Weighted Fractal Networks

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(Dated:)

In this paper we define a new class of weighted complex networks sharing several properties with fractal sets, and whose topology can be completely analytically characterized in terms of the involved parameters and of the fractal dimension. General networks with fractal or hierarchical structures can be set in the proposed framework that moreover could be used to provide some answers to the widespread emergence of fractal structures in Nature.

I. INTRODUCTION

Complex networks have recently attracted a growing interest of scientists from different fields of research, mainly because complex networks define a powerful framework, among others developed in the complex systems sciences, for describing, analyzing and modeling real systems that can be found in Nature and/or society. This framework allows to conjugate the micro to the macro abstraction levels: nodes can be endowed with local dynamical rules, while the whole network can be thought to be composed by hierarchies of clusters of nodes, that thus exhibits aggregated behavior.

The birth of graph theory is usually attributed to L. Euler with his seminal paper concerning the "Königsberg bridge problem"(1736), but it is only in the 50's that network theory started to develop autonomously with the pioneering works of Erdős and Rényi [1]. Nowaday network theory defines a research field in its own [2, 3] and the scientific activity is mainly devoted to construct and characterize complex networks exhibiting some of the remarkable properties of real networks, scale–free [4], small–world [5], communities [6], weighted links [7–9], just to mention few of them.

In recent years we observed an increasing production of papers [10–17] where authors proposed a new point of view by constructing networks exhibiting scale-free and hierarchical structures by adapting ideas taken from fractal construction, e.g. Koch curve or Sierpinski gasket. The aim of the present paper is to generalize these latter constructions and to define a general framework, hereby named *Weighted Fractal Networks*, WFN for short, whose networks share with fractal sets several interesting properties, for instance the self-similarity and the hierarchical structures.

The WFN are constructed via an explicit algorithm and we are able to completely analytically characterize their topology as a function of the parameters involved in the construction. We are thus able to prove that WFN exhibit the "small–world"property, i.e. slow (logarithmic) increase of the average shortest path with the network size, and large average clustering coefficient. Moreover the probability distribution of node strength follows a power law whose exponent is the Hausdorff (fractal) dimension of the "underlying"fractal, hence the WFN are scale–free.

WFN also represent an explicitly computable model for the renormalization procedure recently applied to complex networks [18–21].

The paper is organized as follows. In the next section we will introduce the model, we outline the similarities with fractal sets and the differences with respect to the above mentioned earlier results. In Section III we present the analytical characterization of such networks also supported by dedicated numerical simulations. We then introduce in Section IV a straightforward generalization of the previous theory, and thus we conclude by showing a possible application of WFN to the study of fractal structures emerging in Nature.

II. THE MODEL

According to Mandelbrot [22] "a fractal is by definition a set for which the Hausdorff dimension strictly exceeds the topological dimension". One of the most amazing and interesting feature of fractals is their *self-similarity*, namely looking at all scales we can find conformal copies of the whole set. Starting from this property one can provide rules to build up fractals as fixed point of *Iterated Function Systems* [23, 24], IFS for short, whose Hausdorff dimension is completely characterized by two main parameters, the number of copies $s > 1$ and the scaling factor $0 < f < 1$
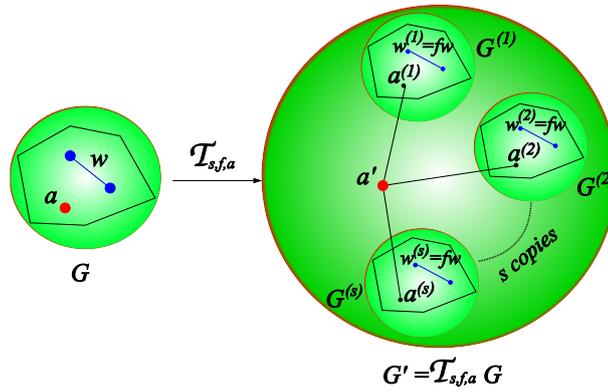
FIG. 1: The definition of the map $\mathcal{T}_{s,f,a}$. On the left a generic initial graph G with its attaching node a (red on-line) and a generic weighted edge $w \in G$ (blue on-line). On the right the new graph G' obtained as follows: Let $G^{(1)}, \ldots, G^{(s)}$ be s copies of G, whose weighted edges (blue on-line) have been scaled by a factor f. For $i = 1, \ldots, s$ let us denote by $a^{(i)}$ the node in $G^{(i)}$ image of the labeled node $a \in G$, then link all those labeled nodes to a new node a' (red on-line) through edges of unitary weight. The connected network obtained linking the s copies $G^{(i)}$ to the node a' will be by definition the image of G through the map: $G' = \mathcal{T}_{s,f,a}(G)$.

of the IFS. Let us observe that in this case this dimension coincides with the so called similarity dimension [24], $d_{fract} = -\log s / \log f$.

