Analysis of semidiscretization of the compressible Navier-Stokes equations

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Abstract

The objective of this work is to investigate the time discretization of two dimensional Navier-Stokes system with the slip boundary conditions. First, the existence of weak solutions for a fixed time step $\triangle t > 0$ is presented and then the limit passage as $\triangle t \to 0^+$ is carried out. The proof is based on a new technique established for the steady Navier-Stokes equations by Mucha P. B. and Pokorný M. 2006 *Nonlinearity* **19** 1747-1768 which enables to estimate the growth of L_∞ norm of the density when $\triangle t$ goes to 0.

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1. Introduction

We investigate a system being time discretization of two dimensional Navier-Stokes equations in the isentropic regime

$$\begin{gathered}\frac{1}{\triangle t}\left(\varrho^k-\varrho^{k-1}\right)+\operatorname{div}(\varrho^k v^k)=0,\\ \frac{1}{\triangle t}\left(\varrho^k v^k-\varrho^{k-1}v^{k-1}\right)+\operatorname{div}(\varrho^k v^k\otimes v^k)-\mu\Delta v^k-(\mu+\nu)\nabla\operatorname{div}v^k+\nabla\pi(\varrho^k)=0,\end{gathered} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^2$ is a fixed domain, $v^k:\Omega\to\mathbb{R}^2$- the velocity field, $\varrho^k:\Omega\to\mathbb{R}_0^+$- the density, $\pi:\mathbb{R}_0^+\to\mathbb{R}$- the internal pressure given by the constitutive relation

$$\pi(\varrho^k)=(\varrho^k)^\gamma,\ \gamma>1.$$

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We assume that the walls of Ω are rigid and that the fluid slips at the boundary

$$\begin{array}{ll} v^k\cdot n = 0 & \text{at } \partial\Omega, \\ n\cdot \mathbb{T}(v^k,\pi)\cdot \tau + f v^k\cdot \tau = 0 & \text{at } \partial\Omega, \end{array} \tag{1.2}$$

where $\mathbb{T}(v^k,\pi) = 2\mu D(v^k) + (\nu \mathrm{div} v - \pi)\mathbb{I}$. By n we denote the outward unit normal to $\partial\Omega$ and τ is the unit tangent vector to $\partial\Omega$.

The conditions (1.2) are known as the Navier or friction relations which means that unlike the case of complete slip of the fluid against the boundary, the friction effects, described by $f \geq 0$, may also be present. The customary zero Dirichlet condition may be understood as a special case of the above, when $f \to \infty$. The main advantage of slip conditions is a possibility to state system (1.1) in terms of the vorticity of velocity $\nabla \times v^k$ as in [10]. In particular, it enables to compute vorticity at the boundary as a function of the tangent velocity if the curvature χ of $\partial\Omega$ is known, i.e.

$$\nabla \times v = (2\chi - \frac{f}{\mu}) v\cdot\tau \quad \text{at } \partial\Omega.$$

We will always assume that our initial conditions ϱ^0, v^0 satisfy

$$\varrho^0 \geq 0 \text{ a.e. in } \Omega, \quad \varrho^0 \in L_\gamma(\Omega), \quad \varrho^0 v^0 \in L_{2\gamma/(\gamma+1)}(\Omega), \quad \varrho^0 (v^0)^2 \in L_1(\Omega). \tag{1.3}$$

The first goal of this paper is to show that for $\triangle t = const.$ and in the case when (ϱ^{k-1}, v^{k-1}) are given functions satisfying conditions specified in (1.3), the solutions of system (1.1)-(1.2) exist in the sense of the following definition

Definition 1. The pair of functions $(\varrho^k, v^k) \in L_\gamma(\Omega) \times W^1_2(\Omega)$, $v^k\cdot n = 0$ at $\partial\Omega$ is a weak solution to (1.1)-(1.2) provided

$$\int_\Omega \varrho^k v^k \cdot \nabla\varphi \, dx = \frac{1}{\triangle t} \int_\Omega (\varrho^k - \varrho^{k-1})\varphi \, dx, \quad \forall \varphi \in C^\infty(\overline{\Omega}),$$

and

$$\begin{aligned} &\frac{1}{\triangle t}\int_\Omega (\varrho^k v^k - \varrho^{k-1} v^{k-1})\varphi \, dx - \int_\Omega \varrho^k v^k \otimes v^k : \nabla\varphi \, dx + 2\mu \int_\Omega \mathbf{D}(v^k) : \mathbf{D}(\varphi)\, dx \\ &+ \nu \int_\Omega \mathrm{div} v^k \mathrm{div}\varphi \, dx - \int_\Omega \pi(\varrho^k)\mathrm{div}\varphi \, dx + \int_{\partial\Omega} f(v^k\cdot\tau)(\varphi\cdot\tau)\, dS = 0, \\ &\qquad\qquad\qquad\qquad \forall \varphi \in C^\infty(\overline{\Omega});\ \varphi\cdot n = 0 \ at\ \partial\Omega. \end{aligned}$$

The first main result reads as follows

Theorem 1 *Let $\Omega \in C^2$ be a bounded domain, $\triangle t = const.$, $\mu > 0$, $2\mu + 3\nu > 0$, $\gamma > 1$, $f \geq 0$. Let $(\varrho^{k-1}, v^{k-1}) \in L_\gamma(\Omega) \times W_2^1(\Omega)$ be given functions satisfying (1.3). Then there exists a weak solution to (1.1)-(1.2) such that*

$$\begin{aligned} &\varrho^k \in L_\infty(\Omega) \quad \text{and } \varrho^k \geq 0, \\ &v^k \in W_p^1(\Omega) \quad \forall p < \infty, \\ &\int_\Omega \varrho^k dx = \int_\Omega \varrho^{k-1} dx, \end{aligned}$$

moreover $\|\varrho^k\|_\infty \leq (\triangle t)^{\frac{-3\gamma}{2(\gamma-1)^2}}$.

The first step in the weak solvability of the time discretized barotropic compressible Navier-Stokes equations is contained in the seminal work of P.L. Lions [5]. It was studied there as a type of stationary problem (for $\triangle t$ fixed) mostly for Dirichlet boundary conditions. The proof was based on compactness of the quantity usually called *effective viscous flux* which provides strong convergence of density in the situation when ϱ belongs to $L_2(\Omega)$. This, in turn, imposes some restrictions upon the exponent γ, i.e., $\gamma > 1$ in two space dimensions and $\gamma \geq \frac{5}{3}$ in three space dimensions. Lions' approach was later on modified [11] to treat smaller values of γ, by adopting Feireisl's concept of oscillation defect measures [13], [2], [4] to the case of steady systems.

It is to be noticed that the weak solution (ϱ, v) constructed in [5] belongs to $L_\infty(\Omega) \times W_p^1(\Omega)$ for each p finite, for $\gamma > 1$ when $N = 2$ and for $\gamma > 3$ when $N = 3$, for the no-slip boundary conditions. The method works also in our case, however, the approach presented here differs already at the level of the approximate system. Namely, it allows for essential reduction of the number of technical tricks and enables to get the required L_∞ bound of density directly from the construction of approximate solutions. But the biggest advantage is the ability to control the growth of $\|\varrho\|_\infty$ in terms of length of time interval $\triangle t$. We will employ the method presented for the first time in [6] for the 2D steady case and then applied for 3D case in [9]. The same method has been recently successfully applied for more complex system of Navier-Stokes-Fourier equations in the steady compressible 3D case [7], [8].

The second result refers to passage to the limit with length of time interval $\triangle t \to 0$. We will show that for such a case our solution tends to the weak solution of evolutionary compressible Navier-Stokes system with a slip boundary condition:

$$\begin{array}{rll} \varrho_t + \mathrm{div}(\varrho v) = 0 & \text{in} & (0,T)\times\Omega, \\ (\varrho v)_t + \mathrm{div}(\varrho v\otimes v) - \mu\Delta v - (\mu+\nu)\nabla \mathrm{div} v + \nabla\pi(\varrho) = 0 & \text{in} & (0,T)\times\Omega, \\ v\cdot n = 0 & \text{at} & \partial\Omega, \\ n\cdot \mathbb{T}(v,\pi)\cdot\tau + f v\cdot\tau = 0 & \text{at} & \partial\Omega, \end{array} \tag{1.4}$$

in sense of the following definition.

Definition 2. We say, the pair of functions $(\varrho, v) \in L_\infty(L_\gamma)\times L_2(W^1_2)$, $v\cdot n = 0$ at $\partial\Omega$ is a weak solution to (1.4) provided

$$\int_0^T\int_\Omega (\varrho\varphi_t + \varrho v\cdot\nabla\varphi)\ dxdt = 0, \quad \forall\varphi\in C^\infty_c([0,T)\times\overline{\Omega}),$$

and

$$\int_0^T\int_\Omega (\varrho v\varphi_t + \varrho v\otimes v : \nabla_x\varphi + \pi(\varrho)\mathrm{div}_x\varphi)\ dxdt =$$
$$= \int_0^T\int_\Omega (2\mu\mathbf{D}(v):\mathbf{D}(\varphi) + \nu\mathrm{div}_x v\ \mathrm{div}_x\varphi)\ dxdt + \int_0^T\int_{\partial\Omega} f(v\cdot\tau)(\varphi\cdot\tau)\ dSdt,$$
$$\forall\varphi\in C^\infty_c([0,T)\times\overline{\Omega});\ \varphi\cdot n = 0 \ at\ \partial\Omega. \tag{1.5}$$

The existence of solutions to the evolutionary system is assured by the following theorem

Theorem 2 *Under the hypotheses of Theorem 1, and for $\gamma>2$, the solution (ϱ^k, v^k) converges to (ϱ, v) as $\triangle t\to 0^+$ weakly (weakly*) in $L_\infty(L_\gamma)\times L_2(W^1_2)$. Moreover ϱ belongs to $L_{\gamma+1}(\Omega\times(0,T))$ and the following energy inequality is satisfied for almost all $t\in[0,T]$*

$$\int_\Omega \varrho v^2(T)dx + \frac{1}{\gamma-1}\int_\Omega \varrho^\gamma(T)dx + \int_0^T\int_\Omega \left(2\mu|\mathbf{D}(v)|^2 + \nu(\mathrm{div} v)^2\right)\ dxdt$$
$$+ \int_0^T\int_{\partial\Omega} f(v\cdot\tau)^2 dx\ dt \le C(\varrho^0, v^0).$$

We enclose the proof of Theorem 2 only for sake of completeness of theory presented here. This is not an optimal result since we require that $\gamma>2$, and it is possible to relax this condition. Already in the book [5] it was shown that the weak renormalized solutions to system (1.4) exist for $\gamma \geq \frac{3}{2}$ when $N = 2$ and $\gamma \geq \frac{9}{5}$ when $N = 3$. The idea consists of a simple modification of the pressure $\pi_\delta(\varrho) = \varrho^\gamma + \delta\varrho^\Gamma$ with suitable large Γ, which provides better *a priori* integrability of the density necessary to employ some compensated compactness arguments. Further extensions of this concept can be found in [13], [4].

The article is organised as follows. In the next section we will show the existence and uniqueness of regular solution to the problem being the new $\epsilon-$approximation scheme for the time-discretized Navier-Stokes equations. Although the proof is based on the standard fixed-point method, we will present most of steps in view of the fact that our approximation affects the nonlinear term too. Our solution $(\varrho^k,\ v^k)$ will be obtained as a weak limit as $\epsilon \to 0^+$ of the sequence $(\varrho_\epsilon^k,\ v_\epsilon^k)$. This limit process will be carried out in Section 3 by using some uniform estimates and the following property of the density sequence

$$\lim_{\epsilon\to 0^+} |\{x \in \Omega : \varrho_\epsilon^k(x) > m\}| = 0$$

for m sufficiently large, which enables to show the convergence of the pressure.

Section 4 is dedicated to the proof of Theorem 2. The central problem is, as usually, to show the convergence of the pressure. We solve it by using, roughly speaking, as a test function in the momentum equation $\phi = (\nabla\Delta^{-1})[\varrho]$ together with several results about the commutators, in the spirit of theory developed in [1], and a concept of renormalized solutions to continuity equation.

We shall make here some remarks concerning notation. We will usually skip $(0, T)$ and Ω in notation of the spaces, for example we will write L_2 instead of $L_2(\Omega)$ and $L_\infty(L_2)$ instead of $L_\infty(0, T; L_2(\Omega))$.

