The anti-self-dual Yang-Mills equations and discrete integrable systems

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To my parents, Carmine e Pia

I, Gregorio Benedetto Benincasa, declare that, to the best of my knowledge, the material contained in this thesis is original work obtained in collaboration with my supervisor, R. Halburd, unless otherwise stated. Parts of Section 3.2 and of Chapters 4 and 6 have already been published as [1]. Chapters 4 and 5 and the Appendix contain material which will appear in a joint paper with R. Halburd [2]. Most of Chapter 6 will appear in a joint paper with R. Halburd [3]. The material in Chapter 7 is work in progress.

Abstract

In this dissertation the Bäcklund-Darboux transformations for the anti-self-dual Yang-Mills (ASDYM) equations and implications of such constructions are studied. After introducing Bäcklund and Darboux type transformations and the anti-self-dual Yang-Mills equations, which are the central objects we are concerned with, two principal themes arising from these are treated. Firstly, we construct a Bäcklund-Darboux transformation for the ASDYM equations and present reductions of this transformation to the transformations of integrable sub-systems embedded in the anti-self-duality equations. We further show how the geometry of the ASDYM equations may be exploited to give a more geometric understanding of the degeneration process involved in mapping one Painlevé equation to another. Our transformation inherits some of this geometry and we exploit this feature to lift the degenerations to the transformation itself.

The second theme deals with a reinterpretation of such structure. We employ the transformation for the construction of a discrete equation governing the evolution of solutions to the ASDYM equations on the lattice. This system is a lattice gauge theory defined over $\mathbb{Z}^2$ and we discuss the properties of such system, including some reductions and continuous limits. A Darboux transformation for this system and an extension of this system to three dimensions is also presented. We conclude with an analysis of the singular structure of the ASDYM equations.

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Contents

Chapter 1

Introduction

The theory of integrable systems studies fundamental classes of non-linear equations either differential (DE) or discrete (dE) which, in principle, can be solved analytically. There is, however, no single characterisation of what it means for systems to be integrable, let alone a definition for this. Therefore, for the purpose of this thesis, we shall consider integrability to be a statement regarding two specific properties which integrable systems share and which make it possible to describe the solutions to the system. The first such feature is the existence of an associated, overdetermined, linear system whose compatibility condition results in the non-linear equation under consideration. This is the so-called Lax pair (or, more generally, zero-curvature) formalism ([4, 5]), a crucial ingredient (though not sufficient to guarantee in itself integrability) in solving a vast class of non-linear equations. The second feature, of central importance to this thesis, is the existence of a very special class of transformations mapping the solution space of the system to the same solution space or to the solution space of a different equation. These transformations, known as Darboux and Bäcklund transformations, originate in the classical differential geometry of surfaces and are crucial solution-generation techniques. However, suitably reinterpreted, the implications of these actions are much broader; these transformations applied to continuous systems may be employed to give rise to difference equations. Thus we find that some functions can be defined both by DEs and dEs. The independent variables of these two equations will not be the same, rather we find a remarkable duality between the parameters appearing in the equation (or in the transformations) and the independent variables. It is worth mentioning that a wealth of other features are common to integrable systems, including the existence of many conserved quantities, the Painlevé property and hierarchies of commuting flows for continuous systems or singularity confinement and multidimensional consistency, for discrete equations, [4] and [6]–[12].

Given the particular features of such systems it should come as no surprise that integrable systems are a rare breed and, in fact, these systems are only a small fraction of non-linear systems. Even though they are of great importance in mathematics due to their nature as systems which are highly non-trivial and yet very tractable, one might contend that the particular investigation of such systems is void of relevance as these are too special and thus possess little to no relevance and applicability. Remarkably, this is not the case. In fact integrable systems possess a universal character making them widely applicable. This astonishing feature of universality (see [13, 14]) is reflected in their appearance in a vast number of physical applications, from non-linear optics to fluid dynamics, from spinning tops to general relativity and particle physics. Integrability is closely related with regularity of the behaviour of the solutions, where this regularity is not a reflection of triviality, but rather depends on some underlying mathematical structure that restricts the dynamics in some non-trivial way. The rich mathematical structure which inhabits these systems has deep connections to important areas of mathematics such as complex analysis, algebraic geometry and differential geometry to name a few.

Among all this, a central role in the subject is played by the anti-self-dual Yang-Mills (ASDYM) equations (a system of non-linear equations for Lie algebra valued functions on $\mathbb{C}^4$). These form the condition that the curvature of a connection be anti-self-dual (ASD) with respect to the Hodge star operator and are, by virtue of the Bianchi identity, also solutions of the Yang-Mills equations. The ASDYM equations are integrable in the sense of possessing an associated linear problem and there exist profound connections between this system and integrability. It itself has been found to be a rich source of integrable systems in the sense that most of the known integrable systems in dimensions three, two and one are embedded in the

ASDYM equations, arising as symmetry reductions of it. The ASDYM equations also play a fundamental role in other areas of mathematics, noteworthy is their use in classifying four dimensional manifolds in the work of Donaldson, [15]. From the perspective of integrable systems the importance of this system is that it is an integrable system in four dimensions which may be viewed as a master system in the sense hinted to above; a welcome feature given that it allows us to put, at least in part, some order in the realm of integrable systems.

1.1 Overview, Motivation and Outline

This thesis is concerned with the construction of a general Bäcklund-Darboux type transformation for the anti-self-dual Yang-Mills equations. The consequences of such a transformation in conjunction with the partial ordering property of integrable systems, whereby the process of reducing an integrable system by imposing symmetry or specialising the parameters leads to another integrable system, are explored. The partial ordering property led to the search for a 'maximal element' [10, 16], that is an integrable system from which all others may be derived. Although such a 'master' system has not been found it is the case that the ASDYM equations are able to yield almost all known integrable systems in one and two dimensions and important systems in three dimensions. Furthermore, having integrable systems arise as reductions of the ASDYM equations allows these to inherit some of the rich geometry of the ASDYM equations. The motivation for the work presented in this thesis relies on the above feature of the ASDYM equations as a sort of 'master' system and on the existence of Bäcklund transformations (BTs) mapping a solution to an integrable equation to another such solution. The latter transformations can be used, among other things, to construct discrete integrable equations from continuous ones thus mapping integrable continuous systems to discrete ones. Such structure provides us with the possibility to develop the following prescription:

- reduce the ASDYM equations to some lower dimensional integrable system,
- construct the Bäcklund transformations for the reduced equation,

- exploit the Bäcklund transformations for the construction of a discrete equation.

In this work we exploit these features in order to unify the various ingredients on which such prescription relies, develop new structure and make the above process more direct (see figure 1.1). For this we construct a general Bäcklund-Darboux type transformation ([1, 2, 3]) for the ASDYM equations capable of yielding meaningful transformations independently of which gauge the system is in. Its generality relies on this important feature of gauge covariance, itself a consequence of two 'transporter' matrices appearing in the Darboux matrix. This is the principal ingredient of our work, a sort of 'master' Bäcklund transform which, under reduction, yields the Bäcklund transforms of the reduced equations (figure 1.1). Finally, equipped with this transformation, we employ it for the construction of a general lattice equation governing the evolution of solutions to the ASDYM equations on a lattice. We find that such a system, which we have called the 'ASDYM Bianchi system', is gauge invariant on the lattice and is itself a rich source of discrete integrable systems. It possesses continuous limits to two-dimensional reductions of the ASDYM equations and reductions to well-known integrable lattice equations. It fails to represent a discrete ASDYM (dASDYM) equation given that it is a system on the $\mathbb{Z}^2$ lattice, but may shed some light as to how to construct (and indeed whether there even exists) a genuine discrete analogue of the ASDYM equations. It is worth noting that in general BTs only form discrete versions of 'simpler' equations.

The material presented in chapters 4, 5, and 6 is based on the papers [1, 2, 3]. Specifically, [1] presents the main ingredients and constructions of chapters 4 and 6. [2] focuses on the ASDYM BT and its reductions in greater detail and includes the work presented in chapter 5. Similarly, [3] will be a more complete treatment of the results presented in chapter 6.

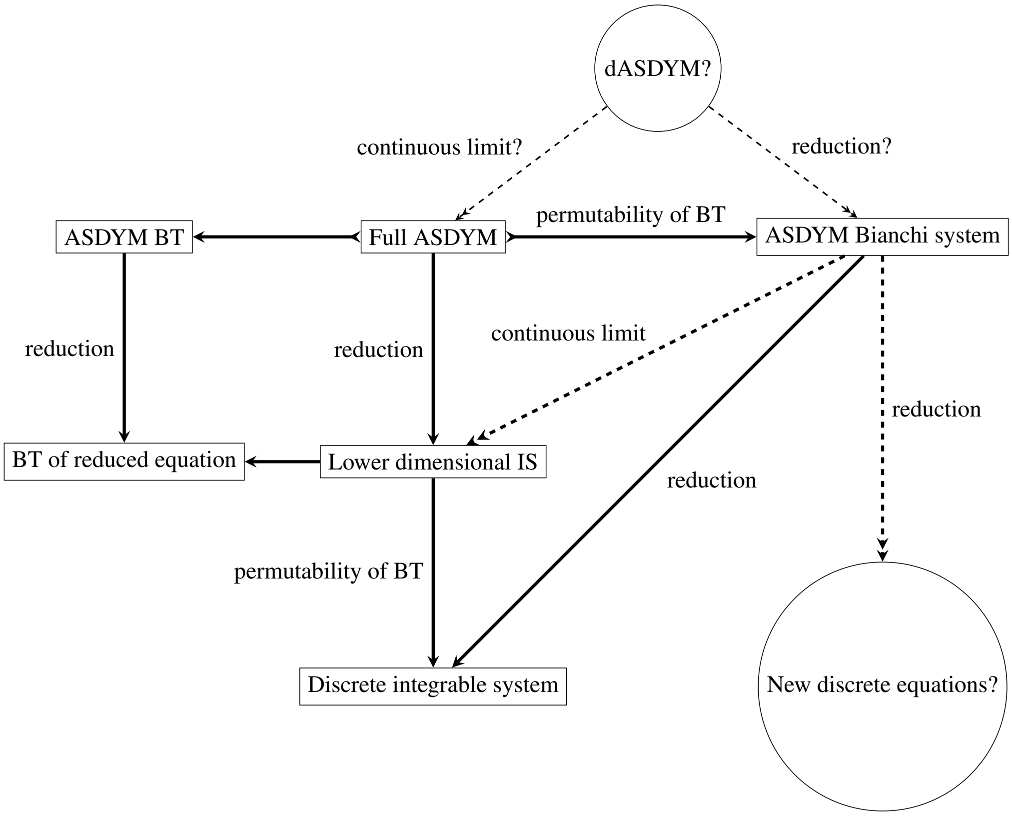

Figure 1.1: Schematic for the construction of a discrete 'master' system.

In chapters 2 and 3 we give brief backgrounds of the material relevant to our studies, starting with a description of Bäcklund and Darboux transformations in chapter 2 and introducing the anti-self-dual Yang-Mills equations in chapter 3. Thus chapter 2 introduces and presents examples of the concept of Bäcklund and Darboux transformations (see 2.1.1) and describes how these can be exploited to obtain discrete equations from the continuous ones. In relation to this, a deep and powerful result due to L. Bianchi is the permutability of Bäcklund transformations. This enables us to construct integrable lattice equations and, suitably reinterpreted, paves the way for the conception of a criterion of integrability for discrete equations known as multidimensional consistency. The chapter ends with a number of examples of how to go from continuous to discrete equations. Chapter 3 introduces the ASDYM equations which will be the main ingredient of this work. After describing the system we discuss its symmetries, its associated linear problem and a potential form of the equations due to Pohlmeyer, [17], which will be crucial for our construction of a Bäcklund-Darboux transformation for the ASDYM equations. In section 3.2 we describe the general reduction process under conformal symmetries and show how this is implemented for reductions to two and one dimensions. Given the importance of the Painlevé equations both in this thesis and in the realm of integrable systems in general, subsection 3.2.3 gives a rapid overview of the Painlevé equations before reproducing the reduction of the ASDYM system to the Painlevé VI equation. Although the description of the ASDYM equations in terms of the connection one-form is the most natural for the implementation of such reductions, we shall require the reduction in the potential formalism due to Pohlmeyer in later chapters. Thus we describe how this is done for each of the examples which we construct the transformation for - to our knowledge this is the first time such reductions in 'J-form' have been presented.

Chapter 4 presents the construction of a Bäcklund-Darboux transformation for the ASDYM equations arising from the action of a Darboux matrix affine in the spectral parameter on the relevant linear problem associated with the anti-self-duality equations. In the past many authors have constructed BTs for the ASDYM equations ([18]–[28]), however these transformations are not convenient for our purpose of constructing a 'master' BT nor for reinterpretation as a discrete system. Moreover, our BT arises so naturally from a simple characterisation of the Darboux matrix in terms of the zero-curvature equations of the ASDYM equations that one might expect this to inherit a large part of the geometry of the self-duality equations. Lastly, the explicit form of the Darboux matrix makes it straightforward to interpret this as the relevant Lax pair for the discrete equation arising from the Bianchi permutability of such BT (see chapter 6). Reductions of the ASDYM BT are performed in section 4.2 showing how to recover the BT of the reduced equations obtained in 3.2.1 directly from the BT for the ASDYM equations. We conclude the chapter (section 4.3) with a calculation reproducing the effect of the action of Möbius transformations on the isomonodromy problem for the Painlevé VI equation (P_{VI}) and describe how this action may be composed with the Schlesinger transformations for P_{VI} to recover the Schlesinger transformation generating shifts in all monodromy exponents from those giving shifts in, for example, the monodromy data for the singular points at 0 and ∞, i.e. θ_0 and θ_∞.

The reduction to the Schlesinger transformations for the remaining Painlevé equations can also be reproduced from the ASDYM BT by following a similar process to the one resulting in the Schlesinger transformations for P_{VI}. However we describe here how, using the framework we have developed, one may circumvent this and recover the relevant transformations from those of P_{VI}. We achieve this by exploiting a feature of the Painlevé equations analogous to that of the classical special functions in conjunction with the geometry and the structure of the ASDYM equations. Specifically, the Painlevé equations possess a type of limiting process, known as the coalescence cascade, [29, 30], through which one can recover one DE from another through a limit under which the singularities of the associated isomonodromy problems coalesce. A classical algebro-geometric result due to F. Klein is the construction of an isomorphism between the complex conformal group and the projective general linear group $\mathrm{PGL}(4,\mathbb{C})$ and, through this, conformal transformations are given by a matrix mapping of a specific form. This structure, coming from the geometry of the ASDYM equations, provides geometric understanding of the coalescence cascade which in turn furnishes a geometric framework to understand the relation among the Schlesinger transformations of the Painlevé equations. In fact the reductions of the ASDYM equations to the Painlevé equations results by imposing invariance of the system under a three dimensional Abelian group of symmetries generated by what Mason and Woodhouse call the 'Painlevé groups' ([16, 31]). These groups are the maximal Abelian Lie subalgebras which are defined as centralisers of regular elements in the Lie algebra of $\mathrm{GL}(4,\mathbb{C})$ and in chapter 5 the confluence mapping one Painlevé equation to another is achieved through the construction of a map from the centraliser of one regular element to that of another, where the regular elements correspond to *adjacent* partitions of 4. Different partitions of 4 correspond to the different Painlevé equations, the correspondence being one between the partition and the structure of the singular points of the associated linear problem. It is important to realise that such structure crucially relies on the Klein isomorphism. The relevant maps, motivated by the work of Aomoto [32] and Gelfand [33] on the general hypergeometric systems, were constructed in [34, 35]. For the degeneration $\mathrm{P_{VI}} \to \mathrm{P_V}$ such constructions are not necessary and we start with the presentation of how the confluence is developed in this case, section 5.1. For the confluences to the remaining Painlevé equations however we shall require the above constructions and thus, after introducing the relevant machinery in section 5.2, we describe how we have implemented these maps in the case of the remaining Painlevé groups and how these actions lift to the Painlevé reductions from the ASDYM equations. We show how to implement these maps in section 5.3 where the process of the confluence $\mathrm{P_V} \to \mathrm{P_{III}}$ is presented in detail. These maps are further exploited to obtain new results whereby the confluence process lifts to the set of Schlesinger transformations. The necessary transformations for the remaining coalescences are given in appendix A. Thus starting from only a few Schlesinger transformations for $\mathrm{P_{VI}}$, our work allows us to reproduce the Schlesinger transformations for $\mathrm{P_V}$—$\mathrm{P_{II}}$ through composition of the Möbius action on the isomonodromy problem and the transformations associated with the confluence to the other Painlevé equations. Therefore this work provides the framework for a more geometrical understanding of how to obtain the Schlesinger transformations for the Painlevé equations and how to map the transformations of one Painlevé equation to another.

In chapter 6 we return to the unreduced ASDYM BT and reinterpret it as a Lax pair for a discrete equation, the compatibility of which results in Bianchi's permutability theorem for the ASDYM equations — an integrable $N \times N$-matrix lattice system over $\mathbb{Z}^2$ governing the evolution of solutions to the ASDYM equations generated from iterated action of the ASDYM BT. We call this the 'ASDYM Bianchi system' and present both the autonomous and non-autonomous versions in sections 6.1 and 6.2, respectively. After taking some time to discuss the system (section 6.2) and its interpretations as a lattice gauge theory, we present both an alternative formulation in 'potential form' (c.f. Pohlmeyer's formulation of the ASDYM equations) and, further, a generalisation of such system to three dimensions. Moreover, we discuss the similarities and differences between the ASDYM Bianchi system and the ASDYM equations and present a number of continuous limits to important two dimensional reductions of the ASDYM equations. Section 6.3 presents the construction of a Darboux transformation for the autonomous ASDYM Bianchi system and this is followed by the final section of the chapter, section 6.4, which gives some examples of reductions of the ASDYM Bianchi system to well-known integrable lattice equations such as the non-autonomous lattice modified KdV (lmKdV) equation (see figure 1.1).

Analysis of the singularity structure of a PDE may be employed to yield the Bäcklund transformations and chapter 7 presents groundwork in this direction for the ASDYM equations. Thus the aim here is to perform singular analysis on the ASDYM equations and recover *our* Bäcklund transformation from such analysis. A result of this type would, in our view, give some closure to the work presented in this thesis. More specifically, among the different 'tests' for integrability there exists one concerning the singular structure of the equation which possesses a long history dating back to the pioneering work of S. Kowalevski ([36]). Integrability, as a concept reliant on the tractability of the system, itself based on the 'moderate' complexity of the solutions, has profound connections with the singular structure the solutions may exhibit. In [37], Ablowitz et al. conjectured that when all the ODEs obtained by exact similarity reduction from a given PDE have the Painlevé property, loosely speaking having solutions whose singularities are at worse poles (see 7.1), then the PDE will be 'integrable' in the sense of being solvable through the inverse scattering transform method, [10]. The conjecture relies on the analysis of the singular structure of solutions of ODEs (roughly, this is the Painlevé test) arising as reductions of the PDE under investigation and as such is indirect and not without complications, [38]. To circumvent this, Weiss, Tabor and Carnevale, [39], proposed a method to apply the Painlevé test to the PDE directly and this elaboration was shown to be able to yield explicit construction of the Bäcklund transformations of these equations. This test is described in 7.1.1 where we also give an example of its application. In [40], Jimbo, Kruskal and Miura applied this test to the ASDYM equations, however the test is sensitive to the specific form the equation is expressed in and in implementing the test the analysis was performed with a choice of expansion resulting in an analytic solution. The singular structure was thus not fully probed and in 7.1.2 we perform the analysis with a modified, more relevant expansion which does probe such structure, finding that indeed such a (local) series solution can consistently be found possessing the correct number of arbitrary functions capable of describing all possible solutions or the general solution. This will be the focus of future work.

Chapter 8 concludes with a summary of this work.

Chapter 2

Bäcklund and Darboux Transformations

Integrable systems possess special nonlinear transformations called Bäcklund transformations (BTs), [41, 42, 43]. These originate with the work of S. Lie and A. Bäcklund (see [44] and references therein) in the late nineteenth century during their studies of parametrised surfaces in $\mathbb{R}^3$. Such transformations are a non-trivial generalisation of the more familiar methods of change of independent variables, change of dependent variables and contact transformations (including hodographic transformation, known to be useful in linearising non-linear PDEs). In studying surfaces and differential equations they realised that there exists an intimate relation between the transformations between special surfaces in terms of coordinates on these surfaces and transformations between solutions of DEs. Specifically the effect of these transformations is to map the solutions space of a system of DEs to another solution space where this can be the solution space of the same equation or the transformation may map the solution of the equation to the solution of a different equation. Such transformations are of critical importance in the theory of integrable systems both for their use in obtaining solutions and as solution generating techniques. What is more, choosing to reinterpret them and their actions from a profoundly different perspective allow us to construct discrete equations interpreted as genuine objects on their own merit.

2.1 Bäcklund transformations

Bäcklund transformations originate from the works of Lie on contact transformations and the geometric study of pseudo-spherical surfaces. We shall describe the notion of BT via some classical examples, in particular through the BT for the sine-Gordon (SG) equation. Our motive for picking this as the working example is three-fold. Firstly, historically this was the first system studied for which Bäcklund and Bianchi recovered the transformation. Secondly, it allows one to appreciate the origin of the Bäcklund parameter. This, as we shall see, reflects the invariance of the SG equation under re-parametrisation of the (asymptotic) coordinates. Hence, the parameter is a result of the invariance of the equation under transformations of Lie type. Thirdly, it is through the BTs for the SG equation that Bianchi developed his theorem on the permutability of such transformations, a result whose importance is difficult to overestimate and which is fundamental to part of the results presented in this thesis.

The defining property of BTs is that they map the solution space, $\mathbb{S}_E$, of the system E to the solution space $\mathbb{S}'$

$$\mathbb{S}_E \mapsto \mathbb{S}', \tag{2.1}$$

where $\mathbb{S}'$ could be the same solution space, i.e. $\mathbb{S}_E$, in which case we call the transformation an *auto* Bäcklund transformation, or $\mathbb{S}'$ could be the solution space to another system, in which case we call it a *hetero* Bäcklund transformation. Furthermore if the transformation involves a parameter then it is called (by some authors) a *parametric* BT, however we shall not make this distinction in this thesis. BTs are often used in connection with integral surfaces of certain nonlinear PDEs and in the *auto* case the invariance under the transformation may be used to generate an infinite sequence of solutions of such equations by purely *algebraic* procedures via a non-linear form of superposition principle (Bianchi's permutability theorem). In the *hetero* case they may be used to relate equations whose properties are well established to others whose properties are less well known or understood. We show both uses through two examples.

2.1.1 Examples

Example 2.1.1 (Liouville's equation)**.** *Here we provide an illustration of a non-auto Bäcklund transformation and its application by working with Liouville's equation ([42])*

$$\frac{\partial^2 u}{\partial x^1 \partial x^2} = e^u. \tag{2.2}$$

Consider the following relations

$$\begin{aligned} \frac{\partial u'}{\partial x'^1} &= \frac{\partial u}{\partial x^1} + \beta e^{\frac{1}{2}(u+u')} := \mathbb{B}'_1(u, u_1; u'), \\ \frac{\partial u'}{\partial x'^2} &= -\frac{\partial u}{\partial x^2} - \frac{2}{\beta} e^{\frac{1}{2}(u-u')} := \mathbb{B}'_2(u, u_2; u'), \\ x'^a &= x^a, \quad a = 1, 2, \end{aligned} \tag{2.3}$$

where $\beta \in \mathbb{R}$, a nonzero constant, is the Bäcklund parameter and $\mathbb{B}'$ denotes the Bäcklund transformation. Applying the integrability condition

$$\frac{\partial \mathbb{B}'_1}{\partial x'^2} - \frac{\partial \mathbb{B}'_2}{\partial x'^1} = 0 \tag{2.4}$$

to $(2.3)_{1,2}$ *produces Liouville's equation* (2.2)*. If instead we rewrite* $(2.3)_{1,2}$ *as*

$$\begin{aligned} \frac{\partial u}{\partial x^1} &= \frac{\partial u'}{\partial x'^1} - \beta e^{\frac{1}{2}(u+u')} := \mathbb{B}_1(u', u'_1; u), \\ \frac{\partial u}{\partial x^2} &= -\frac{\partial u'}{\partial x'^2} - \frac{2}{\beta} e^{\frac{1}{2}(u-u')} := \mathbb{B}_2(u', u'_2; u), \end{aligned} \tag{2.5}$$

then the requirement for integrability

$$\frac{\partial \mathbb{B}_1}{\partial x^2} - \frac{\partial \mathbb{B}_2}{\partial x^1} = 0 \tag{2.6}$$

gives

$$\frac{\partial^2 u'}{\partial x^1 \partial x^2} = 0. \tag{2.7}$$

Therefore, encoded in the set of equations (2.5) *is a link between the non-linear equation* (2.2) *and the linear equation* (2.7)*. It is then a beautiful and intriguing* *result that this connection can be exploited to solve* (2.2)*. Specifically, using the general solution of* (2.7)

$$u' = X^1(x^1) + X^2(x^2) \tag{2.8}$$

where X^1 *and* X^2 *are arbitrary differentiable functions, into the 'Bäcklund relations'* (2.5) *and subsequent integration produces the general solution of Liouville's equation, [42],*

$$u = 2\ln\left[\frac{\exp[(X^1(x^1) - X^2(x^2))/2]}{(\beta/2)\int_{x_0^1}^{x^1}\exp[X^1(\sigma)]d\sigma + (1/\beta)\int_{x_0^2}^{x^2}\exp[-X^2(\tau)]d\tau}\right]. \tag{2.9}$$

This example shows how BTs may be used to generate solutions of nonlinear equations given solutions of an associated (via Bäcklund transforms) linear equation. Such a result is, of course, quite rare.

Example 2.1.2. *The sine-Gordon (SG) equation*

$$\theta_{xt} = \sin\theta, \tag{2.10}$$

is the famous equation governing the evolution of the angle between asymptotic coordinates on pseudo-spherical surfaces (a surface of constant negative Gaussian curvature). This equation has been shown to arise in a number of physical situations including in the theory of crystal dislocation. If θ *is a 'seed' solution to the SG equation* (2.10)*, then* $\hat{\theta}$ *given by the following differential equation is also a solution*

$$\left(\frac{\hat{\theta} - \theta}{2}\right)_x = \beta\sin\left(\frac{\hat{\theta} + \theta}{2}\right), \tag{2.11}$$

$$\left(\frac{\hat{\theta} + \theta}{2}\right)_t = \frac{1}{\beta}\sin\left(\frac{\hat{\theta} - \theta}{2}\right). \tag{2.12}$$

The parameter β *in the above is the Bäcklund parameter and makes an appearance due to the invariance of the SG equation under the scaling of the asymptotic coordinates*

$$x^* = \beta x, \qquad t^* = \frac{t}{\beta}, \qquad \beta \neq 0.$$

2.1.2 Bäcklund transformations in general

More generally the mapping, in the classical literature, takes the following form, [42]: let $u = u(x^a)$, $u' = u'(x'^a)$, $a = 1, ..., n$, represent two surfaces Σ and Σ' (or indeed integral surfaces solutions of PDEs) respectively, in $\mathbb{R}^n$. A set of $2n$ relations

$$\mathbb{B}_i^*(x^a, u, u_a; x'^a, u', u'_a, \beta) = 0, \quad i = 1, ..., 2n, \ \ a = 1, ..., n, \tag{2.13}$$

which connects the surface elements $\{x^a,\ u,\ u_a\}$ and $\{x'^a,\ u',\ u'_a\}$ of Σ and Σ', respectively, is termed a *Bäcklund transformation* and the parameter, β, is what is called the *Bäcklund parameter*.

If (2.13) admits the explicit resolutions

$$u'_i = \mathbb{B}'_i(x^a, u, u_a; u'), \quad i = 1, ..., n, \tag{2.14}$$

and

$$u_i = \mathbb{B}_i(x'^a, u', u'_a; u), \quad i = 1, ..., n, \tag{2.15}$$

together with

$$x'^j = X^j(x^a, u, u_a; u'), \quad j = 1, ..., n \tag{2.16}$$

then, for these relations to transform a surface $u = u(x^a)$ with surface element $\{x^a, u, u_a\}$ to a surface $u' = u'(x'^a)$ with surface element $\{x'^a, u', u'_a\}$, it is required that the following relations be integrable

$$du - \mathbb{B}_a dx^a = 0, \tag{2.17}$$

$$du' - \mathbb{B}'_a dx'^a = 0, \tag{2.18}$$

where we use the Einstein summation convention on repeated indices. Hence, we have the conditions

$$\frac{\partial \mathbb{B}_i}{\partial x^j} - \frac{\partial \mathbb{B}_j}{\partial x^i} = 0, \tag{2.19}$$

$$\frac{\partial \mathbb{B}'_i}{\partial x'^j} - \frac{\partial \mathbb{B}'_j}{\partial x'^i} = 0. \tag{2.20}$$

Application of (2.20) to the Bäcklund relations (2.14) and (2.15) can be shown to lead to a nonlinear equation whose form will depend on which variables are involved while the PDEs resulting from the compatibility conditions (2.19) and (2.20) will have solutions related to each other via a BT.

2.2 Darboux transformations - The Darboux matrix

The linear problem associated with an integrable non-linear PDE (NPDE), that is, the fact that a NPDE can be obtained as the compatibility condition of an associated spectral problem, allows for the construction of a gauge like transformation from which the majority of BTs can be recovered. The procedure starts from taking a seed wave function, Ψ (leading to the dressed one $\hat{\Psi}$), of the spectral operator corresponding to the required field q ($\hat{q}$) and then requiring the existence of a non-singular gauge-like transformation, $\mathbb{D}(q,\hat{q};\zeta)$ between Ψ and $\hat{\Psi}$. The matrix $\mathbb{D}$ we call a 'Darboux matrix' (DM). Thus, the DM is a spectral parameter dependent gauge-like transformation acting on the eigenfunction of the linear problem, [45]. Starting from the generalised version of the Lax pair corresponding to the fields q and $\hat{q}$

$$\Psi_x = U(q,\zeta)\Psi, \qquad \Psi_t = V(q,\zeta)\Psi, \tag{2.21}$$

$$\hat{\Psi}_x = U(\hat{q},\zeta)\hat{\Psi}, \qquad \hat{\Psi}_t = V(\hat{q},\zeta)\hat{\Psi}, \tag{2.22}$$

where U and V are matrix valued functions of the fields q and $\hat{q}$ and of the spectral parameter, ζ, and requiring the existence of a matrix $\mathbb{D}(q,\hat{q};\zeta)$ such that

$$\hat{\Psi} = \mathbb{D}\Psi \tag{2.23}$$

we find that the DM, $\mathbb{D}$, has to satisfy the following:

$$\mathbb{D}_x = U(\hat{q})\mathbb{D} - \mathbb{D}U(q), \tag{2.24}$$

$$\mathbb{D}_t = V(\hat{q})\mathbb{D} - \mathbb{D}V(q), \tag{2.25}$$

which we shall view as BTs at the matrix level. Compatibility of (2.24) and (2.25) gives us

$$U_t(\hat{q}) - V_x(\hat{q}) + [U(\hat{q}), V(\hat{q})] = \mathbb{D}\left(U_t(q) - V_x(q) + [U(q), V(q)]\right)\mathbb{D}^{-1}, \qquad (2.26)$$

and therefore if q is a solution of the nonlinear PDE arising from the compatibility of the operators in (2.21), then $\hat{q}$ will be a solution of the nonlinear PDE arising from the compatibility of (2.22). Note that a DM independent of the spectral parameter is a standard gauge transformation, and therefore the first non-trivial case is obtained by taking $\mathbb{D}$ as a first degree polynomial in ζ. We call this an *elementary* Darboux transformation, [46]. Also, one can easily show from (2.24) and (2.25) the following composition properties which extend, in a nontrivial way, to the BTs, see [47, 48, 49] and [45]. Denoting the set of all possible Darboux matrices corresponding to a given spectral problem by $\mathscr{D}$, we can see that:

- in $\mathscr{D}$ there exists the identity transformation given by $\mathbb{D} = I$,
- in $\mathscr{D}$ there exists the inverse of a given Darboux matrix $\mathbb{D}$ given by $\mathbb{D}^{-1}$,
- in $\mathscr{D}$ one can define the product of two Darboux matrices, $\mathbb{D}_1$ and $\mathbb{D}_2$, given by $\mathbb{D} = \mathbb{D}_2(\hat{q}, \hat{\hat{q}}; \zeta)\mathbb{D}_1(q, \hat{q}, \zeta)$,
- in $\mathscr{D}$ there exists two Darboux matrices, $\mathbb{D}_1$ and $\mathbb{D}_2$, which we may use to construct the sum $\mathbb{D} = \mathbb{D}_1(q, \hat{q}; \zeta) + \mathbb{D}_2(q, \hat{q}; \zeta)$.

It is evident that one can construct $\mathbb{D}$ by using $\mathbb{D} = \hat{\Psi}\Psi^{-1}$. Of course this can only be achieved if one knows what both the seed wave function and the dressed one are, consequently one usually guesses a form for the DM, i.e. uses an ansatz, such as to recover the dressed eigenfunction. We shall concentrate on the simplest nontrivial type, a DM with affine dependence on the spectral parameter

$$\mathbb{D}(x, t; \zeta) = \mathbb{D}_0(x, t) + \zeta\mathbb{D}_1(x, t). \qquad (2.27)$$

This has been shown to be, remarkably, very general and capable of producing a large number of BTs, see [43, 45, 50].

2.2.1 Equivalent Darboux matrices

It seems very natural that Darboux matrices which produce the same transformations should be considered equivalent. Since the linear problem is invariant under the transformations $\Psi \mapsto \Psi C_0$ for any constant non degenerate matrix $C_0 = C_0(\zeta)$, two Darboux matrices, $\mathbb{D}$ and $\mathbb{D}'$ can be considered equivalent if there exists a matrix C such that $\mathbb{D}\Psi = \mathbb{D}'\Psi C$. That is C should commute with Ψ which means that $C = f(\zeta) \in \mathbb{C}$. In conclusion, the equivalence class is given by;

$$\mathbb{D}' \sim \mathbb{D} \ \text{ if } \ \mathbb{D}' = f(\zeta)\mathbb{D},$$

where f is a complex function of ζ, and we then call $\mathbb{D}'$ and $\mathbb{D}$ equivalent.

2.3 Bianchi's permutability theorem

2.3.1 From continuous to discrete via transformations

Discrete systems have historically been used as a powerful tool for implementation of numerical methods to study continuous systems. More recently these systems have gained importance due to attempts by physicists to reformulate the fundamental theories of nature in the framework of discrete space-time as to construct a unified theory of gravity and quantum mechanics. Analogously to continuous integrable systems, there are discrete systems which we can classify as being integrable and just like their continuous counterparts, discrete integrable systems are of great importance in obtaining a more in depth understanding of the behaviour of solutions to discrete equations in general.

The concept of complete integrability for continuous systems, as explained in the previous sections, is not uniquely defined but, rather, is given by a spectrum of definite properties (and possibly working definitions) possessed by those systems for which explicit solutions can be obtained. For discrete systems the concept of integrability follows along similar lines. These properties are not all in direct correspondence with those for continuous systems although some do seem to be discrete analogues of their continuous counterparts, while some are novel [6, 12]. With the exception of multidimensional consistency, we will not discuss the details of these properties in this thesis. Rather, we are satisfied with the view that discrete integrable systems are systems that can be described by ordinary or partial difference equations and which allow exact methods of solution.

Remarkably, it is possible to obtain difference equations by applying transformations to differential equations, meaning then that there exist functions which can be defined both by DEs and dEs, but where the independent variables in the two are different. It is common for functions defined by DEs to possess special transformations, in particular a differential equation containing parameters often possesses transformations acting on it by changing its parameters (for example Weber functions and Bessel functions). It is then known that upon iteration we may obtain a sequence of DEs corresponding to a changing sequence of parameter values. A simple and constructive example was given in [51] through consideration of the ODE

$$xw' = \alpha w, \tag{2.28}$$

with α a constant parameter, whose solution is $w_\alpha(x) = kx^\alpha$, k constant. It is then clear that the transformation which multiplies w_α by x, i.e.

$$w_{\alpha+1} = kx^{\alpha+1} = xw_\alpha \tag{2.29}$$

changes α and therefore maps the equation with parameter α to one with parameter $\alpha+1$, that is it maps neighbouring terms in a sequence of equations. We are now free to change our perspective and view (2.29) as a discrete or difference equation with α the independent (discrete) variable and x a mere parameter. The fascinating aspect of this is that now both (2.28) and (2.29) describe the same function $w_\alpha(x)$ but from radically different points of view.