The main goal of this paper is to generalize such ideas to networks, aimed at constructing weighted complex networks [37] with some a priori prescribed topology, that will be described in terms of node strength distribution, average (weighted) shortest path and average (weighted) clustering coefficient, depending on the two main parameters: the number of copies and the scaling factor [38]. Moreover taking advantage of the similarity with the IFS fractals, some topological properties of the networks will depend on the fractal dimension of the IFS fractal.

Let us fix a positive real number $f < 1$ and a positive integer $s > 1$ and let us consider a (possibly) weighted network G composed by N nodes, one of which has been labeled *attaching node* and denoted by a. We then define a map, $\mathcal{T}_{s,f,a}$, depending on the two parameters s, f and on the labeled node a, whose action on networks is described in Fig. 1.

So starting with a given initial network G_0 we can construct a family of weighted networks $(G_k)_{k \geq 0}$ iteratively applying the previously defined map: $G_k := \mathcal{T}_{s,f,a}(G_{k-1})$.

Because of its general definition, the map $\mathcal{T}_{s,f,a}$ could be used to cast the various models of fractal or hierarchical networks recently proposed [10–17] into a unified scheme, instead of using "ad hoc" constructions and computations, and so to obtain informations about some relevant topological quantities, such as average shortest path and nodes strength, in a straightforward way using the proposed framework and thus prove, for instance, the small–world character of WFN. In this framework we are also able to construct scale–free networks with any desired power–law just by setting s and f, thus overcoming some limitations of [12].

Let us observe that the main differences between WFN and the above mentioned models reside in the tree–like structure, instead of a star–like [10, 11, 13], that the former acquire because of the growth process, although WFN will exhibit high clustering coefficient, and the absence of a preferential attachment mechanism weight dependent [7, 9, 14].

For the sake of completeness we present numerical results for two WFN: the *Sierpinski* one (see Fig. 2) and the *Cantor dust* (see Fig. 3).

Given G_0 and the map $\mathcal{T}_{s,f,a}$ we are able to completely characterize the topology of each G_k and also of the limit network G_∞, defined as the fixed point of the map: $G_\infty = \mathcal{T}_{s,f,a}(G_\infty)$. Let us observe that the topology of G_∞ will not depend on the initial "seed" G_0 as it happens for the IFS fractal sets.

Thus the WFN undergo through a growth process strictly related to the inverse of the renormalization procedure [18, 19]; at the same time G_∞ will be infinitely renormalizable.