2. Approximation

In this section we present a scheme of approximation being a modification of the one introduced by Mucha, Pokorný [6] for the steady case. We want to investigate the issue of existence of solutions when the time step $\triangle t$ is fixed and less then 1. We will focus on proving the existence of a regular solution in the k-th moment of time, while disposing a sufficient information for the

density and velocity in the previous time step. Although for further purposes there is a necessity to keep trace of the dependence on these quantities in almost all estimates.
Denote:

$$\begin{array}{c} \alpha = \frac{1}{\triangle t}, \\ h = \varrho^{k-1}, \quad \varrho = \varrho^k, \quad v = v^k, \quad g = v^{k-1}. \end{array} \tag{2.1}$$

The objective of this part of work will be then to examine the following approximative system:

$$\begin{array}{r} \alpha\left(\varrho - hK(\varrho)\right) + \mathrm{div}(K(\varrho)\varrho v) - \epsilon\Delta\varrho = 0 \\ \alpha\left(\varrho v - hg\right) + \mathrm{div}(K(\varrho)\varrho v \otimes v) - \mu\Delta v - (\mu+\nu)\nabla \mathrm{div} v + \nabla P(\varrho) + \epsilon\nabla v \nabla\varrho = 0 \\ \frac{\partial\varrho}{\partial n} = 0 \quad \text{at} \quad \partial\Omega, \\ v \cdot n = 0 \quad \text{at} \quad \partial\Omega, \\ n\cdot T(v, P(\varrho))\cdot\tau + fv\cdot\tau = 0 \quad \text{at} \quad \partial\Omega, \end{array} \tag{2.2}$$

we will write simply ϱ, v instead of $\varrho_\epsilon, v_\epsilon$ when no confusion can arise. The other denotations are the following:

$$P(\varrho) = \gamma \int_0^\varrho s^{\gamma-1} K(s) ds, \tag{2.3}$$

where

$$K(\varrho) = \begin{cases} 1 & \varrho \le m_1, \\ 0 & \varrho \ge m_2, \\ \in (0,1) & \varrho \in (m_1, m_2), \end{cases}$$

and

$$K(\cdot) \in C^1(\mathbb{R}) \quad K'(\varrho) < 0 \text{ in } (m_1, m_2),$$

for some constants $m_1,\ m_2$. To avoid difficulties connected with the case when $m_1 \to m_2$ we set the difference $m_2 - m_1$ to be constant, equal 1.
The existence of a regular solution is provided due to the following theorem

Theorem 3 *Let $\Omega \in C^2$ be a bounded domain. Let ϵ, α be positive constants. Let $h \in L_\infty$, $h \ge 0$, $hg \in L_{2\gamma/(\gamma+1)}$, $hg^2 \in L_1$. Then there exists a regular solution (ϱ, v) to (2.2), $\varrho \in W^2_p$, $v \in W^2_p$ for all $p < \infty$.*
Moreover

$$0 \le \varrho \le m_2 \quad in\ \Omega, \tag{2.4}$$

$$\int_\Omega \varrho dx \le \int_\Omega h dx. \tag{2.5}$$

PROOF. We assume that ϱ, v are regular solutions to (2.2) and prove some estimates first, after we go on with the existence.
Step 1. *Proof of (2.5).*
Integrating the first equation of (2.2) over Ω one gets

$$\alpha \int_{\Omega} (\varrho - hK(\varrho))dx + \int_{\partial\Omega} K(\varrho)\varrho v \cdot n dS - \epsilon \int_{\partial\Omega} \frac{\partial \varrho}{\partial n} dS = 0,$$

the boundary integrals vanish and due to the definition of $K(\cdot)$ we truly have

$$\int_{\Omega} \varrho dx = \int_{\Omega} K(\varrho) h dx \leq \int_{\Omega} h dx.$$

Step 2. *Non-negativity of ϱ.*
We integrate the first equation of (2.2) over $\Omega^- = \{x \in \Omega : \varrho(x) < 0\}$

$$\alpha \int_{\Omega^-} (\varrho - K(\varrho)h)dx + \int_{\partial\Omega^-} K(\varrho)\varrho v \cdot n dS - \epsilon \int_{\partial\Omega^-} \frac{\partial \varrho}{\partial n} dS = 0,$$

the first boundary integral vanishes since either ϱ or $v \cdot n$ equals 0 at $\partial\Omega^-$. Moreover, we know that $\frac{\partial \varrho}{\partial n} \geq 0$ at $\partial\Omega^-$, hence

$$\int_{\Omega^-} \varrho dx \geq \int_{\Omega^-} K(\varrho) h dx \geq 0,$$

but this leads to conclusion that $|\Omega^-| = 0$ and consequently $\varrho \geq 0$ in Ω.
Step 3. *Upper bound for ϱ.*
This time we integrate the approximate continuity equation over $\Omega^+ = \{x \in \Omega : \varrho(x) \geq m_2\}$

$$\alpha \int_{\Omega^+} (\varrho - K(\varrho)h)dx + \int_{\partial\Omega^+} K(\varrho)\varrho v \cdot n dS - \epsilon \int_{\partial\Omega^+} \frac{\partial \varrho}{\partial n} dS = 0,$$

At $\partial\Omega^+$ we have $\frac{\partial \varrho}{\partial n} \leq 0$ and either $K(\varrho)$ or $v \cdot n$ equals 0. Thus, in the similar way as previously, the observation

$$\int_{\Omega^+} \varrho dx \leq \int_{\Omega^+} K(\varrho) h dx = 0$$

implies that $\varrho \leq m_2$ in Ω.
Step 4. *Existence.*

In accordance with our notation the proof of existence of approximate solutions is almost identical to the one presented in [6]. In the first step we define for $p \in [1,\infty]$:

$$M_p = \left\{ w \in W^1_p; w \cdot n = 0 \ at \ \partial\Omega \right\}.$$

and claim that the following proposition, which is the analogue of Proposition 3.1. from [6], holds true.

Proposition 4 *Let assumptions of Theorem 3 be satisfied. Then the operator $S : M_\infty \to W^2_p$, where*

$$\begin{aligned} S(v) &= \varrho, \\ \alpha\varrho + \operatorname{div}(K(\varrho)\varrho v) - \epsilon\Delta\varrho &= \alpha h K(\varrho) \quad in \quad \Omega \\ \frac{\partial\varrho}{\partial n} &= 0 \quad at \quad \partial\Omega \end{aligned}$$

is well defined for any $p < \infty$. Moreover

- *$\varrho = S(v)$ satisfy*
$$\int_\Omega \varrho dx \leq \int_\Omega h dx.$$
- *If $h \geq 0$ then $\varrho \geq 0$ a.e. in Ω.*
- *If $\|v\|_{1,\infty} \leq L, \ L > 0$ then*
$$\|\varrho\|_{2,p} \leq C(\epsilon, p, \Omega)(1 + L)\|h\|_p, \quad 1 < p < \infty. \tag{2.6}$$

The only difference in the formulation and the proof with respect to the one presented in [6] relates to the fact that h is not a constant parameter any more, but the information about the solution (h, g) in the $(k-1)$-th moment of time, in particular assumption that $h \in L_\infty$ allows to estimate the norm of h in L_p for all $1 \leq p \leq \infty$.
In the next step of proof of existence we will consider the Lamé operator

$$\mathcal{T} : \ M_\infty \to M_\infty$$

defined as follows: $w = \mathcal{T}(v)$ is a solution to the problem

$$
\begin{aligned}
-\mu\Delta w - (\mu+\nu)\nabla \mathrm{div} w = \alpha h g - \alpha\varrho v - \mathrm{div}(K(\varrho)\varrho v\otimes v) - \nabla P(\varrho) - \epsilon\nabla v\nabla\varrho = \\
= F(\varrho, v, h, g) \\
w\cdot n = 0 \quad \text{at} \quad \partial\Omega, \\
n\cdot(2\mu D(w) + \nu \mathrm{div} w I)\cdot\tau + f v\cdot\tau = 0 \quad \text{at} \quad \partial\Omega
\end{aligned}
\tag{2.7}
$$

Employing the Leray-Schauder fixed point theorem for the operator $\mathcal{T}$ we can almost rewrite the proof of analogous fact from [13] or [6]. The only part that deserves more careful study is the energy estimate which provides some information about solutions, uniformly with respect to ϵ and α necessary to carry out the limit process.
First, observe that $(2.7)_1$ with $w = v$ and $\varrho = S(v)$ can be tested with the solution itself, therefore

$$
\begin{aligned}
\alpha\int_\Omega \varrho v^2 + \int_\Omega \mathrm{div}(K(\varrho)\varrho v\otimes v)v - \mu\int_\Omega(\Delta v)v - (\mu+\nu)\int_\Omega(\nabla\mathrm{div} v)v + \int_\Omega \nabla P(\varrho)v \\
+\frac{\epsilon}{2}\int_\Omega \nabla v^2\nabla\varrho = \alpha\int_\Omega hgv.
\end{aligned}
$$

Next, integrating by parts and using condition on the boundary

$$
\begin{aligned}
\alpha\int_\Omega \varrho v^2 + \frac{1}{2}\int_\Omega \mathrm{div}(K(\varrho)\varrho v)v^2 + 2\mu\int_\Omega |D(v)|^2 + \nu\int_\Omega \mathrm{div}^2 v + \int_{\partial\Omega} f(v\cdot\tau)^2 \\
-\frac{\gamma}{\gamma-1}\int_\Omega \mathrm{div}(K(\varrho)\varrho v)\varrho^{\gamma-1} - \frac{\epsilon}{2}\int_\Omega v^2\Delta\varrho = \alpha\int_\Omega hgv,
\end{aligned}
$$

and then including the information contained in $(2.2)_1$ one gets

$$
\begin{aligned}
\frac{1}{2}\alpha\int_\Omega(\varrho + K(\varrho)h)v^2 + 2\mu\int_\Omega|D(v)|^2 + \nu\int_\Omega \mathrm{div}^2 v + \int_{\partial\Omega} f(v\cdot\tau)^2 \\
+\frac{\gamma}{\gamma-1}\alpha\int_\Omega\varrho^\gamma - \frac{\gamma}{\gamma-1}\alpha\int_\Omega \varrho^{\gamma-1}K(\varrho)h + \gamma\epsilon\int_\Omega \varrho^{\gamma-2}|\nabla\varrho|^2 = \alpha\int_\Omega hgv.
\end{aligned}
$$

Now we add and subtract $\frac{1}{2}\alpha\int_\Omega hg^2$ and $\frac{1}{\gamma-1}\alpha\int_\Omega h^\gamma$

$$
\begin{aligned}
\frac{1}{2}\alpha\int_\Omega(\varrho v^2 - hg^2) + \frac{1}{2}\alpha\int_\Omega h|v-g|^2 + 2\mu\int_\Omega|D(v)|^2 + \nu\int_\Omega\mathrm{div}^2 v + \int_{\partial\Omega} f(v\cdot\tau)^2 \\
+\frac{1}{\gamma-1}\alpha\int_\Omega(\varrho^\gamma - h^\gamma) + \frac{1}{\gamma-1}\alpha\int_\Omega\big((\gamma-1)\varrho^\gamma + h^\gamma - \gamma\varrho^{\gamma-1}K(\varrho)h\big) + \frac{4\epsilon}{\gamma}\int_\Omega|\nabla\varrho^{\frac{\gamma}{2}}|^2 \leq 0.
\end{aligned}
\tag{2.8}
$$

Note that since $\varrho,\ h \geq 0$ and $K(\varrho) \leq 1$ we have that $(\gamma-1)\varrho^\gamma + h^\gamma - \gamma\varrho^{\gamma-1}K(\varrho)h \geq 0$ for all $\gamma > 1$, therefore the following bound is valid

$$\|\varrho\|_\gamma^\gamma + \|\varrho v^2\|_1 \leq C(h, g, \gamma, \Omega), \tag{2.9}$$

in particular the constant C is independent of k, ϵ and α, moreover

$$\int_\Omega \left[|v-g|^2 + (\gamma-1)\varrho^\gamma + h^\gamma - \gamma\varrho^{\gamma-1}K(\varrho)h\right] \leq C. \tag{2.10}$$

Additionally we have

$$\|\mathbf{D}v\|_2^2 \leq \alpha C$$

and by the Korn inequality

$$\|v\|_{1,2}^2 \leq \alpha C. \tag{2.11}$$

Here the constant C depends also on μ and ν.
Finally we also get

$$\|\nabla(\varrho)^{\frac{\gamma}{2}}\|_2^2 \leq \frac{\alpha}{\epsilon} C. \tag{2.12}$$