When dealing with integrable systems, BTs present us with a powerful technique which allows us to map a continuous integrable system to a discrete one. This important result in the study of discrete integrable systems came from the work by Levi and Benguria [52] but was probably somewhat understood already by the classical geometers of the 19th century. In fact a result from classical differential geometry dating back to the 19th century and due to Luigi Bianchi is the existence of a *permutability property* for BTs which was originally applied to the sine-Gordon equation. The permutability theorem states that given two Bäcklund transforms $\omega^1 = \mathbb{B}_{\beta_1}(\omega)$ and $\omega^2 = \mathbb{B}_{\beta_2}(\omega)$ of an original seed solution, ω, of the SG equation, there then exists a fourth solution Ω such that: (i) ω^1 and ω^2 are also Bäcklund transforms of Ω, and (ii) Ω can be expressed as an *algebraic* function of ω, ω^1, and ω^2 hence allowing one to avoid the integration of the Bäcklund transforms relating Ω and ω^1 or Ω and ω^2. The situation may be represented schematically by, and best understood through, a *Bianchi diagram* as given in figure 2.1.

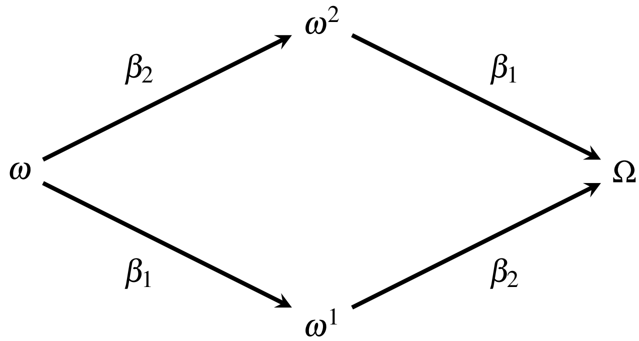

Figure 2.1: Bianchi diagram representing the permutability of BTs. Here ω^1, ω^2 are transforms of ω and similarly Ω is the transform of both ω^1 and ω^2. Stated differently, ω^1 and ω^2 are the inverse transforms, with parameters β_1 and β_2, respectively, of Ω. Bianchi's theorem states that there exist parameters β_1 and β_2 such that the transforms of ω^1 and ω^2 yield the same solution, Ω.

The permutability theorem gives us a non-linear superposition of solutions analogous to that existing for linear systems. This result allows one to construct an infinite sequence of solutions going through purely algebraic steps: once the first Bäcklund transforms have been integrated, repeated application of the permutability property moves us along the lattice of solutions. It is worth stressing that it is not at all obvious such transformations should commute. The theorem *requires* parameters and solutions to exist such that this be valid. More specifically, (2.14)—(2.16) define Bäcklund transformations up to a constant (the initial data for the solution u) and therefore we can restate the permutability theorem as follows: *among the family* *of Bäcklund transformations with parameter β_1 obtained by transforming the solution ω^2 and among the Bäcklund transformations with the parameter β_2 obtained by transforming the solution ω^1, there exist a common solution $\Omega \in \mathbb{S}_E$ given by an algebraic function of $\omega,\ \omega^1,\ \omega^2,\ \beta_1,\ \beta_2$ of the form*

$$Q(\Omega, \omega^1, \omega^2, \omega; \beta_1, \beta_2) = 0. \tag{2.30}$$

Transition to lattice equations requires us to reinterpret BTs and associated non-linear superposition principles as, in turn, integrable differential-difference and pure-difference versions of their continuous counterparts, respectively. Thus, one re-interprets the transformed solutions as fields shifted on a lattice, where now the Bäcklund parameters take the role of independent variables while the original, continuous, independent variables are relegated to simple parameters of the system. That is, by iterating the BTs with 2 different parameters we obtain, from one seed solution ω an entire lattice of solutions. Introducing

$$\omega_{m,n} := \mathbb{B}^m_{\beta_1} \circ \mathbb{B}^n_{\beta_2}[\omega],$$

we may then view the permutability principle as a difference equation with shifts along the lattice $\omega_{m,n} \mapsto \omega_{m+1,n}$ and $\omega_{m,n} \mapsto \omega_{m,n+1}$ corresponding to $\mathbb{B}_{\beta_1}$ and $\mathbb{B}_{\beta_2}$ respectively. It is at this point that we change our perspective; at each elementary plaquette of lattice of solutions we have a relation (2.30) and we now choose to elevate these equations as being the equations of interest, with the original independent variables x^μ now taking a back seat as mere parameters in the equation and β_1, β_2 as the new variables. The Bäcklund parameters appearing in the equation we now interpret as the lattice parameters - they play a central role in the theory. These parameters represent the width of the underlying lattice grid and allow us to recover, through continuous limits, a wealth of other equations both semi-continuous (differential-delay) as well as fully continuous (PDEs).

Let us show, via two explicit examples, how this superposition can be reinterpreted as a pure difference equation to get a lattice equation from the KdV and the SG equations.

2.3.2 Lattice KdV from Bianchi's permutability

The KdV equation may be written in conservation form as

$$u_t + (6u^2 + u_{xx})_x = 0. \tag{2.31}$$

Let us introduce the potential function w given by $u = -w_x$. A new solution, $u_1 = -w_{1,x}$, to (2.31) is then given by the BT [53],

$$\begin{aligned} w_{1,x} &= -w_x - \beta^2 + (w_1 - w)^2, \\ w_{1,t} &= -w_t + 4[\beta^2 u_1 + u^2 - u(w_1 - w)^2 - u_x(w_1 - w)], \end{aligned} \tag{2.32}$$

where β is the Bäcklund parameter. This allows us to construct a 'ladder' of solutions to the KdV equation by recursive application of the BT to *any* starting solution. Each step in the generation of solutions involves a new, more complicated nonlinear system whose integration becomes increasingly arduous. However, by virtue of Bianchi's permutability theorem, integration is required only for the first step from any starting solution. One can reach succeeding steps by purely algebraic operations through the nonlinear superposition principle given by Bianchi.

Specifically, consider a sequence of two successive transformations induced by $(2.32)_1$, starting from an *arbitrary* solution w. First transform to w_1 via the parameter β_1,

$$w_{1,x} + w_x = -\beta_1^2 + (w_1 - w)^2, \tag{2.33}$$

and then to w_{12} via the parameter β_2,

$$w_{12,x} + w_{1,x} = -\beta_2^2 + (w_{12} - w_1)^2. \tag{2.34}$$

Note that the subscripts are also used to denote the parametric dependences; i.e. $w_1 = w_1(\beta_1)$, $w_{12} = w_{12}(\beta_1, \beta_2)$. Performing the transformations in the opposite order, imposing commutativity of the transformations, i.e. $w_{12} = w_{21} = \Omega$, and eliminating the $w_{1,x}$ and $w_{2,x}$ terms by use of (2.32) one obtains:

$$\Omega = w + \frac{\beta_2^2 - \beta_1^2}{w_2 - w_1}. \tag{2.35}$$

If the above nonlinear superposition is reinterpreted as an equation on a lattice prescribed by the Bäcklund parameters, that is, if we view the original, continuous, independent variables as simple parameters and the Bäcklund parameters as the new independent variables on a lattice, then the above results in a pure difference equation which we call the lattice KdV equation. Grammaticos et al. [54], have observed that the above makes an appearance in numerical analysis via the so-called ε-algorithm

$$x_n^{k+1} = x_n^{k-1} + \frac{1}{x_{n+1}^k - x_{n-1}^k}, \tag{2.36}$$

where x_n denotes a member of a sequence and x_n^k denotes its kth iteration in an expansion. We see then that (2.36) is *mutatis mutandis* a version of the nonlinear superposition principle (2.35) for the KdV. There, the parameters β_1 and β_2 are chosen so that $\beta_1 - \beta_2 = 1/2$. Thus the algorithm (2.36) is equivalent to the permutability theorem encoded in the Bianchi diagram as given in 2.1.

2.3.3 Bianchi permutability for the SG equation

Suppose that θ is a 'seed' solution to the SG equation $\theta_{xt} = \sin\theta$ (see 2.1.2) and that θ_1 and θ_2 are the Bäcklund transforms of θ via $\mathbb{B}_{\beta_1}$ and $\mathbb{B}_{\beta_2}$ respectively, i.e. $\theta_i = \mathbb{B}_{\beta_i}(\theta)$, $i = 1,2$. Further let $\theta_{12} = \mathbb{B}_{\beta_2} \circ \mathbb{B}_{\beta_1}(\theta)$ and $\theta_{21} = \mathbb{B}_{\beta_1} \circ \mathbb{B}_{\beta_2}(\theta)$. Thus

$$\theta_{1,x} = \theta_x + 2\beta_1 \sin\left(\frac{\theta_1 + \theta}{2}\right), \tag{2.37}$$

$$\theta_{2,x} = \theta_x + 2\beta_2 \sin\left(\frac{\theta_2 + \theta}{2}\right), \tag{2.38}$$

$$\theta_{12,x} = \theta_{1,x} + 2\beta_2 \sin\left(\frac{\theta_{12} + \theta_1}{2}\right), \tag{2.39}$$

$$\theta_{21,x} = \theta_{2,x} + 2\beta_1 \sin\left(\frac{\theta_{21} + \theta_2}{2}\right). \tag{2.40}$$

If we now ask for permutability by imposing

$$\theta_{12} = \theta_{21} = \Theta,$$

the operations (2.37) − (2.38) + (2.39) − (2.40) yield

$$\beta_1 \left\{ \sin\left(\frac{\theta_1+\theta}{2}\right) - \sin\left(\frac{\Theta+\theta_2}{2}\right) \right\} = \beta_2 \left\{ \sin\left(\frac{\theta_2+\theta}{2}\right) - \sin\left(\frac{\Theta+\theta_1}{2}\right) \right\}.$$

Introducing $w_{m,n} = e^{\frac{i}{2}\mathbb{B}^m_{\beta_1}\circ\mathbb{B}^n_{\beta_2}(\theta)}$ the above becomes

$$\beta_1\left(w_{m+1,n+1}w_{m,n+1} - w_{m+1,n}w_{m,n}\right) = \beta_2\left(w_{m+1,n+1}w_{m+1,n} - w_{m,n+1}w_{m,n}\right) \quad (2.41)$$

which we recognise as an equation of the form (2.30).

2.3.4 From continuous $\mathrm{P_{II}}$ to alternate discrete $\mathrm{P_I}$ ($\mathrm{dP_I}$)

An alternative way to construct discrete equations without resorting to Bianchi's theorem is obtained by simply combining BTs in such a way as to eliminate the derivatives in favour of the old and new solutions. We show this below applying the process to the second Painlevé equation ($\mathrm{P_{II}}$). This calculation was performed by Fokas et al. in [55].

Let $w(z,\alpha)$ be a solution of $\mathrm{P_{II}}$

$$w'' = 2w^3 + zw + \alpha \quad (2.42)$$

with parameter α. The starting point is the BT of $\mathrm{P_{II}}$, which has the following form

$$w(z,\tilde{\alpha}) = F(w'(z,\alpha), w(z,\alpha), z).$$

One follows by finding another value $\hat{\alpha}$ such that

$$w(z,\hat{\alpha}) = F(w'(z,\alpha), w(z,\alpha), z)$$

exists and then eliminates $w'(z,\alpha)$ between the two.

Concretely, we have the following BTs for P_{II}

$$\begin{aligned} \mathbb{B}_1: \quad & w(z,-\alpha) = -w(z,\alpha), \\ \mathbb{B}_2: \quad & w(z,\alpha+1) = -w(z,\alpha) - \frac{1+2\alpha}{2w^2+2w'+z}. \end{aligned}$$

We then compose the BTs as $\mathbb{B}_1 \circ \mathbb{B}_2 \circ \mathbb{B}_1[w(z,\alpha)]$ to obtain $w(z,\alpha-1)$. Eliminating w' between the expressions for $w(z,\alpha+1)$ and $w(z,\alpha-1)$ results in

$$\frac{\eta_n}{w_{n+1}+w_n} + \frac{\eta_{n-1}}{w_{n-1}+w_n} = -(2w_n^2+z), \tag{2.43}$$

where $w_n = w(z,\alpha)$ and $\eta_n = \alpha + \frac{1}{2}$. Fokas et al. proceed in showing that a continuous limit of this equation gives the first Painlevé equation (P_I). We note that (2.43) appeared in a 2D model of Quantum Gravity, [56].

2.4 Multidimensional consistency

Multidimensional consistency is a very powerful criterion for integrability of discrete equations. The idea of consistency is at the heart of integrability, where complete integrability of a Hamiltonian flow in the Liouville-Arnold sense corresponds to having the flow included into a complete family of commuting Hamiltonian flows, [57]. For discrete integrable systems (DIS) the concept of consistency, more specifically multidimensional consistency, is also at the heart of integrability and this crucial property is nowadays taken as a characteristic feature of DIS. Let us very briefly describe this important criterion of integrability for *quadrilateral* lattice equations (quad equation) such as (2.30), [58, 59]. As we have seen, for each independent discrete variable m,n on which the solution depends we associate a Bäcklund parameter which we may think of as describing the width of the grid in the direction associated with m and n. The relevant notion of integrability crucially relies on the presence of such parameters; instead of considering the parameters as chosen and fixed, thus specifying a specific equation, we look at the equation as defining a whole parameter-family of equations and, furthermore, we look at them all together. For this interpretation we therefore attach each parameter to some specific discrete variable, $\beta_1 \to m$, $\beta_2 \to n$, $\beta_3 \to p$ etc. Bianchi's permutability naturally gives us a way to assign to each parameter a direction in a now infinite dimensional lattice of Bäcklund transformations $\omega = \omega_{m,n,p,..} = \omega(m,n,p,...;\beta_1,\beta_2,\beta_3,...)$, where here the parameter β_1 is associated with the direction m etc.. Integrability then becomes a statement of commutativity of such transformations. Said differently, the quad equation (2.30) is integrable if the infinite parameter family of partial difference equations (PΔEs) represented by a lattice equation such as (2.30) is compatible, i.e. in each quadrilateral sublattice of the infinite dimensional lattice we can consistently impose a copy of the PΔE in terms of the relevant discrete variable and associated with a corresponding lattice parameter. That is, we require the iteration to be unambiguous.

To see how this is implemented recall what compatibility under shifts in two independent directions means, figure 2.2. The values of the fields at each vertex

$$\begin{tikzcd}
x_1 \arrow[r, "\beta_2"] & x_{12} \\
x \arrow[u, "\beta_1"] \arrow[r, "\beta_2"'] & x_2 \arrow[u, "\beta_1"']
\end{tikzcd}$$

Figure 2.2: Elementary sublattice.

(which we now label x_i for clarity) are related and uniquely defined by the integrable system

$$Q(x, x_1, x_2, x_{12}; \beta_1, \beta_2) = 0. \tag{2.44}$$

Follow by extending this into a third dimension, or, more precisely, consider a copy of this and add edges connecting the vertices to their copy and view them as shift under a third parameter (which can be viewed as a spectral parameter), β_3. Doing this results in a 3-dimensional cube as shown in the figure 2.3. Now we have three equations of the form (2.44) which uniquely determine x_{31}, x_{12} and x_{23}. Then, the same equation for the remaining sub-lattices produce three, a priori different rela-

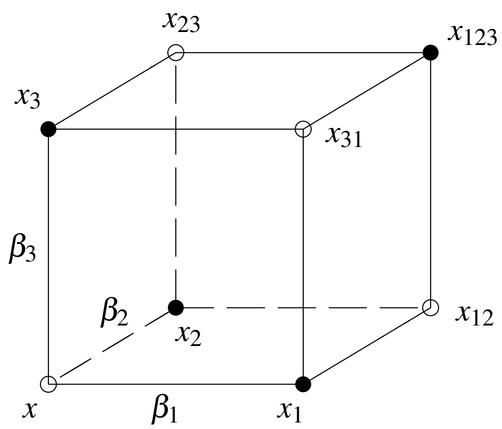

Figure 2.3: Three-dimensional consistency.

tions, for the value of the field x_{123}

$$\begin{aligned} Q(x_1,x_{31},x_{12},x_{123};\beta_1,\beta_2,\beta_3) &= 0, \\ Q(x_3,x_{23},x_{31},x_{123};\beta_1,\beta_2,\beta_3) &= 0, \\ Q(x_2,x_{23},x_{12},x_{123};\beta_1,\beta_2,\beta_3) &= 0. \end{aligned} \tag{2.45}$$

If the three values for x_{123} coincide for any value of the initial data we say that the system, (2.44), is 3-dimensionally consistent. The statement is then: *a discrete system is called integrable if it is consistent*.

A generalisation of the principle of consistency is straightforward:

- A d-dimensional equation is called consistent if it can be embedded in a $d+1$-dimensional lattice such that the system is valid for *all* d-dimensional sub-lattices of the $d+1$ dimensional lattice.

A taste of the importance of this statement can be inferred by the work in [60] where it was shown that integrability as relying on a zero curvature representation *follows*, for two dimensional systems, from three dimensional consistency. Furthermore, in [61] the authors speculate that both the differential geometry of the integrable classes of surfaces (e.g. pseudospherical surfaces) and their transformation theory can systematically be recovered from multidimensional lattices of consistent Bäcklund transformations.

It is simple to check that both the lattice equations in the above examples ((2.35) and (2.41)) are multidimensionally consistent on the cube.

Chapter 3

The anti-self-dual Yang-Mills equations

In this chapter we introduce the anti-self-dual Yang-Mills (ASDYM) equations and describe various ways of expressing the ASD condition, first as a system of first-order equations on the components of the gauge potential, and then as the commutativity condition on a pair of Lax operators. We then look at a second-order form which involves a matrix potential that has become known as Yang's J-matrix. The ASDYM system will be the central object of study in the rest of the thesis. The prescription for performing the symmetry reductions is outlined and examples of reductions are given.

3.1 The anti-self-dual Yang-Mills equations

The ASDYM equations are a special class of first-order reductions of a more general system called the Yang-Mills equations. These first order reductions are the condition that the curvature two-form over space-time, F, be anti-self-dual, that is

$$\star F = -F \tag{3.1}$$

where $F = \frac{1}{2}F_{\mu\nu}dx^{\mu} \wedge dx^{\nu}$ is the curvature two-form and $\star$ is a linear map from the space of r forms over some manifold $\mathscr{M}$ (which will be either $\mathbb{R}^4$ or $\mathbb{R}^{2,2}$), $\Omega^r(\mathscr{M})$, in m dimensions to the space of $(m-r)$ forms, $\Omega^{(m-r)}(\mathscr{M})$. This map is called the Hodge operator, $\star : \Omega^r(\mathscr{M}) \to \Omega^{m-r}(\mathscr{M})$ whose action on a basis of Ω^r is defined by

$$\star(dx^{\mu_1} \wedge dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_r}) = \frac{\sqrt{|g|}}{(m-r)!} \varepsilon^{\mu_1 \ldots \mu_r}{}_{\nu_{r+1} \ldots \nu_m} dx^{\nu_{r+1}} \wedge \ldots \wedge dx^{\nu_m},$$

where $\varepsilon_{\mu_1 \ldots \mu_m}$ is the totally anti-symmetric tensor and $|g| = \det[g]$. Thus, on the four-dimensional manifold $\mathscr{M}$, the Hodge operator takes any two-form $X = \frac{1}{2} X_{\mu\nu} dx^\mu \wedge dx^\nu \in \Omega^2(\mathscr{M})$ to the *dual* two-form $\star X = \frac{1}{2} \varepsilon_{\mu\nu}{}^{\gamma\delta} X_{\gamma\delta} dx^\mu \wedge dx^\nu$, where $\varepsilon_{\mu\nu\gamma\delta}$ is the totally antisymmetric tensor in 4 dimensions normalised as $\varepsilon_{0123} = 1$, and the metric on $\mathscr{M}$ is used to raise and lower indices.

Rather than considering the Euclidean ($\mathbb{R}^4$) and the Hyperbolic ($\mathbb{R}^{2,2}$) cases separately, as this would entail considering a different form of the metric in each case, it will prove convenient to deal with these cases in a common framework by allowing the coordinates to take complex values. We do this as follows: we shall think of these two spaces as being embedded in complexified Minkowski space, $\mathbb{CM}$. For this purpose we work in *double null coordinates*, from which we may recover the various real spaces by imposing the required reality conditions on the double null coordinates $(z, w, \tilde{z}, \tilde{w})$ (see [16] for the specific choices). In these coordinates the metric on $\mathbb{CM}$ is

$$ds^2 = 2(dz d\tilde{z} - dw d\tilde{w}), \tag{3.2}$$

and the volume element is $\nu = dw \wedge d\tilde{w} \wedge dz \wedge d\tilde{z}$. More concretely, let D be a connection on a rank-n vector bundle E over some region U in real or complex space-time, G be a Lie group and let F be the curvature two-form which, in a local trivialisation, takes values in the Lie algebra of G. If $D = d + A$, then $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$, where the components of the curvature two-form are given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{3.3}$$

Then the condition for the curvature two-form to be anti-self-dual takes the form

$$F_{zw} = 0, \quad F_{\tilde{z}\tilde{w}} = 0, \quad F_{z\tilde{z}} - F_{w\tilde{w}} = 0, \tag{3.4}$$

or, in component form

$$\begin{aligned} \partial_z A_w - \partial_w A_z + [A_z, A_w] &= 0, \\ \partial_{\tilde{z}} A_{\tilde{w}} - \partial_{\tilde{w}} A_{\tilde{z}} + [A_{\tilde{z}}, A_{\tilde{w}}] &= 0, \\ \partial_z A_{\tilde{z}} - \partial_{\tilde{z}} A_z - \partial_w A_{\tilde{w}} + \partial_{\tilde{w}} A_w + [A_z, A_{\tilde{z}}] - [A_w, A_{\tilde{w}}] &= 0. \end{aligned} \tag{3.5}$$

Rewriting the covariant derivative in components

$$D_\mu = \partial_\mu + A_\mu, \tag{3.6}$$

the conditions (3.4) are

$$[D_z, D_w] = 0, \quad [D_{\tilde{z}}, D_{\tilde{w}}] = 0, \quad [D_z, D_{\tilde{z}}] - [D_w, D_{\tilde{w}}] = 0. \tag{3.7}$$

3.1.1 Symmetries

The ASDYM equations possess a very rich system of symmetries, indeed this feature lies at the heart of the relevance of the ASDYM equations to integrable systems. These symmetries are of two types; gauge transformations reflecting frame invariance of the system and conformal symmetries of the base manifold. The first type of symmetry makes the equations invariant under *gauge transformations*, that is, a smooth pointwise change of basis such that

$$A_\mu \mapsto g^{-1} A_\mu g + g^{-1} \partial_\mu g, \quad g \in G, \tag{3.8}$$

under which the components of the curvature two-form transform as

$$F_{\mu\nu} \mapsto g^{-1} F_{\mu\nu} g. \tag{3.9}$$

This symmetry acts at the level of the fibres, effectively reflecting the invariance of the system under change of local trivialization, that is under change of basis. Specifically, when the basis is changed the connection one-form undergoes a *gauge transformation* as given by (3.8).

The second group of symmetries arises due to invariance of the Hodge operator in four dimensions when acting on two-forms. In four dimensions the Hodge star operation on two-forms is invariant under conformal transformations of space-time. That is, conformal transformations map one anti-self-dual (ASD) connection to another. It is therefore natural to consider ASD connections which are invariant under a subgroup of the conformal group. This is the consideration when 'reducing' the ASDYM equations. The gauge symmetry of the system, together with the conformal symmetry of the reductions, is responsible for the huge freedom of the system which can be exploited to recover integrable sub-systems. In fact, interest in the ASDYM equation in the context of integrable systems is centred around the concept of symmetry reduction. It was Ward who noticed ([62]) that many of the known integrable systems can be realised as symmetry reductions of the ASDYM equation in four dimensions. Thus, symmetry reductions of the ASDYM equation provides a natural framework for a classification of integrable systems and for a study of relations among them. This feature lies at the heart of the work presented in this thesis.

In discussing the reductions of the ASDYM equations and in particular the reduction to the Painlevé equations and the confluence of Painlevé equations at the ASDYM level, we shall be using the generators of conformal symmetries, each of which generates a conformal Killing vector. In particular, the reductions to the Painlevé equations are given by the so-called Painlevé groups [31]. In view of this we here give in detail the explicit construction of the generators of the conformal group, see [16, 63, 64].

Recall that a diffeomorphism $\rho : \mathscr{M} \to \mathscr{M}$ is a conformal transformation if it preserves the metric up to scale:

$$\rho^* g_{\rho(p)} = e^{2\sigma} g_p,$$

with σ a function over $\mathscr{M}$ and $X,\ Y \in T_p\mathscr{M}$, that is elements of the tangent space at p. In components this means that

$$g_{\rho(p)}(\rho_* X, \rho_* Y) = e^{2\sigma} g_p(X,Y).$$

The identity map and the inverse of a conformal transformation are also conformal transformations and therefore the conformal transformations of $\mathscr{M}$ form a group called the conformal group, or Conf($\mathscr{M}$). Given a conformal transformation there exists a special set of vector fields which generate this symmetry and thus represent the direction of symmetry of $\mathscr{M}$, that is the one-parameter group of transformations generated by the vector field X under pulling back, preserves the metric up to scale. Specifically, let $\mathscr{X}(\mathscr{M})$ be the space of vector fields on $\mathscr{M}$. Given a vector field $X \in \mathscr{X}(\mathscr{M})$ and the infinitesimal displacement εX generating a conformal transformation, then X is a conformal Killing vector field. Thus if $x^\mu \mapsto x^\mu + \varepsilon X^\mu$ then

$$\frac{\partial(x^\kappa + \varepsilon X^\kappa)}{\partial x^\mu} \frac{\partial(x^\lambda + \varepsilon X^\lambda)}{\partial x^\nu} g_{\kappa\lambda}(x + \varepsilon X) = e^{2\sigma} g_{\mu\nu}.$$

Expanding to first order in ε and setting $\sigma = \frac{\varepsilon\psi}{2}$ we find that

$$X^\eta \partial_\eta g_{\mu\nu} + \partial_\mu X^\kappa g_{\kappa\nu} + \partial_\nu X^\lambda g_{\mu\lambda} = \psi g_{\mu\nu},$$

which we can write also in terms of the Lie derivative as $L_X g_{\mu\nu} = \psi g_{\mu\nu}$, and when the metric is constant the condition reduces to

$$\partial_\mu X^\nu + \partial_\nu X^\mu \propto g_{\mu\nu},$$

which we write as

$$\partial_{(\mu} X_{\nu)} \propto g_{\mu\nu}.$$

There is an issue arising when we consider inversions and reflections. Since inversions and reflections map light-cones (directions along which the vectors are null) to infinity which does not belong to $\mathbb{C}^4$, the only conformal transformations defined globally on $\mathbb{C}^4$ are combinations of dilations and isometries (constant rescaling of the coordinates). Thus, in order for us to have a well defined group action on space-time we must compactify by adding what is termed, in a manner analogous to the case of the one point compactification leading to the Riemann sphere, a light cone at infinity to obtain compactified Minkowski space-time, $\mathbb{C}^{4\#}$, [16, 64]. With this construction, every proper conformal transformation $\rho : U \mapsto \rho(U)$, where U is open in $\mathbb{C}^4$ or in one of its real slices, extends uniquely to a global transformation $\mathbb{C}^{4\#} \mapsto \mathbb{C}^{4\#}$, and every conformal Killing vector on $\mathbb{CM}$ extends to and determines an element of the Lie algebra of the conformal group. A classical construction due to Klein identifies $\mathbb{C}^{4\#}$ with what is known as a Klein quadric in $\mathbb{CP}_5$[1], this is achieved via an isomorphism from the complex conformal group to the projective general linear group $\mathrm{PLG}(4,\ \mathbb{C}) = \mathrm{GL}(4,\ \mathbb{C})/\mathbb{C}^*$. That is, the conformal group of complexified Minkowski space is isomorphic to the projective general linear group $\mathrm{PGL}(4,\ \mathbb{C})$ and here we shall use this specific isomorphism without describing it further.

Recall that in double null coordinates the metric is $dzd\tilde{z} - dwd\tilde{w}$. Let $x = (x^{\alpha\beta}),\ \alpha,\beta = 0,\ 1,\ 2,\ 3$, be the skew-symmetric complex singular matrix given by:

$$x = \begin{pmatrix} 0 & s & -w & \tilde{z} \\ -s & 0 & -z & \tilde{w} \\ w & z & 0 & 1 \\ -\tilde{z} & -\tilde{w} & -1 & 0 \end{pmatrix}, \tag{3.10}$$

where $s = z\tilde{z} - w\tilde{w}$. We then have that the metric is proportional to $\varepsilon_{\alpha\beta\gamma\delta}dx^{\alpha\beta}dx^{\gamma\delta}$, that is $\lambda(dzd\tilde{z} - dwd\tilde{w}) = \varepsilon_{\alpha\beta\gamma\delta}dx^{\alpha\beta}dx^{\gamma\delta}$, where $\varepsilon_{\alpha\beta\gamma\delta}$ denotes the four-dimensional alternating symbol and λ is a constant scalar. Conformal transfor-

[1]Complex projective space in 5 dimensions.

mations of $\mathbb{C}^4$ are then induced by mappings of the form

$$x \mapsto y = gxg^T, \tag{3.11}$$

where $g \in \mathrm{GL}(4,\mathbb{C})$. In fact the mapping $x \mapsto y/y^{23}$ generates conformal transformations of $\mathbb{C}^4$ as it maps metrics of the form (3.10) to itself. Note also that since the nonzero multiples of g all induce the same transformation, there is no loss of generality in taking $\det g = 1$. Via this construction one can compute the generators of the conformal transformations, that is the conformal Killing vectors.

Let us then consider the one-parameter family of transformations given by $g = \exp(\varepsilon K) = I + \varepsilon K + O(\varepsilon^2)$, where $K \in \mathfrak{gl}(4,\ \mathbb{C})$. Applying this transformation we get

$$y = \left[I + \varepsilon K + O(\varepsilon^2)\right] x \left[I + \varepsilon K^T + O(\varepsilon^2)\right] \implies y = x + \varepsilon(Kx + xK^T) + O(\varepsilon^2).$$

For simplicity consider the situation in which K has a 1 in the $\mu\nu$ entry and 0 otherwise, i.e. $K^{ij} = \delta^{i\mu}\delta^{j\nu}$ and $(K^T)^{ij} = K^{ji} = \delta^{j\mu}\delta^{i\nu}$, in components

$$\begin{aligned} y^{\alpha\beta} &= x^{\alpha\beta} + \varepsilon\left((Kx)^{\alpha\beta} + (xK^T)^{\alpha\beta}\right) + O(\varepsilon^2) \\ &= x^{\alpha\beta} + \varepsilon\left(K^{\alpha m}x^{m\beta} + x^{\alpha n}K^{\beta n}\right) + O(\varepsilon^2) \\ &= x^{\alpha\beta} + \varepsilon\left(\delta^{\alpha\mu}\delta^{m\nu}x^{m\beta} + x^{\alpha n}\delta^{\beta\mu}\delta^{n\nu}\right) + O(\varepsilon^2) \\ &= x^{\alpha\beta} + \varepsilon\left(x^{\alpha\nu}\delta^{\beta\mu} - x^{\beta\nu}\delta^{\alpha\mu}\right) + O(\varepsilon^2), \end{aligned} \tag{3.12}$$

where for the last equality we have used the skew-symmetry of the metric x, i.e. $x^{ij} = -x^{ji}$. From $x^{\alpha\beta} \mapsto \hat{x}^{\alpha\beta} = y^{\alpha\beta}/y^{23}$ we find

$$\begin{aligned} \hat{x}^{\alpha\beta} &= \left[x^{\alpha\beta} + \varepsilon\left(x^{\alpha\nu}\delta^{\beta\mu} - x^{\beta\nu}\delta^{\alpha\mu}\right) + O(\varepsilon^2)\right]\left[I + \varepsilon\left(x^{2\nu}\delta^{3\mu} - x^{3\nu}\delta^{2\mu}\right)\right]^{-1} \\ &= x^{\alpha\beta} + \varepsilon\left(x^{\alpha\nu}\delta^{\beta\mu} - x^{\beta\nu}\delta^{\alpha\mu} + x^{\alpha\beta}q^{\mu\nu}\right) + O(\varepsilon^2), \end{aligned} \tag{3.13}$$

where $q^{\mu\nu} = x^{3\nu}\delta^{2\mu} - x^{2\nu}\delta^{3\mu}$. From this, by considering the relevant components of $(x^{\alpha\beta})$, we have the relevant transformations of $(z, w, \tilde{z}, \tilde{w})$ under the conformal transformation given by

$$\begin{aligned}
w \mapsto \hat{w} &= w - \varepsilon\left(x^{0\nu}\delta^{2\mu} - x^{2\nu}\delta^{0\mu} - wq^{\mu\nu}\right) + O(\varepsilon^2),\\
\tilde{w} \mapsto \hat{\tilde{w}} &= \tilde{w} + \varepsilon\left(x^{1\nu}\delta^{3\mu} - x^{3\nu}\delta^{1\mu} + \tilde{w}q^{\mu\nu}\right) + O(\varepsilon^2),\\
z \mapsto \hat{z} &= z + \varepsilon\left(x^{2\nu}\delta^{1\mu} - x^{1\nu}\delta^{2\mu} + zq^{\mu\nu}\right) + O(\varepsilon^2),\\
\tilde{z} \mapsto \hat{\tilde{z}} &= \tilde{z} + \varepsilon\left(x^{0\nu}\delta^{3\mu} - x^{3\nu}\delta^{0\mu} + \tilde{z}q^{\mu\nu}\right) + O(\varepsilon^2).
\end{aligned}$$

Consequently, we may read off the conformal Killing vector $X_{\mu\nu}$ associated with the generator K given above. For instance, the 00 component leads to the transformations

$$\hat{w} = w + \varepsilon w, \qquad \hat{\tilde{w}} = \tilde{w}, \qquad \hat{z} = z, \qquad \hat{\tilde{z}} = \tilde{z} + \varepsilon\tilde{z},$$

and therefore the associated Killing vector is

$$X_{00} = w\partial_w + \tilde{z}\partial_{\tilde{z}}.$$

It follows that there are 16 Killing vectors given by $X_{\mu\nu}$ where $\mu,\nu = 0,1,2,3$

$$\begin{aligned}
X_{00} &= w\partial_w + \tilde{z}\partial_{\tilde{z}}, & X_{02} &= \partial_{\tilde{z}},\\
X_{10} &= \tilde{z}\partial_{\tilde{w}} + w\partial_z, & X_{12} &= \partial_{\tilde{w}},\\
X_{01} &= z\partial_w + \tilde{w}\partial_{\tilde{z}}, & X_{03} &= \partial_w,\\
X_{11} &= z\partial_z + \tilde{w}\partial_{\tilde{w}}, & X_{13} &= \partial_z,\\
X_{20} &= -w\tilde{w}\partial_z + \tilde{z}^2\partial_{\tilde{z}} - \tilde{z}\tilde{w}\partial_{\tilde{w}} - w\tilde{z}\partial_w, & X_{22} &= -\tilde{w}\partial_{\tilde{w}} - \tilde{z}\partial_{\tilde{z}},\\
X_{30} &= -z\tilde{z}\partial_w - w\tilde{z}\partial_{\tilde{z}} - wz\partial_z - w^2\partial_w, & X_{32} &= -z\partial_w - w\partial_{\tilde{z}},\\
X_{21} &= -z\tilde{z}\partial_w - \tilde{w}^2\partial_{\tilde{w}} - \tilde{z}\tilde{w}\partial_{\tilde{z}} - z\tilde{w}\partial_z, & X_{23} &= -\tilde{z}\partial_w - \tilde{w}\partial_z,\\
X_{31} &= -w\tilde{w}\partial_{\tilde{z}} - z\tilde{w}\partial_{\tilde{w}} - zw\partial_w - z^2\partial_z, & X_{33} &= -z\partial_z - w\partial_w,
\end{aligned} \tag{3.14}$$

however only 15 of these are independent since $X_{00}+X_{11}+X_{22}+X_{33} = 0$, giving the correct dimension for the conformal group.

