - 1 + 1 - 1 + ...

became the centre of much controversy when different finite arithmetic manipulations yielded different results, such as:

(1 - 1) + (1 - 1) + (1 - 1) + (1 - 1) + ... which resulted in zero;

1 - (1 - 1) - (1 - 1) - (1 - 1) - ... which resulted in one; or, for instance,

by using the formula[9] $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$, and making x = 1, the result became 1/2.

This example is interesting because it shows how, at this stage, an adequate operatory field and methodology for infinite objects was still lacking (in particular the definition of convergence or divergence of a series was still absent). Leibniz (1646-1716) himself used the argument that the above sum had as value 1/2 because this was the average of (in modern terms) the values of the partial sums: 1, 0, 1, 0, 1, ... In another example of the ways in which 17th and 18th century mathematicians dealt with the infinite when working with infinite series, Euler (1707-1783), by using the formula $\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \ldots$, and making x = -1, obtained (in a way which also shows how infinity (∞) was treated as a number) the result:

[8] It should be noted that it was long known (Aristotle himself admitted this possibility) that some infinite series had a finite sum (i.e. that they converged, using modern language), but it was not until this time (16-17th centuries) that they were expressed and incorporated into mathematical language.

[9] Euler obtained the same result by using the similar formula $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$ and making x = -1.

$1 + 2 + 3 + 4 + ... = \infty$ (a),

which seemed quite natural. But when he used the formula $\frac{1}{1-x} = 1 + x + x^2 + x^3 + ...$ with $x = 2$, he obtained a totally contradictory result:

$1 + 2 + 4 + 8 + ... = -1$ (b).

He then reasoned that since the left hand side of (b) is larger than that of (a) which is equal to ∞, then ∞ must be a sort of limit between positive and negative numbers in a similar fashion to zero. It is thus, as explained by Kline (1972), that mathematicians such as Newton (1642-1727), Leibniz, Euler, and even Lagrange (1736-1813) perceived infinite series as extensions of the finite polynomial algebra without initially being aware of the problems that arose when sums were extended to infinite terms. These issues were not resolved until 1821, when Cauchy (1787-1857) defined the concept of limit and pointed out that the algebra of finite quantities could not automatically be extended to infinite processes.

From an educational perspective the problems faced by these mathematicians and their "intuitive form of thought" in dealing with infinite series — involving extending methods of the finite to the infinite — could point to an area of similar difficulties (as confirmed by some of the researchers reviewed in Chapter 3) for students who have not yet learned to differentiate the behaviour of convergent and divergent series and to work within the formal "rules" (e.g. the area of "mathematical limits") which were developed in order to deal with these situations.

As Maor (1987) points out, the concepts of *convergence* and *limit* were central to the development of the calculus, and with these concepts at hand it became at last possible to resolve the ancient paradoxes of infinity which had intrigued mathematicians since the times of Zeno.

Infinitesimals and the development of the calculus.

The ideas of movement and continuity which were debated by Aristotle and the Greeks were at the basis of the development of the calculus in the seventeenth century with the works of Newton and Leibniz. In this context, the concept of infinitesimal re-emerged. For example, in the problem of finding the instantaneous velocity of a moving body, space and time are considered as continuously varying quantities. As Rucker (1982) explains, to calculate the velocity at some instant t_o, one has to imagine measuring the speed over an infinitely small time interval *dt*; thus the speed at t_o is given by $f'(t_o) = \frac{f(t_o + dt) - f(t_o)}{dt}$.

Rucker (ibid.; p.7) explains the rules that govern infinitesimals:

> "The quantity *dt* is called an *infinitesimal,* and obeys many strange rules. If *dt* is added to a regular number, then it can be ignored and treated like zero. But, on the other hand, *dt* is regarded as being different enough from zero to be usable as a denominator....Adding finitely many infinitesimals together just gives another infinitesimal. But adding infinitely many of them together can give either an ordinary number, or an infinitely large quantity."

Leibniz defended the use of infinitesimals, which he considered as practical tools; in 1690, he wrote:

> "It is useful to consider quantities infinitely small such that when their ratio is sought, they may not be considered zero, but which are rejected as often as they occur with quantities incomparably greater."[10]

As this quote shows, unlike the conception held by the ancient Greeks, here the concept of infinitesimal involved a *relative* approach: that is, the way in which an infinitesimal quantity is considered is relative to how it relates or compares with other quantities. This notion of infinitesimal was central in the origins of calculus. For instance, one of the organising principles of the 1696 text of L'Hôpital (1661-1704) on differential calculus, *Analyse des infiniments petits*, is the following:

> "A quantity which increases or decreases by an infinitely small quantity can be considered to remain the same."

In other words, given a quantity *A* and an infinitesimal quantity *b*, the following can be written:

$$A + b = A$$

where the " = " is a *criterion of substitution*, not of equality in the ordinary sense.

[10] English quote by Kline (1972), p. 384, from Leibniz' *Mathematische Schriften,* 4, 63.

The relativity of the notion of infinitesimal was pushed to the limit by Euler, who based his calculus on the introduction of zeros of differential orders, something which was not well accepted by other mathematicians. In his *Differential Calculus* of 1755, he wrote:

> "...there exist infinite orders of infinitely small quantities, which, though they all = 0, still have to be well distinguished among themselves, if we look at their mutual relation, which is explained by a geometrical ratio."[11]

The use of infinitely small and infinitely large numbers in calculus was soon replaced by the conception of limit first introduced by D'Alembert (1717-1783). As Rucker (op.cit.) points out, the limit process allowed calculus to advance without the use of the actually infinite[12]. But the concept of limit was not clearly defined until 1821 when Cauchy (who also produced the first clear definitions of a *convergent sequence* and of a *continuous function*), in his *Cours d'Analyse,* wrote :

> "When the successively attributed values of the same variable indefinitely approach a fixed value, so that they differ from it by as little as desired, the last is called the *limit* of all the others."[13]

According to Grabiner (1983b, p. 204) "Cauchy understood 'limit' differently than had his predecessors. Cauchy entirely avoided the question of whether a variable ever reached its limit; he just didn't discuss it." For instance, for Cauchy an infinitesimal is conceived as a variable which tends to zero. Cauchy expressed it as follows:

> "One says that a variable quantity becomes infinitely small when its numerical values decrease indefinitely in such a way as to converge to the limit zero."[14]

Although modern calculus has found a way around infinitesimals, in practice it is probably common to think in terms of infinitesimals, e.g., every time we round a number or truncate an infinite decimal expansion, equating that which we disregard to zero because it is so small. In fact, Cornu (1991) explains that the idea of an 'intermediate state' between that which is nothing, and that which is not nothing, is

[11] English quote by Struik (1967), p.125, from Euler, *Opera Omnia,* 1st ser., Vol. 10.
[12] However, Rucker (ibid.) also observes that it is unlikely that the calculus could ever have developed so rapidly if mathematicians had not been willing to think in terms of actual infinities.
[13] English quote by Grabiner (1983a), p.185.
[14] English quote by Boyer (1954), p.273.

frequently found in modern students. For instance, they often view the symbol ε as representing a number which is not zero yet is smaller than any positive number. He adds that similarly individuals may believe that 0.999... is the "last number before 1" yet is not equal to one.

From the use made of infinitesimals throughout history, it seems that infinitesimals are perhaps very intuitive, even if they lead to possible difficulties (e.g., when can they be considered as zero, and when not), and they are still a current issue in mathematics. In fact, in 1966, Abraham Robinson, in his theories of non-standard analysis, presented a rigorous construction of infinitesimals, and work still continues in this field trying to define consistently and make use of infinitesimals[15].

Paradoxes of infinite sets (paradoxes of the infinitely big).

One of the paradoxes involving infinite sets is the problem that if a line includes infinitely many points, a length twice as long as another should include a *larger* infinity of points than the latter. However, these points could also be put into what we now call a one-to-one correspondence. Thus, there seem to be two infinities which are simultaneously different and equal (see Figure 2.2):

[15] For instance, Edward Nelson, in the branch of internal set theory, has defined an infinitesimal as a number that lies between zero and every positive standard number (that is, infinitesimals are less than any number which can ever be conceived to be written).

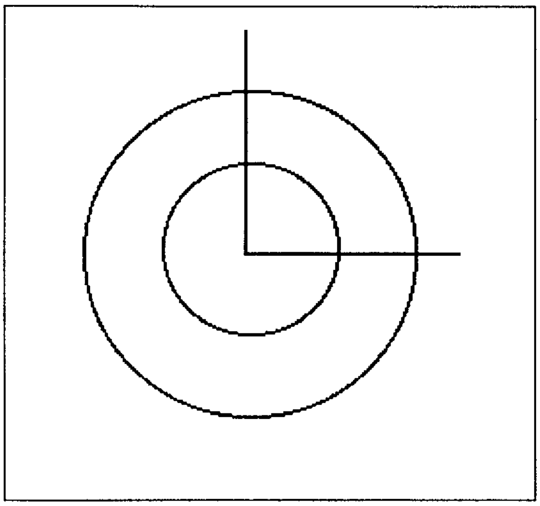


Figure 2.2. Correspondence between two infinite sets of points (circles).

The idea that what is infinite can have proper parts which are also infinite and therefore seem as great as the whole, is one of the reasons why Aristotle refused to accept the actual infinity (see Moore, 1991). The problem with this argument is that it involves thinking of the infinite sets using a finite framework. Galileo, in his *Dialogues Concerning Two New Sciences* (1638), recognised that the infinite cannot be thought of in the same way as we think of the finite:

> "...let us remember that we are dealing with infinities and indivisibles, both of which transcend our finite understanding, the former on account of their magnitude, the latter because of their smallness. In spite of this men cannot refrain from discussing them, even though it must be done in a roundabout way...
> "... since it is clear that we may have one line greater than another, each containing an infinite number of points, we are forced to admit that within one and the same class, we may have something greater than infinity, because the infinity of points in the long line is greater than the infinity of points in the short line. This assigning to an infinite quantity a value greater than infinity is quite beyond my comprehension.
> "This is one of the difficulties which arise when we attempt, with our finite minds, to discuss the infinite, assigning to it those properties which we give to the finite and limited; but this I think it is wrong, for we cannot speak of infinite quantities as being the one greater or less than or equal to another." (op.cit., p.26)

The problem of infinite "quantities" was not confined to the geometric. In fact, Galileo used in his arguments the correspondence between, for instance, the set of natural numbers and that of the squares of natural numbers. Thus, as Rucker (1982, p.6) points out, in Galileo we have the first signs of a modern attitude towards the actual infinity in mathematics: "If infinite sets do not behave like finite sets, this does not mean that infinity is an inconsistent notion. It means, rather, that infinite numbers obey a different 'arithmetic' from finite numbers." Galileo was setting the ground for Bolzano and Cantor 250 years later.

III. Defining the concept of infinity in mathematics.

There are several instances of attempts to talk of the infinite from within mathematics, and there have been several sources of difficulties. It can be said that this is because infinity needed to be defined as a mathematical object; in order for infinity to be incorporated into mathematics it needed to have its own operatory field.

Bolzano and the set-theoretic approach.

Bolzano (1781-1848), in his *Paradoxes of Infinity* (1851), dedicated his efforts to unveil the mystery surrounding the term infinity. He was the first to positively accept the actual infinity, and to introduce infinity into mathematics as an object of study. He evolved a concept of infinity in order to solve the considerable number of paradoxes that had been produced in his time. Bolzano essentially showed that the way in which infinity could be incorporated into mathematics was as a property of *sets*. As Moreno and Waldegg (1991) point out, Bolzano's work led to a new approach that would, in turn, transform infinity into an object with an operational domain. "With this new meaning it was possible for infinity to be *assimilated* into mathematics" (ibid., p.215). Bolzano, in his *Paradoxes*, took as a fact the idea of being able to put into a on-to-one correspondence the elements of an infinite set with those of one of its (infinite) subsets. This would form Dedekind's definition of an infinite set.

Weierstrass, and non-differentiable continuous functions.

Cauchy's work on the convergence of infinite series set the ground for the arithmetization of calculus and the work of Weierstrass (1815-1897). Cauchy's definitions took for granted the numerical continuum. Weirstrass realised (as did

Bolzano) that the ideas introduced by Cauchy could not be developed without a rigorous construction of the real numbers. With Weierstrass new criteria of rigour emerged, and he warned against the dangers of relying on geometrical intuition. This was best exemplified with the discovery of non-differentiable continuous functions.

Up to the 1870s, mathematicians such as Lagrange, Ampère, and many others had tried to prove that continuous functions were differentiable everywhere except at most for a finite number of points. As Chabert (1990) explains, this conviction seems to have been based on a geometric intuition — thinking that a continuous curve has well defined tangents in all but a few points. But by 1872 several counterexamples (e.g. by Riemann) had been found of continuous functions which were non-differentiable in an infinite number of points, and in that same year Weierstrass announced the discovery of a continuous function[16] which was not differentiable for any of its values. Interestingly, these examples were reached through purely analytic methods and, as Chabert (ibid.) points out, they are far from the intuition of hand drawn graphs. This period marks a turning point: from here on continuity becomes a property that is described and verified through *analytic* terms. Thus began the arithmetization of mathematics.

Cantor's Set Theory.

Following Weierstrass, another important development for the mathematics of infinity took place. As expressed by Lévy (1987), the *properly mathematical* history of infinity started in the nineteenth century with Cantor, who formalised the idea of infinite number. Cantor and Dedekind developed the mathematics of the infinite, and a new theory which integrated the infinite with the finite and accepted the duality of infinity as both potential and actual. That is, the theory integrated the two ways of looking at infinity: as a *process* or as an *entity*. As Moreno and Waldegg (1991) explain, in the first, infinity appears as something which *qualifies* the process, whereas in the latter it is an *attribute* or property of a set. Prior to Cantor's work, examples are

[16] $g(x) = \sum_{n=0}^{\infty} b^n \cos(a^n \pi x)$ where a is an odd integer, $0<b<1$ and $ab > 1 + \frac{3}{2}\pi$

found of the difficulties with infinity in the attempts to see it *either* as a process *or* as an object (e.g. the theory of the indivisibles).

Cantor used the one-to-one correspondence as a way of comparing infinite sets, and he used this to show, in 1873, that there are degrees (*powers*) of infinity: the set of points of the real line constitutes a higher infinity than the set of all natural numbers, the real numbers being non-countable. In 1877, Cantor also proved that continuous curves, areas and volumes have the same power. His work also provided an instrument for differentiating between *density* and *continuity*: Cantor showed for example that the rational numbers, which are dense (between any two, there exists another rational number), are, unlike the real numbers, countable.

Peano's space-filling curve.

Cantor's correspondence between continuous curves and areas was brought to another level in 1890, when Peano (1858-1932) constructed a "space-filling" curve, i.e., a parametrical continuous curve which goes through every point in a square. It is noteworthy that Peano did not use any geometric reference when he exhibited his result. Unlike him, Hilbert, a year later, presented an analogous result but used the geometric construction as basis for his proof.

Von Koch's curve and other fractals.

The new finding of nowhere differentiable continuous functions was not an easy result to accept intuitively. Nor was Peano's space-filling curve. And by the early twentieth century there were those who began speaking against the excessive arithmetization of mathematics, claiming that some of the intuition given by geometry was necessary. Among them was Helge von Koch (1870-1924), who, in 1904[17], wrote:

> "Until Weierstrass constructed a continuous function not differentiable at any value of its argument, it was widely believed in the scientific community that every continuous curve had a well determined tangent (except at some singular points). It is known that, that from time to time, some geometers had tried to establish this, no doubt based on the graphical representation of curves.

[17] In his paper "On continuous curves without tangents constructed through elementary geometry".

"Even though the example of Weierstrass has corrected this misconception once and for all, it seems to me that his example is not satisfactory from the geometrical point of view since the function is defined by an analytic expression that hides the geometrical nature of the corresponding curve and so from this point of view one does not see why the curve has no tangent. ...
"This is why I have asked myself — and I believe that this question is of importance also as a didactic point in analysis and geometry — whether one could find a curve without tangents for which the geometrical aspect is in agreement with the facts. The curve that I found and which is the subject of this paper is defined by a geometrical construction sufficiently simple, I believe, that anyone should be able to see through 'naive intuition' the impossibility of the existence of a tangent."
(von Koch, in Edgar, 1993; p.25-26).

The construction of what is now known as the Koch curve begins with a basic figure (see Figure 2.3):

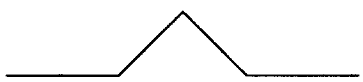


Figure 2.3. First stage in the construction process of the Koch curve.

In the next stage, each segment of the previous figure, is substituted by a copy of the original figure, which yields Figure 2.4:

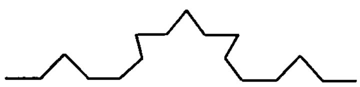


Figure 2.4. Second stage in the construction process of the Koch curve.

The process of substituting each segment by a copy of the original figure is then repeated *ad infinitum.* The figure which is the limit of such a process is the Koch "curve". It is thus that a *self-similar* fractal[18] figure emerged. Rucker (1982, p.8) writes the following: "The Koch curve is found as the limit of an infinite sequence of approximations ... If we take infinity as something that can, in some sense be attained, then we will regard the limit of this infinite process as being a curve actually existing, if not in physical space, then at least as a mathematical object."

[18] The term "fractal" was defined much later in 1975 by B. Mandelbrot — fractals being those figures which have fractional Hausdorff dimension.

More fractals.

When they first appeared, non-differentiable continuous functions, space-filling curves, and what would later be called fractal figures, were considered as "pathological" and were called "mathematical monsters". Koch's and Peano's constructions were followed by many other "monsters". Among them we can find Cantor's set or "dust", the "Devil's Staircase", and Sierpinski's Triangle, shown later in this work, which can be obtained by "erasing" the central half-sized triangle inside the original triangle and repeating this process for each of the remaining smaller triangles. Sierpinski's triangle can also be obtained through a curve which is the limit of polygonal lines.

Today, fractals are an everyday word and they are at the core of modern developments in mathematics. It is interesting to quote Dyson (1978,p. 677-678):

> "A great revolution of ideas separates the classical mathematics of the 19th century from the modern mathematics of the 20th. Classical mathematics had its roots in the regular geometric structures of Euclid and the continuously evolving dynamics of Newton. Modern mathematics began with Cantor's set theory and Peano's space-filling curve. Historically, the revolution was forced by the discovery of mathematical structures that did not fit the patterns of Euclid and Newton. These new structures were regarded... as 'pathological', ... as a 'gallery of monsters'... The mathematicians who created the monsters regarded them as important in showing that the world of pure mathematics contains a richness of possibilities going far beyond the simple structures that they saw in Nature. Twentieth-century mathematics flowered in the belief that it had transcended completely the limitations imposed by its natural origins..... Now, as Mandelbrot points out, ... Nature has played a joke on the mathematicians.... The same pathological structures that the mathematicians invented to break loose from the 19th century naturalism turn out to be inherent in familiar objects all around us."

As Moreno (1995) points out, the situation thus described is characteristic of the history of mathematics, which shows a permanent tension between concrete and abstract. What is abstract at one level of historical developmental becomes concrete at a later one.

IV. Key considerations for the study of the concept of infinity.

As this historical review shows, infinity can be considered a highly "abstract" mental construct that tends to have a contradictory nature. It is a concept that depends — perhaps more so than any other mathematical idea — on the context and point of view we adopt, as has been noted by many of the researchers reviewed in Chapter 3 (e.g. Fischbein et al., 1979; Nuñez, 1993). In this respect, David Tall (1980; p.281) points out that "our interpretation of infinity is relative to our schema of interpretation, rather than an absolute form of truth." Thus, the perspective adopted, and the context in which the infinite is presented, are likely to have a determinant role on how it is conceived. However, it may be possible to take advantage of the situation(s) in which the infinite is presented to make it more "concrete". As discussed in Chapter 3, Wilensky (1991) suggests that abstract objects can become concrete if we have multiple modes of engagement with them. In the same way, the study of the infinite may be facilitated by helping the learner to experience various contexts in which infinity occurs and to build connections between them (see also Noss & Hoyles, 1996). Thus, I identify below some key aspects of the infinite which need to be considered for this purpose:

a.- Types of infinity:

(i)The dual nature of infinity: potential infinity and actual infinity.

On the one hand the infinite can be seen as the result of a *process*, and a process implies change (e.g. change over *time,* movement). This is the idea of *potential infinity*: the idea that you can always add one afterwards. The other view is of considering an infinite *object* (e.g. infinite sets) as a *state.* In this case we have an *actual infinity.*

The evolution of the definition of an infinite set reflects the two perspectives: From the first perspective, a set A is considered (potentially) infinite if the following statement is true: "If $x \in A$ then $x+1 \in A$". The definition evolved to: "A is (actually) infinite, if there exits B, a proper subset of A, such that there exists a one-to-one correspondence between A and B." In the second case, the "process" is finished (it is *outside* time), thus the infinite set can exist as a whole.

(ii) The infinitely big and the infinitely small.

(iii) Powers of infinite sets, e.g. denumerable vs. non-denumerable sets.

b.- Mathematical setting and context:

Infinity is found in a variety of settings and mathematical areas: Geometry, Sequences and Series, Set theory, Limits... As discussed above, one's interpretation of the infinite will depend on the situation in which it is presented. For instance, the dual nature of the concept of infinity can be thought of as resulting from the context and perspective adopted when dealing with the infinite as seen in section a. above.

Thus, it should be taken into account that the infinite can be approached through *visual/geometrical* models, or through processes defined in purely *symbolic/algebraic* terms (as in notations such as $\lim_{x \to \infty} f(x)$). In some circumstances, an infinite process can be contained within *finite bounds*. This is a contextual variable which can be expected to cause difficulties. Furthermore, as some researchers indicate (e.g. Nuñez, 1993), even when the same problem is isomorphically constructed in different contexts, the context affects the way in which the problem is conceived.

c.- Iteration and recursion:

The idea of indefinite repetition is fundamental in the development of the concept of infinity. And iteration is, mathematically speaking, related to recursion. A recursive algorithm intrinsically contains an indefinite number of iterations: it is *potentially* infinite. In fact, infinite objects have *self similar* characteristics resulting from their recursive structure (e.g. fractals are by definition self-similar; the definition of an infinite set also describes a self-referral property in that an infinite set is such that it can be put in a one-to-one correspondence with at least one of its infinite proper subsets).

d.- Nature of the mathematical object.

Two mathematical areas which relate to the infinite should be discriminated: the *discrete* (e.g. the Natural numbers), and the *continuous* (e.g. the Real Number Line = the continuum). The relationship between these two has been a source of conflict since the ancient Greeks. In particular, a distinction should be made between *cardinality*, and spatial *measure* (for instance, the number of nines in 0.999... is a cardinal infinity, while the perimeter of the Koch curve is an infinite spatial measure).

The above aspects should all be taken into account for the study of infinity. In addition to these, the historical review has pointed to :

- different areas and ways in which the infinite appears; and

- the problems and issues which can arise when dealing with the infinite, and which may be resonant with students conceptualisations.

These are all important considerations in the design of a study for the exploration of infinity, and, particularly, in the design and choice of activities to be included. Furthermore, the historical problems and areas of difficulty, together with the key aspects stated above, should be taken into account in the analysis of the results of the study.

In the following chapter, I present a review of aspects of the psychological and pedagogical literature, and, in particular, a review of educational research in the areas of calculus, infinity and limits. The ideas from that chapter complement the considerations which have emerged here. How the ideas from both chapters were implemented in the design of the study is the subject of Chapter 5.

Chapter 3:

Representations, Domains of Abstraction, and Infinity: A review of the literature

I. Representation Theory and Visualisation.

There are two aspects of the literature which are of particular concern. The first is the interplay between visual means of representation and other ones, particularly symbolic forms of representation. The second is the question of mediation and the ways in which representations mediate the construction of mathematical knowledge.

First, however, it is important to clarify the term *representation*.

Defining the term "representation".

Denis (1991) points out that "*representation* can refer to both a process and the outcome of this process" (emphasis added, p.1) where the former is an *activity* generating objects or entities, and the latter "representation" refers to "the *entities* themselves rather than the activity which produced them" (ibid. p.1), and which can be either physical objects or cognitive entities. A drawing of a physical object is a new object which also exists as a physical entity, but is different from a *mental image* of that same thing. Both evoke the original object, but "in the second case, the process of representation results in a specific psychological event, a transient cognitive reality which is not directly observable by others" (ibid., p.2-3). He is thus referring to two types of outcome of a representational process which should be distinguished: "mental" ("cognitive entities") and "external" representations ("physical objects").

Dreyfus (1993) points out that representations perform an important function in mathematics. He defines *external representations* as what we use when communicating about mathematics, such as formulas, graphs, etc. On the other hand, *mental representations* are that which we have in mind when we think of a mathematical object or process, and they "may be vastly different for different people" (ibid. p.123). This idea, also used by Cornu (1991), is developed by Tall and Vinner (1981) who employ the term *concept image* to help elaborate what it means to have an idea of a concept. They define it as that which describes "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p.152). This, Tall and Vinner point out, differs from the *concept definition* in that the latter is just a "form of words used to specify that concept" (ibid.). I would argue that thus defined, the concept definition is a form of external representation which serves to communicate the idea in a formal way. But Tall and Vinner also point to the idea that the only way in which we can approach the formal object is from the corresponding cognitive structures, a view elaborated by Sfard (1991).

Representations and the construction of meanings.

Many researchers have focused on the nature and "adequacy" of mental representations. For instance, von Glasersfeld (1987) discusses the issue of how well cognitive structures match what they are intended to represent, and suggests changing the emphasis from the idea of representing to that of construing, or how we make sense of the world. In fact, Mason (1987) argues that the term "representation" may not be a sensible or consistent way to describe what goes on inside a person, because it is the inner experiences which *are* a person's world, and not merely representations of the world. Nevertheless, it is common to think in terms of how people internally conceive or represent a mathematical knowledge: Davis (1984) argues that in order for any mathematical concept to be present in the mind, it has to be *represented* in some way; thus, representations are in some way the "ideas" one has of a specific concept. Davis adds that for problem solving, *efficient* representations are needed. Some researchers (such as Tall and Vinner, 1981; Sierpinska, 1987; Cornu, 1986) have thought in terms of a conflict (sometimes referred to as *misconceptions)* between the concept image — i.e. what goes on in the student's mind — and the concept definition that could give rise to cognitive obstacles[1] that may impede learning.

Dreyfus (1993) argues that success in mathematics depends on rich mental representations which involve many linked aspects of a given concept. He adds that several mental representations of a concept may be present simultaneously and be called up in different situations, though they may complement each other and eventually become integrated allowing the subject to use them simultaneously and to switch efficiently between them as required by the situation or problem with which he/she is faced with.

I would add that objects are not the only ones to be represented; the actions on these objects can also be represented. Thus a *cognitive structure* can be thought of as an organised system of mental representations supplied with an operation for working on those representations: there is a strong interaction between mental representations and external representations. How can we communicate, or work with a mathematical concept, if not through its (external) representations? The mental representations or conceptions that one has of a particular mathematical idea or concept are determined by one's particular individual experiences and interactions with external representations: that is, by working and (re-)constructing representations and models of instantiations of a concept. Thus the construction of a concept requires an intense use of different representational systems (visual, algebraic, etc.). It is only through the use of external representations that a shared meaning can be constructed. In fact, the *meaning* of the mathematical object under study is the result of the articulation of the different representations of the object. In one way or another, this is the case with every mathematical concept.

Gardiner (1984; p.24) points out: "Connections control meaning. So if we want to endow new ideas with suitable meaning, we must establish appropriate connections between these new ideas and students previous experience". It can be said that these connections are constructed representations of (inter)actions or relationships between the objects or ideas. Wilensky (1991) uses this idea of building connections in his

[1] Some of these obstacles are related to the wider notion of 'epistemological obstacles' which was introduced by Bachelard (1938) and developed in the area of mathematics education by Brousseau (1983).

discussion of what makes knowledge abstract or concrete. Wilensky dismisses the standard definition of "concrete" as something particular which can be visualised or *sensorised.* He explains that "concreteness is not a property of an object but rather a *property of a person's relationship to an object"* (p.198), and points out that the "formal is often abstract because we haven't yet constructed the connections that will concretize it" (ibid., p.202). Thus, an abstract concept can become concrete by relating to it in as many ways as possible. As he puts it:

> "The more connections we make between an object and other objects, the more concrete it becomes for us. The richer the set of representations of the object, the more ways we have of interacting with it, the more concrete it is for us. Concreteness then is that property which measures the degree of our relatedness to the object ...
> "This view will lead us to allow objects not mediated by the senses, objects which are usually considered abstract — such as mathematical objects — to be concrete; provided that we have *multiple modes of engagement with them and a sufficiently rich collection of models to represent them."*
>
> (ibid., p.198-99; emphasis added)

This supports the idea that the learning of a concept is facilitated when the individual has more opportunities of *constructing* and interacting with as many and varied external representations of a concept as possible. However, as contended by Wilensky, it is generally not enough for the individual to be presented with diverse representations of instantiations of a concept. It is by working and re-constructing external representations and the relationships between them, that the subject constructs his/her own mental representations of the objects, as well as the connections between them, and those which give them meaning in the wider conceptual mesh.

In addition, it should be noted that there is a social/contextual component in the construction of knowledge and its representations. This is succinctly put by Piaget and Garcia (1989, p. 247):

> "...in the experience of the child, the situations she encounters are generated by her social environment and the objects appear within contexts which give them their specific significance. The child does not assimilate "pure" objects defined by their physical parameters only. She assimilates the situations in which objects play a specific role. When the system of communication between the child and her social world becomes more complex and enriched...then what we might call direct experience of

> objects comes to be subordinated...to the system of interpretations attributed to it by the social environment".

Piaget and Garcia call attention to the importance of seeing knowledge as not simply the building up of internal structures in the mind. They emphasise the importance of experience as it colours and mediates how knowledge is constructed.

As is well known, Vygotsky extended this perspective emphasising how the construction of mental representations and meaning results from *action* supported by *mediational tools*, such as external representations and language: mental representations are the internal reconstructions of external action. As Wertsch (1991, p.12) remarks:

> "The most central claim I wish to pursue is that human action typically employs mediational means such as tools and language and that these mediational means shape the action in essential ways"

Following this school of thought, Confrey (1993, p.48) aptly describes knowledge in the following way:

> "knowledge is not a set of descriptions about the world but a set of hard worn realizations of how human beings interact with the world through the use of tools (including language)".

External representations are tools in the construction of meanings. Papert (1993, p.142) points to the importance of the use and construction of external representations[2] in the process of knowledge construction:

> "One of my central mathetic tenets is that the construction that takes place 'in the head' often happens especially felicitously when it is supported by construction of a more public sort 'in the world' — a sand castle or a cake, a legohouse or a corporation, a computer program, a poem, or a theory of the universe. Part of what I mean by 'in the world' is that the product can be shown, discussed, examined, probed, and admired. It is out there."

On visual and symbolic representations.

Some representations are of visual form (e.g. the graph of a function); others are purely symbolic or algebraic, lacking a graphical aspect. The terms "visual" and "symbolic" are commonly used to denote two types of representations, even though

[2] Note: From this point onwards, when I talk of *representations*, I refer mainly to *external* representations.

visual representations are also a form of symbolising. Some researchers prefer to use "graphical" instead of visual, and "analytic" or "algebraic" instead of symbolic (e.g. Artigue, 1990, uses "algebraic" and "graphical"). The terms "sentential" or "propositional" (see Dreyfus, 1995) are also sometimes used to contrast with the visual forms. Sutherland (1995, p.72-3) uses Dreyfus's (1995) categories which she summarises as follows:

> "*visual systems* include pictures, icons, mathematical objects with some link to reality (for example a sphere), mathematical objects with no link to reality (for example graphs)
> "*sentential systems* include natural language, algebraic systems and algebra-like programming languages".

Since it is difficult to find the most adequate terms, I will continue to use the terms *visual* and *symbolic* which essentially correspond to the above categories.

The visual (graphical) representation of a mathematical situation gives a global view (as explained for instance by Larkin & Simon, 1987), while, on the other hand, the symbolic representation involves more local analysis. A graph can be analysed locally; yet it is a visual representation of the entire situation. An algebraic representation is one that has to be travelled linearly (see Chevallard, 1985), one aspect at a time, and the image of the whole tends to be out of focus. Whatever the representational form, however, it is necessary to decode it (to analyse it). Thus, visual and symbolic representations are complementary, each representation holding a different form of interpreting the information. An integration of both types of representations appears to be essential for constructing a richer meaning of the mathematical object.

What is visualisation?

Visualisation is often used in literature as referring to mental imagery; yet it is important also to refer to the process of visualisation through the use of external visual models. Zimmerman & Cunningham (1991, p.3) give the following definition:

> "Mathematical visualization is the process of forming images (mentally, or with pencil and paper, or with the aid of technology) and using such images effectively for mathematical discovery and understanding".

They point out that mathematical visualisation differs from the use of the term "visualisation" — in areas such as psychology — as meaning "to form a mental image", in that images are not constrained to being manipulated purely on the mental level; adding that what is of interest is precisely the use of graphical representations (with pencil and paper, or with a computer) to represent a mathematical concept or problem. As Sutherland (1995) argues, mental symbols cannot be communicated and developed without some form of external support — language, diagrams on paper, a computer program etc. She also emphasises the importance of the mediational aspect of these external representations and adds that what is of interest is not so much the mental processes but the person *acting*[3] with mediational means.

It is important to point out that visualisation does not merely refer to ways of thinking visually of a symbolically defined concept. As Noss & Hoyles (1995, p.200) point out: "Perhaps we should not think in terms of 'visualising' a (symbolic) mathematical idea, but rather to consider the visualisation itself as part of the mathematical idea." Davis (1994) argues that mathematical education should consider the inclusion of what he calls "visual theorems", that is "theorems" which come from visual mathematical intuition and reasoning. He points out that calculus is full of visual theorems (e.g. a local extremum of a smooth function occurs where the derivative is zero); but Davis advocates in particular graphical results of computer programs which, through the eye, are organised into a coherent whole which inspires mathematical questions or understanding of some mathematical situation. An example that Davis gives are fractal graphics, for which he points out: "Aspects of the figures can be read off (visual theorems) that cannot be concluded through non-computational mathematical devices" (ibid., p. 339).

Visualisation is recognised as an important process in the construction of mathematical ideas. In fact, mathematicians' reliance on visual reasoning has long been noted (see for instance Hadamard, 1945), many times preceding symbolic formulations. Similarly, Hallett (1991, p.121) points out that "visualisation is a big part of understanding," and he adds: "students who are operating with few mental pictures are not really learning mathematics... the efforts put into this kind of [algorithmic] teaching and learning perpetuate the idea that math involves doing

[3] Emphasis added.

calculations rather than thinking" (ibid.). Further evidence of the value of emphasising visual representation in all aspects of mathematical thinking, is recounted by Bishop (1989), who also describes how the visualisation process has a highly individual nature, which needs to be taken into account by mathematics educators.

Visualisation has been recognised as difficult for many students. For instance, it has been found that pupils tend to prefer symbolic manipulation to visual interpretation, perhaps because the latter requires moving towards a higher cognitive level — for 'decoding' the visual information (see Dreyfus & Eisenberg, 1990; Eisenberg & Dreyfus, 1986; 1991). Many students have difficulties in reading diagrams, and one of the things that has been observed is that students do not easily make links between visual representations and analytical thought (see for instance Artigue 1990, Presmeg 1986, Hillel & Kieran 1987).

The study of how visual models are interpreted and the ability to translate linearly symbolic information into images, have been considered by a number of researchers. Dreyfus et al. (1990) include a discussion related to the concept of function: although it is surmised that visualisation helps students form more complete concept images of a function, they explain the difficulties students have in visualising various aspects of a function graphically, and of interpreting information given in a graphical way. Visualisation seems to be very rare in calculus, "and if it occurs the cognitive link between the visual/graphical and the analytic/algebraic representation is a major point of difficulty" (ibid., p.125). Their findings seem to point out that the procedures of calculus are often learnt on a purely algorithmic level. The authors conclude that students of calculus construct incomplete concept images related to the algebraic/analytic formalism they have learned, lacking visualisation and abstraction. Eisenberg and Dreyfus (1991) discuss the reluctance of students to visualise and give three main reasons for this avoidance. The first is cognitive, deriving from the belief that the visual is more difficult; but they also point to a sociological reason in that there is a lack of visual emphasis in the teaching of mathematics, perhaps because it is more difficult to teach. Third, they blame the widespread belief that the visual does not constitute formal mathematics. In this respect they add that many mathematicians are reluctant to accept visual approaches in their finished work, despite the fact that most of them readily exploit visual exploration and argument in their mathematical activities.

Although most researchers agree on a need for more emphasis on visualisation and the linking of different kinds of representations, visualisation has often been neglected in mathematics education, in particular in areas such as calculus, where the analytic (symbolic) approach has been prevalent for the last decades, and generally students are not sufficiently exposed to visualisation methods. As Cuoco & Goldenberg (1992) point out: "geometry represents virtually the *only visually oriented mathematics that we offer our students*" (p.181, original emphasis), all other areas of mathematics being presented through linguistic symbolism, even though a visual approach seems to be absolutely essential for some students. Schools thus appear to fail to develop visual skills that are a powerful part of mathematical thinking. In addition, this leads to a failure to connect different branches of mathematics, a process which Cuoco and Goldenberg claim helps students "see order and systematicity in place of the lists of disconnected facts and procedures they otherwise experience" (ibid., p.182).

The need for including more of the visual aspect in mathematics education is evident, particularly in contexts which link it to the numerical and symbolic aspects of mathematics. But as Noss & Hoyles (1995, p.191) point out, "we should not take for granted that building links *between* representations is straightforward, or that the more representations which are available, the better it is for learning". Thus, it is important to take into account the context, as well as the social situation, in which the student searches for meaning, as these will have a strong influence on the resulting conceptions.

Computer-based representations and experimental mathematics.

One way in which the mathematics education community has attempted to integrate visual reasoning is through the use of computers and computer-based environments, discussed in more detail in the following section. The computer seems to offer a number of advantages for integrating several types of representations: Several representations can be produced by the same program (which entails a single description of a process or situation); thus the computer has the capability to represent the same situation through simultaneous different representations. This is the property which Kaput (1995) has relied upon for creating environments which offer different representational "windows" on a same general situation. Through his software the user can manipulate one of the particular representations of a situation and observe the consequences in the other representations simultaneously displayed; Kaput argues that these experiences may help students link more familiar and concrete representations to more abstract ones.

Another advantage of the use of computers is that it allows the user to experience representations of processes in a dynamic way, to view them as they "unfold" in time. Vitale (1992) has explored the integration of computer representations of processes of *time* and *change* into mathematics education. But Vitale's approach, using the Logo computer language, emphasises in particular the value of *programming* (an important idea discussed further in section II. below) for defining the role of variables, parameters and initial conditions in a process, as well as for helping in the conceptualisation and representation of a process (through the logico-mathematical structures of the algorithm being used).

Additionally, the exploratory nature offered by computer-based environments — explained further in the next section — together with the visual capabilities of the computer, may allow students to build models which they can use to construct the meaning of mathematical results (or derive results through those computational and visual means, such as Davis's(1994) visual theorems, discussed earlier). In other words, the computer could be used as a sort of mathematical laboratory. This is one of the main ideas given by Mandelbrot (1992) who emphasises that the computer, and computer graphics, are bringing back the idea of experimentation into mathematics and giving a renewed importance to the role of visual representations. He explains:

> "experimental mathematics means injecting experiment back into core parts of mathematics that need not – at present – have any contact with science...it underlines the reality of an essential distinction between mathematical facts and mathematical proofs... the use of the computer is now in the process of changing the role of the eye... computer graphics is bringing it back as an integral part of the very process of thinking, search and discovery".

Schwartz (1995) describes the potential of computer technology for creating environments in which students can build general knowledge through the exploration of particular cases. Schwartz shares the Piagetian belief that learners must play an active role in the construction of their knowledge. He explains that in the case of mathematics this active role involves building on what is known (the particular) by posing and exploring conjectures and hypotheses; in his paper, Schwartz discusses how computer environments can offer students the opportunity to formulate and explore their *own* hypotheses and conjectures in particularly productive ways.

Visual and computer-based "proofs".

Tall (1991b) advocates the effective use of visualisation and exploratory mathematics to give intuition for formal proof by building up an overall picture of the relationships involved. Although he is aware of the downside, in that pictures can suggest false theorems, he explain that in some cases this can be due to inadequate experience with the concept to provide appropriate intuitions. Among his suggestions is the use of the computer to generate numerical solutions, which are generally not precise since computers give real numbers as rational approximations; he believes this "inaccuracy" may promote a need for formal proof in students.

> "By introducing suitably complicated visualizations of mathematical ideas it is possible to give a much broader picture of the ways in which concepts may be realized, thus giving much more powerful intuitions than in a traditional approach. It is possible to design interactive software to allow students to explore mathematical ideas with the dual role of being both immediately appealing to students and also providing foundational concepts on which the ideas can be built. By exploring examples which work and examples which fail, it is possible for students to gain the visual intuitions necessary to provide powerful formal insights. Thus intuition and rigour need not be at odds with each other. By providing a suitably powerful context, intuition naturally leads into the rigour of mathematical proof." p.118.

There are some researchers who advocate a re-evaluation of the role in proof of visual representations. Barwise & Etchemendy (1991) claim that "visual forms of representations can be important not just as heuristic and pedagogic tools, but as legitimate elements in mathematical proofs" (p.9), without saying that these should replace linguistic forms, nor that mathematical proofs should be anything less than rigorous. But putting aside the question of whether visual or computer-generated proofs can be considered rigorous enough to be mathematically accepted, I would like to place emphasis on the role that these forms of visual and computational means can play in the process of discovery and acceptance of mathematical results. Thurston (1994), who analyses the nature of proof and of mathematics itself, explains that it is a search for *understanding* which is at the basis of the exploration and logical processes leading to a proof; he also advocates the use of computers for exploration and discovery of mathematical ideas, and gives priority to what he calls "humanly understandable" proofs over formal proofs.

Among those who have attempted to develop a sense of proof through visual and computer-based explorations are Cuoco & Goldenberg (1992). They explain that for mathematicians, the activity of constructing proofs is a research technique where conjectures arise through the combination of experimentation and deduction. In their project with fractals and recursively defined geometric constructions, described further below, one of their aims was to allow students to experience that process, encountering, for instance, mathematical induction in a visual context.

II. Domains of abstraction: Windows and Microworlds (computer-based learning environments).

With reference to the above discussion on representation, I want to draw particular attention to the role of external tools and representations in the construction of meanings. The fundamental idea is that external representations can be used to express, articulate and make manifest one's own perceptions and ideas: external representations (either one's own, or somebody else's) can be used as tools to think with, through *manipulation and expression* (see Mason, 1987), and in this way they can simultaneously reveal a person's inner world.

These are ideas which can be taken into the context of computational systems. For example, as pointed out by Noss & Hoyles (1996), there is extensive research into the idea that writing a computer program — where relationships need to be articulated

— provides a means for sketching half-understood ideas. Much of the original research in this area involved programming with Logo (e.g. Feurzeig, Papert et al., 1969), and such work still continues; an alternative approach, more general in the choice of language, has been pursued under the name of "algorithmics" (e.g. Johnson, 1991). In this section, I define in particular the idea of "microworlds", computer-based learning environments which are designed to provide tools and means for the learner to explore and articulate his/her ideas, and which can simultaneously provide a window for the researcher to observe the learner's thinking processes.

Microworlds: domains of abstraction.

Hoyles (1993) defines microworlds as "computational worlds where mathematical ideas are expressed and developed" (p.1). She gives a detailed account of the genesis of the meaning of a mathematical microworld which is relevant in that it illustrates the new conception of the meaning of a microworld. She explains:

> "*the goal for microworlds has shifted —from teaching computers to solve problems to designing learning environments for the appropriation of knowledge and, as a consequence of this change in focus, the transitional object takes a central role.*" (p. 2, her emphasis)

where "transitional objects" — a term coined by Papert (1987) — are those standing between the concrete and directly manipulable, and the formal and abstract.

Weir (1987) explains the origins of the term "microworld":

> "the term *microworld,* ... was first used by artificial intelligence workers to describe *a small, coherent domain of objects and activities implemented in the form of a computer program* and corresponding to an interesting part of the real world. Since the real-world counterparts were typically very complex, the microworlds of those early days were simplified versions of reality, acting as experiments to test out theories of behaviour...
> "... Papert then went on to use the same term to describe the computer-based environments he was building, since they function in essentially the same way for the child as those earlier microworlds did for their creators. They are places "to get to know one's way around" a set of concepts, problem situations, activities; places in which the student and teacher can test out ideas in a subject domain of interest... Microworlds are clearly in the *discovery-learning* tradition." (Weir, 1987, p.12, original emphasis).

Weir explains how computational environments can serve to build bridges by linking the intuitive understanding with the act of programming, as well as linking the programming with the central ideas of some subject matter:

> "A computational environment should be a place where the *learner's intuitions, her current explanations for phenomena, are evoked during the process of learning about some subject matter via programming activity.*" (Ibid., p.15; original emphasis)

Over the years researchers have used the term microworld to designate a computer software or environment. For instance, Thompson (1987) defined mathematical microworlds as a system composed of objects, relationships among objects, and operations that transform the objects and the relationships. He added that what is essential is that it contains operations by which new objects can be made, pointing out that that is what makes the microworld "mathematical": constructing relationships and taking those relationships as new objects to be operated upon. Thus the focus of a mathematical microworld is on the construction of meaning and relationships which serves as a model of a formal system:

> "The goal which mathematical microworlds serve is to provide students with opportunities to create mental models that reflect the structure and composition of the formal systems". (p. 85)

The idea of microworld forms the basis of Papert's *Mindstorms* (1980), where he emphasises the importance of the exploratory nature of a microworld and of children being in charge of their activities (thus learning what it is to *be* mathematicians):

> "Children get to know what it is like to explore the properties of a chosen microworld undisturbed by extraneous questions. In doing so they learn to transfer habits of exploration from their personal lives to the formal domain of scientific theory construction" (p.117).

Hoyles & Noss (1987a) took the idea of microworld a step further by considering "the didactical situation in which the interaction takes place" (p.587), and pointing out that the definition of a microworld must take into account the learner, the teacher, the setting and the activity which is in itself shaped by the past experiences and intuitions of the learner together with the aims and experiences of the teacher. They also explain that "the objective of pupil programming is to provide the learner with a tool with which to model the mathematics under discussion" (ibid., p.582), and that Logo programming, in particular, can serve as "a means of engaging in mathematical activity..., as an environment for *doing* mathematics" (ibid., p.583).

Hoyles and Noss thus defined a microworld as having four components: the pupil component (concerned with *the existing understandings and partial conceptions which the child brings to the learning situation*); the technical component (constituted by *the software or programming language and a set of tools which provides the representational system for understanding a mathematical structure or a conceptual field*); the pedagogical component (*all the didactical interventions that take place during the programming activity*); and the contextual component (*the social setting of the activities*).

diSessa (1995; p.341) adds:

> "The ideal microworld should have the following properties: (1) It ought to have a fairly simple interactional form students can easily learn. (2) It ought to provide a wide range of self-motivated activities. (3) It ought reliably to bring students into contact with fundamental mathematical or scientific ideas."

More recently, Noss & Hoyles (1996) have emphasised the mediating role of the computer. They propose the notion of *situated abstraction,* as a way of describing how learners can develop mathematical meanings.

> "We intend by the term *situated abstraction* to describe how learners construct mathematical ideas by drawing on the webbing of a particular setting which, in turn, shapes the way the ideas are expressed." (p.122)

Following Wilensky, they suggest that learners engaged in microworld-based activities would be "abstracting *within*, not *away from*, the situation" (ibid., p. 125). Thus computational environments offer a setting where the objects and relationships can become *meaningful* through actions within the microworld and students can generate and articulate mathematical relationships which are general to the computational situation in which they are working. Although this could be seen as a step towards the corresponding formalised mathematical structures, a situated abstraction is conditioned by the technology and language involved. What is relevant from the educational point of view is that the student who constructs a situated abstraction may not have access to the semantics (and syntax) of the official mathematical language.

The microworld as a window for studying thinking in change.

Noss and Hoyles (1996) also point to the role of the microworld, not only as an exploratory learning environment incorporating the computer, but also as a research tool where mathematics educators can study learning behaviour:

> "Our central heuristic is to take the idea of the computer as a window on knowledge, on the conceptions, beliefs and attitudes of learners, teachers and others involved in the meaning-making process.... By offering a screen on which we and our students can paint our aspirations and ideas, the computer can help make explicit that which is implicit, it can draw attention to that which is often left unnoticed." (ibid., p. 5).

Noss and Hoyles give as one of the key issues the fact that the computer demands the user to *express* him/herself in a semi-formal way. The authors explain that in this sense the computer provides a screen on which learners can express their thinking, simultaneously giving the observer the possibility to glimpse their thoughts. Weir (1987) had previously pointed out the value of computational environments as empirical windows for the researcher: "The computer activity serves to catalyse the surfacing of the learner's intuitions. We can observe how students react to seeing the effect of their actions on the screen and the wide range of responses that they make to these effects"(p.19). Noss and Hoyles (1996) explain how the computer can serve as a means to study what they describe as *thinking-in-change:* instead of attempting to take a snapshot of a person's mental state, the idea is to set thinking in motion and investigate the changes that occur when, for instance, new notions are introduced, and the ways the learner makes connections and constructs meaning. For instance, the computer can serve as a tool to explore the interplay between the creation of visual and symbolic meanings (see also Noss, Healy & Hoyles, in press).

Following this line of thought, Goldenberg (1995) also argues that through the observation of students manipulating multiple representations, "students juggling the interaction among representations" (p. 155), we can get a glimpse of the rich internal models that they construct in their attempt to understand. He explains that this facilitates our understanding of *understanding,* which can hardly be accomplished by looking at a student's handling of one representation in isolation.

III. Research on infinity and limits.

The research into the concept of infinity is quite varied and involves many different points of view and approaches. There are two main mathematical areas where research has been focused: one refers to notions of infinity found in calculus, mainly referring to the concepts of limit and continuity. The other is related to the concept of infinity as found in set theory, and involves ideas such as the infinitely big, the infinitely small, and comparisons between different types of infinities.

Research on the concept of infinity.

One of the first investigations into the concept of infinity was carried out by Fischbein et al. (1979). It focused on what the authors call *the intuition of infinity.* The authors argue that "the concept of infinity (and specifically of infinite divisibility) is intuitively contradictory" (ibid., p.6). Through a questionnaire applied on a large population, Fischbein et al. attempted to study the aspects of infinity which were counter-intuitive, with particular attention on the effects of different figural contexts in the way problems were solved. The authors took care to separate two levels of infinity, one corresponding to denumerable infinite sets, and another to the continuum. From their analysis of the results, the authors confirm the contradictory nature of infinity which they explain by arguing that logical schemes are naturally adapted to a finite reality. Thus, "finitist" interpretations tend to prevail. The authors also find that for geometrical infinite processes, intuitions are very sensitive to the context in which they are presented. Finally they claim that formal mathematics teaching does not modify students conceptions and intuitions of infinity, a result shared by Waldegg (1988); furthermore, they argue that teaching can encourage logical but rigid thinking, which perhaps explains the result of a larger percentage of erroneous and finitist interpretation in pupils with more mathematical training.

Waldegg (1988) looked at Mexican students' conceptions of infinity through their responses in a series of questionnaires, together with an in-depth critical analysis of the historical development of mathematical infinity. In her research, these two parts are intimately related, each used to reinterpret the other. For instance, with reference to the conceptual evolution of the actual mathematical infinity (further described in Moreno & Waldegg, 1991) students' responses are similar to those given by mathematicians throughout history up to the time of Bolzano. At the core of her work is the problem of the extrapolation of properties of the finite to the infinite (such as the idea that the whole is always bigger than the parts) which leads to contradictory situations. Another finding is that the context and form of representation are very influential in the type of responses the students give: if a geometric set is bounded, this may become an obstacle for its infinite quantification; continuous sets are also a potential source of conflict since the "counting" methods used for discrete sets need to be modified. The reasoning methods used in a geometric situation are different from those used in the numeric context. However, in a context which combines numerical and geometrical contexts through the use of algebraic language, Waldegg claims that some of the obstacles observed in previous cases seemed to have disappeared. This is an important finding which supports the idea that by building connections between different types of representations (in this case through algebraic language) some of the difficulties which arise when working in a single context can be diminished.

More recently, Nuñez-Errázuriz (1993), carried out a most interesting study of the psychocognitive aspects underlying the concept of infinity in mathematics. He considers the idea of *iteration* as central in the construction of the concept of infinity and focuses on the following additional aspects:

- The distinction between two types of iteration: divergent and convergent, which are related to the infinitely big and the infinitely small.

- The *nature* of the content of the iteration where cardinality and spatial measure should be distinguished.

- The study of the coordination of those different types of iterations within a situation.

- The study of the figural and conceptual aspects of the context in which a situation is presented.

Nuñez's empirical research — which involved the use of questionnaires and clinical interviews with children between the ages of 8 and 14 — uses simple two-dimensional geometric figures (mainly circles and quadrilaterals) which are sequentially transformed through indefinite repetitions of the same process (in the same way as many fractal figures are constructed). Thus, he points out, his observations were made in a world between the concrete and finite and the potential infinity, not involving the actual infinite.

Nuñez concludes that although iteration is a fundamental idea of the concept of infinity, procedural and arithmetical iteration are not the most important types for the conception of the infinitely big; he believes that non-arithmetical types of iterations are more fundamental.

With regard to the distinction between divergent and convergent iterations (related to the idea of subdivision), Nuñez points to an essential difference between the ways that the infinitely big and the infinitely small are conceived. He observed that when children around the age of 12 start to develop intuitions of convergent iterative processes, they manifest a great number of doubts, hesitations and changes in their opinions. His study of how the different types of iterations were coordinated allowed him to observe that when children start to understand convergence, they begin to ignore the effect of divergent iterations which are simultaneously present. He considers this an epistemological obstacle, but believes that his study on the nature of the iterations (spatial measure and cardinality) offers an explanation, since convergent iterations (where the measure of the partial results decreases) are always implicitly accompanied by divergent iterations (i.e. the number of steps) of a different nature. Thus he argues that convergent iterations have an additional cognitive complexity in that iterations of different type and nature must be coordinated.

Nuñez also observes that the figural (e.g. the form or scale of the figures) and conceptual (e.g. areas or distances) contexts in which the problems were presented have a very important role. He observed that students would have completely different responses to problems which were isomorphically constructed.

The latter finding is shared by Hauchart & Rouche (1987) who investigated the development of the concepts of sequence and series, and limits, including periodic decimals and recurrence, mainly through students' responses to 25 problems set in a variety of contexts: numerical, geometrical, kinetic, concrete settings and philosophical questions, and included some explorations of numeric sequences using programmable calculators, and geometric models of sequences.

Conceptions and obstacles of infinity and limits in calculus.

The area of calculus and limits is one area related to infinity which has received considerable attention, due to the difficulties experienced by students. Robert (1982) looked at more than a thousand school and university students' conceptions related to the limit of a number sequence, through the use of a questionnaire. She points to the following types of mental models that students have for the definition of a convergent sequence:

1. *Primitive* models which are classified as

stationary: "the final terms always have the same value",

barrier: the values cannot pass a certain value, and

monotonic (and *dynamic monotonic*): a sequence is convergent if it is increasing and bounded above (or decreasing and bounded below).

2. *Dynamic* models, which reflect the *process* of approaching the limit; it includes phrases such as "the values approach a number more and more closely" and the expression "tends to".

3. *Static* models, where the terms of the sequence are grouped in an interval near the limit; and

4. a mixture of the above.

The majority of students (35%) had dynamic models, which is consistent with the results of other studies investigating students' spontaneous conceptions of limits. A limit is generally conceived as that which cannot or should not be passed and is associated with terms such as "tends to", "approaches" or "gets close to". This has been observed by many workers such as Schwarzenberger & Tall (1978), Tall & Vinner (1981), Salinas (1985), Mamona-Downs (1987), Sacristán (1991), Cornu (1991) and Tall (1992). The way in which the idea of limit is verbally expressed through these terms, within and outside mathematics, involves a *potential* view of an infinite *process* leading to the limit, and thus carries the implication that the limit can never be reached (also seen as a *dynamic* type of perception of the concept of limit). Many researchers have studied how these spontaneous conceptions can act as epistemological obstacles. Cornu (1983), described by Tall (1986), investigated this in the context of the development of the "proper" conception of limit. Cornu points to four types of epistemological obstacles:

1. The metaphysical aspect of the idea, which refers to the mysterious nature of infinity where the limit cannot be obtained through simple algebraic calculations.

2. The infinitely small and the infinitely large. Students seem to think in terms of very small numbers, which are smaller than all the real numbers and yet not zero; and similarly they seem to think of a number larger than all others but not quite infinite.

3. Is the limit is reached? Students use different expressions according to whether the limit is reached or not. And the expression "tends to" is reserved for when the limit is not reached.

4. The passage to the limit which refers to the passage from the finite to the infinite. The limit, or "that which happens at infinity", seems to be isolated from the dynamic limiting process. It acts as an obstacle to the view that what happens in the finite allows us to predict what happens at infinity.

Cornu (1991) points to these as major obstacles which have also appeared in the history of the limit concept. To these he also adds, by referring to the ancient Greeks, the failure to link geometry with numbers.

It should be mentioned that many researchers have focused on the problem that a limit — or infinity — has the dual property of being a process and an object — potential and actual infinity — which can be a source of difficulty for students. For instance, Duval (1983) studied the problem of what he refers to as the "splitting" *(dédoublement)* of mathematical objects[4] and how this becomes an obstacle for the learning of infinity.

Work centring on epistemological obstacles related to limits was also carried out by Sierpinska, who investigated students' attitudes towards mathematics and infinity (Sierpinska, 1987) and how these obstacles emerge (Sierpinska & Viwegier, 1989). She identifies four areas which are sources of obstacles for limits: scientific knowledge, infinity, function, and real number. In an attempt to explore means for overcoming these obstacles, Sierpinska chose the context of infinite series, aiming "(1) to make a link in students' minds between the idea of convergence and decimal or other expansions; (2) to show that properties of infinite operations cannot be directly transferred to the infinite ones; (3) to show that in some cases it is possible to speak about the sum of an infinite number of terms, while in others it is not" (ibid. 1987; p.374). The first aim relates to the area of the real numbers and the structure of decimal expansions. With the second aim she is focusing upon the contradictions that emerge when extending manipulations of the finite to the infinite observed both historically and by researchers in mathematics education (see Fischbein, 1979; Waldegg, 1988, McDonald, 1992). The third aim relates specifically to the convergence of infinite series (i.e. to the existence of a limit). In the results of her exploration none of the obstacles were overcome, although she did find changes and the emergence of conflicts.

In my own previous work, described in Sacristán (1991), I investigated, through the use of two questionnaires (see Appendix 1 for the main questionnaire), 17 and 18 year old students' spontaneous conceptions and potential epistemological obstacles of concepts related to infinity such as limits and infinite sets. I studied in particular the following: i.) The (infinite) decimal expansions of real numbers, which includes the relationship *number-line* and the relationships *fraction-decimal* and *irrational number-decimal*; ii.) Limits of infinite sequences and series; iii.) Comparisons and one-to-one

[4] This idea of a (static) conceptual object which is simultaneously a (dynamic) process has been studied by many researchers with varying perspectives, terminologies, and cognitive implications, far beyond particular mathematical objects. For instance, Dubinsky (1991) talks of 'encapsulation' of a process; Kaput (Harel & Kaput, 1991) talks of the process of 'entification'; Sfard (1989) uses the idea of 'reification'; and Gray & Tall (1994) coined the term 'procept' to refer to a symbol which represents both a process and the result of that process. These polarities are well documented in the literature but I have chosen not to focus on these here.

correspondences between infinite sets. Among my findings were the facts that *depending on the context* and on previous school instruction, students exhibited different behaviours. It was apparent that for most students their concept images of real number limits and infinite processes were very fragmented: the "same" problem given in different contexts evoked different responses, a result found by other researchers (see for instance Hauchart & Rouche, 1987 ; Tsamir & Tirosh, 1994)

Espinoza & Azcárate (1995) consider the following epistemological and didactic obstacles for the learning of the concept of limit:

> "(a) In order to formulate the concept of limit the concept of real number is required, but in order to define the real number the concept of limit is required as well. Limit (convergence) refers to items which do not yet exist since they have not been defined.
> "(b) From an epistemological standpoint the idea of limit cannot be conceived as unlinked from the idea of real number; both ideas were formalised almost simultaneously..." (p.15)

Much of the difficulty concerning limiting processes in the context of the real numbers is related to the properties of this set and of the continuum. For instance, many researchers have investigated the difficulties of decimal and irrational numbers.

Monaghan (1986) (quoted in Monaghan, Sun & Tall (1994)) investigated adolescent students' conceptualisations of real number, limit and infinity. He found that students showed insecurities regarding infinite decimals which they regarded as "improper" and described as "infinite numbers". Thus $\sqrt{2}=1.414...$ does not indicate that $\sqrt{2}$ is the exact limit of an infinite decimal expansion; the latter is seen only as an approximation (see also Sacristán, 1991). Infinite decimals are seen as a potentially infinite set of instructions for finding the point on the number line, a process which can never be completed; thus infinite decimal expansions are perceived as inaccurate.

Tall & Schwarzenberger (1978) looked at conflicts that occur in the learning of real numbers and limits. They described examples of conflicts between "decimal" and "limit", between "decimal" and "fraction", between "number" and "limit", and between "sequence" and "series". They focused in particular on the question "Is 0.999... = 1?", and found that many of the answers contained infinitesimal concepts with most students considering 0.999... "just less than one", because it can never "reach" one: it can perhaps get infinitely close, but not equal. They also point to the fact that some students cannot accept the fact that two decimal representations could correspond to the same number (as also found by Romero i Chesa & Ascárate, 1994). Similarly, Cornu (1983) studied students beliefs in relation to this question, one of which was that "0.9, 0.99, ... tends to 0.9999... but has limit one because it cannot surpass it. Thus the limit is viewed as a boundary, rather than as the value at infinity. Sierpinska (1987, 1994) relates how students who were able to accept the arguments used for converting recurring decimals to fractions, refused to accept them when it came to showing that 0.9999... = 1. She describes how some students think of 0.999... in terms of a construction process that never ends ("potential infinity"), that is, in terms of a sequence rather than as its limit.

Ferrari et al. (1995) investigated the acquisition and development of the concept of infinity in secondary school students. First they looked at situations, not necessarily mathematical, where students can encounter this concept, and at some of the historic-philosophic stages in the development of the concept of infinity from a didactic perspective. They then investigated the students' conceptions of infinity through the use of a questionnaire covering various mathematical areas: dense numerical sets; periodical numbers; discrete infinity; infinitesimals (related to the process of potentially infinite subdivision); the geometrical continuum; and one-to-one correspondences. As have other researchers, Ferrari et al. found that most students have difficulty with periodic decimals. They also point to difficulties with the idea of density, which they relate to students' problem in accepting that a bounded set can contain infinite points: for instance some students cannot conceive a segment as possibly having an infinite number of points. They also found confusions between measure and cardinality: e.g. the length of a segment would seem to be proportional to the number of points. Ferrari et al. corroborate that infinity is not a primary intuition although spontaneous ideas do form and they conclude that there is a need for proper guidance in order for the concept of infinity to be adequately learned.

Much research has been carried out on students' conceptions of the real continuum and its relationship to the concept of limit, since it is considered one of the main sources of difficulty: For instance, Schwarzenberger (1980) claims that calculus cannot be made easy because of the complex characteristics of the real line.

Mamona (1987) focused her research on finding explanations as to why students respond the way they do in relation to central ideas of mathematical analysis including limits and infinity, comparing two groups of students: Greek and British. She investigated the limit concept both in the continuous case, and in the discrete case which included infinite recurring decimals, limits of sequences and series, and limits in geometrical contexts. She finds that the root of students difficulties was their conceptualisation of the continuum which seems to be nearer to the "dynamic" one of Leibniz-Cauchy than to the "static" one of Weierstrass, "i.e. the numbers on the real line are not 'deprived' of their infinitesimal neighbourhoods" (ibid., p.244).

> "'Infinity' is perceived vaguely as a very large number, identifiable in its dynamic potentiality. Infinitesimals on the other hand do not constitute numbers. A vague idea is predominant of an incredibly small 'quantity' or of minute, (yet still somehow concrete), neighbourhoods of real numbers on the number line." (Ibid., p.244)

In a more recent study, Mamona-Downs (1994) looked at students' conceptions of the real continuum by asking students if a number exists between the sets A={0.3, 0.33, 0.333..., ...} and B= {0.4, 0.34, 0.334, ...}. She identified the following types of responses:

- Dependence on the symbolic representation of numbers; a decimal between the two sets cannot be constructed since there doesn't exist a digit between 3 and 4.

- Each of the two sets are endlessly under construction, so there is a diminishing interval between the two sets

- Each set has a "last" member of "infinite order: 0.333... and 0.333...4 respectively

- Use of the rule for finding out the limit

- There is no number between A and B since the largest element of A is equal to the smallest of B

- Use of the density and completeness of the real numbers: The number between A and B is the limit of the sequence x_1, x_2, ... where $0.3<x_1<0.4$, $0.33<x_2<0.34$, ...

Romero i Chesa & Ascárate (1994) claim that the results of their questionnaire, which they applied to 74 students of 16-17 years of age, give overwhelming evidence that non-integer numbers are conceived as a different kind of number depending on how they are written: that is, numbers of different written forms seem to belong to different species of numbers. They also observe that the real line is perceived either as a sort of tape, or as a set of points perceived as small disks. They thus classify the subjects as either "continuists" or "atomists". They found that the concept of the continuum is far from intuitive. Students' concept images of the continuum seem to be "a loose aggregate of images and enunciations of properties" (ibid., p.191) and they note a lack of connections made between the geometrical line and numbers as has been noted by Cornu (1991).

Rigo (1994) carried out a comprehensive study on continuity, where she objects to the common assumption in secondary school mathematics teaching that continuity is an evident geometric property of the line, since this does not take into account the fact that continuity is not only seen in the context of the geometrical line but also in numeric domains (the real numbers). Rigo uses Dedekind's work on irrational numbers to show that the construction of the real number line depends on concepts such as infinity and continuity and operations such as continuous variation, which she argues require a notion of the continuum different from the intuitive geometrical one. (Part 1, p.20). She then shows, through the analysis of questionnaires given to a group of teachers, that students in secondary school mathematics have already developed their ideas of what constitutes continuity, and that these intuitive notions are rarely changed by education.