This information allows us to repeat the procedure described in [13] which together with the Proposition 4 yield the existence of regular solutions, and hence complete the proof of Theorem 3. □
Apart from the first *a priori* estimate, the limit passage requires also some others estimates independent ϵ and α. First of them is the estimate for the norm of gradient of the density. Observe that multiplying $(2.2)_1$ by ϱ and integrating over Ω one gets

$$\begin{aligned}
\epsilon\int_\Omega |\nabla\varrho|^2 &= \alpha\int_\Omega hK(\varrho)\varrho - \alpha\int_\Omega \varrho^2 - \int_\Omega K(\varrho)\varrho v\cdot\nabla\varrho \\
&\leq \alpha C m_2 + \int_\Omega v\cdot\nabla\left(\int_0^\varrho K(t)t\, dt\right) = \alpha C m_2 - \int_\Omega \mathrm{div} v\left(\int_0^\varrho K(t)t\, dt\right) \\
&\leq \alpha C m_2 + \int_\Omega |\mathrm{div} v|\varrho^2 \leq \alpha C m_2 + \sqrt{\alpha} C m_2^2.
\end{aligned}$$

This means that $\|\nabla\varrho\|_2$ may blow up as $\epsilon \to 0^+$, however we can provide that $\epsilon\|\nabla\varrho\|_2$ will tend to zero, i.e.

$$\epsilon\|\nabla\varrho\|_2 \leq \sqrt{\epsilon} C(\alpha, m_2), \tag{2.13}$$

for some constant C independent of ϵ.
Now we would like to obtain integrability of the pressure with the power 2, as previously independently of ϵ and, if possible, of m_2. Therefore the choice of an appropriate test function seems to be obvious:

$$\Phi = \mathcal{B}\Big(P(\varrho) - \{P(\varrho)\}\Big) \text{ in } \Omega,$$

where $\mathcal{B}$ is the Bogovskii operator and $\{\cdot\} = \frac{1}{|\Omega|}\int_\Omega (\cdot) dx$. By virtue of the basic properties of the operator $\mathcal{B}$ and the Poincaré inequality we have:

$$\|\Phi\|_{\bar{p}} \leq c(p,\Omega)\|P(\varrho)\|_p, \quad \|\nabla\Phi\|_p \leq c(p,\Omega)\|P(\varrho)\|_p \tag{2.14}$$
$$0 < p < \infty, \quad \bar{p} = \begin{cases} \frac{2p}{2-p} & \text{if} \quad p < 2 \\ \in [1,\infty) & \text{if} \quad p = 2 \\ \infty & \text{if} \quad p > 2. \end{cases}$$

From this testing, the following identity appears:

$$\int_\Omega P(\varrho)^2 = \frac{1}{|\Omega|}\left(\int_\Omega P(\varrho)\right)^2 + \alpha\int_\Omega (\varrho v - hg)\Phi + \mu\int_\Omega \nabla v : \nabla\Phi + (\mu+\nu)\int_\Omega \mathrm{div} v\, \mathrm{div}\Phi$$
$$- \int_\Omega K(\varrho)\varrho v \otimes v : \nabla\Phi + \epsilon\int_\Omega \nabla v \nabla\varrho\Phi = \sum_{i=1}^{6} I_i.$$

Now each term will be estimated separately.
(i) By estimate (2.9) and the definition of P the first one becomes straightforward

$$I_1 = \frac{1}{|\Omega|}\left(\int_\Omega P(\varrho)\right)^2 \leq \frac{1}{|\Omega|}\left(\int_\Omega \varrho^\gamma\right)^2 \leq C.$$

(ii) Relation (2.14) together with estimate (2.9) imply

$$I_2 = \alpha\int_\Omega (\varrho v - hg)\Phi\, dx \leq C\alpha\left(\|\varrho\|_\gamma\|v\|_2 + \|h\|_\gamma\|g\|_2\right)\|P(\varrho)\|_2 \leq C\alpha^{3/2}\|P(\varrho)\|_2.$$

(iii) We also have $\|\nabla\Phi\|_2 \leq \|P(\gamma)\|_2$, thus

$$I_3 + I_4 = \mu\int_\Omega \nabla v\nabla\Phi + (\mu+\nu)\int_\Omega \mathrm{div} v \mathrm{div}\Phi \leq C\|v\|_2\|P(\varrho)\|_2 \leq C\alpha^{1/2}\|P(\varrho)\|_2.$$

(iv) The Hölder inequality and imbedding mentioned above lead to

$$I_5 = \int_\Omega K(\varrho)\varrho v \otimes v : \nabla\Phi \le C\|K(\varrho)\varrho\|_q \|v\|_{1,2}^2 \|P(\varrho)\|_2,$$

for some $q > 2$. By the definition of $P(\varrho)$ and a simple interpolation one gets

$$\|K(\varrho)\varrho\|_q \le \|K(\varrho)\varrho\|_\gamma^{(2\gamma-q)/q} \|K(\varrho)\varrho\|_{2\gamma}^{(2q-2\gamma)/q} \le C\|P\|_2^{(2q-2\gamma)/(\gamma q)}$$

provided additionally that $\gamma < q < 2\gamma$. Therefore the integral I_5 can be now estimated as follows

$$I_5 = \int_\Omega K(\varrho)\varrho v \otimes v : \nabla\Phi \le C\alpha\|P(\varrho)\|_2^{1+\eta},$$

where $\eta = \frac{2(q-\gamma)}{\gamma q} < 1$.
(v) Finally, employing the Hölder inequality we may get that

$$I_6 = \epsilon \int_\Omega \nabla v \nabla\varrho \Phi \le \epsilon \|\nabla\varrho\|_q \|v\|_{1,2} \|P(\varrho)\|_2,$$

for some $q > 2$. To get the estimate for $\|\nabla\varrho\|_q$ we need to interpret the approximate continuity equation as a Neumann-boundary problem

$$\begin{array}{c} -\epsilon\Delta\varrho = \mathrm{div} b \quad in\ \Omega \\ \frac{\partial\varrho}{\partial n} = b\cdot n \quad at\ \partial\Omega, \end{array} \tag{2.15}$$

with the right hand side

$$b = \alpha\mathcal{B}(K(\varrho)h - \varrho) - K(\varrho)\varrho v.$$

From the classical theory we know that if $\partial\Omega$ is smooth enough and if $b \in L_p$, then there exists the unique $\varrho \in W_p^1$ satisfying (2.15) in the weak sense, such that $\int_\Omega \varrho dx = const$. Moreover,

$$\|\nabla\varrho\|_q \le \frac{c(p,\Omega)}{\epsilon}\|b\|_q. \tag{2.16}$$

In our case it is enough to see that the q-norm of b may be estimated as

$$\|b\|_q \le \alpha(\|\varrho\|_\gamma + \|h\|_\gamma) + C\|\varrho\|_\gamma \|v\|_{1,2} \le C\alpha, \tag{2.17}$$

where $\frac{2\gamma}{2-\gamma} > q > 2$ if $\gamma < 2$, otherwise q is arbitrary. Thus the observation (2.16) yields the following estimate of I_6

$$I_6 = \epsilon \int_\Omega \nabla v \nabla \varrho \Phi \leq C\alpha^{3/2} \|P(\varrho)\|_2.$$

Gathering the estimates of terms I_i for $i = 1, \ldots, 6$ one can easily see that

$$\|P(\varrho)\|_2 \leq C\alpha^{\frac{3\gamma q}{4\gamma + 2\gamma q - 4q}}, \tag{2.18}$$

where $q > 2$ and the constant C does not depend on ϵ nor m_2.

Remark 1. Observe that taking $q \to 2^+$ we obtain in the limit that the growth of L_2 norm of $P(\varrho)$ is smaller than $\alpha^{\frac{3\gamma}{4(\gamma-1)}}$.

Now our aim will be to estimate the norm of ∇v in L_q for some $q > 2$. For this purpose we will apply to system (2.7) the following lemma (for the proof, see [6] Lemma 3.3.).

Lemma 5 *Let* $1 < p < \infty$, $\Omega \in C^2$, $F \in (M_{2p/(p+2)})^*$, $\mu > 0, 2\mu + 3\nu > 0$. *Then there exists the unique* $w \in M_p$, *solution to (2.7). Moreover*

$$\|w\|_{1,p} \leq C(p, \Omega) \|F\|_{(M_{p/(p-1)})^*}.$$

If we consider the approximate momentum equation as a part of Lamé system with $w = v$ we will get the estimate for the norm of ∇v in L_q

$$\begin{aligned}\|\nabla v\|_q \leq C(\alpha \|\varrho v\|_{2q/(q+2)} + \alpha \|hg\|_{2q/(q+2)}) + \|K(\varrho)\varrho v \otimes v\|_q + \|P(\varrho)\|_q \\ + \epsilon \|\nabla v \nabla \varrho\|_{2q/(q+2)}).\end{aligned}$$

Recalling $\gamma > 1$, we can choose such $q > 2$ that $q < \gamma + 1$, then by both (2.9) and (2.11) we get

$$\alpha \|\varrho v\|_{2q/(q+2)} + \alpha \|hg\|_{2q/(q+2)} \leq C\alpha(\|\varrho v^2\|_1^{1/2} \|\varrho\|_\gamma^{\gamma/2} \|v\|_{1,2} + \|hg^2\|_1^{1/2} \|h\|_\gamma^{\gamma/2} \|g\|_{1,2}) \leq C\alpha^{3/2}.$$

By the definition of $P(\cdot)$ and the Hölder inequality we also have

$$\|K(\varrho)\varrho v \otimes v\|_q \leq C \|P(\varrho)\|_{q/\gamma}^\gamma \|v\|_{1,2}^2 \leq C\alpha \|P(\varrho)\|_{q/\gamma}^{1/\gamma}.$$

At this step there is a need to include the estimates depending on the parameter m_2, more precisely we will use

$$\begin{aligned}
&\|P(\varrho)\|_q \le \|P(\varrho)\|_\infty^{1-2/q}\|P(\varrho)\|_2^{2/q} \le C\alpha^{\frac{3\gamma}{2\gamma+\gamma q-2q}} m_2^{(1-2/q)\gamma},\\
&\epsilon\|\nabla v\nabla\varrho\|_{2q/(q+2)} \le \epsilon\|\nabla\varrho\|_q\|v\|_{1,2} \le C\alpha^{3/2},
\end{aligned}$$

where the last inequality is obtained by the same argument as in (2.17). Summarising, we have shown that $\|\nabla v\|_q \le C(m_2,\alpha)$ with a constant $C(m_2,\alpha)$ independent of ϵ. Particularly for $2<q<\gamma+1$ we have justified that

$$\|\nabla v\|_q \le C(\alpha^{3/2} + \alpha^{\frac{3\gamma}{2\gamma+\gamma q-2q}} m_2^{(1-2/q)\gamma}). \tag{2.19}$$

Before passing to the zero limit with ϵ we will compute *a priori* estimate of the vorticity

$$\omega = \mathrm{rot} v = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}.$$

Differentiating $n\cdot v=0$ at $\partial\Omega$ with respect to the length parameter and combining it with the last boundary condition in system (2.2) we obtain:

$$\omega = \left(2\chi - \frac{f}{\mu}\right) v\cdot\tau \quad \text{at } \partial\Omega.$$

Taking the rotation of $(2.2)_2$, we get

$$-\mu\Delta\omega = -\alpha\mathrm{rot}(hg-\varrho v) - \mathrm{rotdiv}(K(\varrho)\varrho v\otimes v) - \epsilon\mathrm{rot}(\nabla v\nabla\varrho). \tag{2.20}$$

Denote $\omega=\omega_1+\omega_2$, where $\omega_1,\ \omega_2$ satisfy:

$$\begin{aligned}
&-\mu\Delta\omega_1 = -\mathrm{rotdiv}(K(\varrho)\varrho v\otimes v) \quad \text{in } \Omega,\\
&\omega_1 = 0 \quad \text{at } \partial\Omega,\\
&-\mu\Delta\omega_2 = -\alpha\mathrm{rot}(hg-\varrho v) - \epsilon\mathrm{rot}(\nabla v\nabla\varrho) \quad \text{in } \Omega,\\
&\omega_2 = \left(2\chi - \frac{f}{\mu}\right) v\cdot\tau \quad \text{at } \partial\Omega.
\end{aligned}$$

For the weak solutions ω_1,ω_2 of the above problems one gets the following estimates:

$$\|\omega_1\|_p \le C\|K(\varrho)\varrho v\otimes v\|_q \le C\alpha$$

where for $p<2\gamma$, C is independent of m_2 and for $p>\gamma$, $C=C_0 m_2^{1-\gamma/q}$,

$$\|\omega_2\|_{1,p} \le C(\alpha\|hg\|_p + \alpha\|\varrho v\|_p + \epsilon\|\nabla v\nabla\varrho\|_p) + C(\Omega)\|v\cdot\tau\|_{1-1/p,p,\partial\Omega},$$

thus for $p < \frac{2\gamma}{\gamma+1}$, the Hölder inequality, the imbedding $W_2^{1/2}(\partial\Omega) \subset W_p^{1-1/p}(\partial\Omega)$ and the trace theorem imply

$$\begin{aligned}\|\omega_2\|_{1,p} &\leq C(\alpha\|hg\|_{\frac{2\gamma}{\gamma+1}} + \alpha\|\varrho v\|_{\frac{2\gamma}{\gamma+1}} + \epsilon\|\nabla\varrho\|_{2p(2-p)}\|\nabla v\|_2) + C(\Omega)\|v\|_{1,2} \\ &\leq C\alpha + C(\Omega)\alpha^{1/2},\end{aligned}$$

otherwise we must use m_2-dependent estimates of ϱ or gradient of v

$$\|\omega_2\|_{1,p} \leq C(\alpha, m_2)$$

and the dependence of m_2 is higher then linear.