Reductions of the ASDYM equations are performed by imposing invariance under a subgroup of the full conformal group (section 3.2). For instance, the element K generating a 3-dimensional Abelian subgroup of the conformal group resulting in the reduction to the sixth Painlevé equation ($\mathrm{P_{VI}}$) are those belonging to the conjugacy class ([31]) of the matrix

$$\mathrm{P_{VI}} \qquad \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}, \tag{3.15}$$

and the conformal Killing vectors associated with this Painlevé subgroup are of the form

$$aX_{00} + bX_{11} + cX_{22} + dX_{33},$$

where the (independent) Killing vectors associated to this generator can be chosen as

$$X_{11} = z\partial_z + \tilde{w}\partial_{\tilde{w}}, \quad X_{22} = -\tilde{z}\partial_{\tilde{z}} - \tilde{w}\partial_{\tilde{w}}, \quad X_{33} = -z\partial_z - w\partial_w. \tag{3.16}$$

In section 3.2.3 we show how imposing invariance under the action generated by these Killing vectors yields a (matrix) system equivalent to $\mathrm{P_{VI}}$. Similarly, the other Painlevé equations also arise as a symmetry reduction of the ASDYM equations under some 3-dimensional Abelian subgroup of the conformal group and, as we shall explain in chapter 5, the relevant generators belong to conjugacy classes of matrices of a very special type. These are the centralisers of regular elements of the Lie algebra $\mathfrak{gl}(4,\mathbb{C})$, where, for our purposes, a regular element is a matrix whose Jordan blocks have distinct eigenvalues and the centralisers are those elements which commute with the regular element. Thus for example, in the $\mathrm{P_{VI}}$ case above, the generator (3.15) is the centraliser of a non-degenerate regular element (four distinct eigenvalues) and the four parameters a,b,c,d may be associated with the four Fuchsian singular points associated to the sixth Painlevé equation. We shall have more to say about the significance of these matrices in section 3.2.3 and in chapter 5 where we exploit this structure to reformulate the well known ‘coalescence cascade’ of the Painlevé equations, a special type of limiting process moving us from one Painlevé equation to another, from a more geometric perspective. Our main result is that by constructing a map from one centraliser to another the degeneration of singularities is reinterpreted as the process of degeneration of the Killing vectors associated to the specific equations. This framework, induced by the geometric structure of the ASDYM equations, enables us to lift the degeneration process to the level of the Schlesinger transformations (special BTs for the Painlevé equations) in such a way that the coalescence of singularities yields the degeneration of the transformations themselves.

3.1.2 Associated linear problem for the ASDYM

Crucially the ASDYM equations arise as the compatibility condition for an overdetermined, isospectral[2], linear system — a fundamental concept which underlies the integrability of non-linear equations, [10].

Consider the pair of Lax operators

$$L = D_w - \zeta D_{\tilde{z}}, \qquad M = D_z - \zeta D_{\tilde{w}}, \tag{3.17}$$

where ζ is a complex spectral parameter, the D_μ's are as in (3.6) and L and M act on vector-valued functions of the space-time coordinates. These operators commute for every value of ζ by virtue of the ASDYM equations

$$[L,M] = F_{zw} - \zeta(F_{z\tilde{z}} - F_{w\tilde{w}}) + \zeta^2 F_{\tilde{z}\tilde{w}} = 0. \tag{3.18}$$

Therefore, the ASDYM equations arise as the compatibility condition for the overdetermined linear system

$$L\Psi = 0, \qquad M\Psi = 0, \tag{3.19}$$

where $\Psi = \Psi(z, w, \tilde{z}, \tilde{w}, \zeta)$ is the fundamental matrix solution. Hence, the linear

[2]That is the spectral parameter is a constant.

problem associated to the ASDYM equation is given by

$$(\partial_z - \zeta \partial_{\tilde{w}})\Psi = -(A_z - \zeta A_{\tilde{w}})\Psi, \tag{3.20}$$

and

$$(\partial_w - \zeta \partial_{\tilde{z}})\Psi = -(A_w - \zeta A_{\tilde{z}})\Psi. \tag{3.21}$$

Note that a gauge transformation under which the connection transforms as in (3.8) results from the (pointwise) transformation of the eigenvector given by $\Psi \mapsto g^{-1}\Psi$.

3.1.3 Pohlmeyer form of the ASD Yang-Mills equations

Pohlmeyer in [17] gave an alternative form of the ASDYM equations by noticing that the first two equations in (3.5), i.e. $(3.5)_1$ and $(3.5)_2$, can be written as

$$\begin{aligned} F_{zw} &= [D_z, D_w] = 0, \\ F_{\tilde{z}\tilde{w}} &= [D_{\tilde{z}}, D_{\tilde{w}}] = 0. \end{aligned} \tag{3.22}$$

These equations imply that the pairs of operators D_z, D_w and $D_{\tilde{z}}, D_{\tilde{w}}$ are compatible, that is, there exist (locally) two matrix-valued functions H and K of the space-time coordinates such that

$$\begin{aligned} D_z(H) &= D_w(H) = 0, \\ D_{\tilde{z}}(K) &= D_{\tilde{w}}(K) = 0, \end{aligned} \tag{3.23}$$

where recall that $D_\mu = \partial_\mu + A_\mu$. Exploiting this the coefficients (of the one-form) A_μ's can be expressed in terms of H and K

$$\begin{aligned} \partial_w H + A_w H = 0, \qquad \partial_z H + A_z H = 0, \\ \partial_{\tilde{w}} K + A_{\tilde{w}} K = 0, \qquad \partial_{\tilde{z}} K + A_{\tilde{z}} K = 0, \end{aligned} \tag{3.24}$$

and then

$$\begin{aligned} D_z = H\partial_z H^{-1}, \;\; D_w = H\partial_w H^{-1}, \\ D_{\tilde{z}} = K\partial_{\tilde{z}} K^{-1}, \;\; D_{\tilde{w}} = K\partial_{\tilde{w}} K^{-1}, \end{aligned} \tag{3.25}$$

where juxtaposition of operators indicate an operator product. Therefore $(3.5)_{1,2}$ are the local integrability conditions for the existence of H and K determined uniquely by A up to the gauge freedom

$$H \mapsto H\tilde{P}, \qquad K \mapsto KP, \tag{3.26}$$

where $P = P(z,w)$ and $\tilde{P} = \tilde{P}(\tilde{z},\tilde{w})$.

Under a gauge transformation $\Psi \mapsto g^{-1}\Psi$, where A is replaced by the gauge equivalent potential $g^{-1}Ag + g^{-1}dg$, H and K transform according to $g^{-1}H$ and $g^{-1}K$, thus leaving an important quantity, $K^{-1}H$ which *defines* the *Yang matrix* as $J := K^{-1}H$, unchanged ([65]). This matrix is determined by the connection $D = d + A$ up to $J \mapsto P^{-1}J\tilde{P}$, and it itself determines D since we can write

$$A = -(H_z H^{-1}dz + H_w H^{-1}dw + K_{\tilde{z}}K^{-1}d\tilde{z} + K_{\tilde{w}}K^{-1}d\tilde{w}) \tag{3.27}$$

and under the gauge transformation $\Psi \mapsto H^{-1}\Psi$, i.e. $A \mapsto H^{-1}AH + H^{-1}dH$,

$$A \mapsto H^{-1}(H_{\tilde{z}} - K_{\tilde{z}}J)d\tilde{z} + H^{-1}(H_{\tilde{w}} - K_{\tilde{w}}J)d\tilde{w}. \tag{3.28}$$

So then we see that A is equivalent to

$$J^{-1}\tilde{\partial}J = (J^{-1}\partial_{\tilde{z}}J)d\tilde{z} + (J^{-1}\partial_{\tilde{w}}J)d\tilde{w}, \tag{3.29}$$

where $\tilde{\partial} = d\tilde{z}\partial_{\tilde{z}} + d\tilde{w}\partial_{\tilde{w}}$, and where juxtaposition here denotes the tensor product. Alternatively, under the transformation $\Psi \mapsto K^{-1}\Psi$, i.e. $A \mapsto K^{-1}AK + K^{-1}dK$, we have A equivalent to

$$J^{-1}\partial J = (J^{-1}\partial_z J)dz + (J^{-1}\partial_w J)dw, \tag{3.30}$$

where $\partial = dz\partial_z + dw\partial_w$. That is, one obtains gauge equivalent potentials with vanishing z and w components in the first case and vanishing $\tilde{z}$ and $\tilde{w}$ components in the second case. This 'splitting' will be relevant when we interpret the action of the Darboux transformation, see chapters 4 and 6.

Equations $(3.5)_{1,2}$ are satisfied identically by H and K while the third equation,

$(3.5)_3$, holds if and only if J satisfies *Yang's equation*

$$\partial_z(J^{-1}\partial_{\tilde{z}}J) - \partial_w(J^{-1}\partial_{\tilde{w}}J) = 0. \tag{3.31}$$

We see from the above that Yang's equation is equivalent to the ASDYM equations but also that it does not preserve all the symmetries; from 3.24 (3.27) we see that the system is not covariant under coordinate transformations which change the 2-planes spanned by ∂_z and ∂_w and by $\partial_{\tilde{z}}$ and $\partial_{\tilde{w}}$, that is it does not preserve the $SO(4)$ symmetry. If under this description we perform the gauge transformation $L \mapsto K^{-1}LK$ and $M \mapsto K^{-1}MK$, the Lax pair for the ASDYM equations can be expressed in terms of J in the simple form

$$L = J^{-1}\partial_w J - \zeta\partial_{\tilde{z}}, \;\; M = J^{-1}\partial_z J - \zeta\partial_{\tilde{w}}. \tag{3.32}$$

Yang's form of the ASDYM equations will be of great interest to us for the construction of a gauge invariant Bäcklund-Darboux transformation of the ASDYM equations. It is worth mentioning that the construction of Yang's matrix can be understood from the geometric point of view as the patching matrix (which is the map from the intersection of two frame fields to the general linear group) associated with D, see [16].

When the gauge group is $SU(2)$ one can parametrise J in the following convenient form [17, 26],

$$J = \frac{1}{f}\begin{pmatrix} 1 & g \\ e & f^2 + ge \end{pmatrix}. \tag{3.33}$$

Then (3.31) reads

$$f\Box f = \nabla f \cdot \bar{\nabla} f - \nabla g \cdot \bar{\nabla} e, \tag{3.34}$$

$$f\Box e = 2\nabla f \cdot \bar{\nabla} e, \tag{3.35}$$

$$f\Box g = 2\nabla g \cdot \bar{\nabla} f, \tag{3.36}$$

where the field variables f, g, e are functions of the four independent variables z, w, $\tilde{z}$ and $\tilde{w}$. Also, $\nabla = (\partial_w, \partial_z)$, $\bar{\nabla} = (\partial_{\tilde{w}}, -\partial_{\tilde{z}})$ and $\Box = (\partial_{w\tilde{w}} - \partial_{z\tilde{z}})$. That is, we can express Yang's equation (3.31) as the above set of coupled non-linear equations. These are called Yang's equations in the 'R-gauge', [65].

3.1.4 The solution space

For the purpose of understanding the Bäcklund-Darboux transformation to be constructed in section 4.1 in the spirit of the framework described in 2.1, we here describe what is meant by the 'solution space' of the ASDYM equation. The *solution space* of the ASDYM equations is given by the quotient $\mathscr{M} = \mathscr{C}/\mathscr{G}$, where $\mathscr{C}$ is the set of ASD connections (on a fixed vector bundle), and $\mathscr{G}$ is the group of gauge transformations. Therefore, two connections $D, D' \in \mathscr{C}$ determine the same element of $\mathscr{M}$ whenever they are in the same equivalence class specified by a gauge automorphism.

3.2 Symmetry reductions

Here we give a brief prescription of the general aspects of the reduction process by discussing a list of criteria used for the classification of symmetry reductions of the ASDYM equations, we then give examples of reductions. The majority of reductions find their natural setting in the gauge potential, i.e. 'A', form of the ASDYM equations. Reductions obtained naturally from the J-matrix form are more rare as it is not directly clear how to impose the symmetry on the connection in such a way that the J-matrix will also be invariant. The BT for the ASDYM equation we construct in the next chapter finds its natural formulation in terms of the J-matrix and therefore it has been necessary for us to convert the relevant reductions to the J-formalism to fully implement the reduction of the BT. This conversion is often not straightforward and requires some care. For our work we have had to convert most of the reductions for the implementation of our BT, this is to our knowledge new material not present in the current literature. A systematic exposition of symmetry reductions of the ASDYM equation can be found in [16] where the authors have implemented the program of reducing the ASDYM equations to various integrable equations as proposed by Ward ([66]). Classification of the reductions follows the scheme:

- **Subgroup, H, of the conformal group.** As mentioned above H is a symmetry group of the ASDYM equations because the ASD condition on two-forms is preserved by conformal isometries of space-time. These are in fact the only (see [67]) space-time transformations with this property and thus invariance under point transformations of space-time requires the transformation to be conformal. It should be noted that the reduction process acquires a different character depending on whether or not the subgroup acts freely, that is if the only element of H with fixed points is the identity. In this work we only consider reductions under free actions. In this case the assumption that a connection on the bundle is invariant under the group transformation is equivalent to the assumption that the components of the potential one-form are constant along the generators of the group. It is then important that the matrices of the gauge transformations be independent of the ignorable coordinates also as otherwise one cannot ensure that the connection remains in the invariant gauge.

- **Gauge group or structure group, G.** This is the group of transformations of local trivialisations of the vector bundle on which the Yang-Mills connection is defined. The potential one-form takes values in the Lie algebra of the gauge group, and therefore different gauge groups typically lead to different equations. That is, reductions by subgroups of the conformal group with different gauge groups generically result in equations of different characters. For example, under a reduction by a 3-dimensional subgroup of the conformal group called a Painlevé group, the ASDYM equation with gauge group $\mathrm{SL}(2,\mathbb{C})$ reduces to a Painlevé equation, whereas if the gauge group is one of the Bianchi groups (groups obtained via the exponential map from 3-dimensional Lie algebras, which are classified according to the Bianchi classification) the reduced equations are linear, [66].

- **Conjugacy classes of some of the Higgs fields.** The Higgs fields are auxiliary fields corresponding to connection-like objects for invariant directions. Because of this, they transform by conjugation under gauge transformations and in some cases the conjugacy class of a Higgs field is constant. In these cases we can use gauge symmetry to put the Higgs fields in Jordan normal forms and the different reductions can be distinguished by different choices for the normal form. We shall give examples in later chapters.

Consider as a simple example the situation where the connection is invariant under the subgroup of time translations $(t,x_1,x_2,x_3) \mapsto (t+\alpha,x_1,x_2,x_3)$, that is where the generator of the symmetry is given by $X = \partial_t$. Invariance may then be imposed by requiring the components of the potential one form,

$$A = A_t dt + A_x dx + A_y dy + A_z dz$$

to be independent of the 'ignorable coordinate' t. It is then clear that a general gauge transformation $A \mapsto g^{-1}Ag + g^{-1}dg$, where g depends on t in addition to the other coordinates transforms a potential that is invariant into one that is not, therefore independence of time as a condition on the components of A is a restriction on both the potential and the gauge in which it is presented: only transformations for which g is independent of t preserve the invariance of the gauge. Moreover, in a characterisation of the reduction, one usually makes use of gauge symmetry of the ASDYM equation to put the Higgs fields or the components of the potential one-form in some particular form in which the reduction is done. Still, after all this, the reduced equation may possess an additional coordinate symmetry which can be used to put the reduced equation in a canonical form.

3.2.1 Reductions to 2 dimensions

3.2.1.1 Reduction by group of translations — The NLS and the KdV equations

A two-dimensional translation group H is generated by two constant independent vectors X and Y. Consider the reduction under the symmetry generators

$$X = \partial_w - \partial_{\tilde{w}}, \qquad Y = \partial_{\tilde{z}}.$$

Choosing the linear coordinates $x = w + \tilde{w}$ and $t = z$, which are constant along X and Y and therefore are well defined on the quotient space, we can then reduce the linear system to

$$L = \partial_x + A_w - \zeta Q, \qquad M = \partial_t + A_z - \zeta(\partial_x + A_{\tilde{w}}), \tag{3.37}$$

where the components of A depend solely on x and t and $Q = \iota_Y(A) = A_{\tilde{z}}$ and $P = \iota_X(A) = A_w - A_{\tilde{w}}$ are the Higgs field of Y and X respectively. The compatibility condition $[L, M] = 0$ gives the reduced ASDYM equations

$$\begin{aligned} Q_x + [A_{\tilde{w}}, Q] &= 0, \\ [\partial_x + A_w, \partial_t + A_z] &= 0, \\ P_x + Q_t + [A_w, P] + [A_z, Q] &= 0. \end{aligned} \tag{3.38}$$

The first equation in (3.38) implies that the conjugacy class of Q only depends on t and therefore we gauge transform Q to a normal form in which it depends only on t. Further, we can choose a gauge such that $A_{\tilde{w}} = 0$. The resulting form of the potential, A, is said to be in *normal gauge* and this is the gauge used to obtain the reduction to both the Korteweg-de Vries (KdV) and non-linear Schrödinger (NLS) equations. The remaining gauge freedom is given by $P \mapsto g^{-1}Pg$ and $Q \mapsto g^{-1}Qg$, where g is a constant matrix.

Having chosen the subgroup of the conformal group we now have to make the choice for the structure group. The $\mathrm{SL}(2,\mathbb{C})$ structure group reduces the ASDYM equation, essentially, to the KdV and the NLS equations. In fact this gives a complete classification of the reductions of SL(2, $\mathbb{C}$) by 2-dimensional translation groups, [31]. With gauge group SL(2, $\mathbb{C}$), the Higgs fields are represented by 2×2 matrices and one must analyse the normal forms of these. The semi-simple case gives the NLS while the degenerate one gives the KdV. It becomes more difficult to obtain a complete classification with a larger gauge group.

3.2.1.2 The sine-Gordon equation

Consider the reduction under the symmetry generators

$$X = \partial_w, \qquad Y = \partial_{\tilde{w}}$$

that is, let the ASDYM fields depend on z and $\tilde{z}$ only. Choose a gauge such that $A_{\tilde{z}} = 0$. The ASDYM equations (3.5) then reduce to

$$\begin{aligned} \partial_z A_w + [A_z, A_w] &= 0, \\ \partial_{\tilde{z}} A_z + [A_w, A_{\tilde{w}}] &= 0, \\ \partial_{\tilde{z}} A_{\tilde{w}} &= 0. \end{aligned} \tag{3.39}$$

Furthermore, in the generic case, we may use the remaining freedom to gauge the system such that ([68])

$$A_{\tilde{w}} = k \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and let the remaining components take the form

$$A_z = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}, \qquad A_w = \lambda \begin{pmatrix} 0 & a - ib \\ a + ib & 0 \end{pmatrix}.$$

The field equations (3.39) then result in

$$a_z = 2ibc, \quad b_z = -2iac, \text{ and } c_{\tilde{z}} = 2ikb, \tag{3.40}$$

and the first two of the above equations give us the constant of motion $a^2 + b^2 = \lambda^2$ which we exploit by introducing the parametrization $a = \lambda \cos\theta$ and $b = \lambda \sin\theta$. Then $c = \frac{i}{2}\theta_z$ and therefore the equation for c results in

$$\theta_{z\tilde{z}} = 4k\lambda \sin\theta, \tag{3.41}$$

which is the sine-Gordon (SG) equation in 'moving' frame. That is, the reduction to the sine-Gordon equation is given by $A_{\tilde{z}} = 0$ and

$$A_z = \frac{i\theta_z}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad A_w = \lambda \begin{pmatrix} 0 & \exp(-i\theta) \\ \exp(i\theta) & 0 \end{pmatrix}, \qquad A_{\tilde{w}} = k\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.42}$$

with $\theta \equiv \theta(z, \tilde{z})$.

As pointed out at the beginning of this section the majority of reductions are more natural in the 'A' form, as above for the SG equation. However for the later implementation of the BT which we have derive for the ASDYM equations we shall require the reduction in the framework of Yang's matrix, which is achieved by solving equations (3.24) defining the K and H matrices. We have had to perform these computations for the reductions in our work with the exception of the reduction to the Ernst equation which naturally takes form in the J formalism. Note that this task is often not straightforward and can in fact be fairly non-trivial. To the best of our knowledge these computations are new. For instance, reduction of the fifth Painlevé given the choice of independent generators given in [16] is not at all straightforward. We discuss this a little more in chapter 5.

To obtain the reduction in the J-formalism we must solve equations (3.24) given the above forms of the A_μ's. Thus $\partial_{\tilde{z}} K = 0 \implies K = K(z, w, \tilde{w})$ and then $\partial_{\tilde{w}} K = -A_{\tilde{w}} K^{-1} \implies$

$$K = \exp\left[-k\tilde{w}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] M^{-1}(z, w),$$

which, using the definition of the matrix exponential we can compute as

$$K = \begin{pmatrix} \cosh k\tilde{w} & -\sinh k\tilde{w} \\ -\sinh k\tilde{w} & \cosh k\tilde{w} \end{pmatrix} M^{-1}(z, w). \tag{3.43}$$

Similarly, $\partial_z H = -A_z H^{-1} \implies$

$$H = \begin{pmatrix} e^{\frac{i\theta}{2}} & 0 \\ 0 & e^{-\frac{i\theta}{2}} \end{pmatrix} \tilde{M}(w, \tilde{z}, \tilde{w}),$$

and finally $\partial_w H = -A_w H^{-1} \implies$

$$H = \begin{pmatrix} e^{\frac{i\theta}{2}} & 0 \\ 0 & e^{-\frac{i\theta}{2}} \end{pmatrix} \begin{pmatrix} \cosh\lambda w & -\sinh\lambda w \\ -\sinh\lambda w & \cosh\lambda w \end{pmatrix} \tilde{M}(\tilde{z}, \tilde{w}).$$

The Yang matrix $J = K^{-1}H$ for the SG reduction is then

$$J_{SG} = \begin{pmatrix} \cosh(k\tilde{w}) & \sinh(k\tilde{w}) \\ \sinh(k\tilde{w}) & \cosh(k\tilde{w}) \end{pmatrix} \begin{pmatrix} \exp(i\theta/2) & 0 \\ 0 & \exp(-i\theta/2) \end{pmatrix} \begin{pmatrix} \cosh(\lambda w) & -\sinh(\lambda w) \\ -\sinh(\lambda w) & \cosh(\lambda w) \end{pmatrix}, \tag{3.44}$$

up to gauge $J \mapsto M(z,w) J \tilde{M}(\tilde{z}, \tilde{w})$. Setting $\lambda = k = \frac{1}{2}$ (3.31) $\iff$

$$\theta_{z\tilde{z}} = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) = \sin\theta. \tag{3.45}$$

3.2.1.3 The Ernst equation

Following Witten, [69], we can obtain the Ernst equation via dimensional reduction of the ASDYM equation. In this case we choose a two-dimensional conformal group which is not a translation. Specifically, consider the subgroup generated by one rotation and one translation

$$X = w\partial_w - \tilde{w}\partial_{\tilde{w}}, \qquad Y = \partial_z + \partial_{\tilde{z}},$$

and introduce the coordinates $w = re^{i\theta}$, $\tilde{w} = re^{-i\theta}$, $z = t - x$ and $\tilde{z} = t + x$. Given these one can perform a gauge transformation such that $A = -P\frac{d\tilde{w}}{\tilde{w}} + Qd\tilde{z}$, where P and Q depend only on x and r. Inserting this in (3.5) we obtain the reduced ASDYM equations in the form

$$P_x + rQ_r + 2[Q,P] = 0, \qquad P_r - rQ_x = 0. \tag{3.46}$$

The first implies that there exists a Yang matrix $J(x,r)$ such that

$$P = -\frac{r}{2}J^{-1}J_r, \qquad Q = \frac{1}{2}J^{-1}J_x. \tag{3.47}$$

and the second equation gives us that the J matrix satisfies the equation[3]

$$r\partial_x\left(J^{-1}J_x\right) + \partial_r\left(rJ^{-1}J_r\right) = 0. \tag{3.48}$$

If we parametrise J as

$$J = \frac{1}{f}\begin{pmatrix} \psi^2 + f^2 & \psi \\ \psi & 1 \end{pmatrix}, \tag{3.49}$$

then (3.48) becomes

$$\begin{aligned} f\Delta f &= \nabla f \cdot \nabla f - \nabla\psi \cdot \nabla\psi, \\ f\Delta\psi &= 2\,(\nabla f)\cdot(\nabla\psi), \end{aligned} \tag{3.50}$$

where $\nabla := (\partial_x, \partial_r)$ and $\Delta := \partial_{xx} + \frac{1}{r}\partial_r + \partial_{rr}$. Introducing the Ernst potential ε as $\varepsilon = f + i\psi$ the above can be written as

$$\Re(\varepsilon)\Delta\varepsilon = \nabla\varepsilon\cdot\nabla\varepsilon. \tag{3.51}$$

Equation (3.51) is called the Ernst equation and it describes stationary axisymmetric space-times in general relativity.

3.2.2 The Nahm equations

Suppose that A depends only on $t := w + \tilde{w}$, which means A is invariant under the group of translations parallel to the hyperplanes of constant t, i.e. generated by $X = \partial_w - \partial_{\tilde{w}}$. A gauge can then be found such that $A_w + A_{\tilde{w}} = 0$ and we can write

$$A_z = i(T_2 + iT_3), \qquad A_{\tilde{z}} = i(T_2 - iT_3), \qquad A_w = -iT_1, \qquad A_{\tilde{w}} = iT_1, \tag{3.52}$$

[3] Every solution to this reduced form of Yang's equation determines a stationary axisymmetric ASDYM field [16].

where the T_i's are matrix valued functions of t. The ASDYM equations, (3.5), then reduce to Nahm's equations

$$\dot{T}_1 = [T_2, T_3], \qquad \dot{T}_2 = [T_3, T_1], \qquad \dot{T}_3 = [T_1, T_2].$$

Furthermore, imposing the restriction $T_i(t) = \omega_i(t)\sigma_i \; i = 1,\ 2,\ 3$ we have

$$\dot{\omega}_1 = \omega_2\omega_3, \qquad \dot{\omega}_2 = \omega_3\omega_1, \qquad \dot{\omega}_3 = \omega_1\omega_2, \tag{3.53}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

are the Pauli spin matrices. Equations (3.53) have the first integrals $\omega_1^2 - \omega_2^2 = \mu^2$ and $\omega_1^2 - \omega_3^2 = \lambda^2$ and we recognise these as the algebraic relations satisfied by the Jacobi elliptic functions, thus we have the solution given by

$$\begin{aligned} \omega_1(t) &= \mu \ \mathrm{sn}(\lambda t + \alpha, \lambda^{-1}\mu), \\ \omega_2(t) &= i\mu \ \mathrm{cn}(\lambda t + \alpha, \lambda^{-1}\mu), \\ \omega_3(t) &= -i\lambda \ \mathrm{dn}(\lambda t + \alpha, \lambda^{-1}\mu). \end{aligned}$$

Note that from $(3.53)_1$, say, and the first integrals we see that

$$\dot{\omega}_1^2 = (\omega_1^2 - \mu^2)(\omega_1^2 - \lambda^2),$$

which we recognise as an elliptic curve, thus the solution of the system in terms of elliptic functions is of no surprise.

The above reduction is standard but for the purpose of obtaining the reduction in the J matrix formalism, as for the SG equation, we must solve equations (3.24) using the form of the components of the potential given by (3.52) with $T_i(t) = \omega_i(t)\sigma_i \; i = 1,\ 2,\ 3$, that is

$$A_z = \frac{1}{2}\begin{pmatrix} 0 & \omega_1+\omega_2 \\ \omega_1+\omega_2 & 0 \end{pmatrix}, \quad A_w = \frac{\omega_3}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$A_{\tilde{z}} = \frac{1}{2}\begin{pmatrix} 0 & \omega_1-\omega_2 \\ \omega_1-\omega_2 & 0 \end{pmatrix}, \quad A_{\tilde{w}} = \frac{\omega_3}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.54}$$

So then solving (3.24), in this case, using the first integral $\omega_1^2 - \omega_2^2 = \mu^2$, we get

$$H = \begin{pmatrix} \sqrt{\omega_1(t)+\omega_2(t)} & -\sqrt{\omega_1(t)+\omega_2(t)} \\ \sqrt{\omega_1(t)-\omega_2(t)} & \sqrt{\omega_1(t)-\omega_2(t)} \end{pmatrix}\begin{pmatrix} e^{-\mu z/2} & 0 \\ 0 & e^{\mu z/2} \end{pmatrix}\tilde{M}(\tilde{z},\tilde{w}). \tag{3.55}$$

and similar for K and therefore, up to gauge, Yang's matrix for the Nahm reduction is

$$J_N = \frac{1}{\mu}\begin{pmatrix} \omega_1 e^{-\mu(z-\tilde{z})/2} & -\omega_2 e^{\mu(z+\tilde{z})/2} \\ -\omega_2 e^{-\mu(z+\tilde{z})/2} & \omega_1 e^{\mu(z-\tilde{z})/2} \end{pmatrix}. \tag{3.56}$$

Again we remark that we do not know of this result having been written down before.

3.2.3 The Painlevé equations

Important progress toward the classification of reductions of the ASDYM equations under the above prescription was the success in obtaining the six Painlevé equations when the gauge group is SL(2, $\mathbb{C}$). The Painlevé equations play a fundamental role in the theory of integrable systems, their appearance in studies being virtually synonymous with integrability. They are the non-linear analogues of the classical special functions ([30]), possessing remarkable similarities to the latter. Let us give a brief review of the Painlevé equations. They first appeared during studies of a problem posed by Picard ([70]) who asked; which second order ODEs of the form

$$\frac{d^2u}{dz^2} = F(z,\ u,\ u'), \tag{3.57}$$

where F is rational in u and u' and analytic in z, have the property that solutions have no movable branch points, that is whose location of multi-valued singularities of any of the solutions are independent of the particular solutions chosen — they only depend on the equation itself. This property is now known as the Painlevé property (PP). Painlevé et al. [71] found 50 canonical equations of the form (3.57) with this property, up to Möbius (birational) transformations

$$U(\xi) = \frac{a(z)u + b(z)}{c(z)u + d(z)}, \qquad \xi = \phi(z),$$

with a, b, c, d, ϕ locally analytic functions. Of these 50 equations, 44 are either solved in terms of previously known functions (elementary functions, elliptic or solutions to linear ODEs) or are reducible to six new non-linear ODEs. These six define new transcendental functions ([29]) called the Painlevé equations and their solutions are called the Painlevé transcendents. The six equations are

$$\mathrm{P_I} \qquad u'' = 6u^2 + z, \tag{3.58}$$

$$\mathrm{P_{II}} \qquad u'' = 2u^3 + zu + \alpha, \tag{3.59}$$

$$\mathrm{P_{III}} \qquad u'' = \frac{1}{u}u'^2 - \frac{1}{z}u' + \frac{1}{z}(\alpha u^2 + \beta) + \gamma u^3 + \frac{\delta}{u}, \tag{3.60}$$

$$\mathrm{P_{IV}} \qquad u'' = \frac{1}{2u}u'^2 + \frac{3}{2}u^3 + 4zu^2 + 2(z^2 - \alpha)u + \frac{\beta}{u}, \tag{3.61}$$

$$\begin{aligned} \mathrm{P_V} \qquad u'' =& \left\{ \frac{1}{2u} + \frac{1}{u-1} \right\} u'^2 - \frac{1}{z}u' \\ &+ \frac{(u-1)^2}{z^2}\left(\alpha u + \frac{\beta}{u}\right) + \frac{\gamma u}{z} + \frac{\delta u(u+1)}{u-1}, \end{aligned} \tag{3.62}$$

$$\begin{aligned} \mathrm{P_{VI}} \qquad u'' =& \frac{1}{2}\left\{ \frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-z} \right\} u'^2 - \left\{ \frac{1}{z} + \frac{1}{u-1} + \frac{1}{u-z} \right\} u' \\ &+ \frac{u(u-1)(u-z)}{z^2(z-1)^2}\left\{ \alpha + \frac{\beta z}{u^2} + \frac{\gamma(z-1)}{(u-1)^2} + \frac{\delta z(z-1)}{(u-z)^2} \right\}, \end{aligned} \tag{3.63}$$

where α, β, γ and δ are constants.

First discovered from mathematical considerations, the Painlevé equations have appeared in a multitude of crucial physical applications including statistical mechanics, plasma physics, non-linear waves, quantum gravity and general relativity. Additionally they have attracted renewed interest since they were found to arise as reductions of soliton equations solvable by inverse scattering ([9, 10]), i.e. equations solvable by the inverse scattering transform method reduce, under symmetry reductions, to ODEs with the PP. The converse leads to the so-called Painlevé test which we discuss further in chapter 7.

The Painlevé equations are viewed as non-linear analogues of the classical special functions. Their general solution are irreducible in the sense that they cannot be expressed in terms of previously known functions, for example rational functions or special function solutions. However they do possess many rational solutions and special function solutions for specific values of the parameters (known as classical solutions) and they possess special kinds of BTs, known as Schelsinger transformations, whose action shift the parameters by integer values, [72, 73].

In the theory of isomonodromic deformations a leading role is played by the Painlevé equations. The isomonodromy method was developed precisely to study the Painlevé equations ([74, 75, 76]) and in this sense they are integrable. Isomonodromic deformation studies linear ODEs

$$\frac{d\Psi}{d\lambda} = A(\lambda,\ a_j)\Psi, \tag{3.64}$$

where Ψ is an $N \times N$ matrix of solutions and $A(\lambda,\ a_j)$ is an $N \times N$ matrix coefficient rational in λ and with some constant parameters a_j. Being rational in λ, Ψ will generically have singular points and the values of Ψ at initial, Ψ_i, and final, Ψ_f, points of a closed contour encircling a singular point generally take different values. Since the equation is linear we must have that these values are related as $\Psi_f = \Psi_i M$, where M is the $N \times N$ monodromy matrix. Monodromy matrices and other information regarding Ψ around the singular points form the monodromy data, see for example [76, 77]. It is clear that *a priori* this data will depend on the parameters a_j and the isomonodromic deformation studies ODEs of the form (3.64) parametrised by a_j which have the same monodromy data. Work in this area was initiated by Schlesinger ([78]) and subsequently extended by Jimbo et al. [75, 79, 80]. It was found that for the data to be independent of a_j, Ψ must also satisfy a system of linear PDEs of the form

$$\frac{\partial \Psi}{\partial a_k} = B_k(\lambda,\ a_j)\Psi, \tag{3.65}$$

with B_k an $N \times N$ matrix. The isomonodromy condition is therefore equivalent to the compatibility of (3.64)–(3.65). This compatibility, called the deformation equation, a non-linear DE for A, has the PP. Jimbo and Miwa ([75, 79, 80]) showed that when the matrices are 2×2 and the system has 4 singular points the deformation equations are equivalent to the Painlevé equations.

Finally a striking similarity with the classical special functions is that they possess a coalescence cascade, figure 3.1 ([29, 81]), reminiscent of the special functions.

That is, among the Painlevé equations the 'master' one is $\mathrm{P_{VI}}$ and the others are

$$\begin{tikzcd}
\mathrm{P_{VI}} \arrow[r] & \mathrm{P_{V}} \arrow[r] \arrow[d] & \mathrm{P_{IV}} \arrow[d] & \\
 & \mathrm{P_{III}} \arrow[r] & \mathrm{P_{II}} \arrow[r] & \mathrm{P_{I}}
\end{tikzcd}$$

Figure 3.1: Coalescence cascade for the Painlevé equations

derived from it by 'degeneration' according to the above diagram. Here the arrow $\mathrm{P_{VI}} \to \mathrm{P_{V}}$ means, for example, that we may recover $\mathrm{P_{V}}$ from $\mathrm{P_{VI}}$ by the transformation $(t,u) \mapsto (\hat{t},\hat{u})\ :\ t = 1+\varepsilon\hat{t}, u = \hat{u}$, a suitable change of parameters in $\mathrm{P_{VI}}$ and by taking the limit $\varepsilon \to 0$, [29]. In fact similarities with the classical special functions can be put on a more robust and geometric framework by associating with the singularities of each linear system a partition of n (where n is the number of singular points) and then making a correspondence of this partition with regular elements of $\mathrm{GL}(n,\mathbb{C})$, see [32, 34, 35] for such a programme for both special functions and the Painlevé equations. Originally, the partitions of 4 are associated with the Painlevé equations according to:

For instance, the partition $(1,1,1,1)$ for $\mathrm{P_{VI}}$ means that the linear system of differential equations, whose isomonodromic deformation provides $\mathrm{P_{VI}}$, has 4 *regular* singular points $\lambda = 0,1,t,\infty$. Likewise, the partition $(2,1,1)$ for $\mathrm{P_{V}}$ means that this equation is obtained from the isomonodromic deformation of a linear system with

$$\begin{tikzcd}
(1,1,1,1) \arrow[r] & (2,1,1) \arrow[r] \arrow[d] & (3,1) \arrow[d] \\
& (2,2) \arrow[r] & (4)
\end{tikzcd}$$

Figure 3.2: Each partition of 4 is associated with some Painlevé equation. Note that $\mathrm{P_I}$ and $\mathrm{P_{II}}$ correspond to the same partition.

one irregular singular point of Poincaré rank 1 and two regular singular points. The confluence process can then be reinterpreted as a map between regular elements of the Lie algebra of $\mathrm{GL}(n,\ \mathbb{C})$ (where for the Painlevé equations $n = 4$) associated to different partitions, see for instance [34, 35] for the implementation of this process in the case of the generalised hypergeometric functions (GHF).

