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PAPER

Gaussian entanglement revisited

Ludovico Lami[1] , **Alessio Serafini**[2] and **Gerardo Adesso**[3]

1 Física Teòrica: Informació i Fenòmens Quàntics, Departament de Física, Universitat Autònoma de Barcelona, E-08193 Bellaterra (Barcelona), Spain
2 Department of Physics & Astronomy, University College London, Gower Street, London WC1E 6BT, United Kingdom
3 Centre for the Mathematics and Theoretical Physics of Quantum Non-Equilibrium Systems, School of Mathematical Sciences, The University of Nottingham, University Park, Nottingham NG7 2RD, United Kingdom
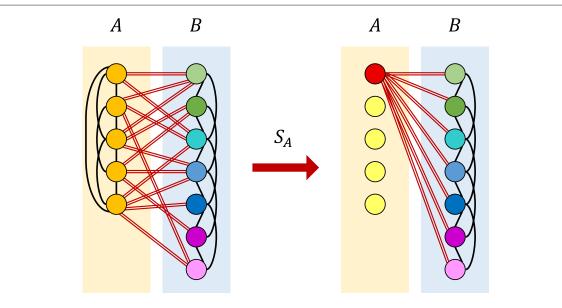
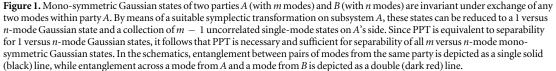**E-mail:** ludovico.lami@gmail.com

Abstract

We present a novel approach to the separability problem for Gaussian quantum states of bosonic continuous variable systems. We derive a simplified necessary and sufficient separability criterion for arbitrary Gaussian states of m versus n modes, which relies on convex optimisation over marginal covariance matrices on one subsystem only. We further revisit the currently known results stating the equivalence between separability and positive partial transposition (PPT) for specific classes of Gaussian states. Using techniques based on matrix analysis, such as Schur complements and matrix means, we then provide a unified treatment and compact proofs of all these results. In particular, we recover the PPT-separability equivalence for: (i) Gaussian states of 1 versus n modes; and (ii) isotropic Gaussian states. In passing, we also retrieve (iii) the recently established equivalence between separability of a Gaussian state and and its complete Gaussian extendability. Our techniques are then applied to progress beyond the state of the art. We prove that: (iv) Gaussian states that are invariant under partial transposition are necessarily separable; (v) the PPT criterion is necessary and sufficient for separability for Gaussian states of m versus n modes that are symmetric under the exchange of any two modes belonging to one of the parties; and (vi) Gaussian states which remain PPT under passive optical operations can not be entangled by them either. This is not a foregone conclusion per se (since Gaussian bound entangled states do exist) and settles a question that had been left unanswered in the existing literature on the subject. This paper, enjoyable by both the quantum optics and the matrix analysis communities, overall delivers technical and conceptual advances which are likely to be useful for further applications in continuous variable quantum information theory, beyond the separability problem.

1. Introduction

Gaussian states have played a privileged role in quantum optics and bosonic field theories, essentially since the very early steps of such theories, due to their ease of theoretical description and relevance to experimental practice. Over the last twenty years, such a privilege has carried over to quantum information science, where Gaussian states form the core of the 'continuous variable' toolbox [1–6]. The analysis of quantum Gaussian states from the information theoretic standpoint brought up new subtle elements and much previously unknown insight into their structure. For, while Gaussian dynamics may essentially be dealt with entirely at the phase space level (typically by normal mode decomposition, so that Gaussian dynamics are often trivialised as 'quasi-free' in field theory), the analysis of quantum information properties requires one to confront the Hilbert space description of the quantum states. Hence, while Gaussian dynamics might well be exactly solvable with elementary tools, the properties of Gaussian states related to the Hilbert space and tensor product structures are far from being equally transparent.

Figure 1. Mono-symmetric Gaussian states of two parties A (with m modes) and B (with n modes) are invariant under exchange of any two modes within party A. By means of a suitable symplectic transformation on subsystem A, these states can be reduced to a 1 versus n-mode Gaussian state and a collection of $m - 1$ uncorrelated single-mode states on A's side. Since PPT is equivalent to separability for 1 versus n-mode Gaussian states, it follows that PPT is necessary and sufficient for separability of all m versus n-mode mono-symmetric Gaussian states. In the schematics, entanglement between pairs of modes from the same party is depicted as a single solid (black) line, while entanglement across a mode from A and a mode from B is depicted as a double (dark red) line.

The problem of *Gaussian separability*, that is, determining whether a bipartite Gaussian state is separable or entangled [3], exemplifies such a situation very well. Necessary and sufficient conditions for Gaussian separability in the general m versus n-mode case are available [7, 8], yet they are recast in terms of convex optimisation problems whose solution (albeit numerically efficient) does not admit, in general, a closed analytical form. For non-separable states, a closely related question is whether their entanglement is distillable or bound [9]. In the case of arbitrary multimode bipartite Gaussian states, while entanglement can never be distilled by Gaussian operations alone [10–12], it is known that entanglement distillability under general local operations and classical communications is equivalent to violation of the positivity of the partial transposition (PPT) criterion [8, 13]. In turn, the PPT criterion, which is as well efficiently computable at the level of covariance matrices for Gaussian states, and is in general only necessary for separability [14], has been proven to be also sufficient in some important cases, notably when the bipartite Gaussian state under examination pertains to a 1 versus n-mode system [7, 15], when it is 'bi-symmetric' [16], i.e. invariant under local permutations of any two modes within any of the two subsystems, and when it is 'isotropic', i.e. with a fully degenerate symplectic spectrum of its covariance matrix [17–19]. Outside of these special families, bound entangled Gaussian states can occur, as first shown in the 2 versus 2-mode case in [7].

In this paper we provide significant advances towards the characterisation of separability and entanglement distillability in Gaussian quantum states. On one hand, we revisit the existing results, providing in particular a new compact proof for the equivalence between PPT and separability in 1 versus n-mode Gaussian states, which encompasses the seminal 1 versus 1-mode case originally tackled by Simon [15] and its extension settled by Werner and Wolf [7]. Key to our proof is the intensive use of Schur complements, which have enjoyed applications in various areas of (Gaussian) quantum information theory [8, 10–12, 18, 20–23], and—as further reinforced by this work—may be appreciated as a mathematical cornerstone for continuous variable quantum technology.

On the other hand, we derive a number of novel results. In particular, a marginal extension of the techniques applied in the aforementioned proof allow us to prove that Gaussian states invariant under partial transposition are necessarily separable, a result previously known only for the partial transposition of qubit subsystems [24]. We then show that the 1 versus n-mode PPT-separability equivalence can be further extended to a class of arbitrary bipartite multimode Gaussian states that we call 'mono-symmetric', i.e., invariant under local exchanges of any two modes on one of the two subsystems (see figure 1). This result, which (to the best of our knowledge) is observed and proven here for the first time, generalises the case of bi-symmetric states studied in [16], providing as a byproduct a simplified proof for the latter as well.

As for isotropic Gaussian states, in the traditional approach the sufficiency of PPT for their separability follows from a well known 'mode-wise' decomposition of pure-state covariance matrices [17–19], and from the fact that the covariance matrix of an isotropic state is just a multiple of the covariance matrix of a pure Gaussian

state. Here, we derive the sufficiency of the PPT criterion for isotropic Gaussian states following a completely different and arguably more direct approach. Main ingredients of this novel proof are advanced matrix analysis tools such as the operator geometric mean, already found to be useful in the context of quantum optics [23].

We also consider the well known class of Gaussian passive operations (i.e., the ones that preserve the average number of excitations of the input state, such as beam splitters and phase shifters), which play a central role in quantum optics [1, 5, 6], and we prove that a bipartite Gaussian state that always remains PPT under such a set of operations must also always stay separable. This novel result complements the seminal study of [25], in that the latter only considered the possibility of turning a PPT state into a non-PPT one through passive operations— essentially, the question of generating distillable entanglement—which is not the same as the question of generating inseparability, because Gaussian PPT bound entangled states do exist [7]. Here we settle the latter, more general and fundamental question.

All the previous results enable us to substantially extend the range of equivalence between Gaussian separability and PPT in contexts of strong practical relevance. Last but not the least—in fact, first and foremost in the paper—we address the separability problem directly, and derive a simplified necessary and sufficient condition for Gaussian separability. For a bipartite state, this requires convex optimisation over marginal covariance matrices on one subsystem only, yielding a significant simplification over the existing criteria, which instead require optimisation on both parties [7, 8, 20].