3. Passage to the limit when $\epsilon \to 0^+$

This section is devoted to the passage with $\epsilon \to 0$ in system (2.2). Recall that so far we have obtained the following estimates:

$$\|\varrho_\epsilon\|_\infty \leq m_2, \quad \|v_\epsilon\|_{1,2} \leq C\alpha, \tag{3.1}$$

$$\|P(\varrho_\epsilon)\|_2 \leq C(\alpha), \tag{3.2}$$

$$\|v_\epsilon\|_{1,q} + \epsilon^{1/2}\|\nabla\varrho_\epsilon\|_2 \leq C(m_2,\alpha,q) \qquad \text{for } 1 \leq q < \infty, \tag{3.3}$$

$$\epsilon^{1/2}\|\nabla v_\epsilon \nabla\varrho_\epsilon\|_q \leq C(m_2,\alpha,q) \qquad \text{for } 1 \leq q < 2. \tag{3.4}$$

Therefore, for an appropriately chosen subsequences we have

$$\begin{aligned}&\varrho_\epsilon \rightharpoonup^* \varrho \quad \text{in } L_\infty(\Omega), \\ &P(\varrho_\epsilon) \rightharpoonup \overline{P(\varrho)} \quad \text{in } L_2(\Omega), \\ &v_\epsilon \rightharpoonup v \quad \text{in } W^1_q(\Omega), \\ &\epsilon\nabla\varrho_\epsilon \to 0 \quad \text{in } L_2(\Omega), \\ &\epsilon\nabla v_\epsilon\nabla\varrho_\epsilon \to 0 \quad \text{in } L_q(\Omega) \text{ for } 1 \leq q < 2,\end{aligned}$$

where the line over a term denotes its weak limit.
These information allow us to pass to the limit in our approximative system:

$$\begin{aligned}\alpha\left(\varrho - \overline{hK(\varrho)}\right) + \operatorname{div}(\overline{K(\varrho)\varrho v}) &= 0 \\ \alpha\,(\varrho v - hg) + \operatorname{div}(\overline{K(\varrho)}\varrho v \otimes v) - \mu\Delta v - (\mu+\nu)\nabla\operatorname{div} v + \nabla\overline{P(\varrho)} &= 0 \\ v\cdot n = 0 \quad at \quad &\partial\Omega, \\ n\cdot\mathbb{T}(v,\overline{P(\varrho)})\cdot\tau + fv\cdot\tau = 0 \quad at \quad &\partial\Omega.\end{aligned} \tag{3.5}$$

To show that we have really found the solution to our initial problem there left several questions that need to find the answer.
Firstly, if we can get rid of $K(\varrho)$ that remains at several places, i.e. if we can prove that $K(\varrho) = 1$ a.e. in Ω. This, as we shall see below, is equivalent with showing that there can be suitably chosen constant m sufficiently large but still sharply smaller than the *a priori* bound for density, such that the measure of the set

$$\{x \in \Omega : \varrho_{\epsilon_n}(x) > m\}$$

tends to zero for some subsequence $\epsilon_n \to 0^+$. Indeed, as for any smooth function η one has

$$\int_\Omega \varrho_{\epsilon_n} K(\varrho_{\epsilon_n})\eta \, dx = \int_\Omega \varrho_{\epsilon_n}\eta \, dx + \int_{\{\varrho_{\epsilon_n} > m_1\}} (K(\varrho_{\epsilon_n}) - 1)\varrho_{\epsilon_n}\eta \, dx,$$

and by taking $m < m_1$ we see that after passing to the limit the last term on the right hand side disappears, and thus we truly have

$$\lim_{\epsilon_n \to 0^+} \int_\Omega \varrho_{\epsilon_n} K(\varrho_{\epsilon_n})\eta \, dx = \int_\Omega \varrho\eta \, dx, \quad \forall \eta \in C^\infty(\Omega).$$

The next difficulty concerns the convergence in the nonlinear term i.e. is it true that $\overline{P(\varrho)} = P(\varrho)$. The positive answer can be obtained in a rather standard way, and at the stage when one already knows that $K(\varrho) = 1$ it reduces to proving the strong convergence for the density sequence.
Finally, what does the condition $(3.5)_4$ mean, in other words, in which sense is it satisfied? Having solved the two previous problem it is quite easy to see that this boundary condition can be recovered while passing to the limit in a weak formulation corresponding to the momentum equation.
Now our aim will be to justify precisely the considerations developed above. For this purpose we will adapt a technique widely used for these type of problems, more precisely we will take advantage of some properties of the effective viscous flux denoted in this paper by G.
Introducing the Helmholtz decomposition of the velocity vector field defined as:

$$v = \nabla\phi + \nabla^\perp A, \tag{3.6}$$

where the divergence-free part $\nabla^\perp A = \left(-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}\right) A$ and the gradient part ϕ are given by:

$$\left\{ \begin{array}{ll} \Delta A = \mathrm{rot} v & \text{in } \Omega \\ \nabla^\perp A \cdot n = 0 & \text{at } \partial\Omega \end{array} \right. , \quad \left\{ \begin{array}{ll} \Delta\phi = \mathrm{div} v & \text{in } \Omega, \\ \frac{\partial\phi}{\partial n} = 0 & \text{at } \partial\Omega \end{array} \right. , \tag{3.7}$$

we can transform the limit equation $(3.5)_2$ into the form:

$$\nabla G = \alpha h g - \alpha \varrho v - \operatorname{div}(\overline{K(\varrho)\varrho v} \otimes v) + \mu \Delta \nabla^{\perp} A, \tag{3.8}$$

where G is defined as

$$G = (2\mu + \nu)\Delta\phi + \overline{P(\varrho)}.$$

Note that due to (3.7), the L_2 norm of G can be estimated by

$$\|G\|_2 \leq C(\|\nabla v\|_2 + \|\overline{P(\varrho)}\|_2) \leq C(\alpha).$$

The next goal is to show that the L_∞ norm of G is bounded. It will follow from the integrability of the gradient of G with a power grater than 2. Indeed, since the mean value of G is controlled we can employ the Poincaré inequality and the Sobolev embedding theorem, which, in the case of two dimensional domain Ω, implies the desired result.

Lemma 6 *For $q > 2$ we have:*

$$\|\nabla G\|_q \leq C(\alpha, m_2). \tag{3.9}$$

Proof. By virtue of (3.8)

$$\|\nabla G\|_q \leq C\alpha\|hg\|_q + \alpha\|\varrho v\|_q + \|\operatorname{div}(\overline{K(\varrho)\varrho v} \otimes v)\|_q + \mu\|\Delta\nabla^{\perp} A\|_q. \tag{3.10}$$

A direct application of (2.4) gives rise to

$$\alpha\|hg\|_q + \alpha\|\varrho v\|_q \leq C\alpha m_2\|v\|_{1,2} \leq C\alpha^{3/2} m_2.$$

Next, the third term on the right hand side of (3.10) can be transformed by use of the limit continuity equation which together with estimates (2.10) lead to

$$\begin{aligned}\|\operatorname{div}(\overline{K(\varrho)\varrho v} \otimes v)\|_q &\leq \|\overline{K(\varrho)\varrho v} \cdot \nabla v\|_q + \alpha\|\overline{hK(\varrho)}v\|_q + \alpha\|\varrho v\|_q \\ &\leq C m_2 \|\nabla v\|_q^2 + C\alpha^{3/2} m_2,\end{aligned}$$

thus, by estimate (2.19) of $\|\nabla v\|_q$ for $q > 2$ we have

$$\|\operatorname{div}(\overline{K(\varrho)\varrho v} \otimes v)\|_q \leq C\left(\alpha^{3/2} m_2 + \alpha^3 + \alpha^{\frac{6\gamma}{2\gamma+\gamma q-2q}} m_2^{1+2(1-2/q)\gamma}\right).$$

The last term in (3.10) is bounded by the same constant, since

$$\|\Delta\nabla^{\perp}A\|_q \le \|\nabla\omega\|_q \le \alpha\|hg\|_q + \alpha\|\varrho v\|_q + \|\mathrm{div}(\overline{K(\varrho)\varrho v \otimes v})\|_q + C\|v\cdot\tau\|_{1-1/q,q,\partial\Omega},$$

where ω is a weak solution to (2.20) with a corresponding boundary condition after passing with ϵ to 0, i.e. it satisfies

$$\begin{aligned} -\mu\Delta\omega &= -\alpha\mathrm{rot}(hg - \varrho v) - \mathrm{rotdiv}(\overline{K(\varrho)\varrho v \otimes v}) \quad \text{in } \Omega, \\ \omega &= \left(2\chi - \frac{f}{\mu}\right) v\cdot\tau \quad \text{at } \partial\Omega. \quad \square \end{aligned}$$

Now we choose q such that $\gamma > 1 + 2(1-2/q)\gamma$ and simultaneously $q > 2$. Collecting all previous estimates we finally get

$$\|G\|_\infty \le C(\alpha^{3/2}m_2 + \alpha^3 + \alpha^{\frac{6\gamma}{2\gamma+\gamma q-2q}} m_2^{\gamma-\delta}), \tag{3.11}$$

with δ sufficiently small.
We will now apply the analogical decomposition for the approximative system (2.2), i.e.

$$v_\epsilon = \nabla\phi_\epsilon + \nabla^{\perp}A_\epsilon.$$

Similarly as previously this leads to relation

$$\begin{aligned} \nabla G_\epsilon &= (2\mu+\nu)\Delta\phi_\epsilon + P(\varrho_\epsilon) \\ &= \alpha hg - \alpha\varrho_\epsilon v_\epsilon - \mathrm{div}(K(\varrho_\epsilon)\varrho_\epsilon v_\epsilon \otimes v_\epsilon) - \epsilon\nabla\varrho_\epsilon\nabla v_\epsilon + \mu\Delta\nabla^{\perp}A_\epsilon. \end{aligned} \tag{3.12}$$

We are then able to prove that if $\epsilon \to 0^+$ the following lemma holds

Lemma 7 *$G_\epsilon \to G$ strongly in L_2.*

Proof. We will use the fact that if

$$\nabla(G_\epsilon - G) \rightharpoonup 0 \quad \text{weakly in } L_2, \text{ then } G_\epsilon - G \to \mathit{const.} \quad \text{strongly in } L_2.$$

This constant is equal to zero as we know that, at least for some subsequence $\epsilon_n \to 0$, we have

$$\int_\Omega (G_\epsilon - G) = \int_\Omega \Delta(\phi_\epsilon - \phi) + \int_\Omega \left(P(\varrho_\epsilon) - \overline{P(\varrho)}\right) \to 0$$

since $\frac{\partial\phi}{\partial n} = \frac{\partial\phi_\epsilon}{\partial n} = 0$ at $\partial\Omega$.
Therefore it suffices to focus on showing the weak convergence of gradients, we can write

$$\begin{aligned}\nabla(G_\epsilon - G) &= \mu\Delta\nabla^\perp(A^\epsilon - A) - \alpha(\varrho_\epsilon v_\epsilon - \varrho v)\\ &\quad - (\mathrm{div}(K(\varrho_\epsilon)\varrho_\epsilon v_\epsilon \otimes v_\epsilon) - \mathrm{div}(\overline{K(\varrho)\varrho v \otimes v})) - \epsilon\nabla v_\epsilon\nabla\varrho_\epsilon. \quad (3.13)\end{aligned}$$