III. RESULTS

The aim of this section is to characterize the topology of the graphs G_k for all $k \geq 1$ and G_∞, by analytically studying their properties such as the average degree, the node strength distribution, the average (weighted) shortest path and the average (weighted) clustering coefficient.
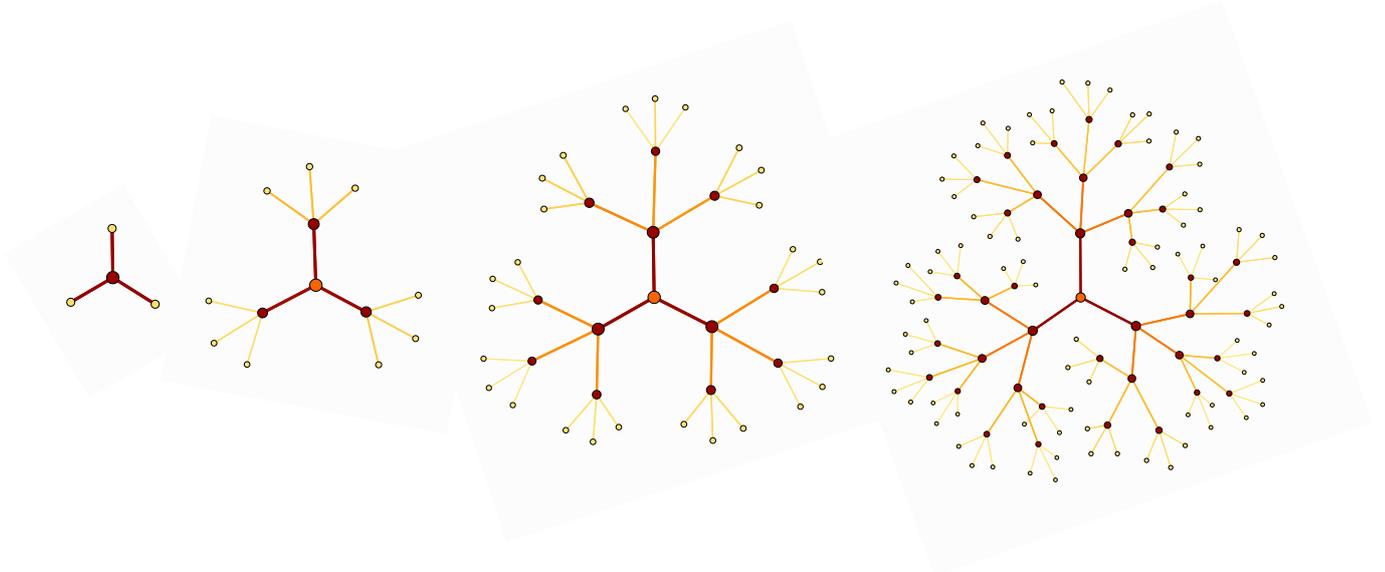
FIG. 2: The "Sierpinski" WFN, $s = 3$, $f = 1/2$ and G_0 is composed by a single node. From the left to the right G_1, G_2, G_3 and G_4. Gray scale (color on-line) reproduces edges weights: the darker the color the larger the weight. The dimension of the fractal is $\log 3 / \log 2 \sim 1.5850$. Visualization was done using Himmeli software [26].
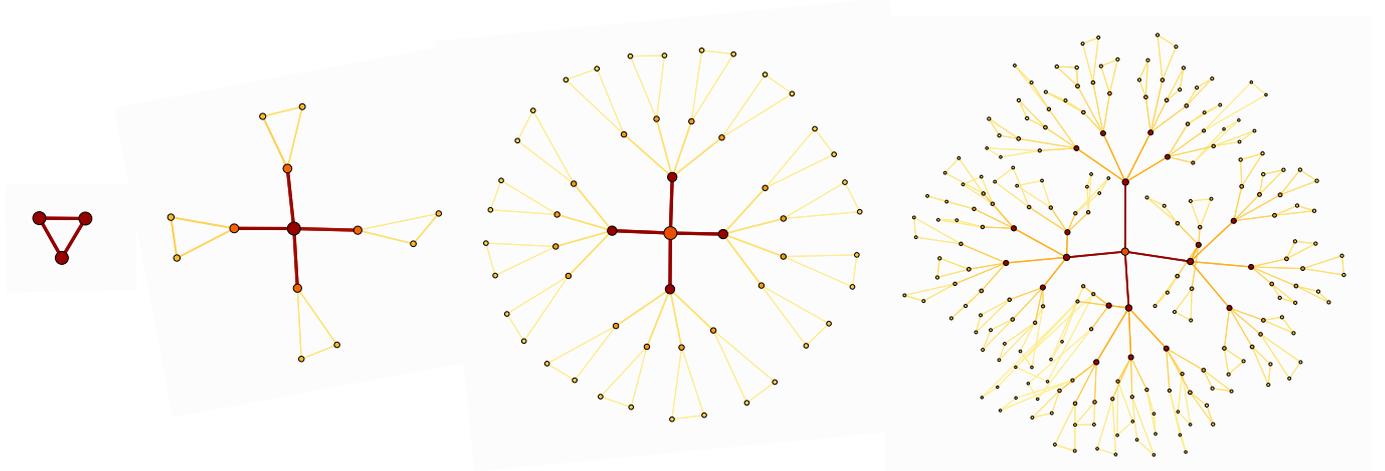

FIG. 3: The "Cantor dust" WFN, $s = 4$, $f = 1/5$ and G_0 is a triangle. From the left to the right G_0, G_1, G_2 and G_3. Gray scale (color on-line) reproduces edges weights: the darker the color the larger the weight. The dimension of the fractal is $\log 4 / \log 5 \sim 0.8614$. Visualization was done using Himmeli software [26].

At each iteration step the graph G_k grows as the number of its nodes increases according to

$$N_k = s^k N_0 + (s^k - 1)/(s - 1), \tag{1}$$

being N_0 the number of nodes in the initial graph, while the number of edges satisfies

$$E_k = s^k E_0 + s(s^k - 1)/(s - 1), \tag{2}$$

being E_0 the number of edges in the graph G_0. Hence in the limit of large k the *average degree* is finite and it is asymptotically given by

$$\frac{E_k}{N_k} \xrightarrow[k \to \infty]{} \frac{s + E_0(s - 1)}{1 + (s - 1)N_0}. \tag{3}$$

Let us denote the weighted degree of node $i \in G_k$, also called *node strength* [8], by $\omega_i^{(k)} = \sum_j w_{ij}^{(k)}$, being $w_{ij}^{(k)}$ the weight of the edge $(ij) \in G_k$; then using the recursive construction, we can explicitly compute the total node strength,

$W_k = \sum_i \omega_i^{(k)}$, and, provided $sf \neq 1$, easily show that

$$W_k = 2s \frac{(sf)^k - 1}{sf - 1} + (sf)^k W_0 \,.$$

Because $f < 1$, we trivially find that the *average node strength* goes to zero [39] as k increases: $W_k/N_k \xrightarrow[k \to \infty]{} 0$.