This paper is organised as follows: in section 2, the definition and basic properties of Schur complements and matrix means that will be used in our derivations are recalled, and the Gaussian notation is set; section 3 contains our main novel finding: a simplified necessary and sufficient condition for Gaussian separability in the general m versus n-mode case; section 4 contains a new proof of the sufficiency of the PPT criterion for separability of 1 versus n-mode Gaussian states achieved in a few, swift Schur complements' manipulation; section 5 shows that invariance under partial transposition implies separability; in section 6 we prove that mono-symmetric states can be reduced, under local unitary operations, to 1 versus n-mode states, which implies that the PPT condition is sufficient for them too (see figure 1); in section 7 we provide the reader with a new proof of the sufficiency of PPT for separability of isotropic Gaussian states, not relying on their mode-wise decomposition; section 8 contains our novel analysis of entanglement generation under passive operations. Section 9 concludes the paper with a brief summary and some future perspectives related to this work.

2. The toolbox: Schur complements, matrix means and Gaussian states

One of the messages of the present paper is to lend further support to the fact that methods based on Schur complements and matrix means can be successfully exployed to derive fundamental results in continuous variable quantum information, following a streak of applications to various contexts including separability, distillability, steerability, entanglement monogamy, characterisation of Gaussian maps, and related problems [8, 10–12, 18, 20–23]. As a divertissement to set the stage, let us present a compact, essential compendium of such methods.

2.1. Schur complements
Given a square matrix M partitioned into blocks as

$$M = \begin{pmatrix} A & X \\ Y & B \end{pmatrix},\tag{1}$$

the *Schur complement* of its (square, invertible) principal submatrix A, denoted by M/A, is defined as

$$M/A := B - YA^{-1}X.\tag{2}$$

A useful reference on Schur complements is the monograph [26]. Here we limit ourselves to stress some of the properties we will make use of in the present paper. As it turns out, Schur complements are the answer [42] to a number of questions that arise pretty naturally in matrix analysis. Many of these applications stem from the fact that the positivity conditions of 2×2 Hermitian block matrices can be easily written in terms of Schur complements.

Lemma 1. *Consider a Hermitian matrix*

$$H = \begin{pmatrix} A & X \\ X^\dagger & B \end{pmatrix}.\tag{3}$$

Then H is strictly positive definite ($H > 0$) if and only if $A > 0$ and $H/A = B - X^\dagger A^{-1}X > 0$. Then, by taking suitable limits, H is semidefinite positive ($H \geqslant 0$) if and only if $A \geqslant 0$ and $B - X^\dagger(A + \varepsilon\mathbb{1})^{-1}X \geqslant 0$ for all $\varepsilon > 0$.

A consequence of this result that will be relevant to us is the following.

Corollary 2. *Let H be a Hermitian matrix partitioned as in* (3). *Then, if* $A > 0$,

$$H/A = \sup\{\tilde{B} = \tilde{B}^\dagger : H > 0 \oplus \tilde{B}\}. \tag{4}$$

Here we mean that the matrix set on the right-hand side has a supremum (i.e. a minimum upper bound) with respect to the Löwner partial order $(X \geqslant Y$ *if and only if* $X - Y$ *is positive semidefinite), and that this supremum is given by the Schur complement on the left-hand side.*

We note in passing that from the above variational representation it follows immediately that H/A is monotone and concave in $H > 0$.

2.2. Matrix means

Somehow related to Schur complements are the so-called matrix means. As one might expect from their name, these are functions taking two positive matrices as inputs and yielding another positive matrix as output. For an excellent introduction to this topic, we refer the reader to [27, chapter 4]. Here, we review only some basic facts that we will find useful throughout the paper. Given two strictly positive matrices $A, B > 0$, the simplest mean one can define is the *arithmetic mean* $(A + B)/2$, whose generalisation from scalars to matrices does not present difficulties. Another easily defined object is the *harmonic mean* [28, 29], denoted by $A!B$ and given by

$$A!B := \left(\frac{A^{-1} + B^{-1}}{2}\right)^{-1}. \tag{5}$$

Incidentally, the harmonic mean can be also defined as a Schur complement, with the help of the identity $A!B = A - A(A + B)^{-1}A = \begin{pmatrix} A & A \\ A & A + B \end{pmatrix}/(A + B)$, which immediately implies that $A!B$ is monotone and jointly concave in A and B, i.e. concave in the pair (A, B).

The least trivially defined among the elementary means is undoubtedly the *geometric mean* $A \# B$ between strictly positive matrices $A, B > 0$ [29, 30], which can be constructed as

$$A \# B := \max\{X = X^\dagger : A \geqslant XB^{-1}X\}, \tag{6}$$

where the above maximisation is with respect to the Löwner partial order, and the fact that the particular set of matrices we chose admits an absolute maximum is already non-trivial. With a bit of work one can show that $A \# B$ is explicitly given by

$$A \# B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}. \tag{7}$$

Having multiple expressions for a single matrix mean is always useful, as some properties that are not easy to prove within one formulation may become apparent when a different approach is taken. For instance, the fact that $A \# B$ is covariant under congruences, i.e. $(MAM^\dagger) \# (MBM^\dagger) = M(A \# B)M^\dagger$ for all invertible M, is far from transparent if one looks at (7), while it becomes almost obvious when (6) is used. On the contrary, the fact that $A \# B = (AB)^{1/2}$ when $[A, B] = 0$ is not easily seen from (6), but it is readily verified employing (7).

As it happens with scalars, the inequality

$$A!B \leqslant A \# B \leqslant \frac{A + B}{2} \tag{8}$$

holds true for all $A, B > 0$. In view of the above inequality, it could be natural to wonder, how the geometric mean between the leftmost and rightmost sides of (8) compares to $A \# B$. That this could be a fruitful thought is readily seen by asking the same question for real numbers. In fact, when $0 < a, b \in \mathbb{R}$ it is elementary to verify that $\sqrt{ab} = \sqrt{\frac{a+b}{2}\left(\frac{1/a+1/b}{2}\right)^{-1}}$. Our first result is a little lemma extending this to the non-commutative case. We were not able to find a proof in the literature, so we provide one.

Lemma 3. *For* $A, B > 0$ *strictly positive matrices, the identity*

$$A \# B = \left(\frac{A + B}{2}\right) \# (A!B) \tag{9}$$

holds true.

Proof. We start by defining $\tilde{A} := (A + B)^{-1/2}A(A + B)^{-1/2}$, $\tilde{B} := (A + B)^{-1/2}B(A + B)^{-1/2}$. It is easy to see that $[\tilde{A}, \tilde{B}] = 0$, for instance because $\tilde{A} + \tilde{B} = (A + B)^{-1/2}(A + B)(A + B)^{-1/2} = \mathbf{1}$. Therefore, the identity $\tilde{A} \# \tilde{B} = (\tilde{A}\tilde{B})^{1/2}$ holds. Now, on the one hand the congruence covariance of the geometric mean implies that

$$\tilde{A} \# \tilde{B} = ((A + B)^{-1/2}A(A + B)^{-1/2}) \# ((A + B)^{-1/2}B(A + B)^{-1/2}) = (A + B)^{-1/2}(A \# B)(A + B)^{-1/2}.$$

On the other hand,

$$\tilde{A}\tilde{B} = (A + B)^{-1/2}A(A + B)^{-1}B(A + B)^{-1/2}$$

$$= (A + B)^{-1/2}(B^{-1}(A + B)A^{-1})^{-1}(A + B)^{-1/2} = \frac{1}{2}(A + B)^{-1/2}(A!B)(A + B)^{-1/2}.$$

Putting all together, we see that

$$(A + B)^{-1/2}(A\#B)(A + B)^{-1/2} = \tilde{A}\#\tilde{B} = (\tilde{A}\tilde{B})^{1/2} = \frac{1}{\sqrt{2}}((A + B)^{-1/2}(A!B)(A + B)^{-1/2})^{1/2}.$$

Conjugating by $(A + B)^{1/2}$, we obtain

$$A\#B = \frac{1}{\sqrt{2}}(A + B)^{1/2}((A + B)^{-1/2}(A!B)(A + B)^{-1/2})^{1/2}(A + B)^{1/2} = \left(\frac{A + B}{2}\right)\#(A!B), \quad (10)$$

where the last step is an application of (7). □

2.3. Gaussian states

In the remainder of this section, we provide a brief introduction to the main concepts of the Gaussian formalism. Quantum continuous variables just describe quantum mechanics applied to an infinite-dimensional Hilbert space equipped with position and momentum operators x_j, p_k ($j, k = 1, \ldots, n$) satisfying the so-called canonical commutation relations $[x_j, p_k] = \mathrm{i}\delta_{jk}$ (in natural units, $\hbar = 1$). Such a Hilbert space describes, for instance, a collection of n quantum harmonic oscillators, or modes of the electromagnetic radiation field. The operators x_j, p_k are often grouped together to form a single vector of $2n$ operators $r := (x_1, p_1, \ldots, x_n, p_n)$. The canonical commutation relations then take the form

$$[r, r^T] = \mathrm{i}\,\Omega := \mathrm{i}\,\omega^{\oplus n}, \quad \omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (11)$$