Each Painlevé equation has particular solutions given in terms of hypergeometric type functions ([81]):

$$\begin{tikzcd}
\text{Gauss} \arrow[r] & \text{Kummer} \arrow[r] \arrow[d] & \text{Hermite} \arrow[d] \\
& \text{Bessel} \arrow[r] & \text{Airy}
\end{tikzcd}$$

Figure 3.3: The classical special functions are particular solutions to the Painlevé equations and posses analogous confluence cascade. Each function is also associated to a partition of 4 corresponding to the singular structure of each differential equation.

and the partitions of 4 in figure 3.2 are already attached to these hypergeometric type functions in the context of general hypergeometric functions (GHF) on the Grassmannian manifold, [34, 35]. In this theory, a partition λ of 4 specifies a type of regular element of $\mathrm{GL}(4,\mathbb{C})$ and the GHF is constructed using the maximal Abelian Lie subalgebra given by the centralisers of the regular element of type λ. Work in this direction, inspired by the works of Aomoto, [32], and Gelfand, [33], has been studied in [34, 35] (see also references within). For the Painlevé equations, this interpretation can also be given (see below) by using the relevant subgroups of the conformal group giving the reduction to the Painlevé equations and this work is described in chapter 5. In fact the relevant subgroups, called the Painlevé groups, [31, 76], are themselves centralisers to regular elements of $\mathrm{GL}(4,\ \mathbb{C})$, and the relevant degeneration map is a map between these centralisers. More details on this will be given in section 5.1 where the explicit maps will further be employed to lift the confluence to the level of the Schlesinger transformations, again this is a new result.

The Painlevé equations correspond to reductions of the ASDYM equations when the Lie algebra is $\mathfrak{sl}(2;\mathbb{C})$. They are reductions under the symmetry generated by elements conjugate to one of the following, [16, 31]

$$
\begin{aligned}
&\mathrm{P}_{\mathrm{I,II}} &&\begin{pmatrix} a & b & c & d \\ 0 & a & b & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix}, \\
&\mathrm{P}_{\mathrm{III}} &&\begin{pmatrix} a & b & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & 0 & c \end{pmatrix}, \\
&\mathrm{P}_{\mathrm{IV}} &&\begin{pmatrix} a & b & c & 0 \\ 0 & a & b & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & d \end{pmatrix}, \\
&\mathrm{P}_{\mathrm{V}} &&\begin{pmatrix} a & b & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}, \\
&\mathrm{P}_{\mathrm{VI}} &&\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}.
\end{aligned}
$$

That is, the conformal Killing vectors of these reductions correspond to the four-parameter subgroups of $\mathrm{GL}(4,\ \mathbb{C})$ given above and may be read off using (3.14).

Mason and Woodhouse call these the '*Painlevé groups*'. These are the centralisers of regular elements of the complex general linear group for a partition of n when $n = 4$. This partition of n can be interpreted as the singular structure of an isomonodromy problem, thus yielding a correspondence between regular elements and the six Painlevé equations. In this framework the geometry of the reductions to the Painlevé equations takes a more pronounced role and, using this perspective we show in chapter 5 how to recover the confluence $\mathrm{P_{VI}} \to \mathrm{P_V}$ and $\mathrm{P_V} \to \mathrm{P_{III}}$ from confluence of the respective elements of the Painlevé groups. The remaining maps giving the confluences for the other coalescences are detailed in Appendix A. Importantly these maps may then be employed, through the Darboux matrix we construct in chapter 4, to obtain a map from the Schlesinger transformation of one Painlevé equation to the Schlesinger transformation of a Painlevé equation lower in the coalescence hierarchy.

Let us show how to obtain the reduction to $\mathrm{P_{VI}}$ which we will later use to obtain the relevant Schlesinger transformations, 4.2.5. We choose the subgroup of $\mathrm{GL}(4,\mathbb{C})$ for $\mathrm{P_{VI}}$ which gives the three generators in spacetime (see (3.16))

$$X = -z\partial_z - w\partial_w, \quad Y = -\tilde{z}\partial_{\tilde{z}} - \tilde{w}\partial_{\tilde{w}}, \quad Z = z\partial_z + \tilde{w}\partial_{\tilde{w}}. \tag{3.66}$$

The above (independent) commuting Killing vectors generate a three-dimensional subgroup of the conformal group and therefore, under this action, the ASDYM equations reduced to an ODE. Introducing local coordinates (p,q,r,t) such that $X = \partial_p$, $Y = \partial_q$ and $Z = \partial_r$ and making a gauge transformation to eliminate the dt component, the (invariant) Yang-Mills potential takes the form

$$A = Pdp + Qdq + Rdr, \tag{3.67}$$

where $P,\ Q,\ R$ are functions of t only and thus under gauge transformation transform through conjugation. These are therefore the Higgs fields of the reduction. The relevant coordinate transformation is $p = -\log w$, $q = -\log \tilde{z}$, $r = \log(\tilde{w}/\tilde{z})$,

$t = (z\tilde{z})/(w\tilde{w})$ and using this we have

$$\begin{aligned}\mathbf{A} &= A_z\,\mathrm{d}z + A_w\,\mathrm{d}w + A_{\tilde{z}}\,\mathrm{d}\tilde{z} + A_{\tilde{w}}\,\mathrm{d}\tilde{w}\\ &= P\mathrm{d}p + Q\mathrm{d}q + R\mathrm{d}r\\ &= -\frac{1}{w}P\mathrm{d}w - \frac{1}{\tilde{z}}Q\mathrm{d}\tilde{z} + R\left(\frac{\mathrm{d}\tilde{w}}{\tilde{w}} - \frac{\mathrm{d}\tilde{z}}{\tilde{z}}\right).\end{aligned}$$

Hence $zA_z = 0$, $wA_w = -P$, $\tilde{z}A_{\tilde{z}} = -(Q+R)$ and $\tilde{w}A_{\tilde{w}} = R$. The ASDYM equations (3.5) reduce to the system of three matrix-valued ODEs

$$P' = 0, \quad tQ' = [R,Q] \quad \text{and} \quad t(1-t)R' = [tP+Q,R], \tag{3.68}$$

where prime denotes differentiation with respect to t. It follows from these equations that the traces of P^2, Q^2, R^2 and $(P+Q+R)^2$ are all constants and are in fact related to the constants α, β, γ and δ appearing in $\mathrm{P_{VI}}$ (3.63). The transcendents themselves are, as one might expect, gauge-invariants of the ASDYM equation and in most cases can be constructed, for example, from the roots of the gauge-invariant quadratic in s ([16, 31])

$$\det([P,\ sQ-R]) = 0. \tag{3.69}$$

Furthermore, with some care the reduction can be extended to the linear problem of the ASDYM equations. The non-trivial aspect in this is that in the reduction the spectral parameter appearing in the linear problem of the ASDYM equation must also be transformed. Thus, on introducing the scaled spectral parameter $\lambda = -\tilde{z}/(w\zeta)$ and taking $\Psi(z,w,\tilde{z},\tilde{w};\zeta) = \Phi(t;\lambda)$, we can extend this reduction to the Lax pair (3.20)–(3.21), giving

$$\begin{aligned}\partial_t\Phi &= -\left(\frac{R}{\lambda-t}\right)\Phi,\\ \partial_\lambda\Phi &= \left(\frac{Q}{\lambda} - \frac{P+Q+R}{\lambda-1} + \frac{R}{\lambda-t}\right)\Phi.\end{aligned}$$

which we recognise as the isomonodromy problem for $\mathrm{P_{VI}}$.

The second equation in (3.68) shows that there is an $\mathrm{SL}(2,\mathbb{C})$-valued function of t, $G(t)$, and a constant $Q_0 \in \mathfrak{sl}(2,\mathbb{C})$, such that

$$Q(t) = G(t)^{-1}Q_0G(t) \text{ and } R(t) = -tG(t)^{-1}G'(t). \tag{3.70}$$

Using the residual gauge freedom we can, in the general case, take the constant matrices P and Q_0 to have the form

$$P = \frac{\theta_\infty}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad Q_0 = \frac{\theta_0}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

To construct J we need to solve (3.24) for H and K. From $(3.24)_1$ we get $\partial_z H = 0$, $w\partial_w H = -PH \implies H = \mathrm{e}^{-P\log w}\tilde{M}(\tilde{z},\tilde{w})$. The remaining two equations, $(3.24)_2$ can be written as

$$\tilde{z}\partial_{\tilde{z}}K = (Q+R)K, \quad \tilde{w}\partial_{\tilde{w}}K = -RK.$$

Summing these and transforming to the (p,q,r,t) coordinates gives

$$\partial_q K = -QK$$

which, upon invoking (3.70) becomes

$$\partial_q(GK) = -Q_0(GK) \implies K = G^{-1}\mathrm{e}^{-qQ_0}\bar{M}(p,\ r,\ t),$$

having used the fact that $G = G(t)$. Inserting this expression into

$$\tilde{z}\partial_{\tilde{z}}K = (Q+R)K$$

we get

$$(t\partial_t - \partial_r)\bar{M} = 0.$$

We solve this via the method of characteristics, thus $dt/ds = t$ and $dr/ds = -1$ and therefore $s = -r$ and $\kappa = \log t + r$ and therefore $\bar{M} = \bar{M}(p,\kappa)$. However

$$\kappa = \log t + r = \log((z\tilde{z})/(w\tilde{w})) + \log(\tilde{w}/\tilde{z}) = \log(z/w)$$

and since $p = -\log w$ we get that $\bar{M} = M^{-1}(z,w)$ from which we conclude that

$$K = G^{-1}\mathrm{e}^{-qQ_0}M^{-1}(z,w)$$

and the form of J is then

$$J = K^{-1}H = \tilde{z}^{-Q_0}G(t)w^P. \tag{3.71}$$

From the last equation in (3.68), we see that the ASDYM equations in this reduction then become

$$(1-t)(tG^{-1}G')' = [tP + G^{-1}Q_0G, G^{-1}G'].$$

So $J_{\mathrm{VI}} = U(\tilde{z})^{-1}G(t)V(w)$, where $t = \dfrac{z\tilde{z}}{w\tilde{w}}$ and

$$U(\tilde{z}) = \tilde{z}^{Q_0} = \begin{pmatrix} \tilde{z}^{\theta_0/2} & 0 \\ 0 & \tilde{z}^{-\theta_0/2} \end{pmatrix}, \qquad V(w) = w^P \begin{pmatrix} w^{\theta_\infty/2} & 0 \\ 0 & w^{-\theta_\infty/2} \end{pmatrix}.$$

The parameters in $\mathrm{P_{VI}}$ (equation (3.63)) are given by $\alpha = \frac{1}{2}(\theta_\infty - 1)^2$, $\beta = -\frac{1}{2}\theta_0^2$, $\gamma = \mathrm{tr}\{(P+Q+R)^2\}$ and $\delta = \mathrm{tr}(R^2) + \frac{3}{2}$.

Chapter 4

Bäcklund-Darboux transformations of the ASDYM equations and some reductions

In this chapter we construct a characterisation of a Darboux matrix (DM) acting on eigenfunctions of the ASDYM linear problem in terms of solutions to the ASDYM equations, [1, 2]. We have seen that the ASDYM equations are a rich source of integrable equations ([16, 62, 63]) and as such it is tempting to view them as a sort of 'master' integrable system. In light of this perspective it is natural to enquire as to whether its BT, if constructed in a sufficiently general set up, might be rich enough that all BT of equations embedded in it can be recovered from it directly. Moreover, from the Bäcklund-Darboux transformation one could attempt to construct a permutability theorem for the ASDYM equations and obtain from it, via reduction, the discrete equations arising as permutability of the BTs of the reduced equations. Such a programme would put, in alignment with the view of the ASDYM equations as a 'master' integrable system, some more order in the study of integrable systems, their BTs and the discrete integrable systems arising from them. Furthermore, such a construction might lead to novel discrete equations and shed some light as to how to obtain a discrete version of the ASDYM equations themselves. We lay out such a programme in the next section by constructing a very general BT for the ASDYM and give examples of its reduction to BTs of the reduced equations presented in chapter 3. In chapter 6 we show, using our ASDYM BT, that a permutability theorem for the BTs of the ASDYM equations can indeed be obtained. Permutability of the transforms is not automatically satisfied but requires three independent algebraic conditions to hold, these we call the '*ASDYM Bianchi system*' - a system of equations governing the evolution of solutions to the ASDYM equations on a lattice which is a rich family of discrete integrable systems possessing lattice gauge invariance, see chapter 6.

Various BTs for the ASDYM equations have been constructed in the past by several authors, ([16], [18]–[23], [25], [82]–[83]), through, for instance, solutions of Riemann-Hilbert problems. Such techniques are extremely powerful but also difficult to implement and, furthermore, the resulting BTs require a specific gauge for the implementation of the transformation itself. This latter feature makes their use inconvenient, especially when attempting to reduce the BT and for their implementation towards a permutability theorem. On the contrary, our result is a gauge invariant BT resulting quite naturally from the constraints on the DM arising from imposition that the transformed equations are to satisfy the same non-linear PDE. A consequence of this is that we are able to recover the Schlesinger transformations for the Painlevé equations through reduction of the ASDYM BT. Furthermore, after constructing the map between regular elements corresponding to the coalescence of singularities for the Painlevé equations, we can combine the two results and recover the Schlesinger transformations for $\mathrm{P_V}$–$\mathrm{P_{II}}$ from those of $\mathrm{P_{VI}}$, see chapter 5.

4.1 Bäcklund-Darboux transformation for the ASDYM equations

We restrict our analysis to a DM with affine dependence on the spectral parameter, see section 2.2. Given the ASDYM Lax pair

$$(\partial_w - \zeta\partial_{\tilde{z}})\Psi = -(A_w - \zeta A_{\tilde{z}})\Psi, \quad (\partial_z - \zeta\partial_{\tilde{w}})\Psi = -(A_z - \zeta A_{\tilde{w}})\Psi, \tag{4.1}$$

consider the affine Darboux matrix, [41, 45, 46, 50],

$$\mathbb{D} = S + \zeta T \tag{4.2}$$

generating the 'dressed' solution $\hat{\Psi} = \mathbb{D}\Psi$ with associated linear problem

$$(\partial_w - \zeta\partial_{\tilde{z}})\hat{\Psi} = -(\hat{A}_w - \zeta\hat{A}_{\tilde{z}})\hat{\Psi}, \quad (\partial_z - \zeta\partial_{\tilde{w}})\hat{\Psi} = -(\hat{A}_z - \zeta\hat{A}_{\tilde{w}})\hat{\Psi}, \tag{4.3}$$

where the $\hat{A}_\mu$ are the transformed components of the connection one form. Using $\hat{\Psi} = \mathbb{D}\Psi = (S + \zeta T)\Psi$ and equating coefficients of powers of ζ (since we require the solutions to be independent of the spectral parameter) gives us the following BTs in the form of DEs for S and T:

$$\partial_w S = SA_w - \hat{A}_w S, \quad \partial_z S = SA_z - \hat{A}_z S, \tag{4.4}$$

$$\partial_{\tilde{z}} S - \partial_w T = SA_{\tilde{z}} - \hat{A}_{\tilde{z}} S - TA_w + \hat{A}_w T, \quad \partial_{\tilde{w}} S - \partial_z T = SA_{\tilde{w}} - \hat{A}_{\tilde{w}} S - TA_z + \hat{A}_z T, \tag{4.5}$$

$$\partial_{\tilde{z}} T = TA_{\tilde{z}} - \hat{A}_{\tilde{z}} T, \quad \partial_{\tilde{w}} T = TA_{\tilde{w}} - \hat{A}_{\tilde{w}} T. \tag{4.6}$$

Recalling that the ASDYM equations guarantee (locally) the existence of matrices H and K such that

$$A_w = -(\partial_w H)H^{-1}, \quad A_z = -(\partial_z H)H^{-1}, \quad A_{\tilde{w}} = -(\partial_{\tilde{w}} K)K^{-1}, \quad A_{\tilde{z}} = -(\partial_{\tilde{z}} K)K^{-1},$$

we define:

$$\tilde{C} = \hat{H}^{-1} S H, \quad C = \hat{K}^{-1} T K, \tag{4.7}$$

thus, $S = \hat{H}\tilde{C}H^{-1}$ and $T = \hat{K}CK^{-1}$. We then have that (4.4) $\iff \partial_w \tilde{C} = \partial_z \tilde{C} = 0$ and, similarly, (4.6) $\iff \partial_{\tilde{z}} C = \partial_{\tilde{w}} C = 0$, that is, $\tilde{C} = \tilde{C}(\tilde{z}, \tilde{w})$ and $C = C(z, w)$.

Hence we have the characterisation $S = \hat{H}\tilde{C}(\tilde{z}, \tilde{w})H^{-1}$, $T = \hat{K}C(z, w)K^{-1}$, which means that the dressed solution is

$$\hat{\Psi} = (\hat{H}\tilde{C}H^{-1} + \zeta\hat{K}CK^{-1})\Psi. \tag{4.8}$$

Alternatively, making use of the gauge $\Psi \mapsto \Phi = K^{-1}\Psi$ this can be re-written as

$$\hat{\Phi} = (\hat{J}\tilde{C}J^{-1} + \zeta C)\Phi. \tag{4.9}$$

Recalling that the solution to Yang's equation is given by $J = K^{-1}H$, we can rewrite $T = \hat{H}\hat{J}^{-1}CJH^{-1}$, and the remaining equations (4.5) give us the required auto-Bäcklund transformation relating the seed solution, J, to the dressed solution, $\hat{J}$, of Yang's equation. That is, if there exists a Darboux transformation for the ASDYM equation corresponding to an affine DM (i.e. equation (4.2)), then the transformation $J \mapsto \hat{J}$ is given by;

$$\hat{J}(\hat{J}^{-1}CJ)_z = (\hat{J}\tilde{C}J^{-1})_{\tilde{w}}J, \qquad \hat{J}(\hat{J}^{-1}CJ)_w = (\hat{J}\tilde{C}J^{-1})_{\tilde{z}}J, \tag{4.10}$$

where $\tilde{C} \equiv \tilde{C}(\tilde{z}, \tilde{w})$ and $C \equiv C(z, w)$.

A few remarks on (4.10) are in order: firstly, as can easily be checked, these are indeed auto-Bäcklund transformations, i.e. if J is anti-self-dual then so is $\hat{J}$. Secondly, the matrices $\tilde{C}$ and C are in general not constant and we shall call them 'transporter' matrices (this choice will be explained later). In fact they are responsible for the gauge invariance of the Bäcklund-Darboux transformation, this can be seen by inserting a gauge transformed solution to Yang's equation (3.31). Specifically, we can talk of gauge transformations of two kinds. The first is that which arises from the integration constants appearing in the construction of the H and K in (3.26). Then J and $\hat{J}$ transform as $J \mapsto P^{-1}J\tilde{P}$ and $\hat{J} \mapsto \hat{P}^{-1}\hat{J}\hat{\tilde{P}}$, where $P = P(z, w)$ and $\tilde{P} = \tilde{P}(\tilde{z}, \tilde{w})$, and therefore $(C, \tilde{C})$ transform according to:

$$C \mapsto \hat{P}^{-1}CP, \qquad \tilde{C} \mapsto \hat{\tilde{P}}^{-1}\tilde{C}\tilde{P}, \tag{4.11}$$

and we see that they both preserve the functional dependence on half of the independent variables. On the other hand, the gauge transformation given by $A \mapsto g^{-1}Ag + g^{-1}dg$ (that is $\Psi \mapsto g^{-1}\Psi$) results in $H \mapsto g^{-1}H$ and $K \mapsto g^{-1}K$ and therefore J

maps to itself while $C = \hat{K}^{-1}TK \mapsto \hat{K}^{-1}\hat{g}Tg^{-1}K$ and $\tilde{C} = \hat{H}^{-1}SH \mapsto \hat{H}^{-1}\hat{g}Sg^{-1}H$ and again, it can be shown that these transformed matrices preserve the functional dependence on half the independent variables.

We see then that $(C,\tilde{C})$ absorb the information resulting from the gauge transformation. What is more, these matrices actually play a double role: they are also responsible for injecting the Bäcklund parameter into the equation, thus reflecting invariance of the system under Lie point symmetries, see 2.1.2 and 4.2.2. We point out that special cases of this same BT was constructed from a different perspective through consideration of certain Riemann-Hilbert problems. The transformation with $(C,\tilde{C})$ constant was obtained by Bruschi et al. in [24] while Sinha, Prasad and Wang in [84] obtained it for $(C,\tilde{C})$ constant multiples of the identity. However the $(C,\tilde{C})$ character as functions is, as we shall see, fundamental for the purpose of reductions in the case of non-translational subgroups of the conformal group, see 4.2.3 and 4.2.5.

To understand the action of the Darboux transformation and the importance of the $(C,\tilde{C})$ matrices we need to look back at section 3.1.3 where we showed that under the gauge transformation with $g = H$ and $g = K$ we could recover the connection with vanishing z and w components in the former case and vanishing $\tilde{z}$ and $\tilde{w}$ components in the latter. In effect this amounts to choosing two distinct frame fields, over the Riemann sphere covered by two open covers V and $\tilde{V}$, where V is the complement of $\zeta = \infty$ and $\tilde{V}$ is the complement of $\zeta = 0$ and the H and K are functions analytic on V and $\tilde{V}$, respectively (see more details in [64, 16]). The patching matrix (which itself determines J) determines the transition from solutions to the linear problem analytic in the neighbourhood of $\zeta = 0$, i.e. on V, and that in the neighbourhood of $\zeta = \infty$, i.e. on $\tilde{V}$, that is on the overlap $V \cap \tilde{V}$. The Darboux matrix $\mathbb{D} = \hat{H}\tilde{C}H^{-1} + \zeta\hat{K}CK^{-1}$ can then be interpreted as involving projections on the subsets V and $\tilde{V}$ by K^{-1} and H^{-1}, mapping these to analogous subsets for the transformed solutions, and then reversing the mappings in the transformed space (see figure 4.1). A more detailed and rigorous analysis is intended as future work, [2].

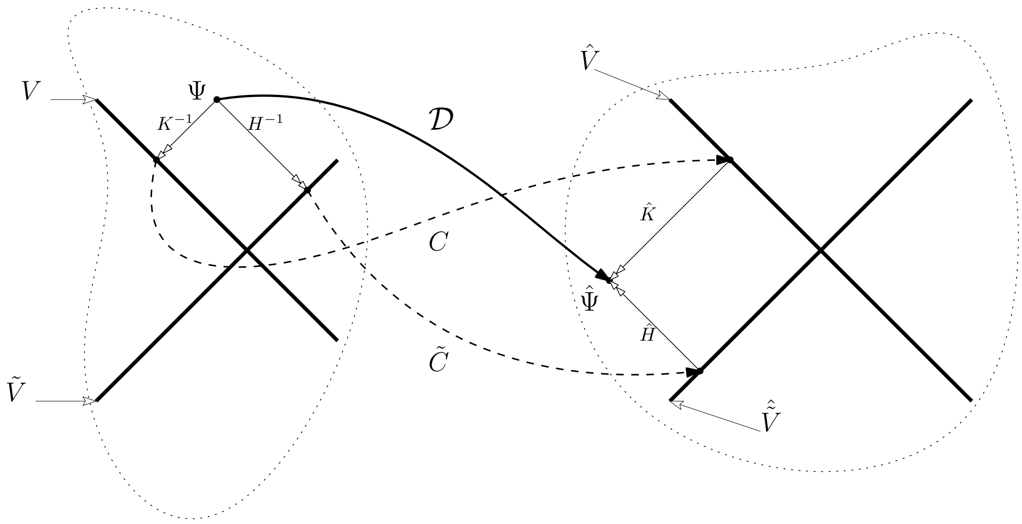

Figure 4.1: The action of element $\mathbb{D} \in \mathscr{D}$ splits, through gauge transformations via H and K respectively, into actions on the relevant subsets V and $\tilde{V}$ by C and $\tilde{C}$.

4.2 Reductions of the ASDYM BT

Here we show how the BTs for the reduced equations given in 3.2 may be obtained by reducing the BT for the ASDYM equations. The reductions are straightforward given the reduced J matrix and some simple ansatz for the 'transporter' matrices $(C, \tilde{C})$. Therefore from (4.10) we are able to determine the BTs for the reduced equations without the need to consider the PDE directly. This will have important implications for the construction of discrete integrable systems from the permutability theorem of the ASDYM BTs as it allows for the possibility of these systems to be recovered directly (see 6.4.1, 6.4.2 and 6.4.3).

4.2.1 The β Bäcklund transformation of Corrigan et al.

In [28] Corrigan et al. obtained the following BT transformation for the $\mathrm{SL}(2,\mathbb{C})$ ASDYM equations. Recall (see 3.1.3) that for $J \in \mathrm{SL}(2,\mathbb{C})$ we can write Yang's form of the ASDYM equations in component form as

$$f\Box f = \nabla f \cdot \bar{\nabla} f - \nabla g \cdot \bar{\nabla} e, \tag{4.12}$$

$$f\Box e = 2\nabla f \cdot \bar{\nabla} e, \tag{4.13}$$

$$f\Box g = 2\nabla g \cdot \bar{\nabla} f, \tag{4.14}$$

where $\nabla = (\partial_w, \partial_z)$, $\bar{\nabla} = (\partial_{\tilde{w}}, -\partial_{\tilde{z}})$ and $\Box = (\partial_{w\tilde{w}} - \partial_{z\tilde{z}})$. The β transformation of Corrigan et al. is then given by ([16, 27, 28])

$$\hat{f} = \frac{1}{f}, \qquad \partial_z \hat{g} = \frac{\partial_{\tilde{w}} e}{f^2}, \qquad \partial_w \hat{g} = \frac{\partial_{\tilde{z}} e}{f^2}, \qquad \partial_{\tilde{z}} \hat{e} = \frac{\partial_w g}{f^2}, \qquad \partial_{\tilde{w}} \hat{e} = \frac{\partial_z g}{f^2}. \tag{4.15}$$

We can recover the above from our BT (4.10) by making a simple choice for the $(C, \tilde{C})$ matrices. Thus, let

$$\tilde{C} = \begin{pmatrix} 0 & \mu \\ 0 & 0 \end{pmatrix},$$

and

$$C = \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix},$$

with μ and γ constants. We then find that

$$\hat{f} = \frac{\kappa}{f}, \qquad \partial_z \hat{g} = \frac{\mu\kappa}{\gamma}\frac{\partial_{\tilde{w}} e}{f^2}, \qquad \partial_w \hat{g} = \frac{\mu\kappa}{\gamma}\frac{\partial_{\tilde{z}} e}{f^2}, \qquad \partial_{\tilde{z}} \hat{e} = \frac{\gamma\kappa}{\mu}\frac{\partial_w g}{f^2}, \qquad \partial_{\tilde{w}} \hat{e} = \frac{\gamma\kappa}{\mu}\frac{\partial_z g}{f^2}, \tag{4.16}$$

where κ is a constant of integration. Choosing $\kappa = 1$ and $\gamma = \mu$ we recover the required transformation (4.15).

4.2.2 The sine-Gordon equation

The reduction to the SG equation

$$\theta_{z\tilde{z}} = \sin\theta,$$

in J form allows us to recover its standard BT, [41, 42]. Again recall that, using $\lambda = k = \frac{1}{2}$ in (3.44), we have obtained the J-matrix corresponding to the SG reduction which takes the form (see 3.2.1.2)

$$J_{SG} = U(\tilde{w})F(z, \tilde{z})V(w),$$

where

$$F = \begin{pmatrix} \exp(i\theta/2) & 0 \\ 0 & \exp(-i\theta/2) \end{pmatrix}$$

and

$$U^{-1}U_{\tilde{w}} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \implies U = U_0\tilde{W} = U_0\begin{pmatrix} \cosh(\tilde{w}/2) & \sinh(\tilde{w}/2) \\ \sinh(\tilde{w}/2) & \cosh(\tilde{w}/2) \end{pmatrix}, \quad (4.17)$$

$$V_wV^{-1} = -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \implies V = WV_0 = \begin{pmatrix} \cosh(w/2) & \sinh(w/2) \\ \sinh(w/2) & \cosh(w/2) \end{pmatrix} V_0. \quad (4.18)$$

The reduction of (4.10) to the Bäcklund transformation of the SG equation will result from inserting the expression

$$\begin{aligned} J_{SG} = \\ U_0 \begin{pmatrix} \cosh(\tilde{w}/2) & \sinh(\tilde{w}/2) \\ \sinh(\tilde{w}/2) & \cosh(\tilde{w}/2) \end{pmatrix} F(z,\tilde{z}) \begin{pmatrix} \cosh(w/2) & \sinh(w/2) \\ \sinh(w/2) & \cosh(w/2) \end{pmatrix} V_0, \end{aligned} \quad (4.19)$$

in (4.10) and requiring that the resulting differential equation be a PDE in z, $\tilde{z}$. Given that the reduction is obtained under generators of translations it is sufficient in this situation to take the matrices C and $\tilde{C}$ to be constants. It will be seen that in other reductions we are not allowed this choice indiscriminately, but, rather, the system will impose the choice of the dependence of these matrices on us. Upon inserting (4.19) in (4.10) we have[1] that $(4.10)_1$ gives

$$\begin{aligned} (\hat{F}^{-1}\hat{F}_{\tilde{z}})W\hat{V}_0\tilde{C}V_0^{-1}W^{-1} - W\hat{V}_0\tilde{C}V_0^{-1}W^{-1}(F^{-1}F_{\tilde{z}}) \\ = [\hat{F}^{-1}\tilde{W}^{-1}\hat{U}_0^{-1}CU_0\tilde{W}F, W_wW^{-1}]. \end{aligned} \quad (4.20)$$

[1]Note that just as in the previous section, we shall take hatted quantities to mean the transformed quantities.

Since we have chosen constant C and $\tilde{C}$ matrices the expressions $\hat{V}_0\tilde{C}V_0^{-1}$ and $\hat{U}_0^{-1}CU_0$ are also constant and we write them as

$$\hat{V}_0\tilde{C}V_0^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \text{and} \quad \hat{U}_0^{-1}CU_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{4.21}$$

To ensure that the above equation is a PDE in the variables z and $\tilde{z}$ only we require that both $W\hat{V}_0\tilde{C}V_0^{-1}$ and $\tilde{W}^{-1}\hat{U}_0^{-1}CU_0\tilde{W}$ be constants. This will be the case if $\alpha = \delta$, $\beta = \gamma$ and $a = d$, $b = c$ and therefore (4.20) gives the transformations

$$\alpha(\hat{\theta} - \theta)_{\tilde{z}} = 2b\sin\left\{\frac{\hat{\theta}+\theta}{2}\right\}, \qquad \beta(\hat{\theta} + \theta)_{\tilde{z}} = 2a\sin\left\{\frac{\hat{\theta}-\theta}{2}\right\}. \tag{4.22}$$

Similarly, $(4.10)_2$ give the transformations

$$b(\hat{\theta} + \theta)_{z} = 2\alpha\sin\left\{\frac{\hat{\theta}-\theta}{2}\right\}, \qquad a(\hat{\theta} - \theta)_{z} = 2\beta\sin\left\{\frac{\hat{\theta}+\theta}{2}\right\}, \tag{4.23}$$

and so we recover the whole set of BT for the SG equation, [42]. Furthermore, differentiating $(4.22)_1$ with respect to z and using $(4.23)_1$ to replace the derivatives with respect to z we find that the compatibility is satisfied. On the other hand, if we replace the derivative with respect to z using $(4.23)_2$ then commutativity of the partial derivatives is only satisfied if $b = \alpha = 0$. Similarly, starting with $(4.22)_2$ and differentiating with respect to z we find that consistency holds through use of $(4.23)_2$ but only holds through use of $(4.23)_1$ if $a = \beta = 0$. Therefore, compatibility of the above transformations implies that either $\alpha = b = 0$ or $\beta = a = 0$ and we recover the standard BTs for the SG equation with Bäcklund parameter β given by the combination b/α or β/a.

4.2.3 Ernst's equation

The Bäcklund transformations for the Ernst equation have been obtained by various authors, [42, 85, 86, 87] while the Darboux transformation can be found in [41] (see also references within). Here we reduce the ASDYM BT to recover the Darboux transformation obtained by Rogers and Schief, [41]. Let $z = t - x$, $\tilde{z} = t + x$, $w =$

$re^{i\theta}$ and $\tilde{w} = re^{-i\theta}$ and consider the parametrization of Yang's equation giving the reduction to Ernst's equation (see 3.2.1.3) [10, 41, 63],

$$J = \frac{1}{f}\begin{pmatrix} \psi^2 + f^2 & \psi \\ \psi & 1 \end{pmatrix}.$$

The Ernst potential is defined by $\varepsilon(x,r) := f(x,r) + i\psi(x,r)$. The BTs $(4.10)_1$ and $(4.10)_2$ then reduce to

$$\hat{J}_x\tilde{C} - \hat{J}\tilde{C}J^{-1}J_x + 2\hat{J}\tilde{C}_{\tilde{z}} = 2C_wJ + \left(\frac{\tilde{w}}{w}\right)^{\frac{1}{2}} CJ_r - \left(\frac{\tilde{w}}{w}\right)^{\frac{1}{2}} \hat{J}_r\hat{J}^{-1}CJ, \tag{4.24}$$

and

$$\left(\frac{w}{\tilde{w}}\right)^{\frac{1}{2}} \hat{J}_r\tilde{C} - \left(\frac{w}{\tilde{w}}\right)^{\frac{1}{2}} \hat{J}\tilde{C}J^{-1}J_r + 2\hat{J}\tilde{C}_{\tilde{w}} = 2C_zJ + \hat{J}_x\hat{J}^{-1}CJ - CJ_x. \tag{4.25}$$

We require these to be PDEs in the independent variables x and r and this will be achieved by exploiting the functional dependence of the $(C, \tilde{C})$ matrices on the independent variables. Thus we assume that (this is one possible ansatz) $\tilde{C}(\tilde{z}, \tilde{w}) = \tilde{l}(\tilde{z}, \tilde{w})\tilde{L}_0$ and $C(z,w) = l(z,w)L_0$, with $\tilde{L}_0$ and L_0 constant matrices given by

$$\tilde{L}_0 = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix},$$

and

$$L_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The transformations (4.24) and (4.25) will then be PDEs in the variables (x,t) if the quantities $\tilde{l}_{\tilde{z}}/\tilde{l}$, $l_w/\tilde{l}$, $\left(\frac{\tilde{w}}{w}\right)^{\frac{1}{2}} l/\tilde{l}$, $\left(\frac{\tilde{w}}{w}\right)^{\frac{1}{2}} \tilde{l}_{\tilde{w}}/\tilde{l}$ and $\left(\frac{\tilde{w}}{w}\right)^{\frac{1}{2}} l_z/\tilde{l}$ are functions of (x,t) only. Solving these we find that

$$\tilde{C} = \tilde{w}^{\kappa} e^{\lambda\tilde{z}}\tilde{L}_0, \qquad \text{and} \qquad C = w^{1-\kappa} e^{\lambda z} L_0, \tag{4.26}$$

where κ and λ are constants of integration and consequentially (4.24)–(4.25) reduce to the system

$$\left(\hat{J}_x - 2\lambda\hat{J}\right)\tilde{L}_0 - \hat{J}\tilde{L}_0 J^{-1}J_x = 2(1-\kappa)r^{-2\kappa}e^{-2x\lambda}L_0 J + r^{(1-2\kappa)}e^{-2x\lambda}\left(L_0 J_r - \hat{J}_r\hat{J}^{-1}L_0 J\right),$$

$$\left(\hat{J}_r + 2\frac{\kappa}{r}\hat{J}\right)\tilde{L}_0 - \hat{J}\tilde{L}_0 J^{-1}J_r = 2\lambda r^{(1-2\kappa)}e^{-2x\lambda}L_0 J + r^{(1-2\kappa)}e^{-2x\lambda}\left(\hat{J}_x\hat{J}^{-1}L_0 J - L_0 J_x\right).$$

The Darboux transformation for the Ernst equation is obtained by taking $\tilde{a} = \tilde{c} = \tilde{d} = a = b = d = 0$ and then compatibility of the above requires $\lambda = 0$ and $\kappa = 1/2$ and $c = \pm i\tilde{b}$. The two choices are equivalent up to an Ehlers transformation [41, 42]. Take $c = i\tilde{b}$, then

$$\begin{aligned}\omega_x &= \frac{-\omega(\omega+\bar{\omega})}{2|\varepsilon|^2(\varepsilon+\bar{\varepsilon})\bar{\omega}}\left(\varepsilon^2(\bar{\varepsilon}_x + i\bar{\varepsilon}_r) + \bar{\varepsilon}^2(\varepsilon_x - i\varepsilon_r)\right),\\ \omega_r &= \frac{-\omega(\omega+\bar{\omega})}{2|\varepsilon|^2(\varepsilon+\bar{\varepsilon})\bar{\omega}}\left(\frac{|\varepsilon|^2(\varepsilon+\bar{\varepsilon})}{r} + \varepsilon^2(\bar{\varepsilon}_r - i\bar{\varepsilon}_x) + \bar{\varepsilon}^2(\varepsilon_r + i\varepsilon_x)\right),\end{aligned} \tag{4.27}$$

where ω is the transformed solution to the Ernst equation.
If we introduce the variables $x^1 = r + ix$ and $x^2 = r - ix$ then the above transformation takes the more symmetric form

$$\begin{aligned}\omega_1 &= -\frac{\omega(\omega+\bar{\omega})}{\bar{\omega}(\varepsilon+\bar{\varepsilon})}\left(\frac{\varepsilon}{\bar{\varepsilon}}\bar{\varepsilon}_1 + \frac{(\varepsilon+\bar{\varepsilon})}{4r}\right),\\ \omega_2 &= -\frac{\omega(\omega+\bar{\omega})}{\bar{\omega}(\varepsilon+\bar{\varepsilon})}\left(\frac{\bar{\varepsilon}}{\varepsilon}\varepsilon_2 + \frac{(\varepsilon+\bar{\varepsilon})}{4r}\right),\end{aligned} \tag{4.28}$$

where $\omega_i = \partial_{x^i}\omega$.