Collel's (1995) research focuses on investigating the connection between the understanding of the limit concept and the understanding of the topological concepts of interval and neighbourhood which consider a "mathematical point" as something which can never be exactly represented since it would imply a potentially infinite dynamical process similar to a limiting process. She claims that these two ideas are related to the completeness of the real numbers. Her research centres on the analysis of Argentinean high school students' responses to two word problems, one which focused on a process of indefinite iteration, the other related to the concept of neighbourhood. She concludes that for most students a point is a "static residue" which results from a *finite* process of subsequent divisions, not conceiving a point as an ideal entity which results from an indefinite process.

On the other hand she found that students seem to accept a potential infinite numerical process, such as the iterative process of finding the half sum between a fixed number and that resulting from the process. She does add however, that students show inconsistencies in relation to the operatory field of the infinite which they use arbitrarily. She observes that the numerical and geometrical representations are not integrated, and argues this is due to the emphasis in schools on numerical representations, even though her research points to a correlation in the learning of the concept of limit with that of the ideas of interval and neighbourhood of the geometrical line.

Very much related to how the continuum is perceived, there is evidence (Cornu, 1983; Tall 1980) that many students have notions that there exist very very small numbers (infinitesimal numbers?), smaller than "real" numbers, but not zero. This is in fact a view that was present in the origins of calculus, and is the basis of what is known as non-standard analysis. Tall (1980) suggests that a more intuitive way for students to perceive the line is as made up of points, not of magnitude zero, but infinitesimally small (the infinite measuring number).

David Tall has often pointed to the mental nature of the concept of infinity and whether we should have rigid views following classical analysis. In Tall (1992, p.506), he says:

> ..."it is also interesting to ask whether the concept of infinity provoked by asking the meaning of '...' (potential infinity) is the same kind of infinity as the number of points in a line segment (cardinal infinity). In order to research the beliefs held by students and to classify those beliefs, it is important first to analyse the concepts concerned and the kind of concept images generated by various experiences without imbuing then with a classical mathematical prejudice".

Another area related to infinity which has been subject to numerous research is that related to infinite sets and comparisons between different sizes of infinity. These include Fischbein et al. (1979); Duval (1983) who studied the problem of one to one correspondence between infinite sets; Borasi (1985) who studied the errors and misconceptions that arise when comparing the "number" of elements in two finite sets, Tirosh, Fischbein & Dor 1985,; Waldegg (1988); Falk and Ben-Lavy (1989) who studied children's conceptions of the "size" of infinite sets and found that children do not initially recognise the difference between an infinite set and very large finite sets; Moreno and Waldegg, (1991); Sacristán (1991); Tsamir & Tirosh (1994) who found that students' evaluations of one-to-one correspondences between infinite sets are largely influenced by the way in which the sets are represented.

The use of the computer for the study of calculus, infinite processes, and limits.

Most of the research described above has not included the use of computers. Only in one of the above cases (Hauchart & Rouche) was there a small use of the calculator for exploring the behaviour of numeric sequences. Research involving the computer in the area of calculus has focused mainly on the study of functions. For instance, Schwarz (1989) (described in Schwarz & Dreyfus (1989)) created a microworld for the study of the concept of function called TRM: Triple Representation Model which included three representations of functions: algebraic, graphical and tabular.

Tall (1992, p.503) mentions that "teaching the notion of limit using the computer has, on the whole, fared badly". This applies to the treatment of numerical ideas of limits which are susceptible to accuracy problems (and which must therefore be combined with discussion of the problems of computer arithmetic), as well as symbolic ones using for instance *Derive*. He suggests providing the student with rich experiences through non-formal approaches -such as his own work (Tall, 1986)- for building "cognitive roots" (rich concept images) upon which the formal theories of Calculus can later be built.

Tall's (1986) *Graphic Calculus* is an attempt to incorporate the computer in the calculus curriculum through what he calls *generic organizers*, that is, environments or microworlds which enable the learner to manipulate and explore both examples and non-examples of a mathematical concept or related system of concepts. Among other features, the *Graphic Calculus* software can magnify graphs of functions, including "fractal" graphs: i.e. graphs of (continuous) functions which are nowhere differentiable and therefore are so wrinkled that they are never "straight" when magnified. For instance, one such function is the Blacmange function. Tall (1991b) emphasises that one of the benefits of using this function is the *recursive way* in which it is defined ("the blacmange can be seen as being the nth approximation with $1/2^n$ size blacmange added"), and he points to the importance of students being exposed to the construction process through dynamic computer graphics: that is by building up partial sums and noticing that new "teeth" no longer add much. He explains how this can become an intuitive proof of the non-differentiability of this function.

Tall also used this to investigate, among other things, the intuitive nature of the limiting process. He found that a spontaneous limit concept did *not* occur to *any* pupil with no calculus experience, indicating that a geometric limit is not an intuitive concept; and he points to some of the problems with the limit concept:

- language which suggests that a limit is "approached" but not reached,.

- unfinished nature of the concept: gets close but never seems to arrive, and

- problems with the quantifiers in the formal definitions

Monaghan, Sun & Tall (1994) studied the effects of the computer algebra system *Derive* on students' conceptions of limit focusing particularly on views of limit as a process and limit as a object. They compared this approach with the traditional one, finding that each approach highlights and suppresses different facets of the concept of limit, the traditional one allowing for a clearer picture of limit as an object but with other difficulties. They suggest that by using a computer algebra system which produces symbolic limits as 'proper" numerical expressions, it may be possible for students to develop a more balanced view of a limit as both a process and a concept.

A group of researchers in the *Seeing Beauty in Mathematics* project (Lewis, 1990; Goldenberg, 1991; Cuoco & Goldenberg, 1992) of the Education Development Center in Massachusetts, have explored ways in which students can encounter, in a geometric (and computer-based) context, ideas such as those of sequences and series, limits, and mathematical induction, among others. They chose to use recursively defined geometric constructions and fractal figures because of their mathematical features (e.g. the links between the geometry of recursive structures and sequences). The main idea was that the investigation of these figures offered an attractive way in which students could get involved with mathematics and mathematical thinking, and provided a way to integrate visual, investigative and motivating aspects into the activities. Lewis (1990) describes a fractal curriculum, using Logo procedures, which includes investigations into "replacement fractals", in particular the study of the perimeter of the Koch curve and the area of the snowflake. He points out that the snowflake curve, which provides (in its limiting case) a case of a figure with an unbounded perimeter and bounded area, is an example which can lead to nice investigations of facts about self-similar figures, and the nature of convergent and divergent sequences and series. He points to the finding that this is an exploration which is accessible, at an appropriate level of sophistication, to most students. Goldenberg (1991) describes other activities based on the construction — through a specifically designed computer software: *The Fractal Explorer* — of fractals or self-similar figures, for exploring and analysing different properties of these objects such as geometrical relationships. These works point to some of the uses of fractal explorations in mathematics education — an important background for some of the activities I used in the research presented in this dissertation — although they constitute more of an open-ended empirical research into exploratory mathematics through fractal figures, rather than a structured investigation.

IV. Relevant ideas and implications for my research.

In the above review, I have touched upon four areas which can be summarised as follows:

- the mediational role of representations in the construction of knowledge,

- computational environments, domains of abstraction, and windows into learning, and

- research related to limits and infinity, and the use of computers in those areas.

Below I draw out some of the main ideas which emerge from the literature in these areas, and that I consider relevant for my research.

Principal implications for my research.

The review on representation theory points to central ideas on how knowledge is constructed and how meanings are created. I would like to adopt the perspective that learning involves the construction of representations. That it is through the construction of models, which serve to represent an observed phenomena, idea or concept, that we make sense of the world, including mathematical objects. Thus models, or representations become *tools* for understanding.

A distinction is made between external representations and mental representations. To summarise, mental representations are in some way the "ideas" one has of a specific concept, the forms taken on by intuitions and conceptualisations of a knowledge being constructed: they are how we conceive, or internalise external forms of representation and action in our minds. External and mental representations are continuously interacting. It seems that the construction of a concept requires an intense use and articulation of different types of representation. A concept may become more meaningful (or, as Wilensky says, more "concrete") as the mental representations become more rich and appropriate connections are established between new ideas or objects, previous experience, and other objects.

Engaging with multiple representations of an object seems to be important, but emphasis should also be placed on the actions and experiences involved in this process since they will also determine the way in which knowledge is constructed. This supports the idea that the *learning of a concept is facilitated when the individual has more opportunities of constructing and interacting with as diverse as possible external representations of a concept* (which follows Papert's idea of "constructionism"; see Harel & Papert; 1991).

It also seems important to include different *types* of representations; that is both visual and symbolic ones since they are complementary and appear to engage different cognitive structures. An integration of both types of representations appears to be essential for constructing a richer meaning of the mathematical object. However, as is evident from the review, this integration is seldom straightforward, and in particular in the learning of calculus it has been observed that establishing cognitive links between visual and symbolic representations is a permanent area of conflict. Thus the importance of fostering an interaction between the two forms of representation, through for instance the mediation of a computer-based activities in a microworld.

It is generally agreed that visualisation is important for understanding, but it also seems that most students have difficulties visualising, and that it is an area neglected in mathematics education which often emphasises the analytic approach. Incorporating visual means of representation is thus important. Computers appear to offer an accessible way of doing this, and furthermore, their visual capabilities also include *unfolding* (dynamic) images. Additionally, it has been suggested, that visual images and computers as exploratory environments can be used as means to construct informal proofs of certain mathematical results.

Another important point relates to the environment in which learning takes place. Learning is mediated and coloured by all the experiences of the student. Computer-based microworlds, as defined by Hoyles and Noss (1987) attempt to take this into account and use the computer as *part* of a learning environment. In theory, in a well designed microworld, the student has the possibility of being actively engaged in the learning environment and has the possibility of exploring, discovering and constructing relationships, and generating his/her own generalisations within the environment (what Noss and Hoyles have called *situated abstractions).* Furthermore, it has been suggested that computer-based microworlds can also be used as a research tool (a window) for looking at students' conceptions and thought processes.

The review of research on limits and infinity points to some epistemological areas of difficulty, some of which had similar manifestations in the historical development of mathematics, as we will see in the following chapter. These will be important to take into account in the design and analysis of the empirical research. It also appears, according to some of the authors reviewed that many students have misconceptions or inappropriate intuitions — spontaneous conceptions based on their finite-world experiences — of infinite processes and of actually infinite (mathematical) objects that can become obstacles for the adequate construction of formalised versions of these concepts.

It seems that one of the difficulties is that most of the representations used in the teaching of calculus are mainly symbolic and have a "static" nature (the connection with the fundamental idea of *movement* is thus lost) and there is evidence of the difficulty in linking the visual/graphical and the analytic/algebraic representations. These representations may not allow an easy exploration of ideas: they often cannot be transformed to produce another object, and it is hard to "move" ideas, to construct meanings with such representational systems. Furthermore, as is evident from the historical review given in Chapter 2, infinity — a fundamental underlying concept to calculus — in particular, is a mental construct, not extractable from a sensory experience. The challenge is thus to make the infinite more accessible (i.e. more concrete) by providing means for students to explore it and make connections. As Tall (1986; p. 51) puts it:

> "Just as the experts in history gained a cognitive belief in mathematical concepts through using them, we may be able to help students gain an insight into the ideas of the calculus by providing an environment in which they can explore and manipulate the ideas to give them a cognitive reality."

If students were to be provided with more accessible tools for constructing and exploring representations of a mathematical ideas related to infinity, tools which allowed them to *express* and explore — and thus develop — their own ideas on the matter, it may help develop a more integrated cognitive structure which may later serve as a basis for the comprehension and internalisation of more advanced and formal representations.

Using a computer environment for making infinity more "concrete".

One possibility that may help make infinity more concrete may be through the implementation of the computer as part of a learning situation. On the one hand, the computer can allow the use of symbolic representations (e.g. through the computer program or code); it can also facilitate access to the visual context and provide unfolding (dynamic) representations. An appropriate computer environment — a microworld which integrates and facilitates the interaction between the symbolic and the visual forms of representations— may help in the construction of more functional and coherent mental representations. In Chapter 2, I presented an overview of the different contexts, situations and forms in which the infinite appears. These different settings and situations, and the relationships between them, should be considered, as well as taken advantage of, for the exploration of the infinite. The possibilities which can be provided by a computer environment of interactively working with diverse representations, as well as for exploring and expressing ideas, may help build connections which, following Wilensky's ideas, are important for giving the student a feeling of "concreteness" and the ability to grasp abstract objects such as those related to infinity. Working in, for instance, a Logo-based environment, the student may be able to explore geometrical situations (such as fractals which are "limit objects") and be assisted by a language that supplies the correspondence between the geometrical objects (a phenomenon on the screen) and the code (a phenomenon on the keyboard). Drawing on a computer screen can be thought of as a process of transference of contexts, from the symbolic code to the visual and conversely. Furthermore, the procedure can be said to be an *active* representation: it is something that generates the geometrical object.

Additionally, the visual aspect of the computer offers the possibility to observe the evolution in time of a process: it allows us to perceive the dynamics of the process — through "unfolding" visual images — eliminating the limitation of only observing the final state, the result of the process. This can provide a feeling for the behaviour and iterative sequential change found in infinite processes.

However, it is important to be aware that the conceptions that students may develop when they are allowed to explore and play with ideas of infinity and infinite processes in a computer-based microworld are mediated and conditioned by this environment, which is why it is also interesting to attempt to use the environment as a window for looking at these (changing) conceptions.

The way in which the above ideas were taken into consideration and used for the design of the study that is the focus of this thesis will be described in Chapter 5, where I give an account of how an infinite processes microworld was developed as the basis for the study. First, however, I present, in the following chapter, the description of the research methodology employed in the investigation.

Chapter 4:

The Iterative Design of the Study and the Research Methodology

I. Methodological Issues.

The focus of the study was to investigate students' developing conceptions of mathematical infinite processes as mediated by the computational environment provided. The style of research was planned as an observation of the ongoing thought and learning processes through the actions within the environment, rather than one that set out to test predetermined hypotheses. It was therefore determined that the methodology to be used should be based on a process-oriented approach which could illuminate the learning processes. The chosen methodology thus follows *illuminative evaluation* techniques derived from anthropological or ethnographic research traditions (Hamilton, 1977; Eisenhart, 1988) which are primarily concerned with description and interpretation rather than measurement and prediction, advocating the examination of a programme as a whole and taking into account the wider context and 'learning milieu' in which it functions (Parlett & Hamilton, 1972). The focus is on activities, rather than intents, being responsive to the *issues* that emerge as the study progresses which are used to understand complex phenomena (Stake, 1977).

Illuminative evaluation concentrates on observation and interview, the latter being crucial for discovering the views of the participants, but it is also combined with questionnaires and analysis of documents to help illuminate problems, issues and significant features (Parlett & Hamilton, op. cit.), and can be integrated with more general research and evaluation strategies as proposed by Smith (1971). Among the additional techniques suggested by Parlett & Hamilton, is to ask participants to write comments or compile work diaries that record their activities over a specific period of time.

In keeping with this tradition, my investigation was based on a participant observation methodology, and used a case-study approach for an increased understanding of the variables and dynamics of the situation under study (MacDonald & Walker, 1974). The data-gathering techniques employed during the participant-observation work are given with the descriptions of each of the phases of the study.

II. The Iterative Design of the Study.

The methodology and design of the study were developed through an iterative process that consisted of three phases: prior to the main study, two preliminary studies — the exploratory and pilot studies — were undertaken. These studies were essential for the overall design of the main study from content to methodological issues. Some of the methodological issues to be tested and/or decided, included: how many students should be observed at a time; how the students should work (individually, in pairs, sharing a computer...); the use of worksheets; the data-gathering techniques. In this chapter I present the methodology used in each study. The timetable and main characteristics of these studies is illustrated in Table 4.1.

	Exploratory Study	**Pilot Study**		**Main Study**	
		Main phase	Complementary phase	Preparatory phase	Main Phase
Programming / Microworld activities	Jan.-Feb. 1992	Nov.-Dec. '92	May 1993	July '94	Aug.-Oct. '94
	8 sessions	4-6 sessions per student or pair	1 session per student or pair	10 sessions	5 sessions per pair of students
Duration of each session	2 hours	1.5 hrs.	2 hrs.	3-4 hrs.	3 hrs.
Number of students	9	5		16	10 (5 pairs)[1]

Table 4. 1. Phases of the empirical research.

[1] Although one of these pairs of students did not complete the study and was therefore excluded from the analysis.

All of the studies were carried out in Mexico, with Mexican students. Each of the sessions from the three phases were carried out in a computer-equipped classroom in the Department of Mathematics Education at the Centre for Research and Advanced Studies (CINVESTAV-IPN) in Mexico.

1. The exploratory study.

The first preliminary study was designed to explore ideas for activities and task domains, as well as to get insights into methodological issues, providing a basis for the design of a more structured study. It was an *informal* exploratory study, with loosely defined goals and incorporating a wide range of activities, as will be described in Chapter 5.

a. The subjects.

For this study I used as subjects a group of Masters students in Mathematics Education in Mexico. I took advantage of the fact that I had to teach[2] an introductory course in computer programming to these students, and used them for the exploratory study. These subjects were all mathematics teachers, generally at high school or college level, so most of them were familiar with mathematics topics such as calculus. Initially, I had a few reservations about using mathematically experienced subjects, but since this study was meant to try out ideas I proceeded anyway. As it turned out, working with these subjects was more interesting than expected since they were experiencing a very different approach to the mathematical ideas explored than that to which they were accustomed; this led me to decide to use some students with this type of background in the studies that followed.

b. Methodology.

This exploratory phase was carried out during a period of four weeks (during January and February, 1992) with two 2-hour sessions per week, for a total of eight sessions. It was held in a computer-equipped classroom, and all of the students participated simultaneously in the sessions. A total of 9 students attended the sessions.

[2] As an appointed lecturer at the Department of Mathematics Education of the CINVESTAV-IPN in Mexico.

Six of them worked as 3 pairs, with the same partner throughout the eight sessions and sharing a computer. Two others worked mostly on their own but sometimes paired up and discussed together some of the tasks, but one of these two students did not attend four of the eight sessions. Another student worked on his own, spending most of his time working on a single procedure; he was completely uninvolved with the other students and did not do any written work, so no data was collected for this student. I had the dual role of being both the teacher — directing and administering the study activities — as well as the researcher — observing and collecting data.

In this phase I considered it best not to inform the students of the specific aims of the study because I did not want that knowledge to affect their behaviour and responses, particularly considering the experienced background of these subjects. At the end of the last session I did ask the students to tell me what they thought the purpose of all the activities had been, and they answered it was to study sequences, series and limits. In retrospect it seemed to me that withholding that information did not have a significant effect on the way the explorations unfolded: the students had started to become aware of the mathematical topic being explored from the beginning, but because the approach was quite different from anything they were accustomed to, I could still gather valuable data from the way they approached and responded to the computer-based activities, even if they looked at a problem from a traditional mathematical perspective

Prior to the first session of this study, I gave the students a few basic introductory lessons in Logo programming, but they had still not acquired full familiarity with this type of programming when the study activities began. In particular they had no previous experience with recursion.

All the activities were introduced and structured through activity sheets (see Appendix 5) which the students followed almost step by step. Although I continuously tried to encourage the students to continue the explorations with variations of the activities proposed and with other related ideas of their own, very few ever did. The students approached the explorations with the attitude of doing only what was asked (on the worksheets) and only that. The lack of initiative on the part of the students made me reflect on the way the activities had been presented, realising that on the sheets the activities were too directed, as in the traditional schooling the students were used to, where the focus is on answering specific questions or solving specific problems. I realised that to foster an attitude of creative exploration, I should not express a detailed direction of exploration on any worksheet used, and perhaps not even in my oral presentation of the activities.

c. Data Collection.

During the exploratory study, the collection of data was carried out in three ways (see Table 4.2): i) Being present in all the sessions, I observed the way in which the students carried out the activities, and made field notes, although it must be remembered that all 9 students were working simultaneously which made it difficult to observe in detail the work of each of them, which is why I relied mainly on the following method of data collection. ii) Each individual student was asked to keep a diary of all his/her activities, thoughts and comments. These diaries contained very detailed descriptions of everything the students had done during the activities, including reasons, justifications and comments. Any other written work (e.g. sheets with computations, etc.) was also kept together with the corresponding diary. iii) Additionally, all the procedures written by each of the students (or pair of students) were kept as Logo files.

Exploratory Study: Data-gathering techniques
• Field notes • Student's written work and diaries. • Logo files of students' procedures.

Table 4.2. Data-gathering techniques used during the exploratory study.

2. The pilot study.

Based on the experiences during the exploratory study, a selection of activities was made, as will be explained in Chapter 5. The purpose of the pilot study was to test the activities and design of the study, as a basis for the final design and implementation of the main study. The pilot study was initially carried out over a period of two months (in November and December 1992) which I have called the main phase of the pilot study (see Table 4.1. above), with four to six work sessions per student or pair of students (see below) during this period, each session lasting approximately 1½ hours. In May 1993, I scheduled an additional session with each of the students to try out an additional activity; in Table 4.1, I call this the complementary phase of the pilot study.

a. The working and research environment.

After analysing the results of the exploratory study it was decided that the best methodology was to work individually with students, in a clinical interview style, as opposed to working simultaneously with an entire group of students. Since the purpose of the study was to examine students' conceptions of infinity and of infinite processes as mediated by the activities and the learning environment, it was easier to observe and interview the students when working with them individually (using the microworld as a window into their thinking processes, as explained in Chapter 3). However, I still had to decide whether it was better to have each student working singly, or a pair of students working together who could interact with each other. It is equally possible to be constantly present and involved throughout the activities of paired students as it is with a single student. I decided to try out both options in the pilot study, sometimes working with one student on an individual basis, and at other times with a pair of students sharing one computer.

b. The students

Having worked exclusively with mathematics teachers in the exploratory study, I was interested in exploring the possibilities of carrying out the study with other types of students as well. Five students of different mathematical and computer literacy backgrounds were used for this study, as summarised in Table 4.3. These students and their participation in the pilot study were as follows:

	Pair 1		Pair 2		Individual
	José	Enrique	Isabel	Rafael	Larha
Age	Early thirties		21	18	18
Computer and Logo Experience	Little	Little	None	None	None
Individual sessions	2	2	1	1	7
Sessions as pairs	4		4		
Home explorations	Yes	Yes	No	No	No

Table 4.3. Students and sessions distribution during the pilot study.

(i) Two male students, José and Enrique, with the same background as those in the exploratory study[3]: they were students of the Master's programme in Mathematics Education, and school teachers with a good mathematical background but were not mathematicians. With these two students I alternated working individually with each of them and pairing them up; I spent two sessions with each separately, and four working with them as a pair. Additionally, between sessions, these students continued the explorations on their own, both together and individually.

(ii) An 18 year-old girl, Larha, who had finished high-school 6 months prior to the study, and was having a year off before entering the university in August 1993. In high-school she studied in the "Physics and Mathematics area"[4], and she liked mathematics, although it was not a subject at which she ever excelled. She remembered having studied limits although she did not recognise the word "convergence"; she also did not remember having studied infinite sequences and series. She had never seen Logo before and had no computer programming experience. At the beginning of the first session I spent a short time showing her some of the basic primitives such as Forward (FD), Right (RT), PRINT, etc., explained about procedures and how to define them, and showed her how to use the Logo editor.

I worked individually with Larha for 7 sessions, and this was the only time she had for the explorations since she had no access at home to a computer. Since I

[3] These two students belonged to a generation subsequent to the students who participated in the exploratory study, and therefore were not involved in that previous study in any way.
[4] In the Mexican school system, in the last three years of high school (usually 15 to 18 years of age), students have to study a very wide variety of subjects (approximately 10 different subjects each year). There is a minimum degree of specialisation in the last year when students choose an area of studies on which there will be more emphasis, although they still have to take courses from all disciplines. In the "Physics and Mathematics area"' the emphasis is on physics, mathematics, and science courses.

worked with her individually, she seemed to create a dependence on me, forcing me to became her "explorations partner", as will be discussed further on.

(iii) A 21-year old female university Architecture student, Isabel, and an 18-year old boy, Rafael, in his last year of high school. Both of these students had the same background and came from the same high school where they both studied in the "Physics and Mathematics Area"[5]. Neither had worked with Logo before, and as with Larha I showed them briefly some basic primitives, the Logo environment, and the use of the editor at the beginning of the first session. I worked with these students as a pair for 2 sessions, and with each of them alone for an additional session and interview.

c . Methodological issues.

<u>(i). Format of the activities.</u>

With all of the students the same format was used for the work sessions. The students were given a worksheet at the beginning of the session which helped set up the activity with which they would work on (see below). During the worksessions I was present at all times and actively involved in the students' work through my interventions which were in the form of suggestions and questions (with single students my presence almost became that of a work partner, as will be discussed later on). At a time that I judged appropriate, depending on the evolution of the session, I presented students with other worksheets so that they could continue their explorations further with new ideas and suggestions.

<u>(ii). On the design and use of the worksheets.</u>

The design and use of the worksheets followed certain intentions and premises such as those given below, although it was subsequently found that when working with students on an individual basis, as was done during this study, some of those suppositions needed to be reconsidered either because they were not as necessary as initially believed, or because alternative ways of working could allow the students

[5] See previous footnote.

[6] Originally, two other pairs of students also began to participate in the study but they did not complete it.

more freedom for exploring and expressing themselves. The original suppositions were:

a.) The worksheets would help set up the range of activities of the microworld. That is, they would offer a starting point for the activities as well a set of directions for those activities within the topic of interest (in this case, infinite processes, sequences, series, and limits) — serving as a guideline for the students, and helping them focus on ideas relevant to the purpose of the activities.

b) Being in written form, they could serve as a reminder for the students of the initial suggestions and purpose of the activities (whereas oral instruction could soon be forgotten).

c) Written worksheets could give the students some autonomy from the instructor (and researcher, in this case), hopefully allowing more spontaneity in the explorations.

d.) The worksheets could assist the students with the programming aspects of the activities (e.g. by giving them some of the initial procedures, procedures which served as exploration tools, or ideas for modifying their procedures.)

During the pilot study, most of the worksheets included a sample procedure with which the students could begin an activity or a set of activities. I had decided to give students the procedures, rather than to have them program from scratch so that they could focus more on the substance of the program, rather than on the programming itself. Furthermore, these initial procedures were meant to incorporate the basic material with which the students could begin their exploratory activities (by modifying these procedures: reflecting on how they worked then making changes of their own).

Whereas in the exploratory study the worksheets were quite detailed, which seemed to limit the students initiative and creativity, during the pilot study the worksheets were kept as simple as possible, giving only one activity or set of activities at a time (thus avoiding confusing the students with too much information). I wanted students to have in each worksheet an easy and accessible suggestion for explorations with the computer, which would help them focus on the activity but at the same time giving them freedom for exploration.

For this study I designed three sets of worksheets, each set a collection of activities for explorations related to a topic (Spirals, The Koch curve and snowflake, the Sierpinski Triangle — see Chapter 5) which did not necessarily have to follow one another. Since the students were meant to have the freedom for creativity in their explorations, it was always possible to alter the order in which the worksheets were used or even to skip some of them.

(iii). On the role of my interventions.

In general I wanted to let students work as much as possible on their own, allowing them to be in control of the explorations, and limiting my interventions to the conditions given below. However, this was not always possible, especially when working with a single student or when students needed extra guidance. But in general I did try to use my interventions as follows:

a.) From a technical perspective, to provide assistance when needed with the programming language and with technical details.

b.) To provide additional suggestions and guidance for the activities.

c.) To ask particular questions with one or both of the following purposes in mind:

i. to direct students' thoughts in a certain direction or towards observing certain aspects relevant to the learning process, arising from the explorations or activities;

ii. from the researcher's point of view, to interview the students informally in order to gain as much insight as possible into the students' conceptions and how these were mediated by the activities.

c. Collection of data.

In addition to the data-gathering techniques used in the previous study — field notes from observations, the students' written work, and Logo files of their procedures — during this study I recorded all of the informal interviews and oral conversations which took place during the activities. All of these recordings were later completely transcribed into written form, and served as the basis for the analysis of this study. The data-gathering techniques used during the pilot study are summarised in Table 4.4:

Pilot Study: Data-gathering techniques
• Audio tapes. • Logo files of students' procedures. • Direct observation, and field notes. • Informal interviews. • Worksheets and other written work.

Table 4.4. Data-gathering techniques used during the pilot study.

3. The main study.

a. The students.

The main study was carried out with 4 pairs of Mexican students[6] of varying ages and backgrounds (in total there were 4 female and 4 male students). Two of these students were as young as fourteen years of age. Some researchers such as Gardiner (1985), advocate introducing young children to infinity-related mathematical ideas. I simply wanted to have a sample of different age groups, although I was interested in observing the conceptions of younger students and the ways in which they worked in the environment, in comparison to older students. All of these students were interested in learning Logo and volunteered for the study. They were suitable for my investigation since the research objectives did not make any particular demands on the sample. Based on observations during the preparatory Logo course which took place a month before the study, I paired up the students by experience and age, also taking into account whether the pairs functioned well together (for instance, I avoided pairs in which one of the students would be too dominating, preventing the other to express his/her ideas). These pairs, summarised in Table 4.3, were as follows:

Verónica and Consuelo: Both 14 year-old school girls and average mathematics students with no particular mathematical inclinations.

[6] Originally, two other pairs of students also began to participate in the study but they did not complete it.

Manuel and Jesús: Manuel, a 17 year-old boy, had finished the first two years of a three-year vocational high-school in the area of electronics, physics and mathematics. Jesús, an 18 year-old boy, had just finished high-school and was about to enter the university to study Electrical-Mechanical Engineering.

Alejandra and Victor: Alejandra, a 27 year-old female, had just finished university[7] studies in Graphic Design; she indicated having always had difficulty with mathematics in school. Victor, a 23 year-old male, was in his third year of a five-year course in Engineering Studies.

Elvia and Martin: Elvia female, Martin male; both mathematics teachers in their thirties, and students in the Masters programme in Mathematics Education.

		Age	Previous computer experience
Pair 1	Verónica	14	None
	Consuelo	14	None
Pair 2	Manuel	17	None
	Jesús	18	Little
Pair 3	Alejandra	27	None
	Victor	23	Little
Pair 4	Elvia	30+	None
	Martin	30+	None

Table 4.3. Students participating in the Main Study.

These students all finished an intensive two week general preparatory Logo course — described below — a month before working in the microworld, so they all were able to write their own procedures, and were confident in making changes to a procedure and exploring variations. Except for Jesús and Victor, who had very limited previous computer experience, none of the students had had any computer experience

[7] Note: In Mexico it is not uncommon for university (college) students to be older than in other countries. Many students study part-time while they work, and all study programmes require a minimum of 4-5 years (if full-time).

prior to the Logo preparatory course. This had the advantage that they could work without major expectations or programming habits, facilitating the exploratory mode.

b. The preparatory course.

Based on the experiences from the exploratory and pilot studies (where I respectively used students with some knowledge of Logo and without), it was decided that the students in the main study should get acquainted with Logo prior to the study so that they would be able to write their own procedures, would have the confidence to make changes to a procedure and explore variations, and would not be restricted in their explorations by technical difficulties. Therefore all the students attended a preparatory Logo course, which I taught, a month before working in the microworld. This course had the following aims:

- to familiarise the students with the Logo environment: the graphic environment (turtle geometry), the use of commands and procedures, knowledge of the most useful primitives, the use of variables, the use of the editor for defining and editing procedures, and its use for exploring variations of procedures;

- to try to induce good programming habits and the use of modularity (including using procedures as tools in other procedures and for investigations; and

- to familiarise students with *exploratory* activities using Logo, and the use of pencil and paper activities as aids and complements to the Logo activities.

The course was attended by 16 students all aged from 13 to 35 years of age. The students worked as pairs with a computer for each pair, as they would during the main study. Worksheets were used for the activities, designed to instruct the students in the use of the diverse Logo primitives and their applications. Each pair worked at their own pace, although there was an effort to keep everyone at more or less the same stage. The course lasted 10 days, with 3-4 hour sessions per day.

The contents of the course included the following:

- Turtle geometry in direct mode.

- Defining procedures and the use of the editor.

- The use of variables.

- Functions, and understanding the OUTPUT idea.

- Recursion, which included generating lists of numbers and playing with the position of the recursive call, and procedures for turtle geometry figures such as rotating polygons and a fractal tree.

- Recursive functions, such as the factorial function procedure.

- Lists.

- A final group project.

The course allowed the students to feel confident with Logo programming and exploration of procedures, as well as helping them to develop a basic understanding of recursion which they would need during the microworld activities.

c. *The study.*

The main study was carried out in Mexico during the late summer of 1994. The main phase of the study consisted of three parts (summarised in Table 4.6).

Questionnaire and Initial Interview	Observation of Microworld activities	Final Interview
1- 1½ hours	±15 hours	1 hour

Table 4.6. Parts of the main study (excluding the preparatory course).

(i). Questionnaire and initial interview

Students were given a questionnaire (see Appendix 1) from a previous study (Sacristán, 1988) intended to observe the ways in which students conceptualise infinity when working in different areas of mathematics. This questionnaire (as well as the interview) was used for getting an insight into the students' initial conceptions of infinity. It also served as the basis for the interview, where students were asked to explain further their answers to the questionnaire, and where I tried to get an insight into their conceptions of infinity.

The questionnaire consisted of three parts on the following topics: (i). The (infinite) decimal expansions of real numbers: the relationship number-line (locating infinite decimal expansions on the real number line), and the relationships rational number (fraction)-decimal expansion and irrational number-decimal expansion. (ii). Limits of infinite sequences and sums (both in algebraic and in geometric contexts). (iii). One-to-one correspondences between infinite sets. Among the ideas touched upon were:

(ii). The computer-based Logo microworld.

The content and design of the microworld activities will be described in Chapter 5. The design of those activities was directly influenced by the previous experiences with the exploratory and pilot studies, as was the case with the way in which the study was set up. The considerations included:

1. The decision that students should work in pairs with one computer. Students working in pairs can share and discuss ideas, potentially making their explorations richer and less dependent on the guidance from the instructor (unlike what had happened with individual students during the pilot study).

2. From a research methodology perspective it was also decided that analysis of students' experiences within the microworld could be best carried out by working with only one pair of students at a time, using a clinical interview style.

3. The latter point also allowed for the activities to be designed in a more open fashion, giving the students more freedom, since I could give necessary guidance if needed. In particular, although some of the procedures I gave to students were printed or in written form, in general I did not use predefined worksheets as I had on the pilot study: one of the findings of the pilot study was that if I was present at all times with only one or two students at a time, my assistance could fulfil some of the intentions I had for the use of worksheets (see section 2.c.: the methodological issues of the pilot study). However, another finding of the pilot study was that, even though the worksheets used in the pilot study were a simplified form of the ones used in the exploratory study, they still enforced a considerably directed approach to the explorations. Giving the students more freedom in their explorations was important in order for them to demonstrate their own initiatives and how they themselves decided

to use the tools of the microworld and express themselves: As explained in Chapter 3, a key issue from the points of view of both the student and the researcher is to use the computer in such a way that the students are able to articulate for themselves; the students thus play an active role which can foster the construction of meanings while simultaneously providing a window for the researcher to get insights into their thinking processes.

I should add that in order to encourage students to feel free to explore and express their ideas, I began the the first microworld-activities session by reminding the students that they had to think of themselves as scientists who are observing and trying to analyse and understand the processes they see through exploration and experimentation.

(iii). Final interview.

Finally, in order to get a small insight into students' conceptions of infinity at the end of the study, I had a final interview with each pair where they were directly asked if they would change any of their answers in the questionnaire, and whether they had new ideas about infinity. They were also presented with the formal definition of the limit of an infinite sequence.

In all the parts, I worked with each pair of students separately in a clinical interview style. Each pair spent 5 sessions of approximately 3 hours each working in the microworld, in addition to two short sessions at the beginning and end for the initial and final interviews. Except when answering the initial written questionnaire which they did individually, each pair of students worked together on all the activities and shared a single computer; this included the pen and paper activities of the microworld where, for example, they filled out tables of values for their explorations. The initial and final interviews were also conducted with both students at the same time.

d. The role of the researcher.

As in the pilot study (see above) my role was that of a participant observer; I therefore formed an active part of the microworld, being present at all times. I had the dual role of being both the researcher — observing and collecting data — and a guide for the students in the microworld, acting in the following ways:

1. Suggesting the field of work (the initial procedures and activities), as well as new ideas for exploration when needed. I allowed students to pursue their own ideas but also kept them from going astray into areas not related to the field of interest.

2. Helping students with technical difficulties, an important role for the smooth functioning of the microworld.

3. Informally interviewing the students on their findings while they were working in the microworld, mainly to gather data, but sometimes also provoking some reflective processes in them.

In general I tried to let students work as much as possible on their own, allowing them to be in control of the explorations, and giving them freedom to explore and express their ideas. I was careful with my interventions, using them only when I considered them necessary, and only under the conditions listed above.

e. Collection of data

The data-gathering techniques used during the main study (summarised in Table 4.7) included the following:

- Observation of the students working on the microworld activities (which included participation when necessary).

- Field notes.

- Complete audio *and* video tapes of every session, including the initial and final interviews. The recording of the microworld activities included a videotape of the students working on the microworld activities and with the computer, as well as a separate videotape of the computer screen to record everything the students typed, produced, and observed on the computer (and which could be matched to the printed record — see below). These tapes were later completely transcribed into text files into which I incorporated the information from the dribble files (see below), from the video tapes (e.g. the graphic outputs), and from notes relating other visual information or actions.

• Complete printed record of every student/computer interaction through the use of Logo "dribble" files which recorded all of the students' keyboard actions and the (text) outputs given by Logo. This was complemented with Logo files of all the procedures written by the students.

• Informal interviews with the students during their work on the microworld activities (in addition to the initial and final interviews), used to clarify their comments, or to ask them to express their opinions and thoughts.

• Students' written work: the students were provided with paper and blank tables to encourage them to write their observations which they could use as part of their explorations.

Main Study: Data-gathering techniques
• Video tapes (of both the students and the computer screen). • Audio tapes. • Dribble files: record of all the student/computer interactions. • Logo files of students' procedures. • Direct observation, and field notes. • Informal interviews. • Student's written work.

Table 4.7. Data-gathering techniques used during the main study.

f. The phases of analysis.

The process of analysis of the data comprised several phases. In a preliminary phase, from the analysis of the tapes, transcripts and other raw data, in-depth descriptions of the work with each of the pairs of students who participated in the study were written. These descriptions formed the basis for developing detailed case studies of each of the pairs of students. A complete case study of the work with one of the pairs of students (Consuelo and Verónica) is presented in Appendix 7.

These case studies were later merged to construct a common account of how the activities with the students were conducted — the microworld in practice — and which is given in Chapter 6.

More importantly, from the case studies, different categories of analysis emerged which formed the framework for the findings described in Chapter 7.

In the following chapter, I present a description of contents of the microworld, explaining the epistemological issues and the iterative construction process (through the exploratory and pilot studies) behind the design of that microworld.

Chapter 5:

Design and Description of the Microworld

I. The design of the activities and the microworld.

In this section I present the rationale and epistemological issues underlying the design of the activities and the microworld used in the study. I explain:

a.- The design considerations for the construction of an infinite processes microworld.

b.- The use of the Logo programming language as the medium of the microworld.

c.- The choice of the mathematical topics for exploration.

d.- The iterative design of the activities chosen for the microworld.

A. An infinite processes microworld.

The concern of the study was to investigate students' developing conceptions of infinity and infinite processes through the computational environment provided. Thus, my aim was to design and implement a computer-based microworld where students could be involved with, and explore a range of ideas related to the concept of infinity and infinite processes. To aid in this process some of the ideas in Chapters 2 and 3 served as pointers, as will be appropriately indicated. This microworld was intended to be an exploratory setting which would involve the use and construction of different types of representations (e.g. symbolic, visual, unfolding — i.e. using movement) serving as tools and means with which to conceptualise an infinite process, as I particularly wanted to investigate their role as mediators in the thinking processes of students.

Thus, I wanted the microworld to include the following aspects:

1. A programming environment:

Following the ideas discussed in Chapter 3, stressing the importance of students playing an active role by, for instance, being able to express and articulate themselves, I wanted students to be in an environment where they could write, create, modify and explore procedures; and where the programming code (a symbolic representation) could serve as a control structure. In other words, I wanted the code to serve as a means for the students to control the process and the structure of the process. Furthermore, by having the students creating, controlling and modifying the procedures, I intended that an active correspondence could be created between the code (in its different versions) — also a representation — and the diverse geometric and numeric representations it generated.

In this way the programming activities were designed to facilitate the cognitive integration (linking) of the different types representations of a mathematical object, by first linking them on an external level. This intent followed the argument presented in Chapter 3, that the construction of meaning involves building connections (Wilensky, 1991). Additionally, the programming aspect of such an environment broadens students' opportunities and means to express themselves and their ideas through the medium, simultaneously providing for the researcher a window into students' thinking, as explained in Chapter 3.

2. Interaction between different types of (re-)constructed representations:

As I have already pointed out, because infinity can be thought of as an abstract concept difficult to grasp, and to help make this concept more concrete (see again the ideas of Wilensky, 1991 presented in Chapter 3), it was important to provide multiple modes of representations with which students could engage and interact. These forms of representations were conceived as mediational tools for the construction of knowledge (see Chapter 3: e.g. Piaget & Garcia, 1989; Vygotsky; Papert, 1993) Thus, the infinite processes were to be explored through the construction of different models for representing them:

- the symbolic code,
- different types of visual models, including unfolding (dynamic) ones, and
- numeric representations to complement and validate the visual observations (taking advantage of the computing capabilities of the computer for reaching high terms in numerical or other types of sequences);

A fundamental premise was for these different types of representations to be explicitly linked one with the other through the first: the procedural code. In fact, the code can act as an isomorphism between the different (visual) models, and serve as a link between all the representations and the subject. For instance, programming the computer to draw on a computer screen can be thought of as a process of interaction between contexts, from the symbolic code to the visual and conversely. The symbolic code of the computer language can serve, within the computer context, to "explain", model, or represent the process and it encapsulates the structure behind the process.

Furthermore, having multiple representations (linked through the student-controlled programming code) was also meant to facilitate the links between the different settings in which the infinite is encountered. As has been found by many researchers (see Chapter 3: e.g. Nuñez, 1993; Moreno & Waldegg, 1991; Sacristán, 1991; Tsamir & Tirosh, 1994), the context and setting in which a situation is presented (see also the end of Chapter 2) strongly influence students' conceptions of infinite processes or objects, and it is thus important to take them into account.

A further premise was that the interactions between the different representations create a setting which facilitates the construction of situated abstractions, as explained in Chapter 3 (Noss & Hoyles, 1996): that is, the process can be understood within the context in which it has been explored, and understanding the symbolic computer code is in this sense achieving a certain degree of conceptual formalisation. (This is why, for the processes and activities chosen for the microworld, several visual models of the same process were obtainable from variations of one type of procedure).

3. Use of visual models.

Following the arguments presented in Chapter 3 that stress the value of and need for visualisation, I considered it important to include visual models of the infinite processes being studied: the use of the visual is an essential element for making infinity more concrete. However, it was also important that models were constructed in co-ordination (and interaction) with other types of representations (see sections 1 and 2 above), since there is evidence (see Chapter 3) that the construction of links between representations is not straightforward.

4. The use of movement: visual unfolding

Additionally, I wanted to take advantage of the use of the computer by incorporating movement into the processes represented (particularly the visual models). As I mentioned at the end of Chapter 2, a process implies *change*; and change over time is related to movement. Through the use of the computer, the processes can be represented as they unfold; instead of viewing only the end result of the process, the process itself and its behaviour can be seen. I considered this particularly useful in the study of infinite processes since the result of the process, which could be said to be the behaviour at infinity, can only be deduced by analysing the behaviour of the process in the finite (this is true even in the formal definitions of a limit). In this way, looking at the process as it unfolds could become a window for looking at the behaviour at infinity.

5. Iterative and recursive structures / recursion:

Another key aspect relevant for the study of infinite processes in particular, was the use of iteration and recursion. As seen in Chapter 2, *iteration* is the building block of infinite processes; *recursive* procedures capture the essence of infinity — they are implicitly endless and they reflect one of the characteristics of infinity: self-similarity. Thus, most of the activities centred on procedures which "called" themselves, whether tail-recursive or fully recursive.

B. The use of Logo.

The microworld was thus designed as a medium for active involvement in which the student could explore visual unfolding representations, together with numerical data, linked and controlled through the symbolic (and analytic) procedural code (the computer program).

The requirements of this microworld made Logo an appropriate programming language to use in the environment for this research. The role and advantages of using Logo in computer-based microworlds has already been adequately made elsewhere in the literature (e.g. Hoyles & Noss, 1992). Nevertheless, I will outline some of the considerations for basing the microworld on this language:

- Logo is easily accessible: thus Logo allows one to carry out the idea of *constructionism* — learning by constructing (in this case through programming);

- Logo allows the learner to build upon or modify original procedures with great ease, and this allows for endless possibilities for investigations and variations in an activity.

- Logo has a built-in visual interface through Turtle Geometry which can be very helpful for the requirements of incorporating the visual representations and creating the interaction with the symbolic code. And because figures have to be created linearly in Turtle Geometry this may help in the observation of the process as it unfolds, fulfilling the requirement of movement and dynamism.

- Finally, recursive programming is fairly straightforward in Logo (although I do not underestimate the difficulties of this kind of programming).

C. The area of study: infinite sequences and fractals.

Some of the areas related to infinity which seemed adequate for the purpose of the study because they could easily be explored in a computer-based environment were those of infinite processes such as sequences and series, and 'limit objects' (e.g. fractals). Thus, the central topic of the microworld was the convergence (and divergence) of infinite sequences and series, and limits, through the use of recursive geometric figures (an idea explored by Cuoco & Goldenberg, 1992 — see Chapter 3). Self-similar figures and fractals — which are the limits of infinite graphical sequences — were used for introducing a different kind of setting for the idea of the limit of a sequence, and for presenting some of the results which sometimes students find paradoxical and that can come about when working with the infinite. Fractals also have an intrinsic *recursive structure.* The activities of the microworld thus involved the visual study of these recursive structures, and the investigation of the building *process* of the geometrical objects involved. By observing the movements of the turtle a sequence of geometrical objects can be seen as *processes* in time; that is, through the geometric representation, the different steps of the process can be seen. These unfolding geometric representations bring back the elements of movement and variation which were fundamental to the origins of calculus (see Chapter 2).

Additionally, the possibility of producing successive levels of a figure as approximations to the "real fractal" can be thought of as a (potentially) infinite process, yet it is an exploration of a geometric object which has already been given through the symbolic code. Progressing through the different levels of the picture can thus be interpreted as approaching an object which "is already there". The construction of fractals becomes a "window" on infinity. The visual behaviour may give students a chance to gain familiarity with a concept that is not realisable in the physical world. Running the code, and reflecting on what is produced, can be seen as an indication that the object is a process.

D. The design process and the activities chosen for the microworld.

As explained in Chapter 4, the design of the study involved several phases: an exploratory study (see Appendices 5 and 6) and a pilot study (Appendices 3 and 4) were used for defining the activities and makeup of the microworld as well as research methodology. During the exploratory study a very wide range of (iterative/recursive) Logo activities were included in order to explore the adequacy of each of these for my research. However, the number of activities was too large for students to be able to carry out appropriate in-depth exploration, as well as restricting the possibilities, from the researcher's point of view, for an adequate qualitative analysis; thus, a selection of the activities was made based on the richness of possibilities of each activity, taking into account those which produced the most interesting results, as well as the simplicity with which they could be approached. I also excluded those activities which did not include geometric representations (e.g. Fibonacci's sequence) since, as already mentioned, I wanted to take advantage of the visual aspect. Fewer activities meant that they could be less structured, thus allowing students greater freedom for exploring and unfolding their ideas and conceptions, something which can give greater insight to the researcher.

As described in Chapter 4, the activities thus needed to be presented and explored in a different format, with changes being made to the worksheets (less directed and more open ended) and the working environment (e.g. working with a pair of students at a time — see Chapter 4). For this reason the pilot study was needed where I could try out the new selection of activities and format. The activities from the main study are essentially the same ones used in the pilot study, although some slight changes in implementation and approach were made based on the results from the pilot study. In particular, as explained in Chapter 4, the activities were approached in an even less directed and less assisted fashion than in the pilot study, allowing the students to do their own programming (which was made possible by teaching the students Logo before the main study). Other changes will be described as appropriate. The activities finally chosen for the microworld in the main study were the following:

a.) Explorations of infinite sequences and their corresponding series.

These included sequences such as $\{1/2^n\}$, $\{1/3^n\}$, $\{(2/3)^n\}$, $\{2^n\}$ etc., and then $\{1/n\}$, $\{1/n^{1.1}\}$,..., $\{1/n^2\}$, and the sequences of their corresponding partial sums, through visual models such as spirals[1], bar graphs, staircases, and straight lines, and the corresponding Logo procedures, with a complementary analysis of the numerical values (their progressions and the apparent limits, if any existed or appeared to exist).

I chose geometric models such as spirals, since they seemed to be a straightforward way of translating arithmetic series into geometric form. For instance, in the 'spiral' type of representation each term of the sequence is translated into a length, visually separated by a turn, so that the total length of the spiral corresponds to that of the sum of the terms (the corresponding series). Thus, for instance for the sequence $\{1/2^n\}$, the visual process and added lengths of the spiral would represent the series: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$, a notation which is descriptive of the process involved — the ellipsis points indicating an indefinite continuity of the process (a *potentially*

[1] I would like to acknowledge the books on infinity by Mason (1988), in particular, and Hemmings & Tahta (1984), which served as inspiration for some of the geometric models of sequences that I used in my work.

infinite process). On the other hand, in the symbolic computer code, the same series can also be represented by a notation corresponding to $\sum_{n=1}^{\infty} \frac{1}{2^n}$, which is an object in itself (an *actual* infinite object), and does not explicitly indicate the infinite process it describes. (This is independent of the convergence or divergence of the series, although when there is convergence it is easier to think of the series as a "complete" object, e.g. when $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$). This could serve to illustrate how the same (mathematical) object can be represented both as a (complete, integral) *object* as well as in terms of a *process*.

There were several important differences in the ways in which these sequences were explored in the main study with respect to how they were studied in the pilot study:

- First was the inclusion of different types of geometric models. In the pilot study, the sequence studies centred around the study of the spiral models, only sometimes stretching the spiral into a line or changing it into a staircase. In the main study the possibility of looking at many different types of models, as listed above, was added. In particular the possibility of looking at the sequence of segments as a *sequence* in a bar graph, was included.

- Secondly, the procedures were modified during the activities — as further described and clarified below — so that the visual models were not simply produced through the process of transforming a previous segment, but rather as actual *models* of a sequence described as a *list* of values. I consider this a significant change for three reasons: (i) First, producing a sequence as a list and *then* modelling it, highlights the idea that the geometric figures are models of a mathematical process which can be symbolically described and independently expressed; whereas before the numerical values of the segments were seen merely in terms of the measure of the segments. It is a *situated* formalisation of the process, in the sense that a more abstract definition of the process is given but still within the context of the microworld and through its language and tools. (ii) Additionally, by adding the idea of producing lists of the values of the sequences, these potentially infinite sequences originally seen as *processes* could then be seen as (actual?) *sets,* which are a conceptually different representation of the same idea. This duality of representation could perhaps be associated with the duality of the infinite as both potential and actual infinities, as Cantor did (see Chapter 2). (iii) Finally, a result of this change was that the explicit description of the sequences as such also helped in the differentiation between the sequence itself and its corresponding series (which was represented as the "total length" of the visual models).

It was hoped that through the observation of the visual (and numeric) behaviour of the models, students would be able to explore the convergence, and the type of convergence, or divergence, of a sequence and its corresponding series, and predict the behaviour at infinity. The different geometric models for the same sequence were meant to provide different perspectives of the same process. But an aspect which was considered important for this, was that the students carried out the transformations of the models themselves by changing the computer code. It was intended that this involvement would help build links between the symbolic representation (in the code) and the different models.

b.) Exploration of fractal figures ("limit objects").

These included the Koch curve and snowflake, and the Sierpinski triangle. The explorations involved the study of their recursive structures (apparent both visually and in the symbolic code), and the apparent paradoxes at infinity, such as a finite area bounded by an infinite perimeter.

With these activities I was hoping to confront students with the idea of "what happens at infinity", by having them "visualise" an infinite process by observing its behaviour through the, albeit finite, computer-based approximations. For all the above activities "measurement" procedures were used for computing numerical data which could complement the visual models. Most of the time, tables were used for structuring this data, becoming an additional representation of the sequences under study.

Besides having the added values of being beautiful, attractive and fun, and of being a more contemporary area of mathematics, these fractal activities also seemed to touch upon some interesting ideas and concepts:

1.- As explained in Chapter 2, fractals are *limit* objects; they "exist" as limits of infinite processes; yet, once "produced" they can also be conceived in terms of sets consisting of infinitely many parts, where each part is also *self-similar* to the whole (thus highlighting the recursive/iterative fundamental nature of the infinite which was also described in Chapter 2). For obvious reasons, these figures provide a rich ground for the exploration of the infinite: of infinite processes and of "infinite" objects.

2.- The apparent contradictions which emerge during the study of fractals, such as the infinite perimeter of the Koch curve being formed by infinitesimal of "zero-sized" segments are reminiscent of the problems discussed by Galileo as described in Chapter 2, which occur when thinking of the infinite from the conceptual framework of the finite (which has also been found to happen in students by among others, Waldegg, 1988 — see Chapter 3). The investigation of the Koch curve touches on other situations found in history (and described in Chapter 2): a non-zero body (length) formed of an infinite set of infinitely small parts is the conception behind Democritus' method of the indivisibles, touching as well on the problem of the nature infinitesimals. Yet, on the other hand, the *construction* process of the Koch curve is more like the method of exhaustion which relies on the *process* of iteration and where the "infinitely small" parts are only thought of in terms of *approaching* zero.

II. Description of the microworld procedures.

The Logo procedures used in the microworld were like the ones given below. These procedures were not given in written form (except in very few occasions such as for the first activity). They were usually programmed by the students using suggestions given by the researcher. Except for the first activity, the only other "worksheets" used were blank sheets of paper and blank tables (see Appendix 2). How students used the pencil and paper was many times determined by suggestions given by the researcher; in general most of the activities included the use of tables with numerical values as

[3] The procedures varied slightly from one set of students to another, since it was the students themselves who did the programming following the researcher's suggestions. The procedures given here are actual procedures used by some of the students; only the names of the procedures were translated into English.

part of the exploratory activities. The microworld procedures were similar to those given below, but varied depending on the students.

- *Study of visual models for sequences such as* $\{1/2^n\}$, $\{1/3^n\}$, $\{(2/3)^n\}$, $\{2^n\}$, *and* $\{1/n\}$, $\{1/n^{1.1}\}$,..., $\{1/n^2\}$.

In every case the activities of the microworld began by giving students the procedure below (in written form through the first worksheet given in Appendix 2). The students were asked to predict its behaviour.

```
TO DRAWING :L
PU
FD :L
RT 90
WAIT 10
DRAWING :L / 2
END
```

This procedure makes the turtle walk through a spiral with arms each having half the length of the previous one (see Figure 5.1). It was a first approach to the infinite sequence {1/2n}, which was chosen because of its simplicity. It should also be noted that this is a tail-recursive procedure without a stop condition, so the procedure could potentially continue indefinitely, in an effectively iterative way. This procedure — which in the beginning does *not* draw but does show the turtle moving — was designed to induce students to reflect on the behaviour of the turtle and of the process itself. The idea of having the pen up had produced interesting results in the pilot study where it forced students' reflection on the process: in that study, the students observed the movements of the turtle (which are made easier to see by the WAIT command) and reconstructed in their minds the actual drawing. In this way they would realise that the turtle was walking half the distance each time it turned and did so without stopping,

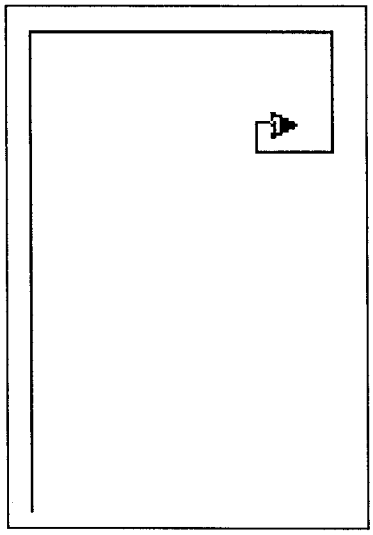


Figure 5.1. Spiral model for the sequence $\{1/2^n\}$.

realising this without the visual obstacle of the computer drawing which after a while seemed to show the turtle staying in the same place (actually in the same pixel).

For the main study this initial activity was to be followed by students modifying the original procedure and ending up with a general procedure and a set of visual models. For instance, the new general procedure could have been[3]:

```
TO DRAWING :L
IF :L < 1 [STOP]
MODEL
DRAWING ( FUNCTION :L )
END
```

where the function, i.e. the transformation carried out at each step of the process, could for instance have been given by:

```
TO FUNCTION :L
OP :L / 2               — or:  OP  1 / POWER :L 2 —
END
```

The visual models are given by MODEL. For instance:

```
TO MODEL
SPIRAL
END
```

where SPIRAL can be replaced by any of the following:

for a spiral:	for a bar graph:	for a staircase:	for a straight line:
TO SPIRAL FD :L RT 90 WAIT 10 END	TO HISTOGRAM JUMP FD :L END TO JUMP PU SETY -100 RT 90 FD 5 LT 90 PD END	TO STEPS FD :L/2 LT 90 FD :L/2 RT 90 END	TO LINE FD :L WAIT 10 END

[3] The procedures varied slightly from one set of students to another, since it was the students themselves who did the programming following the researcher's suggestions. The procedures given here are actual procedures used by some of the students; only the names of the procedures were translated into English.

The Bar graph, Steps and Line models are illustrated in figures 5.2, 5.3, 5.4, respectively.

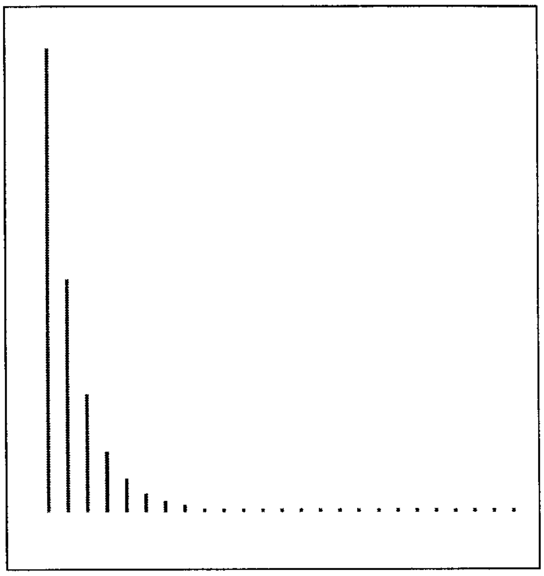


Figure 5.2. Bar graph corresponding to the sequence $\{1/2^n\}$.

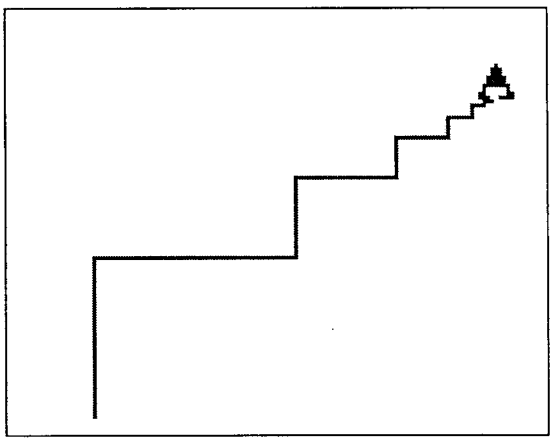


Figure 5.3. Steps model corresponding to the sequence $\{1/2^n\}$.

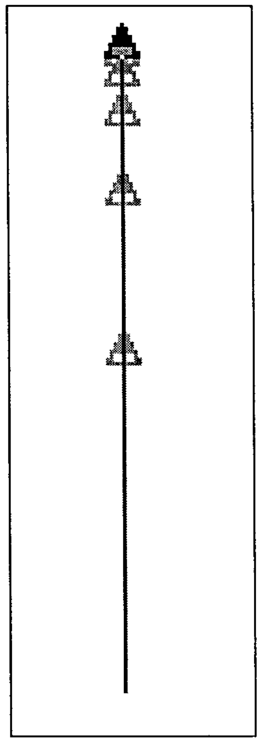


Figure 5.4. Line model corresponding to the sequence $\{1/2^n\}$.

At a later stage, procedures for generating the sequences and storing the first n terms into a list would be used, so the above procedures would be modified to draw visual models of those terms of the sequences. This is useful because it helps create an interaction between the numerical values at each stage and their corresponding visual representations, and assists in the analysis of the behaviour of the overall process (i.e. whether or not the sequence approaches a limit can be seen not only from what the visual model may apparently show, but also from a numerical perspective). With these procedures, the numerical values can also be made independent from the scale used in the drawings. In this new context procedures such as the following are used:

To generate the first N terms of a sequence:

```
TO SEQUENCE :N
IF :N = 1 [OP FN 1]
OP SE (SEQUENCE :N - 1) (FN :N)
END
```

where the sequence is given as a function of N, and not of L. For instance:

```
TO FN :N
OP  1 / POWER 2 :N
END
```

The drawing procedure would then be the following:

```
TO DRAWSEQUENCE :LIST :SCALE
IF :LIST = []  [STOP]
FD :SCALE * FIRST :LIST
MODEL
DRAWSEQUENCE BF :LIST :SCALE
END
```

changing in each model ":L " by ":SCALE * FIRST :LIST".
The scale had the advantage that a same model could be reproduced in different sizes. It thus served as a sort of "zoom" tool.

Finally, the series and partial sums of each sequence would be explored by using the following procedures, using them in combination with SEQUENCE to generate the input list (e.g. typing SUML SEQUENCE <number of terms>). The PARTIALSUMS procedure gives as output the *list* of partial sums of a sequence of *n* terms; while SUML is a procedure which adds all the terms in a list (sequence).

```
TO PARTIALSUMS :LIST
IF :LIST = [] [OP [] ]
OP SE (PARTIALSUMS BL :LIST) (SUML :LIST)
END

TO SUML :LIST
IF :LIST = [] [OP 0]
OP (FIRST :LIST) + SUML BF :LIST
END
```

- *Study of fractal figures*

As already mentioned, the second part of the microworld involved the study of some fractal figures: the Koch curve and snowflake, and the Sierpinski triangle. The purpose of these activities was to explore a limit object of a different kind, including the visual sequence that leads to it, and the (programming) code which reflects its recursive structure, and which each of the steps of the sequence embody.

The Koch curve and snowflake.

For the Koch curve I would draw on paper the first steps of the construction process and the students would write a procedure for generating this figure (see Figure 5.5). The explorations would then involve measuring the perimeter of this curve. The use of tables which included values for the number of segments and size of each segment were used for this purpose. This activity was followed by the exploration of the Koch snowflake (see Figure 5.6), its perimeter and its area. The procedures for these figures would be as follows:

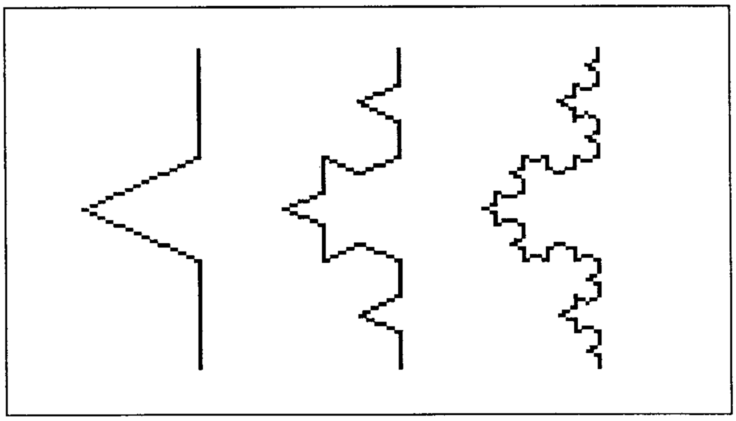


Figure 5.5. Construction process of the Koch curve.

For generating the Koch curve:

```
TO CURVE :L :LEVEL
IF :LEVEL = 1 [FD :L STOP]
CURVE :L / 3 :LEVEL - 1
LT 60
CURVE :L / 3 :LEVEL - 1
RT 120
CURVE :L / 3 :LEVEL - 1
LT 60
CURVE :L / 3 :LEVEL - 1
END
```

And for drawing the snowflake:

```
TO SNOWFLAKE :L :N
REPEAT 3 [CURVE :L :N RT 120]
END
```

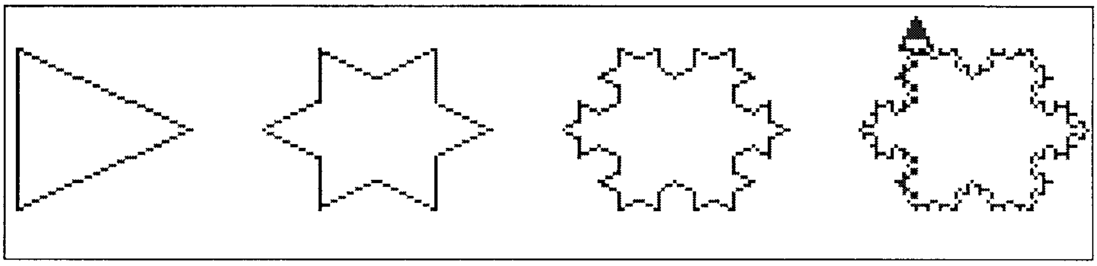


Figure 5.6. Construction process of the Koch snowflake.