An important object one can form is the displacement operator. For any $z \in \mathbb{R}^{2n}$, we define

$$D_z := \mathrm{e}^{\mathrm{i}z^T\Omega r}. \quad (12)$$

It turns out that Nature has a special preference for quadratic Hamiltonians. A prominent example is the free-field Hamiltonian $\mathcal{H}_0 = \frac{1}{2}r^T r$. Not surprisingly, thermal states of quadratic Hamiltonians are extremely easily produced in the lab, in fact so easily that they deserve a special name: Gaussian states [1–6]. As the name suggests, they can be fully described by a real displacement vector $w \in \mathbb{R}^{2n}$ and a real, $2n \times 2n$ *quantum covariance matrix* (QCM) V, defined respectively as $w = \langle r \rangle$ and $V = \langle \{r - w, r^T - w^T\} \rangle$. By QCM we mean a real, symmetric, strictly positive matrix $V > 0$ that moreover satisfies the Heisenberg uncertainty relation [31]

$$V + \mathrm{i}\Omega \geqslant 0. \quad (13)$$

Note that (13) can equivalently be written as $V - \mathrm{i}\Omega \geqslant 0$ upon applying transposition (as $V^T = V$, $\Omega^T = -\Omega$).

The Gaussian state $\rho^G(V, w)$ with QCM V and displacement vector w admits the representation

$$\rho^G(V, w) = \int \frac{\mathrm{d}^{2n}u}{(2\pi)^n}\,\mathrm{e}^{-\frac{1}{4}u^T V u - \mathrm{i}w^T r}D_{\Omega r}, \quad (14)$$

which justifies the alternative definition of Gaussian states as the continuous variable states associated with a Gaussian characteristic function.

Clearly, linear transformations $r \to Sr$ that preserve the commutation relations (11) play a special role within this framework. Any such transformation is described by a *symplectic* matrix, i.e. a matrix S with the property that $S\Omega S^T = \Omega$. Symplectic matrices form a non-compact, connected Lie group that is additionally closed under transposition, and is typically denoted by $\mathrm{Sp}(2n, \mathbb{R})$ [32]. The importance of these operations arises from the fact that for any symplectic S there is a unitary evolution U_S on the Hilbert space such that $U_S^\dagger r U_S = Sr$. Most importantly, such a unitary is the product of a finite number of factors $\mathrm{e}^{\mathrm{i}\mathcal{H}_Q}$, where $\mathcal{H}_Q$ is a quadratic Hamiltonian, and as such it can be easily implemented in laboratory. Under conjugation by U_S, Gaussian states transform as

$$U_S^\dagger \rho^G(V, w) U_S = \rho^G(SVS^T, Sw). \quad (15)$$

It turns out that all Gaussian states can be brought into a remarkably simple normal form via unitary transformations induced by quadratic Hamiltonians. In fact, a theorem by Williamson [33, 34] implies that for all strictly positive matrices $V > 0$ there is a symplectic transformation S and a diagonal matrix $N > 0$ such that

$$S^{-1}VS^{-T} = N := \mathrm{diag}(\nu_1, \nu_1, \ldots, \nu_n, \nu_n). \quad (16)$$

The diagonal elements $\nu_i > 0$, each taken with multiplicity one, are called symplectic eigenvalues of V, and are uniquely determined by V (up to their order, which can be assumed decreasing by convention with no loss of

generality). Accordingly, we will refer to $\vec{\nu} = (\nu_1, \nu_2, \ldots, \nu_n)$ as the *symplectic spectrum* of V. Notably, Heisenberg uncertainty relation (13) can be conveniently restated as $N \geqslant \mathbf{1}$, or equivalently $\nu_i \geqslant 1$ for all $i = 1, \ldots, n$. A Gaussian state $\rho^G(V, w)$ can be shown to be pure if and only if all of its symplectic eigenvalues are equal to 1, which corresponds to the matrix equality $V\Omega V\Omega = -\mathbf{1}$. Correspondingly, a QCM V satisfying $\nu_j = 1$ for all $j = 1, \ldots, n$ (or equivalently $\det V = 1$) will be called a *pure* QCM. Note that pure QCMs V are themselves symplectic matrices, $V = (S^T S)^{-1} \in \mathrm{Sp}(2n, \mathbb{R})$, and are the extremal elements in the convex set of QCMs.

Finally, note that displacement vector w is often irrelevant since it can be made to vanish by local unitaries, resulting from the action of the displacement operator of (12) on each individual mode. Since all the physically relevant informational properties such as purity and entanglement are invariant under local unitaries, all the results we are going to present will not depend on the first moments. Therefore, in what follows, we will completely specify any Gaussian state under our investigation as $\rho^G(V)$ in terms of its QCM V alone.

3. Simplified separability criterion for *M* versus *N*-mode Gaussian states

The QCM V_{AB} of a Gaussian state ρ_{AB}^G pertaining to a $(m + n)$-mode bipartite system AB can be naturally written in block form according to the splitting between the subsystems A and B:

$$V_{AB} = \begin{pmatrix} V_A & X \\ X^T & V_B \end{pmatrix}. \tag{17}$$

According to the same splitting, the matrix Ω appearing in (11) takes the form

$$\Omega_{AB} = \begin{pmatrix} \Omega_A & 0 \\ 0 & \Omega_B \end{pmatrix} = \Omega_A \oplus \Omega_B, \tag{18}$$

with $\Omega_A = \omega^{\oplus m}$ and $\Omega_B = \omega^{\oplus n}$.

The entanglement properties of a bipartite Gaussian state can thus be conveniently translated at the level of QCMs. Recall that, in general, a bipartite quantum state ρ_{AB} is separable if and only if it can be written as a convex mixture of product states, $\rho_{AB} = \sum_k p_k(\sigma_{kA} \otimes \tau_{kB})$, with p_k being probabilities [35]. For a Gaussian state ρ_{AB}^G of a bipartite continuous variable system, we have then the following.

Lemma 4 (**Proposition 1 in** [7]). *A Gaussian state $\rho_{AB}^G(V_{AB})$ with $(m + n)$-mode QCM V_{AB} is separable if and only if there exist an m-mode QCM $\gamma_A \geqslant \mathrm{i}\Omega_A$ and an n mode QCM $\gamma_B \geqslant \mathrm{i}\Omega_B$ such that*

$$V_{AB} \geqslant \gamma_A \oplus \gamma_B. \tag{19}$$

In view of the above result, a QCM V_{AB} satisfying (19) for some marginal QCMs γ_A, γ_B will itself be called *separable* from now on. The criterion in (19) is necessary and sufficient for separability of QCMs, and can be evaluated numerically via convex optimisation [8, 20], however such optimisation runs over both marginal QCMs, hence scaling (polynomially) with both m and n.

The first main result of this paper is to show that the necessary and sufficient separability condition (19), for any m and n, can be further simplified. This result is quite neat and of importance in its own right. In particular, it allows us to recast the Gaussian separability problem as a convex optimisation over the marginal QCM of *one* subsystem only (say A without loss of generality), resulting in an appreciable reduction of computational resources, especially in case party A comprises a much smaller number of modes than party B.

Theorem 5 (**Simplified separability condition for an arbitrary QCM**). *A QCM V_{AB} of $m + n$ modes is separable if and only if there exists an m-mode QCM $\gamma_A \geqslant \mathrm{i}\Omega_A$ such that*

$$V_{AB} \geqslant \gamma_A \oplus \mathrm{i}\Omega_B. \tag{20}$$

In terms of the block form (17) of V_{AB}, when $V_B > \mathrm{i}\Omega_B$ the above condition is equivalent to the existence of a real matrix γ_A satisfying

$$\mathrm{i}\Omega_A \leqslant \gamma_A \leqslant V_A - X(V_B - \mathrm{i}\Omega_B)^{-1}X^T. \tag{21}$$

If $V_B - \mathrm{i}\Omega_B$ is not invertible, we require instead $\mathrm{i}\Omega_A \leqslant \gamma_A \leqslant V_A - X(V_B + \varepsilon\mathbf{1}_B - \mathrm{i}\Omega_B)^{-1}X^T$ for all $\varepsilon > 0$.

Proof. Since both sets of QCMs V_{AB} defined by (19) and (20) are clearly topologically closed, we can just show without loss of generality that their interiors coincide. This latter condition can be rephrased as an equivalence between the two following statements: (i) $V_{AB} > \gamma_A \oplus \gamma_B$ for some QCMs γ_A, γ_B; and (ii) $V_{AB} > \gamma_A \oplus \mathrm{i}\Omega_B$ for some QCM γ_A.