The second term on the right hand side converges to 0 weakly in L_2 owing to the strong convergence of $v_\epsilon \to v$ in L_q for any $0 \le q \le \infty$ and by the boundedness of ϱ_ϵ in L_∞.
The last term converges to zero even strongly in L_2. Now, by the continuity equation, the third term may be written in the form

$$\begin{aligned}\mathrm{div}(K(\varrho_\epsilon)\varrho_\epsilon v_\epsilon \otimes v_\epsilon) - \mathrm{div}(\overline{K(\varrho)\varrho v \otimes v}) &= \alpha h K(\varrho_\epsilon)v_\epsilon - \varrho_\epsilon v_\epsilon + \epsilon\Delta\varrho_\epsilon v_\epsilon\\ &\quad + \alpha\varrho v - \alpha\overline{hK(\varrho)}v + K(\varrho_\epsilon)\varrho_\epsilon v_\epsilon \cdot \nabla v_\epsilon - \overline{K(\varrho)\varrho}v \cdot \nabla v,\end{aligned}$$

due to the argument explained above we need to justify the convergence only for two terms. Firstly note that $\epsilon\Delta\varrho_\epsilon v_\epsilon$ converges to 0 strongly in W_2^{-1}. Secondly, since $\nabla(v_\epsilon - v) \rightharpoonup 0$ weakly in L_2 we obtain the same information for $K(\varrho_\epsilon)\varrho_\epsilon v_\epsilon \cdot \nabla v_\epsilon - \overline{K(\varrho)\varrho}v \cdot \nabla v$.
In order to make sure that the first term in (3.13) also tends to 0 we observe that

$$\Delta\nabla^\perp(A^\epsilon - A) = \nabla^\perp(\omega_\epsilon - \omega), \quad (3.14)$$

and that the function $\omega_\epsilon - \omega$ satisfies the system of equations

$$\begin{aligned}-\mu\Delta(\omega_\epsilon - \omega) &= -\alpha\mathrm{rot}(\varrho_\epsilon v - \varrho v) - \mathrm{rotdiv}(K(\varrho_\epsilon)\varrho_\epsilon v_\epsilon \otimes v_\epsilon - \overline{K(\varrho)\varrho v \otimes v})\\ &\quad -\epsilon\ \mathrm{rot}(\nabla\varrho_\epsilon\nabla v_\epsilon) \quad \textit{in } \Omega\\ \omega_\epsilon - \omega &= \left(2\chi - \frac{f}{\mu}\right)(v_\epsilon - v)\cdot\tau \quad \textit{at } \partial\Omega.\end{aligned}$$

Repeating the same reasoning as in case of ω from previous section and by the above explications we can show that $\nabla(\omega_\epsilon - \omega)$ consists of two parts. One of them converges to 0 strongly in W_2^{-1} and the other converges weakly in L_2. Thus, by (3.14), we get the same for $\Delta\nabla^\perp(A^\epsilon - A)$ and therefore the proof of lemma is complete. $\square$
Provided with these information we can show the final argument for $K(\varrho)$ to be equal 1

Lemma 8 *Let $\kappa > 0$ and let m satisfy*

$$\|G\|_\infty^{1/\gamma} < m < m_1 \quad and \quad \frac{m^{\gamma+1}}{m_2} - \|G\|_\infty - 2\alpha(2\mu+\nu) \geq \kappa > 0$$

then we have

$$\lim_{\epsilon_n \to 0^+} |\{x \in \Omega : \varrho_{\epsilon_n}(x) > m\}| = 0.$$

Proof. The main difference with respect to the Lemma 4.3 from [6] is that the rate of convergence here clearly must depend on α and thus we pass with ϵ to 0 when α is set.

First observe that the assumptions of our lemma are satisfied. Indeed, as the difference $m_2 - \|G\|_\infty^{1/\gamma}$ increases with m_2. Next, we introduce a function $M(\cdot) \in C^1(\mathbb{R})$ given by

$$M(\varrho) = \begin{cases} 1 & \varrho \leq m, \\ 0 & \varrho \geq m+1, \\ \in (0,1) & \varrho \in (m, m+1), \end{cases}$$

where $M'(\varrho) < 0$ in $(m, m+1)$ and $m+1 < m_1$.

We multiply the approximate continuity equation by $M^l(\varrho_\epsilon)$ for some $l \in \mathbb{N}$ and we observe

$$\alpha \int_\Omega M^l(\varrho_\epsilon)\,(\varrho - hK(\varrho))\,dx + \int_\Omega M^l(\varrho_\epsilon)\mathrm{div}(K(\varrho)\varrho v)dx = \epsilon \int_\Omega M^l(\varrho_\epsilon)\Delta\varrho\,dx$$
$$= -\epsilon l \int_\Omega M'(\varrho_\epsilon)M^{l-1}(\varrho_\epsilon)|\nabla\varrho_\epsilon|^2\,dx \geq 0. \quad (3.15)$$

By integrating the second term on the left hand side by parts twice (the boundary terms disappear due to the definition of $M(\cdot)$) one gets

$$\int_\Omega \left(\int_0^{\varrho_\epsilon} tM^{l-1}(t)M'(t)dt \right) \mathrm{div} v_\epsilon\,dx$$
$$\geq \frac{\alpha}{l}\int_\Omega (hK(\varrho_\epsilon) - \varrho_\epsilon)\;dx + \frac{\alpha}{l}\int_\Omega (\varrho_\epsilon - hK(\varrho_\epsilon))\left(1 - M^l(\varrho_\epsilon)\right)\;dx.$$

The first therm on the right hand side cancels due to the Theorem 3. We can replace $\mathrm{div} v_\epsilon$ according to the definition of G_ϵ, then we have

$$\int_\Omega \left(\int_0^{\varrho_\epsilon} tM^{l-1}(t)M'(t)dt \right) (G_\epsilon - P(\varrho_\epsilon))\;dx$$
$$\leq -\frac{\alpha(2\mu+\nu)}{l}\int_\Omega (\varrho_\epsilon - hK(\varrho_\epsilon))\left(1 - M^l(\varrho_\epsilon)\right)\;dx.$$

Since $M'(t)$ is negative, supported in $(m, m+1)$ and $m+1 < m_1 < m_2$ the following inequality holds true

$$
\begin{aligned}
&- m \int_\Omega \left(\int_0^{\varrho_\epsilon} M^{l-1}(t) M'(t) dt \right) P(\varrho_\epsilon)\ dx \\
&\le m_2 \int_\Omega \left| - \int_0^{\varrho_\epsilon} M^{l-1}(t) M'(t) dt \right| |G_\epsilon|\ dx + \frac{\alpha(2\mu+\nu)}{l} \int_\Omega |\varrho_\epsilon - hK(\varrho_\epsilon)| \left(1 - M^l(\varrho_\epsilon)\right)\ dx.
\end{aligned}
$$

The above expression is different from 0 only for a subset of Ω, $\{\varrho_\epsilon > m\}$, thus after integration we come to the following conclusion

$$
\begin{aligned}
&\frac{m}{m_2} \int_{\{\varrho_\epsilon > m\}} (1 - M^l(\varrho_\epsilon)) P(\varrho_\epsilon)\ dx \\
&\le \int_{\{\varrho_\epsilon > m\}} (1 - M^l(\varrho_\epsilon)) |G_\epsilon|\ dx + \frac{\alpha(2\mu+\nu)}{m_2} \int_{\{\varrho_\epsilon > m\}} |\varrho_\epsilon - hK(\varrho_\epsilon)| \left(1 - M^l(\varrho_\epsilon)\right)\ dx.
\end{aligned}
\tag{3.16}
$$

Now, for each $\delta > 0$ we can find such sufficiently large number $l \in \mathbb{N}$, $l = l(\delta, \epsilon)$ that

$$
\|M^l(\varrho_\epsilon)\|_{L_2(\{\varrho_\epsilon > m\})} \le \delta, \tag{3.17}
$$

since $M(\varrho_\epsilon)$ is less then 1 for $\varrho_\epsilon > m$. This allows us to rewrite the inequality (3.16) in the following form

$$
\begin{aligned}
\frac{m^{\gamma+1}}{m_2} |\{\varrho_\epsilon > m\}| &\le \frac{m}{m_2} \|M^l(\varrho_\epsilon)\|_{L_2(\{\varrho_\epsilon > m\})} \|P(\varrho_\epsilon)\|_{L_2(\{\varrho_\epsilon > m\})} \\
&+ C(|\Omega|) \|G - G_\epsilon\|_2 + \|G\|_\infty |(\{\varrho_\epsilon > m\}| + 2\alpha(2\mu+\nu) |(\{\varrho_\epsilon > m\}|,
\end{aligned}
$$

where the term on the left is a consequence of the definition of $P(\cdot)$ and the limits of integration. Due to observation (3.17) and bound from (3.2) we may write

$$
\left(\frac{m^{\gamma+1}}{m_2} - \|G\|_\infty - 2\alpha(2\mu+\nu) \right) |\{\varrho_\epsilon > m\}| \le \frac{C(\alpha) m}{m_2} \delta + C(|\Omega|) \|G - G_\epsilon\|_2.
$$

Under our assumptions, the expression in the brackets is separated from 0. As δ may be arbitrary small and $\alpha = \mathit{const.}$, by Lemma 7, we truly have

$$
\lim_{\epsilon_n \to 0^+} |\{\varrho_{\epsilon_n} > m\}| = 0. \quad \Box
$$

This fact, as it was already mentioned before, completes justification that $K(\varrho) = 1$ a.e. in Ω.
The second problem to solve was to show that $\overline{P(\varrho)} = P(\varrho)$. For this purpose we multiply the approximate continuity equation by the function $\ln \frac{m_2}{\varrho_\epsilon+\delta}$ for $\delta > 0$ and integrate over Ω. Like in the proof of last lemma, we observe

$$\alpha \int_\Omega \ln \frac{m_2}{\varrho_\epsilon + \delta} (\varrho - h)\, dx + \int_\Omega \ln \frac{m_2}{\varrho_\epsilon + \delta} \mathrm{div}(\varrho v) dx \\ = \epsilon \int_\Omega \ln \frac{m_2}{\varrho_\epsilon + \delta} \Delta \varrho \, dx = \epsilon l \int_\Omega \frac{|\nabla \varrho_\epsilon|^2}{\varrho_\epsilon + \delta}\, dx \geq 0. \quad (3.18)$$

Similarly as previously we integrate by parts, pass with $\delta \to 0^+$, substitute G_ϵ from the definition and pass with $\epsilon \to 0^+$ to get

$$\int_\Omega \overline{P(\varrho)\varrho}\, dx + (2\mu + \nu)\alpha \int_\Omega \overline{(\varrho - h) \ln \varrho}\, dx \leq \int_\Omega G\varrho\, dx. \quad (3.19)$$

From now on we will seek to reverse the sign of above inequality. We will use the fact that the limit continuity equation works with any smooth function up to the boundary. To indicate an appropriate one we first introduce the distribution:

$$v \cdot \nabla \varrho = \mathrm{div}(\varrho v) - \varrho \mathrm{div} v.$$

Then let us recall the following lemma (for the proof consult [12]).