A. Node strength distribution.

Let $g_k(x)$ denote the number of nodes in G_k that have strength $\omega_i^{(k)} = x$ and let us assume g_0 to have values in some finite discrete subset of the positive reals, namely:

$$g_0(x) > 0 \text{ if and only if } x \in \{x_1, \ldots, x_m\} \,,$$

otherwise $g_0(x) = 0$. Using the property of the map $\mathcal{T}_{s,f,a}$ we straightforwardly get $g_k(x) = s g_{k-1}(x/f)$ provided [40] $x \neq s$ and $x \neq fs + 1$, from which we can conclude that for all k:

$$g_k(x) = s^k g_0(x/f^k) \,, \quad g_k(fs + 1) = s \quad \text{and} \quad g_k(s) = 1 \,. \tag{4}$$

This implies than the node strengths are distributed according to a power law with exponent $d_{fract} = -\log s/\log f$, that equals the fractal dimension of the fractal obtained as fixed point of the IFS with the same parameters s and f. In fact defining $x_{ik} = f^k x_i$ we get:

$$\begin{aligned}
\log g_k(x_{ik}) &= k \log s + \log g_0(x_i) \\
&= \frac{\log s}{\log f} \log x_{ik} + \log g_0(x_i) - \frac{\log s}{\log f} \log x_i \,,
\end{aligned}$$

namely for large k (see Fig. 4)

$$g_k(x) \sim C/x^{d_{frac}} \,. \tag{5}$$

B. Average weighted shortest path.

By definition the average *weighted shortest path* [3] of the graph G_k is given by

$$\lambda_k = \frac{\Lambda_k}{N_k(N_k - 1)} \,, \tag{6}$$

where

$$\Lambda_k = \sum_{ij \in G_k} p_{ij}^{(k)} \,, \tag{7}$$

being $p_{ij}^{(k)}$ the weighted shortest path linking nodes i and j in G_k.

To simplify the remaining part of the proof it is useful to introduce $\Lambda_k^{(a_k)} = \sum_{i \in G_k} p_{ia_k}^{(k)}$, i.e. the sum of all weighted shortest paths ending at the attaching node, $a_k \in G_k$. One can prove (see Appendix A 1) that for large k the asymptotic behavior of $\Lambda_k^{(a_k)}$ is given by

$$\Lambda_k^{(a_k)} \underset{k \to \infty}{\sim} \frac{N_0(s-1) + 1}{(1-f)(s-1)} s^{k-1} \,. \tag{8}$$

Using the construction algorithm and its symmetry one can prove (see Appendix A 2) that Λ_k satisfies the recursive relation

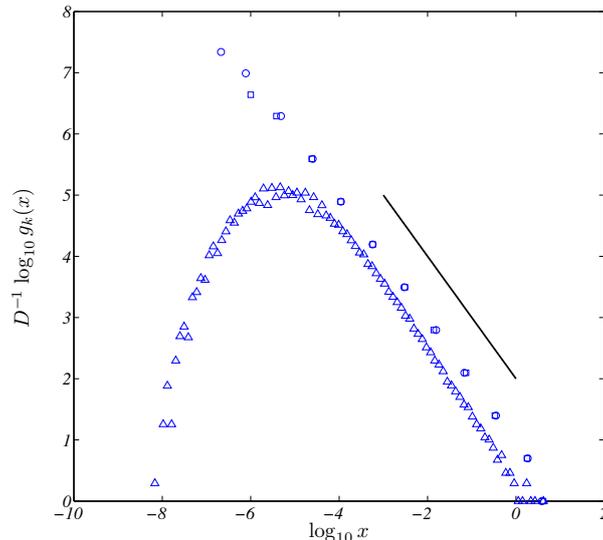$$\Lambda_k = sf\Lambda_{k-1} + 2s[(s-1)N_{k-1} + 1][N_{k-1} + f\Lambda_{k-1}^{(a_{k-1})}] \,, \tag{9}$$