Now, once $\gamma_A < V_A$ is fixed, the supremum of all the matrices γ_B satisfying $V_{AB} > \gamma_A \oplus \gamma_B$ is given by the Schur complement $(V_{AB} - (\gamma_A \oplus 0_B))/(V_A - \gamma_A)$, as the variational characterisation (4) reveals. Therefore,

statement (i) is equivalent to the existence of $i\Omega_A \leqslant \gamma_A < V_A$ such that $(V_{AB} - (\gamma_A \oplus 0_B))/(V_A - \gamma_A) > i\Omega_B$. This is the same as to require $V_{AB} > \gamma_A \oplus i\Omega_B$, as the positivity conditions of lemma 1 immediately show.

Until now, we have proven that the separability of V_{AB} can be restated as $V_{AB} \geqslant \gamma_A \oplus i\Omega_B$ for some appropriate QCM γ_A. Employing lemma 1, we see that this is turn equivalent to (21), or to its ε-modified version when $V_B - i\Omega_B$ is not invertible.   □

Remark 1. It has been recently observed [36] that condition (21) is equivalent to the corresponding Gaussian state $\rho_{AB}^G(V_{AB})$ with QCM V_{AB} being completely extendable with Gaussian extensions. We remind the reader that a bipartite state ρ_{AB} is said to be *completely extendable* if for all k there exists a state $\rho_{AB_1 \cdots B_k}$ that is: (i) symmetric under exchange of any two B_i systems; and (ii) an extension of ρ_{AB} in the sense that $\mathrm{Tr}_{B_2 \cdots B_k} \rho_{AB_1 \cdots B_k} = \rho_{AB}$. When the original state ρ_{AB}^G is Gaussian, it is natural to consider extensions $\rho_{AB_1 \cdots B_k}^G$ of Gaussian form as well. Interestingly enough, the above theorem 5 provides a simple alternative proof of the remarkable fact (also proven in [36]) that Gaussian states are separable if and only if completely extendable with Gaussian extensions.

Remark 2. It is worth noticing that both lemma 4 and theorem 5 extend straightforwardly to encompass the case of full separability of multipartite Gaussian states. In the case of lemma 4, this extension was already formulated in [7, 37]. As for theorem 5, the corresponding necessary and sufficient condition for the full separability of a k-partite QCM $V_{A_1 \cdots A_k}$ would read $V_{A_1 \cdots A_k} \geqslant \gamma_{A_1} \oplus \cdots \oplus \gamma_{A_{k-1}} \oplus i\Omega_{A_k}$ for appropriate QCMs $\gamma_1, \ldots, \gamma_{k-1}$.

4. PPT implies separability for 1 versus N-mode Gaussian states—revisited

We now focus on investigating known and new conditions under which separability becomes equivalent to PPT for Gaussian states, so that the problem of deciding whether a given QCM is separable or not admits a handy formulation.

For any bipartite state ρ_{AB}, recall that the PPT criterion provides a useful necessary condition for separability [14]:

$$\rho_{AB} \text{ is separable } \Rightarrow \rho_{AB}^{T_B} \geqslant 0, \tag{22}$$

where the suffix T_B denotes transposition with respect to the degrees of freedom of subsystem B only. In finite-dimensional systems, PPT is also a sufficient condition for separability when $\dim(A) \cdot \dim(B) \leqslant 6$ [38].

In continuous variable systems, the PPT criterion turns out to be also sufficient for separability of QCMs when either A or B is composed of one mode only.

Theorem 6 (PPT is sufficient for Gaussian states of 1 versus n modes [7, 15]). *Let V_{AB} be a bipartite QCM such that either A or B are composed of one mode only. Then V_{AB} is separable if and only if*

$$V_{AB} \geqslant \begin{pmatrix} i\Omega_A & 0 \\ 0 & \pm i\Omega_B \end{pmatrix} = i\Omega_A \oplus (\pm i\Omega_B), \tag{23}$$

which amounts to the corresponding Gaussian state being PPT, $\rho_{AB}^{G\,T_B} \geqslant 0$.

For completeness, we recall that the partial transpose of an $(m + n)$-mode QCM V_{AB} (i.e., the covariance matrix of the partially transposed density operator $\rho_{AB}^{G\,T_B}$) is given by $V_{AB}^{T_B} = \Theta_B V_{AB} \Theta_B$, where with respect to a mode-wise decomposition on the B subsystem the matrix Θ_B can be written as $\Theta_B := \mathbb{1}_A \oplus (\bigoplus_{j=1}^n \zeta)_B$, with $\zeta := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ [15]. Accordingly, we can say that the QCM V_{AB} is PPT if and only if $V_{AB}^{T_B}$ is a valid QCM obeying (13), which is equivalent to (23).

The original proof of theorem 6 came in two steps. Firstly, Simon [15] proved it in the particular case when both A and B are made of one mode only by performing an explicit analysis of the symplectic invariants of V_{AB}; this seminal analysis is quite straightforward to follow and particularly instructive, but eventually a bit cumbersome, since it requires to distinguish between three cases, according to the sign of $\det X$, where X is the off-diagonal block of the QCM V_{AB} partitioned as in (17). Later on, Werner and Wolf [7] reduced the problem for the 1 versus n-mode case with arbitrary n to the 1 versus 1-mode case; the proof of this reduction is geometric in nature and rather elegant, but also relatively difficult.

Our purpose in this section is to use Schur complements to provide the reader with a simple, direct proof of theorem 6. Before coming to that, there is a preliminary lemma we want to discuss.

Lemma 7. *Let M, N be 2×2 Hermitian matrices. There is a real symmetric matrix R satisfying $M \leqslant R \leqslant N$ if and only if $M \leqslant N$, N^*, where $*$ denotes complex conjugation.*

Proof. The only complex entry in a 2×2 Hermitian matrix is in the off-diagonal element. Suppose without loss of generality that $\Im M_{12} \geqslant 0$ and $\Im N_{12} \leqslant 0$ (both conditions in the statement are in fact symmetric under complex conjugation of M or N). It is easy to verify that a p such that $0 \leqslant p \leqslant 1$ and $\Im(pM + (1 - p)N)_{12} = 0$ always exists, and we see that $R := pM + (1 - p)N$ is a real symmetric matrix. Moreover, since R belongs to the segment joining M and $N \geqslant M$ we conclude that $M \leqslant R \leqslant N$.   □

Remark 3. Lemma 7 admits an appealing physical interpretation which also leads to an intuitive proof. This interpretation is based on the fact that 2×2 Hermitian matrices can be seen as events in four-dimensional Minkowski space–time through the correspondence $x_0 \mathbf{1} + \vec{x} \cdot \vec{\sigma} \leftrightarrow (x_0, \vec{x})$. Furthermore, $M \leqslant N$ translates in Minkowski space–time to 'N is in the absolute future of M', since the remarkable determinantal identity $\det(x_0 \mathbf{1} + \vec{x} \cdot \vec{\sigma}) = x_0^2 - \vec{x}^2$ holds true. Now, the complex conjugation at the matrix level becomes nothing but a spatial reflection with respect to a fixed spatial plane in Minkowski space–time. Thus, our original question is: is it true that whenever both an event N and its spatial reflection N^* are in the absolute future of a reference event M then there is another event R which is: (i) in the absolute future of M; (ii) in the absolute past of both N and N^*; and (iii) lies right on the reflection plane? The answer is clearly yes, and there is a simple way to obtain it. Start from M and shoot a photon to the location of that event between N and N^* that will happen on the other side of the reflection plane. After some time the photon hits the plane, and this event R clearly satisfies all requirements.

Now we are ready to give our direct proof of the equivalence between PPT and separability for 1 versus n-mode Gaussian states, leveraging the simplified separability condition of theorem 5.