This is the Darboux transformation for the Ernst equation ([41]), let us denote it by $\mathbb{D}$. In [41] it is shown that this is related to the Harrison transformation, [87], by $\mathbb{H} = \mathbb{G} \circ \mathbb{D}$, where $\mathbb{H}$ is the Harrison transformation and $\mathbb{G}$ is the Kramer-Neugebauer transformation (see [41, 42] and references therein). We have yet to determine whether our BT is able to yield these individual transformations directly.

4.2.4 Nahm's equations

In the case of Nahm's equations the independent variable is $t = \tilde{w} + w$ and the J matrix (see 3.2.2) is then given by

$$J_N = \frac{1}{\mu}\tilde{Z}WZ, \tag{4.29}$$

where $\mu^2 = \omega_1^2 - \omega_2^2$ is one first integral, $Z = \mathrm{diag}(e^{-\frac{\mu z}{2}}, e^{\frac{\mu z}{2}})$, $\tilde{Z} = \mathrm{diag}(e^{\frac{\mu \tilde{z}}{2}}, e^{-\frac{\mu \tilde{z}}{2}})$ and

$$W = \begin{pmatrix} \omega_1 & -\omega_2 \\ -\omega_2 & \omega_1 \end{pmatrix}.$$

We again make use of the BT (4.10) with $\hat{J}_N = \frac{1}{\hat{\mu}}\hat{\tilde{Z}}\hat{W}\hat{Z}$ and we introduce the following notation: $\Gamma = \hat{Z}\tilde{C}Z^{-1}$, $\tilde{\Gamma} = \hat{\tilde{Z}}^{-1}C\tilde{Z}$, $\sigma_3 = \mathrm{diag}(1,-1)$ and

$$W^{-1}W' = -W_3 = -\begin{pmatrix} 0 & \omega_3 \\ \omega_3 & 0 \end{pmatrix},$$

where W' denoted differentialtion with respect to t. Then (4.10) results in the expressions

$$\begin{aligned} \hat{W}_3\Gamma - \Gamma W_3 &= -\frac{\hat{\mu}}{2\mu}\left(\hat{\mu}\sigma_3\hat{W}^{-1}\tilde{\Gamma}W - \mu\hat{W}^{-1}\tilde{\Gamma}W\sigma_3\right), \\ \hat{W}_3\tilde{\Gamma} - \tilde{\Gamma}W_3 &= -\frac{\mu}{2\hat{\mu}}\left(\mu\hat{W}\Gamma W^{-1}\sigma_3 - \hat{\mu}\sigma_3\hat{W}\Gamma W^{-1}\right), \end{aligned} \tag{4.30}$$

and for these to be ODEs in the variable t we require either:

1.
$$\mu = \hat{\mu} \quad \text{with} \quad \Gamma = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \quad \text{and} \quad \tilde{\Gamma} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \tag{4.31}$$

or

2.
$$\mu = -\hat{\mu} \quad \text{with} \quad \Gamma = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \quad \text{and} \quad \tilde{\Gamma} = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}. \tag{4.32}$$

The BT for the two cases above reduce to:

1.

$$
\begin{aligned}
\hat{W}_3\Gamma - \Gamma W_3 &= \frac{\mu}{2}\left[\hat{W}^{-1}\tilde{\Gamma}W, \sigma_3\right], \\
\hat{W}_3\tilde{\Gamma} - \tilde{\Gamma} W_3 &= \frac{\mu}{2}\left[\sigma_3, \hat{W}\Gamma W^{-1}\right],
\end{aligned}
\tag{4.33}
$$

2.

$$
\begin{aligned}
\hat{W}_3\Gamma - \Gamma W_3 &= -\frac{\mu}{2}\left\{\hat{W}^{-1}\tilde{\Gamma}W, \sigma_3\right\}, \\
\hat{W}_3\tilde{\Gamma} - \tilde{\Gamma} W_3 &= \frac{\mu}{2}\left\{\sigma_3, \hat{W}\Gamma W^{-1}\right\}.
\end{aligned}
\tag{4.34}
$$

where $[,]$ is the usual commutator and $\{,\}$ the anti-commutator.

The above two give equivalent Bäcklund transformations and we therefore here proceed with the first case only. This gives

$$
\begin{aligned}
\delta\hat{\omega}_3 - \alpha\omega_3 &= -\frac{1}{\mu}\left(d\omega_1\hat{\omega}_2 - a\omega_2\hat{\omega}_1\right), \\
\alpha\hat{\omega}_3 - \delta\omega_3 &= -\frac{1}{\mu}\left(d\omega_2\hat{\omega}_1 - a\omega_1\hat{\omega}_2\right), \\
d\hat{\omega}_3 - a\omega_3 &= -\frac{1}{\mu}\left(\delta\omega_1\hat{\omega}_2 - \alpha\omega_2\hat{\omega}_1\right), \\
a\hat{\omega}_3 - d\omega_3 &= -\frac{1}{\mu}\left(\delta\omega_2\hat{\omega}_1 - \alpha\omega_1\hat{\omega}_2\right).
\end{aligned}
\tag{4.35}
$$

and there are eight non-trivial solutions for different choices of the parameters which, modulo choice of parameters, give fundamentally equivalent transformations. One such solution is given by $\delta = d$, $\alpha = a = 0$, finally yielding the transformation

$$
\hat{\omega}_3 = -\frac{1}{\hat{\mu}}\omega_1\hat{\omega}_2, \qquad \text{and} \qquad \hat{\omega}_1 = \hat{\mu}\frac{\omega_3}{\omega_2}, \tag{4.36}
$$

with $\hat{\mu}^2 = \hat{\omega}_1^2 - \hat{\omega}_2^2$. These transformations are nothing but the relevant addition laws for the Jacobi elliptic functions. Using the constant of motion μ^2 we then find the expression for $\hat{\omega}_i$, $i = 1,2,3$ in terms of the seed solutions ω_i as

$$\begin{aligned}
\hat{\omega}_1^2 &= \frac{\dot{\omega}_2^2}{\omega_2^2} - \omega_3^2, \\
\hat{\omega}_2^2 &= \frac{\dot{\omega}_2^2}{\omega_2^2} + \omega_2^2 - \omega_1^2 - \omega_3^2, \\
\hat{\omega}_3^2 &= \frac{\dot{\omega}_2^2}{\omega_2^2} - \omega_1^2.
\end{aligned} \tag{4.37}$$

It is straightforward to check that the transformed solutions satisfy the Nahm equations. Note that from (4.36) we immediately get

$$\hat{\omega}_3 - \omega_3 = -\frac{1}{\hat{\mu}} \left(\omega_1 \hat{\omega}_2 + \omega_2 \hat{\omega}_1\right), \tag{4.38}$$

which is the discrete Euler top of Hirota's and Kimura's discretization, see [88].

4.2.5 Schlesinger transformations for $\mathrm{P_{VI}}$

The sixth Painlevé equation

$$\begin{aligned}
u'' \;=\;& \frac{1}{2}\left\{\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-z}\right\} u'^2 - \left\{\frac{1}{z} + \frac{1}{u-1} + \frac{1}{u-z}\right\} u' \\
&+ \frac{u(u-1)(u-z)}{z^2(z-1)^2}\left\{\alpha + \frac{\beta z}{u^2} + \frac{\gamma(z-1)}{(u-1)^2} + \frac{\delta z(z-1)}{(u-z)^2}\right\},
\end{aligned} \tag{4.39}$$

where α, β, γ and δ are constants, is, as we have shown (see 3.2.3), a reduction of the ASDYM equations with Lie algebra $\mathfrak{sl}(2;\mathbb{C})$ ([16]). This equation, like all Painlevé equations with the exception of $\mathrm{P_I}$, possesses BTs mapping the solution to the equation to solutions of the same equation with different values of the parameters. These transformations are special types of BTs where the parameters are shifted by integer values, we call them *Schlesinger transformations* (STs). $\mathrm{P_{VI}}$ has 12 such transformations which were obtained by Muğan and Sakka in [72] by solving what is known as a Riemann-Hilbert problem. In this section we show how these STs can be recovered by reducing the BT for the ASDYM equation. The resulting DM giving these transformations can be aligned with the transformation matrices acting on the isomonodromy problem obtained by Muğan and Sakka. Recovering the Schlesinger transformations via the BT for the ASDYM equation has the advantage of being a simple, albeit tedious, process involving algebraic manipulations and solutions to some simple first-order PDEs. What is more, given the set of transformations for P_{VI} (in fact it turns out we do not need all 12 transformations; as we show in the next section, 4.3, by combining some Schlesinger transformations with Möbius transformations we can generate the whole set) we may recover, by a process of confluence, the Schlesinger transformations for P_V, P_{III} and the rest. This will be described in the next chapter, 5. Therefore a neat geometric picture for recovering the Schlesinger transformations for the Painlevé equation emerges from this framework. We remark that Masuda had also recovered the BTs of the Painlevé equations P_{II} - P_{IV} from the symmetries of Yang's equation, [26, 27].

Yang's matrix, J, for the P_{VI} reduction is given by (see 3.2.3)

$$J_{\mathrm{VI}} = U^{-1}(\tilde{z})G(t)V(w),$$

where $U^{-1}(\tilde{z}) = \tilde{z}^{-Q_0}$ and $V(w) = w^P$. With this form of J the BTs $(4.10)_1$ and $(4.10)_2$ give

$$\frac{t}{z}\left(\hat{G}^{-1}L\dot{G} - \hat{G}^{-1}\dot{\hat{G}}\hat{G}^{-1}LG\right) + \hat{G}^{-1}L_zG = \tilde{L}_{\tilde{w}} + \frac{t}{\tilde{w}}\left(\tilde{L}G^{-1}\dot{G} - \hat{G}^{-1}\dot{\hat{G}}\tilde{L}\right),$$

and

$$\begin{aligned}\frac{1}{w}\left(t\left(\hat{G}^{-1}\dot{\hat{G}}\hat{G}^{-1}LG - \hat{G}^{-1}L\dot{G}\right) + \left(\hat{G}^{-1}LGP - \hat{P}\hat{G}^{-1}LG\right)\right) + \hat{G}^{-1}L_wG \\ = \frac{1}{\tilde{z}}\left(t\left(\hat{G}^{-1}\dot{\hat{G}}\tilde{L} - \tilde{L}G^{-1}\dot{G}\right) + \left(\tilde{L}G^{-1}Q_0G - \hat{G}^{-1}\hat{Q}_0\hat{G}\tilde{L}\right)\right) + \tilde{L}_{\tilde{z}},\end{aligned}$$

where, as before, the hatted terms denote the transformed quantities. Furthermore we have introduced the quantities $L(z,w,\tilde{z}) = \hat{A}CA^{-1}$ and $\tilde{L}(w,\tilde{z},\tilde{w}) = \hat{B}\tilde{C}B^{-1}$. Now make the ansatz $L(z,w,\tilde{z}) = l(z,w,\tilde{z})L_0$ and $\tilde{L}(w,\tilde{z},\tilde{w}) = \tilde{l}(w,\tilde{z},\tilde{w})\tilde{L}_0$, where l and $\tilde{l}$ are scalar functions and L_0 and $\tilde{L}_0$ are constant matrices and introduce $X = \hat{G}^{-1}L_0G$.

The BTs become

$$t\dot{X}+\frac{zl_z}{l}X=\frac{z\tilde{l}_{\tilde{w}}}{l}\tilde{L}_0+\frac{z\tilde{l}}{\tilde{w}l}\left(\hat{R}\tilde{L}_0-\tilde{L}_0R\right),$$

and

$$t\dot{X}+\hat{P}X-XP-\frac{wl_w}{l}X=\frac{w\tilde{l}}{\tilde{z}l}\left(\left(\hat{R}\tilde{L}_0-\tilde{L}_0R\right)+\left(\hat{Q}\tilde{L}_0-\tilde{L}_0Q\right)\right)-\frac{w\tilde{l}_{\tilde{z}}}{l}\tilde{L}_0,$$

where $R=-tG^{-1}\dot{G}$, $Q=G^{-1}Q_0G$ (see 3.2.3). We require these to be ODEs in t and it is here that the functional form of $(C,\tilde{C})$ plays a critical role (c.f. 4.2.3). In fact for this we need to choose $(C,\tilde{C})$ such that the combinations

$$\frac{zl_z}{l},\qquad\frac{z\tilde{l}_{\tilde{w}}}{l},\qquad\frac{wl_w}{l},\qquad\frac{w\tilde{l}_{\tilde{z}}}{l},\qquad\frac{w\tilde{l}}{\tilde{z}l}.$$

be functions of $t=\frac{z\tilde{z}}{w\tilde{w}}$ only. Thus, solving these we find that $l(z,w,\tilde{z})=z^{\varepsilon}w^{\omega}\tilde{z}^{\kappa}$ and $\tilde{l}(w,\tilde{z},\tilde{w})=w^{\varepsilon+\omega-1}\tilde{w}^{\varepsilon}\tilde{z}^{1+\kappa-\varepsilon}$ (where we have ignored integration constants as these can be absorbed into L_0 and $\tilde{L}_0$) and $L(z,w,\tilde{z})=\hat{A}CA^{-1}=l(z,w,\tilde{z})L_0=z^{\varepsilon}w^{\omega}\tilde{z}^{\kappa}L_0$ and $\tilde{L}(w,\tilde{z},\tilde{w})=\hat{B}\tilde{C}B^{-1}=\tilde{l}\tilde{L}_0=w^{\varepsilon+\omega-1}\tilde{w}^{\varepsilon}\tilde{z}^{1+\kappa-\varepsilon}\tilde{L}_0$. We are now in a position to obtain the Schlesinger transformations for the $\mathrm{P_{VI}}$ equation.

Recall we chose a gauge in the reduction such that

$$P=\frac{\theta_\infty}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix},\qquad Q_0=\frac{\theta_0}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix},$$

and similar for the hatted variables, from which we have that

$$C=z^{\varepsilon}w^{\omega}\tilde{z}^{\kappa}\begin{pmatrix}a\tilde{z}^{\frac{1}{2}(\theta_0-\hat{\theta}_0)}&b\tilde{z}^{-\frac{1}{2}(\theta_0+\hat{\theta}_0)}\\c\tilde{z}^{\frac{1}{2}(\theta_0+\hat{\theta}_0)}&d\tilde{z}^{-\frac{1}{2}(\theta_0-\hat{\theta}_0)}\end{pmatrix},\tag{4.40}$$

and

$$\tilde{C}=w^{\varepsilon+\omega-1}\tilde{w}^{\varepsilon}\tilde{z}^{1+\kappa-\varepsilon}\begin{pmatrix}\tilde{a}\tilde{z}^{\frac{1}{2}(\theta_\infty-\hat{\theta}_\infty)}&\tilde{b}\tilde{z}^{-\frac{1}{2}(\theta_\infty+\hat{\theta}_\infty)}\\\tilde{c}\tilde{z}^{\frac{1}{2}(\theta_\infty+\hat{\theta}_\infty)}&\tilde{d}\tilde{z}^{-\frac{1}{2}(\theta_\infty-\hat{\theta}_\infty)}\end{pmatrix},\tag{4.41}$$

where

$$L_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{and} \quad \tilde{L}_0 = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}.$$

Using the Higgs fields P, R and Q (and corresponding hatted variables) given above, $X = \hat{G}^{-1}L_0G$ and l and $\tilde{l}$ found above, we may re-write the BTs as

$$t\dot{X} = t^{1-\varepsilon}\left[\hat{R}\tilde{C}_0 - \tilde{C}_0R + \varepsilon\tilde{C}_0\right] - \varepsilon X, \tag{4.42}$$

$$\begin{aligned} &\hat{P}X - XP - (\varepsilon+\omega)X \\ &= t^{-\varepsilon}\left[\left(\hat{Q}\tilde{C}_0 - \tilde{C}_0Q - (\kappa+1)\tilde{C}_0\right) + (1-t)\left(\hat{R}\tilde{C}_0 - \tilde{C}_0R + \varepsilon\tilde{C}_0\right)\right]. \end{aligned} \tag{4.43}$$

Differentiating (4.43) and substituting $\dot{X}$ from (4.42) we get the compatibility condition given by

$$\begin{aligned} &(\hat{R}\hat{P}\tilde{C}_0 + \tilde{C}_0PR) + (\hat{R}\tilde{C}_0 - \tilde{C}_0R) - (\hat{P}\tilde{C}_0R + \hat{R}\tilde{C}_0P) \\ &+(\varepsilon+\omega)\left(\varepsilon t^{\varepsilon-1}X - \hat{R}\tilde{C}_0 + \tilde{C}_0R - \varepsilon\tilde{C}_0\right) = \varepsilon\left[(\hat{P}\tilde{C}_0 - \tilde{C}_0P) + t^{\varepsilon-1}(XP - \hat{P}X)\right] \\ &\qquad +\varepsilon t^{-1}\left[(\hat{Q}\tilde{C}_0 - \tilde{C}_0Q) + (1-t)(\hat{R}\tilde{C}_0 - \tilde{C}_0R) - (\kappa+1)\tilde{C}_0\right] \end{aligned} \tag{4.44}$$

which must be satisfied for the BTs to be consistent. From (4.40) and (4.41) we get constraints on ε, κ, and ω depending on how the parameters θ_∞ and θ_0 transform and one then has to check that the system is consistent for the chosen shifts. Since $\tilde{C}$ is a function of $\tilde{z}$ and $\tilde{w}$ only we get that:

$$\begin{aligned} \hat{\theta}_\infty &= \theta_\infty + 2(\varepsilon+\omega-1) \quad &\text{or} \quad \tilde{a} = 0, \\ \hat{\theta}_\infty &= -\theta_\infty + 2(\varepsilon+\omega-1) \quad &\text{or} \quad \tilde{b} = 0, \\ \hat{\theta}_\infty &= -\theta_\infty - 2(\varepsilon+\omega-1) \quad &\text{or} \quad \tilde{c} = 0, \\ \hat{\theta}_\infty &= \theta_\infty - 2(\varepsilon+\omega-1) \quad &\text{or} \quad \tilde{d} = 0. \end{aligned} \tag{4.45}$$

While C being a function of z and w gives:

$$\begin{aligned}
\hat{\theta}_0 &= \theta_0 + 2\kappa \quad &\text{or} \quad a = 0,\\
\hat{\theta}_0 &= -\theta_0 + 2\kappa \quad &\text{or} \quad b = 0,\\
\hat{\theta}_0 &= -\theta_0 - 2\kappa \quad &\text{or} \quad c = 0,\\
\hat{\theta}_0 &= \theta_0 - 2\kappa \quad &\text{or} \quad d = 0,
\end{aligned} \tag{4.46}$$

Before reproducing all 12 Schlesinger transformations let us give a detailed exposition of how the calculation proceeds for the transformation shifting the monodromy exponents corresponding to the singularities at 0 and ∞, that is the DM $\mathbb{D}_{(\theta_0^{+},\theta_\infty^{\pm})}$ corresponding to the shift $(\theta_0,\theta_\infty)\mapsto(\hat{\theta}_0,\hat{\theta}_\infty)=(\theta_0+1,\theta_\infty+1)$.

1. Take $\hat{\theta}_\infty = \theta_\infty - 2(\varepsilon+\omega-1)$, $\hat{\theta}_0 = \theta_0 - 2\kappa$, $a=b=c=\tilde{a}=\tilde{b}=\tilde{c}=0$ and $d,\tilde{d}\neq 0$ (note that if either d or $\tilde{d}$ are zero then the Darboux transformation is just a gauge transformation), that is

$$L_0 = \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}, \quad \text{and} \quad \tilde{L}_0 = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{d} \end{pmatrix},$$

and let

$$G(t) = \begin{pmatrix} x(t) & y(t) \\ u(t) & v(t) \end{pmatrix}, \quad \text{and} \quad \hat{G}(t) = \begin{pmatrix} \hat{x}(t) & \hat{y}(t) \\ \hat{u}(t) & \hat{v}(t) \end{pmatrix}. \tag{4.47}$$

Using the above we find that (4.44) $\Rightarrow \varepsilon = 0$ and the $(1,1)$ component of (4.43) gives $\omega = 1/2$. Furthermore, the $(1,1)$ component of (4.42) gives us $\dot{\hat{y}} = -\hat{y}\frac{\dot{y}}{y}$ which we can solve to get:

$$\hat{y} = \frac{\Omega}{y}, \tag{4.48}$$

with Ω a constant of integration. The (1,2) component of (4.42) yields

$$\dot{\hat{x}} = \eta t\left(x\frac{\dot{u}}{u} - \dot{x}\right) - \hat{x}\frac{\dot{u}}{u}, \tag{4.49}$$

and (2,1) component gives

$$\dot{\hat{v}} = \frac{1}{\eta t}\left(\dot{v} - v\frac{\dot{u}}{u}\right) - \hat{v}\frac{\dot{u}}{u}, \tag{4.50}$$

where $\eta = \tilde{d}/d$. With these the (2,2) component of (4.42) is satisfied. Next look at (4.43) for which the (2,1) component gives

$$\hat{x} = -\frac{\eta}{(\theta_\infty + 1)}\left[\theta_0 x + t(t-1)\left(\dot{x} - x\frac{\dot{u}}{u}\right)\right]. \tag{4.51}$$

We solve the (1,2) and the (2,2) components of (4.43) to give us $\dot{\hat{y}}$ and $\dot{\hat{v}}$ and compare these with the expressions given in (4.48)–(4.51). Imposing that $\dot{\hat{y}}$ be the same gives an expression for $\hat{x}$ which, after comparison with (4.51), yields

$$\hat{\theta}_0 = \theta_0 + 2 + 2\kappa. \tag{4.52}$$

Recall though that from $(4.46)_4$ we have $\hat{\theta}_0 = \theta_0 - 2\kappa$ which, together with (4.52), gives $\kappa = -1/2$ and therefore $\hat{\theta}_0 = \theta_0 + 1$. Finally, equating the expressions for $\dot{\hat{v}}$ and replacing $\kappa = -1/2$, $\hat{\theta}_0 = \theta_0 + 1$ and $\hat{x}$ of (4.51) we get

$$\hat{v} = -\frac{1}{\eta(\theta_0 + 1)}\left[\theta_\infty v - (t-1)\left(v\frac{\dot{u}}{u} - \dot{v}\right)\right]. \tag{4.53}$$

We can then check that $\hat{\theta}_1 = \theta_1$ and $\hat{\theta}_t = \theta_t$ and the transformation $(\hat{\theta}_0, \hat{\theta}_\infty, \hat{\theta}_1, \hat{\theta}_t) = (\theta_0 + 1, \theta_\infty + 1, \theta_1, \theta_t)$ given by the Darboux matrix

$$\mathbb{D}_{(\theta_0^+, \theta_\infty^\pm)} = \tilde{z}^{1/2} w^{-1/2} \begin{pmatrix} 0 & 0 \\ 0 & \tilde{d} \end{pmatrix} + \zeta w^{1/2} \tilde{z}^{-1/2} \hat{G}^{-1} \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} G, \tag{4.54}$$

transforms $G(t)$ as

$$G(t) \mapsto \hat{G}(t) = \begin{pmatrix} \hat{x}(t) & \hat{y}(t) \\ \frac{(\hat{x}(t)\hat{v}(t) - 1)}{\hat{y}(t)} & \hat{v}(t) \end{pmatrix}. \tag{4.55}$$

where $\hat{y}, \hat{x}$ and $\hat{v}$ are given by (4.48), (4.51) and (4.53) respectively. The above transformation is equivalent to the R1 Schlesinger transformation in [72]. Having shown in detail the computation for the above case let us now consider the other cases.

2. When $b = c = d = \tilde{b} = \tilde{c} = \tilde{d} = 0$, (4.44) $\Rightarrow \varepsilon = 0$ and we find that $\kappa = -1/2$, $\omega = 1/2$ and the transformation

$$\mathbb{D}_{(\theta_0^-,\theta_\infty^-)} = \tilde{z}^{1/2}w^{-1/2}\begin{pmatrix}\tilde{a} & 0\\ 0 & 0\end{pmatrix} + \zeta w^{1/2}\tilde{z}^{-1/2}\hat{G}^{-1}\begin{pmatrix}a & 0\\ 0 & 0\end{pmatrix}G, \tag{4.56}$$

acts on the parameters as $(\hat{\theta}_0, \hat{\theta}_\infty, \hat{\theta}_1, \hat{\theta}_t) = (\theta_0 - 1, \theta_\infty - 1, \theta_1, \theta_t)$ corresponding to the R2 transformation in [72]. The other possibilities, i.e. $\tilde{a} = 0$ or $a = 0$, again give a gauge transformation.

3. When $b = c = d = \tilde{a} = \tilde{b} = \tilde{c} = 0$, (4.44) $\Rightarrow \varepsilon = 0$ and we find that $\kappa = -1/2$, $\omega = 3/2$ and the transformation

$$\mathbb{D}_{(\theta_0^-,\theta_\infty^+)} = \tilde{z}^{1/2}w^{-1/2}\begin{pmatrix}0 & 0\\ 0 & \tilde{d}\end{pmatrix} + \zeta w^{1/2}\tilde{z}^{-1/2}\hat{G}^{-1}\begin{pmatrix}a & 0\\ 0 & 0\end{pmatrix}G, \tag{4.57}$$

acts on the parameters as $(\hat{\theta}_0, \hat{\theta}_\infty, \hat{\theta}_1, \hat{\theta}_t) = (\theta_0 - 1, \theta_\infty + 1, \theta_1, \theta_t)$ (R3 in [72]).

4. When $a = b = c = \tilde{b} = \tilde{c} = \tilde{d} = 0$, (4.44) $\Rightarrow \varepsilon = 0$ and we find that $\kappa = -1/2$, $\omega = 1/2$ and the transformation

$$\mathbb{D}_{(\theta_0^+,\theta_\infty^-)} = \tilde{z}^{1/2}w^{-1/2}\begin{pmatrix}\tilde{a} & 0\\ 0 & 0\end{pmatrix} + \zeta w^{1/2}\tilde{z}^{-1/2}\hat{G}^{-1}\begin{pmatrix}0 & 0\\ 0 & d\end{pmatrix}G, \tag{4.58}$$

acts on the parameters as $(\hat{\theta}_0, \hat{\theta}_\infty, \hat{\theta}_1, \hat{\theta}_t) = (\theta_0 + 1, \theta_\infty - 1, \theta_1, \theta_t)$ (R4 in [72]).

5. Let $\kappa = 0 \Rightarrow b = c = 0$ and take $\tilde{a} = \tilde{b} = \tilde{c} = 0$. In this case we get the following options, $\varepsilon = 0$ or $1/2$ and $a = 0$ or $d = 0$. Taking $\varepsilon = 0 \Rightarrow \omega = 1/2$

and chosing $a=0$ we get the transformation

$$\mathbb{D}_{(\theta_\infty^\pm,\theta_1^+)} = \tilde{z} w^{-\frac{1}{2}} \begin{pmatrix} 0 & 0 \\ 0 & \tilde{d} \end{pmatrix} + \zeta w^{\frac{1}{2}} \hat{G}^{-1} \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} G, \tag{4.59}$$

which acts on the parameters as $(\hat{\theta}_0, \hat{\theta}_\infty, \hat{\theta}_1, \hat{\theta}_t) = (\theta_0, \theta_\infty + 1, \theta_1 + 1, \theta_t)$ (R5 in [72]).

6. Let $\kappa = 0 \Rightarrow b = c = 0$ and take $\tilde{a} = \tilde{b} = \tilde{c} = 0$. Again, take $\varepsilon = 0 \Rightarrow \omega = 1/2$ but now letting $d = 0$ we get the transformation

$$\mathbb{D}_{(\theta_\infty^\pm,\theta_1^-)} = \tilde{z} w^{-\frac{1}{2}} \begin{pmatrix} 0 & 0 \\ 0 & \tilde{d} \end{pmatrix} + \zeta w^{\frac{1}{2}} \hat{G}^{-1} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} G, \tag{4.60}$$

which acts on the parameters as $(\hat{\theta}_0, \hat{\theta}_\infty, \hat{\theta}_1, \hat{\theta}_t) = (\theta_0, \theta_\infty + 1, \theta_1 - 1, \theta_t)$ (R7 [72]).

7. Let $\kappa = 0 \Rightarrow b = c = 0$ and take $\tilde{a} = \tilde{b} = \tilde{c} = 0$. Letting $\varepsilon = 1/2$ and $d = 0$ we get that $\omega = 0$ and the transformation

$$\mathbb{D}_{(\theta_\infty^\pm,\theta_t^+)} = \tilde{z}^{\frac{1}{2}} w^{-\frac{1}{2}} \tilde{w}^{\frac{1}{2}} \begin{pmatrix} 0 & 0 \\ 0 & \tilde{d} \end{pmatrix} + \zeta z^{\frac{1}{2}} \hat{G}^{-1} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} G, \tag{4.61}$$

acts on the parameters as $(\hat{\theta}_0, \hat{\theta}_\infty, \hat{\theta}_1, \hat{\theta}_t) = (\theta_0, \theta_\infty + 1, \theta_1, \theta_t + 1)$ (R9 in [72]).

8. Let $\kappa = 0 \Rightarrow b = c = 0$ and take $\tilde{a} = \tilde{b} = \tilde{c} = 0$. Letting $\varepsilon = 1/2$ and $d = 0$ we get that $\omega = 0$ and the transformation

$$\mathbb{D}_{(\theta_\infty^\pm,\theta_t^-)} = \tilde{z}^{\frac{1}{2}} w^{-\frac{1}{2}} \tilde{w}^{\frac{1}{2}} \begin{pmatrix} 0 & 0 \\ 0 & \tilde{d} \end{pmatrix} + \zeta z^{\frac{1}{2}} \hat{G}^{-1} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} G, \tag{4.62}$$

acts on the parameters as $(\hat{\theta}_0, \hat{\theta}_\infty, \hat{\theta}_1, \hat{\theta}_t) = (\theta_0, \theta_\infty + 1, \theta_1, \theta_t - 1)$ (R11 in [72]).

Note that in 4–7 the choice $\tilde{b} = \tilde{c} = \tilde{d} = 0$ would require that $\tilde{a} = 0$ and therefore simply be gauge.

9. Let $\kappa = 0 \Rightarrow b = c = 0$ and take $\tilde{b} = \tilde{c} = \tilde{d} = 0$. Again we get that either $\varepsilon = 0$ or $1/2$ and $a = 0$ or $d = 0$. Taking $\varepsilon = 0$ and $d = 0$ gives us that $\omega = 1/2$ and the transformation

$$\mathbb{D}_{(\theta_\infty^-, \theta_1^-)} = \tilde{z} w^{-\frac{1}{2}} \begin{pmatrix} \tilde{a} & 0 \\ 0 & 0 \end{pmatrix} + \zeta w^{\frac{1}{2}} \hat{G}^{-1} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} G, \tag{4.63}$$

acts on the parameters as $(\hat{\theta}_0, \hat{\theta}_\infty, \hat{\theta}_1, \hat{\theta}_t) = (\theta_0, \theta_\infty - 1, \theta_1 - 1, \theta_t)$ (R6 in [72]).

10. Let $\kappa = 0 \Rightarrow b = c = 0$ and take $\tilde{b} = \tilde{c} = \tilde{d} = 0$. Again take $\varepsilon = 0 \Rightarrow \omega = 1/2$ but now let $a = 0$. This results in the transformation

$$\mathbb{D}_{(\theta_\infty^-, \theta_1^+)} = \tilde{z} w^{-\frac{1}{2}} \begin{pmatrix} \tilde{a} & 0 \\ 0 & 0 \end{pmatrix} + \zeta w^{\frac{1}{2}} \hat{G}^{-1} \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} G, \tag{4.64}$$

which acts on the parameters as $(\hat{\theta}_0, \hat{\theta}_\infty, \hat{\theta}_1, \hat{\theta}_t) = (\theta_0, \theta_\infty - 1, \theta_1 + 1, \theta_t)$ (R8 in [72]).

11. Let $\kappa = 0 \Rightarrow b = c = 0$ and take $\tilde{b} = \tilde{c} = \tilde{d} = 0$. This time take $\varepsilon = 1/2 \Rightarrow \omega = 0$ and let $a = 0$. This results in the transformation

$$\mathbb{D}_{(\theta_\infty^-, \theta_t^-)} = \tilde{w}^{\frac{1}{2}} \tilde{z}^{\frac{1}{2}} w^{-\frac{1}{2}} \begin{pmatrix} \tilde{a} & 0 \\ 0 & 0 \end{pmatrix} + \zeta z^{\frac{1}{2}} \hat{G}^{-1} \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} G, \tag{4.65}$$

which acts on the parameters as $(\hat{\theta}_0, \hat{\theta}_\infty, \hat{\theta}_1, \hat{\theta}_t) = (\theta_0, \theta_\infty - 1, \theta_1, \theta_t - 1)$ (R10 in [72]).

12. Let $\kappa = 0 \Rightarrow b = c = 0$ and take $\tilde{b} = \tilde{c} = \tilde{d} = 0$. Again take $\varepsilon = 1/2 \Rightarrow \omega = 0$ but now let $d = 0$. This results in the transformation

$$\mathbb{D}_{(\theta_\infty^-, \theta_t^+)} = \tilde{w}^{\frac{1}{2}} \tilde{z}^{\frac{1}{2}} w^{-\frac{1}{2}} \begin{pmatrix} \tilde{a} & 0 \\ 0 & 0 \end{pmatrix} + \zeta z^{\frac{1}{2}} \hat{G}^{-1} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} G, \tag{4.66}$$

which acts on the parameters as $(\hat{\theta}_0, \hat{\theta}_\infty, \hat{\theta}_1, \hat{\theta}_t) = (\theta_0, \theta_\infty - 1, \theta_1, \theta_t + 1)$ (R12 in [72]).

Again note that similar to the cases 4–7, in 8–12 the choice $\tilde{a} = \tilde{b} = \tilde{c} = 0$ would require that $\tilde{d} = 0$ and therefore simply be gauge.