FIG. 4: Node Strengths Distribution. Plot of the renormalized node strengths distribution $D^{-1} \log_{10} g_k(x)$, where $D = d_{frac}$ in the homogeneous case, while $D = -\frac{s \log s}{\log(f_1 \cdots f_s)}$ in the non–homogeneous one. Symbols refer to : $\square$ the finite approximation G_{14} with 2391484 nodes of the "Sierpinski" WFN, $s = 3$, $f = 1/2$ and G_0 is formed by one initial node; $\bigcirc$ the finite approximation G_{11} composed by 3495253 nodes of the "Cantor dust" WFN, $s = 4$, $f = 1/5$ and G_0 is made by a triangle; $\triangle$ the finite approximation G_{11} composed by 3495253 nodes of the non–homogeneous "Cantor dust" WFN, $s = 4$, $f_1 = 1/2$, $f_2 = 1/3$, $f_3 = 1/5$, $f_4 = 1/7$ and G_0 is formed by a triangle. The reference line has slope -1; linear best fits (data not shown) provide a slope -0.9964 ± 0.034 with $R^2 = 0.9993$ for the Sierpinski WFN, a slope -1.002 ± 0.064 with $R^2 = 0.9996$ for the Cantor dust WFN and a slope -1.006 ± 0.024 with $R^2 = 0.9976$ for the non–homogeneous Cantor dust WFN.
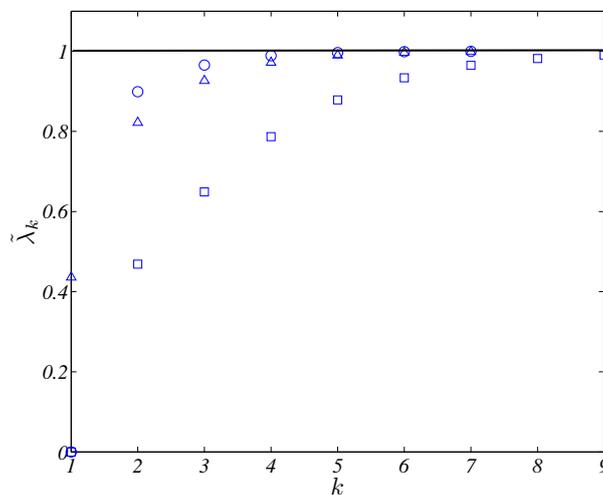

FIG. 5: The average weighted shortest path. Plot of the renormalized average weighted shortest path $\tilde{\lambda}_k$ versus the iteration number k, where $\tilde{\lambda}_k = \lambda_k \frac{(s-F)(s^2-F)}{2s^2(s-1)}$ and $F = f_1 + \cdots + f_s$ for the non–homogeneous case, while $F = sf$ for the homogeneous one. Symbols refer to : $\square$ the "Sierpinski" WFN, $s = 3$, $f = 1/2$ and G_0 is formed by one initial node; $\bigcirc$ the "Cantor dust" WFN, $s = 4$, $f = 1/5$ and G_0 is made by a triangle; $\triangle$ the non–homogeneous "Cantor dust" WFN, $s = 4$, $f_1 = 1/2$, $f_2 = 1/3$, $f_3 = 1/5$, $f_4 = 1/7$ and G_0 is formed by a triangle.
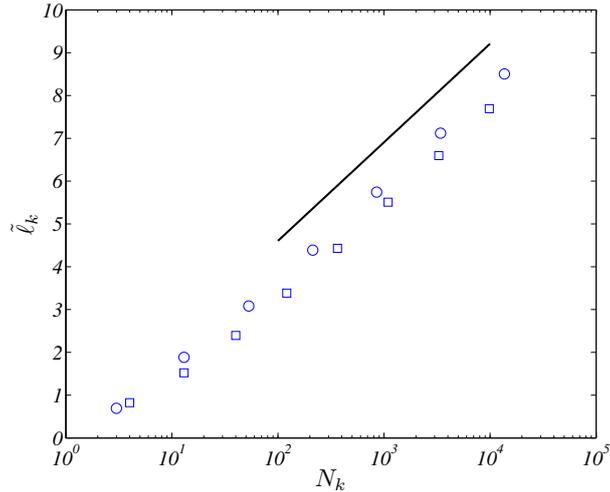
FIG. 6: The average shortest path ℓ_k as a function of the network size (semilog graph). Plot of the renormalized average shortest path $\tilde{\ell}_k$ versus the network size N_k, where $\tilde{\ell}_k = \ell_k \frac{\log s}{2}$. Symbols refer to : $\square$ the "Sierpinski" WFN, $s = 3$, $f = 1/2$ and G_0 is formed by one initial node; $\bigcirc$ the "Cantor dust" WFN, $s = 4$, $f = 1/5$ and G_0 is made by a triangle. The reference line has slope 1. Linear best fits (data not shown) provides a slope 0.9942 ± 0.019 and $R^2 = 1$ for the Sierpinski WFN and a slope 0.9952 ± 0.019 and $R^2 = 1$ for the Cantor dust WFN.

that provides the following asymptotic behavior in the limit of large k (see Fig. 5)

$$\lambda_k = \frac{\Lambda_k}{N_k(N_k - 1)} \xrightarrow[k \to \infty]{} \frac{2(s-1)}{(1-f)(s-f)} . \tag{10}$$

We can also compute the *average shortest path*, ℓ_k, formally obtained by setting $f = 1$ in the previous formulas (6) and (7). Hence slightly modifying the results previously presented we can prove that asymptotically we have

$$\ell_k \underset{k \to \infty}{\sim} 2 \left(k - \frac{s}{s-1} \right) \underset{k \to \infty}{\sim} \frac{2}{\log s} \log N_k , \tag{11}$$

where the last relation has been obtained using the growth law of N_k given by equation (1) (see Fig. 6). Let us remark that the average shortest path is a topological quantity and thus it doesn't depend on the scaling factor, that's why we don't report in Fig. 6 the case of the non-homogeneous WFN.