Proof of theorem 6. Suppose without loss of generality that A is composed of one mode only. As in the proof of theorem 5, since both sets of QCMs V_{AB} defined by (19) and (23) are topologically closed, we can assume that V_{AB} is in the interior of the PPT set, i.e. that $V_{AB} > \mathrm{i}\Omega_A \oplus (\pm \mathrm{i}\Omega_B)$. Our goal will be to show that in this case V_{AB} belongs to the separable set, as characterised by theorem 5. Since $V_B - \mathrm{i}\Omega_B$ is taken to be invertible, the PPT condition reads

$$V_A - X(V_B \mp \mathrm{i}\Omega_B)^{-1}X^T \geqslant \mathrm{i}\Omega_A.$$

Now, define $M = \mathrm{i}\Omega_A$ and $N = V_A - X(V_B + \mathrm{i}\Omega_B)^{-1}X^T$, and observe that $N^* = V_A - X(V_B - \mathrm{i}\Omega_B)^{-1}X^T$. Thanks to lemma 7, we can find a real matrix γ_A such that

$$V_A - X(V_B \mp \mathrm{i}\Omega_B)^{-1}X^T \geqslant \gamma_A \geqslant \mathrm{i}\Omega_A.$$

Choosing the negative sign in the above inequality, we see that the second condition (21) in theorem 5 is met, and therefore V_{AB} is separable.   □

5. Gaussian states that are invariant under partial transpose are separable

As a further example of application of theorem 5, we study here the separability of a special class of PPT Gaussian states, i.e. those that are *invariant* under partial transposition of one of the subsystems. This problem has an analogue in finite-dimensional quantum information, already studied in [24], where it was shown that bipartite states on $\mathbb{C}^2 \otimes \mathbb{C}^d$ that are invariant under partial transpose on the first system are necessarily separable[4]. Here we show that for Gaussian states an even stronger statement holds, in that invariance under partial transposition implies separability for any number of local modes.

Corollary 8. *A bipartite Gaussian state ρ_{AB}^G that is invariant under partial transposition of one of the two subsystems is necessarily separable.*

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[4] The proof reported in [24] is rather long, so here we provide a shorter one, again based on Schur complements. A state on $\mathbb{C}^2 \otimes \mathbb{C}^d$ that is invariant under partial transposition on the first subsystem can be represented in block form as $\rho = \begin{pmatrix} A & X \\ X & B \end{pmatrix}$. By a continuity argument, we can suppose without loss of generality that $A > 0$. Rewrite $\rho = \begin{pmatrix} A & X \\ X & XA^{-1}X \end{pmatrix} + |1\rangle\langle 1| \otimes (B - XA^{-1}X)$. Both terms are positive by lemma 1. Since the second one is explicitly separable, let us deal only with the first one, call it $\tilde{\rho}$. We have $\tilde{\rho} = \mathbf{1}_2 \otimes A^{1/2} \begin{pmatrix} \mathbf{1} & Y \\ Y & Y^2 \end{pmatrix} \mathbf{1}_2 \otimes A^{1/2}$, where $Y := A^{-1/2}XA^{-1/2}$ is Hermitian. Denoting by $Y = \sum_i y_i |e_i\rangle\langle e_i|$ its spectral decomposition, we obtain the following manifestly separable representation of $\tilde{\rho}$:

$$\tilde{\rho} = \mathbf{1}_2 \otimes A^{1/2} \left(\sum_i \begin{pmatrix} 1 & y_i \\ y_i & y_i^2 \end{pmatrix} \otimes |e_i\rangle\langle e_i| \right) \mathbf{1}_2 \otimes A^{1/2}.$$

Proof. Without loss of generality, we can assume that the partial transpose on the *B* system leaves the state invariant. We now show that under the this assumption the separability condition (21) is immediately satisfied, since the rightmost side is already a real, symmetric matrix. In fact, equating the original QCM (17) with the one obtained after partial transpose on the *B* system, we get the identities $X = X\Theta_B$ and $V_B = \Theta_B V_B \Theta_B$, where, as previously set, $\Theta_B = \bigoplus_{j=1}^n \zeta$ according to a mode-wise decomposition of the *B* system, and $\zeta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. As a consequence,

$$X(V_B - \mathrm{i}\Omega_B)^{-1}X^T = X\Theta_B(V_B - \mathrm{i}\Omega_B)^{-1}\Theta_B X^T = X(\Theta_B(V_B - \mathrm{i}\Omega_B)\Theta_B)^{-1}X^T = X(V_B + \mathrm{i}\Omega_B)^{-1}X^T,$$

where we used also $\Theta_B\Omega_B\Theta_B = -\Omega_B$. This shows that $X(V_B - \mathrm{i}\Omega_B)^{-1}X^T$ is equal to its complex conjugate, and is therefore (despite appearances) a real symmetric matrix. Hence the separability condition (21) is satisfied with $\gamma_A = V_A - X(V_B - \mathrm{i}\Omega_B)^{-1}X^T$, which is a legitimate QCM as follows from the bonafide condition (13) together with lemma 1. $\square$

6. PPT implies separability for multimode mono-symmetric Gaussian states

Throughout this section, we show how the PPT criterion is also necessary and sufficient for deciding the separability of bipartite Gaussian states of *m* versus *n* modes that are symmetric under the exchange of any two among the first *m* modes. These states will be referred to as *mono-symmetric* (with respect to the first party *A*). As can be easily seen, this novel result (see figure 1 for a graphical visualisation) is a generalisation of both theorem 6 and of one of the main results in [16], where the subclass of bi-symmetric states was considered instead, bi-symmetric meaning that they are invariant under swapping any two modes either within the first *m* or within the last *n* (that is, they are mono-symmetric in both *A* and *B*).

Theorem 9 (Symplectic localisation of mono-symmetric states). *Let $\rho_{AB}^G(V_{AB})$ be a mono-symmetric Gaussian state of $m + n$ modes, i.e. specified by a QCM V_{AB} that is symmetric under the exchange of any two of the m modes of subsystem A. Then there exists a local unitary operation on A corresponding to a symplectic transformation $S_A \in \mathrm{Sp}(2\,m, \mathbb{R})$ that transforms ρ_{AB}^G into the tensor product of $m - 1$ uncorrelated single-mode Gaussian states $\tilde{\rho}_{A_j}^G\ (\widetilde{V}_{A_j})\ (j = 2, \ldots, m)$ and a bipartite Gaussian state $\tilde{\rho}_{A_1B}^G\ (\widetilde{V}_{A_1B})$ of 1 versus n modes. At the QCM level, this reads*

$$(S_A \oplus \mathbf{1}_B) V_{AB}(S_A^T \oplus \mathbf{1}_B) = \left(\bigoplus_{j=2}^m \widetilde{V}_{A_j}\right) \oplus \widetilde{V}_{A_1B}. \tag{24}$$

The separability properties of V_{AB} and $\widetilde{V}_{A_1B}$ are equivalent, in particular $\rho_{AB}^G(V_{AB})$ is separable if and only if it is PPT.

Proof. We will prove (24) directly at the QCM level, by constructing a suitable local symplectic S_A. By virtue of the symmetry under the exchange of any two modes of subsystem *A*, if we decompose V_{AB} as in (17), the submatrices V_A and *X* have the following structure:

$$V_A = \begin{pmatrix} \alpha & \varepsilon & \ldots & \varepsilon \\ \varepsilon & \alpha & & \vdots \\ \vdots & & \ddots & \varepsilon \\ \varepsilon & \ldots & \varepsilon & \alpha \end{pmatrix}, \qquad X = \begin{pmatrix} \kappa_1 & \kappa_2 & \ldots & \kappa_n \\ \kappa_1 & \kappa_2 & \ldots & \kappa_n \\ \vdots & \vdots & & \vdots \\ \kappa_1 & \kappa_2 & \ldots & \kappa_n \end{pmatrix}, \tag{25}$$

where each one of the blocks α, ε, κ_j in (25) is a 2×2 real matrix, with α and ε symmetric [39].

We can now decompose the real space of the first *m* modes as $\mathbb{R}^{2m} = \mathbb{R}^m \otimes \mathbb{R}^2$. According to this decomposition, we may rewrite V_A and *X* as follows:

$$V_A = \mathbf{1}_m \otimes (\alpha - \varepsilon) + m|+\rangle\langle+| \otimes \varepsilon, \qquad X = \sqrt{m}\, \sum_{j=1}^n |+\rangle\langle j| \otimes \kappa_j, \tag{26}$$

where $|+\rangle = \frac{1}{\sqrt{m}}\sum_{i=1}^m|i\rangle$, with $\{|i\rangle\}_{i=1}^m$ denoting the standard basis for $\mathbb{R}^m$. Observe that the symplectic form Ω_A on subsystem *A* decomposes accordingly as $\Omega_A = \mathbf{1}_m \otimes \omega$. If *O* is an $m \times m$ orthogonal matrix such that $O|+\rangle = |1\rangle$, we easily see that on the one hand $O \otimes \mathbf{1}_2\, \Omega_A\, O^T \otimes \mathbf{1}_2 = \Omega_A$, i.e. $O \otimes \mathbf{1}_2$ is symplectic, while on the other hand

$$O \otimes \mathbb{1}_2 \, V_A \, O^T \otimes \mathbb{1}_2 \ = \ |1\rangle\langle 1| \otimes (\alpha + (m-1)\varepsilon) + \sum_{i=2}^{m} |i\rangle \otimes \langle i| \otimes (\alpha - \varepsilon)$$

$$= \begin{pmatrix} \alpha + (m-1)\varepsilon & 0 & \dots & 0 \\ 0 & \alpha - \varepsilon & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \alpha - \varepsilon \end{pmatrix}, \tag{27}$$

$$O \otimes \mathbb{1}_2 \, X \ = \ \sqrt{m} \sum_{j=1}^{n} |1\rangle\langle j| \otimes \kappa_j = \begin{pmatrix} \sqrt{m}\,\kappa_1 & \sqrt{m}\,\kappa_2 & \dots & \sqrt{m}\,\kappa_n \\ 0 & 0 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}. \tag{28}$$