Lemma 9 *Let $\Omega \in C^{0,1}$, $v \in W^1_q$, $\varrho \in L_p, 1 < p, q < \infty$, $v \cdot \nabla \varrho \in L_s$, $1/s = 1/p + 1/q$. Then there exists $\varrho_n \in C^\infty(\overline{\Omega})$ such that*

$$v \cdot \nabla \varrho_n \to v \cdot \nabla \varrho \text{ in } L_s \quad \text{and} \quad \varrho_n \to \varrho \text{ in } L_p.$$

For such a ϱ_n one gets

$$\int_\Omega \mathrm{div}(\varrho_n v) dx = \int_{\partial\Omega} \varrho_n v \cdot n dS = 0,$$

thus passing with $n \to \infty$ our lemma provides that

$$\int_\Omega \varrho \mathrm{div} v dx = - \int_\Omega v \cdot \nabla \varrho dx.$$

Note that a function $\ln \frac{\delta}{\varrho_n+\delta}$ for $\delta > 0$ is an admissible test function as it follows from the proof of Lemma 9 that $0 \leq \varrho_n \leq m_2$, hence we get

$$\alpha \int_\Omega (h-\varrho) \ln \frac{\delta}{\varrho_n+\delta} = \int_\Omega \varrho v \frac{\nabla \varrho_n}{\varrho_n+\delta}.$$

We may now pass with $n \to \infty$

$$\alpha \int_\Omega (h-\varrho) \ln \frac{\delta}{\varrho+\delta} = \int_\Omega \frac{\varrho v \cdot \nabla \varrho}{\varrho+\delta}.$$

Next we also want to pass with $\delta \to 0^+$, since $\int_\Omega (\varrho - h) \ln \delta \ dx = 0$, the only difficult term is $\alpha \int_\Omega h \ln(\varrho+\delta)$, but it can be solved by the Lebesgue monotone convergence theorem, then we obtain

$$\alpha \int_\Omega h \ln \varrho = \alpha \int_\Omega \varrho \ln \varrho - \int_\Omega v \cdot \nabla \varrho = \alpha \int_\Omega \varrho \ln \varrho + \int_\Omega \varrho \mathrm{div} v.$$

Finally, recalling the definition of G one gets

$$\int_\Omega G\varrho \ dx = (2\mu+\nu)\alpha \int_\Omega (\varrho-h) \ln \varrho \ dx + \int_\Omega \overline{P(\varrho)}\varrho \ dx. \tag{3.20}$$

The information contained in (3.19), (3.20) together imply

$$\int_\Omega \overline{P(\varrho)\varrho} \, dx + (2\mu+\nu)\alpha \int_\Omega \overline{(\varrho-h)\ln\varrho} \, dx \leq (2\mu+\nu)\alpha \int_\Omega (\varrho-h)\ln\varrho \, dx + \int_\Omega \overline{P(\varrho)}\varrho \, dx. \tag{3.21}$$

The convexity of functions $\varrho \ln(\varrho)$ and $-h\ln(\varrho)$ ensure lower semicontinuity of the functional $\int_\Omega (\varrho - h)\ln(\varrho) \ dx$, in other words

$$\int_\Omega (\varrho-h)\ln\varrho \ dx \leq \int_\Omega \overline{(\varrho-h)\ln\varrho} \ dx. \tag{3.22}$$

Therefore (3.21) reduces to

$$\int_\Omega \overline{P(\varrho)\varrho} \ dx \leq \int_\Omega \overline{P(\varrho)}\varrho \ dx. \tag{3.23}$$

By the fact that ϱ^γ is a non-decreasing function of ϱ we have that $\varrho\overline{\varrho^\gamma} \leq \overline{\varrho^{\gamma+1}}$ (see [4] Theorem 10.19). On the other hand, by (3.23) we conclude $\overline{\varrho^\gamma}\varrho = \overline{\varrho^{\gamma+1}}$, which provides that

$$\varrho^\gamma = \overline{\varrho^\gamma}.$$

Since $L_\gamma(\Omega)$ is a uniformly convex Banach space for $\gamma > 1$, $\varrho_\epsilon \rightharpoonup \varrho$ weakly in L_γ and $\|\varrho_\epsilon\|_\gamma^\gamma \to \|\varrho\|_\gamma^\gamma$ we may deduce, that $\varrho_\epsilon \to \varrho$ strongly in L_γ. This in turn implies, that for some subsequence $\varrho_\epsilon \to \varrho$ a.e. in Ω. Next, condition $\|\varrho_\epsilon\|_{L_\infty}$ guarantees the uniform integrability of the sequence $\{\varrho_{\epsilon_n}\}_{n=1}^\infty$, thus the Vitali convergence theorem leads to the strong convergence of the approximate densities to the function ϱ in L_p for any $1 \leq p < \infty$.

Remark 2. The density obtained in the above procedure is bounded by some m as we could see from Lemma 8. Now, by taking κ sufficiently small and m_1, m_2 sufficiently close to m, the assumptions of Lemma 8 and estimate (3.11) imply that this m satisfies

$$m^\gamma \geq C\left(\alpha + \alpha^{3/2}m + \alpha^3 + \alpha^{\frac{6\gamma}{2\gamma+\gamma q-2q}} m^{1+2(1-2/q)\gamma}\right)$$

in particular, for $q \to 2^+$ and for $1 < \gamma < 2$ one gets

$$\|\varrho\|_\infty \leq \alpha^{\frac{3\gamma}{2(\gamma-1)^2}}.$$

Theorem 1 is now proved. $\Box$

4. Passage to the limit when $\triangle t \to 0^+$

In this section we wish to present the proof of Theorem 2, i.e. to demonstrate the passage with $\triangle t \to 0^+$. The two previous sections provide the existence of weak solutions to system (1.1)-(1.2) assuming only that $\gamma > 1$. Here, we will restrict our attention to the case when $\gamma > 2$ in order to illustrate the technique we use in more transparent way. However, as it was already mentioned, it is possible to relax this condition up to $\gamma > \frac{3}{2}$ by introducing a modification of the pressure $\delta\varrho^\Gamma$ for Γ sufficiently large that gives better integrability of the density and disappears in passage with δ to 0 as pointed out in [5].

Our approach will be based on some estimates uniform with respect to the length of time interval $\triangle t$ that we are going to gain here too. The task requires to work in the Bochner Spaces, thus let us introduce a suitable notation:

$$\left.\begin{array}{l}\hat{\phi}(x,t) = \phi^k(x)\\ \tilde{\phi}(x,t) = \phi^k(x) + (t - k\triangle t)(\frac{\phi^{k+1}-\phi^k}{\triangle t})(x)\end{array}\right\} \quad \text{if } k\triangle t \leq t < (k+1)\triangle t. \tag{4.1}$$

This converts our original system into

$$
\begin{array}{l}
\frac{\partial\tilde{\varrho}}{\partial t}+\operatorname{div}(\hat{\varrho}\hat{v})=0 \quad \text{in } \Omega,\\
\frac{\partial\widetilde{\varrho v}}{\partial t}+\operatorname{div}(\hat{\varrho}\hat{v}\otimes\hat{v})-\mu\Delta\hat{v}-(\mu+\nu)\nabla\operatorname{div}\hat{v}+\nabla\pi(\hat{\varrho})=0 \quad \text{in } \Omega,\\
\hat{v}\cdot n=0 \quad \text{at } \partial\Omega,\\
n\cdot T(\hat{v},\pi)\cdot\tau+f\hat{v}\cdot\tau=0 \quad \text{at } \partial\Omega
\end{array}
\tag{4.2}
$$

Moreover, recalling (2.1) we may now repeat the first *a priori* estimate form Section 2. Relation (2.8) now reads

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega \frac{1}{\triangle t}(\varrho^k(v^k)^2-\varrho^{k-1}(v^{k-1})^2)+\frac{1}{2}\int_\Omega \frac{1}{\triangle t}\varrho^{k-1}|v^k-v^{k-1}|^2\\
&+2\mu\int_\Omega |\mathbf{D}(v^k)|^2+\nu\int_\Omega \operatorname{div}^2 v^k+\int_{\partial\Omega} f(v^k\cdot\tau)^2+\frac{1}{\gamma-1}\frac{1}{\triangle t}\int_\Omega \big((\varrho^k)^\gamma-(\varrho^{k-1})^\gamma\big)\\
&\qquad+\frac{1}{\gamma-1}\frac{1}{\triangle t}\int_\Omega \big((\gamma-1)(\varrho^k)^\gamma+(\varrho^{k-1})^\gamma-\gamma(\varrho^k)^{\gamma-1}\varrho^{k-1}\big)=0.
\end{aligned}
\tag{4.3}
$$

Summing from $k=1$ to $k=M$, multiplying by $\triangle t$ and integrating on Ω and $(0,T)$ respectively, we obtain the analogous bounds which can be expressed in our notation in the following way:

$$\hat{\varrho},\tilde{\varrho} \text{ are bounded in } L_\infty(L_\gamma) \tag{4.4}$$

$$\hat{\varrho}\hat{v}^2,\widetilde{\varrho v^2} \text{ are bounded in } L_\infty(L_1) \tag{4.5}$$

$$\hat{v},\ \tilde{v} \text{ are bounded in } L_2(H^1) \tag{4.6}$$

$$\hat{\varrho}\hat{v},\widetilde{\varrho v} \text{ are bounded in } L_\infty(L_{\frac{2\gamma}{\gamma+1}})\cup L_2(L_r) \tag{4.7}$$

for $1\le r<\gamma$, where the last one holds as

$$\|\varrho^k v^k\|_{2\gamma/(\gamma+1)}\le\|\varrho^k\|_\gamma^{1/2}\|\varrho^k(v^k)^2\|_1^{1/2} \quad \textit{and} \quad \|\varrho^k v^k\|_r\le\|\varrho^k\|_\gamma\|v^k\|_{1,2}\ ,$$

and all the bounds depend on the initial conditions (ϱ^0,v^0), but they are independent of $\triangle t$. Furthermore (4.3) gives rise to two more estimates which are of crucial importance for the limit passage, namely to

$$\|\hat{\varrho}-\hat{\varrho}(\cdot-\triangle t)\|^\gamma_{L_\gamma(L_\gamma)}\le\triangle tC, \tag{4.8}$$

and

$$\|\hat{\varrho}|\hat{v}-\hat{v}(\cdot-\triangle t)|^2\|_{L_1(L_1)}\le\triangle tC, \tag{4.9}$$

for some constant C. Indeed, since for $\gamma > 1$ there exists a positive constant δ such that

$$(\gamma-1)(\varrho^k)^\gamma + (\varrho^{k-1})^\gamma - \gamma(\varrho^k)^{\gamma-1} K(\varrho^k)\varrho^{k-1} \geq \delta|\varrho^k - \varrho^{k-1}|^\gamma.$$

Our next aim will be to reconstruct the estimate for the norm of pressure $\pi(\hat{\varrho}) = \hat{\varrho}^\gamma$ in $L_q(\Omega \times (0,T))$ for some $q > 1$, independently of $\triangle t$. Unfortunately, as we have seen in (2.18), such an estimate might not be achievable for $q = 2$, but it turns out to work for $q = 1 + (1/\gamma)$. To show this we test each k-th momentum equation with a function Φ of the form:

$$\Phi^k = \mathcal{B}((\varrho^k) - \{\varrho^k\}) \quad \text{in } \Omega,$$

multiplying them by $\triangle t$, summing over $k = 1, \ldots, M$ and employing our notation we get

$$\int_0^T \int_\Omega \hat{\varrho}^{\gamma+1} = \int_0^T \int_\Omega (\hat{\varrho})^\gamma \{\hat{\varrho}\} - \int_0^T \int_\Omega \hat{\varrho}\hat{v}\otimes\hat{v} : \nabla\hat{\Phi} + \mu \int_0^T \int_\Omega \nabla\hat{v} : \nabla\hat{\Phi} + (\mu+\nu) \int_0^T \int_\Omega \mathrm{div}\hat{v}\,\mathrm{div}\hat{\Phi}$$
$$+ \int_0^T \int_\Omega \frac{1}{\triangle t}(\hat{\varrho}\hat{v} - \hat{\varrho}(\cdot - \triangle t)\hat{v}(\cdot - \triangle t))\hat{\Phi} = \sum_{i=1}^5 I_i. \quad (4.10)$$

We go one with estimations for each of terms separately.
(i) Since $\hat{\varrho}$ is bounded in $L_\infty(L_1)$ and $L_\infty(L_\gamma)$ one gets

$$I_1 = \int_0^T \int_\Omega (\hat{\varrho})^\gamma \{\hat{\varrho}\} = \int_0^T \frac{1}{|\Omega|} \|\hat{\varrho}\|_{L_1(\Omega)} \|\hat{\varrho}\|^\gamma_{L_\gamma(\Omega)} \leq CT.$$

(ii) The Hölder inequality, (4.6) and (4.7) imply

$$I_2 = -\int_0^T \int_\Omega \hat{\varrho}\hat{v}\otimes\hat{v} : \nabla\hat{\Phi} \leq \int_0^T \|\hat{\varrho}(\hat{v})^2\|_1 \|\hat{\varrho}\|^{1/2}_{\gamma+1} \|\nabla\hat{v}\|_2 \|\nabla\hat{\Phi}\|_{\gamma+1} \leq C(T,\Omega)\|\hat{\varrho}\|^3_{L_{\gamma+1}(L_{\gamma+1})}.$$

(iii) Due to the properties of the Bogovskii operator $\|\nabla\Phi^k\|_p \leq c(p,\Omega)\|\varrho^k\|_p$, thus

$$I_3 + I_4 = \mu \int_0^T \int_\Omega \nabla\hat{v} : \nabla\hat{\Phi} + (\mu+\nu) \int_0^T \int_\Omega \mathrm{div}\hat{v}\,\mathrm{div}\hat{\Phi} \leq C(T)\|\hat{\varrho}\|_{L_{\gamma+1}(L_{\gamma+1})}.$$