Representations of the Sierpinski triangle:

The purpose of this activity, as with the previous one, was to present students with a recursive structure, both visually and symbolically, and to confront them with another example of the "behaviour" of mathematical infinity: through a process that "takes away", at each step, one fourth of the area of each part (see Figure 5.7), the area at infinity becomes nil. Since this is a procedure which, although short once written, can be rather tricky to program (from the technical point of view of getting the turtle to draw the inner triangles in the right place), I provided most of the students with its code, given below:

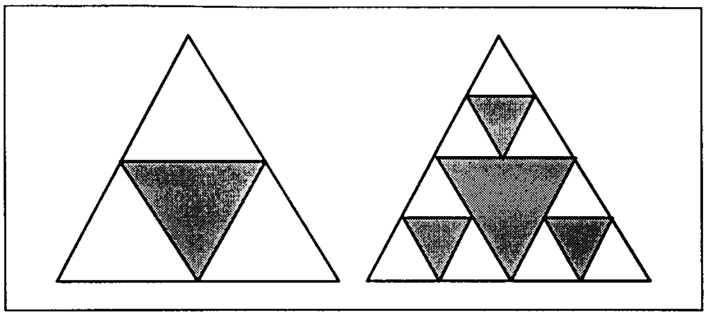


Figure 5.7. Construction process of the Sierpinski triangle.

```
TO TRI :SIDE :LEVEL
IF :LEVEL = 0 [STOP]
REPEAT 3 [TRI :SIDE / 2  :LEVEL - 1  FD :SIDE  RT 120]
END
```

If analysed, the code of this procedure reflects the structure of the procedure since, in each triangle, there are three similar triangles of half the size. As with the Koch explorations, the students explored the area of this figure. Procedures such as the following, used for computing the area of an equilateral triangle, were used as tools for computing the area of the snowflake as well as for the Sierpinski triangle:

```
TO AREATRI :SIDE
OP (POWER :SIDE 2) * (SQRT 3) / 4
END
```

At the end of their explorations I presented the students with another procedure which constructed an open-ended curve describing the points from the Sierpinski triangle (see Figure 5.8):

```
TO CURVE :N :L :P
IF :N = 0 [FD :L STOP]
LT 60 * :P
CURVE :N - 1 :L / 2 (-1 * :P)
RT 60 * :P
CURVE :N - 1 :L / 2 :P
RT 60 * :P
CURVE :N - 1 :L / 2 (-1 * :P)
LT 60 * :P
END
```

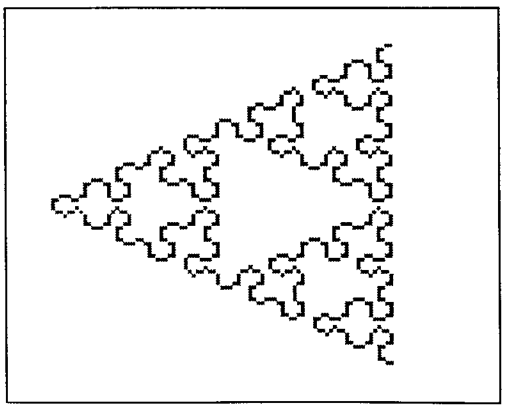


Figure 5.8. Fifth level of the Sierpinski Curve procedure.

The purpose of presenting this procedure was to give students an alternate view of the Sierpinski fractal figure which can be seen as an infinitely twisted curve that never touches itself, and whose points therefore do not describe an area.

III. Areas of research.

A central purpose of the research was to observe how working with the microworld mediated the ways in which students constructed meanings and developed their conceptions. With this objective in mind the microworld was, for example, designed to take advantage of the visual capabilities of the computer and was expected to induce interaction between visual models and the symbolic code. In order to research the process of mediation of the computer-based microworld in the learning processes of students, I aimed to investigate the following areas (most of which are related to the *use* students made of the microworld and the way they *expressed* themselves within it, but are separated as they emphasise different aspects):

1. Conceptualisations of the processes.

2. The value of exploration.

3. Reconstructions of connections between different types of representations.

4. Students "theorems" and situated abstractions.

1. Conceptualisations of the processes.

How students conceptualised the infinite processes under study. One approach for this was to look at the arguments and elements students used to explain the behaviour of the processes in order to get insights into their conceptualisations. Additionally, the way in which they worked in, and with, the microworld — what tools or elements they chose to use or look at, how they expressed themselves when writing procedures or notes, etc. — could provide a window into their thought processes.

2. The value of exploration (prediction and exploration leading to conceptual change).

How students *used* the environment and its tools to make sense of the processes under study: that is, it refers in particular to their initiative to investigate different possibilities through their construction of new representations of a process, variations of that process, or similar processes, and testing their *predictions.*

With reference to the latter, the experience of making predictions, and either confirming them or getting an unexpected result, is an important process in the learning experience, and leads to possible *changes* in students conceptualisations.

3. Reconstructing connections between different types of representations:

How the students connected the different representations in their attempt to make sense of the phenomena they observed. For instance, I looked at their re-actions to a certain visual or numeric output and the ways in which they attempted to explain it: e.g. by returning to the code and re-analysing it, by linking the different types of outputs, or even by using their understanding of the underlying mathematical process.

In this area, the following aspects were of particular interest:

- the use of the computer *code: connecting the code with the graphical*
- the role of *visual* representations
- the role of *movement*, and gradual *unfolding* of the procedure (both visual and numeric)
- the role of the *structure* of the procedures, in particular the iterative or recursive structure, and the relationship to the visual structure.

4. Students "theorems" and situated abstractions.

The ways in which students were able to articulate or express their conclusions and make generalisations or abstractions with reference to some observed behaviour, process or other type of phenomena within the context of the microworld.

The remainder of the thesis.

In the following chapter, I present an account of the microworld in practice, and illustrate the way in which the microworld functioned, bringing together the work done by all the students to provide an overview of the common activities structure. Then, in Chapter 7, the main ideas and key issues that emerge from the empirical research are analysed. The overall conclusions and implications of this research are presented in Chapter 8.

Chapter 6:

The Microworld in Practice: Constructing and coordinating representations.

In Chapter 5, I described the activities and procedures which were designed to form the basis of the microworld investigations. But, as has already been explained, the microworld was meant to allow students freedom to express and explore their own ideas. Students constructed their own variations of procedures or additional procedures to those that were pre-designed; some students also created different graphic models or carried out certain investigations that I had not foreseen. In this chapter I have constructed an account of the actual microworld experiences, putting together information from the four different experiences[1] with each of the four pairs of students presented in Chapter 4. The purpose of this chapter is to illustrate the way in which the microworld functioned, highlighting the role of the exploratory activities and environment, and the importance of each of the tools (procedures, graphic outputs, and tables) as mediators and structuring elements for the processes of discovery and construction of meanings to take place. I illustrate the microworld's constructionist approach and delineate the structure of activities common to all the students. Additionally, I provide some insight into my own role, outlined in Chapter 4 (section II.3.d). As such, the present chapter is essentially *descriptive* in nature, and I postpone a more analytical presentation until the next chapter.

Summing up, the purpose of this chapter is to:

- describe the structure of common activities, specifying when possible my structuring role;

[1] A full account and case study of the experience with one of the pairs of students (Consuelo and Verónica) is given in Appendix 7.

- illustrate, and give a first level of analysis of, the functioning of the microworld, the constructionist approach, and the mediating role of the tools in the construction of meanings;

- present a guide to the data analysed in chapter 7.

I have synthesised as much as possible the work done by all the students, describing the common areas of investigation in the microworld. In what follows, I occasionally refer to "the students" or "most students"; these phrases generally refer to cases where the majority of the students — all except maybe one or two — exhibited a certain behaviour. I employ a greater degree of precision in Chapter 7.

The account given in this chapter follows as much as possible the chronology of the activities with the students. It is therefore presented following the two main parts of the study: the sequence and series studies, and the fractal studies.

Part A. Sequence studies.

1. The initial DRAWING procedure. Sequences of segments defined through operations on those segments (e.g. "taking halves").

As explained in Chapter 5, the activities began by presenting the students with the procedure[2] (see the Initial Worksheet in Appendix 2):

```
TO DRAWING :L
PU
FD :L
RT 90
WAIT 10
DRAWING :L * 1/2
END
```

[2] For the sake of clarity, all of the procedures and dialogue have been translated into English from the original Spanish. For this reason some of the words or expressions used may seem slightly awkward.

a. Making sense of the movements of the turtle by re-analysing the code.

Although the students had been asked to predict the output of the procedure DRAWING, after running the procedure most of the students needed to re-analyse the code in order to make sense of the movements of the turtle, which moved without drawing (the Pen was up) tracing an inward spiral. All of the students then modified the procedure so that the turtle's pen would be down and the drawn spiral would be visible (see Figure 6.1).

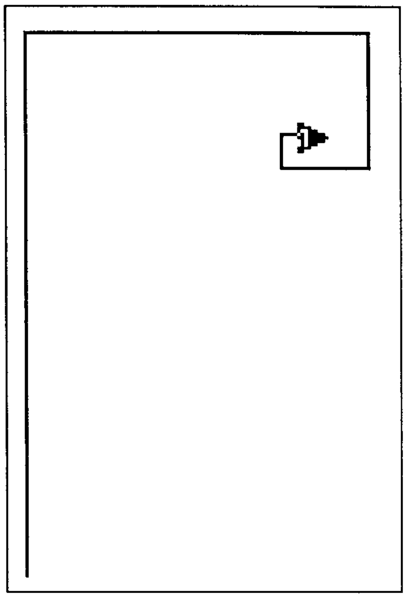


Figure 6.1. Spiral output of the initial DRAWING procedure (representing the sequence $\{1/2^n\}$).

When the students became aware of the recursive structure of the code, and the infinite loop it described, all of them would eventually add a stop condition line to the procedure resulting in a modified procedure such as:

```
TO DRAWING :L
IF :L < 1 [STOP]
FD :L
RT 90
WAIT 10
DRAWING :L * 1/2
END
```

The stop condition was initially added by the students as an instruction to stop the procedure; however, this command would also become a very important investigation tool for all the students as they used it to make sense of the relationship between the code and the graphical output, as well as a research element for the study of the rate of convergence of the sequences under study, as will be described further below and discussed in Chapter 7.

b. Using numeric values to complement the visual output; coordinating the two elements through the code.

Very soon after the investigations with DRAWING began, most students became interested in knowing what the values of the segments forming the spiral were.

Several of the students did mental or pencil-and-paper computations of what the values of the segments should be; Verónica and Consuelo, in particular, recorded this information in a table (see Table 6.1).

Arm [of the spiral]	[Distance] walked
1	100
2	50
3	25
4	12.5
5	6.25
6	3.12
7	1.5
8	0.75

Table 6.1. Table used by Verónica and Consuelo for recording the distance walked by the turtle in each segment of the spiral.

Other students (e.g. Victor and Alejandra) began their numerical investigations by pausing the procedure and asking the computer to type the value of the variable :L, which represents the length the turtle last walked:

e.g. DRAWING ?PR :L
2.273736762E-0011

The "PR :L" command would sooner or later be incorporated inside the DRAWING procedure by all of the students, printing the value of each segment as the turtle drew it. Through this process the students were also able to "see" what the turtle was doing at levels which were no longer visually perceptible. In this way the visual and numeric representations of the same process were linked through the use of the symbolic code — the variable :L has a numeric value but is also what determines the length the turtle draws. This link through the code of the visual and numeric was highlighted as the students had to find the correct position for the "PR :L" line in the procedure (for instance, placing it at the end of the procedure causes the number of values to be different to the number of actual segments drawn, as well as reversing the print order); the students used the *movements* of the turtle to verify the procedure comparing the number of times they could see the turtle turn and walk, with the number of values that appeared on the screen. In this way the students took advantage of the *visual dynamism* (the visual 'unfolding' of the process) provided by the computer to find the correct position for the "PR :L" line and to make sure that the two outputs were correctly linked in the code.

All of the students also decided to investigate the number of segments the turtle drew with varying values in the stop condition. It is interesting that all of the students carried out this exploration on their own initiative. Some students such as Alejandra and Victor, and Jesús and Manuel, simply counted the visible segments, the number of values that were given as output, or used the movements of the turtle to count the number of times the turtle turned — again taking advantage of the visual unfolding of the process. The other students, including the younger students Verónica and Consuelo (see Chapter 7), and the two teachers Elvia and Martin, used a :COUNT variable for counting the segments of the spiral. By adding a "PR :COUNT" command inside the DRAWING procedure, the number of the segments drawn was given as output each time the procedure was run, thus making the workings of the procedure visible. Verónica and Consuelo added this instruction inside the stop condition in the procedure (just before the STOP instruction[3]), thus only getting the final count of segments at the end of the procedure; Elvia and Martin used a procedure such as the following, which, as the turtle drew each segment, would give together the number of the segment and its corresponding value (see Table 6.2):

```
TO DRAWING :L :COUNT
IF :L < 0.1 [STOP]
TYPE :COUNT TYPE " _ " PR :L
FD :L
RT 90
WAIT 10
DRAWING :L * 1/2 :COUNT + 1
END
```

```
1 _ 100
2 _ 50
3 _ 25
4 _ 12.5
5 _ 6.25
6 _ 3.125
7 _ 1.5625
8 _ 0.78125
9 _ 0.390625
10 _ 0.1953125
```

Table 6.2. Numerical output representing the sequence $\{1/2^n\}$: values of each segment drawn preceded by the segment number (count), as programmed by Elvia and Martin.

The numeric representations complemented the visual output, adding a new dimension for the understanding of the process. The numeric values allowed for 'visualising' (i.e. through the numeric values) the deeper levels of a visual sequence: for example, when Verónica and Consuelo were investigating the spiral of $\{1/2^n\}$ the

[3] IF :L<1 [PR :COUNT STOP]

output of numeric values allowed them to "see" the value of segments which were no longer perceptible, and made them aware that the process continued even though the perceivable visual image remained unchanged. The numeric values also served to make the inner workings of a procedure visible: for instance, when Victor noticed a discrepancy between the visual and the numeric outputs, he discovered there was a mistake in the code. Through the code, the visual and numeric representation were automatically linked, both representing the same process in complementary ways.

c. Making sense of the process represented in the code: understanding the relationship between the stop condition and the visual output and creating a table to connect all the elements involved in the process.

Having coordinated the visual and numeric elements through the procedural code, the students would use all of the elements available to carry out their investigations of the process, such as investigating the relationship between the value in the stop condition and the number of segments that the turtle draws before the procedure stops, or simply investigating what happens to the values of the segments as the sequence progresses. At this stage, often following my suggestion, most students constructed tables of values such as the one shown in Table 6.3. These tables became important structuring elements for the investigations and served as additional tools for connecting all the elements.

Scale=100	Value in the Stop Condition :L < ...	Count of segments in the spiral	Size of last segment
	1	7	1.5625
	0.7	8	0.78125
	0.5	8	0.78125
	0.2	9	0.390625
	0.1	10	0.1953125
	0.01	14	0.01220703125

Table 6.3. Example of table used by the students in their explorations of the sequence {$1/2^n$} represented by the spiral produced by the DRAWING procedure.

These investigations also involved a process of exploration and experimentation with the procedure and the visual outputs. For instance, some of the students (most notably the pairs Alejandra/Victor and Manuel/Jesús) experimented with the scale (which corresponds to the initial input of :L), and also with opening up the spiral (by changing the angle; see Figure 6.2), then observed how many segments (or "arms") of the spiral they were able to perceive in each case.

Both the visual investigations and the use of table of values *highlighted the relationship between the stop condition in the code and the visual output*. For example, Alejandra, after running the procedure with different initial inputs[4] observed that "the bigger the scale the larger the number of arms" and added that if the initial input (scale) was smaller than the value in the stop condition then the turtle would not do anything. Most students made similar remarks. The students also became aware of the need to use a constant scale (initial input) for exploring the effects of varying the value in the stop condition.

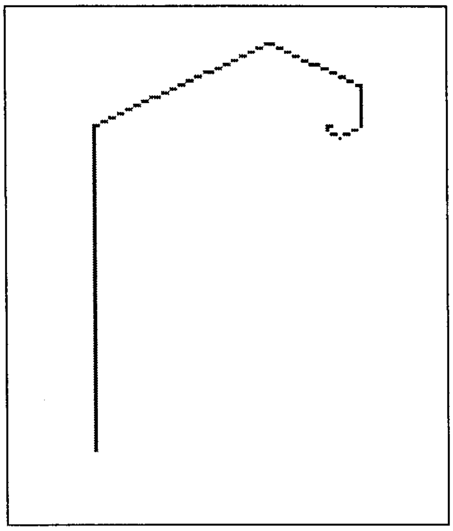


Figure 6.2. Spiral model of the sequence $\{1/2^n\}$ using 60° as rotation angle for the turtle.

As will be discussed in Chapter 7, these initial explorations of the effects of varying the stop condition led students to become aware of the infinitude of the process of taking halves, and of how quickly (the rate of convergence) the values of the sequence it forms become small and get close to zero (without reaching it in a finite amount of time). The numeric explorations were an important element which provided a means for confirming the behaviour — observed or inferred through other representations — of the processes being studied. Additionally, the tables of values which the students created became a structuring element where each of the elements was elucidated and its relationship with the other elements made explicit — this particular feature was specially useful for the fractal studies illustrated in Part B.

[4] E.g. DRAWING 100, DRAWING 300, ...

d. Constructing new ways of looking at the process by transforming the visual model to highlight different characteristics.

With most students, it was at about this stage in the investigations that, in my structuring role, I suggested they could modify the procedures so that the initial visual output of a spiral could be changed into other models — e.g. the Line, Bar Graph and Stairs models presented in Chapter 5. So the students constructed separate subprocedures for each of these models which they could alternately use in DRAWING[5] (usually called SPIRAL, BARS[6], LINE, and STAIRS subprocedures). For example, in order to better appreciate the sequence of segments, the bar graph model was used to separate and compare all the segments in the spiral. The line model, which "stretches" the spiral, was mainly used to investigate what happened with the total length of the spiral (the sum of the sequence of segments). The stairs model has the characteristic that, while preserving the distinction between each of the segments or terms of a sequence (unlike the line model), it still connects all of the segments (unlike the bar graph model) giving a feeling for the behaviour of the sum of segments, as does the spiral model yet providing a different perspective than the latter. An important point here is that because each of the different visual representations was produced through the same procedure (DRAWING) the link between these models was obvious: students were aware of how they could transform one model into another and that the different models all represented the same process.

All of the students seemed to recognise the value and potential of using other types of representations, particularly for observing and confirming the behaviour of the process, how long it continues and how many segments are drawn with a predetermined stop condition. For example, Verónica and Consuelo, in their exploration of the "halving" process (the sequence $\{1/2^n\}$) — described in the DRAWING procedure — had been observing the spiral model and noticing that the

---

[5] For example, the original procedure was usually transformed to the following:

```
TO DRAWING :L                 with
IF :L < 1 [STOP]                  TO SPIRAL :L
SPIRAL :L                         FD :L
DRAWING :L * 1/2                  RT 90
END                               END
```

[6] Manuel and Jesús called this model HISTOGRAM.

computer process continued indefinitely. I then intervened, suggesting that instead of a spiral they could look at a line; this was a model which Consuelo recognised had the potential of showing if the segments kept adding up without bounds:

> Consuelo: If we stretch the spiral we can see if it has stopped going forward, if it is not doing more, if it has stopped..

When these students observed that the line had a finite length (i.e. that it converged) Consuelo suggested separating the spiral into bars (the bar graph model) to study how the segments decreased; Verónica added that this representation would be useful for seeing how many segments were being drawn (relative to the stop value in the procedure). By going back and forth between different models, and by constructing and controlling each of the representations through the computer code, the students kept a link between the models which they used to validate and confirm their conclusions on the behaviour of a process.

As illustrated in the example above, with most students it was not necessary for me to suggest all of the models I had pre-designed. Once I had suggested the idea that the spiral could be transformed into different graphical representations, the students themselves developed their own ideas. The two older students, Elvia and Martin, also constructed another type of visual model (see Figure 6.3), which they called the Squares model defined through the following subprocedure:

```
TO SQUARES :L
REPEAT 4 [FD :L RT 90]
RT 90 FD :L LT 90
END
```

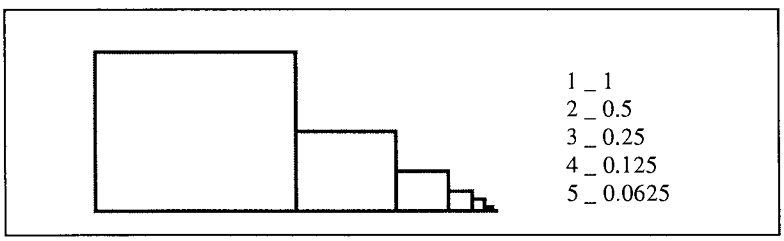


Figure 6.3. The SQUARES model (with numeric output) created by Elvia and Martin; representing here the sequence {1/2^n}.

The possibility to transform visual models of a particular process into other models via the programming code was an important feature of the microworld. The different models became tools for exploring the behaviour and characteristics (e.g. the rate of convergence) of the sequences under study, and students took the initiative to *use* these different models to gather information on the processes being studied.

e. Comparing the process of "taking halves" with other similar processes (e.g. "taking thirds").

Sooner or later, all the students became interested in looking at different processes to that of "taking halves" described in the initial procedure. Most began by modifying the recursive call "DRAWING :L * 1/2" to, for example "DRAWING :L * 1/3" (which only requires changing one digit) thus describing a process of dividing by 3. Modifications such as this allowed the students to engage in a comparative investigation of the different behaviours of different processes. For example, all of the students became aware, *from the unfolding visual representations* (complemented with numeric values) that the sequence produced through the process of dividing by 3, was one which decreased much faster than the previous one (i.e. the segments became smaller more quickly than in the previous case — e.g. see Figures 6.4 and 6.2). The observation of the different visual behaviours prompted students to look for the reason of the differences: they thus became aware of the fact that the process now involved "taking thirds instead of halves" and what this meant, as they re-analysed the code and complemented their comparative explorations with the numeric values of the segments.

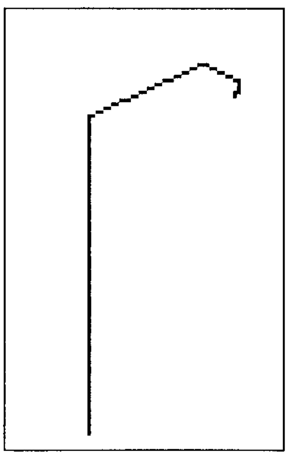


Figure 6.4. Spiral corresponding to the sequence $\{1/3^n\}$, using a 60 degree angle.

2. Modifying the procedure and creating a new symbolisation of the process.

The comparisons between processes highlighted the relationship between the initial input of :L (which also determined the scale) and the values for the segments. In particular, the students realised the importance of keeping that input constant, if they were to make comparisons. As students became aware of this I suggested that the procedures be modified so that the resulting values would be independent of the scale. Making these modifications to the programming approach required a change in the understanding of the process and its relationship with the code: viewing the process in terms of a symbolic formula which could be independent from the computer environment. That is, instead of defining the process in terms of taking a determined value :L (which simultaneously defined the scale of the figure) and operating on that value, the new approach was to use a function which would generate the values of the sequence.

Initially some students (e.g. Victor) had difficulty in conceiving how the change to a sequence independent of the scale could be made, particularly since the original process was described in terms of the original segment. Thus, with some of the students I guided their investigations by suggesting the use of a table (see Table 6.4) to work out a formula: this table acted as a tool which helped the students make the transition to a new type of procedure. It helped them see the scale (represented by :L in the original procedure), as simply a quantifier of the process — each successive value of the process being given by a function[7] ($1/2^N$). Having taken this step, none of the students had difficulty in viewing the process as a sequence of values which could be multiplied by a scale to form a drawing, and using the procedures described in Chapter 5 (see also the Sequence Studies Handouts in Appendix 2): SEQUENCE (which generates a list (sequence) of *n* values from a function — FN), and DRAWSEQUENCE (a modified version of the DRAWING procedure which produces a graphic model of a pre-defined sequence).

[7] As described in Chapter 5, the procedure for this function would be:

```
TO FN :N
OP (1 / POWER 2 :N)
END
```

Arm number	L
The first arm	measures L
2	L/2
3	$L/4 = L/2 \times 1/2 = L/2^2$
4	$L/8 = L/4 \times 1/2 = L/2^3$
5	L/16
6	L/32
7	L/64
N	$1/2^N \times L$

Table 6.4. Table used by Victor and Alejandra for deducing the formula of the sequence generating function $1/2^n$

The new approach allowed students to use different scales to, for instance, look "deeper" into the figure, with the values of the sequence remaining unaffected. The scale thus acted as a magnifying tool that gave the students additional insights into, in particular, the rate of convergence of the sequences under study, while being complemented by the numeric representation (which showed the values that could no longer be perceived on the graph).

3. Comparing sequences of the same type and observing the difference in their rate of convergence through the visual behaviour.

With the modified procedures all of the students engaged in in-depth investigations of different processes of the type $\{1/k^n\}$ and $\{x^n\}$, using all the available tools and graphic models. Among the sequences that the students explored were: $\{1/2^n\}$, $\{1/3^n\}$, $\{1/8^n\}$, $\{0.8^n\}$ (see Figures 6.5 through 6.8), comparing, for example, their corresponding bar graphs. The difference in the *visual* behaviour of the bar graphs (which seemed to form curves) was very apparent, with for instance more "bars" (as opposed to "dots") being visible in sequences which decreased or converged slower, as remarked by Alejandra and most (if not all) of the other students:

Alejandra: There are more bars. It is decreasing but the bars are higher, so it decreases less than the previous ones.

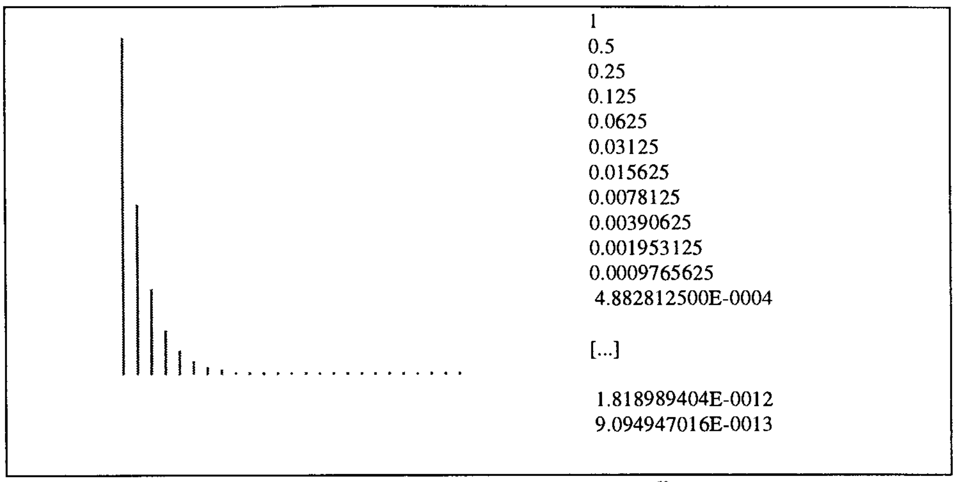


Figure 6.5. Bar graph corresponding to the sequence {1/2^n} with numeric output.

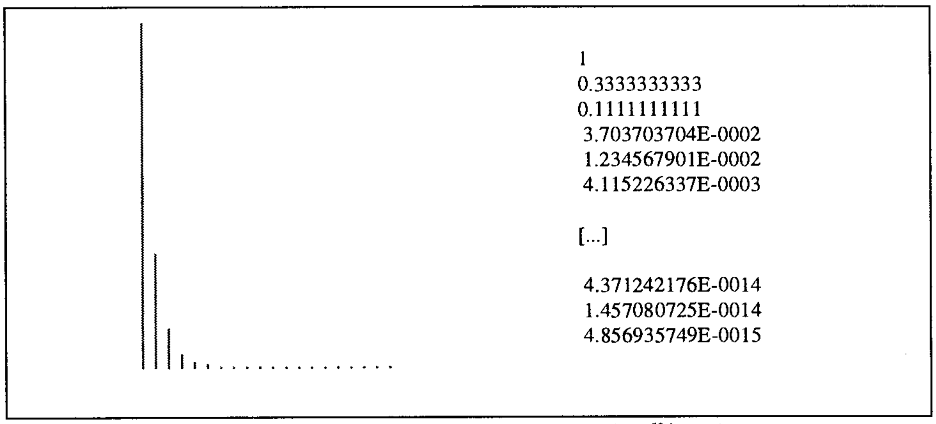


Figure 6.6. Bar graph corresponding to the sequence {1/3^n} with numeric output.

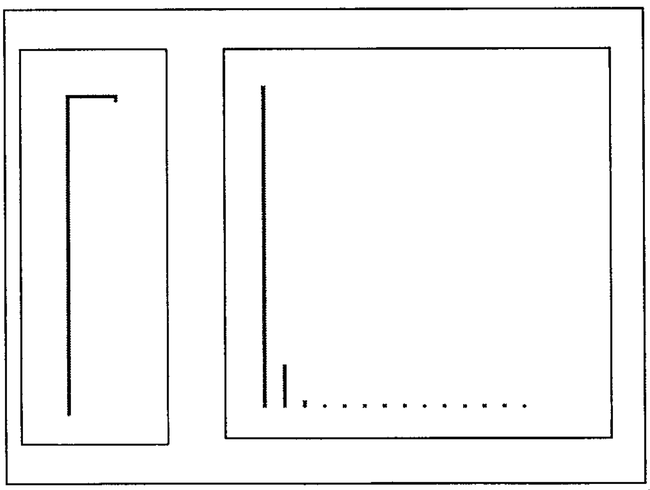


Figure 6.7. Spiral and bar graph models of the sequence {1/8^n}.

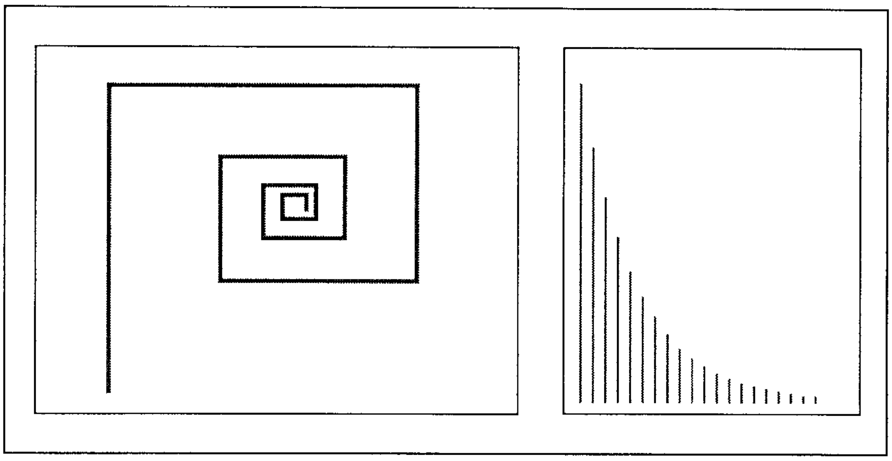


Figure 6.8. Spiral and bar graph corresponding to the sequence $\{(0.8)^n\}$.

4. Exploring other types of functions.

Most students also made changes to the function procedure without necessarily analysing a priori what the procedure would produce or how the resulting sequence would behave, and simply tried it out to see what happened. Below I present some of the sequences explored by the students.

a. Encountering a divergent sequence.

For example, Consuelo had suggested trying a function "that does not divide, that for instance multiplies," so they used[8]: $2 * 5^N$, and looked at the stairs and bar graph models (see Figures 6.9 and 6.10), for which Verónica and Consuelo had to find an adequate (very small) scale as they had not initially realised that they were generating a rapidly increasing sequence which caused the model to soon outgrow the boundaries of the screen. They soon realised from both the visual outputs and the numeric output of the partial sums[9] that in this case there was no limit value for the total length:

> Verónica: This one doesn't have a limit[10].
> Consuelo: It is like it doesn't approach anything, it just goes off...

---

[8] In FN they wrote (2 * POWER 5 :N).

[9] For instance, with a scale of 8, the first three partial sums were "80 480 2480".

[10] It is interesting to note the use of the word *limit* by Verónica, as this was a term I had not used with these students.

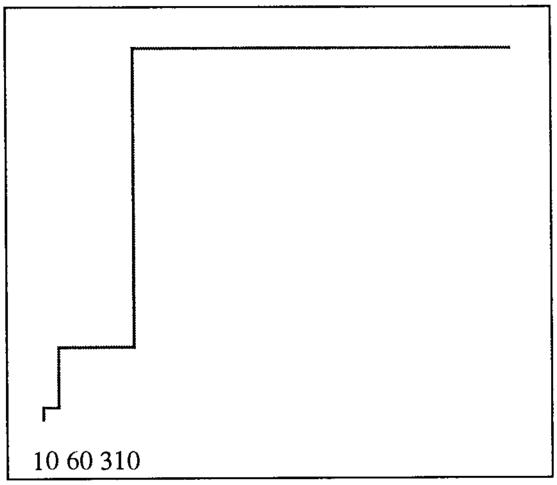


*Figure 6.9. Stairs model of the first 3 terms of the sequence generated by 2 * 5^n, using a scale of 1, with the values of the partial sums*

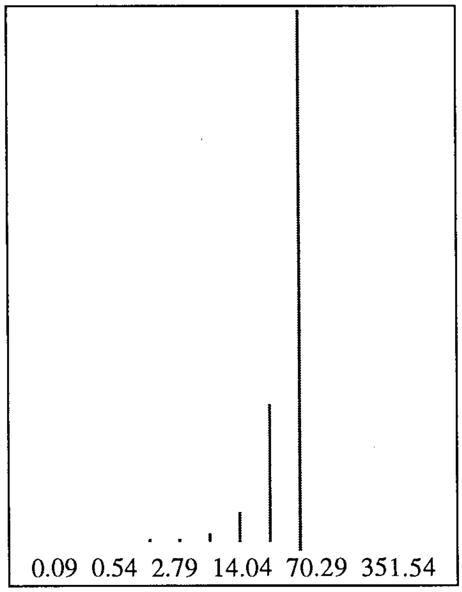


*Figure 6.10. Bars model of the sequence generated by 2 * 5^n, using 0.009 as scale, with the partial sums values.*

b. Looking at the visual representation of a sequence with a constant rate of increase. Making sense of the visual representations by connecting it to the constant growth represented in the symbolic code.

Alejandra and Victor also experimented with changing the code without being aware of the characteristics of the new processes they defined through the code. For example, they changed FN to (:N + 10 * :N) initially looking at the bar graph (Figure 6.11), and later at the spiral model (Figure 6.12); they were surprised by the output and like Consuelo and Verónica in the previous example had to make adjustments to the scale. The *visual* outputs — as well as the numerical values 11, 22, 33, etc.— showed the constant growth of the process, as observed by the students:

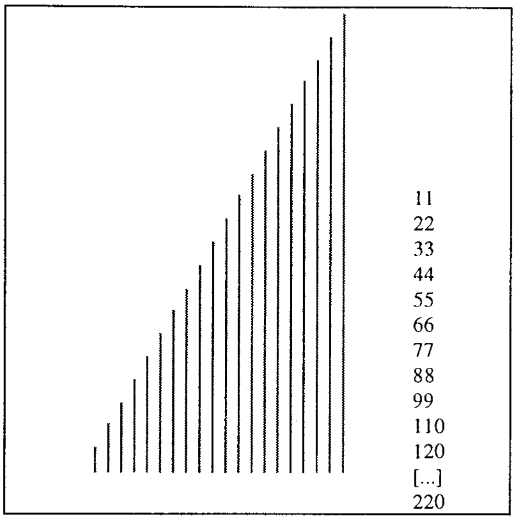


Figure 6.11. Bar graph corresponding to the sequence {n + 10n} (={11n}) with numeric output.

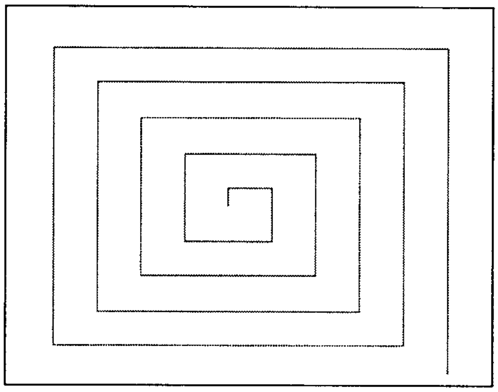


Figure 6.12. Spiral corresponding to the sequence {n + 10n}(={11n}).

Alejandra: The distances between arms, or rather, you could say, between the walls, is constant; that is, the arms that are parallel have the same distance between them.

Victor: Yes, because the growth of the distances is constant, something which in the other spirals we didn't have since each step was a fraction of the previous one.

Thus, by running the procedures they got immediate feedback in the visual/numeric representations which pointed to a characteristic of the sequence that had been overlooked when the students wrote the code, but which could be found in the code. As in other cases, this was a situation which prompted Victor and Alejandra to re-analyse the code and give meaning to the connections between all the elements.

c. Moving to a higher level of complexity: using negative factors, exploring an alternating sequence and making use of the visual dynamism to make sense of the outputs.

Some of the changes in the sequence-generating function were more directed. For example, Alejandra and Victor, by looking at other sequences such as $\{2^n\}$ and $\{(2/3)^n\}$ had coordinated the visual-numeric behaviour of a sequence with the multiplying factor that is present in the symbolic representation of the corresponding process, leading them to the conclusion expressed in the transcript below:

Alejandra: If you multiply by a number larger than 1, it is increasing, and if it is less than 1 then it is decreasing.
And the larger than 1 it becomes the faster it increases.

Victor: The bigger the number the faster it will be.

Alejandra: The closer to 1 it gets the slower it will go. If it is less than 1, it will decrease slowly, and the decrease will be faster the smaller the number gets on that side of 1.

When the students gave the above rule, I intervened — to provoke a reflective process — by asking them if that also applied to negative numbers. This prompted the students to explore using a negative factor and they looked at $\left\{\left(-\frac{2}{3}\right)^n\right\}$[11]. This was a case where they were unable to predict the behaviour. The students' lack of ability to predict the visual output when using a negative factor is hardly surprising: in previous cases such as multiplying by 2/3 each time, it meant taking two thirds of the previous measure at each step; however, when the factor becomes negative the symbolic rule no longer has an obvious link to a process of acting on a *measure* to systematically increase it or decrease it. The connection with a concrete action is no longer evident. This iterative process of multiplying each term by a negative fraction thus involves moving to a higher level of mathematical abstraction, and it is therefore more difficult to (mentally) visualise.

Through the graphical output (a bar graph model; see Figure 6.13) — complemented by the accompanying numerical values — Victor and Alejandra realised that this was an alternating sequence. They tried to make sense of the result by connecting it to the mathematical process where "a negative by a negative will be positive". Their comments also suggest that they were thinking of the process sequentially, since they gave no evidence of analysing the effect of the *power* of (-2/3), but rather seemed to be thinking of "multiplying by (-2/3) each time".

[11] They changed FN to (POWER -(2/3) :N).

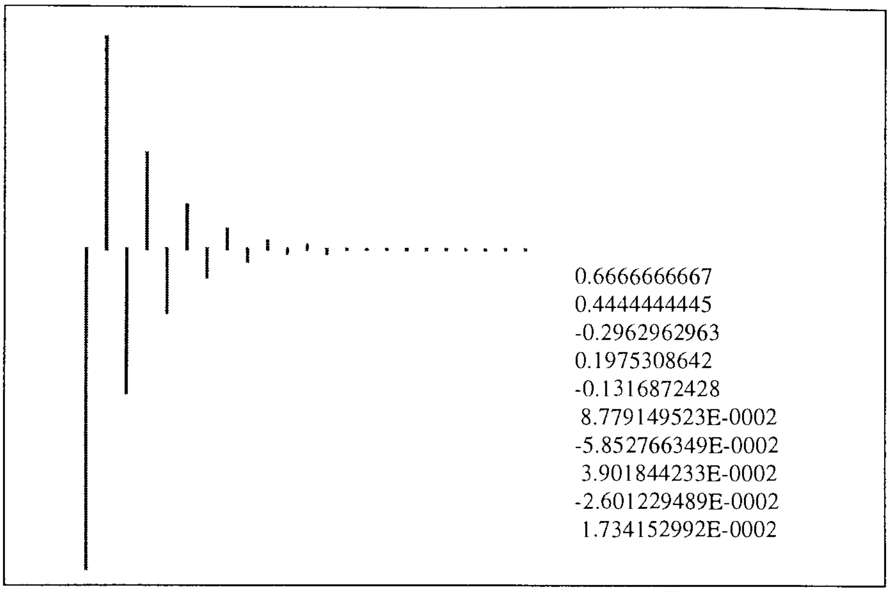


Figure 6.13. Bar graph model corresponding to the sequence $\{(-2/3)^n\}$ with numeric values.

Victor: One is negative and the other positive.
Alejandra: Negative by negative is a positive, so...
Victor: It's the sign rule. So we have one bar going up and the next going down.
Ana: And what happens to the values?
Victor: They get closer to 1, don't they?
Alejandra: Mmm... Let's see it again.
[After running the procedure again with more terms]
Victor: They decrease.
Alejandra: Yes they keep decreasing.

Through the visual output the students were also able to observe that the *measures* (the length of the bars) representing the terms of the sequence decreased. This seemed to allow them to think of the sequence as a decreasing sequence (in *absolute terms*) and to observe its convergence to zero, as Victor pointed out:

Victor: The value of the distance decreases, that is the length decreases, but it is going to decrease in the positive or negative, the sign doesn't matter.

The students wondered what would happen with other types of visual models. Alejandra believed the Spiral model might not be a spiral at all. They were quite surprised when the result was a spiral (see Figure 6.14), which Victor felt did not quite correspond to the idea he got from the bar graph. But by analysing the movements of the turtle they were able to observe that on alternative segments the turtle moved backwards. A connection was then formed between the alternative negative values in the sequence, and the alternative direction of the movements of the turtle:

Victor: It walks backwards! And the spiral is different from the others which started going up and to the right: this one is downward then to the left.

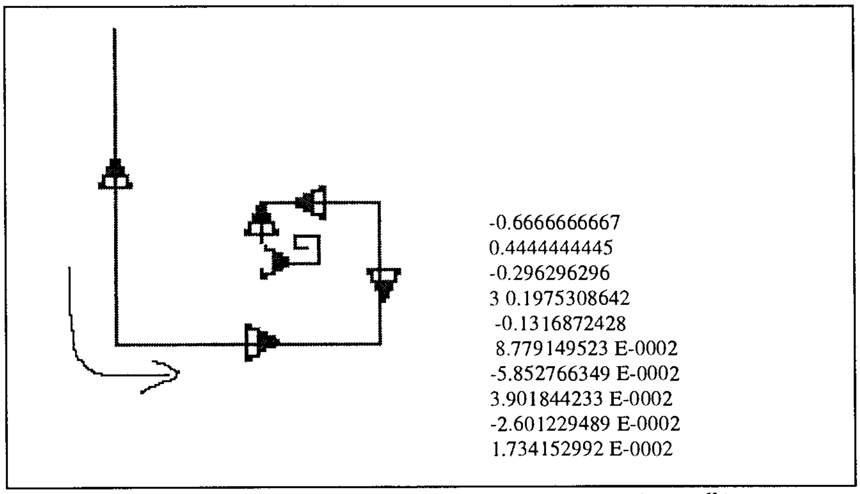


Figure 6.14. Spiral model corresponding to the alternating sequence {$(-2/3)^n$}, with numeric output.

The Stairs model also produced a spiral (see Figure 6.15), and Victor in particular was again able to make sense of the phenomena on the screen by careful observation of the *movements* of the turtle, and connecting the procedural code with the numerical values.

Victor: What happens is that instead of moving forward, half the time it moves backwards... and we get like a spiral.

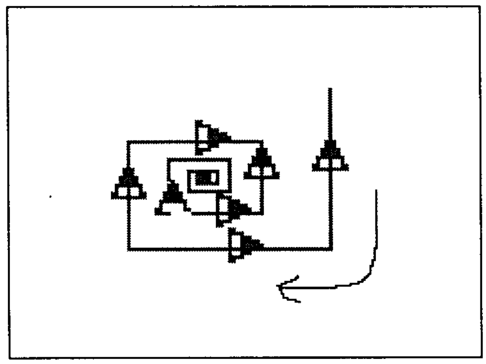


Figure 6.15. Spiral produced using the STAIRS procedure with the alternating sequence {$(-2/3)^n$}, (same scale as previous figure).

The above episode is very illustrative of how the *dynamism* of the computer was a useful tool for making sense of the behaviour of the process. This dynamism allowed the *process* (the sequence) to be viewed as it unfolded in a sequenced manner and as it was being generated. That is, the *construction process* of a sequence could be observed. Furthermore, the visual dynamism was complemented by the simultaneous unfolding of numeric values: each segment (of a sequence) would simultaneously be produced with its corresponding value, thus highlighting the link between the geometric figure and the numerical values.

5. Investigating the behaviour of series.

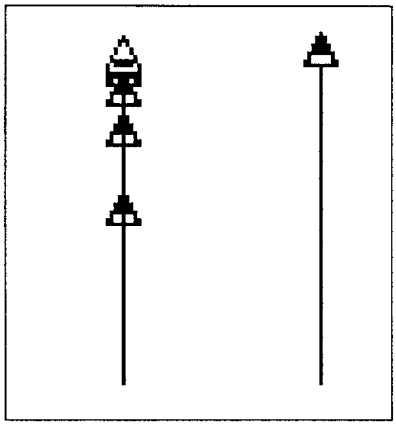


Figure 6.16. The LINE model for $\{L/2^n\}$. The left hand side of the figure depicts the movements of the turtle. The right hand side shows the end result.

An important part of the sequence investigations consisted in the study of the behaviour of the corresponding series. Initially this began with investigations into "the total length of the spiral" or "the sum of all the segments or bars". From the beginning, most students suggested looking at the LINE procedure (see Figure 6.16) which adds up all the bars (it represents the (partial) sum of the sequence). As will be discussed in Chapter 7, many students expected the line to grow in length more and more, as more segments were added. But in the first case the students investigated — that of the series corresponding to the sequence $\{1/2^n\}$ — they all repeatedly observed in the line model how the turtle began vibrating in apparently the same spot with the line not extending after a certain point. This was a surprise to many students (particularly for Verónica and Consuelo), and all the students explained that although the turtle seemed to have stopped, in reality it was still walking imperceptible amounts which were represented by the vibrations of the turtle:

> Verónica: The turtle is blinking..., well, walking. It's because what it is walking is very very small. It keeps walking there, not in the same place, although it looks as if it is staying in the same place.

Some students such as Consuelo, suggested investigating the observed phenomenon numerically by writing a procedure for computing the sum of the "bars" — the procedure SUML, described in Chapter 5 (with other students I suggested this myself). Consuelo was also among the students who suggested writing a procedure — corresponding to the procedure PARTIALSUMS described in Chapter 5 — for generating, in her own words, "a list of all the *partial sums*[12]" to investigate the growth of the sums. An example of the lists of partial sums the students produced is given in

[12] It was interesting that Consuelo actually used the term *partial sums*, as I had not introduced this terminology to these young students.

Table 6.5. For convergent cases like that of the series $\sum \frac{1}{2^n}$, these numeric outputs played an important part in showing the existence and value of the limit, as will be further discussed in Chapter 7.

50 75 87.5 93.75 96.875 98.4375 99.21875 99.609375 99.8046875 99.90234375 99.95117188
99.97558594 99.98779297 99.99389648 99.99694824 99.99847412 99.99923706 99.99961853
99.99980927 99.99990463 99.99995232 99.99997616 99.99998808 99.99999404 99.99999702
99.99999851 99.99999926 99.99999963 99.99999981 99.99999991 99.99999995 99.99999998
99.99999999 99.99999999 100 100 100 100 100 100 100 100 100 100 100 100 100 100
100 100 100 100 100 100 100 100 100 100 100 100 100 100 100 100 100 100 100
100 100 100 100 100 100 100 100 100 100 100 100 100 100 100 100 100 100 100
100 100 100 100 100 100 100 100 100 100 100 100 100 100

Table 6.5. List of the first 100 partial sums of the values of the segments corresponding to the sequence {1/2^n}, with a scale of 100, obtained through the procedure PARTIALSUMS.

As students became more familiar with the limiting behaviour of many of the series, some of them (the pairs Elvia/Martin and Manuel/Jesús) used "comparison lines" to measure the length of the line model. For example, if they predicted that the limit of a series would be 100 taking into account the scale, they would draw a line of length 100 and then generate the line model next to it. They used this to complement the numeric values, and confirm the value of the limit, *through the visual model*, making sure that the line model did not exceed the length of the comparison line.

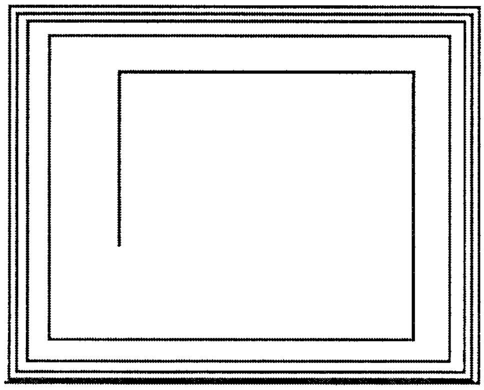
The visual investigations of the series were not limited to the line model. Students got a sense of the behaviour of the series from the other visual representations like the spiral and the stairs, both of which were very useful in the investigation of the divergence of the harmonic series — see Chapter 7. Furthermore, most students (all except Verónica and Consuelo) generated bar graph models of the sequence of partial sums. The pairs Alejandra/Victor[13] and Elvia/Martin[14] also looked at the other models for the partial sums, such as the spiral model. The students who tried this last model found it useful, as the spiral of the partial sums of convergent series formed square frames ("framed spirals") that illustrated the convergent behaviour (e.g. see Figure 6.17). What is interesting from the approach of generating visual models of the list of partial sums, is that students conceptualised the partial

[13] Alejandra and Victor first generated a spiral model of the partial sums accidentally — when they forgot to change the model inside the DRAWSEQUENCE procedure — but then found it very useful.

[14] Elvia and Martin went as far as writing a small procedure — GRAPHSUM — for drawing a model of the partial sums:

```
TO GRAPHSUM :N :SCALE
DRAWSEQUENCE PARTIALSUMS SEQUENCE :N :SCALE
END
```

sums as a sequence (i.e. the series were seen as the limit of a sequence of partial sums), using the list of partial sums as input to the procedure DRAWSEQUENCE (described in Chapter 5).



1
1.353553391
1.54600348
1.67100348
1.760446199
1.828487581
1.882482506
1.92667668
1.963713717
1.995336493
2.022746616
2.046802877
2.0681375
2.087227588
2.104440848
2.120065848
2.134332649
2.147427219
2.159501731
2.170682071

Figure 6.17. Spiral model of the partial sums of the sequence $\{1/n^{1.5}\}$ *(with numeric values). Alejandra and Victor used the "frame" behaviour of this model to speculate that the series would have a limit.*

6. The Harmonic sequence and series.

a. Observing a new behaviour in the visual models of the Harmonic sequence.

An important part of the microworld explorations was the investigation of the Harmonic sequence {1/n} and series, which I proposed to the students (although some students — Verónica and Consuelo — came up on their own with the idea of exploring this sequence). Most students followed the process of looking at each of the different models, beginning with the spiral model. All of the students noticed that the behaviour of the spiral for this sequence was different from other cases they had studied: as Manuel and most students observed, there was a "hole" in the centre, which persisted even though they increased the number of terms, only starting to perceptually disappear with 100 terms; Manuel and Jesús conjectured that this hole would (theoretically) remain even at infinity. All of the students also noticed how the space between the walls of the spiral decreased (see Figure 6.18) as the turtle approached the centre; Verónica observed that she thought the turtle would "not be able to reach the centre of the square" because that space progressively became thinner, so the turtle would tend to stay towards the outside; Victor described this as an "avoidance of the centre".

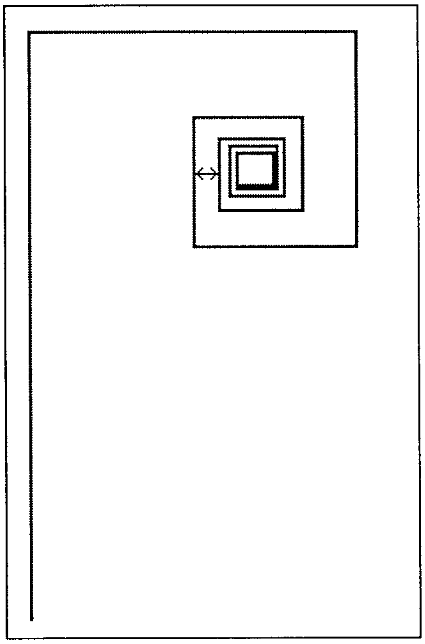


Figure 6.18. Spiral model for the sequence {1/n}.

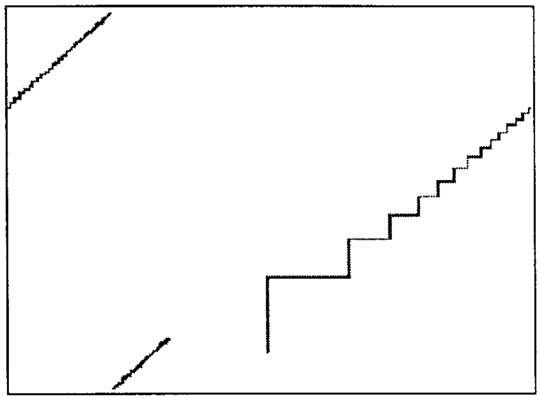


Figure 6.19. Stairs model (wrapping around the screen) corresponding to the Harmonic sequence. (Different scale than Figure 6.18).

In the Stairs model (see Figure 6.19), students observed how the length did not stop growing (wrapping around the screen when given enough terms) no matter how many terms were used, and unlike other sequences they had studied. This was indicative of the slow convergence of this sequence — which was also confirmed through the histogram model (Figure 6.20) where the bars did not become "points" as the sequence progressed — and probable divergence of the corresponding series. Some students, like Manuel and Jesús, found meaning for this slow convergence explaining that in this case the denominator grew much slower than in the other sequences studied.

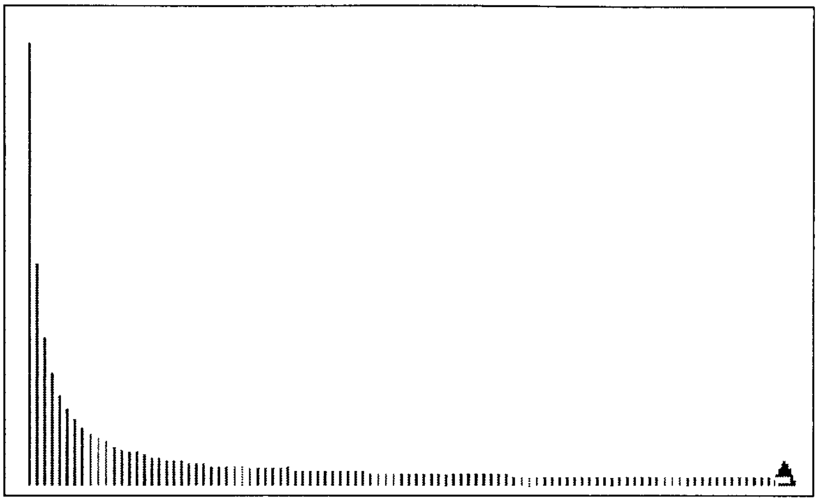


Figure 6.20. Bar graph model of the first 100 terms of the Harmonic sequence {1/n}.

	5.555555556	2.666666666	1.724137931	1.265822785
100	5.405405406	2.631578948	1.709401709	1.257861635
66.66666666	5.263157894	2.597402598	1.694915254	1.25
50	5.128205128	2.564102564	1.680672269	1.242236025
40	5	2.53164557	1.666666667	1.234567901
33.33333334	------------------------	2.5	1.652892562	1.226993865
28.57142858	4.87804878	2.469135802	1.639344262	1.219512195
25	4.761904762	2.43902439	1.62601626	1.212121212
22.22222222	4.65116279	2.409638554	1.612903226	1.204819277
20	4.545454546	2.38095238	1.6	1.19760479
18.18181818	4.444444444	2.352941176	1.587301587	1.19047619
16.66666667	4.347826086	2.325581396	1.57480315	1.183431953
15.38461538	4.255319148	2.298850574	1.5625	1.176470588
14.28571429	4.166666666	2.272727272	1.550387597	1.169590643
13.33333333	4.081632654	2.247191012	1.538461538	1.162790698
12.5	4	2.222222222	1.526717557	1.156069364
------------------------	------------------------	2.197802198	1.515151515	1.149425287
11.76470588	3.921568628	2.173913044	1.503759398	1.142857143
11.11111111	3.846153846	2.150537634	1.492537313	1.136363636
------------------------	3.773584906	2.127659574	1.481481481	1.129943503
10.52631579	3.703703704	2.105263158	1.470588235	1.123595506
10	3.636363636	2.083333334	1.459854015	1.117318436
------------------------	3.571428572	2.06185567	1.449275362	1.111111111
9.523809524	3.50877193	2.040816326	1.438848921	1.104972376
9.09090909	3.448275862	2.02020202	1.428571429	1.098901099
------------------------	3.389830508	2	1.418439716	1.092896175
8.695652174	3.333333334	------------------------	1.408450704	1.086956522
8.333333334	3.278688524	1.98019802	1.398601399	1.081081081
8	3.225806452	1.960784314	1.388888889	1.075268817
------------------------	3.174603174	1.941747573	1.379310345	1.069518717
7.692307692	3.125	1.923076923	1.369863014	1.063829787
7.407407408	3.076923076	1.904761905	1.360544218	1.058201058
7.142857142	3.03030303	1.886792453	1.351351351	1.052631579
------------------------	------------------------	1.869158879	1.342281879	1.047120419
6.896551724	2.985074626	1.851851852	1.333333333	1.041666667
6.666666666	2.94117647	1.834862385	1.324503311	1.03626943
6.451612904	2.898550724	1.818181818	1.315789474	1.030927835
6.25	2.857142858	1.801801802	1.307189542	1.025641026
6.06060606	2.816901408	1.785714286	1.298701299	1.020408163
------------------------	2.777777778	1.769911504	1.290322581	1.015228426
5.882352942	2.739726028	1.754385965	1.282051282	1.01010101
5.714285714	2.702702702	1.739130435	1.27388535	1.005025126
				1

Table 6.6. Output values of the first 200 bar segments with a scale of 200 (printed simultaneously to the corresponding bar segments by the BARS procedure). Consuelo and Verónica observed the increase in the number of values in each integer range as the list progressed, and linked this behaviour with the slow convergence of the sequence.

Verónica and Consuelo coordinated the behaviour in the bar graph of the sequence with the lists of values of the segments of the sequence (given in Table 6.6), noticing the slow convergence by focusing on the number of "bars" or values in each range: Consuelo observed that there are more bars in the 1 value range, than for the 2 value range, and explained that "for every number [range] it extends more and more: each time there are more terms of one value, there are more bars [of the same size]".

The observation of the behaviour of the sequence led students to speculate that the corresponding series (the total length or sum of the bars) would not have a limit; they explored this using the line model, with bar graphs (see Figure 6.21) of the partial sums (an idea which *all* the pairs of students thought of on their own and all found very useful, as they seemed to be able to coordinate the behaviour of the sequence as represented in the bar graph, with that of the corresponding series, also as represented in the bar graph), and through numeric explorations.

The discovery of the divergence of these series will be further described and analysed in Chapter 7, although it is interesting to note the combined methods and tools that the students used. Most students first reached the conclusion that the series did not have a limit by running the procedure several times varying the model, the scale, and the number of terms. The numerical explorations then became an important complementary tool: for instance, by generating lists of partial sums (see Table 6.7) they could observe how the values kept increasing.

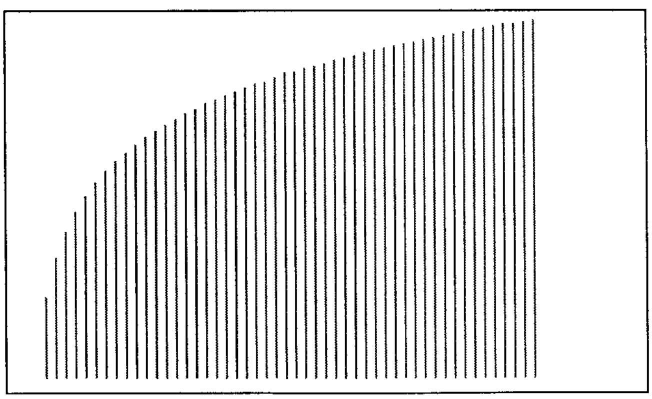


Figure 6.21. Bar graph model of the partial sums of the Harmonic sequence.

1	4.174559197	4.846921265
1.5	4.201586224	4.860810153
1.833333333	4.227902013	4.874508783
2.083333333	4.253543038	4.888022297
2.283333333	4.278543038	4.90135563
2.45	4.302933282	4.914513525
2.592857143	4.326742807	4.927500538
2.717857143	4.349998621	4.940321051
2.828968254	4.372725893	4.952979279
2.928968254	4.394948115	4.965479279
3.019877345	4.416687246	4.977824958
3.103210678	4.437963842	4.99002008
3.180133755	4.458797175	5.002068273
3.251562326	4.479205338	5.013973035
3.318228993	4.499205338	5.02573774
3.380728993	4.518813181	5.037365648
3.439552522	4.53804395	5.048859901
3.495108078	4.556911875	5.060223537
3.547739657	4.575430394	5.071459492
3.597739657	4.593612212	5.082570603
3.645358705	4.611469355	5.093559614
3.69081325	4.629013214	5.104429179
3.734291511	4.646254594	5.115181867
3.775958178	4.663203746	5.125820166
3.815958178	4.679870413	5.136346481
3.854419716	4.696263855	5.146763148
3.891456753	4.712392887	5.157072427
3.927171039	4.728265903	5.167276509
3.961653798	4.743890903	5.177377519
3.994987131	4.759275519	5.187377519
4.027245196	4.774427034	
4.058495196	4.789352407	
4.088798226	4.80405829	
4.11820999	4.818551043	
4.146781419	4.832836758	

Table 6.7. List of the first 100 values of the partial sums of the Harmonic Series.

b. Exploring other sequences of the type $\left\{\frac{1}{n^k}\right\}$ *and their corresponding series.*

After students had explored the Harmonic sequence and series, I suggested they looked at other sequences of the type $\left\{\frac{1}{n^k}\right\}$. Most students (except Elvia and Martin who began by looking at $\left\{\frac{1}{n^{1.1}}\right\}$) first investigated the sequence $\left\{\frac{1}{n^2}\right\}$.

(i) Discovering the convergence of the series $\sum \frac{1}{n^2}$.

In the spiral model of this sequence (see Figure 6.22) the students observed how quickly the spiral closed into centre. This was a model that Manuel and Jesús extensively explored; they felt that this spiral would quickly converge to the centre and be, in their words, a "closed cone", so they investigated this by producing many spirals through opening up the inner angle (see Figure 6.23) and enhancing the scale (see Figure 6.24). After these visual explorations, Manuel and Jesús became convinced that the corresponding series would have a limit.

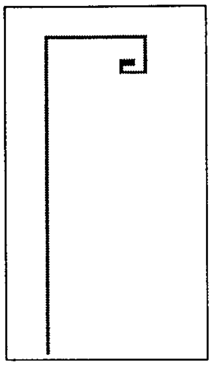


Figure 6.22. Spiral model corresponding to the sequence {1/n²}

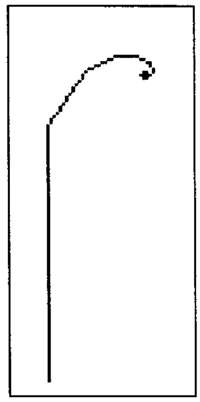


Figure 6.23. Spiral corresponding to the sequence {1/n²} with turning angle 30° and scale of 100.

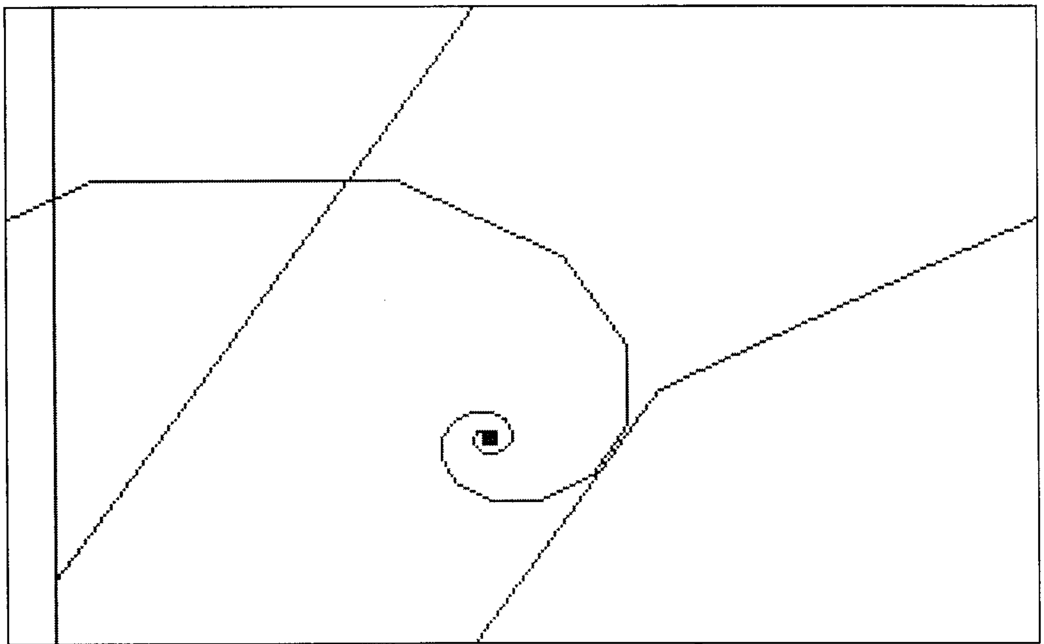


Figure 6.24. Spiral corresponding to the same sequence {1/n²} with a larger scale of 1550 (wrapping around the screen).

All of the students were surprised by the fast rate of decrease of this sequence which they did not expect; this decrease was particularly evident in the bar graph model (Figure 6.25), but was also reflected in the stairs model (Figure 6.26) where the added steps became so small that the stairs stopped extending (as was also observed in the line model). Some students, like Jesús above, took this visual behaviour as indication that the corresponding series converged. In fact, all three pairs Manuel/Jesús, Ver6nica/Consuelo and Elvia/Martin, coincidentally (as it was something I did not suggest in any of the three cases), although with slightly different approaches, decided to compare line models of the partial sums of this sequence (produced by gradually increasing the number of terms) with a fixed line which they believed represented an unreachable boundary for the sums. This investigative process produced a bar graph of the partial sums, and showed how they tended to become constant. It was a visual method for the students to verify that the sums did have a limit.