Therefore, the initial QCM V_{AB} has been decomposed as a direct sum of $m-1$ one-mode QCMs $\tilde{V}_{A_j} = \alpha - \varepsilon$, and of one $(1 + m)$-mode QCM $\widetilde{V}_{A_1 B}$, via a local symplectic operation on subsystem A, given precisely by $S_A = O \otimes \mathbb{1}_2$. This proves (24) constructively. Applying theorem 6, one then gets immediately that the PPT condition is necessary and sufficient for separability in this case.   □

Remark 4. This original result yields a substantial enlargement to the domain of validity of PPT as a necessary and sufficient criterion for separability of multimode Gaussian states, reaching beyond any existing literature. In practice, theorem 9 tells us that, in any mono-symmetric Gaussian state, all the correlations (including and beyond entanglement) shared among the whole m modes of A and the whole n modes of B can be localised onto correlations between a single mode A_1 of A versus the whole B, by means of a local unitary (symplectic at the QCM level) operation at A's side only. Being unitary, this operation is fully reversible, meaning that the correlations with B can be redistributed back and forth between A_1 and the whole set of A modes with no information loss. This also means that quantitative results on any measure of such correlations between A and B encoded in V_{AB} can be conveniently evaluated in the much simpler 1 versus n-mode normal form $\widetilde{V}_{A_1 B}$ constructed in the proof theorem 9, ignoring the $m-1$ uncorrelated modes.

In the special case of V_{AB} being the QCM of a bi-symmetric state, i.e. with full permutation symmetry within both A and B, it is immediate to observe that applying a similar construction by means of a local unitary at B's side as well fully reduces V_{AB} to a two-mode QCM $\widetilde{V}_{A_1 B_1}$, with equivalent entanglement properties as the original V_{AB}, plus a collection of $m + n - 2$ uncorrelated single modes. This reproduces the findings of [16].

Similarly to what discussed in remark 2, the results of theorem 9 can also be straightforwardly extended to characterise full separability and, conversely, multipartite entanglement of arbitrary multimode Gaussian states which are partitioned into k subsystems, with the requirement of local permutation invariance within some of these subsystems. It is clear that, by suitable local symplectic transformations, each of those locally symmetric parties can be localised onto a single mode correlated with the remaining parties, thus removing the redundancy in the QCM. Gaussian states of this sort generalise the so-called multi-symmetric states studied in [40], where local permutation invariance was enforced within all of the subsystems, resulting in a direct multipartite analogue of bi-symmetric states.

7. PPT implies separability for multimode isotropic Gaussian states—revisited

It is well known that the PPT criterion is in general sufficient, as well as obviously necessary, for pure bipartite states to be separable [14]. This may be seen by a direct inspection of the Schmidt decomposition of a pure state. Let us note, incidentally, that a stronger statement holds, namely any bound entangled state (in any dimension) must have at least rank 4 [41].

The Schmidt decomposition theorem is in fact so important that a Gaussian version of it, that is, the determination of a normal form of pure QCMs under local symplectic operations, is of central importance in continuous variable quantum information. As can be shown at the covariance matrix level [17, 18] or at the density operator level [19], every pure bipartite Gaussian state $\rho_{AB}^G(V_{AB})$ can be brought into a tensor product of two-mode squeezed vacuum states and single-mode vacuum states by means of local unitaries with respect to the A versus B partition. In particular, by acting correspondingly with local symplectic transformations, any pure QCM V_{AB} (where pure means $\det V_{AB} = 1$) can be transformed into a direct sum of (pure) two-mode squeezed vacuum QCMs and (pure) single-mode vacuum QCMs. More precisely, at the level of QCMs, one can formulate this fundamental result as follows.

Theorem 10 (Mode-wise decomposition of pure Gaussian states [17–19]). *Let V_{AB} be a bipartite QCM of $m + n$ modes $A_1, \dots, A_m, B_1, \dots, B_n$, assuming $m \leqslant n$ (with no loss of generality). If V_{AB} is a pure QCM, i.e. all its symplectic eigenvalues are equal to 1 (which amounts to $\det V_{AB} = 1$), then there exist local symplectic transformations $S_A \in \mathrm{Sp}(2m, \mathbb{R})$, $S_B \in \mathrm{Sp}(2n, \mathbb{R})$ mapping V_{AB} into the following normal form:*

$$(S_A \oplus S_B) V_{AB} (S_A^T \oplus S_B^T) = \bigoplus_{j=1}^{m} \bar{V}_{A_j B_j}(r_j) \oplus \bigoplus_{k=m+1}^{n} \mathbf{1}_{B_k}, \qquad (29)$$

where $\bar{V}_{A_j B_j} = \begin{pmatrix} c_j \mathbf{1} & s_j \zeta \\ s_j \zeta & c_j \mathbf{1} \end{pmatrix}$ *with* $c_j = \cosh(2r_j)$ *and* $s_j = \sinh(2r_j)$*, for a real squeezing parameter* r_j*, is the pure QCM of a two-mode squeezed vacuum state of modes* A_j *and* B_j*, and* $\mathbf{1}_{B_k}$ *is the pure QCM of the single-mode vacuum state of mode* B_k*. In particular, with respect to the block form* (17)*, for any pure QCM* V_{AB} *the marginal QCMs* V_A *and* V_B *have matching symplectic spectra, given by* $\vec{\nu}_A = (c_1, \ldots, c_m)$ *and* $\vec{\nu}_B = (c_1, \ldots, c_m, \underbrace{1, \ldots, 1}_{n-m})$.

Leaving apart its far-reaching applications, in the context of the present paper this result is mainly instrumental for assessing the separability of so-called *isotropic* multimode Gaussian states. The QCM of any such state of $m + n$ modes is characterised by the property of having a completely degenerate symplectic spectrum, i.e. formed of only one distinct symplectic eigenvalue $\nu \geqslant 1$ (repeated $m + n$ times). This means that the QCM V_{AB} of any isotropic state is proportional by a factor ν to a pure QCM. Hence, theorem 10 tells us that V_{AB} can be brought into a direct sum of two-mode QCMs via a local symplectic congruence (local with respect to any partition into groups of modes A and B), as first observed in [17]. Thanks to theorem 6, this guarantees the following.

Theorem 11. *The PPT criterion is necessary and sufficient for separability of all isotropic Gaussian states of an arbitrary number of modes.*

However, notwithstanding the importance of theorem 7 per se, one could strive to seek a more direct way to obtain theorem 11. Our purpose in this section is in fact to provide an alternative proof of this result, which does not appeal to the mode-wise decomposition theorem at all, and uses directly lemma 4 instead, leveraging matrix analysis tools such as the notions of matrix means introduced in section 2.2.

Note that, in almost all the remainder of this section, for a single system of n modes, we will find it more convenient to reorder the vector of canonical operators as $r := (x_1, \ldots, x_n, p_1, \ldots, p_n)$, corresponding to a position–momentum block structure. The symplectic form Ω appearing in (11) is accordingly rewritten as

$$\Omega := \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}. \qquad (30)$$

We will then write any QCM V, as well as any symplectic operation S acting on it, with respect to this alternative block structure, unless explicitly stated otherwise.

Let us start with a preliminary result, equivalent to proposition 12 of [43] or lemma 13 of [23]. We include a proof for the sake of completeness.