(iv) By the assumption that $\gamma > 2$ we know that $\widehat{\widetilde{\varrho v}} \in L_2(L_2)$ which is the special case of (4.7), hence by the continuity equation

$$
\begin{aligned}
I_5 &= \int_0^T \int_\Omega \frac{1}{\triangle t}(\hat{\varrho}\hat{v} - \hat{\varrho}(\cdot - \triangle t)\hat{v}(\cdot - \triangle t))\hat{\Phi} \\
&= \int_0^T \int_\Omega \frac{\partial}{\partial t}\widetilde{\varrho v \Phi} + \int_0^T \int_\Omega \frac{1}{\triangle t}\hat{\varrho}(\cdot - \triangle t)\hat{v}(\cdot - \triangle t)(\hat{\Phi}(\cdot - \triangle t) - \hat{\Phi}) \\
&\leq \sup_{0 \leq t \leq T} \int_\Omega |\widetilde{\varrho v \Phi}| + \int_0^T \|\hat{\varrho}(\cdot - \triangle t)\hat{v}(\cdot - \triangle t)\|_{L_2(\Omega)} \|\hat{\varrho}(t)\hat{v}(t)\|_{L_2(\Omega)} \\
&\qquad\qquad\qquad \leq C + \int_0^T \|\hat{\varrho}\|_{L_\gamma}^2 \|\hat{v}\|_{L_{2\gamma/(\gamma-2)}}^2 \leq C(\Omega)
\end{aligned}
$$

All together leads to desired conclusion $\|\hat{\varrho}\|_{L_{\gamma+1}(L_{\gamma+1})}^{\gamma+1} \leq C(T, \Omega)\left(1 + \|\hat{\varrho}\|_{L_{\gamma+1}(L_{\gamma+1})}^3\right)$, in particular, since $\gamma + 1 > 3$, one gets

$$
\sum_{k=1}^{M} \triangle t \|\varrho^k\|_{L_{\gamma+1}}^{\gamma+1} < C(T, \Omega). \tag{4.11}
$$

We are now in a position to validate that as $\triangle t \to 0$ the following convergences hold:

$$
[\hat{\varrho} - \hat{\varrho}(\cdot - \triangle t)], [\hat{\varrho} - \tilde{\varrho}] \to 0 \quad \text{in } L_q(L_\gamma) \tag{4.12}
$$

for $q \in [1, \infty)$

$$
[\hat{\varrho}\hat{v} - \hat{\varrho}\hat{v}(\cdot - \triangle t)],\ [\hat{\varrho}\hat{v} - \widetilde{\varrho v}] \to 0 \quad \text{in } L_q(L_r), \tag{4.13}
$$

for $\{q \in [1, \infty),\ r \in [1, \frac{2\gamma}{\gamma+1}]\} \cup \{q \in [1, 2),\ r \in [1, \gamma)\}$,

$$
[\hat{\varrho}\hat{v} \otimes \hat{v} - \widetilde{\varrho v} \otimes \hat{v}] \to 0 \quad \text{in } L_1(L_r) \cap L_q(L_1), \tag{4.14}
$$

for $q \in [1, \infty)$ $r \in [1, \gamma)$.

To see this it suffices to use estimates (4.4, 4.5, 4.6, 4.7) together with the observations (4.8) and (4.9). From what has already been written we deduce that

$$
\begin{aligned}
\hat{\varrho},\ \tilde{\varrho} &\rightharpoonup \varrho \quad \text{weakly}^* \text{ in } L_\infty(L_\gamma),\ \text{weakly in } L_{\gamma+1}((0,T) \times \Omega), && (4.15) \\
\hat{v} &\rightharpoonup v \quad \text{weakly in } L_2(H^1). && (4.16)
\end{aligned}
$$

Remark 3. Since $\tilde{\varrho}$ $\hat{\varrho}$, $\hat{v}$ satisfy continuity equation $(4.2)_1$, thus the sequence of functions $f(t) = \left(\int_\Omega \tilde{\varrho}\phi \, dx\right)(t)$ is bounded and equicontinuous in $C[0,T]$ for all $\phi \in C^\infty(\overline{\Omega})$, $\phi \cdot n = 0$ at $\partial\Omega$. Therefore, the Arzela-Ascoli theorem, the density argument and the convergence established in (4.12) yield the following

$$\hat{\varrho},\ \tilde{\varrho} \to \varrho \quad \text{in } C_{weak}(L_\gamma). \tag{4.17}$$

What is left is to show that we also have the corresponding convergence of the products $\hat{\varrho}\hat{v}$, $\hat{\varrho}\hat{v} \otimes \hat{v}$. This can be done by repeated application of the following lemma.

Lemma 10 *Let g^n, h^n converge weakly to g, h respectively in $L_{p_1}(L_{p_2})$, $L_{q_1}(L_{q_2})$ where $1 \leq p_1, p_2 \leq \infty$ and*

$$\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} = 1.$$

Let assume in addition that

$$\frac{\partial g^n}{\partial t} \ \textit{is bounded in } L_1(W_1^{-m}) \ \textit{for some } m \geq 0 \ \textit{independent of } n \tag{4.18}$$

$$\|h^n - h^n(\cdot + \xi, t)\|_{L_{q_1}(L_{q_2})} \to 0 \ \textit{as } |\xi| \to 0, \ \textit{uniformly in } n. \tag{4.19}$$

Then $g^n h^n$ converges to gh in the sense of distributions on $\Omega \times (0,T)$.

For the proof we refer the reader to [5].

For our case, since $\frac{\partial\tilde{\varrho}}{\partial t}$ is bounded in $L_\infty(W^{-1}_{2\gamma/(\gamma+1)})$ and $\frac{\partial\widetilde{\varrho v}}{\partial t}$ is bounded in $L_\infty(W_1^{-1}) + L_2(H^{-1})$, the condition (4.18) is satisfied for $g^n = \widetilde{\varrho}, \widetilde{\varrho v}$ and $m = 1$ respectively. Additionally, we have that since $h^n = \hat{v}$ is bounded in $L_2(H^1)$ the condition (4.19) also holds true.
Hereby, we get that $\widetilde{\varrho}\hat{v}$ converges weakly/weakly* in $L_\infty(L_{2\gamma/(\gamma+1)})$ and in $L_2(L_r)$ for $r \in [1,\gamma)$ to ϱv and that $\widetilde{\varrho v} \otimes \hat{v}$ converges weakly in $L_1(L_r) \cap L_q(L_1)$, for $q \in [1,\infty)$ $r \in [1,\gamma)$ to $\varrho v \otimes v$. Thus, relations (4.13) and (4.14) cause that we actually have

$$\hat{\varrho}\hat{v} \rightharpoonup \varrho v \quad \textit{weakly in } L_q(L_r) \tag{4.20}$$

for $\{q \in [1,\infty),\ r \in [1, \frac{2\gamma}{\gamma+1}]\} \cup \{q \in [1,2),\ r \in [1,\gamma)\}$,

$$\hat{\varrho}\hat{v} \otimes \hat{v} \rightharpoonup \varrho v \otimes v \quad \text{weakly in } L_1(L_r) \cap L_q(L_1), \tag{4.21}$$

for $q \in [1,\infty)$ $r \in [1,\gamma)$.
Having this we can pass to the (weak,weak*) limit as $\triangle t \to 0^+$ in system (4.2) everywhere expect in the term corresponding to the pressure:

$$\begin{array}{l} \frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho v) = 0 \quad \text{in } \Omega, \\ \frac{\partial \varrho v}{\partial t} + \operatorname{div}(\varrho v \otimes v) - \mu \Delta v - (\mu+\nu)\nabla \operatorname{div} v + \nabla \overline{\pi(\varrho)} = 0 \quad \text{in } \Omega, \\ v \cdot n = 0 \quad \text{at } \partial\Omega, \\ n\cdot T(v,\pi)\cdot \tau + f v \cdot \tau = 0 \quad \text{at } \partial\Omega \end{array} \tag{4.22}$$

The proof of strong convergence of $\pi(\varrho^k) = (\varrho^k)^\gamma$ in $L_1(\Omega \times (0,T))$ is based on some properties of the double Riesz transform, defined on the whole $\mathbb{R}^2$ in the following way

$$\mathcal{R}_{i,j} = -\partial_{x_i}(-\Delta)^{-1}\partial_{x_j},$$

where the inverse Laplacian is identified through the Fourier transform $\mathcal{F}$ and the inverse Fourier transform $\mathcal{F}^{-1}$ as

$$(-\Delta)^{-1}(v) = \mathcal{F}^{-1}\left(\frac{1}{|\xi|^2}\mathcal{F}(v)\right).$$

We will be using general results on such operators as continuity but also some facts concerning the commutators involving Riesz operators, being mostly the consequence of the Coifman-Mayer lemma [1], [3].
To take advantage of what we mentioned, there is a need to extended system (4.2) to the whole $\mathbb{R}^2$, as this is where the definition of the operator Δ_x^{-1} makes sense. We first observe that it can easily be done so for the continuity equation as $\hat{\varrho}\hat{v} \cdot n = 0$ at $\partial\Omega$, hence

$$\frac{\partial 1_\Omega \tilde{\varrho}}{\partial t} + \operatorname{div}(1_\Omega \hat{\varrho}\hat{v}) = 0. \tag{4.23}$$

For the momentum equation $(4.2)_2$ we check that

$$\begin{aligned} \hat{\varphi}(t,x) = \psi(t)\zeta(x)\tilde{\phi}, \quad \tilde{\phi} = (\nabla\Delta^{-1})[1_\Omega \tilde{\varrho}], \\ \psi \in C_c^\infty((0,T)),\ \zeta \in C_0^\infty(\overline{\Omega}), \end{aligned}$$

is an admissible test function. This can be seen as a consequence of estimates (4.4, 4.5, 4.6, 4.7, 4.11) and by the fact that the operator $\nabla_x \Delta_x^{-1}$ gives rise to the spatial regularity to its range comparing to its argument of one.

Particularly, later on we will take advantage of that for $\gamma > 2$, the embedding $W^1_\gamma(\Omega) \subset C(\overline{\Omega})$ together with Remark 3 imply

$$(\nabla\Delta^{-1})[1_\Omega\tilde{\varrho}] \to (\nabla\Delta^{-1})[1_\Omega\varrho] \quad \text{in } C([0,T]\times\overline{\Omega}). \tag{4.24}$$

Having disposed of this preliminary step, we can get the following integral identity

$$\int_0^T\int_\Omega \psi\zeta\left(\hat{\varrho}^\gamma\tilde{\varrho} - (2\mu\mathbf{D}\hat{v} + \nu\mathrm{div}\hat{v}) : \nabla\Delta^{-1}\nabla[1_\Omega\tilde{\varrho}]\right)dx\ dt = \sum_{i=1}^5 I_i \tag{4.25}$$

where

$$\begin{aligned}
I_1 &= -\int_0^T\int_\Omega \psi\zeta\left(\widetilde{\varrho v}\partial_t\tilde{\phi} + \hat{\varrho}\hat{v}\otimes\hat{v} : \nabla\Delta^{-1}\nabla[1_\Omega\tilde{\varrho}]\right)\ dx\ dt,\\
I_2 &= -\int_0^T\int_\Omega \psi\hat{\varrho}^\gamma\nabla\zeta\cdot\nabla\Delta^{-1}[1_\Omega\tilde{\varrho}]\ dx\ dt,\\
I_3 &= \int_0^T\int_\Omega \psi(2\mu\mathbf{D}\hat{v} + \nu\mathrm{div}\hat{v}) : \nabla\zeta\otimes\nabla\Delta^{-1}[1_\Omega\tilde{\varrho}]\ dx\ dt,\\
I_4 &= -\int_0^T\int_\Omega \psi\left(\hat{\varrho}\hat{v}\otimes\hat{v}\right) : \nabla\zeta\otimes\nabla\Delta^{-1}[1_\Omega\tilde{\varrho}]\ dx\ dt,\\
I_5 &= -\int_0^T\int_\Omega \partial_t\psi\zeta\widetilde{\varrho v}\cdot\nabla\Delta^{-1}[1_\Omega\tilde{\varrho}]\ dx\ dt.
\end{aligned}$$