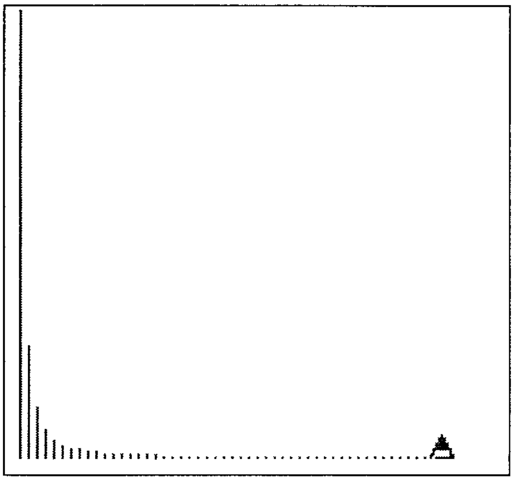


Figure 6.25. Bar graph model of the first 50 terms of the sequence $\{1/n^2\}$.

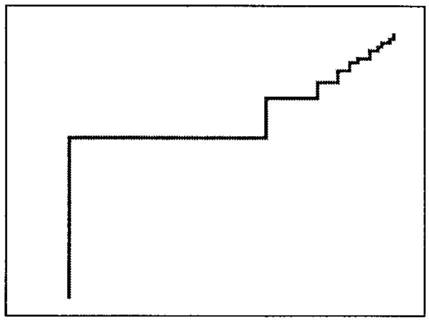


Figure 6.26. Stairs model corresponding to the sequence $\{1/n^2\}$

But the visual behaviour was not enough for all students to believe in the existence of a limit for the sum: Alejandra and Victor, perhaps influenced by the previous study of the Harmonic sequence, disregarded the visual information (particularly that of the stairs and line models where the turtle "got stuck"), and predicted that although the rate of increase would be very slow, the sums of $\left\{\frac{1}{n^2}\right\}$ would tend to be infinite. As will be discussed in Chapter 7, their prediction seemed to reflect a resurfacing of the intuition that if a process is infinite, then the result of the process will also be infinite.

To investigate with more precision the behaviour of the sequence and the corresponding series, as well as the value of the limit most students felt existed, the students carried out extensive numeric investigations (e.g. by using the output of values of the segments, or by printing lists of values of the sequence[15]). and filled out tables of values (Tables 6.8 and 6.9); they all observed that the growth of the series slowed down and settled around 1.64...[16]. Victor and Alejandra initially thought that

[15] E.g. by typing "PR SEQUENCE 100".

[16] The series $\sum \frac{1}{n^2}$ actually converge to $\frac{\pi^2}{6} = 1.64493...$

the sums would eventually go over 1.65 — as they were still dominated by the idea that the sum should grow infinitely since it was an infinite process — and they extensively tested this. But by computing values up to 1500 terms (for which they got 1.644267616), *all* of the students became fairly convinced that the series would have a limit strictly less than 1.65. Consuelo and Verónica for example, concluded that the sums definitely had a limit with value probably less than 1.65, a value they considered "very difficult" to reach because the segments (of the sequence) became very very small. Victor and Alejandra finally also accepted the existence of a limit: As had happened in the earlier sessions (with Verónica and Consuelo in particular), Victor and Alejandra were able to coordinate the infinitude of the process with its convergent behaviour, by using the numerical decimal structure (as is explained and illustrated in Chapter 7).

f = 1/POWER :N 2		Scale: 200
Terms	Last bar	Size without scale
50	0.08	0.0004
100	0.02	0.0001
200	0.005	0.000025

Table 6.8. Verónica and Consuelo's table of the sequence $\{1/n^2\}$.

f = 1/POWER :N 2	
Terms	Sum
25	1.605723404
50	1.625232734
100	1.6349839
500	1.642936066

Table 6.9. Verónica and Consuelo's table of some partial sums of $\{1/n^2\}$.

(ii) Constructing a generalisation for the behaviour of series of the type $\sum\frac{1}{n^k}$.

Having discovered that the series of the sequence $\left\{\frac{1}{n^2}\right\}$ converged but that the harmonic series diverged, all of the students became interested in looking at sequences (and series) of the type $\frac{1}{n^k}$, with $k>1$. Most students felt there was a transition point where the series became convergent. Consuelo and Verónica in particular, predicted that for k>1 the turtle would "get stuck" and that "it would have a limit", with

Consuelo explaining she thought only the sums of {1/n} did not have a limit, arguing that if n was raised to a power, no matter how small, the sums *would* have a limit.

Among the sequences that the students explored were $\left\{\frac{1}{n^{1.1}}\right\}$, $\left\{\frac{1}{n^{1.2}}\right\}$, $\left\{\frac{1}{n^{1.5}}\right\}$, comparing the behaviour of their visual models with those of the Harmonic series, and later filling out comparative tables of values (e.g. see Table 6.10), to analyse the behaviour of the series. Initially students like Manuel and Jesús thought that for example the spiral model of $\left\{\frac{1}{n^{1.1}}\right\}$ (see Figure 6.27) did not look much different from that of the Harmonic sequence. By re-generating this model varying the scale and the inner angle, they noticed that the walls were not as close as in the case of the Harmonic sequence, and that the segments became smaller faster, as they confirmed through the bar graph (see Figure 6.28), where they noticed that the terms decreased much quicker than in the sequence $\sum\frac{1}{n^2}$:

Manuel:	Yes, it gets smaller more quickly...
Jesús:	Yes, in comparison with the other one [the bar graph of $\left\{\frac{1}{n}\right\}$] which was higher around here [pointing].
Manuel:	Yes, exactly. The other one looked like an asymptote.
Jesús:	And this one doesn't. This one even looks diagonally symmetrical: we could fold it and it would coincide; but the other didn't.

Manuel and Jesús concluded that in this case the spiral would "close-up" at infinity[17], like a cone which for them was indicative that the sums would probably be convergent.

[17] Manuel: If there were infinite terms, it would be a cone.

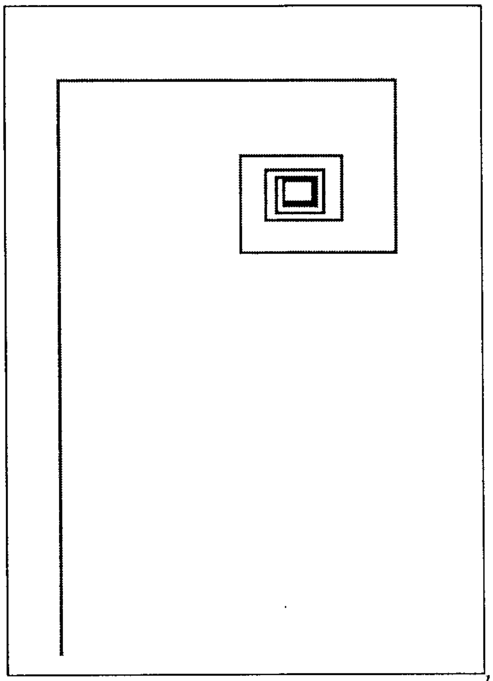


Figure 6.27. Spiral model for the sequence $\{1/n^{1.1}\}$.

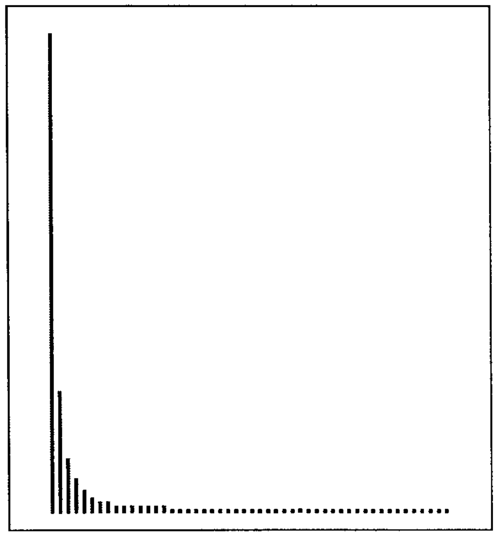


Figure 6.28. Bar graph of the sequence $\{1/n^{1.1}\}$.

x	$\sum_{N=1}^{x}\frac{1}{N}$	$\sum_{N=1}^{x}\frac{1}{N^{1.1}}$	$\sum_{N=1}^{x}\frac{1}{N^{2}}$
100	5.187377519	4.27802402	1.634983903
200	5.87803095	4.698878679	1.6399465
500	6.792823435	5.213343642	1.642936004
800		5.459802537	
1000	7.485470865		1.643934...
1500			1.644267616...

Table 6.10. Comparative table for the numeric investigation of series of the type $\sum\frac{1}{N^k}$.

The combined investigations of the various visual and numeric representations highlighted to the students the importance of the *rate of convergence* for the behaviour of the series, and allowed them to feel confident that for $k>1$ the series of the type $\sum\frac{1}{N^k}$ would have a limit.

Part B. Fractal studies.

As explained in Chapter 5, the microworld activities included explorations of some fractals such as the Koch curve and snowflake, and the Sierpinski triangle. I began the activity by showing the students the first three levels of the Koch curve on a blackboard or paper, and explaining how each new level is derived from the previous one. The students themselves then analysed how to write the procedure for this process.

1. The Koch curve

a. Writing the procedure for the Koch curve: making sense of the self-similarity of the figure and linking it to the recursive characteristic of the procedure.

Most students began by writing a procedure that would generate the "peak" representing the first level of the Koch curve (Figure 6.24). Most students recognised the recursive (self-similar) structure of the fractal figure, and suggested having the procedure call itself in order to draw the following stages of the curve: for example, Verónica suggested replacing each of the "FD lines" with a recursive call because in the figure each segment was to be replaced with a "PEAK".

```
TO PEAK :L
FD :L / 3
LT 60
FD :L / 3
RT 120
FD :L / 3
LT 60
FD :L / 3
END
```

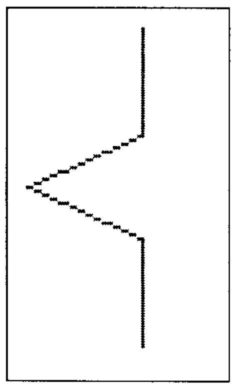


Figure 6.24. First stage of the Koch curve.

Some of the students took the approach of writing a second procedure — PEAKS — which would call PEAKY:

```
TO PEAKS :L
PEAKY :L / 3
LT 60
PEAKY :L / 3
RT 120
PEAKY :L / 3
LT 60
PEAKY :L / 3
END
```

Thus, by typing PEAKS 100, the next stage in the construction process of the Koch curve was created (see left part of Figure 6.25) which could also be seen as superimposed on the previous stage of the curve (right part of Figure 6.25).

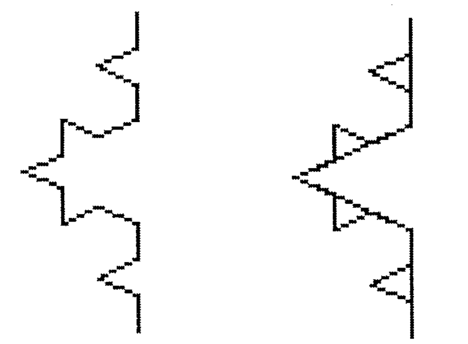


Figure 6.25. Second stage of the Koch curve, with its superimposition on the previous stage.

The structure of this new procedure is identical to the original procedure (PEAK) it calls. The students realised that they could write a new procedure with the same structure for each stage of the geometrical process, by using the same structure and calling the procedure for the previous stage. By noticing this similarity in the procedures the students confirmed that a single (recursive) procedure could be used: (e.g. by having PEAKS calling PEAKS).

This construction process for a recursive procedure is not new — Harvey (1985), for example, calls it the "combining method" — but what is interesting is that this iterative construction process (i.e. writing a series of procedures each identical to the previous one) reflects the self-referral geometrical construction process. Furthermore, this method involved giving a symbolic definition (through the code) of the process by defining the stage *n* in terms of the stage *n-1*.

Eventually all the students constructed a procedure for the Koch curve which had the appropriate (recursive) structure (see Table 6.11), even if it lacked the stop condition with a drawing (FD :L) instruction. By running it they would soon realise the need for these instructions (as without them nothing would happen and the procedure would enter an infinite loop).

```
TO PEAK :L          TO PEAK :L
                                  <—>IF :L < 1 [FD :L  STOP]
FD :L / 3   <—>     PEAK :L/3
LT 60               LT 60
FD :L / 3   <—>     PEAK :L/3
RT 120              RT 120
FD :L / 3   <—>     PEAK :L/3
LT 60               LT 60
FD :L / 3   <—>     PEAK :L/3
END                 END
```

Table 6.11. Construction of the Koch curve procedure.

It is interesting that when the students ran the completed procedure, most were surprised by the depth of the level and intricacy of the figure they obtained (see Figure 6.26). Even though most students had been theoretically aware and had predicted that each part of the figure would contain a similar figure to the whole, it was clear that some had not grasped the full import of the recursive / self-similar characteristic. The visual result served to reiterate what could be seen from the code: that, in Victor's words "PEAKS calls PEAKS, so it is calling the same figure", that each part is a repetition of the same: the little peaks are just like the big peak (the whole figure), so the figure would endlessly be full of "little peaks". Or, as Consuelo expressed it: "it is supposed to be a big one, with smaller ones here, and smaller ones here..., everything is a third by a third, by a third..."

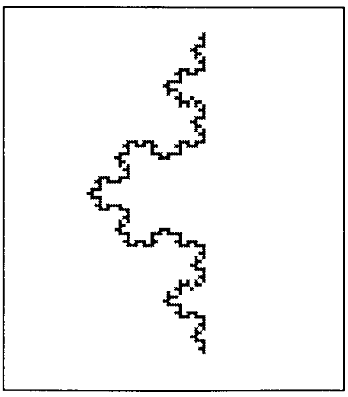


Figure 6.26. The Koch curve.

b. Explorations into the perimeter of the Koch curve

Once the students had written a procedure for the Koch curve, they began investigating the length of that curve, i.e. how much the turtle walked when drawing that curve. Verónica and Consuelo first defined that length as follows:

Verónica: A third of :L, plus a third, plus a third....
Consuelo: Or rather, a third, plus a third of a third, plus a third of a third of a third...

At this stage students realised they needed to be able to determine the level of the figure or how many segments had been drawn, in order to determine how much the turtle had walked, which was needed in order to carry out a methodical analysis of the exact measure of that perimeter. The procedure was therefore modified to include a variable :N for the level:

```
TO PEAK :L :N
IF :N = 1 [FD :L STOP]
PEAK :L / 3  :N - 1
LT 60
PEAK :L / 3  :N - 1
RT 120
PEAK :L / 3  :N - 1
LT 60
PEAK :L / 3  :N - 1
END
```

Some students, like Consuelo, recognised in the code the process of "taking thirds" (which they had studied in the sequence activities) and concluded that the length of each segment was given by $\frac{1}{3^{:L}}$, a value which approached zero as :L increased:

Consuelo: They will be a third of each, so it is 1 over 3 to the Lth power.
Ana: And what would happen if the level, that is, L, is very big?
Consuelo: It's going to be very small, it's going to reach a limit, zero... No, it will not reach zero. It will be 0.000...9 or 0.0000...1...

At this stage, some of the students (Alejandra and the pair Verónica/Consuelo) believed that the total length of the curve would become almost constant, with the prevailing conception being that "what is added is too small", although there were three influential factors here: (i) each of the segments added is very small; (ii) the figure is *visually* invariant after a certain stage; and (iii) the entire figure is bounded in a finite area.

Consuelo: It is going to grow little by little until it reaches a point where it keeps growing but so little, that although it [the growth] is not zero, it [the length] will be the same.

With the other students the prevailing conception was that if the number of segments added tended to be infinite, then the length of the curve would tend to have infinite length. However, with Manuel and Jesús a paradox arose when they reflected on the idea of an infinite length made up of segments with length measure zero. The episode with Manuel and Jesús will be discussed in more detail in Chapter 7.

With all the students, the discovery and acceptance of what happened to the perimeter would come after extensive numerical explorations. All the students constructed tables of values such as Table 6.12, analysing what happened to the length at each level. Through the visual figures the students were able to observe the numbers of segments in each case and noticed how each segment was being replaced by four new segments (which eventually led to the conclusion that the number of segments was a power of 4). Then, by working through the table the students gradually constructed generalisations for the number of segments (4^{n-1}) and the size of those segments ($L/3^{n-1}$) in function of the level *n*.

For L = 100 Level	Side of each segment	Number of segments	Total distance (perimeter)	= Total
1	100	1	100	100
2	100/3	4	(100/3) * 4	$133.3\bar{3}$
3	100/9 (3^2)	4 x 4 = 16	(100/9) * 16	$177.7\bar{7}$
4	100/27 (3^3)	16 x 4 = 4^3= 64	(100/27) * 64	$237.\overline{037}$
5	100/81	256	(100/81) * 256	316.0493827
N	$\frac{100}{3^{N-1}}$	4^{N-1}	——	
	$\frac{L}{3^{N-1}}$		——	
100			233848680765595.64783	

Table 6.12. Table used by the students in their study of the Koch curve.

The students then translated the formula for the length of the perimeter into a procedure (PERIMETER[18]) which allowed them to calculate the perimeter of the

18
```
TO PERIMETER :L :N
OP ( POWER ( 4 / 3 ) ( :N - 1 ) ) * :L
END
```

curve for any level (:N). Using this procedure the students were able to extend the numeric investigations which allowed them to confirm or discover the divergence of the length (at the 100th level for example the value of the perimeter was very, very large, as shown on the table).

c. Explorations of the Koch Snowflake

The investigation of the area of the Koch snowflake[19] which followed was carried out in a similar way to that of the perimeter, through two complementary methods and tools which helped discern each of the elements involved in the process structure their relationships and the way they progressed; these were: (i) repeated visual observations gradually increasing the level (see Figure 6.27) — including overlapping several levels in one drawing (Figure 6.28) — and (ii) the use of a table (Table 6.13).

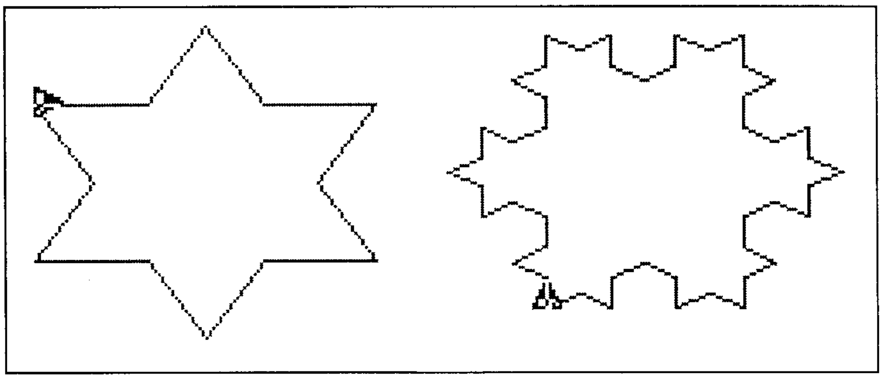


Figure 6.27. Levels 1 and 2 in the construction of the Koch Snowflake.

[19] The snowflake was produced using the procedure:

```
TO SNOWFLAKE :S :L
REPEAT 3 [PEAK :S  :L   RT 120]
END
```

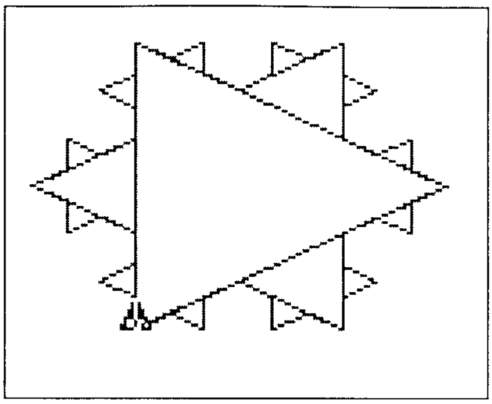


Figure 6.28. Overlapping levels (1, 2, and 3) of the Snowflake, showing the way in which the area increases.

The snowflake				Scale = 100	
Level	Number of triangles added to the previous level	Side and the smallest	Area of triangle	Total area	Perimeter
1	1	100	A_1	4330.12702	300
2	3	100/3	481.15942...	A_2 = 5773.502694	
3	4 x 3			6415.002993	533 . 3 3
4				6700.114237	
5				6826.830345	
6				6883.148615	
n	4^{n-2} x 3	$100/3^{n-1}$		x (area smallest triangle)	
7				6908.178957	
20				6928.202702	70950.7924
50				6928.20323	397300716.733
100				6928.20323	7.01 x 10^{14}

Table 6.13. Table used for investigating the area of the Koch snowflake.

As before, the generalisations obtained through the table led to a formula (see Table 6.13), which was translated into a (recursive) procedure AREASNOWFLAKE for computing the values of the area of the snowflake at any level :N (using the procedure AREATRI[20] which computes the area of an equilateral triangle with side :S):

---

[20]

```
TO AREATRI :S
OP ( POWER :S 2) * ( SQRT 3 ) / 4
END
```

```
TO AREASNOWFLAKE :SIDE :N
IF :N = 1 [OP AREATRI :SIDE]
OP ( AREASNOWFLAKE :SIDE :N - 1 ) + ( POWER 4 :N - 2 ) * 3
      * AREATRI (:SIDE / POWER 3 :N - 1)
END
```

Using this procedure for the numerical investigations, students were able to observe the very rapid convergence of the area, in spite of the divergence of the perimeter. The numeric information indicating the existence of a limit complemented the visual one, where students noticed that the figures of, for example, levels 6 and 7 hardly differed:

Consuelo: I think it's also going to have a limit. [At any level after 6] they look almost the same, because the little segments that are added are so small that they can't be seen. And so the area is almost the same.

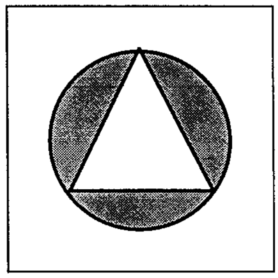


Figure 6.29. Area between the circumscribing circle and the triangle from which the Koch snowflake is generated.

An additional element in the study of the behaviour of the area was to consider the area between the snowflake and the circumscribing circle to the original triangle (see Figure 6.29) as the levels progressed. Some students initially thought that the snowflake would fill up that circle, but the question prompted them to reflect further on the behaviour of the growth of the area. For example, Consuelo used drawings on paper to explain that the snowflake would always stay within the circle, even though little "peaks" kept being added; she then added: "That is why the area has a limit". She had found another explanation for the convergent behaviour of the area she had already observed.

Some students (e.g. Jesús; Consuelo; Alejandra) even though they accepted that the perimeter of the snowflake tended to be infinite but was contained in a finite area — which they confirmed through both the visual observation of the figure and the numerical values — found the conjunction of these two elements quite surprising and counter-intuitive. Intuitive paradoxes such as this one which arise when dealing with the infinite will be further analysed and discussed in Chapter 7. In Jesús's case (see transcript below) the dilemma was solved when his partner Manuel pointed out the significance of the *shape* of the figure as the determinant factor: the figure's shape is such that the perimeter simply folds up as it increases, not letting the area grow any further.

Jesús:	It is incredible, incredible, that it has an infinite perimeter and that it comes to a point where the area is limited.
Manuel:	Well, not so incredible since...
Jesús:	Well, it is unusual. What other figure do you know that has an infinite perimeter with a limited area?
Manuel:	Well, what happens is that the perimeter is growing and growing but it is somehow folding inside the [circumscribing] circle, and that is why the area is almost constant, and looking at it in that way I don't find it so incredible...

2. Explorations with the Sierpinski triangle.

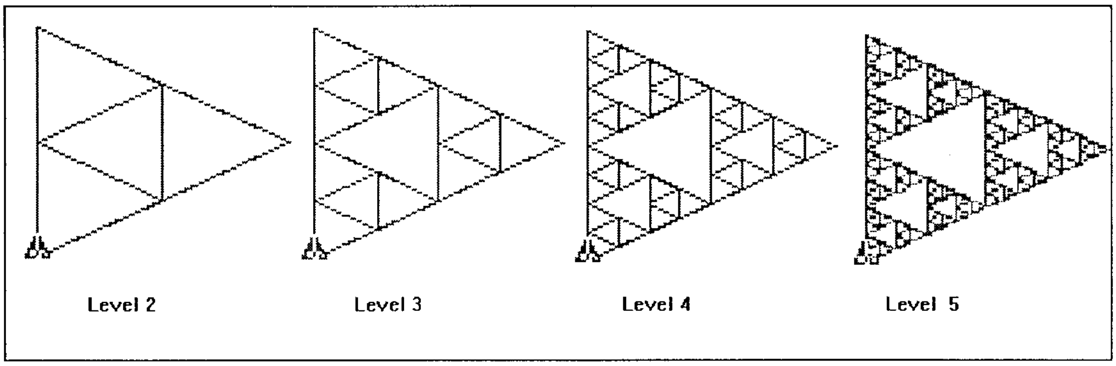


Figure 6.30. Construction of the Sierpinski triangle.

The experience with the Sierpinski triangle[21] was very similar to that with the Koch curve and snowflake. I had asked the students to imagine they were removing the central triangle of each triangle (see Figure 6.31), and then consider what would happen to the remaining area. Some students (e.g. Alejandra and Victor) predicted from the beginning that the remaining area would tend to zero, but this was not always the case. Consuelo and Verónica initially suggested that if they rearranged the remaining areas they might get a triangle the size of the central triangle, which would be a fourth of the original area. However, when Verónica reflected on the fact that after level 7, all the subsequent figures *looked* the same but had *less* area, thinking aloud, she exclaimed that it would be the entire triangle which would be removed.

[21] The procedure for the Sierpinski triangle was given in Chapter 5.

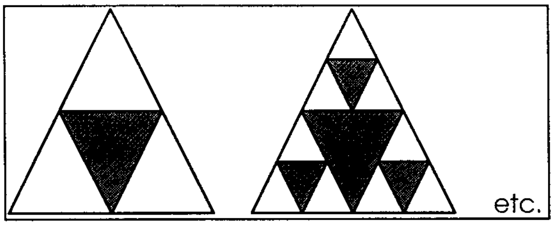


Figure 6.31. Area removed at each step in the construction process of the Sierpinski triangle.

As in previous activities, the students numerically explored the behaviour of the area and constructed tables of values (see Table 6.14). From visual observations they noticed that at each step a fourth of the area of each triangle was being removed, a process they described through a formula (see last row of Table 6.14) and translated into a procedure (AREAREM[22]) for computing the remaining area. As in other cases, through these explorations the students discovered and accepted the convergence of the area to zero.

Level	Side of the smallest triangle	Area of the smallest triangle	Number of remaining small triangles	Total remaining area
1	100	4330.127...	1	4330.127...
2	100/2	1082.5...	3	3247.5952...
3	$100/2^2$		$9 = 3^2$	2435.6964...
4				1826.772...
5				1370.67925...
6				1027.55943...
10				325.126228...
50				0.000326...
100				0.00000000185...
1000				$6.648... \times 10^{-122}$ = 0.(-121 zeros-)6...
n	$100/2^{n-1}$	AREATRI[23] $100/2^{n-1}$	3^{n-1}	3^{n-1} x AREATRI $100/2^{n-1}$

Table 6.14. Table used (by Verónica and Consuelo) for investigating the area of the Sierpinski triangle (using a scale of 100).

I would like to add an interesting episode that happened with regard to the initial observations of the Sierpinski triangle: when some of the students produced a figure with half the scale, they noticed that the resulting figure was a "part of the bigger triangle": a third of the full-scaled figure. This was an experience that highlighted the

22
```
TO AREAREM :S :L
OP ( POWER 3 :L - 1 ) * AREATRI :S / ( POWER 2 :L - 1 )
END
```

23 I remind the reader that AREATRI is a procedure that outputs the area of an equilateral triangle taking as input the length of the side of the triangle.

self-similarity of the figure, and which the students could coordinate with the self-similar recursive structure of the procedure (as for example Consuelo did when she explained: "everything is similar to everything else, because TRI calls TRI"). With regard to the visual image most students maintained that at an infinite level the figure would look pretty much like the one of level 7. Some students (e.g. Consuelo; Jesús and Manuel) conceived the figure at infinity to be made up of, in Consuelo's words, just "little points" forming the triangular patterns, with no area. This conception would later be compatible with the limit of the figures produced by the CURVE procedure in the activity that followed.

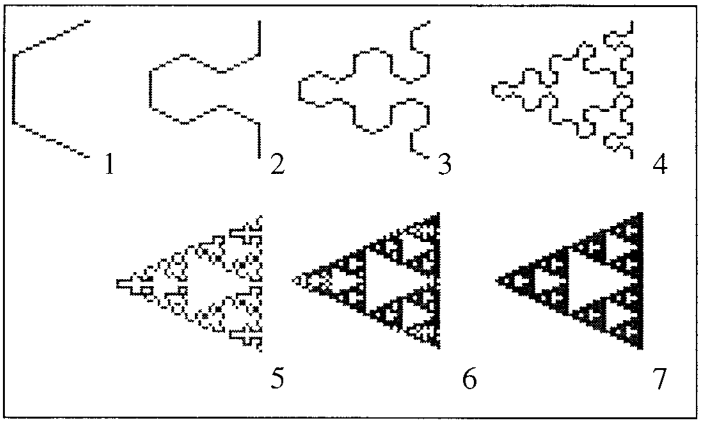


Figure 6.32. First seven stages of the "Sierpinski curve".

When I gave the students the CURVE[24] procedure, described in Chapter 5, all the students recognised that it was a recursive procedure and most therefore predicted it would produce another self-similar figure where "the part resembles the whole"; Consuelo even predicted it would produce something "like the previous triangle". Most students were therefore confused when they saw the first levels of the process defined by this procedure (see Figure 6.32), but they soon were able to detect how the sequence of images was produced with each part being replaced by a (smaller-scaled)

24 (where the input of :P is always 1)

```
TO CURVE :L(evel) :S(cale) :P
IF :L = 0 [FD :S STOP]
LT 60 * :P
CURVE :L - 1 :S / 2 ( -:P )
RT 60 * :P
CURVE :L - 1 :S / 2 :P
RT 60 * :P
CURVE :L - 1 :S / 2 ( -:P )
LT 60 * :P
END
```

figure of the previous level, and also observed that the new procedure indeed generated a figure similar to the Sierpinski triangle produced by the TRI procedure.

All the students then compared on the same screen (see Figure 6.33) the figures generated by both procedures — TRI and CURVE. From a *visual* perspective the two figures looked identical and it did seem they both converged to the same set of points. The students then reflected if at an infinite level the figures would be identical. Some found arguments in favour: Alejandra felt the two figures would be identical "because they keep the same area empty"; others, like Consuelo and Manuel, concluded that both procedures "marked" the same points. For these students two different (infinite) geometric sequences converged to the same image. But for Victor, the awareness that the two figures had been produced through different methods prevented them from accepting that the two could be identical. (Although it is possible, since Victor always maintained the conception that the limit is never reached, that he thought of the slight differences that were bound to occur between the two figures "before the limit".)

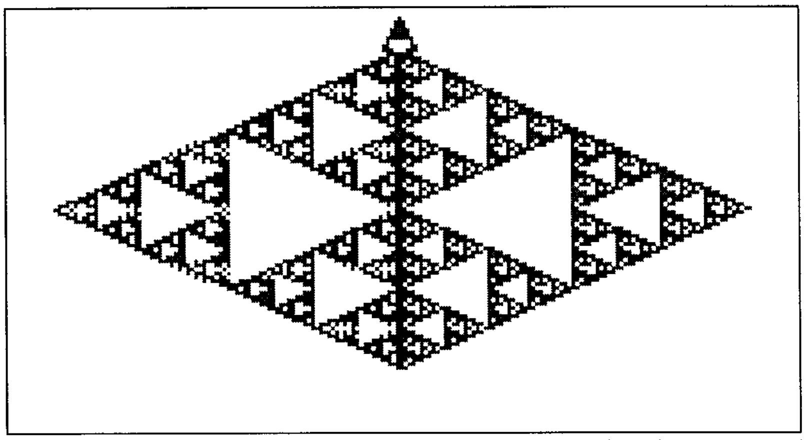


Figure 6.33. Comparison of the "Sierpinski triangles" produced, on the left, by "CURVE 100 7", and on the right by "TRI 100 6".

3. Exploring the Cantor set.

Two of the four pairs of students, Manuel/Jesús and Consuelo/Verónica, explored another fractal that was not in the pre-design of the study: the Cantor set or "dust". In the case of the first pair, Manuel was reminded of this set by the experience with the Koch Snowflake and the idea that "something infinite" was spatially bounded (even though the students had previously implicitly encountered this characteristic:

e.g. the convergent infinite sequences they had previously explored; the number of points in a line segment):

Manuel:	It reminds me of something I saw about sets, where there was an infinite number of figures in a finite space: by dividing a segment in three, and then again in three, and so on infinitely, and you get an infinite number of figures in a finite space...

Recognising the recursiveness of the process, and using the PEAK procedure as a guide, none of the two pairs had difficulty writing a procedure (which Verónica and Consuelo called BITS) for generating the Cantor set (see Figure 6.34):

```
TO BITS :L :N
IF :N = 1 [FD :L STOP]
BITS :L / 3  :N - 1
PU  FD :L / 3  PD
BITS :L / 3  :N - 1
END
```

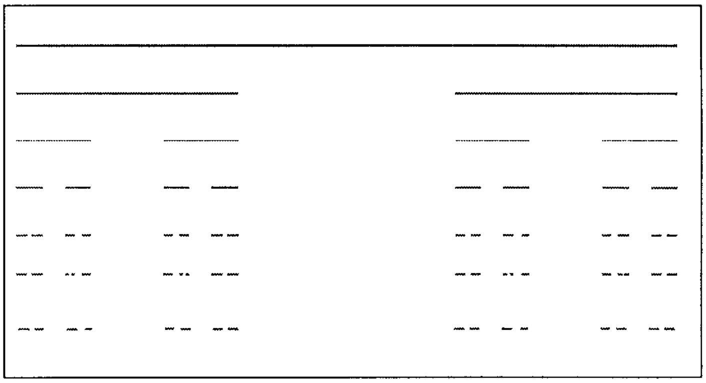


Figure 6.34. First 7 levels of the Cantor set, produced using the BITS procedure.

It became very clear that all of these students linked the behaviour of the Cantor sequence with the sequence $\{1/3^n\}$ which they knew tended to zero. This led to the conclusion that the segments would become like points, although this last point would later be the cause of some debate:

Verónica:	We are dividing by 3, so it will be 300[25], then 300 over 3 which is 100, then 100 over 3, then over 3, and over 3.....
Consuelo:	It's 300 over 3 to the N. And that is like the one we saw which got closer to zero.
Ana:	So what is going to happen to each of these bits?
Verónica:	They are going to get close to zero.
Consuelo:	It seems like there will not be anything, but there will be very small little points.

Through the observation of the images on the screen and the recursive structure of the code, the students realised that this was another self-similar figure explaining that each part was "like the whole", which as the level tended to infinity would look the same even though, in Consuelo's words, "each segment [would] become almost zero, becoming like very very small points".

[25] They had used 300 as the scale, and thus that was the length of the initial segment.

Jesús: They would be points in the same order as all the figure, with the same pattern.

But the fact that the pattern of the figure was preserved and did not "disappear", even though the length of the segments tended to zero, caused Verónica and Consuelo to emphasise that the segments would still be segments although they would look like points "because they would still have a measure". Manuel faced a related dilemma: if the segments became zero, then also the spaces between the segments would tend to zero, and then, instead of having discrete points, the points would "stick together" creating segments! Manuel's conception at this stage of the Cantor set will be analysed and discussed in Chapter 7, as it touches upon issues related to the nature of the real line. But in spite of his doubts, at the end Manuel did become convinced of the self-similar pattern of the figure, after he suggested modifying the procedure so that it "raised the spaces" between the segments of the set (see Figure 6.35). Manuel concluded that the "base-segments" would eventually become points (and added they would be infinite in number):

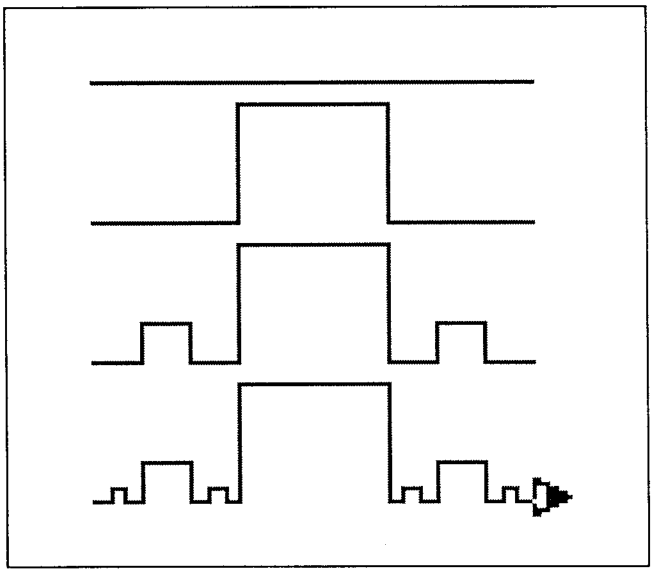


Figure 6.35. First four levels in the construction of the "raised" Cantor Set.

Manuel: These [he points to the base segments] are points.
Yes, and if it gets to an infinite level, there will be an infinite number in a finite space which is what I said at the beginning.

Final remarks.

Through the structure of the common exploratory activities described above, I illustrated how the students made use of all the elements in the microworld to construct meanings. I would like to point out that, although the overall structure of the activities was similar for all the students, the *way* in which each of the pair of students worked was unique for each of them. This is not surprising since all the students had different backgrounds, experience and mathematical knowledge. An interesting finding is that the younger students (e.g. Consuelo and Verónica) focused more on open-ended explorations, while the more mathematically experienced students (e.g. Martin and Elvia) seemed more interested in making connections with the formal or learned mathematical knowledge.

In any case, the facilities provided by the tools of the microworld (procedures, direct Logo commands which were used to express and compute values, variations of graphical outputs, and tables) proved to be an important aspect in the discovery process; these tools provided a means for the students to structure their explorations, form and express relationships and generalisations. The back and forth process of coordinating all the elements was the key element which allowed students to form connections and construct meanings. In the next two chapters I analyse, discuss and summarise the key elements and tools involved in the discovery and how their interactions formed connections leading to the construction of meanings.

With regard to the activities presented in this chapter, it is worth adding that all of these activities proved appealing to the students (all of them said the activities were fun), motivating them to investigate as much as possible every situation. The students also appreciated all the elements of the microworld and the connection between the graphics and the procedures.

Chapter 7:

Constructing Meanings for the Infinite.

In Chapter 6, I presented an overview of the microworld in practice, describing the common activity structures but also illustrating its constructionist aspect by highlighting *in general* the ways in which students used the tools of the microworld, constructing and coordinating different representations — symbolic (e.g. through the code), visual and numeric. In this chapter, I look in particular at the ways in which students used the environment and its tools *to construct meanings for the infinite.* Here I present, illustrate and discuss the key issues and findings of the study: the main aim being to investigate the mediating role of the computer-based microworld in the construction of conceptions related to infinity and infinite processes. I make use of specific examples to illustrate and analyse some of the ways in which students used and coordinated the elements of the exploratory medium to construct meanings for the infinite. I have divided the findings into three categories:

1. The construction of meanings through programming.

2. The use of the medium as a "mathematical laboratory". Discovering and testing patterns and relationships, constructing generalisations, situated abstractions, and "situated proofs".

3. The relationship between the activities and tools of the environment and students' conceptions of the infinite.

I. Constructing meanings through programming.

In this section I illustrate how the microworld gave the students the means to make sense of what they saw on the screen *via the programming code:* the interactions between the code and its outputs. This is in marked contrast to the normal function of the symbolic representation which is often seen as an end-point rather than as a means to an end.

As is evident from the research reviewed in Chapter 3, visual objects need to be read and interpreted, something which is not always straightforward. My premise was that because the production of the graphical (and numerical) representations necessitated the construction of the code, it provided students with a means for interpreting these representations. I was therefore interested in observing the ways in which students constructed connections between the symbolic code and the visual output. In particular, I was interested in looking at the role of the *structure* of the procedures, particularly the iterative or recursive structure, and its relationship to the visual structure. There were two facets to this phenomenon:

- the link of the endlessness of the process represented on the screen, with the iterative structure of the code; and
- the use of the symbolic recursive structure of the code to *visualise the self-similar visual behaviour.*

a. Endless movement and the link with the recursive (iterative) structure of the code.

In the first microworld activity, as was described in Chapter 6, (section A.1.a.) although students had been asked to predict the behaviour of the turtle prior to running the initial DRAWING procedure, most did not expect to see the turtle endlessly spinning without leaving a trace (the turtle's pen was up). In order to explain to themselves this unexpected behaviour and make sense of why the turtle was endlessly spinning, the students had to re-examine the procedural code. Victor was one student who immediately remarked that the procedure would never stop because the recursive structure of the code represented an infinite process. He explained it was because the procedure called itself without anything telling it to stop, so it never would stop; the process of turning and walking half the previous distance would continue repeating itself and would never stop:

Victor: It [the procedure DRAWING] is never going to stop, because it is calling DRAWING and it is repeating the process, but there isn't a point where it says "stop if you get to certain point". So the distance we gave it as input will always be divided: it will divide the 100; it will walk, then turn 90, then walk half, then again turn, then walk a half of a half...

By analysing the code Victor was able to connect to it the behaviour of the visual output (in this case the *movements* of the turtle) since he correctly predicted the outcome and was able to justify that visual behaviour through the code. He *linked the recursive structure of the code with the infinitude of the process.*

A modified procedure[1] (with the Pen down) produced an inward spiral with the turtle then turning endlessly in its centre. Victor pointed out that although the turtle seemed to just be turning in the same spot, in reality there was "a variation". His partner, Alejandra, also said that she thought the turtle was still walking something. There were two factors here: a) the turtle kept turning, and b) the turtle turned at the same spot. The first factor could have served as an indicator that the process continued, but it was the fact that the students seemed to be able to disregard the *visual appearance* of the turtle — spinning in apparently the same spot — that is evidence that they understood that the underlying (mathematical) process continued, and that they were able to link the output with the code and the process. Later in the activities, when the students modified the procedure to give out numeric values (see Chapter 6, section A.1.b.), they would confirm the continuation of the process by still getting an output of values, even when the turtle seemed stuck:

> Alejandra: Apparently it is stopping on the screen, but it is still walking because we are still getting the values.

Alejandra and Victor were able, via a process of experimenting backwards and forwards from code to figure, to make sense of the behaviour of the turtle, which seemed to be spinning on the same spot, by realising that the amount that the turtle moved each time was halved. The key point here is that the analysis of the code allowed them 1) to recognise in the recursive structure a potentially infinite process; and 2) to quantify the movement, to explain that although the turtle seemed to be turning without moving forward, in reality there *was* a variation. Thus by coordinating the visual and symbolic — in the direction visual to symbolic to visual — and later complementing it through the numeric, their understanding of the process became

---

1
```
TO DRAWING :L
FD :L
RT 90
WAIT 10
DRAWING :L * 1/2
END
```

integrated, and potentially misleading visual appearances could be ignored. This interplay between the code and its outputs, which led Victor and Alejandra to make sense of what they observed by linking all the elements, is illustrated in Figure 7.1:

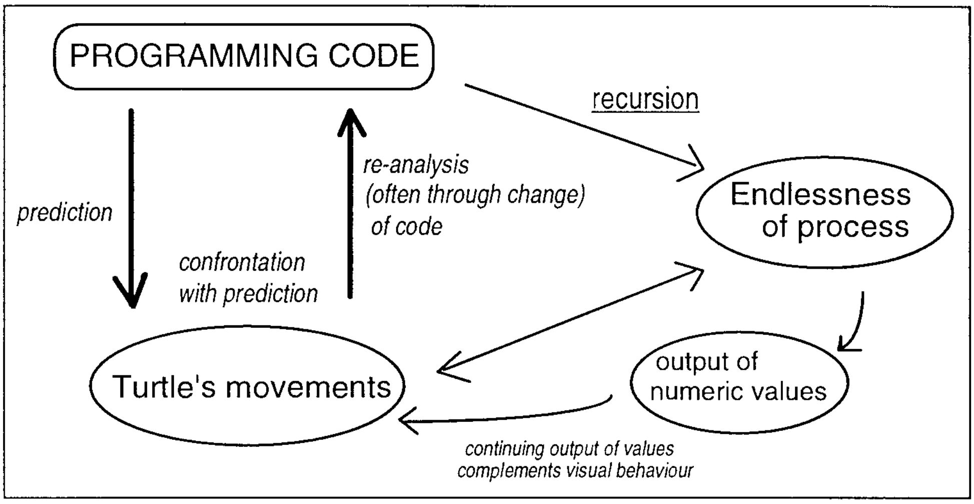


Figure 7.1. The interplay between the code and its output to make sense of the endless movement: the graphical image gains meaning from the symbolic representation.

b. Using the recursive *structure of the code to predict a self-similar visual structure: the code "encapsulates" the process.*

Whereas the recognition of the (tail-) recursive structure in the code explained the endlessness of the process, which involved going from the visual to the symbolic, this recursive structure also served to predict and *visualise* the figure produced by the code — a process from symbolic to visual. For instance, when the students were unable to see the deeper levels in the visual representation of the sequence under study (e.g. they noticed that the centre of the spiral model looked like a point), some students blamed this on the resolution and were able to compensate for the deficiencies of the screen by using the information provided in the symbolic structure of the code to visualise those levels. Victor, for example, in the first activity (described in the section above) explained that even though the centre of the spiral looked like a point, the figure did *not* become a point, and would always have the same spiral shape, even at its centre. Another student, Martin, explained this same point as follows:

Martin: What happens is that there is a part that our eyes can no longer perceive. Inside [the spiral] it continues the same way, because it is the same process that continues... If we used a magnifying glass and looked at that little square there, we would see like all this part [the full spiral].

This is a key issue: the visual self-similar structure is a reflection of the recursive structure of the code (and of the infinite iterative nature of the process). And the programming code thus serves for "visualising" beyond the visual image. The programming code can thus be said to embody or "encapsulate" the entire process. An infinite process would, of course, take infinitely long to be generated; but the symbolic code (which generates it) holds the entire process in *latent* form, as would a mathematical formula, and its structure reflects the structure of the process.

This connection between the structure of the code and that of the figure was of course more obvious in the fractal explorations. For example, as described in Chapter 6, section B.1., first, the construction of the structure of the Koch curve was based on the theoretical structure of the visual figure — the structure of the procedure mirroring the way the process would be (visually) constructed. Then as the Koch curve was generated, the students would make sense of the figure (the fact that each part contains the whole) by relating it back to the structure of the code (which calls itself). The same correspondence between code and figure was found for all the fractal figures investigated, and, as in the case of the Sierpinski Curve described in section B.2 of Chapter 6, students were able to predict a recursive visual structure from the observation of the structure of the code.

It is interesting to point out that most (if not all) of the students explicitly recognised the value of having a recursive code which defined, and therefore was connected to, the process both visually and in general. As they pointed out, the self-similar/recursive structure of the code allowed them to have an idea of what was going to happen subsequently, particularly since the figure can only have so much resolution. This was best explained by Jesús during the final interview when he pointed out that it was the recursive structure of the procedure which helped them realise, and reflected the fact, that the figure would repeat itself in a self-similar way. He added that it was the procedure (i.e. the code) which helped to understand what happens at infinity:

Jesús: I would say that our most powerful weapon is recursion, which is what allows us to be aware of the details...
In those shapes that repeat themselves, it is the same part which is the basis...
The recursion is recorded there as the same figures are repeated, and at the end it does the same figure but bigger.
The same happened with the Spiral and the Histogram.
And because we understand the language, it was the recursion which helped us understand better, and do a better analysis... going from the figure to the

> procedure, or from the procedure to the figure...
> In any case, once I knew that the procedure was recursive, I more or less had an idea of what would happen later, because the drawings by themselves do not have enough resolution...
> The Logo procedures give support and help define what happens at infinity. That is, they include the notion of infinity, and that is really helpful. They help convince us, or confirm the ideas we may have....
> This type of procedures is very helpful for understanding what happens at infinity... that is, by using the sequences as a basis.

II. Using the medium as a "mathematical laboratory".

There were several levels in the way the microworld was used as mathematical laboratory, each of which is illustrated below. At a first level, students made observations and discoveries *situated within* the medium of the microworld. By playing in, interacting with, and working within the microworld, the students could express their perceptions and ideas within the medium, through the tools, activities and forms of symbolism built into the environment. This idea will be partially illustrated in section 1 below, since this example centres in particular on the *discovery* process within the environment.

At a second level, some students were able to abstract and articulate their findings in a way that could be taken beyond the medium in which they were constructed, and they consciously exploited the tools of the microworld for discovery, exploration, and "proof" of mathematical relationships or "theorems". This is illustrated in section 2 below.

1. Employing the microworld as a domain of abstraction. Finding patterns and relationships within the environment.

Verónica and Consuelo had been investigating the effect of changing the value in the stop condition of the DRAWING procedure (see section A.1.a. in Chapter 6), and its relationship with the behaviour of the turtle and the graphic. The students constructed their own investigative approach which was to look at the number drawn before the value in the stop condition became true and had added a :COUNT variable to the DRAWING procedure[2] (see Chapter 6, section A.1.b) to count the number of segments the turtle drew, thus modifying the procedure to help them in their purpose. Later Verónica thought of looking for a *pattern* which linked the value in the stop condition with the number of segments — a new area of investigation — thinking that for each smaller value they used, one more segment would be drawn:

> Verónica: The smaller the value is... No, the *bigger* the value gets [the number of segments] is reduced by 1. That is, it is reduced by 1 because for 0.5 it's 8, then for 1 it's 7, and for 2 it's 6. So it's doing one arm less.
> [...] If we put zero point zero... it is going to go on with 11, 12, 13, 14,...

But when they obtained the same number of segments for different stop values (0.15 and 0.1), this situation made them reflect on the relationship between the two factors (the number of segments and the value in the stop condition), and the students realised that Verónica's conjecture was incorrect. Consuelo then suggested that they should investigate the variation of the number of segments between different stop values (taken at constant intervals):

> Consuelo: And if we looked to see if there is a rule here for the difference in the count?

They chose to vary the value in the stop condition by 10 decimal places each time. The values were recorded in Table 7.1:

Value in the Condition: :L < 10^{-*}	COUNT (Scale (initial :L) = 100)	Difference with previous count
0.0000000001 (10 digits)= 10^{-10}	40	
1	7	33
0.1	10	
10^{-10}	40	
0.0...01 (20 digits) = 10^{-20}	74	34
10^{-30}	107	33
10^{-40}	140	33
10^{-50}	173	33
10^{-210}	705	
10^{-220}	738	33

Table 7.1. Table used by Verónica and Consuelo in their exploration of the relationship between the value in the stop condition and the number of segments drawn; corresponding to the sequence $\{L/2^n\}$.

2
```
TO DRAWING :L :COUNT
IF :L < 1 [PR :COUNT STOP]
BARS :L
DRAWING :L * 1 / 2 :COUNT + 1
END
```

Through the table of values they soon discovered that the variation in the count of segments or bars tended to be a constant of 33, which they also tested for very small stop values (10^{-210}). Although the students did not make conjectures as to why a pattern had emerged, their discovery did seem to show that the process behaved in a constant manner. They repeated the same steps — looking for a similar pattern — for the process corresponding to the sequence $\{1/3^n\}$, using some of the same stop values, and recording their findings in Table 7.2. They then noticed that the difference in the number of segments changed from 33 to 21 when the procedure took thirds instead of halves. Consuelo then *connected the smaller difference in the number of segments to the faster decrease of the segments in the sequence* $\{L/3^n\}$ (the faster rate of decrease being evident in particular from the observation of the bar graphs), a relationship which was articulated *with reference to the medium* in which it appeared:

Consuelo: It does less because it is now taking a third, and it did more because it was taking halves. Because when dividing by 3 the bars get smaller faster.

Condition: :L < ...	COUNT	Scale = 100 Difference [with previous count]
1	5	-
0.0...-10 zeros-...01 = 10^{-10}	26	21
0.0...-20 zeros-...01 = 10^{-20}	47	21
0.0...-30 zeros-...01 = 10^{-30}	68	21

Table 7.2. Table of values used by Verónica and Consuelo in their explorations of the relationship between the value in the stop condition and the number of segments drawn; corresponding to the sequence $\{L/3^n\}$.

It was thus that: a) The record in a table of the number of segments through *constant* variations led to the discovery of the sought-after pattern: the number of segments also *increased in a constant manner*. This was a relationship which was discovered (and then tested) *within* the context in which the processes were presented. b) The students then used the tools of the microworld to test their observations and investigate if similar results appeared when they modified the process, having the initiative to compare the behaviours of two different processes (e.g. that of "taking thirds" vs. "taking halves"). c) Consuelo was then able to coordinate all the evidence, which pointed to the fact that when "taking thirds" the process decreased faster than when "taking halves": she saw that the difference, recorded in the tables, in the number of segments was complemented visually by the behaviour of the bar graphs.

She used the elements provided to "web" (Noss & Hoyles, 1996) meanings for her observations.

2. "Situated proofs": using the tools of the microworld for discovery and "proof" of mathematical results.

Manuel and Jesús were among the students who discovered that series of the type $\sum_{n=1}^{\infty}\frac{1}{k^n}$, where the integer[3] k>1 converges to $\frac{1}{k-1}$. These students had been exploring and comparing the sequences $\{1/2^n\}$ and $\{1/3^n\}$, and began to discover a pattern in the behaviour of the corresponding series: Manuel observed that as they increased the denominator value k in the sequences of the type $\left\{\frac{1}{k^n}\right\}_n$, then the limit of the corresponding series was smaller and in fact seemed to have as value $\frac{1}{k-1}$. They explicitly constructed a generalisation for this mathematical result (which they would later call "the theorem of Manuel and Jesús") and used it to *predict* the probable behaviour of other sequences and series of the same type:

Manuel:	Look, if you subtract 1 from the number that is the base in the denominator, and you divide 1 by that number, then that is the number to which it will approach. If we do it with 3, 3 minus 1 is 2, and it tends to a half... So if it was $1/2000^N$, the sum must approach 1/1999...
Jesús:	Yes, the bigger the base in the denominator, the smaller the limit.
Manuel:	But now we have a method for knowing to where it approaches. ... We saw that $1/2^N$ became small very quickly, but the one of $1/3^N$ decreases much, much more quickly. And we saw that its series didn't tend to 1 like the previous one, that it approached a half, so we noticed a more or less regular behaviour, so if we wanted to know to how much the series of $1/2000^N$ would be we would only have to reduce it by a number, and it would tend to 1/1999.

Manuel and Jesús then employed the medium and its tools to *test out* their predictions They began by changing the sequence generating function to $1/4^N$, predicting that the corresponding series would tend to 1/3. They used all the resources available to explore this sequence and its series, looking at all the available graphic models (the Spiral, Stairs, Bar Graph and Line models). With the spiral they were amazed at how quickly the values of the sequence decreased, something they

[3] Manuel and Jesús seemed to implicitely consider k as a positive integer larger than 1, although they did not make this condition explicit.

confirmed with the stairs and bar graph where, no matter how large they made the scale — they tried up to 9999, the computer's limit — not more than 7 terms (larger than 1) were visible. From the rapidly decreasing behaviour of the sequence they deduced that the series converged, choosing the line model to verify this by observing how the turtle stopped going forward at the predicted length (in this case 1/3 of the scale). Although the visual explorations were enough to convince the students of the validity of their conjecture, they complemented these with a numeric exploration of the partial sums (using the procedure PARTIALSUMS, described in Chapter 5. They observed that the 20th partial sum printed out to be 0.3333333333, confirming further their hypothesis. A final test of their conjecture was carried out by exploring the sequence $\{1/13^n\}$, through visual and numeric representations — which showed the much more rapid decrease of this sequence (this time not more than 3 terms were ever visible with the largest scale) — again verifying that the corresponding series tended to the predicted value of 1/12.

For Manuel and Jesús there was now no doubt that their conjecture was true, although it is interesting to see the extent to which they wanted to make sure it was valid: Manuel worried that this mathematical generalisation would not hold if the value of *k* was infinite, until he found favourable arguments:

Manuel:	Listen... there might be a contradiction in our assumption: if we did one over infinity... ah, but infinity minus any number is still infinity..., so we are right. It tends to zero.
Jesús	The limit of one over infinity tends to zero...
Ana:	And what is it that you are concluding here?
Manuel:	Well, infinity, if you take away from it, there remains infinity, because the infinite always keeps going on, and if you subtract from it, it can always be infinite, it is always going to go up to infinity, and so if one over infinity tends to zero, then also one over infinity-minus-one, because infinity minus one is infinity, then it also tends to zero. And here we have that the bigger the base of the denominator gets..., the smaller the series....

It is worth noting that most students, including the younger students Verónica and Consuelo, discovered the rule for the behaviour of the series of the type $\sum \frac{1}{k^n}$, which they tested and then generalised. Manuel and Jesús were more experienced mathematically, which was reflected in the way they expressed the rule, but Verónica and Consuelo also constructed the generalisation within the context of the activity — a *situated abstraction* — expressing it relative to the inputs used by the procedure (e.g. the scale):

> Consuelo: So the sum of the bars for 1/3 it's one half [the scale], and for 1/4 it would be 1/3 [of the scale], and for a fifth: 1/4 [of the scale], and so on.

The possibility to work with many different cases (different sequences of the same type), and use diverse resources (different visual models and different complementary types of representations), provided the students with a means: (i) to infer their own generalisation through the discovery of a pattern, and (ii) to validate and *confirm* their predictions and generalisation (becoming *convinced* of the general validity of their conjecture). The results may not have been formally proven, and the students were aware of this, but the process of repeatedly observing different variations, cases, and situations, was enough to convince the students of the validity (or in other cases falseness) of their conjectures. I have called these experiences *situated proofs*. These convincing experiences resulted from the combination of all the elements which the students used in their attempts to confirm their conjectures. All the representational forms were coordinated and used in a complementary manner in the search for proof. Thus, one of the key characteristics of these situated proofs is their tool-dependency.

III. Student's conceptions of the infinite as mediated by the environment.

Some of the ways in which students conceptualised or explored the infinite nature of a process have already been mentioned above, such as the relationship between the endlessness of the process with a) the recursive structure of the code, and b) the numeric measures which quantify it. Here, I present some of the other means the students used in their explorations of the infinite and some of the conceptions that emerged.

1. Looking at the behaviour of the process.

One of the advantages of the microworld was that the *behaviour* of the process could be observed, rather than the end result, as is usually the case in traditional school mathematics. Observing the behaviour, such as the rate of convergence, played a very important role in giving meaning and finding explanations as to why in a particular instance a process converged or diverged. The exploration of the behaviour was done in several ways, including

(i) the observation of the process through its unfolding visual and numerical behaviour,

(ii) the possibility to compare different sequences and models,

(iii) the coordination of the stop value with the outputs,

(iv) in the case of series, coordinating the behaviour of the series with that of the corresponding sequences.

An example which illustrates most of the above points, and where all the elements, visual and numeric, needed to be coordinated to determine the behaviour of the process, is found in the way Manuel and Jesús determined the divergence of the Harmonic series (see Chapter 6, section A.6.). They went through the following steps in their discovery process:

- from the observation in the visual spiral model of a "hole" at its centre as well as the apparent avoidance of that centre, there was an initial realisation that the Harmonic sequence behaved differently to other cases studied;

- the mathematical analysis of the formula showed that $\lim_{N \to \infty} \frac{1}{N} = 0$, i.e., that the Harmonic sequence converged to zero;

- a return to the visual model led students to presume that the convergence of the sequence must be very slow;

- by choosing to look at another visual model: the stairs model, they observed how this new model depicted a persistent growth (corresponding to the behaviour of the series);

- a search to explain the observed behaviour highlighted the very slow *rate* of convergence of the Harmonic sequence as compared to other sequences such as $\{1/2^n\}$, and why this is the case;

- the observation of the histogram model showed visually and confirmed the slow convergence of this sequence;

- the collected observations led Manuel and Jesús to predict the behaviour of the line model (illustrating the behaviour of the series) was one which would persistently extend towards an infinite measure.

- Two explanations for the divergence of the series emerged:

(i) the process is infinite: something is always added.

(ii) the very slow convergence of the sequence.

- A temporary conflict emerged from the realisation that (i), above, is always true even in cases where the series converges. A new doubt emerged when they focused their attention on the idea that when N is very big, the growth is very small, almost insignificant, and therefore an infinite value of the sum would seem unlikely.

- However, a structured numerical analysis, using a table of values, pointed to the absence of a limit (for the series). This was complemented by a visual bar graph (histogram) model of the partial sums, and Manuel and Jesús were finally convinced of the slow divergence of the series which they explained and related to the slow convergence of the sequence.

As we saw, several elements were highlighted through the explorations of the Harmonic sequence and series: (i) the *slow* convergent behaviour of the sequence; (ii) the divergent behaviour of the series, and its independence from the infinitude of the process; and (iii) the effect of the *rate* of convergence of the sequence in the behaviour of the corresponding series.

The students had to investigate how these elements related and were coordinated in order to solve the intuitive contradictions that emerged, particularly between the convergence of the sequence and the divergence of the corresponding series. This involved an intensive back and forth process of exploration using all the resources available, each having a valuable complementary role: the different visual models, the exploration of the numerical values, as well as an analysis of the mathematical formulae. Through the coordination of all of these elements and experiences, the students were not only convinced of the divergence of the series, but were also able to find some explanation for this in the behaviour of the sequence. This story thus points to the importance of the study of the *behaviour* of the sequence and its relationship with the behaviour of the series. As other students pointed out:

Victor: It is possible to know more or less to where things are going by watching how it grows or decreases... its behaviour.

Alejandra: Sometimes we see a fast increase at the beginning but them we notice that it slows down.

a. Visualising limits through graphic representations and numerical values.

As is evident from the descriptions given above, students discovered and explored limiting (or divergent) behaviours first through the graphical representations and then carrying out a back and forth process between these representations and numeric values; only in the case of Manuel/Jesús and Elvia/Martin was there some degree of a more traditional "mathematical" analysis of the formula. The main visual element which gave students indication of the existence of a limit was the *visual invariance* through several stages. This was particularly true in the fractal explorations, such as in the case of the Koch snowflake, where the visual image conveyed the boundaries of the area, highlighting its independent behaviour from the infinite perimeter that delineates it.

At a second level students would use numeric values, organised into tables, to complement and confirm the observed visual behaviour and give an indication of the value of the limit or divergence of the sequences.

b. Gaining insights into the rate of convergence through comparative analysis.

A method which that played a significant role for observing the behaviour of the process was the possibility to compare different sequences with one another, particularly by comparing them through the same visual model. This comparative analysis was described in Chapter 6, section A.3. The comparisons between, for instance, the bar graph of $\{1/2^n\}$ and that of $\{1/3^n\}$ — observed as they visually unfolded — highlighted the *rate* of convergence of each of these sequences. Elvia and Martin, who were mathematics teachers, particularly appreciated this feature of the microworld. Although they generally knew whether a sequence was convergent or divergent, they had never before investigated the *behaviour* of, for instance, sequences of the type $\{1/k^n\}$. It was through the microworld explorations, testing and looking at several cases of sequences of the above type (by varying the parameter *k*) that they became aware that, in Martin's words, "the bigger the denominator [k] the sooner [the sequence] decreases". These two students also learned to coordinate the behaviour of a sequence with that of the corresponding series, using in particular the visual models.

Another example of the value of comparative analysis is found in the exploration of the divergent Harmonic series, described above and in Chapter 6 (section A.6); students compared the spiral model in this case (Figure 7.2) with others they had seen (which corresponded to convergent series) as well as with those corresponding to other series of the type $\sum \frac{1}{n^k}$ (which also converged; see Figure 7.3). These comparisons highlighted a characteristic feature not found in other cases: the turtle seemed to avoid going to the centre of the spiral, delineating a "hole" at the centre. These observations were complemented by using comparisons using the other visual models (e.g. the bar graph) and through the numeric values (see Table 6.10 of Chapter 6), contrasting the divergence of the Harmonic series with the different rates of convergence of other series.

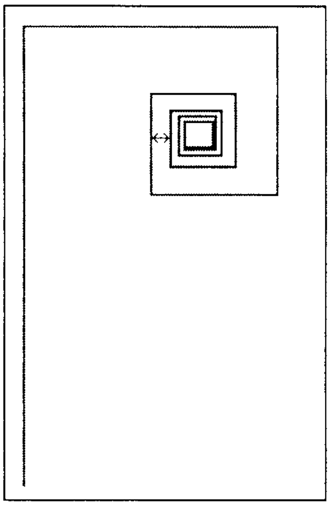


Figure 7.2. Spiral model of {1/n}.

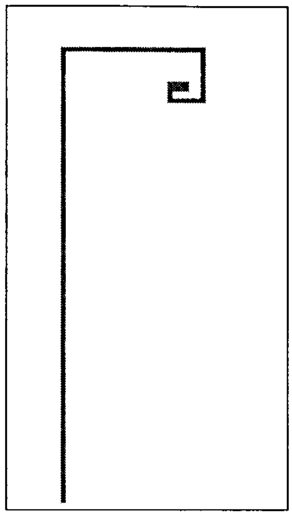


Figure 7.3. Spiral model of {1/n²}.

c. Coordinating the behaviour of a sequence with its corresponding series.

An important result of the study was the way in which sequences and their corresponding series were coordinated. An example highlighting the relationship between the *rate* of convergence of the sequence and the behaviour of the corresponding series is given above in Manuel and Jesús' study of the Harmonic sequence and series. But there is an additional point to be made here: it has often been found in students that confusions emerge between the behaviour of a sequence and that of its corresponding series (see Tall & Schwarzenberger, 1978). However, in the design of the microworld the series were approached first as "the total length of the spiral/line/stairs model"; then through the partial sums (as the limits of a sequence of partial sums). Thus, as described in section A.5. of Chapter 6, students investigated the series (i) through the behaviour of the models of the sequences and (ii) by directly exploring the sequences of partial sums using in particular the bar graph and spiral models. The possibility to represent a sequence in different *visual* ways highlighted the distinction between a sequence and its series: through the bar graph, the sequence was seen as a sequence of bar segments, whereas the series were seen as the sum of all the bars (e.g. the total length of the line model).

Additionally, the step up from thinking of "the total length" to looking at the sequence of partial sums happened in a natural way. Because the procedural code defined the sequence with a finite number of terms, students began comparing what happened with the sum (or total length) as they increased the number of segments. As described in section A.5 of Chapter 6, this led Verónica and Consuelo to want to investigate the *sequence of partial sums,* using the procedure PARTIALSUMS which takes as input a sequence and outputs the partial sums as a *sequence.* Elvia and Martin were among those students who then routinely began looking at visual models of the partial sums, treating them as a sequence, but aware of the connection with the original sequence. For Verónica and Consuelo, who initially only *visually* conceived the series as the total length of the spiral or line models (even though they numerically extensively explored the sequence of partial sums), through the comparison of line models with different numbers of terms as shown in Figure 7.4, they realised they could directly construct a bar graph for the partial sums which would show the behaviour of the series.