Lemma 12. *Let* $V > 0$ *be a positive matrix. Then* V *is a pure QCM if and only if* $V = \Omega V^{-1} \Omega^T$*, and it obeys* (13) *if and only if*

$$V \geqslant V \# (\Omega V^{-1} \Omega^T). \qquad (31)$$

Proof. Let the Williamson form of V be given by (16), where (in the convention of this section) $N = \Lambda \oplus \Lambda$, with $\Lambda := (\nu_1, \ldots \nu_n)$. Then we can write

$$\Omega V^{-1} \Omega^T = \Omega S^{-T}(\Lambda^{-1} \oplus \Lambda^{-1}) S^{-1} \Omega^T = S\Omega(\Lambda^{-1} \oplus \Lambda^{-1}) \Omega^T S^T = S(\Lambda^{-1} \oplus \Lambda^{-1})\Omega\Omega^T S^T = S(\Lambda^{-1} \oplus \Lambda^{-1}) S^T,$$

where we used in order: (i) the identities $\Omega S^{-T} = S\Omega$, $S^{-1}\Omega^T = \Omega^T S^T$, all consequences of the defining symplectic identity $S\Omega S^T = \Omega$; (ii) the fact that Ω commutes with $\Lambda^{-1} \oplus \Lambda^{-1}$; and (iii) the orthogonality relation $\Omega\Omega^T = \mathbf{1}$. Now the first claim becomes obvious, since $\Lambda = \Lambda^{-1} = \mathbf{1}$ if and only if V is a pure QCM. In general, as it can be seen from the above expression, V and $\Omega V^{-1} \Omega^T$ are brought in Williamson form by simultaneous congruences with the same symplectic matrix (i.e. S^{-1}, in the convention of (16)). Hence, the covariance of the geometric mean under congruence ensures that

$$V \# (\Omega V^{-1} \Omega^T) = \left(S \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix} S^T \right) \# \left(S \begin{pmatrix} \Lambda^{-1} & 0 \\ 0 & \Lambda^{-1} \end{pmatrix} S^T \right) = S\left(\begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix} \# \begin{pmatrix} \Lambda^{-1} & 0 \\ 0 & \Lambda^{-1} \end{pmatrix} \right) S^T = SS^T,$$

where the last passage is an easy consequence of the fact that $A \# A^{-1} = \mathbf{1}$ for all $A > 0$. By comparison with (16), we see that the Heisenberg uncertainty relation $\Lambda \geqslant \mathbf{1}$ can be rephrased as $V = S(\Lambda \oplus \Lambda) S^T \geqslant SS^T = V \# (\Omega V^{-1} \Omega^T)$, which reproduces (31), proving the second claim. $\square$

Remark 5. From the above proof it is also apparent how, for any positive $A > 0$, the matrix $A \# (\Omega A^{-1} \Omega^T)$ is a pure QCM (independently of the nature of A).

Now we are ready to explain our direct argument to show separability of PPT isotropic Gaussian states, alternative to the use of the mode-wise decomposition.

Proof of theorem 11. We start by rewriting the PPT condition (23) for a QCM V_{AB} as

$$\Theta_B V \Theta_B, \geqslant i\Omega,$$

where in the convention of this section $\Theta_B = \mathbf{1}_A \oplus \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}_B$. Thanks to lemma 12, this becomes in turn

$$\Theta_B V \Theta_B \geqslant (\Theta_B V \Theta_B) \# (\Omega \Theta_B V^{-1} \Theta_B \Omega^T)$$

and finally

$$V \geqslant V \# (\Theta_B \Omega \Theta_B V^{-1} \Theta_B \Omega^T \Theta_B) = (gV) \# (Z\Omega \, (gV)^{-1} \, \Omega^T Z)$$

after conjugating by Θ_B, applying once more the covariance of the geometric mean under congruences, introducing a real parameter $g > 0$ (to be fixed later), and defining $Z := \mathbf{1}_A \oplus (-\mathbf{1}_B) = \Theta_B \Omega \Theta_B \Omega^T$. Now, we apply lemma 3 to the above expression, obtaining

$$V \geqslant \left(\frac{gV + Z\Omega \, (gV)^{-1} \, \Omega^T Z}{2} \right) \# ((gV)!(Z\Omega \, (gV)^{-1} \, \Omega^T Z)).$$

Although it is not yet transparent, we are done, as the right-hand side of the above inequality is exactly of the form $\gamma_A \oplus \gamma_B$ when V is the QCM of an isotropic Gaussian state. In fact, let $g > 0$ be such that gV is a pure QCM, satisfying $gV = \Omega \, (gV)^{-1} \, \Omega^T = \begin{pmatrix} P & Y \\ Y^T & Q \end{pmatrix}$, where we have now reverted to a block decomposition with respect to the A versus B splitting. Then on the one hand since $Z = \mathbf{1}_A \oplus (-\mathbf{1}_B)$ we find

$$\frac{gV + Z\Omega \, (gV)^{-1} \, \Omega^T Z}{2} = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix},$$

while on the other hand

$$(gV)!(Z\Omega \, (gV)^{-1} \, \Omega^T Z) = 2((gV)^{-1} + Z\Omega \, (gV)\Omega^T Z)^{-1} = 2\Omega(\Omega(gV)^{-1}\Omega^T + Z(gV)Z)^{-1}\Omega^T$$

$$= 2\Omega(gV + Z(gV)Z)^{-1}\Omega^T = \begin{pmatrix} \Omega P^{-1}\Omega^T & 0 \\ 0 & \Omega Q^{-1}\Omega^T \end{pmatrix},$$

where we used the definition (5) of harmonic mean and the fact that $[Z, \Omega] = 0$. Putting all together, we find

$$V \geqslant \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \# \begin{pmatrix} \Omega P^{-1}\Omega^T & 0 \\ 0 & \Omega Q^{-1}\Omega^T \end{pmatrix} = \begin{pmatrix} P\#\Omega P^{-1}\Omega^T & 0 \\ 0 & Q\#\Omega Q^{-1}\Omega^T \end{pmatrix} = \gamma_A \oplus \gamma_B.$$

Since we already observed that $P\#\Omega P^{-1}\Omega^T$ is a QCM for any $P > 0$ (and analogously for Q), a direct invocation of lemma 4 allows us to conclude the proof.   □

8. Entangling Gaussian states via passive optical operations

Throughout this section, we finally complete the solution of a problem posed in [25] and there addressed under some additional constraints. Let us start by recalling that symplectic operations can be divided into two main categories, namely those such as squeezers that require an exchange of energy between the system and the apparatus, called *active*, and those that can be implemented using only beam splitters and phase plates, called *passive*. A symplectic matrix K represents a passive transformation if and only if it is also *orthogonal*, meaning that $KK^T = \mathbf{1}$ (it may be worth adding that symplectic orthogonal transformations form the maximal compact subgroup of the symplectic group). As it turns out, symplectic orthogonal matrices can be represented in an especially simple form if we resort to a position-momentum block decomposition. Namely, one has the parametrisation [6]

$$K = W^\dagger \begin{pmatrix} U & \\ & U^* \end{pmatrix} W, \tag{32}$$

where

$$W := \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1} & i\mathbf{1} \\ \mathbf{1} & -i\mathbf{1} \end{pmatrix}$$

and U is a generic, $n \times n$ unitary matrix, with U^* denoting its complex conjugate.

Since the implementation of passive operations is so inexpensive in quantum optics and entangled states so useful for quantum technologies, the question first posed in [25] was a natural one: 'What bipartite Gaussian

states are such that they can be entangled via a global, passive operation?' However, in this full generality the problem was left unanswered in [25]. Instead, another related question was investigated and answered there, namely whether *distillable* Gaussian entanglement can be produced in the same fashion. For Gaussian states, as mentioned in the Introduction, distillability is well known to be equivalent to non-positivity of the partial transpose [8, 13], so the authors of [25] proceeded to identify the class of Gaussian states that can be made to violate the PPT condition with a passive transformation. However, it is important to realise that since PPT and separability are not the same for general multimode Gaussian states, the two questions are *a priori* different. Here we show that the answer to the original question above turns out to be yet another situation where the PPT condition is necessary and sufficient to ensure separability of Gaussian states. In other words, we will prove that a bipartite Gaussian state that can not be made distillable (i.e.non-PPT) via passive operations is necessarily separable, and thus it stays separable under the application of said passive operations. Let us start with a technical lemma that we deduce from recent results obtained in [44].

Lemma 13. *Let $A > 0$ be a strictly positive $2n \times 2n$ matrix. Let $\nu_i(A)$ and $\lambda_i(A)$ denote its symplectic and ordinary (orthogonal) eigenvalues, respectively, arranged in non-decreasing order. Then*

$$\nu_1(A)^2 \geqslant \lambda_1(A)\lambda_2(A). \tag{33}$$

In particular, every positive matrix whose two smallest eigenvalues satisfy $\lambda_1 \lambda_2 \geqslant 1$ is automatically a legitimate QCM.

Proof. From [44, equation (71)] we deduce $\prod_{j=1}^{k} \nu_{n-j+1}(A)^2 \leqslant \prod_{j=1}^{2k} \lambda_{2n-j+1}(A)$ for all $k = 1, \ldots, n$, with equality for $k = n$, when both terms equal the determinant of A. We can use this observation to deduce that $\prod_{j=1}^{k} \nu_j(A)^2 \geqslant \prod_{j=1}^{2k} \lambda_j(A)$ for all $k = 1, \ldots, n$. The special case $k = 1$ yields the claim. $\qquad \square$

Now, we are ready to present our strengthening of [25, proposition 1].