4.3 Automorphisms of the Riemann sphere and Schlesinger transformations

The above Schlesinger transformations act on the system by shifting the monodromy exponents at the different singular points by integer values. In this section we present a construction from which all Schlesinger transformations can be recovered by using Möbius transformations to permute the singular points and composing this with, say, the $\mathbb{D}_{(\theta_0^\pm, \theta_\infty^\pm)}$ Schlesinger transformations, which are those shifting the exponents corresponding to the Fuchsian singularities at 0 and ∞. That is, the aim is to obtain all Schlesinger transformations via the composition of Möbius transformations whose action permutes singular points and the Schlesinger transformation associated to the action shifting the monodromy exponent at specific, fixed, singular points. An example is given by the diagram below, figure 4.2, where the schematic shows how to obtain the Schlesinger transformation shifting the monodromy exponents at 1 and ∞ via composition of Möbius transformations and the Schlesinger transformation acting on the monodromy exponent associated with the singularities at 0 and ∞. This would allow one to recover all Schlesinger transformations of P_{VI} from those shifting the monodromy exponents at, say, $(\theta_0, \theta_\infty)$.

$$
\begin{tikzcd}
(\theta_0,\ \theta_1,\ \theta_t,\ \theta_\infty) \arrow[r, "\mathbb{D}_{(\theta_1^+,\ \theta_\infty^\pm)}"] \arrow[d, "\mathbb{M}"'] & (\theta_0,\ \theta_1+1,\ \theta_t,\ \theta_\infty+1) \\
(\hat{\theta}_0,\ \hat{\theta}_1,\ \hat{\theta}_t,\ \hat{\theta}_\infty) \arrow[r, "\mathbb{D}_{(\hat{\theta}_0^+,\ \hat{\theta}_\infty^\pm)}"'] & (\hat{\theta}_0+1,\ \hat{\theta}_1,\ \hat{\theta}_t,\ \hat{\theta}_\infty+1) \arrow[u, "\mathbb{M}^{-1}"]
\end{tikzcd}
$$

Figure 4.2: Schematic of the composition $\mathbb{D}_{(\theta_1^+,\ \theta_\infty^\pm)} = \mathbb{M}^{-1} \circ \mathbb{D}_{(\hat{\theta}_0^+,\ \hat{\theta}_\infty^\pm)} \circ \mathbb{M}$, with $(\hat{\theta}_0,\ \hat{\theta}_1,\ \hat{\theta}_t,\ \hat{\theta}_\infty)$=$(\theta_1,\ \theta_0,\ \theta_t,\ \theta_\infty)$.

More generally we can represent the process, for arbitrary transformations, as the composition

$$\mathbb{D}_{(\theta_i^{\pm},\ \theta_j^{\pm})} = \mathbb{M}^{-1}_{(i\leftrightarrow 0,\ j\leftrightarrow\infty)} \circ \mathbb{D}_{(\hat{\theta}_i^{\pm}, \hat{\theta}_j^{\pm})} \circ \mathbb{M}_{(i\leftrightarrow 0,\ j\leftrightarrow\infty)}. \tag{4.67}$$

We start by describing the relevant Möbius transformations that we shall need, listing them below with the resulting action on four points $\{0,\ 1,\ t,\ \infty\} \in \mathbb{CP}^1$

1. $\zeta \mapsto \hat{\zeta} = 1 - \zeta$: $\quad (0,\ 1,\ t,\ \infty) \mapsto (1,\ 0,\ \hat{t} = 1 - t, \infty)$,
2. $\zeta \mapsto \hat{\zeta} = \frac{\zeta}{\zeta - 1}$: $\quad (0,\ 1,\ t,\ \infty) \mapsto (0,\ \infty,\ \hat{t} = \frac{t}{t-1}, 1)$,
3. $\zeta \mapsto \hat{\zeta} = \frac{t-\zeta}{t-1}$: $\quad (0,\ 1,\ t,\ \infty) \mapsto (\hat{t} = \frac{t}{t-1},\ 1,\ 0, \infty)$.

The induced action on the Higgs fields is obtained by implementing the transformation on the spectral part of the $\mathrm{P_{VI}}$ iso-monodromy problem

$$\Phi_\zeta = \left[\frac{Q}{\zeta} - \frac{(P+Q+R)}{\zeta - 1} + \frac{R}{\zeta - t}\right]\Phi, \tag{4.68}$$

re-expressing it in the same functional form and finally equating the Higgs fields. We perform this for each of the above transformations.

1. (4.68) becomes

$$\Phi_{\hat{\zeta}} = \left[-\frac{(P+Q+R)}{\hat{\zeta}} - \frac{-Q}{\hat{\zeta} - 1} + \frac{R}{\hat{\zeta} - \hat{t}}\right]\Phi, \tag{4.69}$$

   and therefore $\hat{P} = P$, $\hat{Q} = -(P+Q+R)$, $\hat{R} = R$.

2. (4.68) becomes

$$\Phi_{\hat{\zeta}} = \left[\frac{Q}{\hat{\zeta}} - \frac{-P}{\hat{\zeta} - 1} + \frac{R}{\hat{\zeta} - \hat{t}}\right]\Phi, \tag{4.70}$$

   and therefore $\hat{P} = -(P+Q+R)$, $\hat{Q} = Q$, $\hat{R} = R$.

3. (4.68) becomes

$$\Phi_{\hat{\zeta}} = \left[\frac{R}{\hat{\zeta}} - \frac{(P+Q+R)}{\hat{\zeta} - 1} + \frac{Q}{\hat{\zeta} - \hat{t}}\right]\Phi, \tag{4.71}$$

and therefore $\hat{P} = P,\ \hat{Q} = R,\ \hat{R} = Q$.

Knowledge of how the Higgs fields transform makes it possible to determine the effect of each of the above transformations on the first integrals and, therefore, on the parameters. For instance, the effect of the first transformations is to permute the monodromy exponents as

$$\begin{aligned} \theta_0^2 \mapsto \hat{\theta}_0^2 = 2\,\mathrm{Tr}(\hat{Q}^2) = 2\,\mathrm{Tr}((P+Q+R)^2) &\implies \hat{\theta}_0^2 = \theta_1^2, \\ \theta_1^2 \mapsto \hat{\theta}_1^2 = 2\,\mathrm{Tr}((\hat{P}+\hat{Q}+\hat{R})^2) = 2\,\mathrm{Tr}(Q^2) &\implies \hat{\theta}_1^2 = \theta_0^2, \\ \theta_t^2 \mapsto \hat{\theta}_t^2 = -2\,\mathrm{Tr}(\hat{R}^2) = -2\,\mathrm{Tr}(R^2) &\implies \hat{\theta}_t^2 = \theta_t^2, \\ \theta_\infty^2 \mapsto \hat{\theta}_\infty = 2\,\mathrm{Tr}(\hat{P}^2) = 2\,\mathrm{Tr}(P^2) &\implies \hat{\theta}_\infty^2 = \theta_\infty^2, \end{aligned} \tag{4.72}$$

which is precisely the transformation we were looking for, permuting the singular values $t = 0$ and $t = 1$.

For the implementation of the process outlined in figure 4.2, or, more generally, in (4.67) we must solve for the transformed G in each case.

1. $\hat{Q} = -(P+Q+R)$ and $\hat{Q} = \hat{G}^{-1}\hat{Q}_0\hat{G}$. Moreover $\hat{Q}_0$ diagonal means $\hat{G}$ is the modal matrix diagonalising the sum $-(P+Q+R)$, that is

$$\hat{Q}_0 = \hat{G}\,[-(P+Q+R)]\,\hat{G}^{-1}. \tag{4.73}$$

We know that $\hat{Q}_0 = \mathrm{diag}(\frac{\hat{\theta}_0}{2}, -\frac{\hat{\theta}_0}{2}) = \mathrm{diag}(\frac{\theta_1}{2}, -\frac{\theta_1}{2})$, thus letting

$$S = -(P+Q+R) = \begin{pmatrix} A & B \\ C & -A \end{pmatrix},$$

we can find the eigenvectors associates with the eigenvalues $\frac{\theta_1}{2}$ and $-\frac{\theta_1}{2}$ which are

$$v_{\frac{\theta_1}{2}} = \begin{pmatrix} B \\ \frac{\theta_1}{2} - A \end{pmatrix}, \tag{4.74}$$

and

$$v_{-\frac{\theta_1}{2}} = \begin{pmatrix} -B \\ \frac{\theta_1}{2} + A \end{pmatrix}, \tag{4.75}$$

from which we may then recover $\hat{G}$

$$\hat{G} = \frac{1}{\sqrt{\theta_1 B}} \begin{pmatrix} B & -B \\ \frac{\theta_1}{2} - A & \frac{\theta_1}{2} + A \end{pmatrix}. \tag{4.76}$$

Note that we have normalised the eigenvectors as we require the condition $|\hat{G}| = 1$.

2. In this case we know that $\hat{Q} = Q$ and therefore we obtain $\hat{G}$ by solving the equation $\hat{Q}_0\hat{G} - \hat{G}\left(G^{-1}Q_0G\right) = 0$. If we parametrise G as

$$G = \begin{pmatrix} x & y \\ \frac{xv-1}{y} & v \end{pmatrix},$$

and similar for $\hat{G}$ with the entries replaced by hatted entries, we find that

$$\hat{G} = \begin{pmatrix} \frac{x\vartheta}{y} & \vartheta \\ \frac{xv-1}{\vartheta} & \frac{yv}{\vartheta} \end{pmatrix}, \tag{4.77}$$

where $\hat{y} = \vartheta$ is a free parameter reflecting the residual gauge freedom in the parametrisation.

3. The final case is analogous to the first one. The algebraic relation to be solved can be retrieved from the expression for the transformation of the Q field, or $\hat{Q} = R$ implying that $\hat{Q}_0\hat{G} - \hat{t}(\hat{t}-1)\hat{G}G^{-1}\dot{G} = 0$. Once again since $\hat{Q}_0$ is diagonal this means that $\hat{G}$ diagonalises the matrix $R = -tG^{-1}\dot{G}$. Thus

$$\hat{Q}_0 = \hat{G}\left[-tG^{-1}\dot{G}\right]\hat{G}^{-1}. \tag{4.78}$$

We know that $\hat{Q}_0 = \mathrm{diag}(\frac{\hat{\theta}_0}{2}, -\frac{\hat{\theta}_0}{2}) = \mathrm{diag}(\frac{\theta_t}{2}, -\frac{\theta_t}{2})$, thus letting

$$S = -\left(tG^{-1}\dot{G}\right) = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}$$

we can find the eigenvectors associates with the eigenvalues $\frac{\theta_t}{2}$ and $-\frac{\theta_t}{2}$ which are

$$v_{\frac{\theta_t}{2}} = \begin{pmatrix} B \\ \frac{\theta_t}{2} - A \end{pmatrix}, \tag{4.79}$$

and

$$v_{-\frac{\theta_t}{2}} = \begin{pmatrix} -B \\ \frac{\theta_t}{2} + A \end{pmatrix}, \tag{4.80}$$

from which we may then recover $\hat{G}$

$$\hat{G} = \frac{1}{\sqrt{\theta_t B}} \begin{pmatrix} B & -B \\ \frac{\theta_t}{2} - A & \frac{\theta_t}{2} + A \end{pmatrix}. \tag{4.81}$$

Note that once again we have normalised the eigenvectors such that $|\hat{G}| = 1$ is satisfied.

Composing the relevant transformations as in (4.67) it is then possible to obtain the various Schlesinger transformations from, for instance, $\mathbb{D}_{(\hat{\theta}_0^{\pm},\, \hat{\theta}_\infty^{\pm})}$ and the Möbius transformations presented above.

Chapter 5

On confluence of Painlevé group elements

The Painlevé equations correspond to reductions of the ASDYM equations when the Lie algebra is $\mathfrak{sl}(2,\mathbb{C})$ and the conformal Killing vectors of these reductions are associated to the four-parameter subgroups of $\mathrm{GL}(4,\mathbb{C})$ called the 'Painlevé groups' written down in 3.2.3 which generate a three dimensional group of conformal symmetries. Importantly, these groups arise as centralisers of regular elements (roughly these are matrices expressible as direct sums of Jordan blocks with distinct eigenvalues, see 5.2) of the Lie algebra of the complex general linear group (representing conformal transformations) and each regular element can itself be associated to a specific partition of 4. This partition of 4 can be interpreted as the singular structure of the isomonodromy problem thus yielding a correspondence between centralisers of regular elements (which are the Painlevé groups) and the six Painlevé equations. For example $\mathrm{P_{VI}}$ is associated to the partition $(1,1,1,1)$ while $\mathrm{P_{IV}}$ is associated to the partition $(3,1)$, see figure 3.2. The Painlevé equations, just like the classical special functions, are related through a coalescence cascade, figure 5.1, a type of limit process which allows us to obtain one DE from another starting from the Painlevé VI and linking all equations through to the Painlevé I. For example if in $\mathrm{P_{II}}$, (3.59), we make the transformation

$$
\begin{aligned}
u(z;\alpha) &= \varepsilon U(\xi) + \frac{1}{\varepsilon^5}, \\
z &= \varepsilon^2 \xi - \frac{6}{\varepsilon^{10}}, \\
\alpha &= \frac{4}{\varepsilon^{15}},
\end{aligned}
$$

then

$$
\frac{d^2U}{d\xi^2} = 6U^2 + \xi + \varepsilon^6\left(2U^3 + \xi U\right).
$$

Note that the coefficients depend holomorphically on ε at $\varepsilon = 0$ and in the limit as $\varepsilon \to 0$, $U(\xi)$ satisfies $\mathrm{P_I}$, (3.58), with $z = \xi$.

$$
\begin{tikzcd}
P_{\mathrm{VI}} \arrow[r] & P_{\mathrm{V}} \arrow[r] \arrow[d] & P_{\mathrm{IV}} \arrow[d] & \\
 & P_{\mathrm{III}} \arrow[r] & P_{\mathrm{II}} \arrow[r] & P_{\mathrm{I}}
\end{tikzcd}
$$

Figure 5.1: $\mathrm{P_V}$-$\mathrm{P_I}$ are obtained from $\mathrm{P_{VI}}$ by a process of confluence of singularities. This schematic shows the coalescence cascade relating them.

Given that each element of the Painlevé group determines a DE, we find that the process of confluence can be understood as arising from the geometry of the set of regular elements of the Lie algebra of $\mathrm{GL}(4,\mathbb{C})$. Again, the reductions arise due to invariance under a group generated by elements of the Painlevé groups which themselves arise as centralisers of regular elements. We can therefore interpret the equations as systems associated with the centralisers of regular elements of $\mathrm{GL}(4,\mathbb{C})$, i.e. each equation is determined by the centraliser of a regular element of $\mathrm{GL}(4,\mathbb{C})$. In this chapter we recover the relevant singular transformation on both the independent variables and the Higgs fields giving the confluences and from this we are able to lift the confluence process to the Schlesinger transformations. Stated differently, we explicitly construct the limit process by which all Painlevé equations for the partitions $\lambda \neq (1,1,1,1)$ can be obtained from the partition $\lambda = (1,1,1,1)$ ($\mathrm{P_{VI}}$); this lifts to provide confluence on the level of the Schlesinger transformations. We therefore are able to recover all Schlesinger transformations (STs) for all the Painlevé equations from those for $\mathrm{P_{VI}}$, given that the $\mathrm{P_{VI}}$ STs belong to the most general DE in the coalescence hierarchy. To do this we use a map in the space of regular elements and the realisation of this map to map between centralisers of these regular elements. This yields a process of confluence among elements of the Painlevé group.

We start by presenting the process for the confluence $\mathrm{P_{VI}} \to \mathrm{P_V}$, [2]. For this we do not require any particular machinery and therefore we can present the process without the need to introduce any further theory - this will also furnish a good example of what we are trying to achieve. However, for the other confluences we require some further machinery to implement these maps and we shall therefore introduce such theory before presenting the confluence $\mathrm{P_V} \to \mathrm{P_{III}}$, [2]. Details of the remaining limits are given the appendix, where we construct the relevant maps and obtain the transformations of the independent variables but the full details are not made explicit.

5.1 Painlevé VI degeneration to Painlevé V

To study the confluence process let us start with the element of the four-dimensional Abelian Lie subalgebra giving the reduction to $\mathrm{P_{VI}}$, i.e. the generator of the symmetry given by

$$\mathfrak{P}_{\mathrm{VI}} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}. \tag{5.1}$$

The four distinct eigenvalues correspond to the four Fuchsian singular points $(0,\ 1,\ t,\ \infty)$ of the isomonodromy problem. Given this one constructs the relevant element of the conformal group by the usual exponentiation process, thus the reduction from the ASDYM to $\mathrm{P_{VI}}$ is given by the transformation (3.11)

$$x \mapsto y = g_{\mathrm{VI}} x g_{\mathrm{VI}}^T, \tag{5.2}$$

where $g_{\mathrm{VI}} = \exp(\tau\mathfrak{P}_{\mathrm{VI}}) \in \mathrm{GL}(4;\mathbb{C})$, $\tau \in \mathbb{R}$. The reduction to the $\mathrm{P_V}$ equation arises as a reduction via the symmetry generated by the element

$$\mathfrak{P}_{\mathrm{V}} = \begin{pmatrix} a & b & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}, \tag{5.3}$$

and the idea is to recover $\mathfrak{P}_{\mathrm{V}}$ as some singular limit of a function of $\mathfrak{P}_{\mathrm{VI}}$. For this perturb $\mathfrak{P}_{\mathrm{V}}$ to obtain a diagonalisable matrix function holomorphic in the perturbation parameter ε

$$A(\varepsilon) = \mathfrak{P}_V + \varepsilon A_1 + \cdots$$

and then diagonalise via the modal matrix X constructed from its eigenvectors, that is compute

$$D(\varepsilon) = X(\varepsilon)^{-1}A(\varepsilon)X(\varepsilon).$$

It is clear that the diagonal matrix is nothing other than $\mathfrak{P}_{\mathrm{VI}}$, thus $D = \mathfrak{P}_{\mathrm{VI}}$. Of course the modal matrix, X, will be unbounded at $\varepsilon = 0$ since for $\varepsilon = 0$ two eigenvectors degenerate and therefore do not span the space and in the singular limit $\varepsilon \to 0$, $\mathfrak{P}_{\mathrm{VI}} \to \mathfrak{P}_{\mathrm{V}}$. Writing the element of the conformal group, g_{VI}, giving the reduction to $\mathrm{P_{VI}}$ as

$$g_{\mathrm{VI}} = \exp(\tau\mathfrak{P}_{\mathrm{VI}}) = \exp(\tau X^{-1}AX) = X^{-1}\exp(\tau A)X$$

the transformed metric may be determined. This is

$$\begin{aligned} x \mapsto y = (X^{-1}\exp(\tau A)X)x(X^T\exp(\tau A^T)(X^T)^{-1}) \\ \implies \qquad XyX^T = \exp(\tau A)[XxX^T]\exp(\tau A^T) \\ \implies \qquad \hat{y} = \exp(\tau A)\hat{x}\exp(\tau A^T), \end{aligned}$$

where $\hat{y} = XyX^T$ and $\hat{x} = XxX^T$. In the limit $\varepsilon \to 0$, the metric, given by $\hat{x}$, gives the metric expressed through the coordinates for the $\mathrm{P_V}$ reduction and therefore, by simply reading off the elements of this matrix, we have the transformation relating the coordinates for the $\mathrm{P_{VI}}$ reduction to those for $\mathrm{P_V}$.

Specifically we choose to perturb $\mathfrak{P}_{\mathrm{V}}$ as (chosen as to simplify the computations which follow)

$$A(\varepsilon)=\begin{pmatrix} b & \beta-\alpha & 0 & 0\\ 0 & b & 0 & 0\\ 0 & 0 & c & 0\\ 0 & 0 & 0 & d\end{pmatrix}+\varepsilon\begin{pmatrix} \beta & 0 & 0 & 0\\ 0 & \alpha & 0 & 0\\ 0 & 0 & c & 0\\ 0 & 0 & 0 & d\end{pmatrix} \tag{5.4}$$

from which $\hat{x}(\varepsilon)=X(\varepsilon)^{-1}x(X(\varepsilon)^T)^{-1}$ gives

$$\begin{pmatrix} 0 & \hat{S} & -W & \tilde{Z}\\ -\hat{S} & 0 & -Z & \tilde{W}\\ W & Z & 0 & 1\\ -\tilde{Z} & -\tilde{W} & -1 & 0\end{pmatrix}=X^{-1}\begin{pmatrix} 0 & s & -w & \tilde{z}\\ -s & 0 & -z & \tilde{w}\\ w & z & 0 & 1\\ -\tilde{z} & -\tilde{w} & -1 & 0\end{pmatrix}(X^T)^{-1} \tag{5.5}$$

where X is the matrix whose columns are the eigenvectors of $A(\varepsilon)$, $(Z,W,\tilde{Z},\tilde{W})$ are functions of ε giving the transformed coordinates on $\mathbb{C}^4$, $\hat{S}=Z\tilde{Z}-W\tilde{W}$ and the determinant of X has a pole at $\varepsilon=0$. Expanding in powers of ε we find that

$$Z=z,\quad \tilde{Z}=\frac{\tilde{w}}{\varepsilon}+\tilde{z},\quad W=\frac{z}{\varepsilon}+w,\quad \tilde{W}=\tilde{w}. \tag{5.6}$$

Notice that, as expected, in the $\varepsilon\to 0$ limit the above transformation is singular and this is precisely the coordinate transformation which gives us the confluence process $P_{\mathrm{VI}}\to P_{\mathrm{V}}$. To see this we look at how the conformal Killing vectors transform under the above change of variables. Recall that the Killing vectors associated with the reduction to the $\mathrm{P_{VI}}$ equation in the $(Z,W,\tilde{Z},\tilde{W})$ coordinates are

$$\begin{aligned}\mathbb{X}_{\mathrm{VI}}&=-Z\partial_Z-W\partial_W,\\ \mathbb{Y}_{\mathrm{VI}}&=-\tilde{Z}\partial_{\tilde{Z}}-\tilde{W}\partial_{\tilde{W}},\\ \mathbb{Z}_{\mathrm{VI}}&=Z\partial_Z+\tilde{W}\partial_{\tilde{W}},\end{aligned} \tag{5.7}$$

which in terms of $(z, w, \tilde{z}, \tilde{w}, \varepsilon)$ read

$$\begin{aligned}
\mathbb{X}_{\mathrm{VI}} &= -z\partial_z - w\partial_w = -(\mathbb{Y}_{\mathrm{V}} + \mathbb{Z}_{\mathrm{V}}), \\
\mathbb{Y}_{\mathrm{VI}} &= -\tilde{z}\partial_{\tilde{z}} - \tilde{w}\partial_{\tilde{w}} = \mathbb{Z}_{\mathrm{V}}, \\
\mathbb{Z}_{\mathrm{VI}} &= \frac{1}{\varepsilon}(-z\partial_w - \tilde{w}\partial_{\tilde{z}}) + (z\partial_z + \tilde{w}\partial_{\tilde{w}}) = -\frac{1}{\varepsilon}\mathbb{X}_{\mathrm{V}} + (z\partial_z + \tilde{w}\partial_{\tilde{w}}),
\end{aligned} \tag{5.8}$$

showing that in the $\varepsilon \to 0$ limit the Killing vectors degenerate to those generating the three dimensional abelian subgroup giving the Painlevé V equation.

In fact we know that under the action of g_{VI} the combination $t = \frac{Z\tilde{Z}}{W\tilde{W}}$ is invariant (giving the independent variable for the Painlevé VI equation) and expressing this in terms of $z, w, \tilde{z}, \tilde{w}, \varepsilon$ coordinates and expanding in powers of ε we find that

$$t = \frac{Z\tilde{Z}}{W\tilde{W}} = 1 + \varepsilon\left(\frac{\tilde{z}}{\tilde{w}} - \frac{w}{z}\right) + O(\varepsilon^2) = 1 + \varepsilon\hat{t} + O(\varepsilon^2), \tag{5.9}$$

which is, to first order in ε, the change of variables giving the degeneration to $\mathrm{P_V}$, [29].

To relate the two systems at the field equation level we must determine how the local coordinates introduced for the reductions are related. For this purpose recall that for the Painlevé VI equation we choose the local coordinates p, q, r, t such that

$$\mathbb{X}_{\mathrm{VI}} = \partial_p, \quad \mathbb{Y}_{\mathrm{VI}} = \partial_q, \quad \mathbb{Z}_{\mathrm{VI}} = \partial_r.$$

giving

$$\begin{aligned}
p &= -\log W = -\log(w + z/\varepsilon), \\
q &= -\log\tilde{Z} = -\log(\tilde{z} + \tilde{w}/\varepsilon), \\
r &= \log(\tilde{W}/\tilde{Z}) = \log\left(\frac{\tilde{w}}{\tilde{z} + \tilde{w}/\varepsilon}\right).
\end{aligned} \tag{5.10}$$

Making a gauge transformation we then bring the general invariant potential to the form

$$A = Pdp + Qdq + Rdr$$

and the ASDYM equations, (3.5), reduce to the system of ODEs for P, Q and R

given by

$$
\begin{aligned}
t(1-t)\frac{dR}{dt} &= [tP+Q,R],\\
t\frac{dQ}{dt} &= [R,Q],\\
\frac{dP}{dt} &= 0.
\end{aligned} \tag{5.11}
$$

Similarly, let us determine the local coordinates $\hat{p},\ \hat{q},\ \hat{r}$ for the reduction to $\mathrm{P_V}$ such that

$$
-(\mathbb{Y}_{\mathrm{V}}+\mathbb{Z}_{\mathrm{V}}) = \partial_{\hat{p}}, \quad \mathbb{Z}_{\mathrm{V}} = \partial_{\hat{q}}, \quad -\mathbb{X}_{\mathrm{V}} = \partial_{\hat{r}},
$$

giving

$$
\begin{aligned}
\hat{p} &= -\log z,\\
\hat{q} &= -\log \tilde{w},\\
\hat{r} &= -\tilde{z}/\tilde{w}.
\end{aligned} \tag{5.12}
$$

Again, following a prescription similar to that for the $\mathrm{P_{VI}}$ case we gauge to bring the potential to the form

$$
A = \hat{P}d\hat{p} + \hat{Q}d\hat{q} + \hat{R}d\hat{r}
$$

and find the field equations for $\mathrm{P_V}$

$$
\begin{aligned}
\hat{t}\,\frac{d\hat{R}}{d\hat{t}} &= [\hat{R},\hat{P}+\hat{Q}],\\
\frac{d\hat{Q}}{d\hat{t}} &= [\hat{R},\hat{Q}],\\
\frac{d\hat{P}}{d\hat{t}} &= 0.
\end{aligned} \tag{5.13}
$$

Before proceeding further let us return to the issue of obtaining the reduction in J form mentioned during the discussion of reductions in sections 3.2 and 3.2.1.2. The above field equations, (5.13), corresponding to the $\mathrm{P_V}$ are not the ones presented in [16] since there the (independent) symmetry generators chosen were different.

Their choice yields the field equations

$$\begin{aligned} \hat{t}\,\frac{d\hat{R}}{d\hat{t}} &= [\hat{R},\hat{t}\hat{P}+\hat{Q}], \\ \frac{d\hat{Q}}{d\hat{t}} &= [\hat{P},\hat{R}], \\ \frac{d\hat{P}}{d\hat{t}} &= 0, \end{aligned} \tag{5.14}$$

which, given their form and the appearance of the $\hat{t}$ term in the commutator for the evolution of the $\hat{R}$ Higgs field, it is not obvious how one can solve it. However, the construction of the above confluence process furnishes a different choice of (independent) generators which yield a more tractable structure for the field equations. We thus see that constructing the reduction in the J-matrix form is not always straightforward and, moreover, that the degeneration of the $\mathrm{P_{VI}}$ generators to those for $\mathrm{P_V}$ in our construction present us with a more tractable set of equations.

We are now in a position to relate the Higgs fields of the two systems and, therefore the respective field equations. Once this has been done it will be straightforward to recover the necessary transformation of the parameters appearing in the equations and ultimately the correct re-parametrisation of the Schlesinger transformations for $\mathrm{P_{VI}}$ giving, under the limit $\varepsilon \to 0$, the Schlesinger transformations for the $\mathrm{P_V}$ equation. From (5.10) and (5.12) we get that

$$\begin{aligned} p &= \log(\varepsilon)+\hat{p}+\varepsilon(\hat{t}+\hat{r})+O(\varepsilon^2), \\ q &= \log(\varepsilon)+\hat{q}+\varepsilon\hat{r}+O(\varepsilon^2), \\ r &= \log(\varepsilon)+\varepsilon\hat{r}+O(\varepsilon^2), \end{aligned} \tag{5.15}$$

together with $t = 1+\varepsilon\hat{t}+O(\varepsilon^2)$. We can then rewrite the potential for the $\mathrm{P_{VI}}$ reduction in terms of the Higgs fields associated with the reduction to $\mathrm{P_V}$

$$\begin{aligned} Pdp+Qdq+Rdr &= \hat{P}d\hat{p}+\hat{Q}d\hat{q}+\hat{R}d\hat{r}+\hat{T}d\hat{t}, \\ \implies P(d\hat{p}+\varepsilon d\hat{t}+\varepsilon d\hat{r})+Q(d\hat{q}+\varepsilon d\hat{r})+\hat{R}(\varepsilon d\hat{r}) &= \hat{P}d\hat{p}+\hat{Q}d\hat{q}+\hat{R}d\hat{r}+\hat{T}d\hat{t}, \end{aligned}$$

finally giving

$$\begin{aligned}\hat{P} &= P,\\ \hat{Q} &= Q,\\ \hat{R} &= \varepsilon(P+Q+R),\\ \hat{T} &= \varepsilon P.\end{aligned} \tag{5.16}$$

Note that $\hat{T}$ is related to P to $O(\varepsilon)$ and therefore we do not require a second gauge when relating the Higgs of one to the other. This is not always the case, indeed in the confluence $\mathfrak{P}_{\mathrm{V}} \to \mathfrak{P}_{\mathrm{IV}}$ the relation is to $O(1)$ and a gauge transformation will be required. From (5.16) the field equations for P_{VI}, (5.11), transform as

$$\begin{aligned}\frac{dP}{dt} = 0 \quad &\Longrightarrow \frac{d\hat{P}}{d\hat{t}} = 0,\\ t\frac{dQ}{dt} = [R,Q] \quad &\Longrightarrow (1+\varepsilon\hat{t})\frac{d\hat{Q}}{d\hat{t}} = [\hat{R}-\varepsilon\hat{P},\hat{Q}],\end{aligned} \tag{5.17}$$

and finally $t(1-t)\frac{dR}{dt} = [tP+Q,R]$ becomes

$$\hat{t}(1+\varepsilon\hat{t})\frac{d}{d\hat{t}}\left(\varepsilon\hat{P}+\varepsilon\hat{Q}-\hat{R}\right) = [(1+\varepsilon\hat{t})\hat{P}+\hat{Q},\hat{R}-\varepsilon\hat{P}-\varepsilon\hat{Q}]. \tag{5.18}$$

The above equations are holomorphic in ε at $\varepsilon = 0$ and taking the $\varepsilon \to 0$ limit we recover the field equations for the P_{V} reduction, namely equations (5.13).

The invariants for (5.11) are $\mathrm{Tr}(P^2),\ \mathrm{Tr}(Q^2),\ \mathrm{Tr}(R^2),\ \mathrm{Tr}((P+Q+R)^2)$ which, expressed in terms of the Higgs fields for P_{V} obtained from (5.16), become

$$\begin{aligned}\mathrm{Tr}(P^2) &= \mathrm{Tr}(\hat{P}^2),\\ \mathrm{Tr}(Q^2) &= \mathrm{Tr}(\hat{Q}^2),\\ \mathrm{Tr}(R^2) &= \frac{1}{\varepsilon^2}\mathrm{Tr}(\hat{R}^2) - \frac{2}{\varepsilon}\mathrm{Tr}(\hat{R}(\hat{Q}+\hat{P})) + O(1),\\ \mathrm{Tr}((P+Q+R)^2) &= \frac{1}{\varepsilon^2}\mathrm{Tr}(\hat{R}^2),\end{aligned} \tag{5.19}$$

where $\mathrm{Tr}(\hat{P}^2), \mathrm{Tr}(\hat{Q}^2), \mathrm{Tr}(\hat{R}^2)$ and $\mathrm{Tr}(\hat{R}(\hat{Q}+\hat{P}))$ are the first integrals for (5.13).

The above degeneration lifts to the J-matrix level. To see this notice that solving (5.13) and using (5.16) we get that $\hat{P} = P$, and $\hat{Q}_0 = (G\hat{G}^{-1})\hat{Q}_0(G\hat{G}^{-1})^{-1}$. Then from the J-matrix for the $\mathrm{P_{VI}}$ reduction and (5.10) we have

$$\begin{aligned}
J &= \exp(qQ_0)G\exp(-pP) \\
&= e^{\left[\left(\log\varepsilon+\hat{q}+\varepsilon\hat{r}+O(\varepsilon^2)\right)(G\hat{G}^{-1})\hat{Q}_0(G\hat{G}^{-1})^{-1}\right]}Ge^{-\left(\log\varepsilon+\hat{p}+\varepsilon(\hat{t}+\hat{r})+O(\varepsilon^2)\right)\hat{P}} \\
&= \left(G\hat{G}^{-1}\right)\varepsilon^{\hat{Q}_0}e^{\hat{q}\hat{Q}_0}e^{\varepsilon\hat{r}\hat{Q}_0}\hat{G}\varepsilon^{-\hat{P}}e^{-\hat{p}\hat{P}}e^{-\varepsilon(\hat{t}+\hat{r})\hat{P}}
\end{aligned}$$

$$\varepsilon \to 0 \quad \Longrightarrow \quad e^{\hat{q}\hat{Q}_0}\hat{G}e^{-\hat{p}\hat{P}} = J_V,$$

where the fact that the above is the J-matrix for the $\mathrm{P_V}$ reduction can be checked by solving (5.13). Note that in this confluence we have that $G = \hat{G}$, this can be shown by considering how (3.24) behave under the $\varepsilon \to 0$ limit. Therefore, by constructing a holomorphic (in ε) map in the space of centralisers to the regular element of $\mathrm{GL}(4,\mathbb{C})$ corresponding to the partition of 4 (representing the four Fuchsian singular points for the isomonodromy problem of the Painlevé VI equation), we can, by a limiting process, recover the centraliser to the regular element of $\mathrm{GL}(4,\mathbb{C})$ corresponding to the partition representing the singular structure for the isomonodromy problem corresponding to the Painlevé V equation.

Recalling the results from 4.2.5 we see that it is possible to lift this process directly at the Darboux matrix level; that is, we can transform the relevant terms in the Darboux matrix corresponding to some specific Schlesinger transformation for $\mathrm{P_{VI}}$ and take the limit leading us to the corresponding transformation for the $\mathrm{P_V}$ equation. For instance, consider the transformation

$$\mathbb{D}_{(\theta_0^+,\theta_\infty^\pm)} = \tilde{Z}^{1/2}W^{-1/2}\begin{pmatrix}0 & 0\\ 0 & \tilde{d}\end{pmatrix} + \zeta W^{1/2}\tilde{Z}^{-1/2}\hat{G}_{\mathrm{VI}}^{-1}\begin{pmatrix}0 & 0\\ 0 & d\end{pmatrix}G_{\mathrm{VI}}, \tag{5.20}$$

acting on the $\mathrm{P_{VI}}$ parameters as $(\theta_0,\theta_\infty,\theta_1,\theta_t) \mapsto (\theta_0+1,\theta_\infty+1,\theta_1,\theta_t)$. The confluence to $\mathrm{P_V}$ is given by the transformations in (5.6) with $G_{\mathrm{VI}}(t) = G_{\mathrm{V}}(\hat{t})$ which using (5.20) gives

$$\mathbb{D}_{(\theta_0^+,\theta_\infty^\pm)} = \tilde{z}^{1/2}\left(\frac{z}{\varepsilon}+w\right)^{-1/2}\begin{pmatrix}0 & 0\\ 0 & \tilde{d}\end{pmatrix} + \zeta\left(\frac{z}{\varepsilon}+w\right)^{1/2}\tilde{z}^{-1/2}\hat{G}_{\mathrm{V}}^{-1}\begin{pmatrix}0 & 0\\ 0 & d\end{pmatrix}G_{\mathrm{V}}$$

$$= \tilde{z}^{1/2}\left(\frac{\varepsilon^{1/2}}{z^{1/2}}+O(\varepsilon^{3/2})\right)\begin{pmatrix}0 & 0\\ 0 & \tilde{d}\end{pmatrix} + \zeta\left(\frac{\varepsilon^{1/2}}{z^{1/2}}+O(\varepsilon^{3/2})\right)\tilde{z}^{-1/2}\hat{G}_{\mathrm{V}}^{-1}\begin{pmatrix}0 & 0\\ 0 & d\end{pmatrix}G_{\mathrm{V}}.$$

The above provides us with the transformation acting on the $\mathrm{P_V}$ parameters as $(\theta_0,\theta_\infty,\theta_1)\mapsto(\theta_0+1,\theta_\infty+1,\theta_1)$. In the case where the $\mathrm{P_{VI}}$ transformation generates shifts in the θ_t parameter the transformation degenerates, by virtue of the various compatibility conditions to be satisfied (see (4.42)—(4.44)), to a gauge transformation. We then have, through use of the above degeneration process and of the Möbius transformations presented in section 4.3, a prescription for recovering the Schlesinger transformations for the $\mathrm{P_V}$ equation from a small set of transformations for $\mathrm{P_{VI}}$. Below we construct a similar limiting process taking us from $\mathrm{P_V}$ to $\mathrm{P_{III}}$ and in Appendix A we construct the maps for the remaining confluences. In principle then we may recover all Schlesinger transformations for the Painlevé equations through a few transformations of $\mathrm{P_{VI}}$.