Thus, as previously stated, the network grows unbounded but with the logarithm of the network size, while the weighted shortest distances stay bounded.

C. Average clustering coefficient.

The average clustering coefficient [3, 5] of the graph G_k is defined as the average over the whole set of nodes of the local clustering coefficient $c_i^{(k)}$, namely $< c_k > = C_k/N_k$, where $C_k = \sum_{i \in G_k} c_i^{(k)}$. Because of the construction algorithm the network inherits a tree–like structure preventing the inner core from acquiring new triangles, in such a way that the number of possible triangles, hence the local clustering coefficient, at each step increases just by a factor s. Thus after k iterations we will have $C_k = s^k C_0$, being C_0 the sum of local clustering coefficients in the initial graph. We can thus conclude that the clustering coefficient of the graph is asymptotically given by:

$$< c_k > \xrightarrow[k \to \infty]{} \frac{s-1}{s} \frac{< c_0 > N_0}{(s-1)N_0 + 1} . \tag{12}$$

On the other hand, one can use edges' values to weigh the clustering coefficient [27]; hence generalizing the previous relation, we can easily prove that the average *weighted clustering coefficient* of the graph is asymptotically given by:

$$< \gamma_k > \underset{k \to \infty}{\sim} \frac{s-1}{fs} \frac{< \gamma_0 > N_0}{(s-1)N_0 + 1} f^k \underset{k \to \infty}{\sim} \frac{1}{N_k^{1/d_{fract}}} , \tag{13}$$

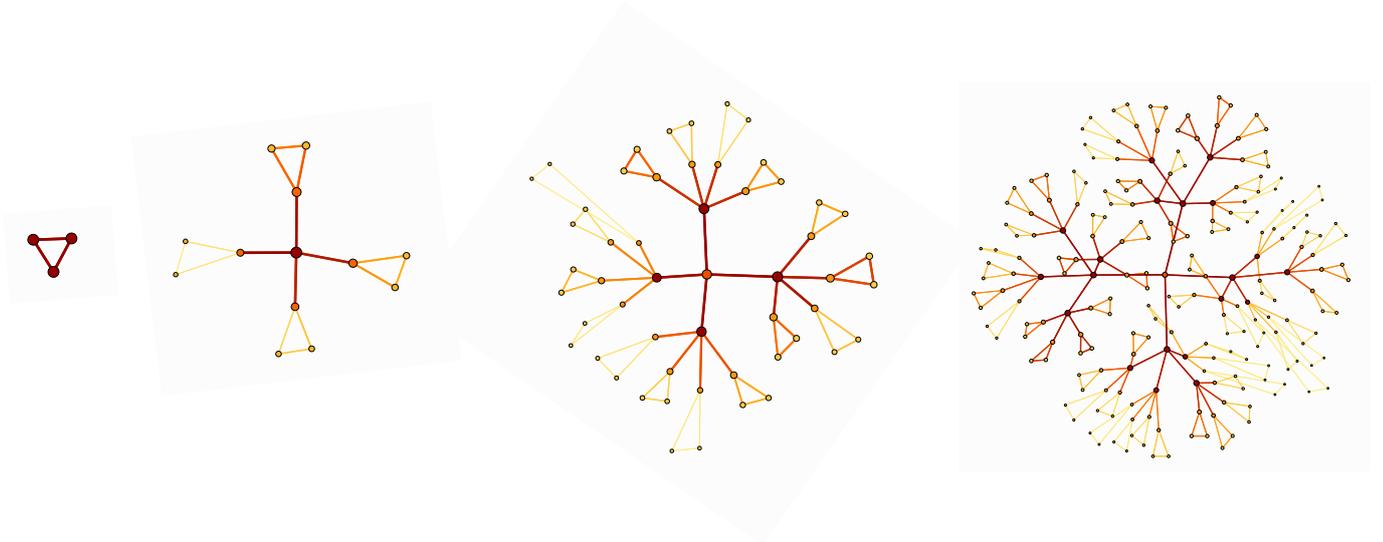where once again, the fractal dimension d_{fract} of the IFS fractal play a relevant role.

FIG. 7: The non–homogeneous "Cantor dust" WFN, $s = 4$, $f_1 = 1/2$, $f_2 = 1/3$, $f_3 = 1/5$, $f_4 = 1/7$ and G_0 is formed by a triangle. From the left to the right G_0, G_1, G_2 and G_3. Gray scale (color on-line) reproduces edges weights: the darker the color the larger the weight. Visualization was done using Himmeli software [26].

IV. NON–HOMOGENEOUS WEIGHTED FRACTAL NETWORKS

The aim of this section is to slightly generalize the previous construction to the case of *non–homogeneous* scaling factors for each subnetwork $G^{(i)}$. So given an integer $s > 1$ and s real numbers $f_1, \ldots, f_s \in (0, 1)$, we modify the map $\mathcal{T}_{s,f,a}$ by allowing a different scaling for each edge weight according to which subgraph it belongs to: if the edge $w^{(j)}$, image of $w \in G$, belongs to $G^{(j)}$, then $w^{(j)} = f_j w$.