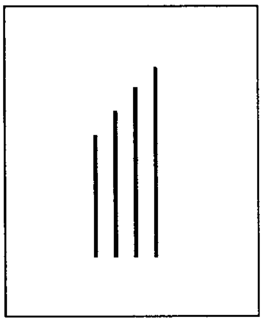


Figure 7.4. Comparison of, respectively, the Line models of 25, 50, 100 and 200 terms of the harmonic sequence.

As illustrated in this section, the relationship and distinction between the sequence and its series was made evident from the way in which the series were visually presented as well as the way in which both the sequence and the partial sums were defined in the procedures. These constructions and representations of the sequences and series highlighted the links between the two, which can be overlooked in a purely symbolic approach. Furthermore, students were also able to connect the *behaviour* of a sequence with that of its corresponding series.

d. Using the stop condition in the code as an instrument for looking at the (convergent) behaviour of a sequence.

One of the elements in the procedures which had significant value was the stopping condition, which is necessary in a recursive code. By varying the stop value in this condition, the students could, for instance, explore its effect on the total number of segments as the stop values decreased (see section A.1.c. in Chapter 6); this type of activity allowed the students to get a feeling for the *rate of convergence.* An example of this is given in section II.a. above, where Consuelo and Verónica investigated the number of segments produced using an arbitrarily small value in the stop condition.

All of the other students also used this approach: for example, Victor and Alejandra, in their exploration of the sequence $\{1/2^n\}$ through the DRAWING procedure (see Chapter 6, section A.1.), explored the effect of changing the value in the stop condition, wondering how small a value they could use (see Table 7.3). They even tried a negative value, using IF :L < -1 as the condition, observing that this condition never became true as the procedure never stopped. Then, using a very small value (0.0000003) in the stop condition, they were able to verify through the numeric (and visual) outputs that after a finite number of steps (31 in this case) the terms of the sequence became smaller than that value.

Scale: 200	**Value in the Stop Condition**	**Arms in the spiral ([Turtle] Turns)**	**Last arm (size)**
	0.2	11	0.1953125
	0.3	11	0.1953125
	0.5	10	0.390625
	0.7	10	0.390625
	0.0000003	31	0.0000001862645

Table 7.3. Table used by Alejandra and Victor to record the differences in the spiral corresponding to the sequence $\{1/2^n\}$, with relation to the value of the stop command in the procedure.

Alejandra and Victor then concluded that the terms of the sequence approached zero in a never-ending (infinite) process, without reaching it, nor going over to the negative values.

Victor: It is going towards zero, but it is never going to get to zero, nor is it going to go over to the negative numbers; it is always going to be to the I-don't-know-how-much [negative] power.

Alejandra: Yes, I think it is not going to get to zero. It gets to very, very small decimals, but it will not get to negative numbers.

Victor: It is never going to stop.

Alejandra: And it is never going to be zero.

Victor: It gets close to zero... but it will never get to zero.

Another pair of students, Elvia and Martin tested the rate of *divergence* of an *increasing* sequence. They had changed the sequence generating function to :N/0.5 (which actually corresponds to the sequence $\{2^n\}$) using "IF :L > POWER 10 2", since they had predicted that it would be an increasing sequence[4]. They then observed that after 7 terms the condition became true, which meant that by the 8th term the value of the sequence became larger than 100. They then increased the value in the stop condition to 1000 (POWER 10 3), and observed that after only 10 terms the condition became true. These experiences highlighted the very fast rate of increase of this sequence, as well as confirming its divergence to infinity:

Martin: It grows indefinitely. The others decreased indefinitely, but the difference is the others had a limit zero, while this one doesn't have a limit, the limit is infinite.

Because the stop value was connected to the measure of the terms (length of the segments) in the sequence, it acted as a primitive situated criteria for determining the way in which a sequence converged or diverged, an approach which is similar to Cauchy's definition of a limit given in Chapter 2. That is, the students were able to stop the process using arbitrarily small (or big) values in the code. They used this to "confirm" limit values through the corresponding visual and numeric outputs. Thus, the code offered the possibility of developing situated methods and criteria for evaluating the convergence of infinite sequences. The use of the stop condition became a "window" for understanding the behaviour of the process.

2. Conceptions of the infinite.

Using the microworld experiences as a window into the thinking processes of the students, the focus of this section is to illustrate some of their shifting conceptions of infinity.

a. Intuition that if process is infinite, then it will diverge.

With some of the students, particularly with those who were less mathematically oriented — e.g., Verónica and Consuelo; and Alejandra and Victor — throughout the study, a common intuition arose: the confusion that if a process is infinite then its value is infinite.

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[4] This investigation followed the exploration of sequences such as $\{1/2^n\}$, $\{1/3^n\}$, and $\{1/1.5^n\}$ where Martin had concluded: "The bigger the denominator the sooner it decreases... And if it's less than one then it will increase, right?"

For example, in Chapter 6 (section A.5), I presented how some students expected the line representing the sum of the segments corresponding to the sequence {1/2n} to grow without bounds, since an infinite number of segments were being added. Verónica, for instance, had predicted that if they "stretched the spiral", and did not use a stop condition, then the resulting line would go all the way past the top of the screen because it would become very long (perhaps infinitely?). She and her partner, Consuelo, were then quite surprised to see that the line got "stuck" at a length twice the scale. Because they were convinced that the line would grow indefinitely, they attempted increasing the scale, but always got a line that eventually "got stuck". Consuelo then observed that they were using a stop condition in the procedure, and believed this was the justification for the turtle stopping at a certain length.

Consuelo: It only stops because it has an IF. It continues straight up. The spiral is stretched. [But it doesn't go all the way] because we have an IF.

But when she removed that instruction the behaviour of the line model was unchanged: the turtle appeared to stop, vibrating in the same place even though the procedure did not stop. The vibrations of the turtle were complemented by a numerical count of segments, and both gave evidence that the process continued even though the visual model seemed to become invariant. Looking for a means to coordinate the two observed factors (the ongoing process vs. an invariant visual model), Consuelo began to realise that as the process progressed the added segments became very small ("it must be that it walks very very little and it can no longer be seen"), something she would confirm by looking at the bar graph model (Figure 7.5) and at numeric values (Table 7.4). As explained in section c. below, the numeric representations played an important role for overcoming the intuition that the result of an infinite process is infinite: they served as a means to justify how the process could in fact continue indefinitely, independently of the result, in that the "number of zeros" in the decimal representations could always increase.

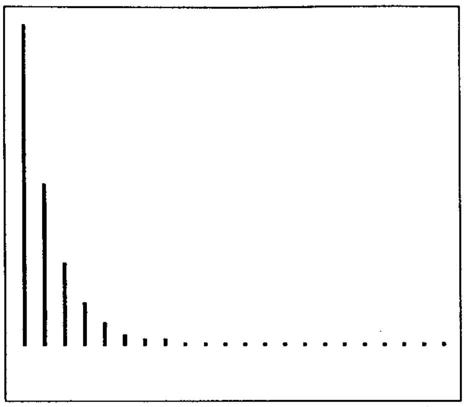

Figure 7.5. Bar graph model for $\{L/2^n\}$.

COUNT	Length walked
95	0.00...-26 zeros-...02524354...
1857	0.0....-557 zeros-...09711...

Table 7.4. Table constructed by Verónica and Consuelo recording the length of the segments corresponding to certain terms (as given by COUNT) of the sequence $\{1/2^n\}$.

However, although the students seemed to have discovered and made sense, *at a first level*, of how a process can *continue* infinitely and not *grow* to be infinite, it would take a much greater deal of experience working with the microworld for the intuition to gradually loose its hold. In fact, when these two students returned for the next session, the intuition had resurfaced and they had partly forgotten what they had discovered and had to repeat the entire process. At the beginning of the next session, both students repeatedly maintained that if the process was allowed to continue indefinitely, the line should continue growing past the edge of the screen. The students again associated the infinite nature of the process with an expectation that the sum of the terms (segments) represented through the line model would show an extended growth. This was a dominant view, even when they perceived otherwise in the visual line models. But the following events led to a change:

First, the visual behaviour (with the turtle vibrating in the same spot) reminded the students that the sequence became very small, as Consuelo confirmed by looking at the list of the values of the segments, although she still thought the line would *eventually* grow past the edge of the screen: "It is going to take a long time, it is going to take a very very very long time, because now it is doing very, very little." Later, the students were surprised when they observed that the partial sums (see Table 6.5 in Chapter 6) eventually became a constant value. This led to an investigation of the values of the last segments and they observed how small those values were with tens of zeros after the point in the decimal expansion, which served as first explanation for the convergence of the sum, even though the existence of a bound or limit was not yet fully realised. Further analysis of the values of the partial sums finally led Consuelo to realise that there was indeed a value which would not be reached nor surpassed, and she was able to explain the on-going nature of the process through the numeric decimal representation where more digits can always be added. For Verónica this realisation would take longer, as the intuition that the line should keep extending, despite all the evidence to the contrary (which she dismissed by saying they were computer "rounding" errors), remained through further explorations. Although the idea was still dominant in Verónica, a change started to take place during her conversation with Consuelo: she started to realise that because the added segments became very small, the sum would not grow much.

Consuelo:	It won't reach 100, by a few digits...
Verónica:	It does pass it, but because there are too many numbers, then it rounds it; the computer cannot write down so many numbers...
Consuelo:	I believe that no, it is not going to pass 100. I think it is always going to stay where it is, because it doesn't pass 100, so the nines are going to keep increasing: nines, nines, nines, and so on...
Verónica:	Right, it is not going to pass 100.
Ana:	So is it going to go off the screen at some point?
Verónica:	It would reach the top of the screen and go off, if it didn't round so much...
Ana:	So you think that if it did not round, it could go off the screen?
Consuelo:	No, it cannot go off the screen... It would have to reach 100 in order to be able to go off the screen... pass 100, and to get to the top of the screen it would have to pass much more..
Verónica:	So, it's impossible?

Verónica seemed to be focusing on the process as indefinite, as a process that goes on and on, and therefore felt that the line should keep growing (go off the screen); Consuelo on the other hand realised that the process could continue without necessarily passing the observed bounds, and she found a numerical explanation for this in terms of "you can always add more nines to the decimal expansion 99.9999... and therefore never reach 100".

The intuition discussed above seems to be a deeply rooted one since it would often re-emerge, and was also observed with other students (e.g. Alejandra and Victor). It is also interesting that this intuition particularly re-emerged after the explorations with the divergent Harmonic series (i.e., where the value of the infinite sum is infinite), which is a case in accordance with that misconception. For example, as described in Chapter 6 (section A.6), after the study of the Harmonic series, Alejandra and Victor were convinced that the series $\sum \frac{1}{n^2}$ would diverge and they spent quite some time trying to find evidence that those series would exceed the value 1.65. When they noticed slight increases in the decimal expansions, they would try to use that to support their intuition:

Alejandra:	It's still increasing...
Ana:	Do you still think that this sum will go off to infinity?
Victor:	**Yes, that is, since there are infinite numbers, the sum can be infinite.**
Ana:	What do you mean by "infinite": Do you mean infinite decimal digits, or that it actually measures infinite?
Victor:	It can measure infinite.
Ana:	Tell me, do you think that the sum will pass, for instance, 10?
Alejandra:	**Yes, it has to pass it, if the number is infinite. Yes, it has to be a very very big number.**
Ana:	And why are you so convinced that it has to pass 10?
Alejandra:	Because although it increases very little, it is still increasing.

But generally, as the students gained more experience, this intuition would gradually lose strength: that is, whereas at the beginning of the study most students relied heavily on the decimal structure of numbers to cope with an infinite process having a finite value (limit), at later stages, even though the intuition would often briefly re-emerge, it would be more easily dismissed than at the beginning of the study. I believe this is due to the fact that at the beginning the dominating idea was "more is bigger", but through their experiences other meanings would be constructed; e.g., the continuity of the process is found in the decimal expansion, not in the total length[5].

The type of intuition discussed in this section has been found by other researchers. Nuñez (1993) — reviewed in Chapter 3 — in particular explains that the problem arises when there are several competing components (processes) present; that is, when two types of iterations, of perhaps different nature (cardinality vs. measure), are confused: the process itself and the divergent process of adding terms to the sequence. Thus, in the case described above where infinite sums are involved the

[5] In the case of the Koch curve (see Chapter 6, section B.1.d) — where the infinite perimeter was bounded within a finite area — the *shape* of the figure was found to be the determinant factor for explaining the infinitude of the process.

intuition appears as: "if an infinite number of terms or elements (*cardinality*) is added then the *measure* of the sum must be infinite, it must pass any preset value."

It is relevant to point out, however, that, whereas in the context of sequences and series, the approach was in the direction: process to visual/numeric — which led to the belief that adding more would imply a larger figure/value — in the context of the fractal explorations, the approach began with the figure. In the latter cases, the dominating factor was the visual image (which was *visually* invariant), particularly with the more inexperienced students (possibly because these students were more susceptible to be influenced by the most dominating factor present). Thus, in the case of the Koch curve for example, the perimeter, which grew at each stage as more segments were being added, was initially thought by Alejandra and Victor to be constant since the segments added were too small (the prevailing conception here was "what is added is too small to make a difference" — see Chapter 6, section B.1.b.). This is another finding which is in accordance with the findings of many other researchers (e.g. Fischbein, 1979; Waldegg, 1988; Nuñez, 1993; Hauchart & Rouche, 1987) which point to the influence of the context in which a situation is presented.

Another interesting finding is that with the more "mathematical" students — Manuel/Jesús and Elvia/Martin — the intuition that "more is bigger" did not dominate: these students had an *a priori* knowledge that certain infinite processes (e.g. the series of $\{1/2^n\}$) were convergent, and they approached the investigations differently.

b. Koch curve 'paradoxes': solving an indetermination by coordinating two simultaneous infinite processes.

In the section above, I discussed a situation that required discerning and coordinating two simultaneous infinite processes: the infinite iterative process of adding or increasing the number of terms, with the behaviour of the process itself (which could be convergent). In this section, I will give another example that also involves the coordination of several simultaneous infinite processes.

During the Koch curve explorations (see Chapter 6, section B.1.b.), for some students the idea of an infinite perimeter formed by an infinite number of "zero-length" segments caused anxiety.

This was particularly the case for Manuel and Jesús: they had initially thought that the perimeter of the Koch curve tended to be infinite, but then confusions arose when they tried to relate this to the segments which tend to zero:

Jesús: Oh... it is zero. Then it could not be infinite, the perimeter, like we said... or can it?

This triggered a very long discussion in an attempt to determine what happened to this curve at infinity. Manuel, focusing on the line as made up of an infinite number of points each measuring zero, first concluded:

Manuel: Then it will evidently be a curve. It wouldn't have segments. It would be a curve or a line... It would be an infinite sequence of points.

On their worksheet they wrote the following arguments as to why the perimeter of the curve tended to infinity, focusing on two notions: the number of segments tends to infinity, and the perimeter is potentially infinite because segments are always being added to it:

" If $N = \infty$, it is an infinite sequence of points, and the perimeter must be infinite.
" If the level is infinite, the perimeter also is (could be) infinite, because it would not stop increasing even if the size of the segments was very small."

The confusions, however, continued and the students realised it was necessary to further analyse this situation, including algebraic and numeric explorations:

Jesús: I was analysing it, and the size of the segment becomes very small; but what always increases is the amount of segments, and that does go off until infinity so there are infinite segments...
Manuel: Which measure zero...
Jesús & Manuel: So they are points.
Ana: What about the perimeter?
Manuel: Well... by watching its behaviour... Why don't we use the formula as a guide?
Jesús: Well..., by observing the numerical behaviour it should give us the idea that the perimeter will become very large.
Ana: How large?
Manuel: Well, if the number of terms is infinite, then it will be infinite.
Jesús: That's according to the numbers, to how it is growing...
But there is a problem: the number of segments increases, but they also become very small.
And in fact we already saw that this function has a limit when it is $1/3^N$, it goes to zero.
Manuel: If there are infinite segments...
Jesús: Then there is no perimeter.
Manuel: It would be 0.
Jesús: There wouldn't be a perimeter. It is like there wasn't any perimeter. It would be zero.

Manuel: What I say is that the segments would no longer be line segments, they would be points, and so it would no longer be a star-like shape, it would form a "curve", to call it something, but I don't know what shape it would have after that....

The transcript above shows how Jesús was aware of the problem of having two types of processes involved in the change of the perimeter: the increase in the number of segments, and the decrease in the size of those segments. He realised that *the behaviour of the numerical values* pointed towards the perimeter becoming very large, while Manuel considered that it tended to infinity. But when they considered that the segments *at infinity* measured zero, this seemed to indicate that *at infinity* the perimeter would measure zero! In fact, by focusing on the latter process Jesús would challenge the idea of the divergence of the perimeter: "The segments are getting smaller... The perimeter cannot be infinite...." But Manuel had a different perspective: he focused more on how the zero-sized segments would affect the *shape* of the figure. Through their line of thought the students were of course dealing with what is defined as an *indetermination* (infinite number of segments of size zero: ∞ x 0). When Manuel decided to go back and look at the process from an algebraic perspective he soon discovered this:

Manuel: Better think of where we are going, an infinite number:
$100/3^{N-1}$, if N is infinite, then it is zero, right?
And 4^{N-1}, if N is infinite, it is infinite. And how much is zero by infinity?...
Oh! How awful! What is infinity times zero?

Initially they were unable to solve this situation, which was a definite source of anxiety:

Manuel: I don't really know about the perimeter: one theory says it will be infinite, and the other that it is zero... I can't even imagine it.
Jesús: The problem is we are multiplying an exageratedly small number by an exageratedly big number...
Manuel: This seems to be beyond the limit of my imagination.
Jesús: Oh my God! I think I am going to be thinking all day long...

When Jesús came to next work session, he brought with him a written list of conjectures for solving the paradoxical situation and was convinced the perimeter would tend to infinity. He wrote:

" 1) The perimeter will be infinite, because the length of the segments will never be equal to zero, and their number increases permanently.
" 2) Because the perimeter is obtained by multiplying the number of segments by their length, then a product is obtained where an extremely small number is multiplied by another that is extremely large, therefore it is always increasing.

" 3) The sequence $1/3^n$ tends to a limit but 4^n does not have one, therefore the perimeter must be infinite.
" 4) Perhaps we need to observe how much [multiplying by] $1/3^n$ reduces 4^n; that is, how many decimal places move to the right after doing the product."

The first argument shows how Jesús conceived the process as only *potentially* infinite, and thus the limit zero of the size of the segments would never actually be reached. His reasoning was then that if the number of segments of non-zero (no matter how small) measure increased, then the total must always increase without bounds; an argument which — although faulty as expressed — *intuitively* helped support his idea that the total length tended to infinity. A similar type of reasoning is shown in his second argument. In his third argument he is perhaps saying that because $1/3^n$ has a limit and 4^n does not, it is like multiplying something which tends to infinity by a finite value which would not change this tendency. But the most interesting is his last observation: he was interested in how each of the factors ($1/3^n$ and 4^n) affected each other. This is a key issue since it is the different rate at which each of the two sequences progress which is determinant in the final outcome. Jesús was aware of this, as shown in his oral explanations below (these explanations also clarify his thoughts behind his second argument above). He used numerical explorations to explore the behaviour of the perimeter, verify his hypothesis, and become convinced of the divergence of the perimeter.

Jesús: Yes, I now have the total conviction that the perimeter of the curve is infinite.
I was analysing what happened with both elements in the product:
On the one hand the length of the segment: even if N is very very large that function, $L/3^{N-1}$, never gets to be equal to zero. It would always be an extremely small number.
And the other element, which is 4^{N-1}, that is going to be a very, but very very large..., too large, way too large, number... So the number of segments is increasing, and it will be multiplied by a very small number, which will reduce it a bit, but **the increase is more than the decrease... so even though the segments are extremely small, the perimeter will always increase.**
So that is why I say that the perimeter will be infinite.

I even did some computations using a scientific calculator, and I was able to get as far as 320 [for N], and that number is already very very big. So with that result and the reasoning I did, I can say that the perimeter will be infinite.

So the segments will be very small, but they are never going to be equal to zero. The limit is supposedly zero, but that is when we divide by infinity or an infinite value, and that cannot be done.

Manuel, on the other hand, still had a conflict between what his intuitions told him, and his attempt to apply (finite) mathematics and "logical" principles ("a number multiplied by zero is zero" vs. "a number multiplied by infinity is infinite"), and his confusions would resurface during the explorations of the Koch Snowflake perimeter:

Manuel:	We didn't really settle what happened with the curve... and if it became zero, then it would only be a point it seems, at least now that I think about it that's what it seems... So **if the number is infinite the perimeter is zero, and what will happen? That all of this will become a point.**
Jesús:	No, I do not agree because the number of segments is always increasing and the perimeter will always be a very large number.
Manuel:	I agree that the number of segments is infinite, but tell me, what size will they be?
Jesús:	Extremely small.
Manuel:	Of size zero.
Jesús:	Well, they would be as if of size zero, but **they wouldn't be exactly zero.**
Manuel:	**No, they would be of size zero.**
Jesús:	Why? Because we are reaching infinity? That's why you say they would have size zero and that they would be points?
Manuel:	According to the formula, yes.
Jesús:	But what I say is that *relative* to the perimeter we know that the length of the segments cannot be zero.
Manuel:	Yes, it is, it is entirely equal to zero.
Jesús:	No, it wouldn't be equal to zero, it would be extremely small, but it would *not* be equal to zero.
...	
Jesús:	What you are saying is that then only a point will remain?
Manuel:	Yes. It becomes very, very small.
Jesús:	No, I disagree. How is it possible that after having something so large then it suddenly becomes that? No!

The paradox faced here by the students is analogous to Zeno's paradoxes discussed in Chapter 2. In this case there are two components present: the *number* of segments, and the *measure* of the segments. As in Zeno's paradoxes, the construction of the Koch curve involves infinite subdivisions of the continuum, and the problem thus touches on many mathematical areas related to the infinite: limits of infinite processes, infinite sets, and the nature of the continuum.

Manuel wanted to conceive the infinite process as completed, considering, at that point, that the segments forming the curve would measure zero, implying a sort of "collapse" of the curve into a point. His dilemma is also reminiscent of the difficulties pointed out by Galileo — quoted in Chapter 2 — which emerge when trying to conceive infinite sets with the conceptual schema of the finite. As has been pointed out in Chapters 2 and 3, the problem of thinking of the infinite with the schema of the finite is common and found both in history and by mathematics education researchers (e.g. Waldegg, 1988).

Jesús on the other hand, as discussed above, did not accept that the segments could ever *equal* zero; for him the segments would become very, very (perhaps infinitesimally) small, but never zero. Jesús had a *potential* view of the process. But was is really important in Jesús's approach is that he takes into account both of the processes present and considers the idea that "relative to the perimeter" the segments would never be zero: he considered that the perimeter's increase was faster than the segments' convergence to zero. Furthermore, he intuitively rejected Manuel's proposition that the curve, after growing very large, would suddenly collapse into a point. But for Manuel it would take a longer process of (particularly numerical) explorations and reflections to become convinced of the divergence of the perimeter, and even then some of his doubts may not have been clearly resolved. In fact, Manuel faced a similar dilemma when he explored the Cantor set, as described in Chapter 6 (section B.3).

It is interesting to observe the influence of the way in which the students conceived the formula for the length of the perimeter — as the size of each segment (determined by $1/3^N$) multiplied by the number of segments (4^N) — in that they did not abstract their reasoning from the resulting formula. That is, they did not consider that from a purely algebraic perspective it can be deduced that $\frac{1}{3^N}.4^N = \frac{4^N}{3^N} = \left(\frac{4}{3}\right)^N$ which solves the indetermination; and, since $\frac{4}{3} > 1$, the length of the perimeter clearly diverges as N tends to infinity.

Whereas limit indeterminations are traditionally solved through algebraic manipulation, in this case Jesús overcame the indetermination through analysis of the *behaviour* of each of the elements involved, observing specifically the difference in the *rate* of divergence or convergence of each of the elements — the rate of decrease in the size of each segment vs. the rate of increase in the number of segments — coordinating the two processes involved. This leads to the question of whether this type of analysis could perhaps help solve the intellectual misgivings that a mere algebraic proof does not resolve.

c. Using the decimal structure of the numeric output for justifying the infinite nature of a bounded process.

As was mentioned in section III.2.a. above, an interesting finding which emerged from the study was the way in which students coped with the infinite nature of the processes, particularly when these processes were bounded (or convergent), by using the decimal structure of the numeric values. Although students may have been more or less convinced of the way a process behaved (for instance that it approached a finite limit even though the process continued endlessly), it was through the numeric values[6], and specifically in their decimal structure, that the students found a way to cope with this infinite nature. The decimal structure provided a means to cope with an apparently paradoxical situation arising from the spontaneous intuition described above: that of a *bounded* infinite process.

For example, when Alejandra and Victor were investigating the behaviour of the series $\sum \frac{1}{n^2}$ (see Chapter 6, section A.6.b.) which they thought would diverge, eventually Alejandra noticed how small the values being added were, and realised that in the decimal expansion of these values the number of zeros would increase more and more as the sequence progressed. This was the turning point which allowed her (and Victor) to find a way in which the process continued although it was bounded:

Alejandra:	So it is adding 0.00000044... So then, no, it is not going to reach 1.65. It has already stayed around 1.64 too long... And here [the zeros in the decimal expansion] are going to keep increasing, aren't they? So this one also has a limit like the other ones, right? That is, it doesn't exceed, doesn't even reach 1.65. We could say that is its limit, but it never reaches that limit. But it does get close to it.
Victor:	Yes... So it grows in the decimals, doesn't it? After the zero point zero zero something...

The decimal structure of numbers is the way they found to cope with a bounded infinite process; it was the means through which two seemingly paradoxical characteristics of the process were integrated: the infinite nature of the process and the bounded result of the process. The key issue here was that *the infinite nature of the process was reflected in the decimal structure of the numeric values.* Students were of course making use of properties of the real numbers and their decimal representations:

[6] I would like to acknowledge here the advantage of the extended computing and numerical capabilities of the computer, which allowed students to explore decimal values with as much precision as wanted (up to 1000 digits in the decimal expansion).

a) the density of the real numbers allows for a value to always be found between the previous one and the limit (or bound) value;

b) the real numbers defined as decimal representations are, as conceived by Cantor (see Chapter 2), an infinite sequence of digits that can be seen separately from the geometry of the real line. We can consider that the students took advantage of two factors related to the decimal representation of the real numbers: i.) seeing the decimal representation as potentially infinite, allowing for an infinite process to take place in the "infinitely small" (see also Chapter 2); and ii.) seeing the process from the point of view of the numeric, temporarily disassociated from the geometry, which allowed the students to cope with the visual boundaries.

It is worth contrasting this finding with that of Ferrari et al. (1995) who found in students difficulties with the idea of density; they related this to students' problems in accepting that a bounded set can contain infinite points. I argue that this is a problem which arises from a failure to coordinate properly geometry and number (Ferrari et al. themselves acknowledge the presence of confusions between measure and cardinality). In the case of the microworld experiences, students had the opportunity to explore a situation initially presented in geometrical form, through linked numerical values, which were also structured in tables, and which led them to discover and make use of the property of density of the real numbers for making sense of the bounded geometrical situation.

The use of the decimal structure of the numeric values by Victor and Alejandra emerged throughout the study (as well as with other students, particularly Consuelo and Verónica), including in the study of the Koch curve where it helped to cope with the fact that the infinite perimeter could be bounded in a finite area:

Victor: The area is constant although the perimeter will always increase. The area is 6928.20323, and it will not go beyond that. It keeps growing, but as we saw last time, it grows to the right of the decimal point, at millionecimals, billionecimals, trillionecimals, I don't know how small...

However, I should note the dependency on the numeric structure also seemed to reinforce the dynamic approach to the limit, with the conception of the limit as unreachable. For example, while Consuelo and Verónica confirmed that the process of "taking halves" (see Chapter 6, section A.1.) could continue indefinitely, as more and more zeros could always be added to the decimal expansion, because this process of adding zeros to the decimal expansion was seen as *potentially* infinite, they concluded that the value of :L would never become zero:

Consuelo:	It's going to keep going, isn't it? Afterwards it will be [in the decimal expansion of the length of the last segment] more zeros and more zeros, and more zeros.... and so it would never get to zero. And so we can use a condition that says that when we get to 5 decimal digits it should stop.
Verónica:	Yes. So it is going to keep increasing each time the zeros to its decimal list. So it is never going to reach zero.
Consuelo:	It is never going to reach zero.

The increase in the number of zeros in the decimal expansion justified the approach of the sequence to the limit zero, but the endlessness of the process (the process seen as *potentially infinite)* dominated, preventing the limit from ever being reached.

d. The link between geometry and number, and students' conceptions of the real number line.

An interesting point to note is that because in the microworld the graphic and numeric representations were intimately linked through the code, students naturally associated both representations. Thus, the structure of the microworld seems to assist in the problem pointed out by, for instance, Cornu (1991) who observed an "obstacle" of students failing to link geometry with numbers.

With reference to this, it is worth looking briefly at the changes in some of the students conceptions of the real number line, although I should clarify that it was not part of my research objectives to "teach" students about these topics or aim to create conceptual changes.

During the initial interview and questionnaire, Jesús's answers reflected that for him irrational numbers did not have corresponding points on the line; for instance, he had answered that $0.\overline{3}$ could not be located on the number line. (Some of the other students also had similar ideas, particularly Victor who explained he thought irrationals were "vibrating" points with no fixed corresponding point on the line.) At the end of the study, however, Jesús referred to the issue as one of which he held a different position. He said that from the beginning he had always had a conflict when he believed that 1/3 had a precise point on the line, but not 0.333.... Through the microworld experience, however, (where for example, the segments of the spirals were cut in thirds) he explained that he had understood that both numerical representations were the same and could be geometrically represented. Finally, he added he now also realised that for every number there exists a corresponding point on the real number line, even [irrational] numbers such as π and $\sqrt{2}$.

Jesús's conceptual change is illustrative of how the microworld experiences — by integrating different forms of representation — formed a basis for the understanding of the relationship between the numerical and geometrical domains, and seemed to help in building a more integral conception of the real number line.

e.Students' conceptions of limit.

It should not be surprising that the procedural focus of the activities, where the processes (and procedures) were presented as potentially endless, was accompanied by the prevalent view that the limits of convergent processes will never be reached. As has been noted by many of the researchers reviewed in Chapter 3, the dynamic model of limit, with the limit as unreachable, tends to be the prevalent conception among students, particularly when the limit is expressed as the result of an infinite *process*. The students who participated in my study were no exception, and most exhibited this dominating perspective from the onset of the study, including during the initial questionnaire and interview.

Espinoza & Azcárate (1995) have noted (see Chapter 3) that there is an intimate link between the concept of limit and the concept of real number. As in the example above, the decimal structure and properties of *density* and *completeness* of the real numbers always allow for "intermediate" values to be found between any value picked as close to the limit as wanted and the limit value, preventing the limit from ever being reached.

i.) The limit is a boundary that is approached but never reached.

It is interesting that Manuel and Jesús, during the final interview, defined limit as that which is approached but never reached, *except* at infinity.

Manuel:	For example, in a sequence [the limit] is the value that the sequence approaches...
Jesús:	But it [the limit] will never be equal to that [the value].
Ana:	Not even at infinity?
Jesús:	At infinity, yes.
Manuel:	At infinity, yes.
Jesús:	Well..., it is not that it *never* gets to be equal, but it only happens at infinity.

These students exhibited a *learned* idea of a limit as that which is reached at infinity, but showed inconsistencies in their conceptions. Despite the above definition, a conflict occurred when I asked them what the limit of the sequence 0.9, 0.99, 0.999, etc. would be. Both students began by saying that the sequence approached 1, but both added that not even at infinity would it become one. Jesús explained:

Jesús:	We would have to add zero, zero, zero, zero,... 1.
Ana:	So at infinity would it be 1?
Manuel:	No.
Manuel:	Simply because infinite nines do not reach 1.
Ana:	So the sequence 0.9, 0.99, 0.999, ..., does it tend to anything?
Jesús:	Yes, it tends to 1. But it never gets there. It will never be equal to 1, no even at infinity.
Manuel:	No, it does not tend to 1, because if it did, then at infinity it would be 1.
Jesús:	O.999999... that would be its limit.
Ana:	That's the limit? but not 1?
Jesús:	No, not 1. If it tended to 1 then at infinity it would be 1, but it does not get there...
Ana:	So zero followed by infinite nines is not 1?
Jesús:	No, it is not 1.

When Manuel realised that saying that the limit of the above sequence was 1, and yet arguing that not even at infinity would it become 1 contradicted their definition of a limit, he changed his mind about 1 being the limit. Jesús then found an exit to the problem by saying that the limit was 0.9999..., which for them clearly differs from 1. This example is illustrative of how deeply rooted is the conception that 0.999... does not equal 1. During the initial questionnaire and interview, both of these students had already given indication that they did not think of 0.999.... as equal to 1 — a finding of my study that is shared by many other researchers reviewed in Chapter 3. This conception seems to be so dominating that it prevailed after all the experiences and discussions.

In fact all the students, except Elvia and Martin who are mathematics teachers (see Chapter 4), affirmed that 0.9999... was *not* equal to 1. This is exemplified in the following transcript which took place when Victor and Alejandra were investigating the series $\sum \frac{1}{2^n}$:

Victor:	It gets close to one: it is .9999999702
Ana:	Does it get to 1?
Alejandra:	No, it gets close but it doesn't get there.
Victor:	It gets close but it doesn't get there.
Ana:	What about in an infinite amount of time? Will it be 1 at infinity?
Victor:	No, it will never reach 1.
Ana:	Not even at infinity?
Victor:	No, not even at infinity
Ana:	Alejandra, what do you think?
Alejandra:	That no; infinitely after the point it is going to be more nines.
Victor:	Yes, nines... so it never gets to 1.
Ana:	So how much do you think it measures at infinity?
Victor:	Absolutely infinite: point, an infinite of nines, and then at last something like 762
Ana:	Wouldn't it just be: point, infinite nines?
Victor:	Well, yes. But it would never reach 1.

ii) Infinitesimal differences:

One way in which students seemed to view the "unreachability" of the limit was in terms of an infinitely small (i.e. an infinitesimal) difference or "gap" separating the limit from the values that approach it. For example, when Alejandra and Victor were studying the decreasing behaviour of the Sierpinski triangle area, by "visualising" the limit zero through the numerical values they explained that this limit would never be reached "because there will always be a gap":

Alejandra:	Yes, it does approach zero.
Victor:	It will approach zero, but it will never get there. It is going to be infinitely small, but it will never get to 0.
Ana:	Why do you think it will never get to zero?
Alejandra:	Because there will always be a gap.

These findings are concordant with those of, for instance, Tall & Schwarzenberger (1978) — see Chapter 3. They point to the evidence discussed by Cornu (1991) that students have infinitesimal notions; notions which were also prevalent throughout history until avoided by formal modern calculus and the formal definition of limit, as discussed in Chapter 2.

iii.) The visualisation of "limits".

It is interesting that although I never explicitly talked in terms of "limit" with Alejandra and Victor, or Verónica and Consuelo, it was a term that all of these students spontaneously used, although they used it indistinctly for both bound and limit. The visual representations seemed to "show" when the sequences (or series) or figures were bounded and "limited", but it was through the combination of the visual and numeric representations that the students were able to "see" the limits. For example, the comment below was spontaneously given by Alejandra at the end of the fractals session, although I had not made any particular reference to that topic. In her comment — without being misled by the language she uses — she seems to be referring to the difficulties in predetermining when a particular process will converge or diverge. While referring to the particular intuitive difficulties of having a figure which is visually (geometrically) bounded, but which has an extension which diverges to infinity, she said:

Alejandra: For limits it is difficult to be able to really define what a limit is. For instance, in these figures, we had a certain limit where the entire extension of the perimeter had to fit in there. So, it is difficult to deal with the concept of limit with so many different cases.

The acceptance of an infinite perimeter bound in a finite area was made possible through the combination of the different representations: the perimeter had been accepted as infinite through the *numeric* explorations, and the area was first seen as finite through the *visual* image, and later confirmed through numeric methods.

I would like to emphasise the role of the graphic representations in the visualisation of limits, as students toward the end of the study used visual criteria for determining the convergent or divergent behaviour of the processes studies. For instance, during the final interview Manuel gave the following method for determining whether a sequence had a limit or not:

Manuel: By looking at the graph, particularly the histogram, because there we could see how it increased and, if afterwards, it reached a point where it looked flat and did not have any noticeable growth.

Jesús: Or a decrease.

Manuel's criterion is based on the search for asymptotic behaviour, and it reflects the influence of the visual representations used in the study.

iv.) Making sense of the formal definition for the convergence of a sequence.

During the final interview I presented students with Cauchy's formal definition of the limit of a sequence (see transcript below). I would like to remark here on how this definition uses the idea that a term in the sequence can always be found as close to the limit as wanted. It reflects the idea of an infinitesimal neighbourhood around the limit, which is how students conceived the "approach" to the limit, as described above. But what is interesting is that most students were able to relate this definition to their experience with the microworld activities. This was the case of Manuel in particular:

Ana:	I will give you a definition: Given an infinite sequence a_1, a_2, etc..., the sequence is said to have a limit L if for any positive number as small as I like, which I'll call "epsilon", there exists a term a_N in the sequence such that for any term after that one in the sequence, the difference between that term and the limit is less than the small number I chose, epsilon. What do you understand of this definition? Does it have anything to do with what you did to determine if something had a limit or not?
Jesús:	Yes
Manuel:	Yes, it is related...
Manuel:	The difference a_N-L, in absolute value... If there is no limit, then that difference could never be smaller than epsilon. Then in the graph we would see how the histogram continues to grow, even if its very little, but it would grow. But if it does have a limit, then because epsilon is as small as wanted, then the difference would be smaller than epsilon. Then, in the histogram we would see a point, a position where it would stop growing (or decreasing) and well, then it would seem to stay constant and would not increase anymore. And then that difference would be smaller than any epsilon that we chose... Yes, and it is also like with the computer, when we verified that [the process] tended to a limit, such as when it tended to 100, then we would draw a line measuring 100, and the two lines looked the same.

The way in which Manuel made sense of the definition by relating it to his experience is illustrative of how connections are formed, and meaning constructed. It exemplifies ways which can help to "make something abstract more concrete", as discussed in Chapter 3.

e. In the infinite the behaviour of things is "weird".

To end this chapter I would like to narrate a conversation, regarding infinity, that I had with Manuel and Jesús at the end of their final interview. First, I asked the students what they thought of how things behaved at infinity. Manuel´s answer: "Weird". Jesús was more specific: "We could say that, in a way, they repeat what was at the beginning, except...." He then explained how the observation of the behaviour of the process *in the finite* provided a glimpse for the behaviour at infinity:

Jesús: But we could say it gave us an idea, because while it was growing in the finite we could see that something was increasing while something else decreased, and it gives us an idea that at infinity it would have a limit.

It is also worth noting that both Manuel and Jesús acknowledged that infinity is different from "a very large number" (a common confusion in children — see chapter 3) and is also not an "amount", even if they were not quite sure how to define it. They also seemed to be clear that the infinite cannot be quantified in the way that finite quantities can (e.g. "we cannot say 'this is half of infinity'"; see below) which indicates their (new?) awareness that finite operativity and logic cannot be applied to the infinite — the same realisation that Galileo had more than 300 years ago (see Chapter 2).

Ana: Can you tell me what do you understand by "infinite"[7]?
Jesús: A larger number than... No, rather an amount. No, it wouldn't be an amount, nor a number...
Manuel: It would be a symbol because...
Jesús: When we talk of the infinite, we are talking of extremes, it is something very extreme, very radical.
Infinity can be something very small or something extremely large...
Manuel: But "very big" is a term, we could say that is finite, and so if we say infinitely big...
Jesús: ...it is a expression that means it doesn't reach any limit...
Manuel: Exactly.
Also, there does not exist any number that can measure the cardinality of the infinite. That is, we cannot say, "this is half of infinity." Or, "this is getting close to infinity."
Jesús: No, it could never be.
Manuel: Exactly. We cannot say: "Well, this is almost reaching infinity," because in that case it is definitively NOT infinite.

[7] In Spanish the same word is used for both "infinite" and "infinity": *infinito.*

IV. Summary of findings.

Summing up, as the above episodes illustrate, the microworld took advantage of the technology to go beyond a purely algebraic/analytic approach. The findings appear to indicate that the microworld served:

a) to highlight the *behaviour* in the finite of the infinite processes: the behaviour in the finite provided clues into the behaviour at the infinite;

b) to allow, through its tools, to carry out "part by part" analysis which allowed students to uncover and coordinate the multiple elements present within an infinite process;

c) to give a means for the students to investigate the relationship between the different elements present and find ways to coordinate these elements, thus assisting in resolving possible paradoxes; and

d) to give students a means to explore, talk, and reflect about the infinite, allowing them to be mathematicians, rather than passive receivers of (often uncoordinated and therefore paradoxical) information about the infinite.

In the next chapter, I present the general conclusions and implications of this research.

Chapter 8:

Conclusions and Implications of the Research

In this concluding chapter, I summarise and bring together the main ideas that emerged during the study. I begin by reviewing the aims and findings of the research. I then discuss three relevant areas of consideration:

a.- The microworld as a set of open tools for constructing meanings.

b.- Shaping understandings of the infinite using the tools of the microworld.

c.- Situated proofs: extending the notion of situated abstraction.

I then briefly comment on some affective issues. Finally, I consider the limitations of the study and state the implications of the research.

I. Summary of the aims and findings of the research.

The main aim of the research was to investigate the ways in which students constructed meanings for the infinite through their involvement in a computer-based microworld which engaged the learner in constructive programming activities and facilitated the interaction between diverse types of representations.

The computer environment provided a window for studying students' shifting conceptions — their *thinking-in-change* — as they used the different tools provided by the microworld to explore and express their ideas. The focus of the research was to observe the ways in which students *made use of those tools and formed connections between different types of representations* in their search for meanings for the processes they observed. The focus was thus on the mediating role of the microworld in the construction of meanings for the infinite, rather than on investigating students' conceptions of infinity and limits from the perspective of formal mathematics, or on finding cognitive obstacles or difficulties for the learning of those concepts.

The design of the microworld which formed the basis for the study drew from several theoretical considerations:

- the idea that learning involves the construction of representations, which are *tools* for understanding, and which mediate the way in which knowledge is constructed;

- the hypothesis that engaging in the construction of multiple modes of representations may be helpful for the construction of richer meanings;

- the premise that the computer-based microworld could act as a *domain of abstraction*, where the learners might abstract and generalise mathematical relationships and properties through the tools of the medium.

Based on the above considerations, the work had the following aims:

a.- To investigate students' conceptions of the infinite as mediated by the different tools and external representations (symbolic, visual, numeric) provided by the microworld.

b.- To probe the ways in which students made use of the environment in order to make sense of the phenomena they observed, and the ways they explored and manipulated ideas in order to make them meaningful.

c.- To study the ways in which the different forms of representations were coordinated and integrated, in particular through their interaction with the procedural code.

One of the aims of the study was to use the computer and its visual and numeric capabilities in a way that could make the infinite more accessible. Part of the difficulty in making sense of the infinite comes from the fact that the infinite can only be accessed through the finite. Whereas in traditional mathematics infinite processes have been studied using a predominantly analytic and algebraic approach that focuses on the result of the process, the microworld incorporated different (interconnected) representational forms that provided a new perspective on the processes.

The data analysed in Chapters 6 and 7 illustrates ways in which the students could use the microworld to explore and gain insights into the behaviour at infinity *through the behaviour in the finite*. The investigation of the infinite processes under study was done in the following ways: (i) through their evolution in time (as the processes *unfolded)* eliminating the limitation of only observing the final state (the result of the process); (ii) providing a visual image of the entire process; and (iii) providing a means to investigate the rate of convergence (divergence) of a process, in particular via the programming activities (e.g. by varying the value in the stop condition). Furthermore, the fine grain tools of the microworld permitted the students to uncover and coordinate the different elements present within an infinite process. This is a key point since many of the paradoxes of the infinite result from confusing and failing to coordinate simultaneous infinite processes or elements involved within a situation.

Three main points emerged as findings of the study:

1. The structure of the microworld was such that it provided adequate tools which students could use to coordinate different types of representations.

2. The environment and its tools shaped students' understandings of the infinite in a way that highlighted, in particular, the case-specific characteristics of the infinite processes, and permitted the students to deal with their complexity by uncovering and coordinating the different elements present.

3. The students were able to draw from the above activities to construct situated abstractions and what I have defined as "situated proofs".

Each of these points is reviewed below.

II. The microworld: A set of open tools for constructing meanings.

A valuable attribute of the microworld was the means it provided for the students to explore and express their ideas through the medium and construct meanings through the coordination of the different elements involved. As was explained in Chapters 4 and 5, following Papert's constructionist paradigm (see

Chapter 3), the microworld was designed in a way that provided means for manipulating and coordinating several types of representations, particularly via programming activities and through the programming code. The findings indicate that the students took advantage of these opportunities, using the tools of the microworld in an interactive way that helped coordinate the different elements involved in the situations they were studying.

I begin by reviewing the main elements and representations involved in the microworld and the ways in which they interacted, and then go on to discuss how the coordination of these representations webbed meanings for the objects and processes under study.

There were three main representational elements involved in the microworld: symbolic (the programming code), visual (geometric figures), and numeric (numerical values). A mathematical process could be represented in each of these complementary forms: symbolically in the code, and visually or numerically by running the code. Thus, the structure of the microworld was such that the process and its different representations were all linked through the computer code, as shown in Figure 8.1.

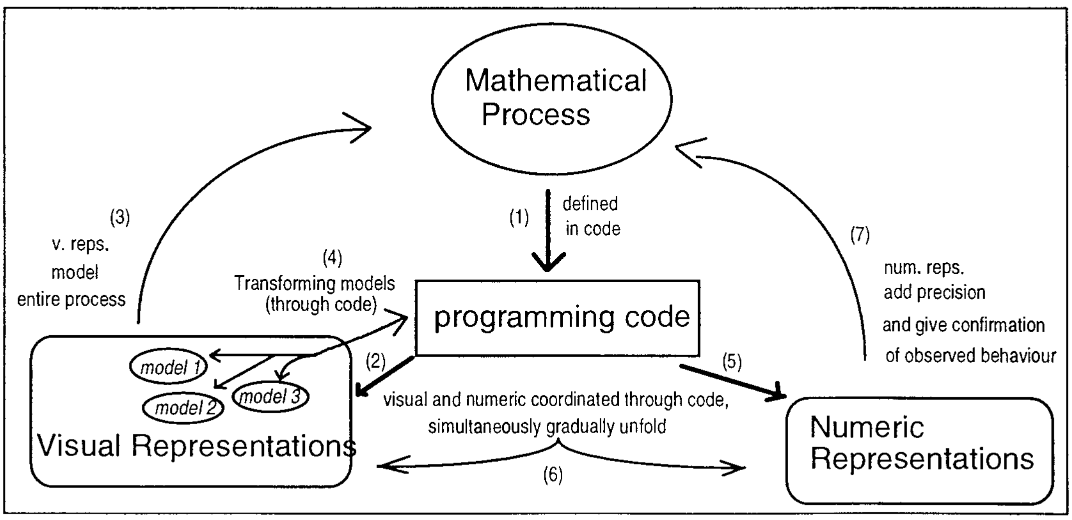

Figure 8.1. The representational elements of the microworld and their interactions.

Figure 8.1 is a schematic of a typical paradigm case of the use of the representational tools of the microworld; in it we see the following (refer to numbers in diagram): (1) The mathematical process is symbolically defined in the code. (2) By running the programming code, a visual representation, which models the entire process (3), is produced. Additionally, the visual representation is produced by the *movements* of the turtle, gradually unfolding: this allows the process to be observed sequentially and for its characteristic *behaviour* to be highlighted. (4) There are different visual models which can be produced, each providing a different visual perspective on the process. The switch between models is done through the code whose structure, and the process it defines, remain invariant: the models are *isomorphically* constructed. (5) Numeric values can also be produced through the same code; this links them to the visual representations and both (6) simultaneously gradually unfold. (Numeric values are also produced through complementary procedures — also representing the same process). (7) The numeric representations serve to add precision and give confirmation of the observed behaviour of the process.

All of the above elements, including the process, and their characteristics are described in more detail below, concluding with a summary in Table 8.1 on page 221.

1.- The mathematical process.

All the microworld activities involved a process of construction of an infinite sequence (this includes the fractal activities, since fractals are constructed through *geometric* sequences). In the case of sequences, the process was first described as an iterative action, involving repeated operation on a variable: e.g. the action of halving an element and repeating this for the resulting process. It was later defined as a symbolic function formula — a Logo function in the programming code (see Chapter 6, section A.2) — which encapsulated the process. In the case of fractal figures, the description (construction) of the figures was done through turtle geometry in a recursive procedure which reflects the self-similar structure of the figure.

2.- The programming activity and code.

The programming activity and the code were the means which integrated all the elements and acted as a "control structure": that is, the students, through the programming activity could coordinate and *act* upon all the elements. As discussed elsewhere, their actions, *via the procedural code*, served to construct meanings for the underlying mathematical process. Here I would like to add that the use of Logo proved to be an adequate choice as it made the programming activity accessible, while providing a direct interface with the visual representations. Several aspects of the programming element are relevant:

(i) The description of the mathematical process (see section 1. above).

(ii) The programming activity, which involves a process of *symbolisation* where the instructions for communicating with the computer are codified. The programming code is thus a symbolic representation which interacts with the other types of representation involved in the microworld.

(iii) The *recursive* structure of the code, which conveys the mathematical[1] idea of *iteration,* and is also reflected in the *self-similar* structure of the visual representations.

(iv) The possibility for exploration and manipulation of the code and its elements. For instance, for the sequence studies, students (a) used variations of the stop condition in the exploration of the convergence and rate of convergence of the sequences under study (see Chapter 7, section III.1.d). Other modifications of the code included: (b) transforming the visual model (see Chapter 6, section A.1.d); (c) changing the defined process (function), particularly for comparison — (see Chapter 7, section III.1.b).

(v) The invariance of the code in the transformation of visual models. That is, the same procedure could be used to generate different visual representations, thus highlighting the process described in the code as the common element which links the different representations. Because the students themselves generated different visual models from the same procedure, it enabled them to become aware that the different models represented the same process. The procedural code acted as an isomorphic function between representations; it *connected* the different models.

3.- The visual/graphical representations.

These representations had several relevant characteristics during the explorations:

[1] For programming, iteration and recursion are different, but from a mathematical point of view they are connected.

(i) First, the visual element was an entry point for the explorations. That is, although the visual figures required the computer procedure, in most cases the realisation of what the procedural code was describing and how it operated did not emerge until after the visual figures were generated and an attempt was made to relate the observed phenomena on the screen with the code that produced it.

(ii) These representations served to "visualise" the behaviour of the processes in two ways:

On the one hand, they provided a *global* view of the processes: the process is synthesised in the figure. For example, when students looked at the "curve" formed by the bar graph model of a sequence, the whole model gave an indication of the behaviour and rate of convergence of that sequence.

On the other hand, the computer provided the added benefit that these visual images *gradually* unfolded, so that each element of the figure (e.g. the visual representations of the terms of the sequence) could be specially observed in relation to the previous ones, giving the sense that a *process* was taking place. This allowed for a local analysis, particularly since the computer simultaneously generated the numeric representations which quantified the process. The gradual unfolding of the process is also related to a further dynamic aspect (see point iii.)

(iii) The element of *movement,* provided information that is difficult to access. This, for example, the turtle turning in the same spot conveyed the idea that the process was continuing even when there was no longer any visible change; and the *direction* the turtle moved was particularly important during the exploration of alternating sequences.

(iv) The visual representations had a *self-similar geometric structure* which resulted from the recursive structure of the code and could be related to the latter, forming an additional link between the two.

(v) Visual models could be *transformed* into others, through the use of the same programming code. Each model had particular characteristics which it appears helped in the understanding of the behaviour of the other models. The different models are *complementary,* each adding its own particular perspective on the process: However, since they were all produced by the same code, the connection between them could be made apparent, facilitating the integration of the information. Particular visual appearances could therefore become less dominating.

(vi) Additionally, a valuable exploration tool for the visual representations was the definition of a scale variable in the code. In a sense the scale acted as a "zoom" function: for instance, it sometimes enabled the students to look "deeper" into the graphical representation (e.g. into the spiral model or the Koch curve). Through this activity they could, for example, appreciate the self-similar characteristic of fractal figures. The use of the scale variable had another value in that its relationship with the figure had to be made explicit through the programming activity. In this sense it could be considered to have an additional function to that of a mere "zoom button".

4.- The numeric values.

The numeric output provided a means for working with the visual and symbolic representations. It was another representation of the mathematical process that added numeric *precision* to both the terms of the sequences and to the limit values. Furthermore, the students could "visualise" *in the numeric values* the limit value of the process or its divergence. The numeric output also provided an additional connection between the visual and the symbolic code acting as a means for quantifying the visual models and confirming the accuracy of the code.

For the sequence studies, the numeric values were used on two levels:

- Initially, the numeric values represented the *measures* of the segments of the spiral. In this case the numeric followed the visual.

- At a later stage, when the process of operating on a measure was changed to the use of a function formula, a procedure for generating the numeric sequences could be constructed. When this happened, an independent numeric sequence emerged, although it remained connected to the other representations by its use in the programming activities.

Because the values were generated *simultaneously* with the visual representation as it unfolded, these values had a concrete connection, rather than being abstract numbers whose meanings could be obscure. Furthermore, the pen and paper activity of *structuring* these values into tables seems to have had an important role since it constituted a precise record of the process and facilitated the observation of the (convergent or divergent) behaviour of the process, complementing the other forms of representations used.

Summary.

The Mathematical Process	The programming activity and code	The visual representation	The numeric values
described in the programming code	describes mathematical process	describes mathematical process giving a synthetic view	linked to visual representation
initially viewed as an action	involves a symbolic representation	dynamic: involves movement and gradually unfolds	complement and validate other representations
operation on a variable can be re-conceptualised and defined in terms of a function formula	code has recursive structure	self-similar geometric structure is related to recursive structure of code	structured as tables: are precise record of process
can be seen through visual representations or through independent numeric values	constructionist aspect: allows manipulation of code (e.g. stop condition)	transforms into other forms (models) through code	show limit value or divergence
	invariant in transformation of visual models	each model gives a different perspective on the process	the decimal structure is tool for coping with the infinite nature of process

Table 8.1. Summary of the main elements involved in the microworld.

The students had the possibility to act in several ways, as described above, and which are summarised in Table 8.2 below. Generally, the main element of action was the modification of the procedure (which is, of course, the means to access the process). The procedures could be modified in many different ways (see section 2. above) which, while maintaining the essential structure and purpose, allowed for the exploration of different features or variations.

• Exploring variations of the same model (for one process).	Repeated observations[2] with - change of scale; - variation in the angle (e.g. of the spiral).
• Exploration of different visual *models* of the same process.	*Transforming* a visual model (e.g. a spiral) into another one (e.g. a bar graph) to highlight different aspects.
• Investigation of numeric values, and construction of tables of values.	+ Investigation into the effects of changing the value in the stop condition.
• Comparing variations of a process (sequence), or different processes of the same type, through their visual and numeric representations	The comparison of different sequences of the same type was a powerful tool for gaining further insight into the behaviour of each sequence and for constructing generalisations regarding the same type of process.

Table 8.1. Methods of action used by the students in their use of the microworld.

These were constructive actions which connected the different elements, playing a mediating role in the exploratory activities for the creation of meanings, as is elaborated in the next section.

I would like to draw to the attention of the reader the importance of the way in which the representational tools are used. The psychological/pedagogical review of Chapter 3, has pointed to the difficulties that students have in linking different types of representations and particularly in analysing visual information (Dreyfus & Eisenberg, 1990b). Findings such as these have led many researchers (e.g. Cuoco and Goldenberg, 1992) to advocate incorporating more representations and types of thinking, particularly visual ones, into school mathematics, although it is clear that the mere presence of multiple representations does not guarantee that the learner will construct cognitive links between them.

It is thus interesting to compare the structure of my microworld with the use of multiple representations illustrated in the work of other researchers such as Kaput (e.g. Kaput, 1995) where, although the different representations are linked in the

[2] For instance, many of the representations of the sequences under study were repeated over and over again by most students. These repetitions, which could include changes in some variable or model, can be thought of as repeated laboratory experiments, supporting the idea of the microworld as a "maths laboratory".

sense that they work simultaneously, the inner workings of the connections between them are not available for the student to manipulate and re-construct. In my study, through interacting with the programming code, the students themselves created and controlled the way in which the multiple representations worked. It is in this sense that — unlike Kaput's multiple representation environments where the tools are fixed — this microworld is what diSessa (1997) would call an *open tool set:* the students were able to reconstruct or redesign the tools, and the links between them, and express themselves through the programming activity.

III. Shaping understandings of the infinite with the tools of the microworld.

The *programming* activities of the microworld provided an opportunity for constructing meanings through *doing*, through action and expression. A *web* (see Noss & Hoyles, 1996) of connections was built into the structure of the environment which served as a set of "navigational signposts" for the user to reconstruct those connections. As the findings indicate, through the (inter)*action* of going back and forth between the different representations in an attempt to make sense of the phenomena they observed, the students seemed to be able to coordinate all the elements, (re-)constructing the connections between them. And in this (re-)constructive process, the students seemed to have been able to uncover how the mathematical processes under study worked within the representation and in coordination with other elements and forms of representations.

In particular, the evidence suggests that sometimes the students developed a sense for how each of the representations would behave from the observation of other forms of representation. Thus, as shown for example in Chapter 7, section I.b, students were often able to *visualise* new processes *from* the analysis of the symbolic code or the numeric representations. The students seemed to be able to coordinate sufficiently all the elements as to attempt to replace, in their minds, the missing elements. The entire *system* of representations thus seemed to facilitate visualisation of the infinite processes and objects under study. The evidence also shows that by the end of the study the students would rarely, if ever, look at one of the representations of the process in isolation, whether it was a visual figure or the symbolic code.

Furthermore, the findings indicate that the students not only coordinated different models and types of representations of a same process, but that they also coordinated representations of different related processes: for instance, they seemed to be able to coordinate the bar graph of a sequence with that of its corresponding series (see Chapter 7, section III.1.c).

The key issue is that students were able to construct meanings by coordinating and building connections between the different elements (see Wilensky, 1991; Noss & Hoyles, 1996). But let me elaborate on this from another perspective by considering diSessa's (1988) view that knowledge is formed by little pieces of information — see also Minsky, 1986 — which need to be connected to become meaningful. Otherwise, the fragmentation of knowledge is what can give rise to difficulties in understanding, and to what other researchers call epistemological obstacles and/or misconceptions (see Chapter 3).

As seen in Chapter 2, the study of infinity is full of paradoxical situations. Most, if not all, of these paradoxes arise because the pieces of information are not coordinated; they may be competing with each other, and some of them have a dominating influence. There are several examples of this which emerged during the present study and which have been found both in history and by other researchers, such as the problem of applying the rules of the finite to the infinite (in that case the knowledge of the finite has a dominating role) or the influence of the context (e.g. the dominating influence of visual appearances, where the visual component is not coordinated properly with other factors present). In Chapter 7 (section III) I discussed in particular a dominating intuition found in students which linked the idea of "more is bigger" with the idea that "things get infinitely big if you add long enough", and which seemed paradoxical in the presence of bounded infinite processes. It seems that this paradox emerges from the failure to coordinate the different components simultaneously present (e.g. in an infinite series two elements are involved: the *number* of terms added — which is infinite — and the *measure* of the sum).

However, the evidence indicates that as the students worked with the different representations they were able to form a connection: for instance, giving meaning to the visual boundaries and the infinite nature of the process through the numeric decimal representations.

I thus argue that a powerful way to help the process of "defragmentation" of knowledge, is to provide tools for expressing and representing in different ways these fragments: it seems that out of this expression occasionally emerges a coordination of the pieces. Richard Noss (personal communication) has suggested calling this process of webbing fragmented pieces of knowledge through the use of representations, *representational moderation.*

Extending on this idea, not only did the environment seem to assist the students in gaining familiarity and understanding about the processes they were studying and the relationship between the forms through which those processes could be represented, but the findings indicate that the students were able to use the environment as a domain of abstraction (see Chapter 3), using it to articulate relationships and build generalisations. Furthermore, the exploratory setting of the activities appeared to enable the students to engage actively in a process of discovery of the properties and characteristics of the processes under study. The environment seems to have provided a language for asking questions, as well as tools for exploring these questions. In particular, the students made predictions and then tested their validity by using all the available tools in the microworld. In this sense, as illustrated in Chapter 7, section II, the microworld became a mathematical laboratory. The fact that the students were able to take advantage of the tools of the microworld can be interpreted to mean that they had a basic understanding of the processes they were studying (e.g. they asked themselves questions) and of the environment (e.g. they decided what aspects of the environment to use and how). The formulation of predictions or conjectures involved a process of reflection and analysis on the part of the students, as they had to, for instance, evaluate the role and relationship of the variables involved. Most of the time the students would make predictions such as of the form the visual models would take, or whether a sequence would diverge or have a limit. They would also make conjectures as to the values of the limits. However, the key point is that, as the findings indicate, the students did not restrict themselves to just predicting and confirming, for instance, the existence of a limit. In many cases the students discovered what seemed like patterns and properties; they would then formulate conjectures which they would test in order to validate them and construct their own generalisations (situated abstractions).

The experiences through which the students validated their conjectures and results were often powerful enough to act as "proofs", as shown in the findings exemplified in section III.2 of Chapter 7. Thus, the analysis of the ways in which students used and coordinated the tools of the microworld suggests a new mechanism for the construction of situated abstractions which I call *situated proofs*[3]. Situated proofs are experiences that lead students to discover and make sense of a mathematical relationship — *convincing* them of its validity. Like situated abstractions, these experiences are dependent on the tools of the medium. They can be thought of as the collection of activities that build meaning for a theorem, before a formal proof is presented. They are examples of the kind of experiences considered useful and advocated by some of the researchers reviewed in Chapter 3 (e.g. Tall, 1991b; Thurston, 1994; Cuoco & Goldenberg, 1992) incorporating exploratory computer and visual activities. The role of exploration through visual and numeric representations, the observation of the behaviour (unfoldment) of the process, and the role of the structure of the code, are all elements which students used to convince themselves of the convergence or divergence of a process, and/or of the existence of a limit. Thus, situated proofs result from the combination of the elements which the students used in their attempts to confirm their conjectures. All the representational forms were coordinated and used in a complementary manner in the search for proof.

Thus, the data of the study indicates that the students, in their explorations of the infinite, seemed to have engaged in a process of reflection —fundamental to any mathematical activity — on the issues surrounding this concept. So, even in cases where the problems and paradoxes which emerged during the study were not totally resolved — which is not surprising, as the infinite is an area which has always been a source of conflict and difficulty throughout history — an important point is that students did seem to engage in a process of analysis and coordination of the elements present. It appears from the evidence that the students learned to take all the elements

[3] I should note that the notion of situated proof draws on a similar idea which we proposed in Moreno & Sacristán (1995), and which we referred to there as "didactic proofs".

into account, gradually being less prone to being misled by initial appearances, analysing each case in a "part by part" fashion.

The microworld provided a means to approach the infinite from a perspective (beyond the traditional purely algebraic/analytic approach) which seemed to allow the students to become aware of its complexity, opening a window on the infinite through the finite tools of the microworld.

IV. A note on affective issues.

Although the following issues were not necessarily part of the main research objectives, they are nevertheless interesting and worth noting:

1. The microworld appeared to be accessible to all the students who participated in the study, even though they had different ages and backgrounds. However, it is clear that the background and experience of each of the students shaped in different ways their interactions with the microworld and the level of organisation of their generalisations and abstractions. Two aspects are worth noting: (i) The younger students appeared to have been able to engage actively in processes of discovery, exploration and generalisation about processes (e.g. those of limits and infinity) which in traditional school mathematics are usually presented at more advanced levels; and, more importantly, these students seemed to be have been able to construct situated abstractions and understandings about those processes, as shown in Chapter 7. (ii) The more experienced students, even when they had previous knowledge of the particular processes studied, were able to explore these processes from new perspectives, uncovering new aspects (e.g. the behaviour of the processes) and developing new connections between the elements. In this sense, the processes seemed to have become much more meaningful for them, as the pieces of knowledge were better integrated.

2. It is worth noting that all the students enjoyed working with the microworld and were deeply interested in the investigations. The motivation factor is one that is commonly ignored, and yet it seems clear that learning is facilitated when students enjoy the activities and are actively involved in them.

V. Limitations and implications of the research.

A. Limitations of the research.

Although, generally speaking, the results of working with the microworld can be considered positive, it is clear that this research had some limitations:

1. The activities chosen were very specific: they dealt only in the area of infinite sequences and series (although also included were *geometrical* infinite sequences in the form of fractal constructions). Furthermore, the sequences studied were quite simple and except for a couple of exceptions, all had monotonic behaviours.

2. There was a dominating procedural approach. The focus was on dynamical processes, which meant that the infinite was more often than not considered from a potential infinity perspective.

3. The research stayed mostly within the domain of the microworld, and did not look deeply to see if students made connections between the situated knowledge and more formal mathematics.

4. The research was carried out with a limited number of students. The work was carried out in rather special circumstances, involving lengthy one-to-one participation with each of the pairs of students on the part of the researcher. It thus remains to be explored how students in general could take advantage of environments such as the one designed here and if they would need as intense an involvement both in terms of time and guidance.

Each of the above limitations open the door to many possibilities for further research.

B. Possibilities for further research.

It is clear that the microworld, as designed, offers many more possibilities for investigation, using, for instance, more complex sequences, such as alternating ones. However, the important point is that this research should be thought of as an example of what can be done to explore infinite processes by integrating and coordinating different types of representations: It is a topic for further research to build on these ideas. For instance, fractals, dynamical systems and chaos are rich areas which offer possibilities for exploring the infinite both as potential and as actual. Another interesting possibility could be the addition of colour as another representational element, particularly for representing the convergent or divergent behaviours of infinite processes, as is done in the case of the Mandelbrot set.

There is, however, at least one area of more immediate research: to investigate how students can develop connections between the situated knowledge (both the situated abstractions and situated proofs) that they constructed through their microworld activities and more general formal mathematics.