Theorem 14. *Let V be a bipartite QCM of an n-mode system. Then the following are equivalent:*

(i) *KVK^T is separable for all Gaussian passive transformations K;*

(ii) *KVK^T is PPT for all Gaussian passive transformations K; and*

(iii) *the two smallest eigenvalues of V satisfy $\lambda_1(V)\lambda_2(V) \geqslant 1$.*

Proof. The implication (i) $\Rightarrow$ (ii) is obvious, while (iii) $\Rightarrow$ (ii) already follows from lemma 13 together with the fact that the partial transpose at the level of QCMs is a congruence by orthogonal transformation and thus does not change the ordinary spectrum. One of the main contributions of [25] is the proof that (ii) and (iii) are in fact equivalent. In view of this discussion, we have just to show that (iii) $\Rightarrow$ (i). To this end, we will assume that V satisfies $\lambda_1(V)\lambda_2(V) \geqslant 1$ and construct two local QCMs γ_A, γ_B that satisfy the hypothesis of the original separability criterion given by lemma 4. Call $\lambda_1(V) = k$ and observe that if $k \geqslant 1$ then $V \geqslant \mathbf{1} = \mathbf{1}_A \oplus \mathbf{1}_B$ and we are done. Otherwise, assume $k < 1$ and denote by $|x\rangle$ the normalised eigenvector corresponding to the minimal eigenvalue of V, i.e. $V|x\rangle = k|x\rangle$ and $\langle x|x\rangle = 1$. Since $\lambda_2(V) \geqslant \frac{1}{k}$ and *a fortiori* $\lambda_i(V) \geqslant \frac{1}{k}$ for all $i \geqslant 2$, we can write

$$V \geqslant k|x\rangle\langle x| + \frac{1}{k}(\mathbf{1} - |x\rangle\langle x|).$$

Now, decompose the vector $|x\rangle$ into its A and B components as $|x\rangle = \begin{pmatrix} \sqrt{p}\,|y\rangle_A \\ \sqrt{1-p}\,|z\rangle_B \end{pmatrix}$, where $0 \leqslant p \leqslant 1$ and $\langle y|y\rangle = 1 = \langle z|z\rangle$. Then, lemma 13 guarantees that the matrices

$$\gamma_A := k|y\rangle\langle y| + \frac{1}{k}(\mathbf{1} - |y\rangle\langle y|)$$

$$\gamma_B := k|z\rangle\langle z| + \frac{1}{k}(\mathbf{1} - |z\rangle\langle z|)$$

are legitimate QCMs. Then showing $V_{AB} - \gamma_A \oplus \gamma_B \geqslant 0$ would complete our proof. By direct computation, we find

$$V_{AB} - \gamma_A \oplus \gamma_B \geqslant -\left(\frac{1}{k} - k\right)\begin{pmatrix} p|y\rangle\langle y| & \sqrt{p(1-p)}|y\rangle\langle z| \\ \sqrt{p(1-p)}|z\rangle\langle y| & (1-p)|z\rangle\langle z| \end{pmatrix} + \frac{1}{k}\begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}$$

$$-\begin{pmatrix} -\left(\frac{1}{k} - k\right)|y\rangle\langle y| + \frac{1}{k}\mathbf{1} & 0 \\ 0 & -\left(\frac{1}{k} - k\right)|z\rangle\langle z| + \frac{1}{k}\mathbf{1} \end{pmatrix}$$

$$= \left(\frac{1}{k} - k\right)\begin{pmatrix} (1-p)|y\rangle\langle y| & -\sqrt{p(1-p)}|y\rangle\langle z| \\ -\sqrt{p(1-p)}|z\rangle\langle y| & p|z\rangle\langle z| \end{pmatrix}$$

$$= \left(\frac{1}{k} - k\right)\left(\sqrt{1-p}\,|y\rangle \quad -\sqrt{p}\,|z\rangle\right)^T\left(\sqrt{1-p}\,|y\rangle \quad -\sqrt{p}\,|z\rangle\right)$$

$$\geqslant 0.$$

$\square$

Remark 6. In some sense, one can think of the question posed in [25] and answered here in theorem 14 as a continuous variable analogue of the *absolute separability* problem in finite-dimensional quantum information, which asks for the characterisation of those spectra $\sigma = (\lambda_1, \ldots, \lambda_{dd'})$ such that every bipartite quantum state on $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ with spectrum σ is separable [45]. For a recent review of the state of the art, we refer the reader to [46]. A suggestive argument concerning this analogy goes as follows. An arbitrary unitary transformation $\rho \mapsto U\rho U^\dagger$ corresponds to an internal time evolution according to some unknown Hamiltonian. Then, the absolutely separable states are exactly those bipartite states whose correlations are so weak that they can not be made entangled by any internal evolution. In the case of continuous variable quantum systems, one may hold the free-field Hamiltonian $\mathcal{H} = \frac{1}{2}r^T r$ as the privileged one, so that it makes sense to restrict oneself to those unitary evolutions that preserve this particular Hamiltonian. If the original state is Gaussian and the unitaries are generated by quadratic Hamiltonians, so that they are represented by symplectic matrices, preserving the free-field Hamiltonian is the defining feature of passive transformations, and one obtains exactly the problem we solved here.

As is often the case, the technical details and the nature of the solution are simpler in the Gaussian realm. We found that the condition for being 'absolutely separable' in the Gaussian sense is expressed by a simple inequality involving only the two smallest ordinary eigenvalues of the QCM, and that there are no 'absolutely PPT' states that are not 'absolutely separable' too. This latter equivalence has been conjectured to hold for the original problem in discrete-variable systems as well, but so far only partial answers are available. Namely, the conditions for absolute PPT-ness can be written explicitly [47], but whether or not they imply absolute separability is in general unknown. However, the answer to this latter question has been shown to be affirmative for the case of two qubits [48] and more recently for qubit-qudit systems [49].

9. Summary and outlook

In this work we advanced the mathematical and physical study of separability and entanglement distillability in Gaussian states of continuous variable quantum systems. Based on the properties of Schur complements and other matrix analysis tools, we obtained a simplified necessary and sufficient condition for the separability of all multimode Gaussian states, requiring optimisation over the set of local covariance matrices of one subsystem only. Exploiting this result, we presented a compact proof of the equivalence between PPT and separability for 1 versus n-mode Gaussian states, a seminal result in continuous variable quantum information theory [7, 15], as well as extended the criterion to multimode classes of so-called mono-symmetric and isotropic Gaussian states, through novel derivations. Furthermore, we completed the investigation of entanglement generation under passive operations by extending seminal results [25] to consider the generation of any, possibly PPT, Gaussian entangled state: in this context we showed that, if passive operations can not turn an initial Gaussian state into a non-PPT one, then no PPT entanglement can be generated through them either. This can be interpreted as establishing the equivalence between absolute separability and absolute PPT-ness in the Gaussian world. Side results of our analysis include a novel proof that Gaussian states invariant under partial transposition are separable, as well as an independent proof of the equivalence between Gaussian separability and complete extendability with Gaussian extensions [36].

In the context of this paper, and with the methods illustrated in this study, it would be interesting to research more general combinations of symmetries and conditions on the symplectic spectra of quantum covariance matrices whereby the sufficiency of the PPT separability criterion might be further extended. For instance, is it possible to obtain a Gaussian analogue of the results in [41], whereby bound entangled Gaussian states can only exist given some simple condition on their symplectic rank? In our studies, both for mono-symmetric and

isotropic states, large degeneracies in their symplectic spectra (for the marginal covariance matrix of one subsystem, and for the global covariance matrix of the bipartite system, respectively) played a key role in proving the sufficiency of PPT for separability. It would be desirable to provide a full systematic characterisation of such requirements, possibly drawing inspiration from and/or shedding new insight on the Gaussian quantum marginal problem [50].

Finally, let us stress how matrix analysis tools such as those heavily hammered in this paper have already been proven useful for qualitative and quantitative analysis of entanglement and other correlations, including Einstein–Podolsky–Rosen steering, in general states of continuous variable systems [8, 10–12, 18, 20–23, 51–53]. Aside from the fact that very powerful analytical results can be proven with relative simplicity using these tools, it is important to remark once more that the characterisations we provided of the separability problem, as well as the variational characterisation of the Schur complement and related problems, can be straightforwardly recast as *semidefinite programms* [20], thus leading to efficient numerical methods to witness inseparability and entanglement distillability in general multimode Gaussian or non-Gaussian states based on covariance matrices. We will explore these and other applications in further studies.

Acknowledgments

ORCID iDs

Ludovico Lami ⓘ https://orcid.org/0000-0003-3290-3557
Gerardo Adesso ⓘ https://orcid.org/0000-0001-7136-3755

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