Let us remark that the construction presented in the former Section II is a particular case of the latter, once we take $f_1 = \cdots = f_s = f$; we nevertheless decided for a sake of clarity, to present it before, because the computations involved in this latter general construction could have hidden the simplicity of the underlying idea. We hereby present some results for the non–homogeneous "Cantor dust" WFN (see Fig. 7).

Using the recursiveness of the algorithm we can, once again, completely characterize the topology of the non–homogeneous WFN, moreover only the weighted quantities will vary with respect to the homogeneous case. For instance, a straightforward, but cumbersome, generalization of the computations presented in the previous Sections allows us to prove that the average weighted shortest path exhibits the following asymptotic behavior (see Fig. 5)

$$\lambda_k \xrightarrow[k \to \infty]{} \frac{2s^2(s-1)}{(s-F)(s^2-F)}\,, \tag{14}$$

where $F = f_1 + \cdots + f_s$. Let us observe that Eq. (14) reduces to Eq. (10) once we set $f_1 = \cdots = f_s = f$ and thus $F = sf$.

Let $g_0(x)$ denote the number of nodes with node strength equal to x in the initial network G_0; then after k steps of the algorithm, all nodes strengths will be rescaled by a factor $f_1^{k_1} \ldots f_s^{k_s}$, where the non-negative integers k_i do satisfy $k_1 + \cdots + k_s = k$. Because this can be done in $k!/(k_1! \ldots k_s!)$ possible different ways, we get the following relation for the node strength distribution for the network G_k:

$$g_k(f_1^{k_1} \ldots f_s^{k_s} x) = \frac{k!}{k_1! \ldots k_s!} g_0(x) \quad \text{with } k_1 + \cdots + k_s = k\,. \tag{15}$$

After sufficiently many steps and assuming that the main contribution arises from the choice $k_1 \sim \cdots \sim k_s \sim k/s$, we can use Stirling formula to get the approximate distribution (see Fig. 4)

$$\log g_k(x) \sim \frac{s \log s}{\log(f_1 \ldots f_s)} \log x\,, \tag{16}$$

so once again the nodes strength distribution follows a power law.

V. CONCLUSIONS

In this paper we introduced a unifying framework for complex networks sharing several properties with fractal sets, hereby named *Weighted Fractal Networks*. This theory, that generalizes to graphs the construction of IFS fractals, allows us to build complex networks with a prescribed topology, whose main quantities can be analytically predicted and have been shown to depend on the fractal dimension of the IFS fractal; for instance the networks are scale–free with exponent the fractal dimension. Moreover the weighted fractal networks share with IFS fractals, the self-similarity structure, and are explicitly computable examples of renormalizable complex networks.

These networks exhibit the *small–world* property. In fact the average shortest path increases logarithmically with the system size (11), hence it is small as the average shortest path of a random network with the same number of nodes and same average degree. On the other hand the clustering coefficient is asymptotically constant (12), thus larger than the clustering coefficient of a random network that shrinks to zero as the system size increases.

The self-similarity property of the weighted fractal networks makes them suitable to model real problems involving generic diffusion over the network coupled with local looses of flow, here modeled via the parameter $f < 1$. For instance electrical grids in the case of mankind artifacts, metabolic networks of living organisms [28, 29] or air flows in mammalian lungs [30–35] in the case of natural networks. In all these studies, assuming some hierarchical fractal structure, scientists could explain some natural laws, such as the allometric scaling or the avalanches in the mammalian respiratory system, although the actual branching numbers could not be explained. Using our framework we suggest a possible analysis for these values; for instance in mammalian lungs where air flows through bronchi–bronchioles, submitted to air vessels' section reduction. Assuming that the recorded avalanches and power–laws, hence the fractal structure, have some functional–biological reason, e.g. a typical time needed to fulfill alveoli with oxygenated air from the primary bronchi, we can relate this time to the average shortest weighted path, which in turn depends of the number of copies s and the scaling factor f. In other words the topology of such networks could have been shaped by evolution in such a way any two nodes can be connected in a finite optimal time, whatever their physical distance is.

Let us finally observe that in this case the fragility of the network with respect to failure of the edges, see [36] and references therein, is a main biological question, that can be restated as follows: how many air vessels should fail before the lung stops to correctly operate ?