C. Concluding remarks.

The results of the study show possibilities for opening new avenues for the study of infinity by incorporating the computer and its tools, in a carefully designed way. But this study can be taken further in that it provides a generic example of how new technologies can be exploited to make accessible difficult areas of mathematics, by creating settings with open tools which involve a) the interaction of multiple representations with b) an active exploratory role of the learner. These last two aspects should be emphasised: the findings of the study confirm the idea that the construction of meanings is facilitated when supported by the construction of external representations (Papert, 1993) and links between them.

Furthermore, it is clear that technology is advancing at a very rapid pace, permeating every area of society and changing our views of the world and our conceptions of knowledge. This is true of mathematics: it seems clear that technological advances will play a role in the way that mathematical objects and processes (including those dealing with the infinite) are conceived. Changes in education are also necessary. But education not only needs to incorporate the new technologies, it should *take advantage* of them. As this research shows, this is possible, although careful consideration of the way in which it is done is necessary. Computers, and other new technologies, can be used in mathematics education to construct carefully designed interactive representational systems that enable students to access mathematical knowledge by building up an awareness of how the various processes and elements are webbed together to create meanings of a whole which is more than the sum of its parts.

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Appendices

Appendix 1: Main study: the Initial Questionnaire.

Appendix 2: Main study: Worksheets and handouts.

Appendix 3: Pilot study: the original worksheets.

Appendix 4: Pilot Study: Description of the worksheets and activities.

Appendix 5: Exploratory study: the activity sheets.

Appendix 6: Exploratory Study: Description of the sessions and activities.

Appendix 7: Main study: Case study of Consuelo and Verónica.

Appendix 1:

Main Study: The Initial Questionnaire.

Note: The following questionnaire was taken from a previous research (Sacristán, 1992). The original questionnaire — as was used in this research — was written in Spanish, but for the sake of clarity for the reader, it is here presented in an English translation.

PART ONE

1. Consider the following two lists of numbers:

A	**B**
0.9	0.1
0.99	0.01
0.999	0.001
.	.
.	.
.	.

If these lists were to be continued indefinitely,
a) Could the last term of list A be added to the last term of list B? (Explain your reasoning)
b) What would be that sum? (Explain your reasoning)

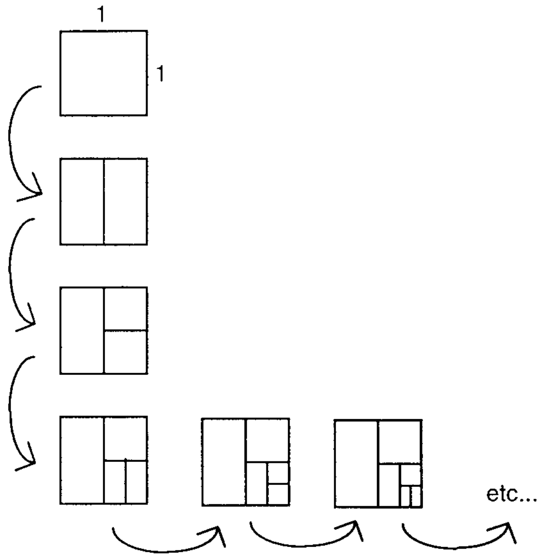
2. What do you think of the following sum?
$1/2 + 1/4 + 1/8 + 1/16 + ... + 1/2^n + ...$; where n is a positive integer.

3. Consider the following square of side 1:

We divide it in half

and we continue dividing each of the halves:

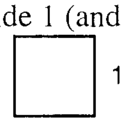
This process is continued indefinitely



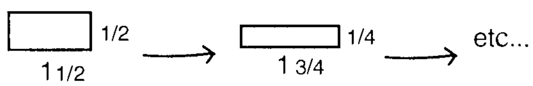
a.) (i) What can you say with regard to the sum of the areas of the resulting rectangles?
(ii) Can this process end? (Explain your answer)
(iii) What would then happen (if the process ended) with the sum of the areas? (Explain your answer)

b.) Can you give a numerical representation for this <u>process</u>? (Explain how).

4. Consider the following square of side 1 (and perimeter 4):



A sequence of rectangles is formed such that the height is progressively smaller , and the length is progressively larger, but the perimeter of 4 units is kept constant:

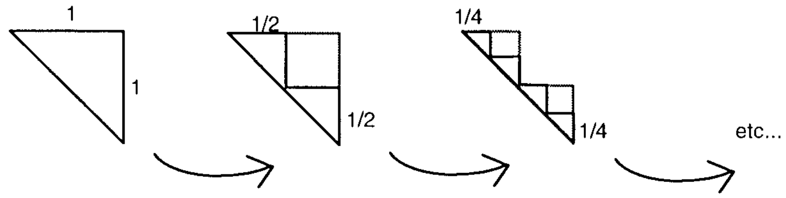


a) Can this process be continued indefinitely, or will it reach a situation in which it has to stop? (Explain your answer).

b) By continuing this process, what happens to the rectangles?

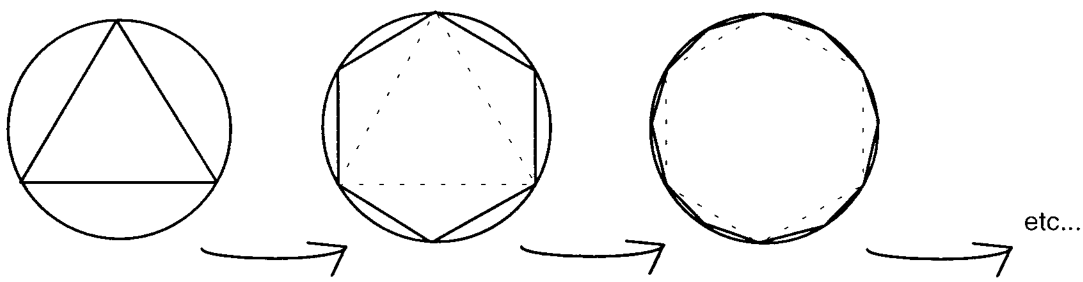
and, what happens to the areas?

5.- Consider the following triangle, and the process resulting from dividing the sides in half:



Imagine that this process is continued indefinitely,
what happens to the length of the stairs?
Can you measure it? If so, what do you get? (Explain your answers).

6.- Consider the following process:



If you continued this process indefinitely, what do you think would happen with the area of the polygons?
Could you measure it? How? (Explain your answers).

PART TWO

1.- Using the set of natural numbers $\mathbb{N} = \{ 1, 2, 3, 4, ...\}$,
can you count the elements in the set of positive odd numbers:

$\{1, 3, 5, 7, 9, ...\}$?

How would you do it?

Would you use all the elements of $\mathbb{N}$? (Explain your answers).

2.- Consider the set of squared numbers: $\{ 1, 4, 9, 16, 25, ...\}$ which results by raising the natural numbers to the squared power:

$1^2 = 1$
$2^2 = 4$
$3^2 = 9$
$4^2 = 16$
$5^2 = 25$
etc...

Using $\mathbb{N}$, can you count the elements of this set?

Would you use all of the elements of $\mathbb{N}$? (Explain your answers).

3.- Using $\mathbb{N}$, can you count the elements of the set of integer numbers:

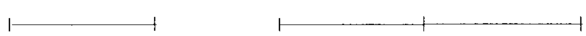
$\mathbb{Z} = \{ ..., -3, -2, -1, 0, 1, 2, 3, ...\}$?

How would you do it? (Explain your answers and write down your comments).

4.- For each of the following cases, compare the <u>points</u> in the sets A and B:

Do you think there are more or an equal number of points in one set as in the other ? (Explain your answer).

(a) A = Segment of length 1 B = Segment of length 2



(b)

A = Half-line

B = Full Line

(c)

A = Segment of length 1

B = Full Line

(d)
A = Segment of length 1 B = Area in a square of side 1

5.- For each of the previous cases,

(i) Do you think there exists a way to associate the points in A with the points in B, in such a way that each point in A corresponds to one, and only one, point in B (without any extra points nor in A, nor in B)?

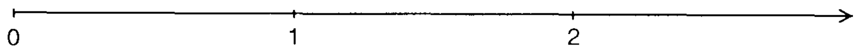
(a) [YES] [NO] Why?
(b) [YES] [NO] Why?
(c) [YES] [NO] Why?
(d) [YES] [NO] Why?

(ii) If you answered YES to any of the previous cases, explain how you would do it.

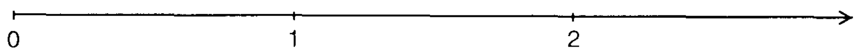
PART THREE

1.- Locate, as precisely as possible, each of the following numbers on the line (explain your method):

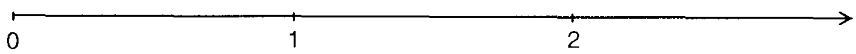
a) 1/3



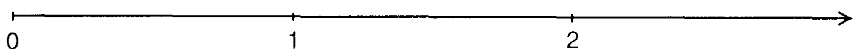
b) 2/3



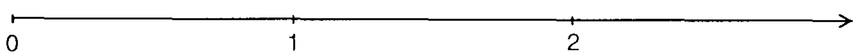
c) 0.3



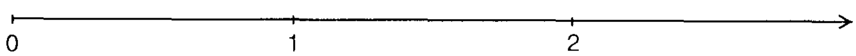
d) 1.9



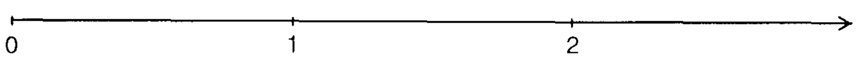
e) 1.4



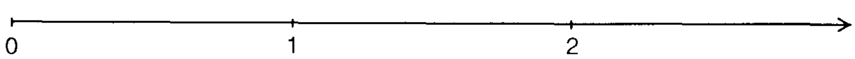
f) 0.33



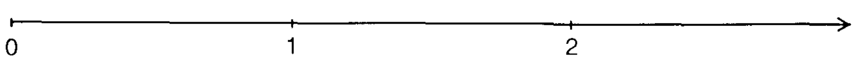
g) 1.99



h) 1.41



i) 0.333



j) 1.414

2.- a) For each of the following, do you think there exists a <u>precise</u> point on the line corresponding to the given number?

(i) $0.\bar{3}=0.333...$ [YES] [NO] [I DON'T KNOW]
Comments:

(ii) $1.\bar{9}=1.999...$ [YES] [NO] [I DON'T KNOW]
Comments:

(iii) $\sqrt{2}$ [YES] [NO] [I DON'T KNOW]
Comments:

(iv) π [YES] [NO] [I DON'T KNOW]
Comments:

b) If you answered YES for any of the given numbers,
Do you think there exists a way for locating on the line, with all precision, the point corresponding to that number?

(i) $0.\bar{3}=0.333...$ [YES] [NO] [I DON'T KNOW]
Comments:

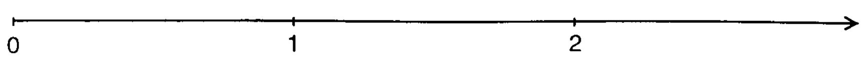
(ii) $1.\bar{9}=1.999...$ [YES] [NO] [I DON'T KNOW]
Comments:

(iii) $\sqrt{2}$ [YES] [NO] [I DON'T KNOW]
Comments:

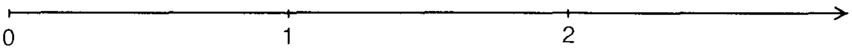
(iv) π [YES] [NO] [I DON'T KNOW]
Comments:

c) If you answered YES for any of the given numbers, can you locate on the line the point corresponding to that number? (Show and explain your method).

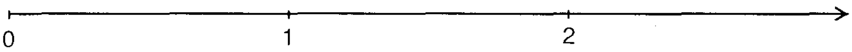
(i) $0.\bar{3}=0.333...$



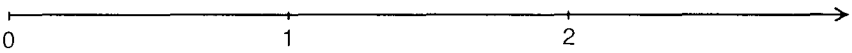
(ii) $1.\overline{9}=1.999...$



(iii) $\sqrt{2}$



(iv) π



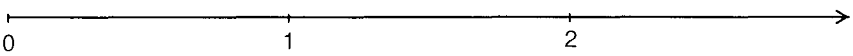
3.- For each of the given numbers, can you write its decimal expression?

(a) $1/3 =$ Comments:

(b) $\sqrt{2} =$ Comments:

(c) $\pi =$ Comments:

4.- Can you precisely locate $0.\overline{6}=0.666...$ on the line?



Comments:

5.- Can you write each of the following numbers as a fraction?

(a) 0.75 Answer: _______

(b) 0.3 _______

(c) $0.\overline{3}=0.333...$ _______

(d) $0.\overline{6}=0.666...$ _______

(e) $1.\overline{9}=1.999...$ _______

(f) 1.4142 _______

Appendix 2:

Main Study: Worksheets and Handouts.

Note: The reader is reminded that not all of the worksheets and handouts were given to every student in the study: many of the ideas or activities contained in these sheets were orally explained and most of the students constructed their own procedures. These worksheets and handouts are presented in a translation from the original Spanish.

The sheets in this appendix, are the following:

I.The Initial Worksheet containing the procedure which traces an "invisible spiral"

II.The handouts with suggestions for procedure modifications and tools for exploration in the sequence studies.

III. Procedures for constructing and exploring the Sierpinski Triangle.

IV. Blank worksheets that I provided the students to encourage them to create tables for their explorations and to record their observations.

Name:____________________________ Date:________________________

Consider the following procedure

```
TO DRAWING :L
PU
FD :L
RT 90
WAIT 10
DRAWING :L*1/2
END
```

What happens when you run **DRAWING 100** ?
In the box below, draw the movements of the turtle:

Describe what the turtle does

____

____

Some suggestions for modifying your procedure:

```
TO DRAWING :L
IF condition [STOP]
FD :L
MODEL
DRAWING  (FUNCTION :L)
END
```

where the *condition* can be: **OP :L <** *some number*

```
TO MODEL
SPIRAL
END
```

```
TO SPIRAL
FD  :L
RT 90
WAIT 10
END
```

```
TO FUNCTION :L
OP   :L / 2
END
```

Some suggestions for modifying your procedure:

CREATING LISTS OF TERMS FROM A FUNCTION

```
TO SEQUENCE :N
IF :N = 1 [OP FN 1]
OP SE (SUCESION :N - 1) (FN :N)
END
```

where, for example, FN can be defined as:

```
TO FN :N
OP 1 / POWER 2 :N
END
```

corresponding to $f(n) = \frac{1}{2^n}$

Some suggestions for modifying your procedure:

DRAWINGS WITH LISTS (SEQUENCES) OF VALUES

```
TO DRAWSEQUENCE :LIST :SCALE
IF :LIST = [] [STOP]
MODEL
DRAWSEQUENCE BF :LIST :SCALE
END
```

NOTE: Do not forget to modify your models by replacing :L with

:SCALE * FIRST :LIST

Try

DRAWSEQUENCE [1 1/2 1/4 1/8] 100

Some suggestions for modifying your procedure:

ADDING THE TERMS OF A LIST

```
TO SUML :LIST
IF :LIST = [] [OP 0]
OP (FIRST :LIST) + SUML BF :LIST
END
```

CREATING A LIST OF PARTIAL SUMS OF THE TERMS OF A LIST

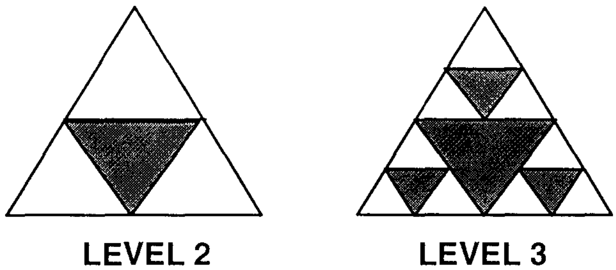
```
TO PARTIALSUMS :LIST
IF :LIST = [] [OP [] ]
OP SE (PARTIALSUMS BL :LIST) (SUML :LIST)
END
```

Consider the following procedure:

```
TO TRI :SIDE :LEVEL
IF :LEVEL = 1 [STOP]
REPEAT 3[TRI :SIDE/2 :LEVEL - 1 FD :SIDE RT 120]
END
```

What can you say about this procedure?

Imagine that the darkened areas in the triangle below are being removed:



What happens to the remaining area as the level is increased?

****

Exploration tool:

The area of an equilateral triangle is given by $A = \frac{\sqrt{3}}{4} L^2$, which can be translated into the following procedure:

```
TO AREATRI :SIDE
OP (POWER :SIDE 2)*(SQRT 3)/4
END
```

Consider the following procedure:

```
TO CURVE :N :L :P
IF :N = 0 [FD :L STOP]
LT 60 * :P
CURVE :N-1 :L/2 (-:P)
RT 60 * :P
CURVE :N-1 :L/2 :P
RT 60 * :P
CURVE :N-1 :L/2 :P
RT 60 * :P
CURVE :N-1 :L/2 (-:P)
LT 60 * :P
END
```

where the initial input of :P must be 1

What happens as the input of the level :N is increased?

Name(s):____________________ Date:____________________

Function(s) (Sequence(s)), or Objects being studied:

Comments:

Name(s):____________________ Date:__________________

Function(s) (Sequence(s)), or Object(s) being studied:

You can write here your observations, comments and conclusions of your explorations (include drawings if you wish to).

Use this column to describe what you did with the computer	Use this column for your observations

Appendix 3:

Pilot Study: Original Worksheets

Note: These worksheets are presented in their original form (in Spanish). They correspond to the following:

Nombre(s):______________________________

Fecha:________________________________

EXPLORACIONES DE CONVERGENCIA DE SUCESIONES
I

Copia el siguiente procedimiento

```
TO DIBUJO :L
PU
FD :L
RT 90
WAIT 10
DIBUJO :L*1/2
END
```

Si quieres parar la ejecución del procedimiento usa CTRL-BREAK

Corre **DIBUJO 100**

Qué hace la tortuga?

Observaciones:

Nombre(s):______________________________

Fecha:_________________________________

EXPLORACIONES DE CONVERGENCIA DE SUCESIONES II

Usa las siguientes sugerencias para modificar tu programa:

```
TO ESPIRAL :L
CS
DIBUJO :L 1 :L
END
```

```
TO DIBUJO :L :CUENTA :DIST
FD :L
RT 90
WAIT 10
IF :L < 5  [PR :CUENTA  PR :DIST  PR :L STOP]
DIBUJO  :L*1/2  :CUENTA+1  (:DIST + :L)
END
```

Modifica este valor en la condición de parada

Usa **ESPIRAL 100** para completar una tabla como la siguiente, variando el valor de la desigualdad de parada:

Valor en la desigualdad de parada	Cuenta	Distancia total recorrida	Ultimo valor de L
5 1 0.1 . . . 0			

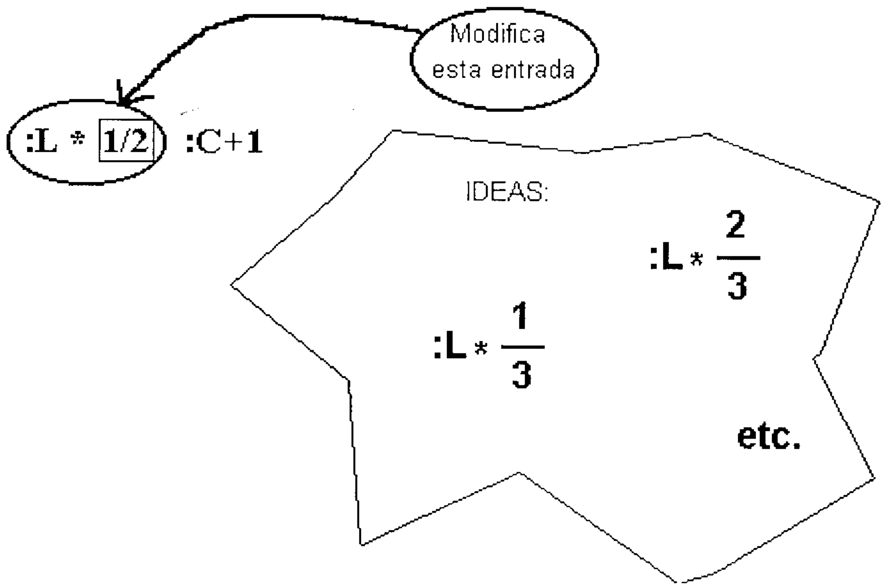
Nombre(s):_______________________________

Fecha:_________________________________

EXPLORACIONES DE CONVERGENCIA DE SUCESIONES III

Explora la convergencia de otras sucesiones
de la misma manera que antes

```
TO DIBUJO :L :C
...
FD :L
RT 90
...
DIBUJO (:L * 1/2) :C+1
END
```



Valor en la desigualdad de parada	Cuenta	Distancia total recorrida	Ultimo valor de L

Nombre(s):______________________________

Fecha:________________________________

EXPLORACIONES DE CONVERGENCIA DE SUCESIONES IV

Escribe un nuevo procedimiento usando las siguientes ideas:

```
TO DIBUJO :X :N

FD :X * 1/:N
RT 90

IF :X * 1/:N  <  ...  [STOP]

DIBUJO :X  :N+1

END
```

Observa la entrada del FD

Usa **DIBUJO 100 1** para explorar lo que esté sucediendo.

Explora también con

$$\frac{X}{N^{1.1}} \text{ y } \frac{X}{N^{2}}$$

Nombre(s):______________________________

Fecha:________________________________

EXPLORACIONES DE CONVERGENCIA DE SUCESIONES V

Usa programas como el siguiente para explorar diferentes series

```
TO SUMA :N
IF :N = 1 [OP 1  STOP]
OP 1/:N + SUMA :N-1
END
```

Sustituye aquí con POWER :N P para otras series

Usa **PR SUMA** x para llenar tus tablas

x	$\sum_{N=1}^{x} \frac{1}{N}$	$\sum \frac{1}{N^{1.1}}$	$\sum \frac{1}{N^2}$	...
1				
2				
3				
4				
5				
10				
20				
50				
100				
200				
500				
1000				

Más sobre series y sucesiones

Usa el siguiente programa para entender lo que sucede con la sucesión 1, 1/2, 1/4, ... y con la serie $1+1/2+1/4+...+ 1/2^n+...$

```
TO TODO
CS
PU
BK 100
PD
CUBOS 100
END
```

```
TO CUBOS :N
IF :N < 1 [STOP]
CUADRO :N
WAIT 10
LLENA
FD :N
WAIT 10
CUBOS :N / 2
END
```

```
TO LLENA
PU
RT 45
FD 1
PD
SETPC 2
FILL
BK 1
SETPC 3
LT 45
END
```

```
TO CUADRO :N
REPEAT 2 [FD :N RT 90 WAIT 5 FD 100 RT 90]
END
```

También prueba cambiando el procedimiento CUADRO a lo siguiente

```
TO CUADRO :N
REPEAT 4 [FD :N RT 90 WAIT 5 ]
END
```

Modifica el programa para explorar otras sucesiones

Nombre(s):______________________________

Fecha:______________________________

EXPLORACIONES CON FRACTALES
I

Copia el siguiente procedimiento

```
TO CURVA :L :NIVEL
IF :NIVEL = 1 [FD :L  STOP]   ←  Añade aqui PR :L para imprimir el tamaño de cada segmento
CURVA :L / 3 :NIVEL - 1
LT 60
CURVA :L / 3 :NIVEL - 1
RT 120
CURVA :L / 3 :NIVEL - 1
LT 60
CURVA :L / 3 :NIVEL - 1
END
```

Corre **CURVA 100 1, CURVA 100 2, CURVA 100 3, CURVA 100 4,** etc...

Explora la longitud de la curva al aumentar el nivel.
Usa una tabla como la siguiente:

Nivel	Tamaño de cada segmento	Número de segmentos	Distancia total recorrida
1	100		
2	100/3		
3			
.			
.			
.			
N	$100/3^{N-1}$		

Nombre(s):____________________________

Fecha:______________________________

EXPLORACIONES CON FRACTALES II

Escribe el programa COPO

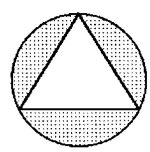
```
TO COPO :N
REPEAT 3 [CURVA 100 :N  RT 120]
END
```

```
TO CURVA :L :NIVEL
IF :NIVEL = 1 [FD :L  STOP]
CURVA :L / 3 :NIVEL - 1
LT 60
CURVA :L / 3 :NIVEL - 1
RT 120
CURVA :L / 3 :NIVEL - 1
LT 60
CURVA :L / 3 :NIVEL - 1
END
```

Qué pasa con el perimetro del copo a medida que aumenta el nivel?

Explora qué pasa con el **área** del copo

Nivel	Area del Copo	Diferencia entre el Area del circulo y la del copo
1 2 3 . . . N		



Circulo que circumscribe al triangulo del primer nivel

Nombre(s):______________________________

Fecha:________________________________

EXPLORACIONES CON EL TRIANGULO DE SIERPINSKI I

Considera el siguiente procedimiento

```
TO TRI :LADO :NIVEL
IF :NIVEL = 1 [STOP]
REPEAT 3[TRI :LADO/2 :NIVEL - 1 FD :LADO RT 120]
END
```

Corre **TRI 100 1, TRI 100 2, TRI 100 3,** etc...

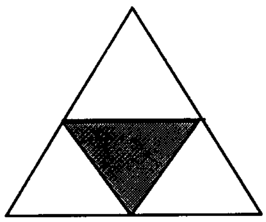
Qué puedes decir sobre este procedimiento?

Nombre(s):______________________________

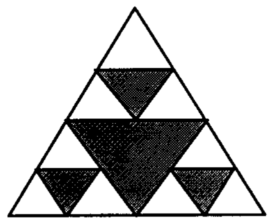
Fecha:________________________________

EXPLORACIONES CON EL TRIANGULO DE SIERPINSKI II

Imagina que vamos quitando las áreas sombreadas del triangulo original.



NIVEL 2



NIVEL 3

Qué pasa con las áreas (la sombreada y la restante) a medida que aumenta el nivel?

Explora qué pasa con el **área** en un triangulo de lado 100.
Puedes usar una tabla como la siguiente:

Nivel	Lado del triángulo más pequeño	Area del triangulo más pequeño	No. de triangulos pequeños	Area Total (no sombreada)
1				
2				
3				
4				
.				
.				
.				
n				

EXPLORACIONES CON EL TRIANGULO DE SIERPINSKI
II - Herramientas

Herramientas para explorar:

El área de un triangulo equilatero está dada por $A = \frac{\sqrt{3}}{4} L^2$

```
TO AREATRI :LADO
OP (POWER :LADO 2)*(SQRT 3)/4
END
```

Otras herramientas utiles que pueden ser utilizadas modificando el procedimiento original y corriendo el procedimiento **TODO:**

```
TO TRI :LADO :NIVEL
IF :NIVEL = 0 [STOP]
IF :NIVEL = 1 [NUMTRI LADOPEQ]
REPEAT 3 [TRI :LADO/2 :NIVEL-1 FD :LADO RT 120]
END
```

```
TO NUMTRI
MAKE "NUM :NUM +1
END
```

```
TO LADOPEQ
MAKE "LPEQ :L
END
```

```
TO TODO
INI
TRI 100 :NIVEL
INFO
END
```

```
TO INI
MAKE "NUM 0
CS
RT 30
END
```

```
TO INFO
PR SE [El lado de los triangulos mas
pequenos es] :LPEQ
PR SE [y el numero de triangulos
pequenos es] :NUMTRI
END
```

Nombre(s):______________________________

Fecha:________________________________

EXPLORACIONES CON EL TRIANGULO DE SIERPINSKI III

Considera el siguiente procedimiento

```
TO CURVA :N :L :P
IF :N = 0 [FD :L STOP]
LT 60 * :P
CURVA :N-1 :L/2 (-:P)
RT 60 * :P
CURVA :N-1 :L/2 :P
RT 60 * :P
CURVA :N-1 :L/2 :P
RT 60 * :P
CURVA :N-1 :L/2 (-:P)
LT 60 * :P
END
```

donde el valor inicial de :P debe ser 1

Qué puedes decir sobre este procedimiento?

Qué pasa con la longitud de la curva?

Appendix 4:

Pilot Study:
Description of the Worksheets and Activities.

1. First set of worksheets: Exploring the convergence of sequences and series through "Spiral Studies".

Worksheet No.1

Through this worksheet the following procedure was given to the students:

```
TO DIBUJO :L
PU
FD :L
RT 90
WAIT 10
DIBUJO :L/2
END
```

This program makes the turtle walk through a spiral with arms each having half the length of the previous one. I believe this is a way in which students can work, in an implicit and informal way, with the infinite sequence

$$1, 1/2, ..., 1/2^n$$

I chose this sequence because I consider it one of the simplest ones there are. I also chose the spiral as the first representation of the sequence for the same reason (on later worksheets and/or activities students can work, also in an informal way, with other representations of this same sequence).

It also is a good way to give them a procedure which they would not normally build in such a way: I am referring in this case to the fact that I had the turtle put the pen up during the procedure. Why? I didn't want the turtle to leave a trace of its movements. I just wanted students to see the movements of the turtle (which are made easier to see by the WAIT command) and reconstruct in their minds the actual drawing: In this way they would realise that the turtle is walking half the distance each time it turns and does so without stopping (which is the reason I did not include a Stop command), and they can realise this without the visual obstacle from the computer drawing which after a while seems to show the turtle staying in the same place (actually in the same pixel).

With this worksheet I wanted students to start exploring the behaviour of the program, of the turtle, and of the sequence. Although it is not written on the sheet I suggested students to use the F5 key to Pause the procedure to type instructions in direct mode.

Worksheet N.2

This worksheet shows how to make some modifications to the program in order to:

- stop the procedure by defining a condition
- add a counter for the number of arms of the spiral = the number of times the program calls itself
- show how to keep a record of the total distance that the turtle has walked (DIST)
- and show how to print these all of these values so that they can be used to explore the behaviour of the sequence and of the series $\sum \frac{1}{2^n}$.

I also included a small procedure (ESPIRAL) that can be run instead of the original one and which gives students the necessary inputs for the DIBUJO procedure so that the output values of the variables being explored are correct. This procedure is also aimed to help students become aware of the modularity of LOGO programming.

I then included a table that I wanted students to complete in order to make them explore the behaviour of the sequence, the series $\sum \frac{1}{2^n}$, with relation to the value they use in the stop command, and to the count. The idea behind this is to have students explore in an implicit manner, how a sequence can be defined to converge. In fact, it could be possible that —by using and exploring different values for the stop command and relating them to the overall behaviour of the sequence— they might implicitly be encountering, situated within the environment, ideas which could be related to the mathematical definition of convergence of a sequence

L is said to be the limit of a sequence $\{a_n\}$, if

$|L - a_n| \rightarrow o$ as $n \rightarrow \infty$

This activity was also designed to help students become aware, again in an implicit manner and situated within the environment, of the *rate* of convergence of a sequence.

Worksheet N.3

The purpose of this worksheet was to have students explore other sequences of the same kind by simply modifying their procedure. I included some simple ideas of other sequences as suggestions of other sequences they can explore.

I also included the same table as before to remind students of the exploration methods they used before.

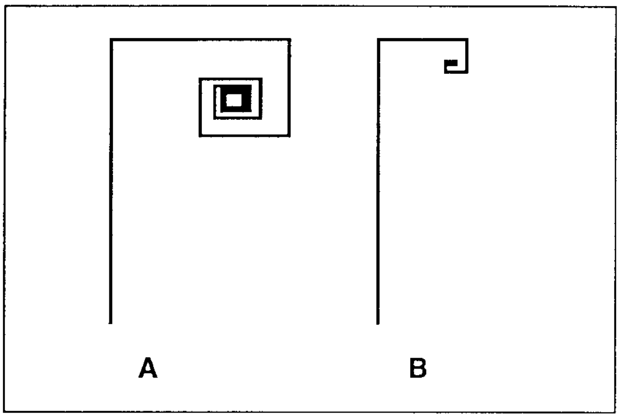
It is possible that through these exploration students can determine, within the environment, some of the conditions which make a series convergent or divergent. For instance, for sequences of the type $\{(k)^n\}$, the series $\sum k^n$, when $n \rightarrow \infty$, diverge if $k>1$, and converge if $k<1$.

Worksheet N.4

This worksheet introduced students to other types of sequences by showing them a new kind of modification to the procedure. In this case the corresponding sequence would be: 1, 1/2, 1/3,.1/4, ..., 1/n. The series corresponding to this sequence is divergent, so I suggested they also explore the series corresponding to the sequence

$\{1/n^{1.1}\}$, which converges although very slowly, and $\{1/n^2\}$ whose series is more apparently convergent.

It is worth mentioning that for all of these sequences a spiral will be drawn, but there is an easily seen visual difference for different sequences that serves as an indicator of the rate of convergence.



The A spiral corresponds to the sequence $\{1/n\}$, while B corresponds to the sequence $\{1/n^2\}$. As mentioned above, the series corresponding to sequence A is divergent, and although the space at the center of the spiral will eventually appear to be filled, in its behaviour the spiral seems to avoid going to a fixed point at the centre. In contrast, in case B where the corresponding series *does* converge, the spiral quickly approaches a central point. This of course does not constitute a proof of the convergence or divergence of these series, but it is an illustration of the role of visual representations as *pedagogic proofs.*

Worksheet N.5

This worksheet was meant to serve as a complement to the other worksheets (particularly to the worksheet N.4) since it shows how to write a program (an operation) for computing directly, without the visual (spiral) representation, the values of partial series such as those of $\sum \frac{1}{n^p}$. This allows the computer to calculate the partial series for higher values without running out of memory. But it also serves as an introduction to "recurrent series" through the symbolic language of LOGO.

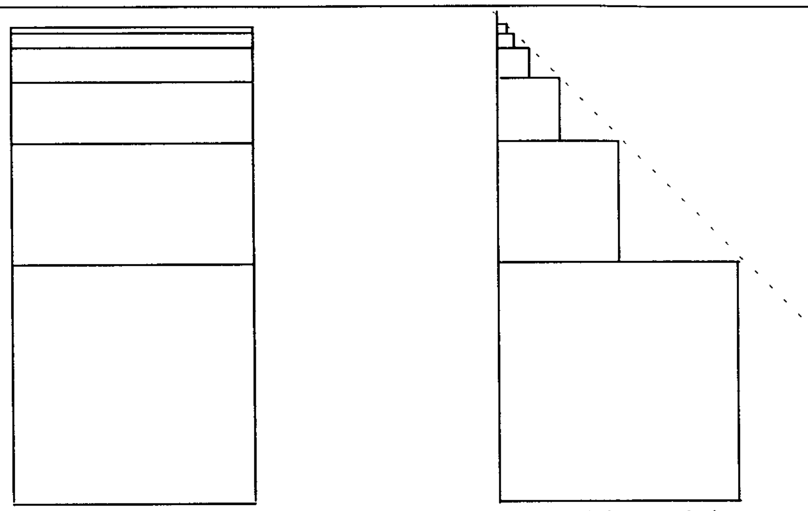


Figure a.) Figure b.)

Additional material

As a supplement to the above worksheets, I presented some of the students with a program which builds another type of representation of the sequence $\{1/2^n\}$ and its corresponding series. It is meant to show students different ways in which they can visualise a particular concept (such as the sequences and series they had been working on).

This program builds a sequence of rectangles, one on top of the other, each having half the height as the previous one (see figure a.). It can also be modified to build a sequence of squares with each square having half the side as the previous one (see figure b.). These representations should help indicate that the series $\sum \frac{1}{2^n}$ is bounded(and has a limit).

2. Second and third set of worksheets: Fractal explorations using the Koch curve and Sierpinski triangle

The purpose of these worksheets was to explore a limit object of a different kind, that is a fractal (in this case the Koch curve or the Sierpinski triangle), including the visual sequence that leads to it, and the (programming) code which reflects its recursive structure, and which each of the steps of the sequence embody.

a. Second set of worksheets: explorations with the Koch curve and snowflake.

The nature of this set of worksheets was of the same kind as the ones in the previous set. I started by giving students the basic procedure for building the Koch curve and gave them a table to help them explore the behavior of the total length of the curve as the sequence progresses, as well as a suggestion on how to measure the length of each "subsegment" of the curve.

In the second worksheet the purpose was to confront students with one of the apparent paradoxes of mathematical infinity: an infinite but *bounded* perimeter that encloses a finite area. The methodology was the same as before: Students are given the procedure for building the Koch snowflake and a table to help them structure their explorations of the area of the object.

b. Third set of worksheets: explorations with the Sierpinski triangle

In the first worksheet students were given a procedure to construct the Sierpinski triangle. In the second worksheet students were asked to explore and compare the areas defined during the construction process with the help of a table (as in previous activities). I also provided students with small additional procedures -tools- to assist them with the computations of the values involved in the exploration. The purpose of this activity, as with the previous one, was to present students with a recursive structure, both visual and symbolically, and to confront them with another example of the "behaviour" of mathematical infinity: by a process that "takes away", at each step, one fourth of the area of each part, the area at infinity becomes nil. The third worksheet in this set gave a procedure which constructed an open-ended curve describing the points from the Sierpinski triangle. The purpose of presenting this procedure was to give students an alternate view of the Sierpinski fractal figure which can be seen as a infinitely twisted curve that never touches itself, and whose points therefore do not describe an area.

3. Final interview with students

After working with the students for several sessions with the computer activities I attempted to present a couple of students with elements from formal mathematics during the course of a final interview and session (where, we should point out, the computer was not used).

These interviews were carried out with one student at a time. During these sessions I introduced students to the formal definition of the limit of sequence. During this interview I also asked the students some of the following questions:

1.) How would you define "infinity" (or "infinite"[1])?

2.) How would you define "limit"? (as well as those of "limit of a sequence" and "limit of a series").

These were presented within the context of the interview dicussion, in order to clarify as much as possible their answers and underlying conceptions.

During these interviews I also tried to reintroduce some of the sequences we had already worked with using the computer; a new discussion of the behaviour of these sequences was carried out in a context devoid of the computer and attemting to put it in the light of the formal mathematics definition of convergence.

At the end of the interview I presented students with the sets of points in a segment of the real number line and of those in a square area, and asked them to compare the cardinality (the "number of points") of each, only to get a different perspective on their conceptions of infinity and of an infinite set.

[1] As explained elsewhere, in Spanish the same word —*infinito*—, is used for denoting both "infinity" and "infinite".

Appendix 5:

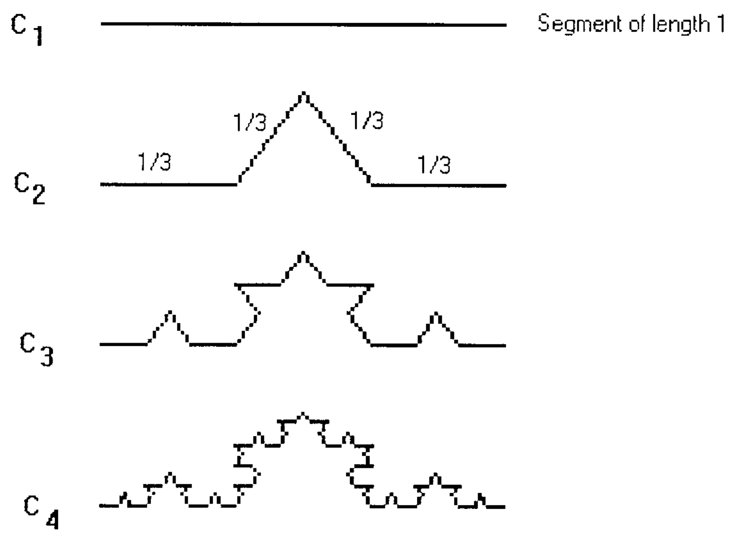
Exploratory Study: the Activity Sheets

Note: The original sheets were written in Spanish, but for the sake of clarity for the reader they are presented here in an English translation.

STUDY OF FRACTAL FIGURES

<u>The von Koch curve</u>...

...is the limit figure of the following sequence of curves



etc...

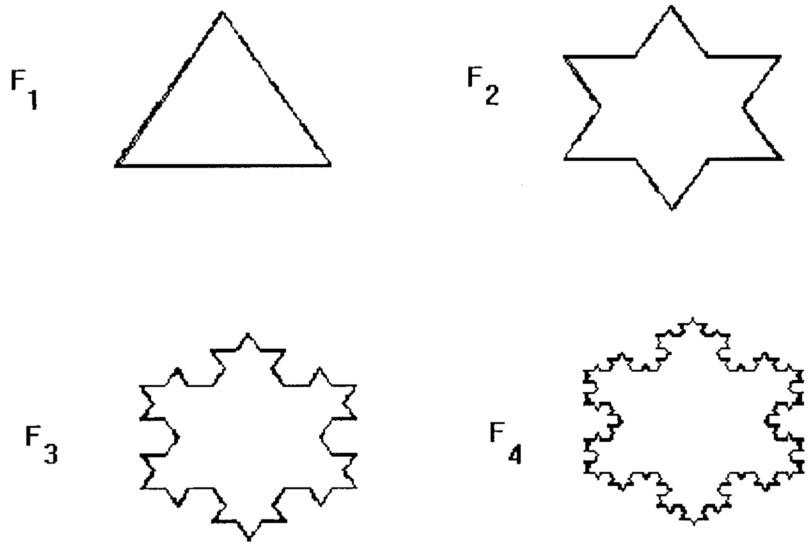
which is obtained by replacing *each* of the line segments of the previous stage, by a figure similar* to that given by curve C_2.

<u>The Snowflake</u>

It is obtained by replacing each of the sides of a triangle by a Koch curve. That is, it is the limit figure of the following sequence:

(continued...)

* Self-similarity is a characteristic of fractal figures



etc..

Activities:

1. Write a LOGO procedure for constructing the von Koch curve, and another one for the snowflake.

2. Study the snowflake fractal figure.*

That is, study the following:

a) i. How does the perimeter of the figures vary from step to step in the sequence.

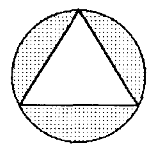
ii. Which will be the perimeter of the limit (fractal) figure.

b) i. How does the inside area of the figures vary from step to step in the sequence.

* NOTE: You can write programs to help you with the computations and the investigations.

ii. Consider the area between the figure and the circumscribing circle to the original triangle.

How does it vary with each step of the sequence?



iii. What happens with the areas studied for the limit (fractal) figure?

EXPLORATORY LOGO ACTIVITIES

From the following list of activities, investigate those which you find most interesting, but try to do as many as you can.

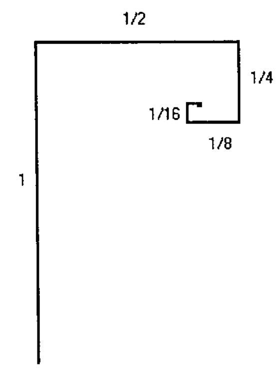
Please remember that these activities are meant to be EXPLORATIONS, which is why you are encouraged to modify the procedures you write for the activities as they are presented here, in order to explore beyond to wherever your imagination might take you.

NOTE: For any of the activities you can write procedures to help you with the computations and the study, or to understand what is happening. Do not forget to write in your diary everything you do or think while you work on the activities.

Spiral Studies

1.

a.) Write a program for drawing a spiral (heading towards the inside) such that each side is half the length of the previous one.



b.) For a given number **N** of sides, calculate the total length of the spiral. That is, write a procedure for calculating the series:

$$k. \sum_{n=0}^{N} \frac{1}{2^n} \quad \text{(k, n positive integers)}$$

c.) What happens when N tends to infinity?
What does the figure show?

2.- Same activity for a spiral that is such that its sides are in proportion to the following sequence:

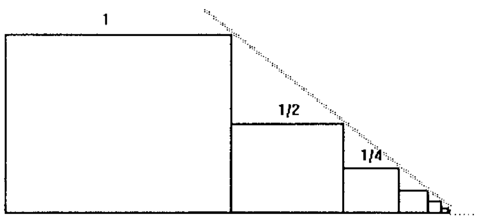
1, 1/2, 1/3, 1/4, ..., 1/n, ... (n positive integer)

What can you say of the sum $\sum \frac{1}{n}$?

(Note: Be careful by what the figure may show!)

Other related studies

1. Write a program for drawing a sequence of squares in the following way:



such that the side of each square is half the side of the previous square.

What can you say about this sequence of squares?
What does the figure show?
Calculate the perimeter and the area of the figure(s).

2.a) Same activity for the following cases:
When the side of each square is

i) 1/3
ii) 2/3

of that of the previous one.

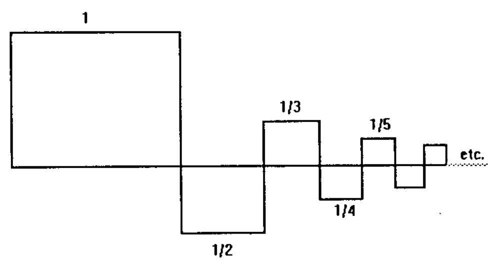
b) Same activity when the sides of the sequence of squares have the proportion of the following sequence:

1, 1/2, 1/3, ..., 1/n (n positive integer)

3. Write a program for calculating $\sum \frac{1}{n}$.
Modify your procedure for calculating $\sum \frac{1}{n^{1.1}}$ and $\sum \frac{1}{n^2}$.
What can you say about these three series?

4. Write a procedure for exploring the value of the following series:
$\sum \frac{1}{n}.(-1)^{n+1} = 1 - 1/2 + 1/3 - 1/4 + 1/5 - ...$

What does the following visual model tell you about that series? (Try to write a program for drawing this model).



5. a) Using activities like the previous ones, try to study other series of your choice, or modifications of the ones we suggested. (Remember it is usually easy to modify your procedures for exploring other things. Some ideas of series to explore are: $\sum\frac{1}{n^5}$, $\sum\frac{1}{2n}$, ...).

b) Can you think of other types of visual models for studying the series you explored?

Exploring recursive sequences

A Fibonacci sequence is defined in the following way:
The n^{th} term of the sequence (S_n) is such that
$S_n = S_{n-1} + S_{n-2}$ (n positive integer)

a) You already should have written a procedure for calculating the n^{th} term of this sequence for the case where $S_0 = 1$ and $S_1 = 1$.

i) Modify your procedure for investigating the case where

$$S_0 = 1 \text{ and } S_1 = \frac{1-\sqrt{5}}{2}$$

What happens to the sequence in this case?

ii) Write a procedure for calculating the values of this sequence, but using the following formula

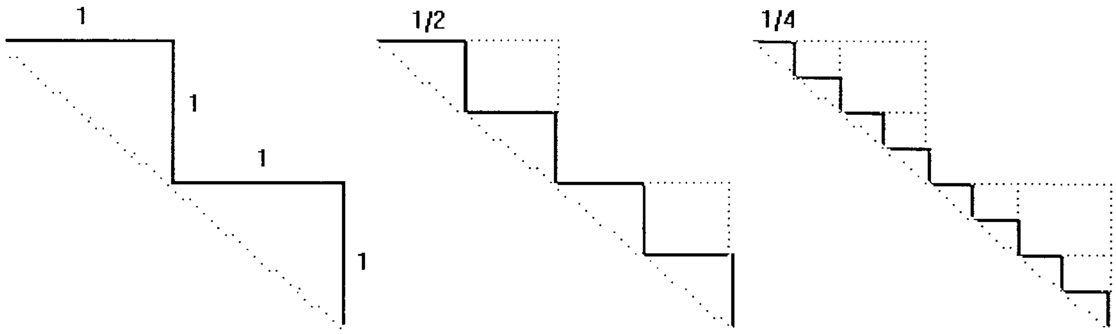
$$S_n = S_{n-1} \cdot \frac{1-\sqrt{5}}{2}, \text{ where } S_0 = 1.$$

(Why are the two formulas theoretically equivalent?)

What happens to the sequence using this method? (Is there some computer limitation for the first method?)

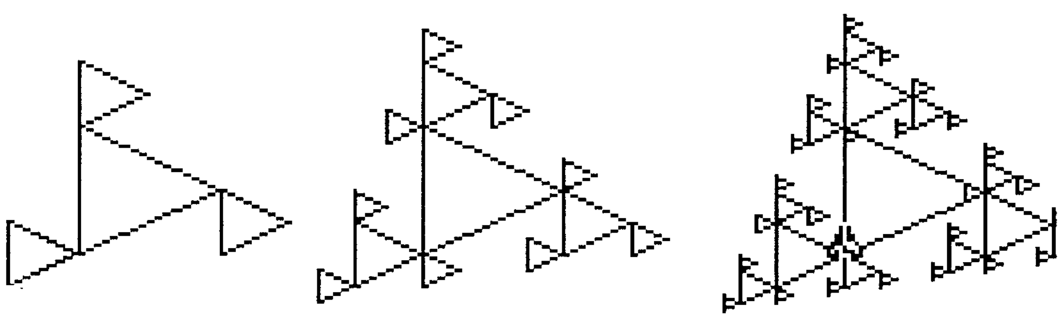
More fractals

1. Write a procedure for generating a staircase in the following way:



What can you say about the length (perimeter) of the steps in the limit?
What does the figure show?
What is happening?

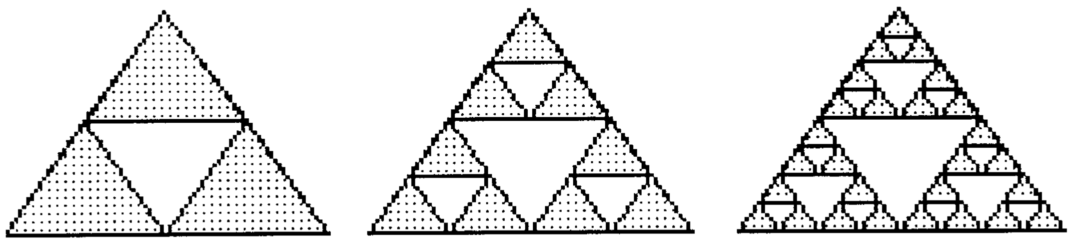
2. The Sierpinski triangle
   a) Write a procedure for generating the following sequence of figures:
   Stage 1 Stage 2 Stage 3 etc...



Investigate this figure by trying out different values for the proportion between the parent triangle and its "children". Is there a value for this proportion for which the border of the "limit figure" has no "gaps"?

b) The figure generated by the previous sequence is known as the Sierpinski triangle, which can also be generated by the following sequence which results from "taking out" the area of the central triangle from each parent (shaded) triangle:

Stage 1 Stage 2 Stage 3 etc.



What will be the (shaded) area remaining in the figure at the limit?

Appendix 6:

The Exploratory Study.
Description of the sessions and activities.

Most of the activities focused mainly on recursive programming. I wanted to use recursion as a central pedagogical element for getting a better insight into certain types of infinite processes (such as some infinite sequences and series), besides being a necessary programming element for most of the activities to be encountered in an infinite processes microworld.

Since the students had never before encountered recursion, it was necessary to introduce them to this type of programming starting with some basic recursive programming activities.

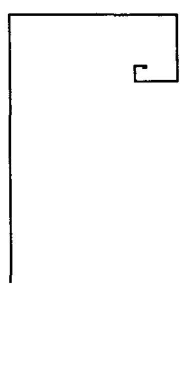
General overview of the sessions.

Session 1:

The purpose of this session was to introduce recursive programming, through the following activities:

Spirals

I asked students to write a program(s) for drawing a spiral such as:



(Fig. 1)

E.g.: An example of a (recursive) procedure for executing this task would be:

```
TO SPIRAL :LENGTH
IF :LENGTH < 1 [STOP]
FD :LENGTH
RT 90
SPIRAL :LENGTH / 2
END
```

General Observations: All of the students were able to program spirals through recursive programming. The exploration of effective ways for programming spirals led them to a natural need to learn and use recursion (realising also the importance of variables in recursion). They soon realised that the execution of the program continued indefinitely[1], and that a stop command was necessary.

The students were asked to change their procedures to explore, for instance, how to invert the heading of the spiral (from outward to inward, or viceversa).
E.g. This change can easily be done by modifying in the above procedure the recursive call to " SPIRAL :LENGTH * 2 " (and, of course, also appropriately modifying the stop command: to e.g. " IF LENGTH > 150 [STOP] "). But it can also be achieved by changing the position of the recursive call within the procedure, thus no longer having a "simple" tail recursive procedure:

```
TO SPIRAL2 :LENGTH
IF :LENGTH < 1 [STOP]
SPIRAL2 :LENGTH / 2
FD :LENGTH
RT 90
END
```

Although the first introduction of simple recursion seemed to occur in an easy manner, I introduced other activities involving simple recursion so that the concept and its use could be better understood.

Polygons

The next activity was to define polygons recursively - initially without stop commands (which again showed how a recursive process essentially continues indefinitely)-, beginning with that of a simple polygon such as that of a square and then generalising.

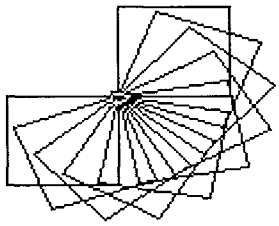
[1] When asked to explain what was happening with the polygon procedure below (before inserting a stop command), some students said that it was repeating itself "**n** times", then they corrected themselves and all the students agreed that the process continued *indefinitely,* some even saying that it was an *infinite* process.

E.g.

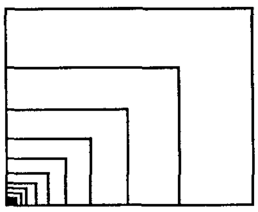
```
TO POLI :SIDE :ANG
FD :SIDE
RT :ANG
POLI :SIDE :ANG
END
```

Clusters and embedded polygons

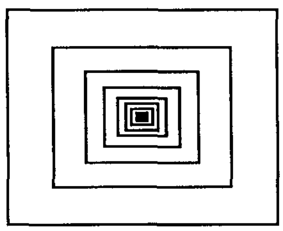
The previous activity was continued by making the polygons rotate (in a recursive procedure) thus making clumps or clusters of polygons. Another activity was to program embedded (nested) polygons (e.g. squares):



(Fig. 2)



(Fig. 3)



(Fig. 4)

Session 2:

Having introduced simple recursion through graphics, the purpose of this session was to introduce numerical recursion and recursive operations with the following activities:

Lists of numbers

The first activity was that of programming a procedure to generate a descending list of numbers: e.g. 10, 9, 8, 7, 6, ... etc.
Most students started with a procedure similar to the following:

```
TO LIST :N
PR :N
LIST :N - 1
END
```

They soon realised that this procedure generates an indefinite list of numbers, so they inserted an appropriate stop command.

The students were then asked to change the position of the recursive call from the end of the program to a position before the print command. They were also asked to experiment with the position of the stop command.
These activities turned out to be a real challenge. As in all activities, the students were asked to predict the outcome of their new procedures, and when the outcome differed from the predicted one, they were asked to try to explain what had actually happened. In this case, all of the students had predictions which differed from the outcome. For instance, in the beginning none of the students was able to explain why the procedure

```
TO NEWLIST :N
IF :N =0 [STOP]
NEWLIST :N - 1
PR :N
END
```

generated a list of numbers in ascending order, such as, in the case of using 10 as the input for N:

```
1
2
3
4
5
6
7
8
9
10
```

Almost the entire session was spent trying to understand the unpredicted outcomes. Most of this time the students reflected and explored on their own and amongst themselves, with the suggestion that they try to think as if they were the computer. After a while only one student was able to understand what was happening.
Finally we played the "Little People Method"[2] to help them understand these recursive procedures.

[2] See Harvey, B. *Computer Programming LOGO style.* MIT Press.

Recursive operations:

At the end of the session, after most students seemed to have a better understanding of the recursive procedures they had dealt with, they were asked to program a procedure for executing the factorial operation.

Session 3:

This session was a continuation of the previous one.

Recursive operations:

Most of the students were unable to solve, between the sessions, the task of programming the factorial operation, so we began the session with this activity. One of the difficulties which most students had with this activity was in realising that the stop command should output 1 when the variable N becomes 0 in a procedure as the following:

```
TO FACT :N
IF :N = 0 [OP 1]
OP :N * FACT :N - 1
END
```

A second difficulty was in the fact that the recursive call does not stand on its own. It is operated upon and is also included within another command (OP).
Most of the students were only able to accomplish and understand this task after considerable guidance.

Sequences and series

The next activity was that of writing a procedure for generating the steps of a series, such as that given by the sum of the first N integers:
$S_N = 1 + 2 + ... + N$
which can be given by a procedure like the following:

```
TO SUM :N
IF :N = 0 [ STOP]
OP (SUM (:N - 1)) + :N
END
```

Another task was to write a program for the Fibonacci sequence, which is such that

$S_0 = 1$

$S_1 = 1$

and $S_N = S_{N-1} + S_{N-2}$ for $N \geq 2$

which can be generated with a procedure such as the following:

```
TO FIBONACCI :N
IF :N = 0 [OP 1]
IF :N = 1 [OP 1]
OP (FIB :N - 1) + (FIB :N - 2)
END
```

For both tasks, most students experienced problems due to omitted or badly placed parenthesis. But a main difficulty seemed to stem from confusions related to the formal mathematical language in which I presented the definition of the Fibonacci series. In other words, many students seemed confused as to what the given formula meant, possibly arising from a lack of understanding of the mathematical concept of how the sequences are generated. However, in the end, the programming activity did seem to help in the understanding of that concept, or at least it seemed to clarify the symbolic notation.

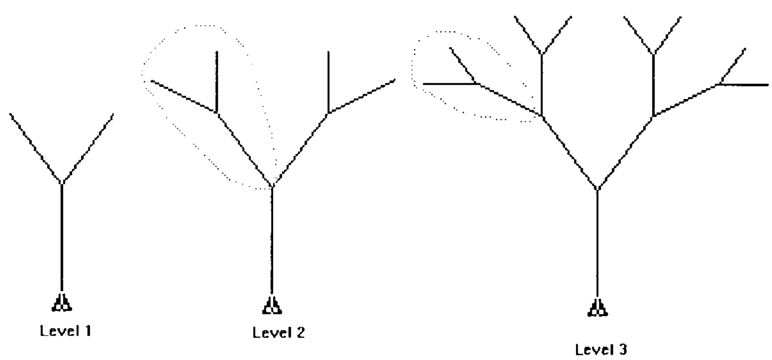
Finally the students were asked to explore how the sequences behaved, by trying out different values, and try to see if those explorations gave some clues into the convergence or divergence of the series at infinity. One problem with this activity was the insufficient memory of the computers, which were unable to compute the values of the steps of the sequences for not very large input values.

<u>Session 4:</u>

The purpose of this session was to introduce students to complex recursion and fractals. So the first activity was to program a tree.

Trees:

All of the students had considerable difficulty with this task, even though the similarity (and proportionality) of each of the branches with the "generating" figure (Level 1 figure)



(Fig. 5)

was pointed out to them several times. This fact was still very obscure and not understood by the student, so the programming task was even more difficult. At the end of the session I tried to give students more hints (see below) into how to solve the problem, but the students were still unable to solve the problem. One of the main ideas used to try to help students understand the problem was to point out how a branch is generated, and the obvious(?) repetitions and similarities within the procedure:

```
        TO BRANCH :LENGTH
        FD :LENGTH
        RT 45
        FD :LENGTH / 2
(-->    To form a tree the procedure for a "subbranch" should be inserted here)
        BK :LENGTH / 2
        LT 90
        FD :LENGTH / 2
(-->    To form a tree the procedure for a "subbranch" should be inserted here)
        BK :LENGTH / 2
        RT 45
        BK :LENGTH
        END
```

Session 5:

This session was entirely spent in trying to program the tree.

Session 6:

Finally, after two entire sessions trying to solve the "tree" problem, we had a group discussion in which most students seemed to finally have understood how the tree could be generated.

More fractals: The Koch curve and the snowflake

The next task was to write a procedure for generating the Koch curve and the snowflake. After having solved the problem of the procedure for a tree, this task did not cause many difficulties.
Examples of procedures for the Koch curve and the snowflake:

```
TO KOCH :LEN :C
IF :C = 1 [FD :LEN STOP]
KOCH :LEN / 3 :C - 1
LT 60
KOCH :LEN / 3 :C - 1
RT 120
KOCH :LEN / 3 :C - 1
LT 60
KOCH :LEN / 3 :C - 1
END
```

```
TO SNO :SIZE :C
REPEAT 3 [KOCH :SIZE :C RT 120]
END
```

Session 7:

The purpose of this session was to study the characteristics of the Koch snowflake: that is, its perimeter and area.

Studying the limits of the perimeter and area of a fractal:

The students were presented with the activity sheet 1 given in Appendix 5. In this activity sheet I defined the Koch curve and the Koch snowflake as the limits of the sequences given by first figures of the construction process. By giving this definition I wanted to point to the fact that a fractal figure "exists" only as the limit of an infinite process. But I didn't say this explicitly because I wanted to see how students interpreted the given definition. I then gave students the activity sheet 1:

Study of the snowflake fractal figure.

That is, I asked them to study the following:

a) i. How does the perimeter of the figures vary from step to step in the sequence.

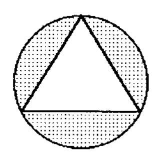
ii. Which will be the perimeter of the limit (fractal) figure.

b) i. How does the inside area of the figures vary from step to step in the sequence.

ii. Consider the area between the figure and the circumscribing circle to the original triangle.

How does it vary with each step of the sequence?

iii. What happens with the areas studied for the limit (fractal) figure?



The purpose of these activities was to make students aware of certain issues related to infinity in this particular context (e.g. unsuspected things such as infinite boundaries containing finite areas; and aspects of convergence and divergence).
The exploration of these activities was quite interesting. Most students realised that their intuitions led them to contradictions. For instance, some thought that if the area was convergent, the perimeter should also be convergent, and were somewhat surprised when they realised the perimeter tended to infinity. Another student was visibly uncomfortable by the fact that the snowflake could be contained within a circle which has a finite perimeter, and yet have an infinite perimeter.
I should remind the reader that many of these students were teachers who taught calculus and similar topics, and yet *two of the most experienced students in advanced mathematics were precisely the ones who were more surprised by their findings.*

Session 8:

The purpose of this session was to try to put together all the previous activities for an exploration of some mathematical ideas such as sequences, series, convergence, divergence, and limits, - all of them concepts involved in infinite processes. So before the session the students were given the activity sheets 2 given in Appendix 5.

Unfortunately the students explored very few of these activities. They mostly only wrote some of the procedures involved but did not stop to think or explore the ideas and questions involved.

Description of Individual Diaries (written by the students to record their activities)[3]

Student 1 (Ru):

Session 1

Spirals: Uses a recursive procedure with a counter, and increases the length by addition for drawing a spiral from the inside out (SPIRAL1). To change the heading of the spiral she changes the position of the recursive call. She explains this by the following: "we just had to imagine that the last value in SPIRAL1 was the starting value in SPIRAL2, and then the inverse process should be executed".

```
TO SPIRAL1 :I :L :A
IF :I = 0 [STOP]
FD :L  RT :A
SPIRAL1 :I - 1  :L + 2  :A
END


TO SPIRAL2 :I  :L
HT
IF :I = 0 [STOP]
SPIRAL2 :I - 1  :L + 2
FD :L   RT 90
END
```

Polygons: She gives a general recursive procedure with two variables: for the number of sides and for the length of the side. She states that she had problems figuring out how to introduce the value of the angle, but then she realised that this could be "determined" by the procedure using 360/number of sides.

```
TO POLY :NUMBER :SIDE
FD :SIDE
RT 360/:NUMBER
POLY :NUMBER :SIDE
END
```

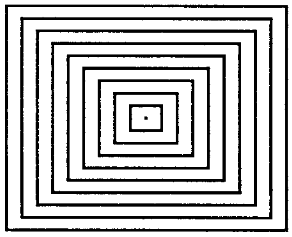
She then also included a counter so that "the polygon (she means the turtle while building the polygon) would not be turning indefinitely".

---

[3] I should point out that the students' procedures and comments which are given in this report have been translated into English for the benefit of the reader. That is - for the procedures - the names of the procedures and variables given by the students have been translated into their English counterparts when applicable. We used an English version of Logo in the study, so with the exception of the chosen names for the procedures and variables, everything else is identical to the original versions.

Embedded squares: The way in which she wrote this program is somewhat unconventional. The recursive call of the "main" procedure (SQUARE) is within a subprocedure (PERSPECTIVE): this is the way she saw it. But in reality she actually had two procedures each one calling the other. And the desired output is obtained regardless of which of the two procedures is called.

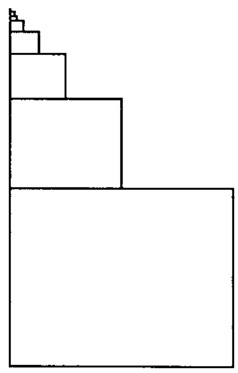
```
TO SQUARE :SIDE
REPEAT 4 [FD :SIDE RT 90]
PERSPECTIVE :SIDE
END

TO PERSPECTIVE :SIDE
IF :SIDE < 0 [STOP]
PU  FD 5  RT 90  FD 5  LT 90  PD
SQUARE  :SIDE - 10
END
```



(Fig. 6)

Other figures using squares:
For constructing the following figure, she first wrote the following procedure:

```
TO SQUARES :L
REPEAT 4 [FD :L RT 90]
FD :L
SQUARES :L - :L / 2
END
```



(Fig. 7)

The way in which she writes the variation on the variable in the recursive call (:L - :L/2, instead of just :L/2) is a reflection of her way of thinking about the procedure (the figure):

to "*reduce* each time the length of the side of the next square in relation to the previous one".

<u>Her comments regarding the first session:</u>

a.

"[Because of my mistakes (or rather carelessness)] I feel that each time I am in contact with the computer (pleasant, of course), my mind becomes a blackboard on which I am doing something (in this case drawing), and I awake to a world where, although I cannot touch a square, I can see it and construct (mentally or step by step) even if it is only graphically."

" [I am] in contact with something I cannot touch but I can perceive it because I did it".

b. She worried about writing the procedures 'the way the teacher wants them', which shows the influence from traditional school systems.

c. She also showed some confusion in the phrase: "having a procedure calling itself", because she interpreted this as meaning that there could not be any operation made on the variables in the recursive call.

Session 2:

She easily wrote the recursive procedure for printing a descending list of numbers :

```
TO NUMBER :N
PR :N
NUMBER :N - 1
END
```

But when she realised that the procedure

```
TO NUMBER2 :N
IF :N = 0 [STOP]
NUMBER2 :N - 1
PR :N
END
```

printed the numbers in ascendant order, she was surprised and asked herself if there was an "intermediate memory between the execution of the program and the printing of the results". She felt it was a "back and forth" calling of the procedure because the printing command was not reached until the stop condition became true.