Analogically, if we test the limit momentum equation by the corresponding test function

$$\varphi(t,x) = \psi(t)\zeta(x)\phi, \quad \phi = (\nabla\Delta^{-1})[1_\Omega\tilde{\varrho}],\ \psi\in C_c^\infty((0,T)),\ \zeta\in C_0^\infty(\overline{\Omega}), \tag{4.26}$$

we get

$$\int_0^T\int_\Omega \psi\zeta\left(\overline{\varrho^\gamma}\varrho - (2\mu\mathbf{D}v + \nu\mathrm{div}v) : \nabla\Delta^{-1}\nabla[1_\Omega\varrho]\right)dx\ dt = \sum_{i=1}^5 I_i \tag{4.27}$$

where

$$
\begin{aligned}
I_1 &= -\int_0^T \int_\Omega \psi\zeta \left(\varrho v \partial_t \phi + \varrho v \otimes v : \nabla\Delta^{-1}\nabla[1_\Omega \varrho]\right)\, dx\, dt, \\
I_2 &= -\int_0^T \int_\Omega \psi \overline{\varrho^\gamma} \nabla\zeta \cdot \nabla\Delta^{-1}[1_\Omega \varrho]\, dx\, dt, \\
I_3 &= \int_0^T \int_\Omega \psi(2\mu \mathbf{D}v + \nu \mathrm{div} v) : \nabla\zeta \otimes \nabla\Delta^{-1}[1_\Omega \varrho]\, dx\, dt, \\
I_4 &= -\int_0^T \int_\Omega \psi\left(\varrho v \otimes v\right) : \nabla\zeta \otimes \nabla\Delta^{-1}[1_\Omega \varrho]\, dx\, dt, \\
I_5 &= -\int_0^T \int_\Omega \partial_t \psi \zeta \varrho v \cdot \nabla\Delta^{-1}[1_\Omega \varrho]\, dx\, dt.
\end{aligned}
$$

The observation (4.24) together with the consequences of lemma 10 justify the convergences of the integrals $I_2, \ldots, I_5$ from (4.25) to their counterparts in (4.27). Moreover by the continuity equation $\partial_t \phi = -\mathcal{R}[1_\Omega \varrho v]$, and the same for the test function in the approximate case, thus we actually have

$$
\begin{aligned}
\lim_{\Delta t \to 0} \int_0^T \int_\Omega & \psi\zeta \left(\hat{\varrho}^\gamma \tilde{\varrho} - (2\mu \mathbf{D}\hat{v} + \nu \mathrm{div}\hat{v}) : \mathcal{R}[1_\Omega \tilde{\varrho}]\right) dx dt \\
&= \int_0^T \int_\Omega \psi\zeta \left(\overline{\varrho^\gamma} \varrho - (2\mu \mathbf{D}v + \nu \mathrm{div} v) : \mathcal{R}[1_\Omega \varrho]\right)\, dx dt \\
&+ \lim_{\Delta t \to 0} \int_0^T \int_\Omega \psi\zeta \left(\widetilde{\varrho v} \mathcal{R}[1_\Omega \hat{\varrho}\hat{v}] - \hat{\varrho}\hat{v} \otimes \hat{v} : \mathcal{R}[1_\Omega \tilde{\varrho}]\right)\, dx dt \\
&- \int_0^T \int_\Omega \psi\zeta \left(\varrho v \mathcal{R}[1_\Omega \varrho v] - \varrho v \otimes v : \mathcal{R}[1_\Omega \varrho]\right)\, dx dt. \qquad (4.28)
\end{aligned}
$$

Now we will show that the two last terms disappear when $\triangle t \to 0$. Indeed, by the properties of the double Riesz transform our task reduces to prove that

$$
\lim_{\Delta t \to 0} \int_0^T \int_\Omega \psi \left(\hat{\varrho}\hat{v} \mathcal{R}[\zeta \widetilde{\varrho v}] - \tilde{\varrho} \mathcal{R}[\zeta \hat{\varrho}\hat{v} \otimes \hat{v}]\right)\, dx dt = \int_0^T \int_\Omega \psi \varrho [v, \mathcal{R}](\zeta \varrho v)\, dx dt. \tag{4.29}
$$

By the triangle inequality applied to he left hand side and in view of (4.12), (4.13) and bounds (4.4), (4.7) we can rewrite

$$
\lim_{\Delta t \to 0} \int_0^T \int_\Omega \psi \left(\hat{\varrho}\hat{v} \mathcal{R}[\zeta \widetilde{\varrho v}] - \tilde{\varrho} \mathcal{R}[\zeta \hat{\varrho}\hat{v} \otimes \hat{v}]\right)\, dx dt = \lim_{\Delta t \to 0} \int_0^T \int_\Omega \psi \tilde{\varrho} [\hat{v}, \mathcal{R}](\zeta \hat{\varrho}\hat{v}).
$$

In order to conclude we refer to the following variant of the Coifman-Mayer lemma about the commutators.

Lemma 11 *Let $V \in W^1_2(\mathbb{R}^2)$ and $U \in L_p(\mathbb{R}^2)$ for $1 < p < \infty$ be given, then for $\frac{1}{s} = \frac{1}{2} + \frac{1}{p}$*

$$\|[V, \mathcal{R}](U)\|_{W^1_s(\mathbb{R}^2)} \leq C(s,p)\|V\|_{W^1_2(\mathbb{R}^2)}\|U\|_{L_p(\mathbb{R}^2)}.$$

Applying this lemma to $V = \hat{v}(t,\cdot)$, $U = \zeta\hat{\varrho}\hat{v}(t,\cdot)$ with $p < \gamma$ we obtain that $[\hat{v},\mathcal{R}](\zeta\hat{\varrho}\hat{v})$ is bounded in $L_1(W^1_s)$ with $\frac{1}{s} > \frac{1}{2} + \frac{1}{\gamma}$, from which it can be deduced that

$$\tilde{\varrho}[\hat{v},\mathcal{R}](\zeta\hat{\varrho}\hat{v}) \rightharpoonup \overline{\varrho[v,\mathcal{R}](\zeta\varrho v)} \quad \text{weakly in } L_1((0,T)\times\Omega). \tag{4.30}$$

In accordance with relations (4.15), (4.16) and by the fact that the operator $\mathcal{R}$ is continuous and linear from $L_p(\mathbb{R}^N)$ to $L_p(\mathbb{R}^N)$ for any $1 < p < \infty$ we are allowed to repeat the procedure used to get (4.20) and (4.21) to justify that for $q < \gamma$ we have

$$[\hat{v},\mathcal{R}](\zeta\hat{\varrho}\hat{v}) \rightharpoonup [v,\mathcal{R}](\zeta\varrho v) \quad \text{weakly in } L_1(L_q). \tag{4.31}$$

Now, the last thing that remains to prove requires to apply the Lions argument from Lemma 10 with $g^n = \tilde{\varrho}$ and $h^n = [\hat{v},\mathcal{R}](\zeta\hat{\varrho}\hat{v})$. In view of boundedness of $[\hat{v},\mathcal{R}](\zeta\hat{\varrho}\hat{v})$ in $L_1(W^1_s)$ with $\frac{1}{s} > \frac{1}{2} + \frac{1}{\gamma}$, of $\tilde{\varrho}$ in $L_\infty(L_\gamma)$ and of $\frac{\partial\tilde{\varrho}}{\partial t}$ in $L_\infty(W^{-1}_{2\gamma/(\gamma+1)})$ one can easily verify that the assumptions of Lemma 10 are satisfied for $m = 1$, $p_1 = \infty$, $p_2 = \gamma$ and $q_1 = 1$, $q_2 = \frac{\gamma}{\gamma-1}$, hence we certainly have

$$\tilde{\varrho}[\hat{v},\mathcal{R}](\zeta\hat{\varrho}\hat{v}) \to \varrho[v,\mathcal{R}](\zeta\varrho v) \tag{4.32}$$

in the sense of distributions on $(0,T)\times\Omega$.
Now, this convergence reduces (4.28) to

$$\lim_{\Delta t\to 0}\int_0^T\int_\Omega \psi\zeta\left(\hat{\varrho}^\gamma\tilde{\varrho} - (2\mu\mathbf{D}\hat{v} + \nu\mathrm{div}\hat{v}) : \mathcal{R}[1_\Omega\tilde{\varrho}]\right)dxdt$$
$$= \int_0^T\int_\Omega \psi\zeta\left(\overline{\varrho^\gamma}\varrho - (2\mu\mathbf{D}v + \nu\mathrm{div}v) : \mathcal{R}[1_\Omega\varrho]\right)\, dxdt \tag{4.33}$$

Observe that by the fact that $\zeta \in C_0^\infty(\overline{\Omega})$ we may integrate by parts the second term on the left hand side and we will get

$$\int_0^T \int_\Omega \psi\zeta(2\mu\mathbf{D}\hat{v} + \nu\mathrm{div}\hat{v}) : \mathcal{R}[1_\Omega\tilde{\varrho}]dxdt = \int_0^T \int_\Omega \psi(2\mu+\nu)\mathrm{div}\hat{v}\tilde{\varrho}\; dxdt$$
$$+ \int_0^T \int_\Omega \psi\Big(\mathcal{R} : [\zeta(2\mu\mathbf{D}\hat{v} + \nu\mathrm{div}\hat{v})] - \zeta\; \mathcal{R} : [2\mu\mathbf{D}\hat{v} + \nu\mathrm{div}\hat{v}]\Big)\tilde{\varrho}\; dxdt \quad (4.34)$$

and similarly for the corresponding term on the right hand side of (4.33). As a direct consequence of smoothness of ζ one gets that after passage we may finally write

$$\int_0^T \int_\Omega \psi\zeta\left(\overline{\varrho^\gamma\varrho} - \overline{\varrho\mathrm{div}_x v}\right) dxdt = \int_0^T \int_\Omega \psi\zeta\left(\overline{\varrho^\gamma}\varrho - \varrho\mathrm{div}_x v\right)\; dxdt,$$

and since the choice of functions ψ and ζ was arbitrary we have

$$\overline{\varrho^\gamma\varrho} - \overline{\varrho\mathrm{div}_x v} = \overline{\varrho^\gamma}\varrho - \varrho\mathrm{div}_x v \quad \text{a.e. in } (0,T)\times\Omega.$$

The monotonicity of the function $f(x) = x^\gamma$ yields $\overline{\varrho^\gamma}\varrho \leq \overline{\varrho^\gamma\varrho}$ and so we conclude this reasoning with the important observation

$$\varrho\mathrm{div}_x v \leq \overline{\varrho\mathrm{div}_x v}. \quad (4.35)$$

Next, we take $\delta > 0$ and multiply the discrete version of the continuity equation by $\ln(\varrho^k + \delta)$. After integrating by parts over Ω one get

$$\frac{1}{\triangle t}\int_\Omega (\varrho^k - \varrho^{k-1})\ln(\varrho^k + \delta) - \int_\Omega \varrho^k v^k \frac{\nabla\varrho^k}{\varrho^k + \delta} = 0.$$

By the Lebesgue monotone convergence theorem we can pass with $\delta \to 0^+$ and then integrate by parts once more to find

$$\frac{1}{\triangle t}\int_\Omega (\varrho^k - \varrho^{k-1})\ln(\varrho^k) + \int_\Omega \mathrm{div}(v^k)\varrho^k = 0.$$

Recall that due to Theorem 1 we have $\int_\Omega \varrho^k = \int_\Omega \varrho^{k-1}$, thus whereas $x\ln(x)$ is a convex function above equality may be changed into

$$\frac{1}{\triangle t}\int_\Omega \left[\varrho^k\ln(\varrho^k) - \varrho^{k-1}\ln(\varrho^{k-1})\right] dx + \int_\Omega \mathrm{div}(v^k)\varrho^k \leq 0. \quad (4.36)$$

Now, we sum (4.36) from $k = 1$ to $k = M$, multiply by Δt and pass to the limit to get

$$\int_\Omega \overline{\varrho \ln(\varrho)}(T)\ dx + \int_0^T \int_\Omega \overline{\varrho \mathrm{div} v}\ dxdt \leq \int_\Omega \varrho \ln(\varrho)(0)\ dx, \tag{4.37}$$

For the limit momentum equation, we take advantage of the fact that it is satisfied in the whole space in sense of distributions, thus the solution is automatically a renormalised solution, i.e. by an appropriate renormalization we may get

$$\int_\Omega \varrho \ln \varrho(T)\ dx + \int_0^T \int_\Omega \varrho \mathrm{div} v\ dxdt = \int_\Omega \varrho \ln \varrho(0)\ dx. \tag{4.38}$$

Consequently, the two results (4.37) and (4.38) give rise to

$$\int_\Omega \overline{\varrho \ln(\varrho)}(T)\ dx + \int_0^T \int_\Omega \overline{\varrho \mathrm{div} v}\ dxdt \leq \int_\Omega \varrho \ln \varrho(T)\ dx + \int_0^T \int_\Omega \varrho \mathrm{div} v\ dxdt.$$

which joined with (4.35) provides the desired information, namely

$$\varrho \ln \varrho = \overline{\varrho \ln \varrho},$$

and finally, by the convexity of function $x \ln x$, we certainly have

$$\hat{\varrho} \to \varrho \quad \text{ a.e. in } (0, T) \times \Omega$$

that completes the proof of Theorem 2. $\Box$

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