Appendix A: Complementary material

1. Computation of $\Lambda_k^{(a_k)}$

Let a_k be the attaching node of the graph G_k. Let us define $\Lambda_k^{(a_k)} = \sum_{i \in G_k} p_{ia_k}^{(k)}$, i.e. the sum of all weighted shortest paths to a_k. Then using the recursive property and the symmetry of the map $\mathcal{T}_{s,f,a_k}$ we can easily obtain a recursive relation for $\Lambda_k^{(a_k)}$:

$$\Lambda_k^{(a_k)} = sf\Lambda_{k-1}^{(a_{k-1})} + sN_{k-1},$$

where N_{k-1} is the number of nodes in G_{k-1}. This recursion can be easily solved to get for all $k \geq 1$

$$\Lambda_k^{(a_k)} = (sf)^{k-1}\Lambda_1^{(a_1)} + \frac{1-f^{k-1}}{1-f}\frac{(s-1)N_0+1}{s-1}s^{k-1} - \frac{s}{s-1}\frac{(sf)^{k-1}-1}{sf-1}, \tag{A1}$$

from which we can conclude, because $f < 1$, that $\Lambda_k^{(a_k)}$ exhibits the asymptotic behavior given by equation (8).

2. Computation of Λ_k

Starting from the definition of the sum of all weighted shortest paths (7), the recursive construction and its symmetry we can decompose the sum Λ_k into three terms:

$$\Lambda_k = s \sum_{ij \in G_k^{(1)}} p_{ij}^{(k)} + s(s-1) \sum_{i \in G_k^{(1)}, j \in G_k^{(2)}} p_{ij}^{(k)} + 2s \sum_{i \in G_k^{(1)}} p_{ia_k}^{(k)} \tag{A2}$$

where the first contribution takes into account all paths starting from and arriving to nodes belonging to the same subgraph, that using the symmetry can be chosen to be $G_k^{(1)}$. The second term takes into account all the possible

paths where the initial point and the final one belong to two different subgraphs, and still using the symmetry we can set them to $G_k^{(1)}$ and $G_k^{(2)}$ and multiply the contribution by a combinatorial factor $s(s-1)$. Finally the last term is the sum of all paths arriving to the attaching node a_k; once again the symmetry allows us to reduce the sum to only one subgraph, say $G_k^{(1)}$, and multiply the contribution by $2s$.

Using the scaling mechanism for the edges, the first term in the right hand side of equation (A2) can be easily identified with

$$\sum_{ij \in G_k^{(1)}} p_{ij}^{(k)} = f \Lambda_{k-1} \,.$$

By construction, each shortest path connecting two nodes belonging to two different subgraphs, must pass through the attaching node, hence using $p_{ij}^{(k)} = p_{ia_k}^{(k)} + p_{a_k j}^{(k)}$ the second term of equation (A2) can be split into two parts:

$$\sum_{i \in G_k^{(1)}, j \in G_k^{(2)}} p_{ij}^{(k)} = \sum_{i \in G_k^{(1)}} p_{ia_k}^{(k)} N_k^{(2)} + \sum_{j \in G_k^{(2)}} p_{a_k j}^{(k)} N_k^{(1)} \,,$$

where $N_k^{(i)}$ denotes the number of nodes in the subgraph $G_k^{(i)}$. Using the symmetry of the construction, the previous relation can be rewritten as

$$\sum_{i \in G_k^{(1)}, j \in G_k^{(2)}} p_{ij}^{(k)} = 2 N_k^{(1)} \sum_{i \in G_k^{(1)}} p_{ia_k}^{(k)} \,.$$

The last term of equation (A2) can be related to $\Lambda_{k-1}^{(a_{k-1})}$ by observing that each path arriving at a_k must pass through $a_k^{(i)}$ for some $i \in \{1, \ldots, s\}$, thus

$$
\begin{aligned}
\sum_{i \in G_k^{(1)}} p_{ia_k}^{(k)} &= \sum_{i \in G_k^{(1)}} (p_{ia_k^{(1)}}^{(k)} + p_{a_k^{(1)} a_k}^{(k)}) = N_k^{(1)} + \sum_{i \in G_k^{(1)}} p_{ia_k^{(1)}}^{(k)} \\
&= N_k^{(1)} + f \Lambda_{k-1}^{(a_{k-1})} \,,
\end{aligned}
\tag{A3}
$$

Observing that $G_k^{(1)}$ has as many nodes as G_{k-1} we can conclude that $N_k^{(1)} = N_{k-1}$ and finally to rewrite equation (A2) as:

$$\Lambda_k = s f \Lambda_{k-1} + 2s[(s-1)N_{k-1} + 1][N_{k-1} + f \Lambda_{k-1}^{(a_{k-1})}] \,.$$

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[37] We hereby present the construction for undirected networks, but it can be straightforwardly generalized to directed graphs as well.

[38] A straightforward generalization will be presented in the next Section IV. See also [25] where the WFN theory will be generalized as to include a stochastic iteration process.

[39] Let us observe that the same is true if $sf = 1$; in this case, in fact W_k grows linearly with k, thus slower than N_k.

[40] Without loose of generality we can assume that for all integers $i, j \in \{1, \ldots, m\}$ and $k > 0$ we have $f^k x_j \neq x_i$ and $f^k(fx_j + 1) \neq x_i$.