[She did not keep a diary for sessions 3-5, and for sessions 6-8 she gave very little information:]

Session 6

Fractals:
She gives the following procedure for the Koch curve:

```
TO FRACTAL :L :N
IF :N = 1 [FD :L STOP]
FRACTAL :L / 3  :N - 1
LT 60
FRACTAL :L / 3  :N - 1
RT 120
FRACTAL :L / 3  :N - 1
LT 60
FRACTAL :L / 3  :N - 1
END
```

Session 7:

Perimeter of the snowflake:
She gives the following comment:
"The perimeter of the fractal has no limit. That is, when a new fractal is being formed by adding figures to the previous ones, the perimeter increases with a [length] proportional to the previous one, making this [length] larger each time, and we can observe a 'wrinkled' figure in its border, and where the smaller figures cannot be seen with the naked eye."

Session 8:

The only activity she recorded in her diary was the procedure for the spiral

```
TO SPIRAL :L
IF :L < 0.01 [STOP]
FD :L  RT 90
SPIRAL :L/2
END
```

but I have no indications that she reflected or explored the ideas suggested in the sheet.
However, I would like to point out the change in the Spiral procedure from the one she programmed in the first session: Probably caused by the way in which I presented the activity, she changed the way in which the variable is modified in the recursive call, from using addition to multiplying by a ratio. She also did not use a counter this time.

Finally, her only other comment was that, up to then, she had not been able to write a procedure for calculating the sum $\sum \frac{1}{n}$.

Student 2 (J): This student paired up with student 1.

Session 1:

The procedures he gives for the spirals, the polygons and the embedded squares are exactly the same as his partner's. But he gives the following comments:
Spirals: He gives as main difficulties for this activity the following:
- Finding out how to stop the drawing from continuing indefinitely
- How to invert the heading of the spiral which was finally achieved by changing the position of the recursive call within the procedure. (Functional knowledge, but probably not structural knowledge ?).

Polygons: In this case the change to a recursive procedure for drawing polygons was quite easy. He commented on the visual interest of the fact that the turtle would continue indefinitely.

In general he was surprised by the power of recursion: the power that a very simple procedure by calling itself can have.

Session 2:

He again gave the same procedures as his partner. He too was surprised to find out that by changing the position of the recursive call the order in which the numbers are printed changes. He realises that "in some way the n-1 [calls] are stored in memory" and recalled the last first when the n^{th} call is reached.
It is interesting that although the same principle applied to his spiral procedure when he modified it to change the heading he did not seem to realise what was happening then.

[For the rest of the sessions he only gave the procedures (same as his partner's) without any comments].

Student 3 (S) (who paired up with ***Student 4 (Ra)***)

Session 1:

Spirals:
This student comments on how this task made her realise the importance of recursion. She and her partner had no difficulty in applying recursion in this task, as well as later using recursion for creating a "fan" of squares and the embedded squares program. However, the procedures this student wrote were many times very complex. For instance, here are the procedures used for generating the spiral:

```
TO SPIRAL :P :V
TURN1 :P
TURNS 300 238 :V - 1
END


TO TURN1 :P
FD 300
RT 90
FD 238
RT 90
FD 300
RT 90
FD 238 - 238 * :P / 200
END
```

```
TO TURNS :L :A : V
RT 90
FD :L - :L * :P / 200
RT 90
FD :A - :A * :P / 100
RT 90
FD :L - :L * :P / 100
RT 90
FD :A - 2 * :A * :P  / 100
IF :V - 1 = 0 [] [TURNS :L - :L * :P / 100  :A - 2 * :A * :P / 100  :V - 1]
END
```

(!)

"Fan" of squares:
Interestingly she gave a very simple procedure for this figure:

```
TO FAN :L
REPEAT 4 [FD :L RT 90]
RT 20
FAN :L
END
```

Embedded squares:

```
TO SQUARES :P :L
POSITION  :L / 2
REPEAT 4 [FD :L  RT  90]
PU
SETPOS [0 0]
PD
SQUARES :P  :L - :L * :P / 100
END

TO POSITION :S
PU
FD :S
RT 90
BK :S
PD
END
```

<u>Session 2:</u>

Lists of numbers:
She thinks of this task in algebraic (general) terms, as writing a procedure for printing a list of numbers in the following way:

N
N - 1
N - 2
.
.

and has no problem in writing the following procedure:

```
TO WRITE :N
PRINT :N
WRITE :N - 1
END
```

She notices that this procedure doesn't stop on its own, so she changes it to the following:

```
TO WRITE :N
PRINT :N
IF :N - 1 = 0 [] [WRITE :N - 1]
END                                          ... (1)
```

which she explains prints a list of numbers in the following manner:

```
N
N - 1
N - 2
...
1
```

When asked to change the PRINT command in (1) to a position after the recursive call, she explains that she originally thought that the PRINT command would never be executed, because "the first instruction would be returning to itself again and again. Therefore, it would not print anything except 1 (the last one) because N - 1 = 0 and [] is executed and it continues to PRINT :N which only prints a 1 (because N has as value 1)."
But then she and her partner were surprised to discover that the list 1, 2, 3,..., N-1, N was generated.
They realised that their idea was correct, except for the fact that the PRINT instructions, which come after the recursive call, were "put on hold".
In order to verify their theory, this pair of students wrote other procedures by modifying the original one, such as:

```
TO WRITE2 :N
IF :N - 1 = 0 [] [WRITE2 :N - 1]
PRINT :N
PRINT 2 *:N
PRINT " (a blank space)
END
```

which would output for, for instance, WRITE2 5 the following:

```
1
2

2
```

4

3
6

4
8

5
10

Session 3:

This student commented that (so far) she had had no difficulties with recursion. However she explained that sometimes the clues or guidelines given by us (the teachers) made her try to solve the task in a "predetermined" way, and she considered this as a limitation to creativity.

Recursive operations:
She gave the following procedure for executing the factorial operation:

```
TO FACT :N
IF :N = 0 [OP :N + 1] [OP (:N) * (FACT :N - 1)]
END
```

She explains that she arrived at this procedure after "playing with LOGO" for a while, to make sure of "that LOGO understood" her instructions.

For the task of writing a procedure for calculating the sum of the first N natural numbers, she quickly realised that it was the same idea used for the factorial, and that she used the "idea of recursion" in the following way":

$$\sum_{i=1}^{N} i = N + \sum_{i=1}^{N-1} i\,, \quad \text{but} \quad \sum_{i=1}^{N-1} i = (N-1) + \sum_{i=1}^{N-2} i$$

so $$\sum_{i=1}^{N} i = n + (N-1) + \sum_{i=1}^{N-2} i$$

and so, by following this process, the sum N + (N-1) + ... + 1 can be calculated. That is, in order to calculate the sum of N, "it first has to calculate SUM N-1, but for this it first has to calculate SUM N-2", and so forth until it reaches SUM 0, when it can finally end:

```
TO SUM :N
IF :N = 0 [OP 0] [:N + SUM :N - 1]
END
```

In the same manner, she was easily able to write recursive procedures for the Fibonacci series:

```
TO FIB :N
IF :N<2 [OP 1] [OP (FIB :N - 1) + (FIB :N - 2)]
END
```

and for printing the first n terms of this series:

```
TO SERIES :N
IF :N < 0 [STOP] [SERIES :N - 1]
PR FIB :N
END
```

It seems clear that this student not only had a functional knowledge of the procedures, but had a good structural understanding as well.

Sessions 4, 5, and 6:

Trees:

Like for most of the other students this task was very challenging for this student. During the first of the two sessions she was only able to write a (non-recursive) procedure for drawing a branch:

```
TO BRANCH :SIZE :ANG
LT :ANG
FD :SIZE
BK :SIZE
RT 2 * :ANG
FD :SIZE
BK :SIZE
LT :ANG
END
```

During the second session she was still not able to write a procedure for the tree. The best she could do was to write a recursive procedure which drew a "one-sided" tree:

```
TO TREE :S(ize) :A(ngle) :L(evel)
IF :L = 0 [STOP]
BRANCH :S :A
LT :A
FD :S
IF :L - 1 = 0 [] [BRANCH :S/2 :A]
BK :S
RT 2 * :A
FD :S
IF :L - 1 = 0 [] [BRANCH :S/2 :A]
TREE :S/2 :A :L - 1
END
```

The problem with this procedure is that she uses only one recursive call at the end of the procedure. In other words she wrote a simple tail recursive procedure.
At this point she gave up on this task, and tried the von Koch snowflake task.

The Koch snowflake:

Surprisingly, she didn't have any difficulties for solving this task and was quickly able to solve it by reasoning in the following way:

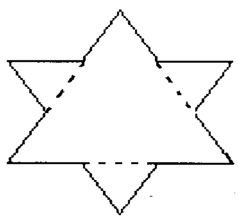
> "I saw this problem as that of building an equilateral triangle with 'weird' sides. If we had a procedure for making those 'sides', we could build the 'triangle' in the following way:

```
TO TRIAN :S :N
RT 30
SIDE :S :N
RT 120
SIDE :S :N
RT 120
SIDE :S :N
END
```

> "SIDE has to be a procedure which builds, according to :N (the level), the equal sides of an equilateral triangle. Thus,
> if :N = 1, the side will be a straight line measuring :S
> if :N = 2, the side should have this shape

(Fig. 8)

> "If we built a triangle with sides having this shape using the TRIAN procedure we would end up with the following figure:



(Fig. 9)

> which corresponds to level 2. In this figure we can 'see' the triangle of level 1."

She was able to write a procedure for SIDE, which is really a procedure for the von Koch curve:

```
TO SIDE :SIZE :LEVEL
IF :LEVEL = 1 [FD :SIZE STOP]
SIDE :SIZE / 3 :LEVEL - 1
LT 60
SIDE :SIZE / 3 :LEVEL - 1
RT 120
SIDE :SIZE / 3 :LEVEL - 1
LT 60
SIDE :SIZE / 3 :LEVEL - 1
END
```

It is interesting that she had no difficulty for writing this procedure which involves four recursive calls, and on the other hand was not able to solve the tree task which is very similar.
She went back to the tree task after solving this one, "motivated by her success". However, it still took her a few failed procedures, before finally achieving the following program involving three procedure, but using recursion correctly in her main procedure:

```
TO BRANCH :S :A
FD :S
RT :A
FD :S / 2
BK :S / 2
LT 2 * :A
FD :S / 2
BK :S / 2
RT :A
BK :S

END
```

```
TO POINTS :S :A
FD :S
RT :A
BRANCH :S / 2  :A
LT 2 * :A
BRANCH :S / 2  :A
RT :A
BK :S
END
```

```
TO TREE :S :A :N
IF :N = 0 [STOP]
IF :N = 1 [BRANCH :S :A STOP]
POINTS :S :A
FD :S
RT :A
TREE :S / 2  :A  :N - 1
LT 2 * :A
TREE :S / 2  :A  :N - 1
RT :A
BK :T
END
```

It is interesting to note that her BRANCH procedure actually draws a tree of what we could call a level 1 tree, and her POINTS procedure draws one of what we could call a level 2 tree, but she seems to be unable to see that with only one procedure she can include those first stages as well as building a more complex tree.

Session 7:

This student did explore the questions asked in the activity sheet for the Koch snowflake.
By algebraically analysing the perimeter at each step she gave the following formula:
$P_n = \frac{4}{3} P_{n-1}$ for the perimeter at stage n, with $P_1 = 3L$ where L is the length of the side of the triangle.
That is, at each stage, the perimeter is 4/3 of the perimeter from the previous stage.
So at stage n the perimeter would be:

$$\frac{4^{n-1}}{3^{n-2}} \cdot L$$

She first imagined that the figure, the higher the stage, the more it would tend to look as the circle. That is, in her words, that the perimeter of the figure would "converge" to the perimeter of the circumscribing circle.
But then she calculated the limit, as n tends to infinity, of P_n. That is,

$$\lim_{n \to \infty} \frac{4^{n-1}}{3^{n-2}} \cdot L = \left(\frac{9}{4}\right) L . \lim_{n \to \infty} \left(\frac{4}{3}\right)^n \to \infty$$

When she discovered that the perimeter tended to infinity, she thought that the snowflake had to get bigger than the circle. But she was surprised to find out that this did not happen!
So she thought that it was the area of the snowflake which converged to the area of the circle. She explained that having a non-converging perimeter but a converging area, was something which she found very puzzling, and she thought she had made a mistake somewhere although she could not find any.
She then tried to calculate the area of the figure. She arrived at the following formulas for the first two stages

at stage 1 $A_1 = \frac{\sqrt{3}}{4} . L^2$

and at stage 2: $A_2 = A_1 + 3.A_1$

This is as far as she got and she didn't calculate the limit of the area as the stage tended to infinity. But seeing her formula at stage 2 made her wonder if also the area didn't converge.

Student 5 (Ch):

Session 1:

Polygons:
He gave the following recursive procedure:

```
TO POLYGON :L :N
FD :L
RT 360 / :N
POLYGON :L :N
END
```

He explains that although using recursion for this procedure was very "elementary" it gave him a new insight into what recursion is.

For making the polygon rotate, he first wrote the following recursive procedure:

```
TO ROTAPOLY :L :N :A
POLYGON :L :N
RT :A
ROTAPOLY :L :N :A
END
```

But he discovered that the procedure generated a neverending cycle (that is, in his definition: "when in a program one or more instructions are repeated an infinite number of times without having the possibility of getting out"), and therefore the next command was never executed. He realised that a counter was necessary to stop the procedure, but he opted for changing the procedure from recursive to iterative (by using a REPEAT command:

```
TO POLYGONS :L :N
REPEAT :N (FD :L  RT 360 / :N)]
END                              ).
```

He explains that he didn't use a counter because "the variable that keeps the count has to be <u>external</u> to the procedure, that is, it must be a global variable, and not a local one" but he claimed that it was un-advisable to use global variables. He later commented how he realised that this was a mistake after having worked with recursive procedures longer.

His comments show that he had a fair understanding of how recursion works and that the non-conditioned recursive procedure gave him a sense of what an infinite process is. They also show his understanding of global and local variables and how they behave within recursive procedures.

Spirals:

He gives the following recursive procedure for a spiral, where the variable is modified by subtraction in the recursive call:

```
TO SPIRAL :L
REPEAT 2 [FD :L RT 90]
IF :L < 0 [STOP]
SPIRAL :L - 10
END
```

By now, it seems that the use of a stop command has become immediate.

For inverting the heading of the spiral he does not change the position of the recursive call. Instead he uses the method of modifying both the stop call and the recursive call so that the length of each new side of the spiral increases instead of decreasing:

```
TO SPIRAL2 :L
REPEAT 2 [FD :L RT 90]
IF :L > 120 [STOP]
SPIRAL2 :L + 10
END
```

Session 2:

This student had no problem for writing the recursive procedure for printing a downward list of numbers:

```
TO COUNT :N
PRINT :N
IF :N = 0 [STOP]
COUNT :N - 1
END
```

However, when asked to change the PRINT command to a position after the recursive call, he was surprised by the resulting output. He had expected that the program would never print anything, but to his surprise it printed the numbers in ascending order. It took quite a while to understand that the program was reaching the PRINT command after the last recursive call was executed.

Session 3:

He did not attend this session, and this was later reflected in the fact that he never programmed his procedures as operations, using the MAKE command instead.

Sessions 4 and 5:

Trees:
As almost all the other students, it took this student quite a bit of time to finally understand how to write the procedure for generating a tree:

```
TO TREE :X :N
IF :N = 0 [STOP]
LT 45
FD :X
TREE :X / 2 :N - 1
BK :X
RT 90
FD :X
TREE :X / 2 :N - 1
BK :X
LT 45
END
```

He explains that his main difficulty was wanting to use only one recursive call, and that he had the idea that another procedure needed to be called and not just the same one. This made him realise that he still had not completely grasped how recursion works.

Session 6:

The Koch curve:
He explains that he was able to write this procedure by using the tree procedure as a model. The fact that he used the previous procedure as a model is clear when, unlike other students, he doesn't write a procedure which uses a variable for controlling the stage of the fractal generating process. Instead his procedure automatically generates an approximation of the fractal by stopping when the segments become fairly small:

```
TO KOCH :X
IF :X < 5 [FD :X  STOP]
KOCH :X / 3
LT 60
KOCH :X / 3
RT 120
KOCH :X / 3
LT 60
KOCH :X / 3
END
```

Other activities:
This student had no problem in writing procedures for most of the activities seen in the rest of the sessions. He seems to have grasped the use of recursion and understood the role of local and global variables. He even learned how to use the MAKE command which I did not introduce (using it instead of programming recursive operations which, as I noted above, he never learned) . He used this command in all of the "series" procedures, like for instance the one for $\Sigma\ 1/2^n$:

```
TO SUM1 :N
IF :N<0 [STOP]
MAKE "S 0
SUM1 :N - 1
MAKE "S  :S + 1 / (POWER 2 :N)
PRINT :S
END
```

He was also able to write the procedures for the Sierpinski triangle first without the children triangles, and then using that method.
Unfortunately, this student completely omitted all of the pencil and paper and other exploratory activities that I suggested around the procedures, and designed for leading them to reflect on infinite processes, limits, etc. He completely concentrated on the programming activities.

Appendix 7:

Main Study: The case of Consuelo and Verónica.

This case study includes some detailed descriptions on the way in which the activities evolved because I feel it is the best way to illustrate the students interaction with the microworld and the *constructionist* aspect: how, through carrying out and exploring their ideas through the microworld, the students could build connections between the elements involved, abstracting and generalising properties *within* the context of the microworld.

The students in this case study are the two fourteen year old girls, Consuelo and Verónica, both of whom had just finished Mexican secondary school (the 7th- 9th years of studies) and were beginning the first year of the Mexican "preparatory school" (years 10-12 of studies). As described in Chapter 4, neither had any previous computer experience (before the Logo course given before the study) and both said they were average mathematics students, with no particular inclination for that field.

Part A. Sequence studies.

1. Explorations with the sequences $1/2^n$ and $1/3^n$.

a. Consuelo and Verónica connect the never-ending procedure with the infinitude of the underlying process by observing how the procedure runs and complementing it with the numeric values.

The students had been running the procedure[1]:

[1] As in the other cases studies, for the sake of clarity, the names of the procedures have been translated from Spanish into English.

```
TO DRAWING :L
FD :L
RT 90
WAIT 10
DRAWING :L * 1 / 2
END
```

which produced a spiral such as the one in Figure 1.

The students had noticed that the turtle kept turning in the centre of the spiral, but were initially unable to explain such behaviour. They did recognise however, that the process would continue indefinitely because it was a recursive procedure and no Stop condition had been given.

At my suggestion, they computed the values the turtle walked each time filling out a table with those values (see Table 1 further below).

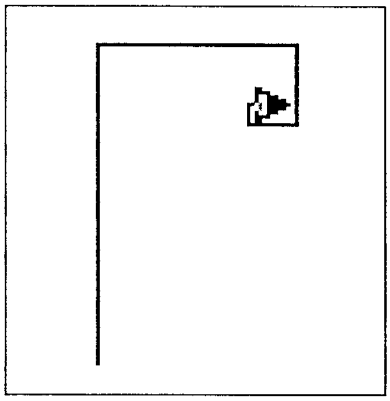


Figure 1: Spiral produced by the DRAWING procedure, corresponding to the sequence $\{L/2^n\}$

Consuelo:	Oh..., its 100 *over* 2. Then it is going to be 50. And the third would be 25, and then 12.5, and then 6.25... But then, why does it start turning? Why is it that after the 6th, it starts turning in the same place?
Verónica:	It is very small,
Consuelo:	There we can only see 6 arms. The others can no longer be perceived.
Ana:	They cannot be perceived, but do they exist or not?
Consuelo:	Yes, because the turtle keeps going.

Consuelo became aware that the process of halving segments of the spiral continued, though her observation of the turtle's continuing movements. The process was seen as decreasing and implicitly convergent to zero, although initially it was not seen as infinite, as is evident from the transcript below. They would have to continue filling out the table and re-observing the behaviour of the turtle.

Verónica:	It could be that it gets to zero, or to less than zero?, and then it won't walk anymore...
Ana:	Do you think it gets to zero?
Consuelo:	Yes; that it gets to a number... that it becomes smaller, and smaller, until it gets to zero.
Ana:	And when would it get to zero?
Consuelo:	Well, after the 6th step.

Arm [of the spiral]	[Distance] walked
1	100
2	50
3	25
4	12.5
5	6.25
6	3.12
7	1.5
8	0.75

Table 1: Table used by Verónica and Consuelo for recording the distance walked by the turtle in each segment of the spiral

Initially both students were confused when they observed that after a certain number of steps, the value of the segment had integer part 0: they thought that a value of "*0. ...*" meant that zero had been reached and "surpassed". After a brief discussion they realised their mistake and that zero could not be surpassed as the number would then be negative. This was a significant event which led them to realise that the values were becoming very very small, and they now started to doubt as to whether zero would ever be reached:

Consuelo:	It seems after the 7th step it will pass zero
Ana:	What is it in the 7th step?
Consuelo:	It's 1.5. And after that it will be less than zero.
Verónica:	It's 0.75
Ana:	Is that less than zero?
Consuelo:	No.
Verónica:	Yes, it is less than zero, isn't it? Because it is *zero* point 75, right?
Ana:	And is that less than zero?
Verónica:	Yes
Ana:	Is 0.3 less than zero?
Verónica:	No... Oh! It has to be negative...
Ana:	And is this at some point going to be negative?
Consuelo:	No, because it is like it continues with the decimals.
Ana:	So, is it not going to be negative?
Consuelo:	No, it becomes very very small
Ana:	And does it become zero?
Verónica:	mmmm...

They then noticed that the procedure had kept running all this time, with the turtle still turning in the same spot. This fact that the procedure continued without apparently having an end, seemed to influence a change in their conceptualisation of the process: at least Verónica now seemed to realise that the turtle was theoretically still drawing the spiral:

Verónica:	It's doing more lines, but because the turtle doesn't fit in that small place, well, it keeps turning in the same spot.

Consuelo suggested doing an "amplified drawing". With twice the scale, they still could only perceive 7 "arms" of the spiral, although they were aware that the turtle kept walking. They now perceived the process as (potentially) infinite with the sequence of values of the segments never reaching their limit.

Consuelo:	No, it's never going to reach zero.
Verónica:	It's not going to reach zero because it has a lot of decimals.
Ana:	So, is it going to stop someday?
Verónica:	It's going to go on forever.

As we saw,

- the students initially did not understand why the turtle kept turning in the centre of the spiral, although they were aware that the procedure was recursive and would continue since it did not have a stop condition;

- by computing the values the turtle walked they seemed to be able to coordinate the decreasing values which approached zero with the decreasing segments and the behaviour of the turtle which appeared to turn in the same spot (the diverse representations were coordinated);

- by again noticing that the procedure continued without stopping, with the turtle still turning in the same spot, and observing the increasingly smaller values in the numeric output, they started to realise that zero would not be reached (in a finite time). That is, they began to connect the continuing procedure with the infinitude of the corresponding process;

- by running the procedure again with a bigger scale they were able to confirm their observations, and they now used the numerical representation (the structure of the decimal numbers) as a means for justifying the infinite nature of the process: "It's not going to reach zero because it has a lot of decimals... It's going to go on forever". They found *meaning* for their observations in the decimal numeric structure of the values.

b. Recognising the potential of constructing new ways of looking at the process by modifying the procedure and the graphic models.

Having realised the never-ending characteristic of the procedure, Consuelo would then suggest adding a Stop condition[2]. Later on, when I suggested modifying the procedures (see the Sequence Studies Handouts in Appendix 2), and I explained that the MODEL has so far been a spiral, but that they could do other types of things such as a simple straight line, Consuelo interrupted and said:

> Consuelo: If we stretch the spiral we can see if it has stopped going forward, if it is not doing more, if it has stopped..

Consuelo also suggested separating the spiral into bars — a bargraph model; this was an idea Verónica liked and said that in that way they would be able to see how many segments were being drawn. Even before they had tried the new models, the students seemed to realise the potential of these two other types of representations for observing and confirming the behaviour of the process, how long it continues and how many segments are drawn with a predetermined stop condition. The students were thus actively involved in the construction of new representations and observational approaches for the process(es) under study.

c. Looking for a relationship between the number of segments drawn and the value in the stop condition.

In the course of their explorations, the students became interested in examining how many segments would be drawn in the (spiral) model before the procedure stopped (that is, before the stop condition became true). They modified the procedure to include a variable (:COUNT) for counting the number of segments the turtle drew[3]: thus, when they investigated the spiral produced through the process of "taking thirds"

2
```
TO DRAWING :L
IF :L < 2 [ STOP]
FD :L
RT 90
WAIT 10
DRAWING :L * 1 / 2
END
```
3 That is, they modified the stop condition in DRAWING to "IF :L < 2 [PR :COUNT STOP]".

which had 4 segments (with a scale of 100), they also got as output the value "4". Consuelo, seemingly aware that the process was indefinite and only stopped through an arbitrary command, commented that the turtle only drew 4 "arms" because of the condition, and suggested comparing it with the procedure which used "1/2" (i.e. which involved the process of "taking halves"). In the latter case they obtained 6 segments in the spiral using the same scale.

(i) Experimenting with the value in the stop condition: coordinating this value with the behaviour of the turtle and graphic.

The students then investigated changing the value in the stop condition (the results of which they recorded in the table shown in Table 2), and Consuelo, who was now beginning to coordinate the value in the stop condition to the underlying numerical process, soon realised and commented that if they used "50" in the Stop condition they would only get 2 segments, and for "25", three. She soon confirmed these suppositions. When they used 0 as the value in the stop condition, the students now also became aware that the process could continue infinitely but the segments (the value of :L) would never be less than zero. In this case Verónica commented: "it is with 0 with which it keeps going on and on, that [the procedure] will never stop"; in the table they wrote "it will continue infinitely".

Value in the inequality [:L<...] in the Stop Condition	Number of arms drawn before the procedure stops
2	6
50	2
25	3
0	It continues infinitely
1	7
0.5	8
0.25	9
0.1	10
0.15	10
0.015	13

Table 2. Table used by the students in the exploration of the model for $L/2^n$, with an initial input of 100 for L

(ii) Finding a pattern for linking the stop value with the number of segments.

In their continuing explorations, Verónica tried to find a pattern which linked the value in the stop condition with the number of segments, thinking that for each smaller value they used, one more segment would be drawn:

Verónica: The smaller the value is... No, the *bigger* the value gets [the number of segments] is reduced by 1. That is, it is reduced by 1 because for 0.5 it's 8, then

for 1 it's 7, and for 2 it's 6. So it's doing one arm less.
[...] If we put zero point zero... it is going to go on with 11, 12, 13, 14,...

I suggested they tried 0.15 as the value in the stop condition; with this value they got the same number of segments as for 0.1, and this made them reflect on why this was so. Later Consuelo suggested stopping the procedure when the value of :L (the segment) became very very small, when it became smaller than a decimal with 500 zeros in its decimal expansion[4]. She also again became interested in the variation of the number of segments, and I pointed out the importance of using constant intervals:

Consuelo: And if we looked to see if there is a rule here for the difference in the count, would it be helpful?
Ana: OK. But you must take constant intervals.

They chose to vary the value in the stop condition by 10 decimal places each time and recorded the values in Table 3:

Value in the Condition: :L < 10^{-*}	COUNT (Scale (initial :L) = 100)	Difference with previous count
0.0...01 (at the 500 position)	1668	
0.0...01 (at the 100 position)	339	
0.0...01 (at the 25 position)	90	
0.0000000001 (10 digits)= 10^{-10}	40	
1	7	33
0.1	10	
10^{-10}	40	
0.0...01 (20 digits) = 10^{-20}	74	34
10^{-30}	107	33
10^{-40}	140	33
10^{-50}	173	33
10^{-210}	705	
10^{-220}	738	33

Table 3. New table used by the students in their exploration of the model $L/2^n$, with initial input 100

They quickly discovered that the variation in the count of segments or "bars" tended to be a constant of 33, which they also tested for large exponents (10^{-210}) in the stop value. Although the students did not make conjectures as to why a pattern had emerged, their discovery did seem to show a constant behaviour in the way the process behaved.

[4] For this they used IF :L < POWER 1 -500.. as the stop condition.

Later on, when they studied the process corresponding to the sequence $\{1/3^n\}$, the students again wanted to look for a similar pattern for that case. By using a bargraph (BARS) model (see Figure 2), they investigated, through Table 4, if a constant pattern in the number of bars emerged when they varied the value (10^{10} times smaller each time) of the smallest bar length (:L) in the stop condition "IF :L < ..." of the procedure below.

```
TO DRAWING :L :COUNT
IF :L < 1 [PR :COUNT STOP]
BARS :L
DRAWING :L * 1 / 3 :COUNT + 1
END
```

Figure 2: Bar graph model corresponding to the sequence $\{1/3^n\}$.

Scale = 100

Condition: :L < ...	COUNT	Difference [with previous count]
1	5	-
0.0...-10 zeros-...01 = 10^{-10}	26	21
0.0...-20 zeros-...01 = 10^{-20}	47	21
0.0...-30 zeros-...01 = 10^{-30}	68	21

Table 4: Table of values used by the students in their explorations of the sequence $\{L/3^n\}$

Using the same values as in the case of the procedure which halved the values, they noticed that the difference in the number of segments changed from 33 to 21 when the procedure took thirds instead of halves. Consuelo was able to connect the smaller difference in the number of segments to the faster decrease of the segments in the sequence $\{L/3^n\}$ (something they had already observed in a previous worksession):

Consuelo:	It does less because it is now taking a third, and it did more because it was taking halves.
Ana:	Why?
Consuelo:	Because when dividing by 3 the bars get smaller faster.

In the above account the following points are worth noting:

- The students constructed their own exploration approaches: they became interested in looking at the number of segments drawn before the stop value was reached, and carried out modifications in the procedure which would help them in their purpose.

- By constructing and observing a table of values, Consuelo was able to coordinate the value of the stop condition with the values of the sequence represented in the graphic: she discovered the meaning of the stop condition value in terms of the number of segments that would be drawn. Verónica as well discovered the meaning of using zero as the stop value: the procedure would go on indefinitely because zero is never reached.

- The students designed another area of investigation: that of finding a *pattern* in the number of segments as they varied the stop condition. The situation of obtaining the same number of segments for different stop values was one which made them reflect on the relationship between the two factors.

- The record in table of the number of segments through constant variations led to the discovery of the sought-after pattern: the number of segments also increased in a constant manner. This was a relationship which was discovered (and then tested) *within* the context in which the processes were presented.

- The students then used the tools of the microworld to test their observations and investigate if similar results appeared when they modified the process; they had the initiative (from the start of these investigations) to compare the behaviours of two different processes (e.g. that of "taking thirds" vs. "taking halves"). Consuelo then seemed able to coordinate all the evidence which pointed to the fact that when "taking thirds" the process decreased faster than when "taking halves": the numerical difference (recorded in the tables) in the number of segments in a same range, and the behaviour of the bar graph. These constituted *situated* observations.

d. Discovering the "convergence" of the sequence $\{L/2^n\}$ to zero (and of the series $L + L/2 + L/4 + L/8 + ...$ to 2L).

When the students began investigating the sequence $\{L/2^n\}$ using the line model, Verónica had predicted that if they "stretched the spiral", and did not use a stop condition, then the resulting line would go all the way to the top of the screen because "it would be very big". The students then noticed that with a scale of 50, in 12 steps (segments) the turtle had drawn a line with approximate length of 100 turtle steps (see Figure 3). The students attempted increasing the scale, but always got a line that eventually "got stuck". Consuelo, who seemed aware that the process was potentially never-ending, found justification for the turtle stopping at a certain length in the stop condition; so she suggested changing the value in the stop condition to 0 which she now knew was equivalent to eliminating that instruction.

Consuelo: It only stops because it has an IF. It continues straight up. The spiral is stretched. [But it doesn't go all the way] because we have an IF.

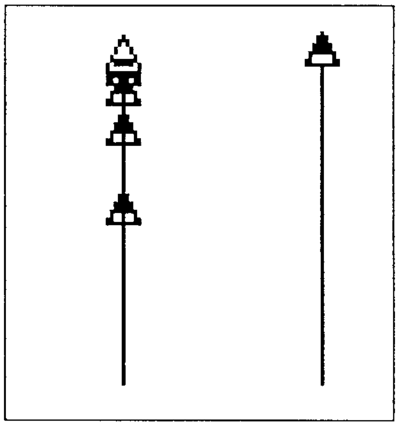


Figure 3: The LINE model for $\{L/2^n\}$. The left hand side of the figure depicts the movements of the turtle. The right hand side shows the actual result.

But when they ran the procedure again the turtle appeared to stop, vibrating in the same place even though the procedure did not stop. Verónica explained that because the stop condition could never become true, then "the turtle [kept] walking but in the same place" (as shown by the vibrations). I then suggested they paused the procedure a few times and printed the value of :COUNT to see how many segments had been drawn so far. At the first pause they got 96 as the value of :COUNT (the number of segments); they then let the procedure run again, and in the next pause they got 126 for the count. The count of the segments helped the students ascertain that in fact the turtle continued "walking" although imperceptibly small amounts.

Consuelo: It must be that it walks very very little and it can no longer be seen.

Consuelo then suggested generating the bar graph model in order to look at how small the segments became. In the resulting figure (Figure 4) they observed how the histogram decreased to the point where the turtle started drawing points endlessly (the value in the stop condition was still 0). When Verónica finally paused the procedure, the count of segments she printed was 545. The last exploration confirmed that the turtle indeed kept walking, and led Consuelo to conclude that even though the turtle looked as if it only vibrated in the same place, it was moving forward even if only very little.

Consuelo: So it does keep walking but it is very very little. So it wasn't in the same place [referring to the turtle in the LINE model].

Ana: How could we determine how little it is walking?

Consuelo: Like with the count, [printing] the value of L. But it is always going to be half.

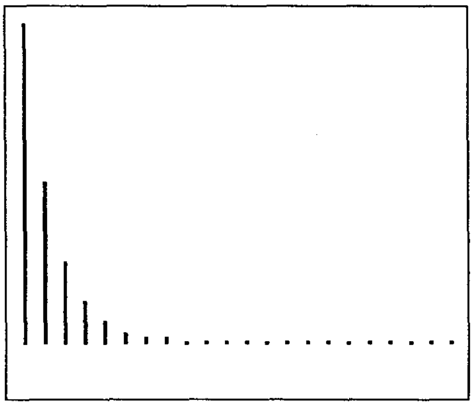


Figure 4:. Bar graph model for $\{L/2^n\}$.

Consuelo then suggested changing the stop condition to having :L *equal* to zero ("IF :L = 0"), but there was no change when they ran the procedure (which continued without stopping). Verónica said the procedure would never stop and Consuelo observed this was because the stop condition did not become true since the length would never become zero. They then paused the procedure and printed the value of :L and of :COUNT (which allowed them to see how many segments it took for :L to get so small), the values of which they recorded in Table 5. After letting the procedure run a bit longer, they paused it again to get the new value of :L (and :COUNT):

COUNT	Length walked
95	0.00...-26 zeros-...02524354...
1857	0.0....-557 zeros-...09711...

Table 5: Table with the length of the segments corresponding to certain terms (as given by COUNT) of the sequence $\{1/2^n\}$.

Already the first value they observed of :L served as confirmation for Consuelo that the process did indeed continue indefinitely as she linked the infinitude of the process with the decimal representation, realising that the process could continue indefinitely as more and more zeros could always be added to the decimal expansion. Because both students viewed the process of adding zeros to the decimal expansion as potentially infinite, they concluded that :L would never become zero; Consuelo also realised that this meant that the stop condition would never become true (she suggested using instead something like "IF :L < 0.0001").

Consuelo:	It's going to keep going, isn't it? Afterwards it will be [in the decimal expansion of the length of the last segment] more zeros and more zeros, and more zeros.... and so it would never get to zero. And so we can use a condition that says that when we get to 5 decimal digits it should stop.
Verónica:	Yes. So it is going to keep increasing each time the zeros to its decimal list. So it is never going to reach zero.
Consuelo:	It is never going to reach zero.

But although the students concluded that the length of the segments could never become zero, both did point out how it was indeed getting *very* close to zero. Within the context of the microworld and through the representational systems provided (lengths of segments also viewed in terms of decimal representations), the students discovered and concluded that the process of "halving lengths" was one which a) could continue indefinitely; with b) the length of the segments approaching zero, even if zero was never reached; and c) the decrease of the segments to almost zero explaining that the Line model, where the segments are added, became visually invariant. The process which led to these conclusions can be summarised as follows:

- The students initially predicted that the Line model — which they viewed as "stretching the spiral" and therefore understood as a "sum of segments" — would be very long (perhaps indefinitely?) as they translated the ongoing (infinite) nature of the process into the behaviour of the graphic model.

- However, the observation of the visual model gave evidence that the Line did *not* go on. The students then made use of the flexibility of the microworld tools (e.g. changing the scale) for testing the initial observation and looking for evidence to the contrary .

- In an attempt to find meaning on how an infinite "sum of segments" could form a limited (bounded) segment, Consuelo looked for justification in the stop condition of the procedure and tested the effect of overriding this command.

- The vibrations of the turtle, later complemented by a numerical count of segments, gave evidence that the process continued even though the visual model seemed to become invariant. Looking for a means to coordinate the two observed factors (the ongoing process vs. an invariant visual model) Consuelo began to realise that as the process progresses the added segments became very small.

- Consuelo suggested a new approach: looking at a bargraph model in order to look at the behaviour of the segments as the sequence progressed. Through this activity the students were able to confirm what they had begun to suspect: (i) the process continued, but (ii) the segments became very small.

- The observation of the behaviour of the procedure when Consuelo changed the stop condition to *equal* to zero, led them to reflect that even though the segments became very small and approached zero, zero would not be reached.

- Their observations were complemented by printing and recording in a table the numerical values of the segments, confirming how small and close to zero they became.

- The numerical representation then served as a means to justify how the process could in fact continue indefinitely, in that the "number of zeros" in the decimal representations could continue growing with the values never actually reaching zero.

2. Modifying the procedures: from "operating on a segment" to sequence lists.

a. Finding the mathematical formula for generating the sequence.

As described in Chapter 5, one of the planned activities in the microworld involved changing from a procedure that was based on a direct operation on segments, to the process of generating an independent sequence of values which could then be linked or transformed into visual segments. With this purpose in mind, I had asked the students if they could write down how much the turtle was walking each time and then in any given step *N*. The students seemed to understand what I meant by the latter, and Consuelo was able to translate the process into an algebraic formula:

Consuelo:	So it walks 100, the second time 50, and then half. So it is a half, then a fourth, then an eighth. So it is one over N to the...
Ana:	OK. What is a fourth equal to? It is equal to 1/2 by 1/2.
Verónica:	Yes
Ana:	And what is that equal to?
Consuelo:	To 1 over 2 square.
	Then it's 8... so it's one half cubed. So... then it is one over 2 to the N.

They wrote "In the Nth step the turtle walks $1/2^n$". Consuelo added that for the process of dividing by 3 each time, the formula in terms of N would be: "One over 3 to the N". Having deduced these formulae, translating them into a function (FN) procedure[5] was then straightforward.

One of the first interesting events using the formula was that when they tested the FN procedure they realised the input :N was similar to the count they had been using before. That is, when they typed "PR FN 95", they noticed that the resulting value (2.524354897E-0029) had the same digits as the length of the 95th segment found before (see Table 5), except it had two extra zeros in the decimal expression; after discussing it for a while they realised that when they used the DRAWING procedure they had used as input of 100, so the values were 100 times the ones given by the FN procedure[6]. In this way the students were led to connect the two procedures and made sense of some of the differences, particularly the fact that the FN procedure did not include the scale.

I had then showed the students how they could generate a list of *n* values from FN 1 to FN *n*, through the procedure SEQUENCE below:

5

```
TO FN :N
OP 1 / POWER 2 :N
END
```

6 They verified this by typing

```
?PR 100 * FN 95
2.524354897E-0027
```

```
TO SEQUENCE :N
IF :N = 1 [OP FN 1]
OP SE (SEQUENCE :N - 1) (FN :N)
END
```

The students did not seem to have any difficulty in relating the output of this procedure to the segments produced through the process of dividing lengths they had been exploring before. In fact, when they typed "PR SEQUENCE 10", they immediately noticed that the output (below) corresponded to the first 10 values of the sequence 1/2, 1/4, 1/8, etc.:

0.5 0.25 0.125 0.0625 0.03125 0.015625 0.0078125 0.00390625
0.001953125 0.0009765625

The DRAWING procedure was then modified into a procedure DRAWS[7] for "draw sequence", which the students confirmed produced the same models as DRAWING.

```
TO DRAWS :LIST :SCALE
IF :LIST = [ ] [STOP]
BARS (:SCALE * FIRST :LIST)
DRAWS BF :LIST :SCALE
END
```

Later, when the students decided to change the function to $1/3^n$, Consuelo suggested using 300 as the scale in DRAWS, explaining that because now they were dividing by 3, it was best to use a multiple of 3. It seems that Consuelo easily linked the new approach of a sequence generated through the function representation with the iterative process of dividing by 3. Furthermore, when I asked the students to explain in writing the visual output of the procedure DRAWS (see Figure 5) , they reproduced the picture on paper, writing 100, 100/3, 100/3/3, 100/3/3/3 under each of the corresponding bars with Consuelo explaining: "It's 100... Then, over 3...,which is like 33.333333..."

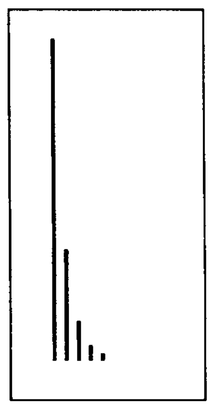


Figure 5: Bar graph model corresponding to the first 5 terms of the sequence $\{1/3^n\}$ produced by typing DRAWS SEQUENCE 5 300

[7] To use this procedure they typed for instance:
?DRAWS SEQUENCE 10 100
which gave as output a bar graph of 10 bars, and if FN corresponded to $1/2^n$ then the first bar measured 50 turtle steps (half the scale).

At this stage the students decided they also wanted to write a procedure which would give out the values of the sequence *with* the scale, so that they would be easier to see; this led to the procedure SEQSCA (short for SEQuence-with-SCAle):

```
TO SEQSCA :N :SCALE
IF :N = 1 [OP FN 1 :SCALE]
OP SE ( SEQSCA :N - 1 :SCALE ) ( FN :N * :SCALE )
END
```

As we saw, the transformation of the procedures from an "operating on segments" approach to the drawing of a function generated sequence list was not too difficult for the students, most probably because of the steps involved in the process which included:

- an observation and listing of what the turtle walked at each step;

- deducing what the turtle would walk in the *n*th step, which was written as a mathematical (function) formula;

- translating the algebraic formula into a function procedure;

- through the numerical results, discovering the connection between the count of the number of segments in DRAWING with the input of the function procedure, obtaining further insight into the relationship of the two procedures;

- using the SEQUENCE procedure to produce a list of the values from FN 1 to FN *n;*

- correlating the output of SEQUENCE with the process of halving: they recognised in the decimal representations the sequence 1/2, 1/4, etc.;

- modifying the DRAWING procedure to a procedure that instead of taking a segment and dividing it, could take as input a sequence which would then be transformed into segments;

- and finally also writing a procedure which would output the actual values (i.e. with scale) of the sequence of segments.

Thus, the transformation in how the processes and sequences would be approached was made in a seemingly straightforward manner, but it constituted an important change since it implied a (situated) formalisation of the process.

b. Observing the rate of convergence of the sequence $\{1/3^n\}$ by looking at how the difference in consecutive terms of the sequence tends to zero as the sequence progresses (i.e. as n gets bigger)

Among the first investigations with the new type of procedure was when Consuelo had explained she wanted to look at "the relationship in the heights of the bars", at the difference between consecutive values. So they wrote a procedure (DIF[8]) to compute the difference between two consecutive bars or terms which they used it to fill a table they had created (see Table 6). Although initially Consuelo was disappointed that she could not find a pattern in the differences (she would later discover that in fact each difference was a third of the previous difference), she did point out that each time "there [was] less difference between one bar and the next" and that these differences were "getting closer to zero" (as did Verónica). But then Consuelo also added that this difference could not get to zero, since *it could not happen that the bars at some point became the same size.* That last statement indicates Consuelo's awareness of what a "zero-difference" between bars would imply: she was conscious that each bar was a third of the previous one and therefore could not equal (since the bars were non-zero).

[8] The procedure DIF, computes the difference between two consecutive elements (:M and :M+1) in any list:

```
TO DIF :LIST :M
OP (ITEM :M :LIST) - (ITEM (:M + 1) :LIST)
END
```

To compute the difference between, for instance, the seventh and eight sequence segments (using a scale of 300) the students would type "PR DIF SEQSCA 10 300 7" where "SEQSCA 10 300" is the list of the first 10 terms of the sequence given with a scale of 300.

$F(n) = 1/3^n$	Scale = 300
between...	Difference
1 & 2	66.6666...
2 & 3	22.2222...
3 & 4	7.40707...
4 & 5	2.46...
5 & 6	0.8230...
6 & 7	0.274348...
7 & 8	0.09144...
8 & 9	0.03048...
9 & 10	0.010161
10 & 11	0.0033870...
15 & 16	0.0000139...
19 & 20	0.00000017207831...

Table 6: Table used by Verónica and Consuelo for recording the difference in the values of consecutive terms of the sequence $\{1/3^n\}$.

Consuelo: From one number to the next, it's like it gets closer to zero. And it gets closer to zero than the one for a half.
Ana: Faster you mean?
Consuelo: Yes.

Furthermore, as the above piece of transcript indicates, although Consuelo did not calculate the differences in the bars or values of the sequence $\{1/2^n\}$ she seems to have been able to give enough meaning to the differences in Table 6 and her previous experience with the two sequences — $\{1/2^n\}$ and $\{1/3^n\}$ — to realise that the faster decrease of the latter sequence also implied that the difference between its consecutive terms would also decrease faster than in the case of the former.

This entire episode is interesting in that it points to some of the ways in which the students investigated the *convergence* (and *rate* of convergence) of the sequence *within* the context of the situation, by observing how the values (segments) tend to become very close. Although these observations were not made explicit in a formal way, it is interesting that the approach which the students themselves constructed is similar to the Cauchy's definition of convergence of a sequence which is based on how small the difference between terms in a sequence can become.

3. Investigating the behaviour of series.

a. Making sense of the convergence of the series $\sum \frac{1}{2^n}$ *by using the decimal representation to explain the continuation of the process.*

In another part of the investigations, the students became involved in the study of "the sum of all the segments or bars". From the very beginning of this activity, Consuelo had suggested they should write a program for computing the sum of the "bars" (the procedure SUMM[9], which used as input a sequence list given by SEQUENCE or SEQSCA). Verónica had also suggested they should start by looking at the LINE procedure which adds up all the bars; Consuelo had agreed pointing out that they could then correlate the results of the SUMM procedure with their visual observations. Thus, through the DRAWS procedure, they began by exploring the LINE model representing the (partial) sum of the sequence $\{1/2^n\}$, with varying numbers of terms.

Although, in an earlier session, the students had already investigated this Line model (see page 334), their intuitions of expecting the line to grow more and more, resurfaced. At the beginning both students repeatedly maintained that, if the process was allowed to continue indefinitely, the line should continue growing past the edge of the screen. But as they began to observe the first visual line models, Consuelo emphasised that it would be a very slow process: "It is going to take a long time, it is going to take a very very very long time, because now it is doing very very little" which she justified through the first list of numeric values (and corresponding sum) she printed:

```
?SHOW SEQSCA 10 100
[50 25 12.5 6.25 3.125 1.5625 0.78125 0.390625 0.1953125 0.09765625]

?PR SUMM SEQSCA 10 100
99.90234375
```

From the visual outputs the students repeatedly observed how the turtle began vibrating in apparently the same spot; they explained those vibrations by remarking (as

---

9

```
TO SUMM :LIST
IF :LIST = [] [OP 0]
OP (FIRST :LIST) + (SUMM BF :LIST)
END
```

they had also done before) that although the turtle seemed to have stopped, in reality it was still walking imperceptible amounts.

Verónica: The turtle is blinking..., well, walking. It' s because what it is walking is very very small. It keeps walking there, not in the same place, although it looks as if is staying in the same place.

In order to investigate the growth of the sums, Consuelo suggested writing a procedure (PSUM[10]) for generating "a list of all the *partial sums*" (actually using the terms "partial sums" even though I had not introduced that terminology to them).

When they obtained the first list of partial sums (see Table 7), the students were surprised at how, after a certain point, all the sums had the same value of 100:

50 75 87.5 93.75 96.875 98.4375 99.21875 99.609375 99.8046875 99.90234375 99.95117188
99.97558594 99.98779297 99.99389648 99.99694824 99.99847412 99.99923706 99.99961853
99.99980927 99.99990463 99.99995232 99.99997616 99.99998808 99.99999404 99.99999702
99.99999851 99.99999926 99.99999963 99.99999981 99.99999991 99.99999995 99.99999998
99.99999999 99.99999999 100 100 100 100 100 100 100 100 100 100 100 100 100 100
100 100 100 100 100 100 100 100 100 100 100 100 100 100 100 100 100 100 100
100 100 100 100 100 100 100 100 100 100 100 100 100 100 100 100 100 100 100
100 100 100 100 100 100 100 100 100 100 100 100 100 100

Table 7: List of the first 100 partial sums of the values of the segments corresponding to the sequence $\{1/2^n\}$, with a scale of 100, obtained through the procedure PSUM.

Verónica: Something is happening, isn't? Because how can it go on with just 100?
Consuelo: It would have to be walking the same, wouldn't it? Zero.

Consuelo realised that having a constant value in the sums would imply that all the last segments added would be zero; so she looked at what the turtle was walking in the last segments through, at my suggestion, typing:

```
?PR FN 99 * 100
 1.577721810E-0028
```

Verónica: It's 28 zeros [after the decimal point] then 1. No, 27...
Consuelo: It's a lot of zeros. So it is not taking into account what comes after the point because it is a lot.
Verónica: It has too many zeros....

[10] The procedure for generating a list of the partial sums of a sequence (given as the input list) was:

```
TO PSUM :LIST
IF :LIST = [] [OP []]
OP SE (PSUM BL :LIST)(SUMM :LIST)
END
```

They also wrote a procedure ALL which produced the drawing and then printed the partial sums, with scale. (With INI being a simple procedure for blanking the screen and setting the turtle in the bottom left-hand corner):

```
TO ALL :N :SCALE
INI
DRAWS SEQUENCE :N :SCALE
PR PSUM SEQSCA :N :SCALE
END
```

Ana:	And do you still think it is going to keep growing if I leave it walking?
Verónica:	Yes, because the zeros are going to be increasing. The value keeps getting smaller but it is going to continue, because for instance here the number of zeros is going to keep increasing, but...
Consuelo:	...after the zeros there are going to be numbers
Ana:	So does it keep on going?
Consuelo:	It goes forward less...
Verónica:	If we use more terms, I think it is going to go off the screen.

When the students saw that the last segment walked had 27 zeros after the point in the decimal expansion, they realised the computer was taking that as zero and found an explanation for the "constancy" of the partial sums. However, at least Verónica still maintained that the line model (the sum of the terms) would eventually grow past the edge of the screen, not accepting the bound of 100: both students reasoned that the fact that "more zeros" could be added to the decimal representations of the segment values proved that the process continued but then Verónica erroneously inferred that this also meant that the line would eventually outgrow the bounds they had so far observed. They attempted modifying Logo's PRECISION to 20 (decimal digits) and noticed how the decimal values gradually filled with nines until the last few partial sums were given as 100.

Ana:	Do you still think it is going to go off the screen?
Verónica:	Yes, although it is going to take too long...
Ana:	And what about the values of the sums?
Consuelo:	Well, the nines are going to go on indefinitely.
Ana:	And, if I let it running an infinite time, what number would I get?
Verónica:	Well it would just keep having only nines, wouldn't it? It would be point nine, nine, nine, nine and it would go on like that. [She writes down ".99__________"]
Consuelo:	It will measure 99.9..., because they would be far too small bits... right?
Verónica:	Oh, yes... With 10000 terms it would be something like this [showing a segment of about 10 cm with her fingers], right? The turtle would only get to about here because the terms would be so small that it wouldn't walk much... The larger the number, the smaller the turtle steps will be.

Although the idea that the line should keep extending (beyond bounds) because more segments were being added, was still dominant in Verónica, a change started to take place during her conversation with Consuelo: she started to realise that because the added segments became very small, the sum would not grow much. Nevertheless, when she tried using 200 terms and increasing the scale to 200 (obtaining a similar visual result as before with the turtle vibrating at a length of about 200), Verónica explained the graphical results as well as the values of the partial sums (see Table 8), both of which indicated the convergence to 200 of the series, as rounding errors, as she expected that value to be surpassed.

```
100  150  175  187.5  193.75  196.875  198.4375  199.21875  199.609375  199.8046875  199.90234375
199.951171875  199.9755859375  199.98779296875  199.993896484375  199.9969482421875
199.99847412109375  199.999237060546875  199.9996185302734375  199.99980926513671875
199.99990463256835938  199.99995231628417969  199.99997615814208984  199.99998807907104492
199.99999403953552246  199.99999701976776123  199.99999850988388062  199.99999925494194031
199.99999962747097015  199.99999981373548508  199.99999990686774254  199.99999995343387127
199.99999997671693564  199.99999998835846782  199.99999999417923391  199.99999999708961695
199.99999999854480848  199.99999999927240424  199.99999999963620212  199.99999999981810106
199.99999999990905053  199.99999999995452527  199.99999999997726263  199.99999999998863132
199.99999999999431566  199.99999999999715783  199.99999999999857892  199.99999999999928946
199.99999999999964473  199.99999999999982236  199.99999999999991118  199.99999999999995559
199.9999999999999778  199.9999999999999889  199.99999999999999445  199.99999999999999722
199.99999999999999861  199.99999999999999931  199.99999999999999965  199.99999999999999983
199.99999999999999991  199.99999999999999996  199.99999999999999998  199.99999999999999999
200  200  200  200  200  200  200  200  200  200  200  200  200  200  200  200  200  200  200
200  200  200  200  200  200  200  200  200  200  200  200  200  200  200  200  200
```

Table 8: Output of the first partial sums of the values of the segments corresponding to the sequence {1/2^n} using a scale of 200.

Consuelo: It won't reach 200, by a few digits...

Verónica: It does pass it, but because there are too many numbers, then it rounds it; the computer cannot write down so many numbers...

Consuelo: I believe that no, it is not going to pass 200. I think it is always going to stay where it is, because it doesn't pass 200, so the nines are going to keep increasing: nines, nines, nines, and so on...

Verónica: Right, it is not going to pass 200.

Ana: So is it going to go off the screen at some point?

Verónica: It would reach the top of the screen and go off, if it didn't round so much...

Ana: So you think that if it did not round, it could go off the screen?

Consuelo: No, it cannot go off the screen... It would have to reach 200 in order to be able to go off the screen... pass 200, and to get to the top of the screen it would have to pass much more..

Verónica: So, it's impossible?

Verónica seemed to be focusing on the process as indefinite, as a process that goes on and on, and therefore felt that the line should keep growing (go off the screen); Consuelo on the other hand realised that the process could continue without necessarily passing the observed bounds, and she found a numerical explanation for this in terms of "you can always add more nines to the decimal expansion 199.9999... and therefore never reach 200". By using more terms (1000) and observing the same results as before, Consuelo confirmed her conclusion that "it is never going to walk beyond 200". Consuelo did add that the 1000 segment line, by having more segments, would be longer than the one with 200 terms, although she explained that the latter would be "smaller by tenths [i.e. decimal numbers]... because neither is ever going to reach 200". Then, when they compared the obtained line with a line of 200 steps[11], Consuelo observed that the two lines looked the same only because of the rounding, because in reality the one produced as the sum of the sequence "should be smaller" and could never reach nor pass the length of the other.

Summing up, as had happened in an earlier session the students had associated the infinite nature of the process with an expectation that the sum of the terms (segments) represented through the line model would show an extended growth. This was a dominant view, even when they perceived otherwise in the visual Line models. But the following events led to a change:

- first, the visual behaviour (with the turtle vibrating in the same spot) reminded the students that the sequence became very small as Consuelo confirmed by looking at the list of the values of the segments;

- later, the students were surprised when they observed that the partial sums eventually became a constant value.

- This led to an investigation of the values of the last segments and they observed how small those values were with tens of zeros after the point in the decimal expansion, which served as first explanation for the convergence of the sum, even though the existence of a bound or limit was not yet fully realised.

- Further analysis of the values of the partial sums finally led Consuelo to realise that there was indeed a value which would not be reached nor surpassed, and she was able to explain the on-going nature of the process through the numeric decimal representation where more digits can always be added.

b. Discovering a rule for the value of the "sum of the bars" for sequences of the type $\{1/k^n\}$.

The students continued their investigations of the "sums of the bars" by using different functions. Using $1/3^n$, they observed through the Line model that the sum

[11] They compared the Line model with a 200 step line, by jumping (with JUMP) and typing "FD 200" after the Line model was produced.

was less than in the previous case and seemed to approximately become half the scale, which they then confirmed by looking at the values of the partial sums (see Table 9):

Consuelo: It's walking less.
Verónica: It is going forward less.
Ana: What is less?
Consuelo: The sum of the bars.
Verónica: It's half the scale, right? Almost a half...

100	133.3333333333333333	144.4444444444444444	148.148148148148148 15
149.38271604938271605	149.79423868312757202	149.93141289437585734	149.97713763145861911
149.99237921048620637	149.99745973682873546	149.99915324560957849	149.99971774853652616
149.99990591617884205	149.99996863872628068	149.99998954624209356	149.99999651541403119
149.99999883847134373	149.99999961282378124	149.99999987094126041	149.99999995698042014
149.99999998566014005	149.99999999522004668	149.99999999840668223	149.99999999946889408
149.99999999982296469	149.99999999994098823	149.99999999998032941	149.99999999999344314
149.99999999999781438	149.99999999999927146		

Table 9: First 30 partial sums of {1/3^n} with a scale of 100.

Verónica then suggested exploring $1/8^n$ as the sequence generating function. When they started to discuss what type of drawing model they would like to use first, I reminded them that they could also use something different like a staircase. Liking that idea, they wrote a STAIRS procedure[12] (by modifying the SPIRAL), which they included in DRAWS. They tried several scales beginning with 100 which proved to be too small. With a scale of 300 they observed how the turtle drew one small step and it then seemed to be stuck vibrating in the same spot. They decided to look at the partial sums values (see Table 10) and observed how these values reflected the behaviour of the turtle in that they stayed in the same range from the second value (they would observe this quick convergence of the series with bigger scales as well):

37.5	42.1875	42.7734375	42.8466796875	42.8558349609375	42.8569793701171875
42.857122421264648438	42.857140302658081055	42.857142537832260132	42.857142817229032517		
42.857142852153629065	42.857142856519203633	42.857142857064900454	42.857142857133112557		
42.857142857141639070	42.857142857142704884	42.857142857142838111	42.857142857142854764		
42.857142857142856846	42.857142857142857106				

Table 10. First partial sums of {1/8n} with a scale of 300.

They gradually increased the scale (up to 800, with which they were able to see a small third step — see Figure 6), recording each corresponding value of the sum in Table 11. When Verónica noticed that the bigger the scale the bigger the value of the

12
```
TO STAIRS :L
FD :L / 2
RT 90
FD :L / 2
LT 90
WAIT 5
END
```

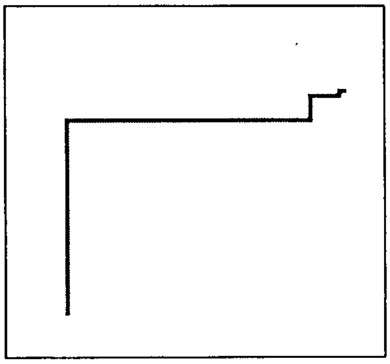


Figure 6. Stair model of the sequence {1/8ⁿ}.

sum, I suggested they explored the ratio between the scale and the sum. They computed this ratio to be 7 (using a calculator); Consuelo claimed it should be the same with any scale. By looking at the Line model they visually confirmed that the length seemed to be a seventh of the scale, something they numerically verified and recorded in the table. Consuelo then predicted that with a scale of 700 the total sum would be "about a 100", "a seventh of the scale", but she emphasised it would always be *less* than 100 and wrote 99.999... for the sum in the table, even though the computer had given as output 100 for all the partial sums after 22 terms. Both Consuelo and Verónica explained the sum of the bars got *close* to a seventh of the scale, but it would never pass it.

$f(n) = 1/8^n$			
Scale	Total Sum	Scale/Sum	[Number of] Terms
300	42.857142...	7.000	20
600	85.711285....	7.00000	20
800	114.285714...	7	20
800	114.285714...	7	100
700	99.9999...		100

Table 11. Verónica and Consuelo's table recording the partial sums of {1/8ⁿ} relative to the scale.

The students also decided to look at other visual models for this sequence such as the Spiral , and the Bars models (see Figure 7), where they noticed again the very fast decrease of the terms of this sequence (which they confirmed by printing a list of values of the sequence). Verónica observed how much faster this sequence became small as compared to the other cases explored.

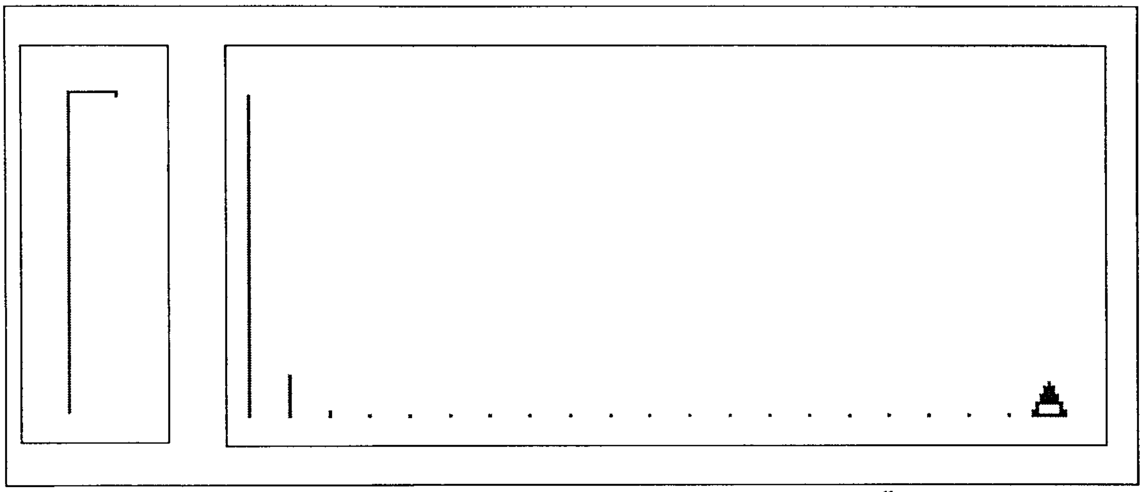


Figure 7: Spiral and Bargraph models of the sequence {1/8^n}.

Consuelo then suggested exploring the function $1/7^n$ and predicted that the sum of the bars was "going to get close to a sixth". Verónica added they should use a scale of 600 to make it is easier to see if the sum was in fact a sixth of the scale. Through both the visual and the numeric values of the partial sums the students then confirmed that the sum in fact tended to a sixth (i.e. to 100 with the chosen scale).

Thus, by looking at several sequences of the same type — $\{1/k^n\}$ — varying the dividing factor, the students:

- *empirically* discovered a rule for the value of "the sum of the bars" of sequences of that type;

- then used that knowledge to predict the behaviour for the series of another sequence of the same type and even picked a scale which would be suitable for the observation of the result.

- When their prediction proved correct the students felt confident that they could generalise the rule. Consuelo expressed this rule in terms of the scale; she remarked: "So for $1/3^{[n]}$ it's 1/2, and for $1/4^{[n]}$ it would be 1/3... and for a fifth 1/4, and so on". In this way the students were able to construct a generalisation which was made within the context of the activity and relative to the inputs used (e.g. the scale).

4. Exploring other types of functions.

a. The students encounter a divergent sequence.

Consuelo had suggested trying a function "that does not divide, that for instance multiplies". The first function they explored was[13]: $2 * 5^N$, beginning with the STAIRS model. The first problem they encountered was in keeping the model for this sequence within the boundaries of the screen, forcing them to finally use very small scale of 1 and only 3 terms (see Figure 8). They soon realised from both the visual output (where, with a scale of 8, the second step filled the screen) and the numeric output of the partial sums[14] that in this case there was no limit value for the total length; Verónica spontaneously said: "This one doesn't have a limit, right?" — interestingly using the term "limit", which I had not used with these students.

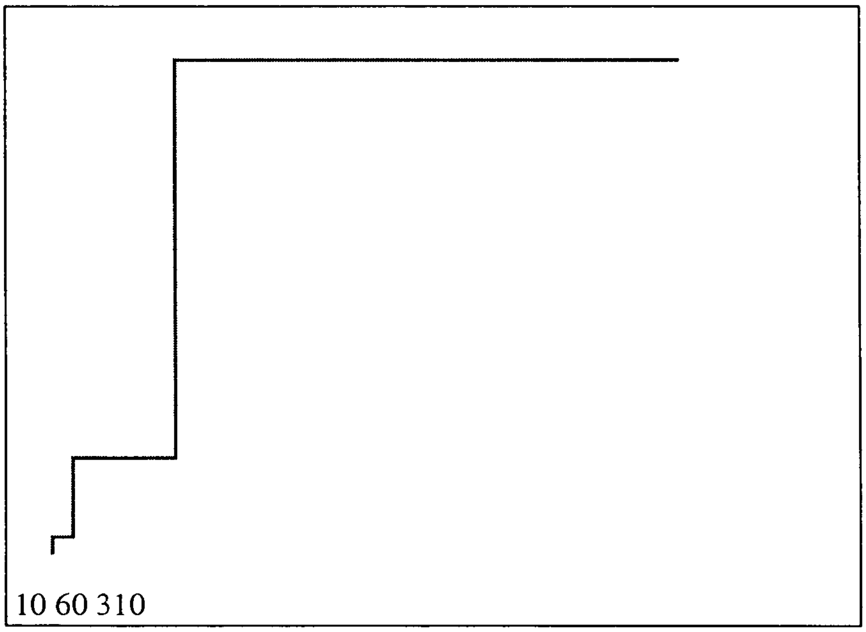


Figure 8: Stairs model of the first 3 terms of the sequence generated by $2 * 5^n$, using a scale of 1, and with the output of the partial sums.

[13] In FN they wrote (2 * POWER 5 :N).

[14] For instance, with a scale of 8, the first three partial sums were "80 480 2480".