5.2 Construction of confluence

Here we briefly describe the necessary theory to construct the map between the centralisers of the Lie group. In the case of the confluence $\mathrm{P_{VI}}\to\mathrm{P_V}$ we saw that this machinery was not necessary, however for the other limits we need the theory developed in [34, 35] where a stratification of the set of regular elements of $\mathrm{GL}(4,\mathbb{C})$ is obtained and the relevant map arises as the specific relation of adherence among the strata of this set. The details of the geometry of this construction will be limited as much as possible; here we attempt to introduce the minimum amount of theory necessary to describe and implement the map. We present the theory for regular elements of $\mathrm{GL}(n,\mathbb{C})$, however we are interested in the Painlevé equations and shall therefore make explicit use of the maps for the case when $n=4$.

Let $b\in\mathfrak{gl}(n,\mathbb{C})$ be a regular element which for the purpose of this thesis means that if b has l distinct eigenvalues $b^{(0)}, \ldots, b^{(l-1)}$ of multiplicities $\lambda_0, \ldots, \lambda_{l-1}$ with $\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{l-1}$ and $\lambda_0 + \ldots + \lambda_{l-1} = n$, then it can be expressed as

$$b = (\mathrm{Ad}g_b)\left(\left(b^{(0)}I_{\lambda_0} + \Lambda_{\lambda_0}\right) \oplus \cdots \oplus \left(b^{(l-1)}I_{\lambda_{l-1}} + \Lambda_{\lambda_{l-1}}\right)\right), \tag{5.21}$$

for some $g_b \in \mathrm{GL}(n,\mathbb{C})$ where I_m denotes the identity matrix of size m for any integer m and $\Lambda_m = (\delta_{i+1,j})_{0\leq i,j<m}$ is the upper triangular shift matrix of size m. Its centraliser, namely the set $\{X \in \mathfrak{gl}(n,\mathbb{C}) \mid [b,X] = 0\}$, will be denoted by $\mathfrak{h}_b$ and is given by

$$\mathfrak{h}_b = (\mathrm{Ad}g_b)\left(\mathfrak{j}(\lambda_0) \oplus \cdots \oplus \mathfrak{j}(\lambda_{l-1})\right), \tag{5.22}$$

with

$$\mathfrak{j}(m) = \left\{ \sum_{0\leq i<m} x_i \Lambda_m^i \mid x_i \in \mathbb{C} \right\}, \tag{5.23}$$

and the i superscript denotes the ith power.
The idea is to consider the sequence $\lambda := (\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{l-1})$ as a partition of n (that is the multiplicities of the eigenvalues correspond to a specific partition of n). Denote by $\mathbf{Y}_n$ the set of partitions of n and let l be the length of λ, denoted by $l(\lambda)$. Note then that if we let

$$\mathfrak{h}_\lambda := \left(\mathfrak{j}(\lambda_0) \oplus \cdots \oplus \mathfrak{j}(\lambda_{l-1})\right), \tag{5.24}$$

then $\mathfrak{h}_b = (\mathrm{Ad}g_b)\,\mathfrak{h}_\lambda$. If we let B denote the set of regular elements of $\mathfrak{gl}(n,\mathbb{C})$ and B_λ for $\lambda := (\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{l-1}) \in \mathbf{Y}_n$ the subset of B whose elements have l distinct eigenvalues (the specific values do not matter) of multiplicities $\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{l-1}$, then B can be described as the disjoint union of all such B_λ, i.e.

$$B = \bigsqcup_{\lambda\in\mathbf{Y}_n} B_\lambda = \bigcup_{\lambda\in\mathbf{Y}_n} \{(b_\lambda,\lambda) \mid \lambda \in \mathbf{Y}_n\}, \tag{5.25}$$

where b_λ is a regular element with partition on n given by λ.

The elements of the Painlevé groups have a little more structure in the sense that they form a group and are therefore non-singular. They are centralisers of regular elements but additionally we describe them as maximal Abelian subgroups of $\mathrm{GL}(n,\mathbb{C})$ (when $n=4$) where the maximal Abelian subgroup associated to $b \in B_\lambda$ is denoted by H_b and given by

$$H_b = (\mathrm{Ad}g_b)\,(J(\lambda_0)\times \cdot\cdot\cdot \times J(\lambda_{l-1}))\,, \tag{5.26}$$

where $J(m)$, for any positive integer m, is the matrix group

$$J(m) = \left\{ \sum_{0\le i<m} h_i \Lambda_m^i \mid h_i \in \mathbb{C}, h_0 \neq 0 \right\} \subset GL(m,\mathbb{C}), \tag{5.27}$$

called the Jordan group of size m and $g_b \in \mathrm{GL}(n,\mathbb{C})$. H_b is then a Lie group of $\mathfrak{h}_b$ and we can write $H_b = (\mathrm{Ad}g_b)H_\lambda$ where

$$H_\lambda = J(\lambda_0)\times \cdot\cdot\cdot \times J(\lambda_{l-1}). \tag{5.28}$$

The relevant confluence will rely on a map from one regular element to another, thus a curve in the space of regular elements B and this map is constructed by exploiting what is called the ‘fibration structure’ of B which we do not describe here. Introduce an injective mapping ([34]) ι_b from H_b to the space of n dimensional row vectors $\iota_b : H_b \to \mathbb{C}^n$. When $g_b = I_n$, $\iota_\lambda := \iota_b$ is defined as

$$\iota_\lambda(h) = (h_0^{(0)}, \ldots, h_{\lambda_0-1}^{(0)}, \ldots, h_0^{(l-1)}, \ldots, h_{\lambda_{l-1}-1}^{(l-1)}) \tag{5.29}$$

for $h = \oplus_{0\le k<l}\sum_{0\le i<\lambda_k} h_i^{(k)}\Lambda_{\lambda_k}^i \in H_\lambda$. What this mapping does is to construct an n-dimensional vector from the n distinct components defining the relevant centraliser (a regular element is one for which the centraliser has dimension n) arranged in sequence according to each Jordan group. The usefulness of this mapping is that it is easier to find the relevant action to deform the vector such that once the inverse mapping is applied we find ourselves in a different maximal Abelian subgroup, that is one corresponding to a regular element associated with a different partition, rather than mapping regular elements associated to different partitions directly. More concretely, view the space of regular elements as being parametrised by $\mathbb{C}^n$ such that the map ι^{-1} takes us to the fibre of a vector in $\mathbb{C}^n$ which corresponds to B_λ for some λ. The map between regular elements is then obtained via construction of a map between vectors in $\mathbb{C}^n$. In the case of a general $b \in B_\lambda$, i. e. one for which $g_b \neq I_n$, the mapping is defined as

$$\iota_b = R_{g_b}^{-1} \circ \iota_\lambda \circ (\mathrm{Ad} g_b)^{-1}, \tag{5.30}$$

where R_{g_b} denotes right multimplication by g_b.

Recall figure 5.1 representing the coalescence cascade linking the different DEs. Each arrow connects different DEs in such a way that moving along each arrow the structure of the singularities changes in a way that only two singularities, whether Fuchsian or not, amalgamate into a new (non-Fuchsian) singularity. In terms of partitions this means that each limit maps *adjacent* partitions, where we define adjacent partitions as follows:

Definition 5.2.1. *We say that $\mu \in \mathbf{Y}_n$ is adjacent to $\lambda \in \mathbf{Y}_n$ and denote this by $\lambda \to \mu$ if*

1. *$l(\mu) = l(\lambda) - 1$, where $l(\cdot)$ denotes the length of a partition,*
2. *there exists $0 \leq j < l(\mu)$, $0 \leq j_2 < j_1 < l(\lambda)$ with $\mu_j = \lambda_{j_1} + \lambda_{j_2}$ such that $\{\mu_k\}_{0 \leq k < l(\mu), k \neq j} = \{\lambda_k\}_{0 \leq k < l(\lambda), k \neq j_1, j_2}$.*

What this says is that we view two partitions as being adjacent if their length differs by one and the non-amalgamated singularities remain the same, that is if the remaining string in the partition, as a set, remains the same. For example the confluence of Painlevé equations given in figure 5.1 represent adjacent relations among elements of $\mathbf{Y}_4$, that is of partitions of 4.

Let $\mu \in \mathbf{Y}_n$ be adjacent to $\lambda \in \mathbf{Y}_n$. We will state a theorem which allow us to realise a family of mappings σ_ε ($\varepsilon \neq 0$) from B_μ to B_λ such that $\sigma_\varepsilon(b)$ is holomorphic in ε and $\lim_{\varepsilon \to 0} \sigma_\varepsilon(b) = b$ for any $b \in B_\mu$. Similarly a Lie algebra isomorphism Ψ_ε from $\mathfrak{h}_b$ to $\mathfrak{h}_{\sigma_\varepsilon(b)}$, $b \in B_\mu$ can be constructed and finally this isomorphism lifted to

$$
\begin{tikzcd}
 & & (3,1) \arrow[dr] & \\
(1,1,1,1) \arrow[r] & (2,1,1) \arrow[ur] \arrow[dr] & & (4) \\
 & & (2,2) \arrow[ur] &
\end{tikzcd}
$$

Figure 5.2: Adjacent partitions of 4. Note that $\mathrm{P_{II}}$ and $\mathrm{P_I}$ correspond to the same partition of 4

the maximal Abelian subgroup. The theorem will rely on crucial lemmas which introduce matrices $g_{\lambda\to\mu}(\varepsilon)$ and $g(\varepsilon)$ dictating the relevant mappings on $\mathbb{C}^n$. Details can be found in [34, 35], here we use the results to obtain the relevant maps.

Let $p,q \in \mathbb{Z}^+$. Introduce the matrix $g(\varepsilon) \in \mathrm{GL}(p+q,\mathbb{C})$ depending holomorphically on $\varepsilon \in \mathbb{C}^\times$ given by

$$
g(\varepsilon) = \begin{pmatrix} I_p & G_{12}(\varepsilon) \\ 0 & G_{22}(\varepsilon) \end{pmatrix}, \tag{5.31}
$$

where the $p\times q$ matrix $G_{12} = G_{12}(\varepsilon)$ and the $q\times q$ matrix $G_{22} = G_{22}(\varepsilon)$ are defined by

$$
\begin{pmatrix} G_{12} \\ G_{22} \end{pmatrix} = D_{p+q}(\varepsilon) \begin{pmatrix} \binom{0}{0} & \binom{0}{1} & \cdots & \binom{0}{q-1} \\ \binom{1}{0} & \binom{1}{1} & \cdots & \binom{1}{q-1} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \binom{p+q-1}{0} & \binom{p+q-1}{1} & \cdots & \binom{p+q-1}{q-1} \end{pmatrix} D_q(\varepsilon^{-1}), \tag{5.32}
$$

with $D_m(\varepsilon) = \mathrm{diag}(1,\varepsilon,\varepsilon^2,\ldots,\varepsilon^{m-1})$, for any $m \in \mathbb{Z}^+$, and $\binom{i}{j}$ are the binomial coefficients which by convention are 0 if $i<j$. Note that $\det g(\varepsilon) = \varepsilon^{pq}$ and therefore $g(\varepsilon)$ is nonsingular when $\varepsilon \neq 0$. Now for any $X = \sum_{0\le i<p+q} x_i \Lambda^i_{p+q} \in \mathfrak{h}_{(p+q)}$ define

$X(\varepsilon) \in (\mathrm{Ad}g_b)\,\mathfrak{h}_{(p,q)}$ as:

$$\big(y_0(\varepsilon),\,\ldots,y_{p+q-1}(\varepsilon)\big) := \big(x_0,\,\ldots,x_{p+q-1}\big)\,g(\varepsilon),$$

$$Y(\varepsilon) := \left(\sum_{0\le i<p} y_i(\varepsilon)\Lambda_p^i\right) \oplus \left(\sum_{p\le i<p+q} y_i(\varepsilon)\Lambda_q^{i-p}\right) \in \mathfrak{h}_{p,q}, \tag{5.33}$$

$$X(\varepsilon) := (\mathrm{Ad}g_b)\,Y(\varepsilon),$$

which we may re-write as

$$X(\varepsilon) = \Big((\mathrm{Ad}g_b)\circ \iota_{(p,q)}^{-1}\circ R_{g(\varepsilon)}\circ \iota_{(p+q)}\Big)\,(X)\,. \tag{5.34}$$

The point here is that X is associated with the partition $(p+q)$ while $X(\varepsilon)$ is associated with the partition (p,q) for all $\varepsilon \neq 0$, thus we have a map from one partition to an adjacent one.

Lemma 5.2.2. *For any $X \in \mathfrak{h}_{(p+q)}$, $X(\varepsilon) \in (Adg(\varepsilon))\,\mathfrak{h}_{(p,q)}$ is holomorphic in ε in a neighbourhood of $\varepsilon = 0$ and*

$$\lim_{\varepsilon\to 0} X(\varepsilon) = X.$$

Extension to general, adjacent, partitions is as follows: let $\mu \in \mathbf{Y}_n$ be adjacent to $\lambda \in \mathbf{Y}_n$ with $\mu_j = \lambda_{j_1} + \lambda_{j_2}$ for some $0 \le j < l(\mu)$, $0 \le j_2 < j_1 < l(\lambda)$ and define $g_{\lambda\to\mu}(\varepsilon) \in \mathrm{GL}(n,\mathbb{C})$ via

$$g_{\lambda\to\mu}(\varepsilon) = \big(I_{\mu_0+\,\cdots\,+\mu_{j-1}} \oplus g^j(\varepsilon) \oplus I_{\mu_{j+1}+\,\cdots\,+\mu_{j-2}}\big) \tag{5.35}$$

where $g^j(\varepsilon) \in \mathrm{GL}(\mu_j;\mathbb{C})$ was defined in (5.31) with $p = \lambda_{j_1}$ and $q = \lambda_{j_2}$. Now for $X \in \mathfrak{h}_\mu$ define $X(\varepsilon) \in \mathfrak{h}_\lambda$ by

$$X(\varepsilon) = \Big((\mathrm{Ad}g_b)\circ \iota_\lambda^{-1}\circ R_{g_{\lambda\to\mu}(\varepsilon)}\circ \iota_\mu\Big)\,(X)\,, \tag{5.36}$$

for all $\varepsilon \neq 0$. We then have

Lemma 5.2.3. *For any $x \in \mathfrak{h}_\mu$, $X(\varepsilon)$ is holomorphic in ε in a neighbourhood of $\varepsilon = 0$ and satisfies*

$$\lim_{\varepsilon \to 0} X(\varepsilon) = X. \tag{5.37}$$

Through this lemma we have a way to construct a map describing convergence of regular elements, centralisers and maximal Abelian subgroups. The relevant theorem is given and proved in [34]:

Theorem 5.2.4. *Suppose $\lambda \to \mu$, $\lambda, \mu \in \boldsymbol{Y}_n$ and let $b \in B_\mu$ with $b \in (\mathrm{Ad} g_b)\, \mathfrak{h}_\mu$. Then*

$$\sigma_\varepsilon(b) := \left(\left(\mathrm{Ad} g_b g_{\lambda\to\mu}(\varepsilon)\right) \circ \iota_\lambda^{-1} \circ R_{g_{\lambda\to\mu}(\varepsilon)} \circ \iota_\mu \circ (\mathrm{Ad} g_b)^{-1} \right)(b), \tag{5.38}$$

is an element of $B_\lambda \cap \left(\mathrm{Ad}_{g_b g_{\lambda\to\mu}(\varepsilon)}\right) \mathfrak{h}_\lambda$ for any ε with $0 < |\varepsilon| \ll 1$, and is holomorphic in ε in a neighbourhood of 0 and satisfies $\lim_{\varepsilon\to 0} \sigma_\varepsilon(b) = b$.

Similar constructions apply for elements of the maximal Abelian subgroup which are the centralisers of a regular element. In (5.51) we obtain this mapping for the confluence $\mathrm{P_V} \to \mathrm{P_{III}}$ and from this recover the relevant transformations of the dependent variables and the Higgs fields. In section 5.1 we obtain the confluence $\mathrm{P_{VI}} \to \mathrm{P}_V$ without the explicit need for such a map. The details for the remaining confluences can be found in the appendix, A.

5.3 Painlevé V degeneration to Painlevé III

For this confluence we use the machinery presented in 5.2. The element corresponding to $\mathrm{P_V}$ is associated to the partition $\lambda = (\lambda_0, \lambda_1, \lambda_2) = (2, 1, 1)$, thus

$$\mathfrak{P}_\mathrm{V} = \begin{pmatrix} a & b & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} = (aI_2 + b\Lambda_2^1) \oplus (cI_1) \oplus (dI_1), \tag{5.39}$$

while that for P_{III} has partition $\mu = (\mu_0, \mu_1) = (2,2)$, i. e. $\mu_1 = \lambda_1 + \lambda_2 = 1+1$ and $p = \lambda_1 = 1 = \lambda_2 = q$,

$$\mathfrak{P}_{III} = \begin{pmatrix} h_0^{(0)} & h_1^{(0)} & 0 & 0 \\ 0 & h_0^{(0)} & 0 & 0 \\ 0 & 0 & h_0^{(1)} & h_1^{(1)} \\ 0 & 0 & 0 & h_0^{(1)} \end{pmatrix} = (h^{(0)} I_2 + h_1^{(0)} \Lambda_2^1) \oplus (h_0^{(1)} I_2 + h_1^{(1)} \Lambda_2^1). \quad (5.40)$$

The main ingredient is the matrix $g_{\lambda\to\mu}(\varepsilon)$ which we can obtain from

$$g_{\lambda\to\mu}(\varepsilon) = (I_{\mu_0=2} \oplus g^{(1)}(\varepsilon)), \quad (5.41)$$

where $g^{(1)}(\varepsilon) \in GL(\mu_1 = 2, \mathbb{C})$. We then have

$$g^{(1)}(\varepsilon) = \begin{pmatrix} I_1 & G_{12} \\ 0 & G_{22} \end{pmatrix} \quad (5.42)$$

and

$$\begin{pmatrix} G_{12} \\ G_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix} \quad (5.43)$$

and therefore $G_{12} = 1$ and $G_{22} = \varepsilon$ from which we finally have that

$$g_{\lambda\to\mu}(\varepsilon) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & \varepsilon \end{pmatrix}. \quad (5.44)$$

From this matrix, which gives us the map at the vector space level, we are able to construct the relevant map for the convergence of the Lie algebras. Start by mapping this element to $\mathbb{C}^4$ via $\iota_\mu[\mathfrak{P}_{\text{III}}] = (h_0^{(0)}, h_1^{(0)}, h_0^{(1)}, h_1^{(1)})$ and from this

$$
\begin{aligned}
R_{g_{\lambda\to\mu}(\varepsilon)} \circ \iota_\mu[\mathfrak{P}_{\text{III}}] &= (h_0^{(0)}, h_1^{(0)}, h_0^{(1)}, h_1^{(1)}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & \varepsilon \end{pmatrix} \\
&= (h_0^{(0)}, h_1^{(0)}, h_0^{(1)}, h_0^{(1)} + \varepsilon h_1^{(1)}).
\end{aligned}
$$

We can now map back to the maximal abelian subgroup via ι^{-1}

$$
b_\varepsilon = \iota_\lambda^{-1} \circ R_{g_{\lambda\to\mu}(\varepsilon)} \circ \iota_\mu[\mathfrak{P}_{\text{III}}] = \begin{pmatrix} h_0^{(0)} & h_1^{(0)} & 0 & 0 \\ 0 & h_0^{(0)} & 0 & 0 \\ 0 & 0 & h_0^{(1)} & 0 \\ 0 & 0 & 0 & h_0^{(1)} + \varepsilon h_1^{(1)} \end{pmatrix},
$$

and then

$$
\Psi_\varepsilon(\mathfrak{P}_{\text{III}}) = \left(\mathrm{Ad}_{g_{\lambda\to\mu}(\varepsilon)}\right)[b_\varepsilon] = \begin{pmatrix} h_0^{(0)} & h_1^{(0)} & 0 & 0 \\ 0 & h_0^{(0)} & 0 & 0 \\ 0 & 0 & h_0^{(1)} & h_1^{(1)} \\ 0 & 0 & 0 & h_0^{(1)} + \varepsilon h_1^{(1)} \end{pmatrix}, \tag{5.45}
$$

which in the limit $\varepsilon \to 0$ gives precisely $\mathfrak{P}_{\text{III}}$. The map satisfies the limit $\lim_{\varepsilon\to 0} \Psi_\varepsilon(\mathfrak{P}_{\text{III}}) = \mathfrak{P}_{\text{III}}$ where

$$
\Psi_\varepsilon(\mathfrak{P}_{\text{III}}) = g_{\lambda\to\mu}\left[\left(\iota_\lambda^{-1} \circ R_{g_{\lambda\to\mu}} \circ \iota_\mu\right)[\mathfrak{P}_{\text{III}}]\right] g_{\lambda\to\mu}^{-1}, \tag{5.46}
$$

which therefore generates the symmetry element

$$
g = \exp(\tau \Psi_\varepsilon(\mathfrak{P}_{\text{III}})) = g_{\lambda\to\mu}\left[\exp(\tau b_\varepsilon)\right] g_{\lambda\to\mu}^{-1}. \tag{5.47}
$$

Making use of the conformal transformation $x \mapsto y = gxg^T$ on the metric x we have

$$\begin{aligned} x &\mapsto y = gxg^T \\ &= g_{\lambda\to\mu}\ \exp(\tau b_\varepsilon)\ g^{-1}_{\lambda\to\mu}\ x\ \left(g^{-1}_{\lambda\to\mu}\right)^T (\exp(\tau b_\varepsilon))^T\ g^T_{\lambda\to\mu} \end{aligned} \tag{5.48}$$

from which we get that

$$\hat{y} = e^{\tau b_\varepsilon}\ \hat{x}\ \left(e^{\tau b_\varepsilon}\right)^T, \tag{5.49}$$

where

$$\hat{x} = (\mathrm{Ad}_{g^{-1}_{\lambda\to\mu}(\varepsilon)})x, \tag{5.50}$$

and similar for $\hat{y}$. The 'bare' metric x corresponds to that for the $\mathrm{P_{III}}$ reduction while the transformed one, $\hat{x}$, corresponds to that for $\mathrm{P_V}$. Comparing entries in the bare and transformed metric we find the relevant transformations

$$\begin{aligned} Z &= z + \frac{\tilde{w}}{\varepsilon}, \quad \tilde{Z} = \frac{\tilde{z}}{\varepsilon}, \\ W &= w + \frac{\tilde{z}}{\varepsilon}, \quad \tilde{W} = \frac{\tilde{w}}{\varepsilon}, \end{aligned} \tag{5.51}$$

from which we finally get the transformation of the independent variable:

$$t_{\mathrm{V}} = \frac{\tilde{Z}}{\tilde{W}} - \frac{W}{Z} = \varepsilon\left(\frac{z\tilde{z} - w\tilde{w}}{\tilde{w}^2}\right) + O(\varepsilon^2) = \varepsilon t^2_{\mathrm{III}} + O(\varepsilon^2). \tag{5.52}$$

We see that via the change of variables (5.52) the singular points $(0,\ 1,\ \infty)$ become the singular points $(0,\ \frac{1}{\varepsilon^{1/2}},\ \infty)$ and as ε tends to 0, the singular points $\hat{t} = \frac{1}{\varepsilon^{1/2}}$ and $\hat{t} = \infty$ coalesce and therefore the third Painlevé equation only has the two (irregular) Fuchsian singular points at 0 and ∞.

As in the limit $P_{\mathrm{VI}} \to P_{\mathrm{V}}$, to relate the two systems at the field equation level we must determine how the local coordinates introduced for the reduction are related, following this we can then relate the Higgs fields. The $\mathrm{P_V}$ field equations arose from the choice of local coordinates[1] $p,\ q,\ r,\ t$ such that

$$\mathbb{X}_{\mathrm{V}} = -Z\partial_Z - W\partial_W = \partial_p, \quad \mathbb{Y}_{\mathrm{V}} = -\tilde{Z}\partial_{\tilde{Z}} - \tilde{W}\partial_{\tilde{W}} = \partial_q, \quad \mathbb{Z}_{\mathrm{V}} = -Z\partial_W - \tilde{W}\partial_{\tilde{Z}} = \partial_r,$$

[1]Remove the ˆ from the variables in (5.12).

giving

$$\begin{aligned} p &= -\log Z, \\ q &= -\log \tilde{W}, \\ r &= -\tilde{Z}/\tilde{W}. \end{aligned} \tag{5.53}$$

In terms of the space-time coordinates $(z, w, \tilde{z}, \tilde{w})$ for the $\mathrm{P_{III}}$ reduction the $\mathrm{P_V}$ generators read

$$\begin{aligned} \mathbb{X}_{\mathrm{V}} &= -z\partial_z - w\partial_w + \frac{1}{\varepsilon}(-\tilde{w}\partial_z - \tilde{z}\partial_w) = \frac{1}{\varepsilon}\mathbb{X}_{\mathrm{III}} - z\partial_z - w\partial_w, \\ \mathbb{Y}_{\mathrm{V}} &= -\tilde{z}\partial_{\tilde{z}} - \tilde{w}\partial_{\tilde{w}} + \frac{1}{\varepsilon}(\tilde{z}\partial_w + \tilde{w}\partial_z) = -\frac{1}{\varepsilon}\mathbb{X}_{\mathrm{III}} - \tilde{z}\partial_{\tilde{z}} - \tilde{w}\partial_{\tilde{w}}, \\ \mathbb{Z}_{\mathrm{V}} &= -z\partial_w - \tilde{w}\partial_{\tilde{z}} = -\mathbb{Y}_{\mathrm{III}}. \end{aligned} \tag{5.54}$$

We are free to take any linear combination of these, and therefore to make the generators linearly independent we choose $\mathbb{X}_{\mathrm{V}} + \mathbb{Y}_{\mathrm{V}} = \mathbb{Z}_{\mathrm{III}} = -z\partial_z - w\partial_w - \tilde{z}\partial_{\tilde{z}} - \tilde{w}\partial_{\tilde{w}}$ yielding the choice for $\mathrm{P_{III}}$ generators

$$\begin{aligned} \mathbb{X}_{\mathrm{III}} &= -\tilde{w}\partial_z - \tilde{z}\partial_w = \partial_{\hat{p}}, \\ \mathbb{Y}_{\mathrm{III}} &= z\partial_w + \tilde{w}\partial_{\tilde{z}} = \partial_{\hat{q}}, \\ \mathbb{Z}_{\mathrm{III}} &= -z\partial_z - w\partial_w - \tilde{z}\partial_{\tilde{z}} - \tilde{w}\partial_{\tilde{w}} = \partial_{\hat{r}}. \end{aligned} \tag{5.55}$$

The coordinates for the $\mathrm{P_{III}}$ reduction are then:

$$\begin{aligned} \hat{p} &= -z/\tilde{w}, \\ \hat{q} &= \tilde{z}/\tilde{w}, \\ \hat{r} &= -\log \tilde{w}, \end{aligned} \tag{5.56}$$

in addition to the invariant quantity $\hat{t} = \tilde{w}(z\tilde{z} - w\tilde{w})^{1/2}$. Given (5.51) and (5.53) the two sets of coordinates are related as

$$
\begin{aligned}
p = -\log Z = -\log(z + \frac{\tilde{w}}{\varepsilon}) &= \log\varepsilon + \hat{r} + \varepsilon\hat{p} + O(\varepsilon^2), \\
q = -\log\tilde{W} = -\log\left(\frac{\tilde{w}}{\varepsilon}\right) &= \log\varepsilon + \hat{r}, \\
r = \frac{\tilde{Z}}{\tilde{W}} &= -\hat{q}, \\
t &= \varepsilon\hat{t} + O(\varepsilon^2).
\end{aligned} \tag{5.57}
$$

Rewriting the potential for the $\mathrm{P_V}$ reduction in terms of the Higgs fields associated with the reduction to $\mathrm{P_{III}}$

$$
\begin{aligned}
Pdp + Qdq + Rdr &= \hat{P}d\hat{p} + \hat{Q}d\hat{q} + \hat{R}d\hat{r} + \hat{T}d\hat{t}, \\
\implies P(d\hat{r} + \varepsilon\hat{p}) + Q(d\hat{r}) + \hat{R}(-d\hat{q}) &= \hat{P}d\hat{p} + \hat{Q}d\hat{q} + \hat{R}d\hat{r} + \hat{T}d\hat{t},
\end{aligned}
$$

yields

$$
\begin{aligned}
\hat{P} &= \varepsilon P, \\
\hat{Q} &= -R, \\
\hat{R} &= P + Q, \\
\hat{T} &= 0.
\end{aligned} \tag{5.58}
$$

From these the field equations for $\mathrm{P_V}$, (5.13), transform to

$$
\begin{aligned}
\frac{dP}{dt} = 0 \quad &\implies \frac{d\hat{P}}{d\hat{t}} = 0, \\
\frac{dQ}{dt} = [R,Q] \quad &\implies \frac{1}{2\hat{t}}\frac{d\hat{R}}{d\hat{t}} = -\varepsilon[\hat{Q},\hat{R}] + [\hat{Q},\hat{P}], \\
t\frac{dR}{dt} = [R,P+Q] \quad &\implies \hat{t}\frac{d\hat{Q}}{d\hat{t}} = 2[\hat{Q},\hat{R}],
\end{aligned} \tag{5.59}
$$

which in the limit $\varepsilon \to 0$ yield the field equations for the third Painlevé equations. The invariants for $\mathrm{P_V}$ are $\mathrm{Tr}(\hat{P}^2), \mathrm{Tr}(\hat{Q}^2), \mathrm{Tr}(\hat{R}^2)$ and $\mathrm{Tr}(\hat{R}(\hat{Q}+\hat{P}))$ which, expressed in terms of the Higgs fields for P_{III} obtained from (5.58), become

$$\begin{aligned} \mathrm{Tr}(P^2) &= \frac{1}{\varepsilon^2}\mathrm{Tr}(\hat{P}^2), \\ \mathrm{Tr}(Q^2) &= \frac{1}{\varepsilon^2}\mathrm{Tr}(\hat{P}^2) - \frac{2}{\varepsilon}\mathrm{Tr}(\hat{P}\hat{R}) + O(1), \\ \mathrm{Tr}(R^2) &= \mathrm{Tr}(\hat{Q}^2), \\ \mathrm{Tr}(R(Q+P)) &= -\mathrm{Tr}(\hat{Q}\hat{R}), \end{aligned} \tag{5.60}$$

where $\mathrm{Tr}(\hat{P}^2), \mathrm{Tr}(\hat{Q}^2), \mathrm{Tr}(\hat{P}\hat{R})$ and $\mathrm{Tr}(\hat{Q}\hat{R})$ are the first integrals in the reduction to P_{III}. This limit can be lifted to the J-matrix level as was done in 5.1 and therefore, from the DM representing the relevant Schlesinger transformations for P_{V} we may recover those corresponding to the Schlesinger transformations for P_{III}. Consequently, through a sequence of limiting processes it is possible, given this framework, to obtain the Schlesinger transformations for any of the Painlevé equations P_{II} – P_{V} as degenerations of the relevant Schlesinger transformation for P_{VI}. If in addition we compose such sequence of transformations with the action of the Möbius transformation as presented in section 4.3, it then becomes possible to recover all Schlesinger transformations for the Painlevé equations starting with a basic set of transformations acting on the sixth Painlevé equation.

Chapter 6

Bianchi permutability theorem for the ASDYM equations

In the spirit of Bianchi we look to impose permutability of the Darboux transformation for the ASDYM equations such as to recover a non-linear superposition of solutions for the ASDYM equations analogous to that first obtained by Bianchi for solutions to the SG equation. From this system, which we call the 'ASDYM Bianchi system', we may obtain, via reduction, a number of integrable lattice equations. Moreover this system possesses gauge invariance on the lattice and we therefore interpret it as a lattice gauge theory. Some continuous limits to 2-dimensional reductions of the continuous ASDYM are performed. The material in this chapter reproduces the original work presented in [1, 3].

6.1 Autonomous ASDYM Bianchi system

Suppose Ψ is a seed solution of the linear problem associated with the ASDYM equations and that Ψ^1 and Ψ^2 are the Bäcklund transforms of Ψ via[1] $\mathbb{D}_0^1$ and $\mathbb{D}_0^2$, that is, $\Psi^1 = \mathbb{D}_0^1\Psi$, $\Psi^2 = \mathbb{D}_0^2\Psi$. Let $\Psi^{21} = \mathbb{D}_1^2\Psi^1$ and $\Psi^{12} = \mathbb{D}_2^1\Psi^2$. Following Bianchi, we enquire whether there are any circumstances under which the commutativity $\Xi = \Psi^{12} = \Psi^{21}$ applies, see figure 6.1. We therefore impose that $[\mathbb{D}_1^2\mathbb{D}_0^1, \mathbb{D}_2^1\mathbb{D}_0^2]\Psi = 0$, where $\mathbb{D}_i^j = S_i^j + \zeta T_i^j$ with $S_i^j = \hat{H}^{ji}\tilde{C}^j(H^i)^{-1}$ and $T_i^j = \hat{K}^{ji}C^j(K^i)^{-1}$, where C^j, $\tilde{C}^j$ denoted shift by the operators in the jth direction. Equating again coefficients of

[1] We here use the notation that the subscript denotes the solution on which we act and the superscript denotes the resulting 'dressed' solution. Thus, $\Psi^j = \mathbb{D}_i^j\Psi^i$.

powers of ζ gives

$$S_1^2 S_0^1 = S_2^1 S_0^2, \tag{6.1}$$

$$S_1^2 T_0^1 + T_1^2 S_0^1 = S_2^1 T_0^2 + T_2^1 S_0^2, \tag{6.2}$$

$$T_1^2 T_0^1 = T_2^1 T_0^2. \tag{6.3}$$

$$\begin{tikzcd}
& \Psi^1 \arrow[dr, "\mathbb{D}_1^2"] & \\
\Psi \arrow[ur, "\mathbb{D}_0^1"] \arrow[dr, "\mathbb{D}_0^2"'] & & \Xi \\
& \Psi^2 \arrow[ur, "\mathbb{D}_2^1"'] &
\end{tikzcd}$$

Figure 6.1: Bianchi diagram for the ASDYM equations

Using the above expressions for S_i^j and T_i^j, (6.1)–(6.3) give us the following conditions on the matrices $\tilde{C}^j$ and C^j, for $j = 1,2$, which must be satisfied if permutability is to hold

$$\left[C^1, C^2\right] = 0, \qquad \left[\tilde{C}^1, \tilde{C}^2\right] = 0, \tag{6.4}$$

and

$$\Omega \left[\tilde{C}^1 (J_2)^{-1} C^2 - \tilde{C}^2 (J_1)^{-1}) C^1\right] J = C^2 J_1 \tilde{C}^1 - C^1 J_2 \tilde{C}^2, \tag{6.5}$$

where $\Omega = J_{12} = J_{21}$. Notice how (6.5) is now an equation relating four solutions of Yang's equation not involving any derivatives. In fact, this is a system relating the four solutions obtained under 'shifts' by the matrices $\tilde{C}^i$ and C^i on some multi-dimensional lattice. We call (6.5) the 'ASDYM Bianchi system'; it is the equation governing the evolution, on a 2-dimensional lattice, of the solutions to the ASDYM equations obtained by iterated action of the Bäcklund transformation. Chau and Chinea [83] previously considered the Bianchi permutability for the special Bäcklund transformations derived by Prasad, Sinha and Wang in [84].