The exploration of the Bars model then illustrated how large the individual segments became as most of the bars went off the screen, and Verónica had to suggest using "0.00-something" as the input for the scale (they tried with 0.009; see Figure 9). In this way the students also concluded that this sequence (as well as its series) did not have a limit with Consuelo pointing out: "Here it's like it doesn't approach anything, it just goes off... [while making a gesture raising her arm up]".

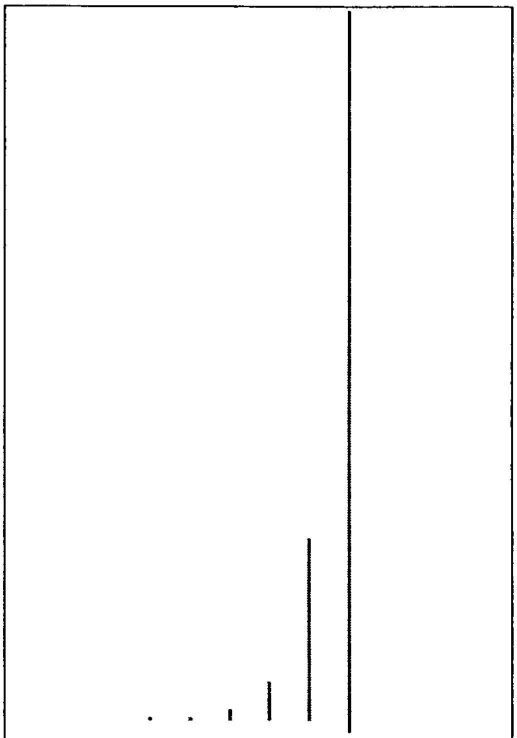


*Figure 9: Bars model of the sequence generated by 2 * 5^n, using 0.009 as scale, with the partial sums values.*

This was an experience which helped the students discover a different type of behaviour (that of a divergent sequence), and through the visual (and numeric) models led them to conclude that both the total length (the series) and the sequence did not have limits.

b. Explorations of the Harmonic sequence and series.

<u>(i). Observing a different behaviour in the visual models.</u>

Verónica had suggested trying "4 over N..." and changed the function to 4 / :N. They began by looking at STAIRS model (see Figure 10) and, from this visual output, the students quickly realised that in this case the behaviour was different than anything they had encountered so far, particularly because although the sequence decreased, the steps in the model never became small enough for the turtle to "get stuck":

Consuelo:	The stair is different. It has too many [steps]....
Verónica:	In this one we are able to see the smaller steps; in the other ones we couldn't, we couldn't see the steps being formed, although the turtle was making the steps...
Consuelo:	...in the same place.
Verónica:	It's decreasing but [the stair] is very long. It is in contrast with the other ones where we used larger numbers [of terms] but the turtle stayed in the same place doing steps that could not be seen.

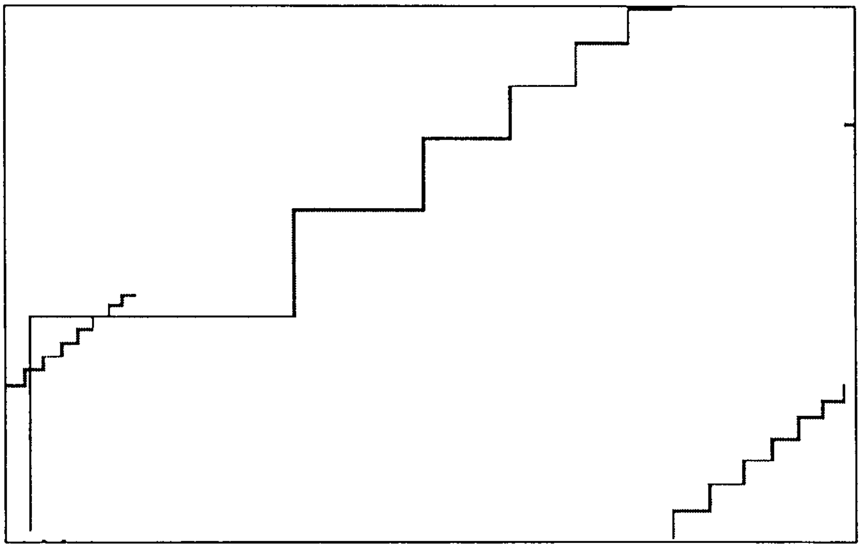


Figure 10:. Stairs model of the sequence {4/n}.

When the students looked at the SPIRAL model with a scale of 100 they were surprised by how big it was (see Figure 11). I took advantage of their comments to point out that using 4/N was like using 1/N with four times the scale; they thus decided to change the function to 1/:N.

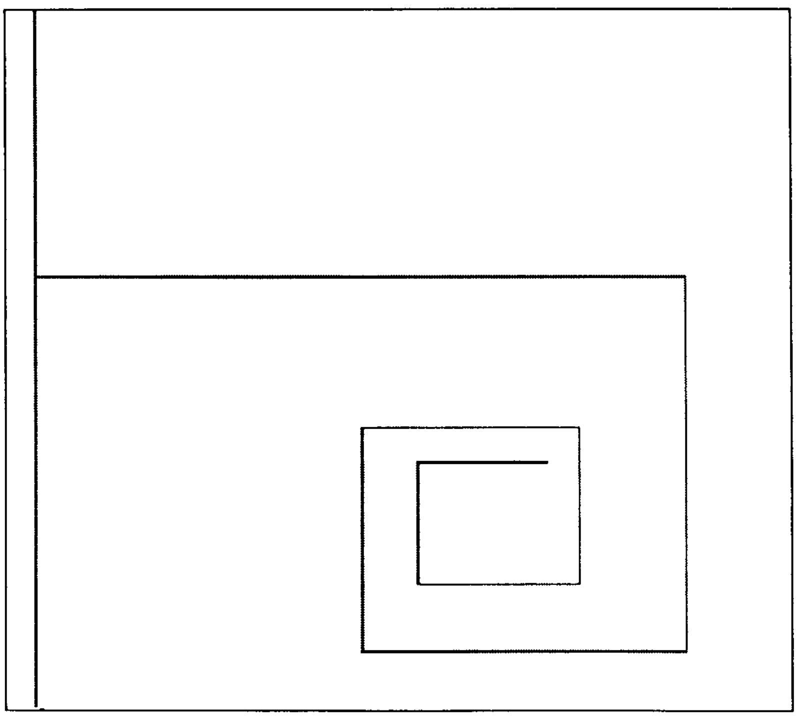


Figure 11: Spiral model of the first 10 terms of the sequence {4/n} (the first segment wrapping around the screen).

Consuelo reflected on why the models for this sequence looked different, and started to consider that maybe in this case the total length did not have a limit (something she would start to repeat more and more as the exploration of the harmonic sequence and series continued). *Spontaneously,* she asked:

Consuelo:	Could it be because the other one had a limit?
Ana:	What do you mean?
Consuelo:	That is... when we used a certain scale, it was never going to reach 200. The nines kept increasing... and in this one they don't.
Ana:	What makes you think that?
Consuelo:	Well, because... the stairs... because it keeps going on, and only stops when it gets to where it has to get.
Ana:	Where do you think it has to get to?
Consuelo:	Well..., that is... I don't know... The other one didn't get to 200, it had a bit left, it walked in the same place and didn't reach 200. And the screen didn't get all filled up, in contrast with this one where the screen can get filled up.
Ana:	And that's why you think that this one doesn't have a limit?
Consuelo:	It could be.

When they generated new Spirals gradually increasing the number of terms (segments), and the scale (up to 1000), they remarked how different the behaviour in this case was as compared with what they had seen before. Verónica noticed how the new segments at the inside of the spiral seemed to be "pushed back" allowing for all the segments drawn to be visible, with the central "square" never filling up. Verónica explained that she thought the turtle would "not be able to reach the centre of the square" because the space between the lines of the spiral progressively became thinner (see Figure 12), so the turtle would tend to stay towards the outside. Consuelo also noticed there was a slower rate of decrease of this sequence as compared to other cases, observing that even with large numbers of terms the length of the last segment was still relatively "big".

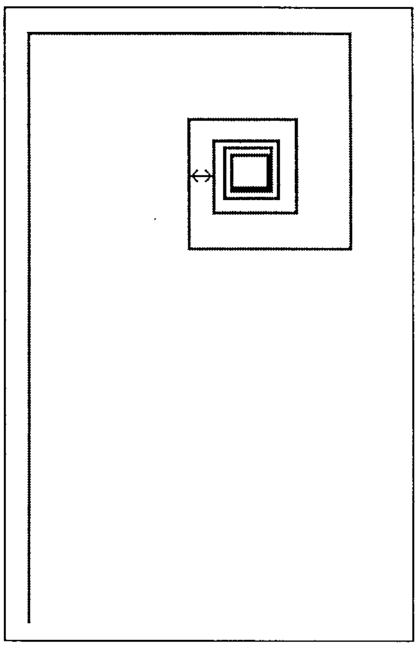


Figure 12:. Spiral model for the sequence {1/n}.

Verónica:	It's becoming smaller... as if it were going to the back. It looks as if it were sinking towards the inside.
Consuelo:	This one doesn't decrease as fast. The other ones decreased so fast that it then became so small we could not see how much it walked.
Verónica:	Yes... in the other ones it went too fast, and in this one we see how it creates the squares; and in the other ones we just saw the turtle standing doing lots of spirals but in the same place; and in this one we do see how it makes the spiral.

....

Verónica:	It's turning and doing all the terms we asked it to do, and we can see them, but in the other one we couldn't see, in the other spiral we couldn't see it making all of the ones we had asked it to do.
Consuelo:	The turtle is still walking something big.
Verónica:	Yes.

Verónica had then suggested creating a table (Table 14, further below, which they only filled out for a couple of values here, but would return to later) with "the terms [number of terms], and what all the terms add up to", since she suspected that the sums would be very large. I should point out that when the students first began the numeric explorations of this sequence, I had suggested they used values independent from the scale (e.g. by using the procedure SEQUENCE, instead of SEQSCA)[15]. Consuelo had appreciated this suggestion remarking: "Then every time we run the procedure we will get the same values, no matter what scale we use".

(ii). Observing the very slow decrease of the sequence.

Both students had then suggested looking at the Bars model of this sequence in order to look at the behaviour of the segments (see Figure 13), adding a PRINT command to BARS[16] in order to see the length of each segment as it was drawn (although they would later decide to create a table (Table 12) recording the values of the sequence each of bars corresponded to). The observations of this model helped the students confirm that although the segments did decrease, this decrease was very slow; Consuelo remarked: "at the beginning it decreases very much but at the end it decreases less and less each time". Verónica added that unlike the other cases studied, the segments would not become dots, that the graph would "fill up with bars" — as she would later confirm when they increased the number of terms (narrowing the distance between bars; see Figure 14):

[15] With this change the students also chose to write a procedure PRSUM (PRint SUMs) which printed the partial sums (independently from the scale):

```
TO PRSUM :N
PR PSUM SEQUENCE :N
END
```

[16]

```
TO BARS :L
FD :L  PR :L
JUMP
END
```

Verónica:	The bars decrease, but in this one we are able to see them, because in the other ones after about the fifth one all we could see were dots.
Ana:	And that won't happen here?
Verónica:	No. It's going to fill up with bars.
Consuelo:	It takes longer to make them smaller.

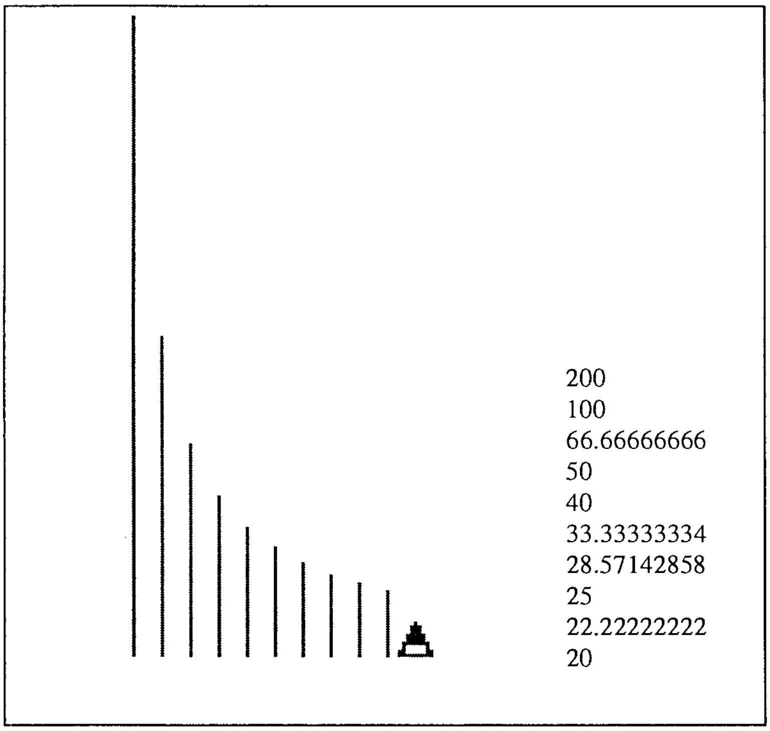


Figure 13: Bargraph model of the first 10 terms of the sequence {1/n}, with the respective values of each bar, using a scale of 200.

Scale=200		BARS model
TERM	SIZE	REAL SIZE
1	200	1
2	100	0.5 = 1/2
3	66.66...	0.33...
4	50	0.25
5	40	0.2
6	33.33...	0.6667...
7	28.57142858	0.428...
8	25	0.125
9	22.222....	0.111...
10	20	0.1
50	4	0.02
100	2	0.01
200	1	0.005

Table 12: Table created by the students for relating the size of the segments drawn (with scale) with the independent values of the terms of the sequence.

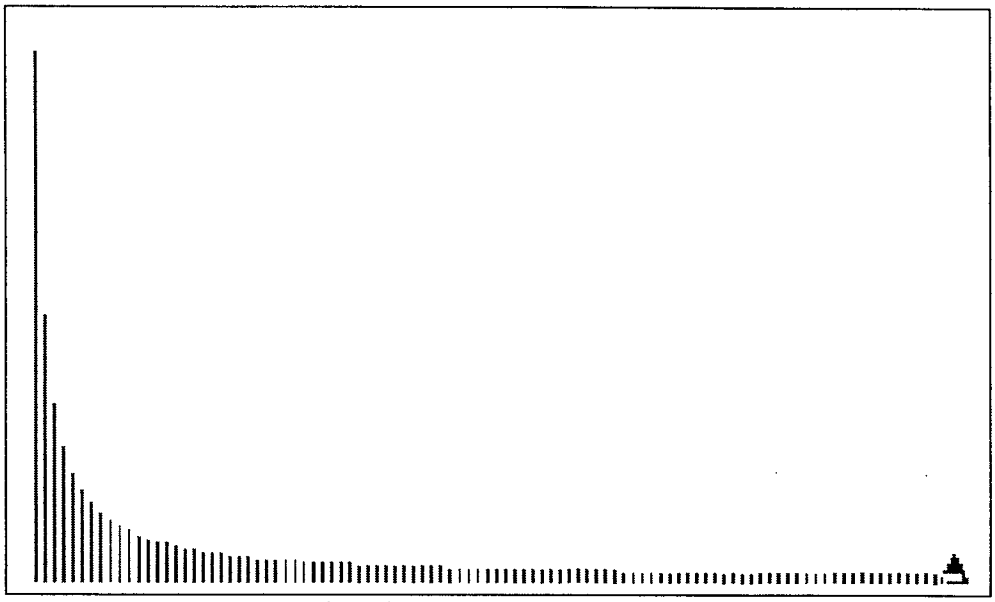


Figure14: Bargraph model of the first 100 terms of the sequence {1/n}.

Both students then pointed out that there were a lot of "bars" in the 2 range. Consuelo then predicted that: "if for 2 it did so many, with 1 it's going to be much more" and explained that "for every number [range] it extends more and more". With 200 terms (see Table 13) they confirmed that the last values were "all the same size, of 1 and a bit" When I asked the students if they thought that at some point these values would became "almost zero or zero", both students denied this possibility even with an extremely large number of terms, with Consuelo adding: "Each time there are more terms of one value, there are more bars [of the same size]"

00	4.87804878	2.469135802	1.652892562	1.242236025
100	4.761904762	2.43902439	1.639344262	1.234567901
66.66666666	4.65116279	2.409638554	1.62601626	1.226993865
50	4.545454546	2.38095238	1.612903226	1.219512195
40	4.444444444	2.352941176	1.6	1.212121212
33.33333334	4.347826086	2.325581396	1.587301587	1.204819277
28.57142858	4.255319148	2.298850574	1.57480315	1.19760479
25	4.166666666	2.272727272	1.5625	1.19047619
22.22222222	4.081632654	2.247191012	1.550387597	1.183431953
20	4	2.222222222	1.538461538	1.176470588
18.18181818	3.921568628	2.197802198	1.526717557	1.169590643
16.66666667	3.846153846	2.173913044	1.515151515	1.162790698
15.38461538	3.773584906	2.150537634	1.503759398	1.156069364
14.28571429	3.703703704	2.127659574	1.492537313	1.149425287
13.33333333	3.636363636	2.105263158	1.481481481	1.142857143
12.5	3.571428572	2.083333334	1.470588235	1.136363636
11.76470588	3.50877193	2.06185567	1.459854015	1.129943503
11.11111111	3.448275862	2.040816326	1.449275362	1.123595506
10.52631579	3.389830508	2.02020202	1.438848921	1.117318436
10	3.333333334	2	1.428571429	1.111111111
9.523809524	3.278688524	1.98019802	1.418439716	1.104972376
9.09090909	3.225806452	1.960784314	1.408450704	1.098901099
8.695652174	3.174603174	1.941747573	1.398601399	1.092896175
8.333333334	3.125	1.923076923	1.388888889	1.086956522
8	3.076923076	1.904761905	1.379310345	1.081081081
7.692307692	3.03030303	1.886792453	1.369863014	1.075268817
7.407407408	2.985074626	1.869158879	1.360544218	1.069518717
7.142857142	2.94117647	1.851851852	1.351351351	1.063829787
6.896551724	2.898550724	1.834862385	1.342281879	1.058201058
6.666666666	2.857142858	1.818181818	1.333333333	1.052631579
6.451612904	2.816901408	1.801801802	1.324503311	1.047120419
6.25	2.777777778	1.785714286	1.315789474	1.041666667
6.06060606	2.739726028	1.769911504	1.307189542	1.03626943
5.882352942	2.702702702	1.754385965	1.298701299	1.030927835
5.714285714	2.666666666	1.739130435	1.290322581	1.025641026
5.555555556	2.631578948	1.724137931	1.282051282	1.020408163
5.405405406	2.597402598	1.709401709	1.27388535	1.015228426
5.263157894	2.564102564	1.694915254	1.265822785	1.01010101
5.128205128	2.53164557	1.680672269	1.257861635	1.005025126
5	2.5	1.666666667	1.25	1

Table 13: Output values of the first 200 bar segments with a scale of 200 (printed simultaneously to the corresponding bar segments by the BARS procedure).

<u>(ii). Confirming the divergence of the "sum of the bars" (the Harmonic series).</u>

At this point the students decided to return to the study of the sum of the bars with both students suggesting they looked at the LINE model which Verónica reminded represented the sum. In their first attempt (with 100 terms and a scale of 200) they observed how the turtle drew a line that wrapped itself around the screen several times, with the turtle always continuing its forward progression.

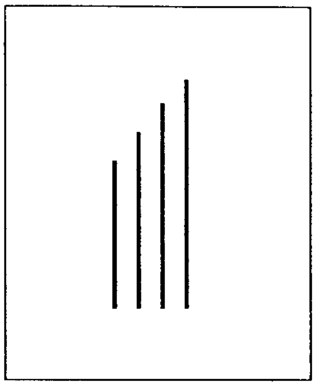


Figure15: Comparison of, respectively, the Line models of 25, 50, 100 and 200 terms of the harmonic sequence.

The students decided to compare lines of sums (see Figure 15) with different numbers of terms[17] (using a constant scale of 10), with Consuelo suggesting they looked at the partial sums in each case, recording each of the final values in Table 14. They noticed how each line was significantly longer than the previous ones and Consuelo remarked the contrast with the other cases studied where after a certain point the Line no longer grew. Verónica then predicted that with more terms the line would still extend. When they looked at the numeric values of the sums, Consuelo began to suspect that the sums would not have a limit:

Consuelo: It seems that this one doesn't have a limit, because it keeps on going; in the other one it would suddenly round to 100, 100, 100..., but this one keeps on going. In the other one there was an increase in the nines [in the decimal digits]... but in this one there isn't: it starts with 3, then 4, then 5...

f=1/n	
[Number of] Terms	Sum
25	3.815958178
50	4.499205338
100	5.187377519
120	5.36886829
200	5.87803095

Table 14: Table created by the students to record the values of the partial sums of {1/n}.

Thus, from both the visual experiences and their observation of the values of the partial sums, both students concluded that there would not be a point where the sums

[17] This was done by using the JUMP procedure between the Line models drawn, instead of clearing up the screen.

would stop increasing like they had seen in other cases. Verónica explained: "It's going to go on. The more terms we use the more the sums will increase".

(iii). Summary and comments

First of all, it is interesting that the students themselves suggested the exploration of using 4/N as the sequence generating function. By this time the students had developed enough familiarity with the tools of the microworld to suggest their own explorations. Also, that same familiarity (in particular the fact that in the DRAWS procedure, the scale multiplied the values of the sequence) allowed to easily accept the change to 1/N.

Then following events are relevant in the above exploration:

• Through the visual models the students immediately realised that this sequence behaved differently than other cases they had seen; they observed that

- the Stairs model seemed to extend without bounds;

- in the Spiral model the inner segments were "pushed" towards the outside, with the inner "square" remaining empty, which illustrated that the segments were not becoming too small.

• Those *visual behaviours* gave enough evidence for Consuelo to propose that the sums would not have a limit. She was able to use and interpret the visual models to form hypotheses on the behaviour of the sequence and its series.

• The experience with those models led the students to want to investigate two things:

- how the sum of the segments grew, using in particular the numerical values; and

- the behaviour of the segments as the sequence progressed through both a bar graph and an analysis of the numerical values (with and without scale) of the terms.

• For the latter, in looking at the bars and then at the list of all the values, the students discovered that the slow decrease of this sequence was reflected in how many terms there were *within a same range of values.* They implicitly used this as a *situated* criteria for determining how slow the sequence decreased.

• For the exploration of the series, in the comparison of lines with different number of terms, the students constructed a method for visually observing how much the sums increased, which they then complemented with a table of values. These explorations complemented their previous observation of the Stairs and Line models which seemed to extend without bounds and helped the students conclude that the sums did not seem to have a limit.

c. Looking at $\left\{\frac{1}{n^2}\right\}$ *and* $\left\{\frac{1}{n^{1.2}}\right\}$, *and developing a generalisation for the existence of a limit of the series of* $\left\{\frac{1}{n^k}\right\}$.

i. Exploring the behaviour of $\left\{\frac{1}{n^2}\right\}$ and its corresponding series.

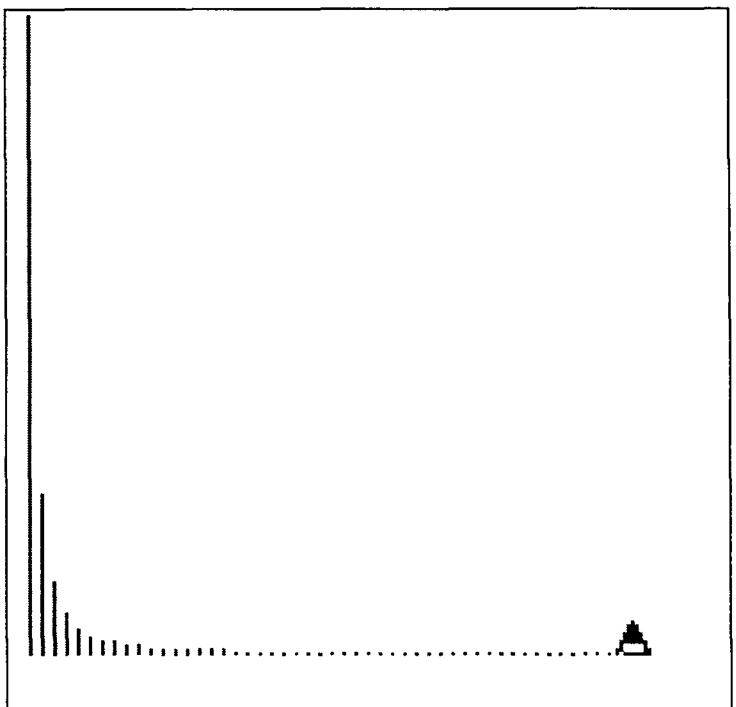


Figure 16. Bar model of the first 50 terms of the sequence {1/n²}.

The exploration which followed, that of $\left\{\frac{1}{n^2}\right\}$, was similar to previous explorations, although both students would remark that the behaviour of this sequence and its sums, which they discovered had a limit, had been a surprise. The students began by looking at the Bars model (see Figure 16) and were surprised at how fast the bars decreased (which clearly became much smaller and closer to zero than in the case of { 1/n}); this observation was complemented with a numerical analysis (see Table 15) using the output of values of the segments and by printing lists of values of the sequence[18]. The students soon concluded that this was a sequence which, in Verónica's words, "wants to get close to zero", although as in previous occasions they emphasised that it would not actually reach zero because the zeros increased after the decimal point but had "numbers afterwards"

f = 1/POWER :N 2		Scale: 200
Terms	Last bar	Size without scale
50	0.08	0.0004
100	0.02	0.0001
200	0.005	0.000025

Table 15. Table used for recording some of the values of the sequence {1/n²}.

The students then explained they could infer from their observations of how very small the bars became what the behaviour of the other graphic models would be: the Stairs (Figure 17) — "it will do a few steps and then will stay in the same place" — and Spiral (Figure 18) — "it will go to the centre fast" with "the centre square becoming very small".

[18] E.g. by typing "PR SEQUENCE 100".

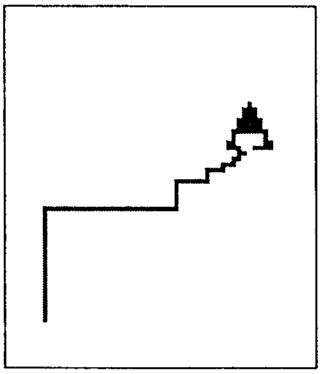


Figure 17. Stairs model corresponding to the sequence {1/n²}.

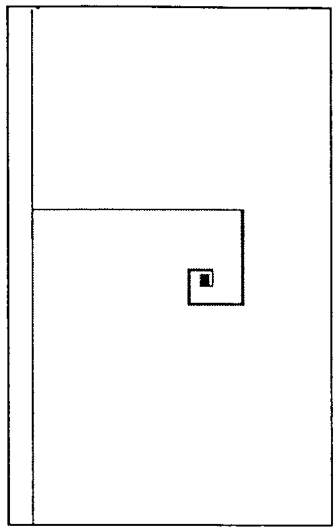


Figure 18. Spiral model corresponding to the sequence {1/n²}.

At their own initiative the students continued with the exploration of the sum of the segments using the Line model and a table of values (Table 16). They compared Line models with different numbers of terms, observing that the length was almost constant which led them to presuppose the existence of a limit; Consuelo pointed out: "it seems this one has a limit which it will not surpass" and predicted it would be less than 2. The "line" explorations were simultaneously complemented with numerical explorations which included printing and analysing lists of partial sums. Through these,

- first, they observed how the increase was less each time, with more sums staying in the same number range (for instance, Consuelo pointed out that the partial sums "stay longer in the 1.58 range that in the 1.57 one.");
- then they were able to conclude that the sums definitely had a limit, probably less than 1.65 which they considered "very difficult" to reach because the segments became very very small.

f = 1/POWER :N 2	
Terms	Sum
25	1.605723404
50	1.625232734
100	1.6349839

Table 16. Table created by the students recording some partial sums of {1/n²}.

ii. The sequence $\left\{\frac{1}{n^{1.2}}\right\}$ and a generalisation for the behaviour of the sums of other sequences of the same type.

In the activity that followed the students made an interesting intuitive prediction: I had asked the students what they though would happen if in the (sequence generating) function (of the type $\frac{1}{N^k}$), the power of :N was something between 1 and 2, and both students immediately said that in that case the turtle would "get stuck" and that "it would have a limit". Consuelo explained she though only the sums of {1/N} did not have a limit, arguing that if N was raised to a power, no matter how small, the sums *would* have a limit.

They decided to test this prediction with an investigation of $\left\{\frac{1}{n^{1.2}}\right\}$. They looked at the Bars model which behaved as Consuelo had predicted, with the bars decreasing slower than for $\left\{\frac{1}{n^2}\right\}$, but faster than for $\left\{\frac{1}{n}\right\}$. They then explored the LINE model, which in the first attempt was almost twice as long as that depicting $\sum\frac{1}{n^2}$, but Consuelo still maintained these sums *would* have a limit although it would be a "larger limit than the one for [one over] N squared". Further explorations, however, temporarily led to some doubts of the existence of a limit as the Line kept extending as they increased the number of terms, until they observed that with even more terms the Lines increased less and less and then Verónica remarked: "Yes, maybe it does get stuck.... Yes, because it's very little what its walking now". An analysis of the numerical values followed, where they noticed that (from the 27th sum) the values very soon stayed in the 3+ range with the 300th partial sum being 3.994228858. At this point the students were pretty much convinced of the existence of a limit which they thought would probably be less than 4 (they presumed that it was likely that more nines would be added to the decimal expansion). The prediction of a limit led Verónica to predict that in the Spiral model (Figure 19) they would see the central square getting "filled", and she was happy to see it was so. On the other hand, the students did *not* expect the behaviour they observed in the Stairs model (Figure 20) which seemed to keep extending. This last event made them

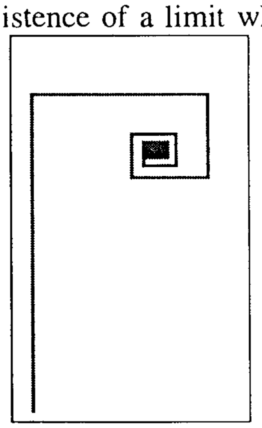


Figure 19. Spiral model for the sequence {1/n$^{1.2}$}.

question whether the limit would indeed be less than 4, with Consuelo pointing out that the sum "with a very big number of terms, may reach 4". Nevertheless, because of their observations with the other models and the numerical values, the students assumed that the stairs would eventually "get stuck", convinced that the sums would have a limit.

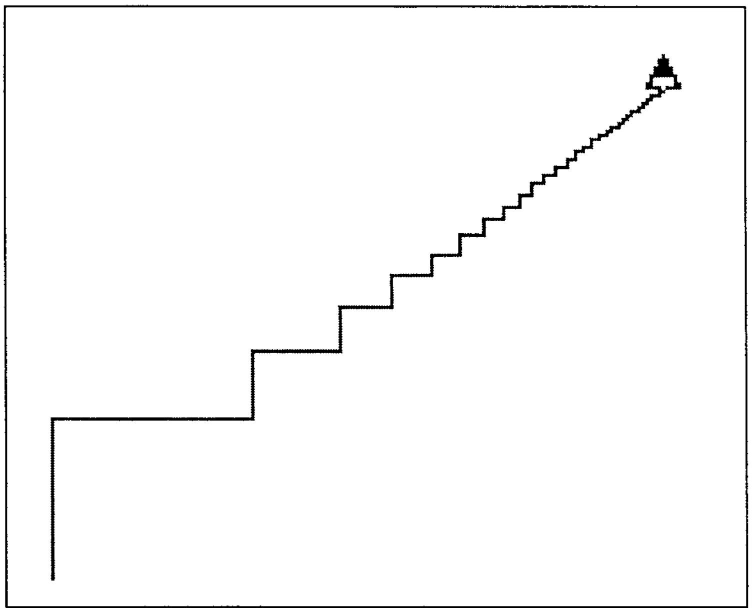


Figure 20. Stairs model for the sequence $\{1/n^{1.2}\}$.

Furthermore, this experience convinced the students that they were right in their prediction that when using (sequence generating) functions of the type $\frac{1}{N^k}$, with a power of N larger than 1, the corresponding series *would* have a limit.

As this last activity showed, by the end of the sequence explorations the students knew how to employ all the tools provided in the microworld with a good understanding of what each of the procedures did and could be used for. They had developed methods of exploration and criteria — relative to the visual models, and the other tools provided by the microworld (e.g. the possibility to look at the lists of values of the sequence and its partial sums) — for determining the eventual behaviour of both the sequence and the series, and the existence (or lack of) limits or bounds. It is clear that the students learned to combine all the elements they looked at and used (both graphic and numeric), going back and forth between them. In particular they were able to make use of the behaviour of the sequence, and to coordinate it with the behaviour of the sums: for instance, they were able to link the behaviour observed in the Bars model with that of the Line or Stairs models. Among some of the elements which they used and relied upon as indicators of the existence of a limit for the series were:

- the behaviour at the centre of the Spiral model;
- the behaviour of the sequence, particularly the rate at which it decreased (or increased), which was clearly seen in the Bars model;
- comparisons of Line models with varying numbers of terms;
- the numerical analysis, looking in particular at how many terms (or sums) were within a certain range of values which indicated, for instance, the rate of decrease of a sequence, or how the sums slowed down as they approached the limit.

Although the graphical models were very valuable, as the students themselves commented:

Consuelo: With figures it is much nicer, and we can realise various things. For instance, the limit of the numbers... just like that it is more difficult, but with the bars and the lines it is easier to see.

Verónica: And we can understand better with the drawings. With drawings and numbers, that is, with both things.

the students also made extensive use of the numerical approach, and in fact it was through the numerical decimal representations that they were able to explain how the processes could continue even if they had a limit.

Part B. Fractal studies.

1. The Koch curve

a. Writing the procedure for the Koch curve: making sense of the self-similarity of the figure and linking it to the recursive characteristic of the procedure.

In this section I want to briefly describe the way in which the students constructed the procedure for the Koch curve in order to illustrate some of the insights which occurred during this process. I had begun the activity by showing the students the first three levels of the Koch curve on a blackboard and explaining how each new level is derived from the previous one. The students reflected on how they could program this picture, and began by writing the procedure shown at left of Table 17. Verónica later suggested replacing each of the "FD lines" with a recursive call because in the figure each segment was to be replaced with a "PEAK". When the modified procedure did not produce anything, I explained the need for a stop condition with a Forward (FD) command, so they added "IF :S = 0 [FD :S STOP]" at the beginning of the procedure. When they ran the procedure and nothing happened (the computer ran out of memory), the students realised the condition would never become true because the length of the side never reached zero: .

> Verónica: Oh, we need to use "less than"... [because for it to reach zero] it would take an infinite amount of time.

so they changed the condition to "IF :S < 10" and produced Figure 21.

```
                        TO PEAK :S
TO PEAK :S               <— IF :S = 0  [FD :S STOP]    <—>IF :S < 10 [FD :S  STOP]
FD :S / 3    <—>        PEAK :S/3
LT 45                   LT 45
FD :S / 3    <—>        PEAK :S/3
RT 90                   RT 90
FD :S / 3    <—>        PEAK :S/3
LT 45                   LT 45
FD :S / 3    <—>        PEAK :S/3
END                     END
```

Table 17. Construction of the Koch curve procedure.

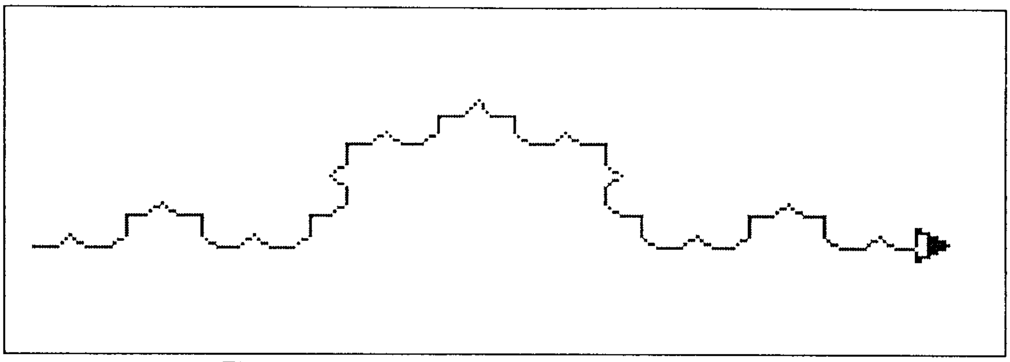


Figure 21. Koch curve produced by typing PEAK 200

At first Verónica was surprised that there were little PEAK everywhere as she seemed to be expecting only a level 2 curve, but Consuelo assured her that the figure was correct since "it is supposed to be a big one, with smaller ones here, and smaller ones here...". Then, when they ran the procedure with many different scales they noticed that the one with a scale of 50 was exactly like the first fourth of the one of scale 200; Consuelo pointed out that this should not be a surprise since *in the procedure* "everything is by a third, a third, a third" so they should be alike. She would later express the self-similar characteristic of the figure by remarking that each part looked like the whole figure, "as if using scales". (Later in the session Verónica would also comment on the self similarity, saying: "It is as if this part here was the whole big figure.") The students also experimented changing the value in the stop condition (from 50 to 1) with Consuelo pointing out (and then testing and confirming) that *the smaller the value in the conditional, the smaller the segments would be*; she thus verbally expressed the meaning and effect of the condition, relating the value :S in the condition with the size of the segments forming the curve.

Thus, by constructing the curve procedure themselves, the students were able to make some sense of how it worked, rather than it being some "mysterious" procedure:

- When Verónica herself suggested the use of the procedure inside itself (recursive calls) she was expressing the idea that each part of the figure contained a similar figure to the whole. However, it is clear that at the first she had not grasped the full import of the recursive / self-similar characteristic, as she was surprised by the intricacy of the output. It was Consuelo who made sense of the results by going back and forth between the code and the resulting figure, becoming aware of the deep self-similarity of the figure (of each part containing a smaller part), though the recursive characteristic of the procedure, and linking it to their observation of how the smaller scaled figure not only represented the entire figure but was also a part of the larger figure (even though she did not yet connect the structure of the code with the ratio between the two figures).

- The addition of a stop condition which did not become true, led Verónica to realise (as in the sequence studies) that the construction of the curve involved an infinite process, and that although the value :S (representing the length of the segments) became smaller and smaller, it could not become zero in a finite time.

- Consuelo's understanding of the potentially infinite depth of the figure and of its recursive structure, was further made evident in a later conversation (after the procedure had been modified to include a "level" variable — see further below):

Ana	If I didn't have an IF and I still were able to do the figure, how many levels would I have?
Consuelo:	Infinite.
Ana:	And what would it look like?
Consuelo:	It would look the same... but it wouldn't be the exact same: it would have more little peaks.

b. Explorations into the perimeter of the Koch curve

The description of the investigation of the perimeter of the Koch curve serves to illustrate how the students discovered relationships, and expressed and generalised them through the microworld, as well as the way in which they uncovered and explained the behaviour of each of the elements involved.

When faced with the question of how much the turtle walked when drawing the Koch curve, the students had first expressed it as follows:

Verónica:	A third of :L, plus a third, plus a third....
Consuelo:	Or rather, a third, plus a third of a third, plus a third of a third of a third...

The students were then interested in carrying out a methodical analysis of the exact measure of that perimeter but they realised the difficulties in determining how much the turtle had walked when they did not know the level of the figure or how many segments had been drawn; at Verónica's initiative the procedure was therefore modified to include a variable :L for the level:

```
TO PEAK :S :L
IF :L = 1 [FD :S STOP]
PEAK :S / 3  :L - 1
LT 60
PEAK :S / 3  :L - 1
RT 120
PEAK :S / 3  :L - 1
LT 60
PEAK :S / 3  :L - 1
END
```

Consuelo recognised the process of "taking thirds" and concluded that the length of the segments was given by $\frac{1}{3^{:L}}$ which she remembered became very close (though not equal) to zero (as :L increased):

Consuelo: They will be a third of each, so it is 1 over 3 to the Lth power.
Ana: And what would happen if the level, that is L, is very big?
Consuelo: It's going to be very small, it's going to reach a limit, zero... No, it will not reach zero. It will be 0.000...9 or 0.0000...1...

This observation led her to conclude that for the total length of the curve "there will come a point in which it will be almost the same" which she explained would be when "the little segments reach the limit of not going past zero". She added:

Consuelo: It is going to grow little by little until it reaches a point where it keeps growing but so little, that although it [the growth] is not zero, it [the length] will be the same.

The students then carried out a detailed investigation of what happened to the length, through the use of Table 18, and by running the procedure repeatedly through different levels. Through the visual figures the students were able to observe the numbers of segments in each case and noticed how each segment was being replaced by four new segments (which eventually led to the conclusion that the number of segments was a power of 4). Then, by working through the table the students gradually constructed generalisations for the number of segments (4^{n-1}) and the size of those segments ($L/3^{n-1}$) in function of the level n. At first they computed the lengths (perimeters) of the curves using the information in the table for each level; for instance, for computing the length of level 4 they used:

```
?PR 64 * 100 / POWER 3 2
711.1111111
```

Level	Scale = 300 Length	Number of segments	Size [of the segments]
1	300	1	300
2	400	4	100 = 300/3
3	533.333...	16 = 4 X 4	100/3 =
4	711.1111...	$16 X 4 = 4^3 = 64$	$100/3^2 = 300/3^3$
5	948.1481472		
6	1264.197529	1024	
n	$4^{n-1} X 300/3^{n-1}$ $= (4/3)^{n-1} X L$	4^{n-1}	$300/3^{n-1} = L/3^{n-1}$
8	2247.462273	16384	

Table 18. Table used in the investigations of the perimeter of the Koch curve.

But as they progressed through the generalisations they found a formula for the total length which they translated into the procedure below (which, after they tested was accurate, used for computing the length at higher levels):

```
TO LENGTH :S :L
OP ( POWER ( 4 / 3 ) ( :L - 1 ) ) * :S
END
```

At first, although Consuelo observed that the perimeter continued to increase unlike what she had predicted, she still felt that at higher levels the increase of the perimeter should become much smaller,

> Consuelo: I know that [it will stop increasing] when they become smaller, when the little peaks are smaller, when we get to a bigger level.

but as they computed the lengths for higher values she began to realise it could be otherwise, and began to find a justification for this behaviour in *how the number of parts increased in the figure*:

> Consuelo: Maybe I am wrong. This one [level 6] is almost 300 longer than the previous one. The length is growing a lot, even more. Because we are having more little peaks here, and before we didn't have as many peaks.

Then, when they looked at the graphical outputs of higher levels — in particular through the attempt to generate the curve of level 10, where the turtle turned endlessly drawing in almost the same place — they became aware, through the behaviour of the turtle, of the large amount of turns and small "peaks" that formed the curve.

Consuelo: It is doing such small peaks, and from the beginning they are so small, that it seems to be turning in the same place.
Verónica: The length is going to be very big.

The visual observations led Verónica to infer that the total length would grow to be very big with "too many..., infinite" number of segments. These experiences led the students to go back to an analysis of the formula for the number of segments (given by 4^{N-1}) which Consuelo pointed out would tend to be infinite. After an examination of the growth in the numeric values of both the number of segments and the total length, both students then concluded that both of these would tend to be infinite, whereas the segments, in Consuelo's words, "get close to zero, but they won't get there."

In the above story, there are two issues of different categories which are of interest:

(i) the method of investigation which involved a discrimination of each of the elements involved, as well as an abstraction into algebraic formulas for describing how each of these elements evolved ; and

(ii) the events that led the students to uncover the behaviour of, not only the perimeter of the curve changes, but of each of the elements involved in the changes of that perimeter

In the case of (i) the investigation involved the following construction process:

- the use of visual investigations through a gradual increase in the level, to observe how the perimeter changed and discern the elements involved: the level, the size of the segments, and the number of segments;

- a back and forth process between the visual analysis and the construction of the table which separated but also related each of the elements;

- more visual analysis through which the students deduced how each of the elements in the table evolved (e.g. each segment was replaced by four others), and then

- gradually abstracting the visual observations into the table, first through arithmetic description, followed by algebraic formulas which could be translated into Logo procedures.

In this way, the table, complemented by visual observations, became an aid with an important role of structuring and mediating the discovery (e.g. of relationships) and generalisation processes.

In relation to (ii), as we saw, there were several phases and events during the investigation:

- First, the initial observations into the behaviour of the segments forming the curve, which tended to zero, led Consuelo to predict that the total length would have a limit, since the segments added would be too small to make a difference. She was basing her deduction by focusing on a single element: the size of the segments.

- However, once they constructed formulas and procedures for determining how the total length grew, and were able to carry out a numerical investigation, Consuelo gradually became aware that her prediction had been wrong: the perimeter did not seem to stop its increase. This was a turning point which forced Consuelo to look for an explanation and shift her attention from what happened to the *size* of the segments, to what happened with the *number* of segments or of "small peaks"; she thus discovered the influence of a previously neglected element in the behaviour of the perimeter.

- The number of segments thus was discovered to be the determinant factor in the (divergent) behaviour of the perimeter above the influence of the size of the segments.

d. Explorations of the Koch Snowflake

The investigation of the area of the Koch snowflake[19] was carried out in a similar way to that of the perimeter: they used repeated visual observations gradually

[19] The snowflake was produced using the procedure, constructed by the students:

```
TO SNOWFLAKE :S :L
REPEAT 3 [PEAK :S :L  RT 120]
END
```

increasing the level (see Figure 22) — including overlapping several levels in one drawing (Figure 23) — and the use of a table (Table 19), both of which helped discern each of the elements, structure their relationships and the way they progressed.

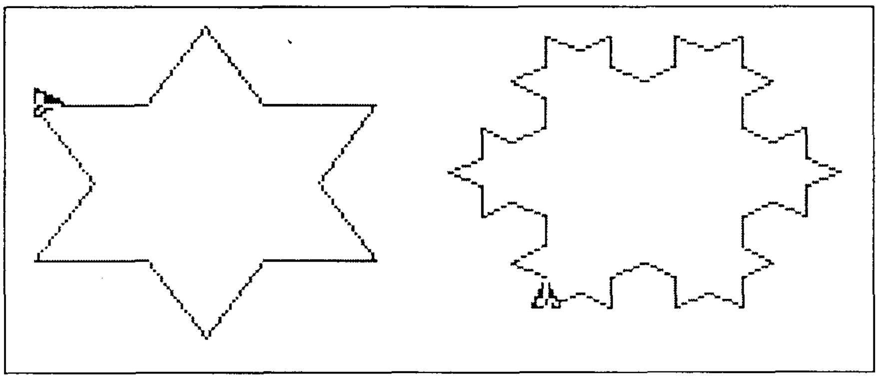


Figure 22. Levels 1 and 2 in the construction of the Koch Snowflake.

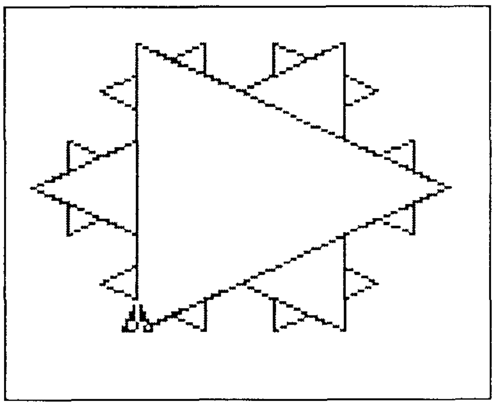


Figure 23. Overlapping levels (1, 2, and 3) of the Snowflake, showing the way in which the area increases.

The snowflake			Scale = 100	
Level	Number of triangles added to the previous level	Side and the smallest	Area of triangle	Total area
1	1	100	A_1	4330.12702
2	3	100/3	481.15942...	A_2 = 5773.502694
3	4 x 3			6415.002993
4				6700.114237
5				6826.830345
6				6883.148615
n	4^{n-2} x 3	$100/3^{n-1}$		$A_n = A_{n-1} + 4^{n-2}$ x 3 x (area smallest triangle)
7				6908.178957
20				6928.202702
50				6928.20323
100				6928.20323

Table 19. Table used for investigating the area of the Koch snowflake.

As before, the generalisations obtained through the table led to a formula, and then a procedure[20], for computing the values of the area of the snowflake at any level :L:

```
TO AREASNOWFLAKE :S :L
IF :L = 1 [OP AREATRI :S]
OP ( AREASNOWFLAKE :S :L - 1 ) + ( POWER 4 :L - 2 ) * 3
      * (AREATRI :S / POWER 3 :L - 1 )
END
```

At the beginning of this investigation the students had predicted that the area would increase, in Consuelo's words, "because each time new bits are clumped together; more triangles are added to the central triangle." However, once they had constructed the procedure for computing the values of the area at higher levels they noticed that soon the increase in the area became very small; they complemented this

[20] The AREASNOWFLAKE procedure used a procedure AREATRI for computing the area of an equilateral triangle of side :S:

```
TO AREATRI :S
OP ( POWER :S 2) * ( SQRT 3 ) / 4
END
```

observation with a visual comparison of the figures at levels 6 and 7 (which hardly differed), and inferred the existence of a limit:

Verónica:	The areas are increasing very little, so there will come a time when it will be very small, right? and so it will stop, or get stuck....
Consuelo:	But I think it's also going to have a limit.
Ana:	And what would that limit be?
Consuelo:	Well, around level 7, or 6, it is already getting stuck.

They also realised that at any level up to infinity, the figure would look practically the same, although the students were aware that the new areas added were not zero:

Consuelo:	[At any level after 6] they look almost the same, because the little segments that are added are so small that they can't be seen. And so the area is almost the same.
Ana:	So, are the areas that are added are almost zero?
Verónica:	No.
Consuelo:	No, they only get close to it.

The visual observations were complemented with a numerical analysis. Though they had expected the area of the snowflake to have a limit, they were surprised by how the value for this area quickly became a constant; this led them to analyse the values of the areas added and Consuelo remarked that those small areas "become very small, only changing in the decimals", becoming something like 0.0...00001, and so the total area was "not really increasing anymore...."

At this point, Verónica suggested computing the perimeter of the snowflake. They noticed they had already partly done this[21], in Table 18, but complemented it by calculating the length of a "side" of the snowflake for the level 100, which was 233848680765595.64783. Verónica remarked that the perimeter would tend to be infinite, as the level increased infinitely. Consuelo agreed explaining she based her decision on the behaviour of the turtle and from the numerical values, although she added that she did "not quite understand the formula when the level is infinite" and felt that having an infinite perimeter around an area that "gets stuck" was "weird". However, when asked to consider what happened to the area between the snowflake and the circumscribing circle to the original triangle (see Figure 24), as the levels progressed Consuelo immediately answered that this area would become smaller, and

[21] The perimeter of a snowflake of scale 100 is three times the perimeter of a curve of the same scale, which is equal to the perimeter of the curve with scale 300, something they had already calculated.

explained (using drawings on paper) that the snowflake would always stay within the circle, even though little "peaks" kept being added; she then added: "That is why the area has a limit". She had found another explanation for the behaviour of the area she had already observed.

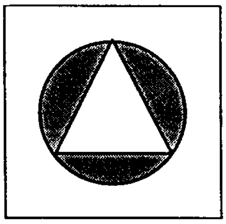


Figure 24. Area between the circumscribing circle and the triangle from which the Koch snowflake is generated.

Through this story we see another example of the role of the exploratory activities and environment, the importance of each of the tools (procedures, graphic outputs, and tables) as mediators and structuring elements for the discovery process to be able to take place, and the way in which the students go back and forth between the elements, with one discovery leading them to look back at other elements, then make sense and express the relationships between the elements.

2. Explorations with the Sierpinski triangle.

a. Explorations into the self-similarity of the figure: discovering that every part is similar to the whole and relating it to the recursive structure of the procedure.

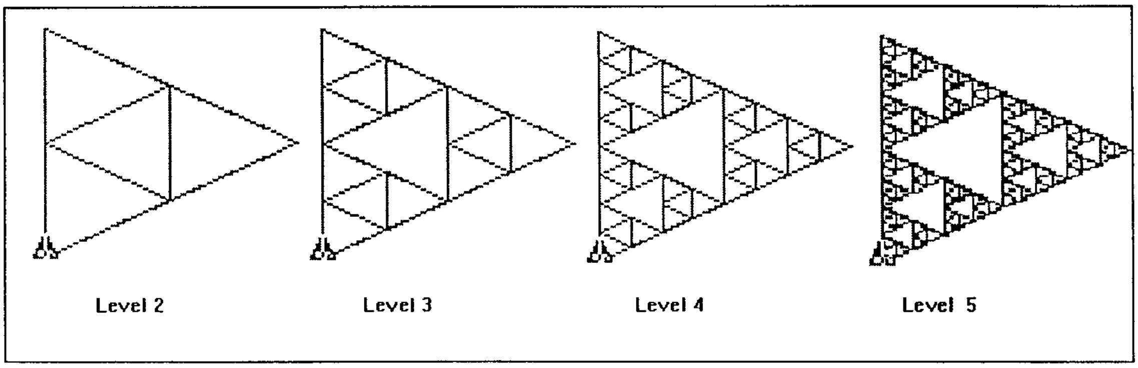


Figure 25. Construction of the Sierpinski triangle.

The experience with the Sierpinski triangle[22] was very similar to that with the Koch curve and snowflake. After looking at the visual images, Verónica described the

[22] The Sierpinski triangle was produced using the procedurebelow which I had given to the students:

```
TO TRI :SIDE :LEVEL
IF :LEVEL = 0 [STOP]
REPEAT 3 [TRI :SIDE / 2 :LEVEL - 1 FD :SIDE RT 120]
END
```

process as each triangle having a triangle inside of it. And when they produced a figure with half the scale, the students noticed and explained that the resulting figure was a "part of the bigger triangle", a third of the full-scaled figure; Consuelo explained this through the self-similar recursive structure of the procedure and the figure, saying "everything is similar to everything else, because TRI calls TRI". Verónica then added that any small part of the figure "would be the same".

b. Explorations into the area of the Sierpinski triangle.

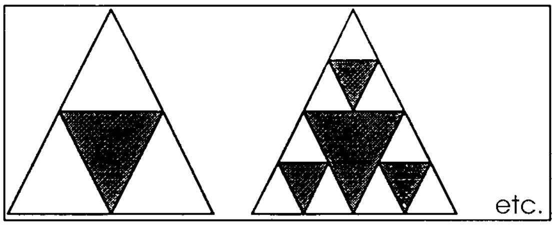


Figure 26. Area removed at each step in the construction process of the Sierpinski triangle.

I had asked the students to imagine they were removing the central triangle of each triangle (see Figure 26), and then consider what would happen to the remaining area. Consuelo suggested that if, for instance in level 7, they rearranged the remaining areas they might get a triangle the size of the central triangle, which would be a fourth of the original area. Verónica agreed with her. I had then asked what they though would happen at an infinite level, and Verónica explained the figure would look almost identical that of level 7, but that the area would be less; then, thinking aloud, she added that it would be the entire triangle which would be removed. Consuelo on the other hand said she believed the remaining area would still be a fourth of the original, or maybe even slightly more. This discussion prompted them to investigate numerically what happened to the area (using always a scale of 100), through the use of Table 20., and through visual observations where they noticed that at each step, a fourth of the area of each triangle was being removed. As in other cases the students deduced a formula (see last row of Table 20) and a procedure (AREAREM) for describing and computing the remaining area:

```
TO AREAREM :S :L
OP ( POWER 3 :L - 1 ) * AREATRI :S / ( POWER 2 :L - 1 )
END
```

Level	Side of the smallest triangle	Area of the smallest triangle	Number of remaining small triangles	Total remaining area
1	100	4330.127...	1	4330.127...
2	100/2	1082.5...	3	3247.5952...
3	$100/2^2$		$9 = 3^2$	2435.6964...
4				1826.772...
5				1370.67925...
6				1027.55943...
10				325.126228...
50				0.000326...
100				0.00000000185...
1000				$6.648... \times 10^{-122}$ = 0.(-121 zeros-)6...
n	$100/2^{n-1}$	AREATRI $100/2^{n-1}$	3^{n-1}	3^{n-1} x AREATRI $100/2^{n-1}$

Table 20. Table used for investigating the area of the Sierpinski triangle (using a scale of 100).

After computing the value of the area at level 6, Consuelo pointed out that the area "decreased very fast", but she also suggested it could be settling down. But after obtaining the value for level 10 (and then for higher levels), she changed her mind stating: "No, no, it doesn't stop". She explained she had been wrong to think the remaining area was a fourth of the original, which she now saw would be very small, "in the decimals". Then, when I asked the students if they thought that the area would sometime reach zero, Verónica explained it would not, "because the zeros [in the decimal expansion] are increasing". And when they looked at the value of the area at the 1000th level, Consuelo (and then also Veronica) commented that the area would be "a little more than zero", only bigger by "decimals" or "digits". Verónica explained that as the level increased there would be "infinite zeros" in the decimal expansion, but as they both explained it would not be zero "because after the zeros there will be numbers", although Consuelo expressed some doubts as to whether all those zeros could be infinite in quantity.

As had happened in earlier occasions, we see again how the students explained a continuing, but limited, infinite process through the changes in the decimal expansion. For both students the decimal expression could progress to having an almost infinite number of zeros after the decimal point, but this value would always be greater than zero because there would be non-zero digit(s) after all those zeros.

With regard to the visual image, the students maintained until the end that at an infinite level the figure would look pretty much like the one of level 7. So I asked them:

Ana:	So, [at an infinite level] what is it that we are looking at?
Consuelo:	Little points.
Verónica:	But it will still have the triangles...
Ana:	What do you mean?
Verónica:	The black triangles, that is the empty ones...
Ana:	You mean holes?
Verónica:	Yes.

c. Observing the self-similarity of the figure through the recursive characteristic of the procedure

When I gave the students the procedure CURVE below (which by typing commands such as, for level 2, "CURVE 2 100 1", produced the images shown in Figure 27), Consuelo predicted it would produce something "like the previous triangle", and although she could not really explain why she thought that, she pointed out that it was a procedure which also called itself. Verónica added:

Verónica: It is going to be like the other ones in which a small part resembles the whole, isn't it?

```
TO CURVE :L(evel) :S(cale) :P   (where the input of :P is always 1)
IF :L = 0 [FD :S STOP]
LT 60 * :P
CURVE :L - 1 :S / 2 ( -:P )
RT 60 * :P
CURVE :L - 1 :S / 2 :P
RT 60 * :P
CURVE :L - 1 :S / 2 ( -:P )
LT 60 * :P
END
```

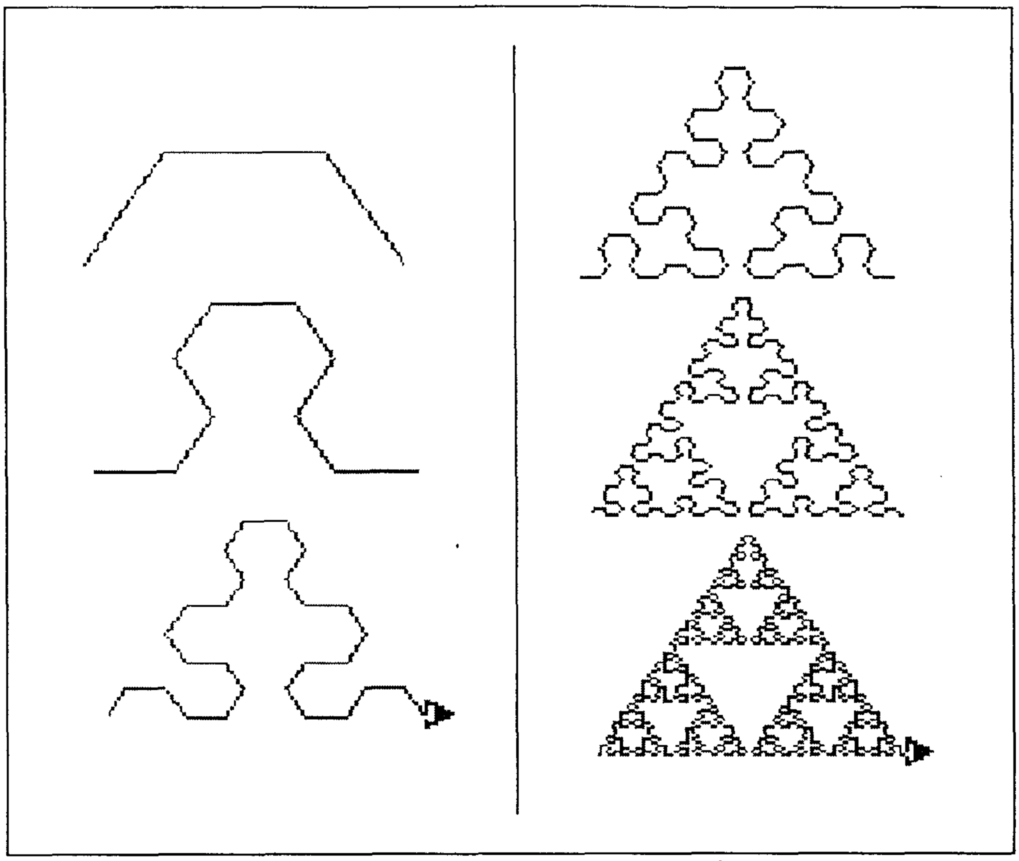


Figure 27. Levels 1-6 of the CURVE procedure.

After producing the figure for level 3, Verónica explained how the sequence of images was produced pointing out:

Verónica: Each part is being replaced by the figure of the previous level but with a lesser scale, right?

After the fifth level the students remarked that this curve was indeed like the figure produced through the previous procedure. I then asked:

Ana: What do you think will happen with this curve in an infinite level?
Consuelo: It is going to look very much or almost the same as the other one.
Verónica: It is going to be the same, isn't it?
Ana: And what do you think will happen to the length of this curve at an infinite level?
Verónica: The same as with the other triangle, it's same length: Infinite.

The students then compared on the same screen (see Figure 28), the figures generated by both procedures — TRI and CURVE. The students concluded that both procedures indeed produced the same figure, that at an infinite level the figures would most likely be identical, although Consuelo did wonder why this happened, and finally concluded that both procedures "marked" the same points. In this way, through the visual images, the students discovered that two different (infinite) geometric sequences converged to the same image.

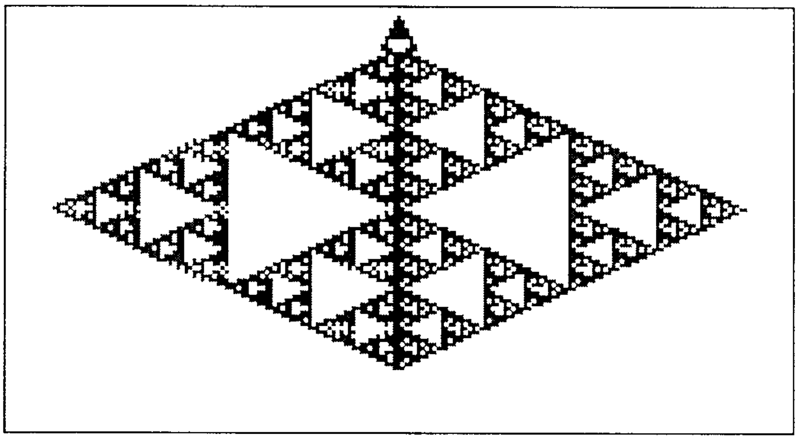


Figure 28. Comparison of the "Sierpinski triangles" produced, on the left, by "CURVE 100 7", and on the right by "TR1 100 6".

3. Exploring the Cantor set.

As a final activity I had shown the students (on paper) the first stages of the construction of the Cantor set, and the students had decided to write a procedure — BITS (using the PEAK procedure to guide themselves) for generating that sequence (see Figure 29):

```
TO BITS :S :L
IF :L = 1 [FD :L STOP]
BITS :S / 3 :L - 1
PU FD :S / 3 PD
BITS :L / 3 :L - 1
END
```

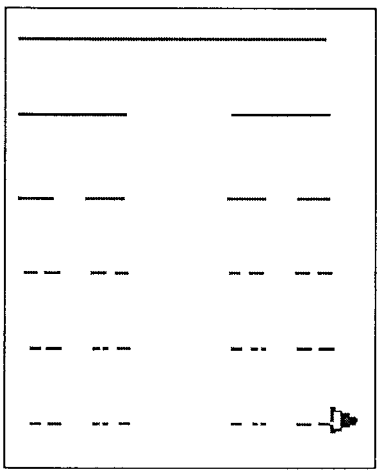


Figure 29. First 6 stages of the Cantor set, produced using the BITS procedure.

I then asked:

Ana:	What do you think will happen when :L [the level] is infinite? What will there be left?
Verónica:	It's going to be very small lines.
Ana:	And what do you think each of those lines will measure?
Consuelo:	Mmm..
Verónica:	We are dividing by 3, so it will be 300[23], then 300 over 3 which is 100, then 100 over 3, then over 3, and over 3.....
Consuelo:	It's 300 over 3 to the N. And that is like the one we saw which got closer to zero.
Ana:	So what is going to happen to each of these bits?
Verónica:	They are going to get close to zero.
Ana:	And what do you think we will have in the end? if we did this up to an infinite level?
Consuelo:	It seems like there will not be anything, but there will be very small little points.

As shown in the transcript above, Consuelo linked the behaviour of the Cantor sequence with the sequence {1/3n} which they knew tended to zero. This led to the conclusion that the segments would become like points, not leaving "anything". But by observing the images produced in higher levels (e.g. level 8), Consuelo realised the figure did not "disappear". The students realised this was another self-similar figure explaining that each part was "like the whole", which as the level tended to infinity would look the same even though "each segment [would] become almost zero"; Consuelo began by saying that these segments would be like "very very small points", although they added:

Verónica:	They would be segments but they would look like points, right?
Ana:	Why do you say they would still be segments?
Consuelo:	Because they would still have a measure.

4. Final comments.

Through the fractal explorations (as well as the sequence studies) described above, I attempted to illustrate how the students made use of all the elements in the microworld to form their own generalisations and conclusions although relative to the context they were working in. An important aspect in this discovery process was the facilities that the tools of the microworld (procedures, direct Logo commands which

[23] They had used 300 as the scale, and thus that was the length of the initial segment.

were used to express and compute values, variations of graphical outputs, and tables) which provided a means for the students to structure their explorations, form and express relationships and generalisations, and through a back and forth process combining all the elements led the students to reach their conclusions.

Furthermore, these activities proved appealing to the students, motivating them to investigate as much as possible every situation. At the end of the last session both students commented on this point, mentioning that they had found the activities "fun and pretty". The students also appreciated all the elements of the microworld and, at least Consuelo expressed the connection between the graphics and the procedures: she explained that the procedures reflected what they saw because "they call on themselves". She added that both graphics and procedures "helped us to see the limits" by showing what happened at infinity.