6.2 Non-autonomous ASDYM Bianchi system

For the purpose of this construction we shall slightly modify our Darboux matrix such as to express it explicitly in terms of the seed solution J and the dressed solution $\hat{J}$. For this we transform the fundamental solution of the linear problem via the gauge transformation $\Psi \mapsto \Phi = K^{-1}\Psi$. Then we can write the Darboux transformation

$$\hat{\Psi} = \left[\hat{H}\tilde{C}H^{-1} + \zeta\hat{K}CK^{-1}\right]\Psi,$$

as

$$\hat{\Phi} = \left[\hat{J}\tilde{C}J^{-1} + \zeta C\right]\Phi,$$

where $J = K^{-1}H$ and $\mathbb{D} = \hat{J}\tilde{C}J^{-1} + \zeta C$. We now choose to reinterpret this transformation as a discrete Lax pair and, furthermore, we de-autonomize the system by allowing the 'transporters', $(C,\tilde{C})$, to depend on the lattice location (the name 'transporters' will be justified below). The result is the Lax pair given by

$$\begin{aligned}
\Phi_{m+1,n} &= \left(J_{m+1,n}\tilde{C}^{1}_{m,n}J^{-1}_{m,n} + \zeta C^{1}_{m,n}\right)\Phi_{m,n} = M_{m,n}\Phi_{m,n},\\
\Phi_{m,n+1} &= \left(J_{m,n+1}\tilde{C}^{2}_{m,n}J^{-1}_{m,n} + \zeta C^{2}_{m,n}\right)\Phi_{m,n} = N_{m,n}\Phi_{m,n},
\end{aligned} \tag{6.6}$$

whose compatibility is satisfied modulo

$$\begin{aligned}
J_{m+1,n+1}&\left[\tilde{C}^{2}_{m+1,n}J^{-1}_{m+1,n}C^{1}_{m,n} - \tilde{C}^{1}_{m,n+1}J^{-1}_{m,n+1}C^{2}_{m,n}\right]\\
&- \left[C^{1}_{m,n+1}J_{m,n+1}\tilde{C}^{2}_{m,n} - C^{2}_{m+1,n}J_{m+1,n}\tilde{C}^{1}_{m,n}\right]J^{-1}_{m,n} = 0,
\end{aligned} \tag{6.7}$$

and where the $(C^{i}_{m,n},\tilde{C}^{i}_{m,n})$ $i = 1,2$ satisfy a more general form of commutation relation than in the autonomous case

$$C^{1}_{m,n+1}C^{2}_{m,n} = C^{2}_{m+1,n}C^{1}_{m,n}, \qquad \text{and} \qquad \tilde{C}^{1}_{m,n+1}\tilde{C}^{2}_{m,n} = \tilde{C}^{2}_{m+1,n}\tilde{C}^{1}_{m,n}. \tag{6.8}$$

This system possesses a local gauge invariance on the lattice $\Phi_{m,n} \mapsto \Lambda_{m,n}\Phi_{m,n}$ under which the terms transform as

$$\begin{aligned} C^1_{m,n} &\mapsto \Lambda_{m+1,n} C^1_{m,n} \Lambda^{-1}_{m,n}, \quad & \tilde{C}^1_{m,n} &\mapsto \Lambda_{m+1,n} \tilde{C}^1_{m,n} \Lambda^{-1}_{m,n} \\ C^2_{m,n} &\mapsto \Lambda_{m,n+1} C^2_{m,n} \Lambda^{-1}_{m,n}, \quad & \tilde{C}^2_{m,n} &\mapsto \Lambda_{m,n+1} \tilde{C}^2_{m,n} \Lambda^{-1}_{m,n} \end{aligned} \tag{6.9}$$

and $J_{m,n} \mapsto \Lambda_{m,n} J_{m,n} \Lambda^{-1}_{m,n}$.

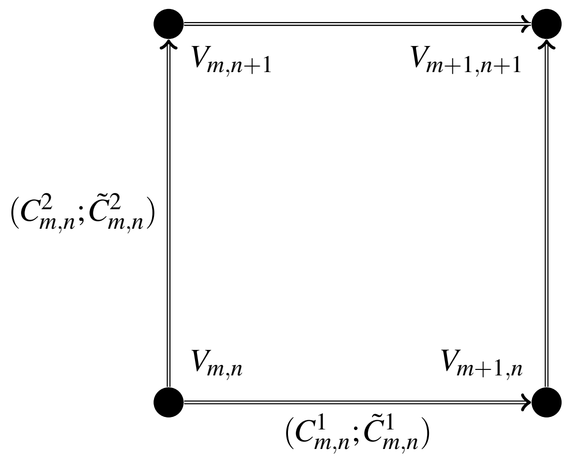

Figure 6.2: Geometric interpretation of the Bianchi system.

To interpret the ASDYM Bianchi system one must view it as a lattice gauge theory; we are led to this interpretation through the transformation properties (6.9). The main features of the description which follow are not novel, indeed a similar interpretation was given for the discrete Nahm equations obtained by Braam and Austin ([89]) and whose integrability was studied by Murray and Singer ([90]) and for the discrete Hitchin equations obtained by Ward, see [91]. Thus let $(m,n) \in \mathbb{Z}^2$ where m is the horizontal direction and n is the vertical one and assume we have a complex k-dimensional vector space $V_{m,n}$ attached to each $(m,n) \in \mathbb{Z}^2$. The transformation properties tell us how to interpret the individual terms. Given that it transforms under conjugation by Λ evaluated at the same lattice site, we naturally interpret $J_{m,n}$ as an endomorphism of V. On the other hand, $(C^1_{m,n},\tilde{C}^1_{m,n})$ and $(C^2_{m,n},\tilde{C}^2_{m,n})$ which transform in a manner dictated by values of the gauge at neighbouring sites we interpret as mapping adjacent vector spaces to each other, that is we interpret them as parallel transporters on the lattice where $C^1_{m,n}$ and $\tilde{C}^1_{m,n}$ map $V_{m,n}$ to $V_{m+1,n}$

whereas $C^2_{m,n}$ and $\tilde{C}^2_{m,n}$ map $V_{m,n}$ to $V_{m,n+1}$. In light of this we assign to each oriented link the pair $(C,\tilde{C})$. Write $(C^1_{m,n},\tilde{C}^1_{m,n})$ for the m-link and $(C^2_{m,n},\tilde{C}^2_{m,n})$ for the n-link and assign the field $J_{m,n}$ to the (m,n) vertex, see figure 6.2. We can push this further and get an interpretation of the Lax pair if we view the space as a vector bundle over $\mathbb{Z}^2$. Thus given $\mathbb{Z}^2$ we choose to consider a vector space $V_{m,n}$ as forming a vector bundle, $\mathbf{V}$, over discrete space $\mathbb{Z}^2$. Then $J_{m,n}$ becomes a section of the corresponding endomorphism bundle (by definition a section of the bundle of endomorphism is the endomorphism of the section of the original bundle) while $(C^i_{m,n},\tilde{C}^i_{m,n})$ $i=1,2$ are discrete analogues of the parallel transport operators (see [90, 92]). Now let $\Gamma(\mathbf{V})$ denote the space of sections of the bundle $\mathbf{V}$, i.e. the set of *sequences* $\Psi_{m,n}$ with $\Psi_{m,n}\in V_{m,n}$, $\forall(m,n)\in\mathbb{Z}^2$. We can then make sense of the formulae for $M_{m,n}$ and $N_{m,n}$ as operators acting on $\Gamma(\mathbf{V})$. This action being

$$
\begin{aligned}
(M\Psi)_{m,n} &= \left(J_{m,n}\tilde{C}^1_{m-1,n}J^{-1}_{m-1,n}+\zeta C^1_{m-1,n}\right)\Psi_{m-1,n},\\
(N\Psi)_{m,n} &= \left(J_{m,n}\tilde{C}^2_{m,n-1}J^{-1}_{m,n-1}+\zeta C^2_{m,n-1}\right)\Psi_{m,n-1}.
\end{aligned}
\tag{6.10}
$$

The condition that $[M,N]=0$, that is the condition for simultaneous eigensections for M and N, for all values of ζ is then equivalent to the non-autonomous ASDYM Bianchi system. In this interpretation we are able to define the curvature on the lattice: on each plaquette the curvature is given by

$$
\begin{aligned}
\Omega &= (C^1_{m,n})^{-1}(C^2_{m+1,n})^{-1}C^1_{m,n+1}C^2_{m,n},\\
\tilde{\Omega} &= (\tilde{C}^1_{m,n})^{-1}(\tilde{C}^2_{m+1,n})^{-1}\tilde{C}^1_{m,n+1}\tilde{C}^2_{m,n},
\end{aligned}
\tag{6.11}
$$

and the two lattice equations (6.8) are $\Omega=1$ and $\tilde{\Omega}=1$. Note though that this is valid for $(C^i_{m,n},\tilde{C}^i_{m,n})$ invertible and we have seen that this is not always the case when interpreted as terms forming the relevant BT (c.f Schlesinger transformation for $\mathrm{P_{VI}}$). However we are here free to consider this system as a discrete equation in its own right, much like was discussed in section 2.3. We then have the gauge invariant quantities given by the trace of a product of link variables along any closed path, see (6.9). The most elementary one is given by the trace of the curvature on a fundamental plaquette, that is $\mathrm{Tr}(\Omega)$ and $\mathrm{Tr}(\tilde{\Omega})$. An obvious observation is that we seem to get two transporters for each direction, a possible interpretation of this is that we view (C^1, C^2) as parallel transporters in some trivialization and $(\tilde{C}^1, \tilde{C}^2)$ in another and these are related as, for example, $C^1_{m,n} = J_{m+1,n}\tilde{C}^1_{m,n}J^{-1}_{m,n}$ (c.f. $A \mapsto g^{-1}Ag + g^{-1}dg$) in which case we should think of the system as in figure 6.3. Equation (6.7) then tells us that there is compatibility of these transformations on a multilayer lattice. This correspondence between the (C^1, C^2) and $(\tilde{C}^1, \tilde{C}^2)$ implies that (6.7) holds identically and also the two equations in (6.8) degenerate into one equation. When this is the case we can give an 'additive potential' form of the equations by letting $C^1_{m,n} = \chi_{m+1,n} - \chi_{m,n}$ and $C^2_{m,n} = \chi_{m,n+1} - \chi_{m,n}$ such that $C^1_{m,n+1}C^2_{m,n} = C^2_{m+1,n}C^1_{m,n}$ yields

$$(\chi_{m+1,n+1} - \chi_{m,n+1})(\chi_{m,n+1} - \chi_{m,n}) = (\chi_{m+1,n+1} - \chi_{m+1,n})(\chi_{m+1,n} - \chi_{m,n}) \tag{6.12}$$

where $\chi_{m,n}$ is a matrix function on the lattice.

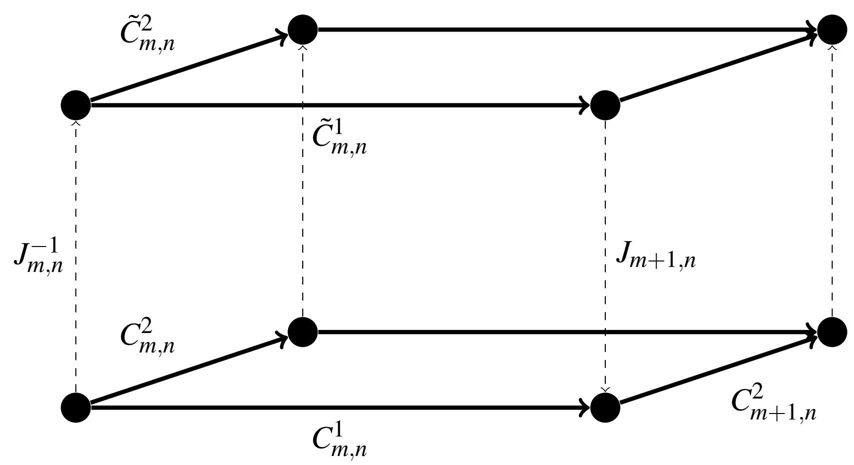

Figure 6.3: Here the J is interpreted linking 'dual' plaquettes. We are still working over $\mathbb{Z}^2$.

More generally consider for instance the M operator (the Darboux matrix) written as

$$M = H_{m+1,n}\tilde{C}^1_{m,n}H^{-1}_{m,n} + \zeta K_{m+1,n}C^1_{m,n}K^{-1}_{m,n},$$

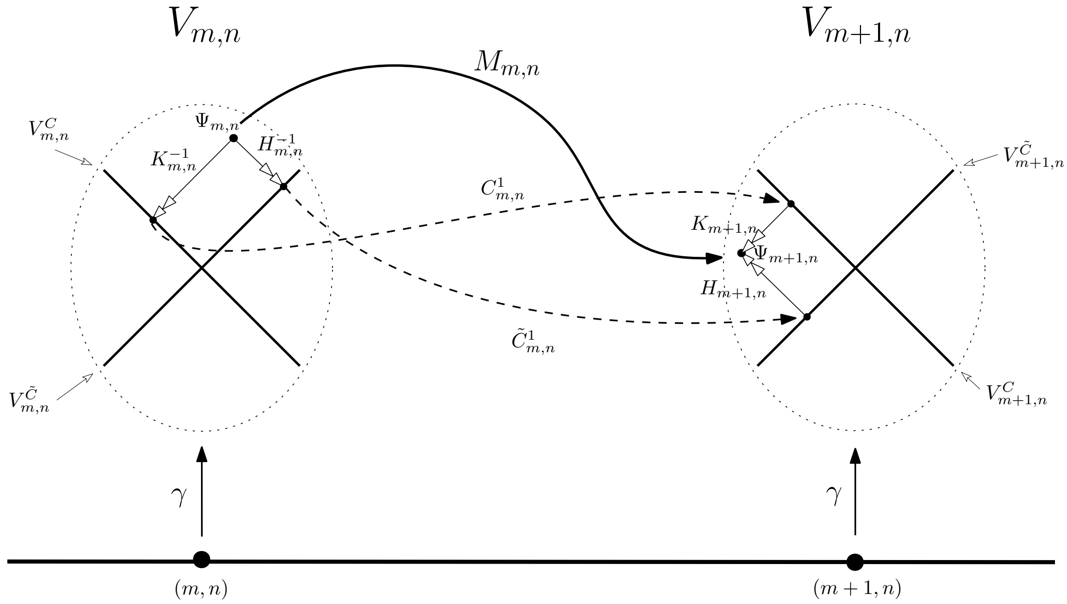

Figure 6.4: Given the description of the action of the Darboux matrix presented in chapter 4 (figure 4.1), we may interpret the action of the Lax equations in the discrete case is a similar fashion.

and recall the discussion of the interpretation of the DM in chapter 4. The gauge $\Psi \mapsto H^{-1}\Psi$ moves us to a gauge in which the potential has no z and w components and then the transformation $H^{-1}\Psi \mapsto \tilde{C}H^{-1}\Psi$ preserves this gauge. Finally gauging with H returns us to a general gauge. Similarly if we originally transform to the gauge $\Psi \mapsto K^{-1}\Psi$ then the action of the transformation given by C preserves the gauge with vanishing $\tilde{z}$ and $\tilde{w}$ components. Looking at figure 6.2 an interpretation of the action of the M operator on the sections of the bundle may be given whereby an element of the fibre at (m,n) is split over subspaces $V^{C}_{m,n}$ and $V^{\tilde{C}}_{m,n}$ on which $\tilde{C}$ and C act, respectively.

Alternatively, since $J \in \mathrm{End}(\mathbf{V})$ one might consider interpreting J as a Higgs field, i.e. an auxiliary field at each lattice location. In this case it would be natural to extend this reasoning and ask what degree of freedom can be introduced such as to view J as a transporter itself. Here again we may be guided by the transformation properties if we generalise the equations (6.9). We do this by observing that the system is invariant under the more general gauge transformation

$$\begin{aligned} C^1_{m,n} &\mapsto \Lambda_{m+1,n} C^1_{m,n} \Lambda^{-1}_{m,n}, \qquad \tilde{C}^1_{m,n} \mapsto \tilde{\Lambda}_{m+1,n} \tilde{C}^1_{m,n} \tilde{\Lambda}^{-1}_{m,n} \\ C^2_{m,n} &\mapsto \Lambda_{m,n+1} C^2_{m,n} \Lambda^{-1}_{m,n}, \qquad \tilde{C}^2_{m,n} \mapsto \tilde{\Lambda}_{m,n+1} \tilde{C}^2_{m,n} \tilde{\Lambda}^{-1}_{m,n} \end{aligned} \tag{6.13}$$

$$J_{m,n} \mapsto \Lambda_{m,n} J_{m,n} \tilde{\Lambda}^{-1}_{m,n}. \tag{6.14}$$

If from this more general gauge invariance we view the $\tilde{}$ as a shift in an independent direction then we see that J itself now transforms as a parallel transporter. In this spirit let us reformulate the system over $\mathbb{Z}^3$ where $J_{l,m,n} : V_{l,m,n} \to V_{l+1,m,n}$ transforms as $J_{l,m,n} \mapsto \Lambda_{l,m,n} J_{l,m,n} \Lambda^{-1}_{l+1,m,n}$.

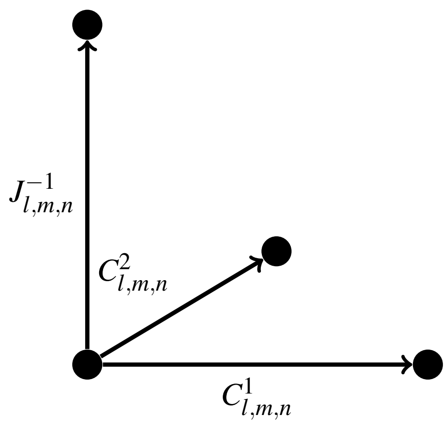

Figure 6.5: J interpreted as a transporter along a new direction. The system (6.15) is now one living on $\mathbb{Z}^3$.

The Bianchi system then becomes

$$\begin{aligned} J_{l+1,m+1,n+1} &\left[C^2_{l+1,m+1,n} J^{-1}_{l,m+1,n} C^1_{l,m,n} - C^1_{l+1,m,n+1} J^{-1}_{l,m,n+1} C^2_{l,m,n} \right] \\ &- \left[C^1_{l,m,n+1} J_{l+1,m,n+1} C^2_{l+1,m,n} - C^2_{l,m+1,n} J_{l+1,m+1,n} C^1_{l+1,m,n} \right] J^{-1}_{l,m,n} = 0, \end{aligned} \tag{6.15}$$

where the $(C^1_{m,n}, C^2_{m,n})$ satisfy a zero curvature type condition on each layer of the lattice

$$C^1_{l,m,n+1} C^2_{l,m,n} = C^2_{l,m+1,n} C^1_{l,m,n}, \qquad \forall\, l \in \mathbb{Z}. \tag{6.16}$$

(6.15) is now a three dimensional discrete system over $\mathbb{Z}^3$. We are yet to study this system further in terms of what reductions it might have and what continuous limits it might possess. One interesting question is whether this system contains the Hirota-Miwa (or discrete KP) equation, [93, 94], a 3-dimensional integrable discrete equation.

Returning to the original 2-dimensional system (6.7)–(6.8), it is possible to rewrite it in an alternative form by noting that (6.8) imply that we can (generically) express the $(C^i_{m,n}, \tilde{C}^i_{m,n})$ matrices as

$$\begin{aligned} C^1_{m,n} &= \alpha^{(1)} h_{m+1,n} h^{-1}_{m,n}, \quad & \tilde{C}^1_{m,n} &= \beta^{(1)} k_{m+1,n} k^{-1}_{m,n}, \\ C^2_{m,n} &= \alpha^{(2)} h_{m,n+1} h^{-1}_{m,n}, \quad & \tilde{C}^2_{m,n} &= \beta^{(2)} k_{m,n+1} k^{-1}_{m,n}, \end{aligned} \tag{6.17}$$

with $\alpha^{(i)}, \beta^{(i)}$, $i = 1, 2$ scalar constants. If $\Phi_{m,n}$ is transformed into the gauge equivalent eigenfunction $\Lambda_{m,n}\Phi_{m,n}$, then $h_{m,n}$ and $k_{m,n}$ can be replaced by $\Lambda_{m,n}h_{m,n}$ and $\Lambda_{m,n}k_{m,n}$. Defining the gauge invariant quantity $\Theta_{m,n} = h^{-1}_{m,n} J_{m,n} k_{m,n}$ equation (6.7) takes the form

$$\frac{\alpha^{(1)}}{\beta^{(1)}} \left[\Theta_{m+1,n+1}\Theta^{-1}_{m+1,n} - \Theta_{m,n+1}\Theta^{-1}_{m,n}\right] = \frac{\alpha^{(2)}}{\beta^{(2)}} \left[\Theta_{m+1,n+1}\Theta^{-1}_{m,n+1} - \Theta_{m+1,n}\Theta^{-1}_{m,n}\right] \tag{6.18}$$

or in conservation form as

$$\alpha^{(1)}\beta^{(2)}\Delta_m \left(\Theta_{m,n+1}\Theta^{-1}_{m,n}\right) = \alpha^{(2)}\beta^{(1)}\Delta_n \left(\Theta_{m+1,n}\Theta^{-1}_{m,n}\right) \tag{6.19}$$

with the linear problem easily obtained by modification of the operators in (6.6) and where Δ_i is the difference operator in the ith direction.

Let us remark that the discrete system (6.7)–(6.8), or (6.18), have a formal similarity with the ASDYM field equations (3.5), or the Pohlmeyer form given by (3.31), respectively. In fact we see that the two zero curvature conditions, $(3.3)_{1,2}$, have formal similarities with (6.8):

$$F_{zw} = 0, \quad F_{\tilde{z}\tilde{w}} = 0, \tag{6.20}$$

$$C^1_{m,n+1} C^2_{m,n} = C^2_{m+1,n} C^1_{m,n}, \qquad \text{and} \qquad \tilde{C}^1_{m,n+1} \tilde{C}^2_{m,n} = \tilde{C}^2_{m+1,n} \tilde{C}^1_{m,n}. \tag{6.21}$$

and the remaining, mixed curvature condition $(3.3)_3$ with (6.7):

$$F_{z\tilde{z}} = F_{w\tilde{w}}, \tag{6.22}$$

$$\begin{aligned} J_{m+1,n+1}\left[\tilde{C}^2_{m+1,n}J^{-1}_{m+1,n}C^1_{m,n} - \tilde{C}^1_{m,n+1}J^{-1}_{m,n+1}C^2_{m,n}\right] \\ = \left[C^1_{m,n+1}J_{m,n+1}\tilde{C}^2_{m,n} - C^2_{m+1,n}J_{m+1,n}\tilde{C}^1_{m,n}\right]J^{-1}_{m,n}. \end{aligned} \tag{6.23}$$

The linear operators also share this similarity with (3.32) corresponding to (6.6)

$$L = J^{-1}\partial_w J - \zeta\partial_{\tilde{z}}, \qquad M = J^{-1}\partial_z J - \zeta\partial_{\tilde{w}}. \tag{6.24}$$

$$M_{m,n} = J_{m+1,n}\tilde{C}^1_{m,n}J^{-1}_{m,n} + \zeta C^1_{m,n}, \;\; N_{m,n} = J_{m,n+1}\tilde{C}^2_{m,n}J^{-1}_{m,n} + \zeta C^2_{m,n}. \tag{6.25}$$

And finally in the potential form due to Pohlmeyer, the local integrability conditions (3.24) are formally analogous to (6.17)

$$\begin{aligned} \partial_w H + A_w H = 0, &\qquad \partial_z H + A_z H = 0, \\ \partial_{\tilde{w}} K + A_{\tilde{w}} K = 0, &\qquad \partial_{\tilde{z}} K + A_{\tilde{z}} K = 0, \end{aligned} \tag{6.26}$$

$$\begin{aligned} C^1_{m,n} = \alpha^{(1)} h_{m+1,n}h^{-1}_{m,n}, &\quad \tilde{C}^1_{m,n} = \beta^{(1)} k_{m+1,n}k^{-1}_{m,n}, \\ C^2_{m,n} = \alpha^{(2)} h_{m,n+1}h^{-1}_{m,n}, &\quad \tilde{C}^2_{m,n} = \beta^{(2)} k_{m,n+1}k^{-1}_{m,n}, \end{aligned} \tag{6.27}$$

each of which allows us to re-write each system in ‘potential’ form

$$\partial_z(J^{-1}(\partial_{\tilde{z}}J)) = \partial_w(J^{-1}(\partial_{\tilde{w}}J)), \tag{6.28}$$

$$\alpha^{(1)}\beta^{(2)}\Delta_m\left(\Theta_{m,n+1}\Theta^{-1}_{m,n}\right) = \alpha^{(2)}\beta^{(1)}\Delta_n\left(\Theta_{m+1,n}\Theta^{-1}_{m,n}\right), \tag{6.29}$$

respectively.

We would like to have a direct correspondence between these (being able to go from discrete to continuous via some continuous limit) but it is clear that this is problematic. While the ASDYM equation are equations in four independent (continuous) variables, the above only have two, or three (if we reinterpret the equation as given in (6.15)) (discrete) independent variables. Therefore, although there are strong similarities between the discrete equations and the ASDYM equations, both given their form and possible interpretation as two zero-curvature conditions and a mixed curvature condition, we are unable to obtain the desired correspondence. It would be interesting, in future work, to study whether there is any twistor-like interpretation of this system where the two discrete zero-curvature conditions correspond to vanishing curvature on α-planes and the third direction, determined by the $J_{m,n}$ matrix, corresponds to the spectral direction. What it means to have a twistor space associated to a discrete space is, however, not at all clear to us and such a search could likely yield no interesting result. Not all is lost however as it is clear that even though the discrete system is not of the correct dimensions to have a limit to the full ASDYM equation it does have the required dimensions to possibly be a discrete version of some 2-dimensional reduction of the ASDYM equations. In which case it would be feasible to conjecture that this be a discrete version of the ASDYM equations in 2 dimensions, that is a reduction of the ASDYM equations by *any* two dimensional subgroup of the conformal group. In this direction let us compute some continuous limits, two continuous limits on (6.18) for two different choices of the parameters $\alpha^{(i)}, \beta^{(i)}$ and a continuous limit on (6.7)–(6.8).

Set $x = hm$ and $y = hn$ and write

$$\Theta_{m,n} = J(x,y).$$

in (6.18). If $\alpha^{(1)} = \alpha^{(2)} = \beta^{(2)} = 1$, $\beta^{(1)} = -1$ then (6.18) becomes

$$\Theta_{m+1,n+1}\Theta^{-1}_{m+1,n} - \Theta_{m,n+1}\Theta^{-1}_{m,n} = \Theta_{m+1,n}\Theta^{-1}_{m,n} - \Theta_{m+1,n+1}\Theta^{-1}_{m,n+1} \tag{6.30}$$

and in the limit $h \to 0$ we recover the chiral equation, [62]:

$$\partial_x\left(J^{-1}\partial_y J\right) + \partial_y\left(J^{-1}\partial_x J\right) = 0. \tag{6.31}$$

Similarly, under the same reduction but with $\alpha^{(1)}\beta^{(2)} = \alpha^{(2)}\beta^{(1)}$ we recover

$$\partial_x\left(J^{-1}\partial_y J\right) - \partial_y\left(J^{-1}\partial_x J\right) = 0, \tag{6.32}$$

which is the field equation of the topological chiral model. For a discussion of the different groups giving the above reductions and a discussion on their differences see [16]. If instead we work with the system (6.7)–(6.8) and again set $x = hm$ and $y = hn$ and write $J = I$ (we can take J constant and then gauge such that $J \mapsto I$) and

$$C^1_{m,n} = hQ(x,y), \quad \tilde{C}^1_{m,n} = I - hA_x(x,y),$$
$$C^2_{m,n} = hP(x,y), \quad \tilde{C}^2_{m,n} = I - hA_y(x,y),$$

and take the limit $h \to 0$ the result is

$$\begin{aligned} F_{xy} = 0, \quad [Q,P] = 0, \\ \partial_y Q - \partial_x P + [P,A_x] + [A_y,Q] = 0, \end{aligned} \tag{6.33}$$

which corresponds to a two dimensional reduction of the ASDYM equation containing both the Boussinesq equation and its generalizations and the n-wave equation ([68]), in addition to the topological chiral model (6.32). In [16] these are reductions by the subgroup denoted H_{SD}. Note that if we set $\Phi_{m,n} = \Phi(x,y)$, then $\Phi_{m+1,n} = \Phi(x,y) + h\partial_x\Phi(x,y) + O(\varepsilon^2)$, $\Phi_{m,n+1} = \Phi(x,y) + h\partial_y\Phi(x,y) + O(\varepsilon^2)$ and in the limit $h \to 0$ the discrete Lax pair (6.6) becomes the Lax pair

$$\begin{aligned} L &= \partial_x + A_x - \zeta Q, \\ M &= \partial_y + A_y - \zeta P, \end{aligned} \tag{6.34}$$

whose compatibility yields the equations (6.33).

6.3 Darboux transformation for the ASDYM Bianchi system

It is natural, given the similarity of the above systems to the continuous one (their Lax pairs being analogous) to ask whether the discrete system itself possesses Darboux transformations. This proceeds along similar lines to the construction of the Darboux matrix for the ASDYM equation. Starting from the discrete linear problem

(6.6)

$$\begin{aligned} m_+\Phi_{m,n} &= \Phi_{m+1,n} = M_{m,n}\Phi_{m,n}, \\ n_+\Phi_{m,n} &= \Phi_{m,n+1} = N_{m,n}\Phi_{m,n}, \end{aligned} \tag{6.35}$$

where m_+ and n_+ denote the shifts in the m and n direction respectively, we look for a Darboux matrix transformation with affine dependence on the spectral parameter transforming the eigenfunction as

$$\hat{\Phi}_{m,n} = \mathbb{D}_{m,n}\Phi_{m,n},$$

with $\mathbb{D}_{mn} = S_{m,n} + \zeta T_{m,n}$ and the hatted term is the transformed quantity. As for the continuous case we ask that the equations for the transformed quantity satisfies a linear problem of the same functional form to (6.35), that is

$$\begin{aligned} m_+\hat{\Phi}_{m,n} &= \hat{M}_{m,n}\hat{\Phi}_{m,n}, \\ n_+\hat{\Phi}_{m,n} &= \hat{N}_{m,n}\hat{\Phi}_{m,n}, \end{aligned}$$

and therefore the Darboux matrix satisfies

$$\begin{aligned} \mathbb{D}_{m+1,n}M_{m,n} &= \hat{M}_{m,n}\mathbb{D}_{m,n}, \\ \mathbb{D}_{m,n+1}N_{m,n} &= \hat{N}_{m,n}\mathbb{D}_{m,n}, \end{aligned} \tag{6.36}$$

which we must solve for $\mathbb{D}_{m,n}$. $M_{m,n}$ and $N_{m,n}$ are nothing but the operators governing the evolution of the discrete system in the m and n direction respectively, explicitly given by

$$\begin{aligned} M_{m,n} &= H_{m+1,n}\tilde{C}^1_{m,n}H^{-1}_{m,n} + \zeta K_{m+1,n}C^1_{m,n}K^{-1}_{m,n}, \\ N_{m,n} &= H_{m,n+1}\tilde{C}^2_{m,n}H^{-1}_{m,n} + \zeta K_{m,n+1}C^2_{m,n}K^{-1}_{m,n}, \end{aligned}$$

from which, upon insertion into (6.36) and equating the coefficients of various powers of ζ to zero we retrieve the six equations

$$S_{m+1,n}H_{m+1,n}\tilde{C}^1_{m,n}H^{-1}_{m,n} = \hat{H}_{m+1,n}\hat{\tilde{C}}^1_{m,n}\hat{H}^{-1}_{m,n}S_{m,n}, \tag{6.37}$$

$$T_{m+1,n}K_{m+1,n}C^1_{m,n}K^{-1}_{m,n} = \hat{K}_{m+1,n}\hat{C}^1_{m,n}\hat{K}^{-1}_{m,n}T_{m,n}, \tag{6.38}$$

$$\begin{aligned}S_{m+1,n}K_{m+1,n}C^1_{m,n}K^{-1}_{m,n} &+ T_{m+1,n}H_{m+1,n}\tilde{C}^1_{m,n}H^{-1}_{m,n}\\ &= \hat{H}_{m+1,n}\hat{\tilde{C}}^1_{m,n}\hat{H}^{-1}_{m,n}T_{m,n} + \hat{K}_{m+1,n}\hat{C}^1_{m,n}\hat{K}^{-1}_{m,n}S_{m,n},\end{aligned} \tag{6.39}$$

$$S_{m,n+1}H_{m,n+1}\tilde{C}^2_{m,n}H^{-1}_{m,n} = \hat{H}_{m,n+1}\hat{\tilde{C}}^2_{m,n}\hat{H}^{-1}_{m,n}S_{m,n}, \tag{6.40}$$

$$T_{m,n+1}K_{m,n+1}C^2_{m,n}K^{-1}_{m,n} = \hat{K}_{m,n+1}\hat{C}^2_{m,n}\hat{K}^{-1}_{m,n}T_{m,n}, \tag{6.41}$$

$$\begin{aligned}S_{m,n+1}K_{m,n+1}C^2_{m,n}K^{-1}_{m,n} &+ T_{m,n+1}H_{m,n+1}\tilde{C}^2_{m,n}H^{-1}_{m,n}\\ &= \hat{H}_{m,n+1}\hat{\tilde{C}}^2_{m,n}\hat{H}^{-1}_{m,n}T_{m,n} + \hat{K}_{m,n+1}\hat{C}^2_{m,n}\hat{K}^{-1}_{m,n}S_{m,n}.\end{aligned} \tag{6.42}$$

To solve the above we first define the functions

$$O_{m,n} = \hat{H}^{-1}_{m,n}S_{m,n}H_{m,n}, \quad \text{and} \quad P_{m,n} = \hat{K}^{-1}_{m,n}T_{m,n}K_{m,n},$$

which, from (6.37), (6.38) and (6.40), (6.41), give us

$$\begin{aligned} O_{m+1,n}\tilde{C}^1_{m,n} &= \hat{\tilde{C}}^1_{m,n}O_{m,n}, \quad & O_{m,n+1}\tilde{C}^2_{m,n} &= \hat{\tilde{C}}^2_{m,n}O_{m,n},\\ P_{m+1,n}C^1_{m,n} &= \hat{C}^1_{m,n}P_{m,n}, \quad & P_{m,n+1}C^2_{m,n} &= \hat{C}^2_{m,n}P_{m,n},\end{aligned} \tag{6.43}$$

respectively.

Consider the autonomous case, that is when $(C^i_{m,n},\tilde{C}^i_{m,n}) \equiv (C^i,\tilde{C}^i)$, $i = 1,\ 2$. In this case, noting that we may use the commutation relations (6.4) to permute the matrices, (6.43) is solved by

$$O_{m,n} = \left(\hat{\tilde{C}}^1\right)^m \left(\hat{\tilde{C}}^2\right)^n \Upsilon \left(\tilde{C}^1\right)^{-m} \left(\tilde{C}^2\right)^{-n}, \tag{6.44}$$

$$P_{m,n} = \left(\hat{C}^1\right)^m \left(\hat{C}^2\right)^n \Gamma \left(C^1\right)^{-m} \left(C^2\right)^{-n}, \tag{6.45}$$

if $(C^i, \tilde{C}^i)$ are invertible, or by

$$O_{m,n} = \left(\hat{\tilde{C}}^1\right)^m \left(\hat{\tilde{C}}^2\right)^n \Upsilon \left(\tilde{C}^1\right)^m \left(\tilde{C}^2\right)^n, \tag{6.46}$$

$$P_{m,n} = \left(\hat{C}^1\right)^m \left(\hat{C}^2\right)^n \Gamma \left(C^1\right)^m \left(C^2\right)^n, \tag{6.47}$$

where $(C^i, \tilde{C}^i)$ are idempotent, i.e. $(C^i)^2 = C^i$ $i = 1,\ 2$ and similar for $\tilde{C}^i$. The remaining equations, (6.39) and (6.42) result in

$$\hat{J}_{m+1,n}\left[O_{m+1,n}J^{-1}_{m+1,n}C^1 - \hat{\tilde{C}}^1\hat{J}^{-1}_{m,n}P_{m,n}\right] = \left[\hat{C}^1\hat{J}_{m,n}O_{m,n} - P_{m+1,n}J_{m+1,n}\tilde{C}^1\right]J^{-1}_{m,n}, \tag{6.48}$$

and

$$\hat{J}_{m,n+1}\left[O_{m,n+1}J^{-1}_{m,n+1}C^2 - \hat{\tilde{C}}^2\hat{J}^{-1}_{m,n}P_{m,n}\right] = \left[\hat{C}^2\hat{J}_{m,n}O_{m,n} - P_{m,n+1}J_{m,n+1}\tilde{C}^2\right]J^{-1}_{m,n}, \tag{6.49}$$

respectively. These are the equations giving the transformations of the discrete system in the autonomous case.

As for the continuous Darboux transformations we may ask for permutability of the above transformations, thus let us view the above discrete Darboux matrix as a discrete Lax pair, that is let

$$\begin{aligned}
\hat{\Phi}_{m,n} &= \left(\hat{H}_{m,n}\hat{O}_{m,n}H^{-1}_{m,n} + \zeta\hat{K}_{m,n}\hat{P}_{m,n}K^{-1}_{m,n}\right)\Phi_{m,n},\\
\bar{\Phi}_{m,n} &= \left(\bar{H}_{m,n}\bar{O}_{m,n}H^{-1}_{m,n} + \zeta\bar{K}_{m,n}\bar{P}_{m,n}K^{-1}_{m,n}\right)\Phi_{m,n},
\end{aligned} \tag{6.50}$$

and ask for $\hat{\bar{\Phi}}_{m,n} = \bar{\hat{\Phi}}_{m,n}$. The result are equations analogous to the original Bianchi system, in fact the equations take the same form but where now the $(C^i, \tilde{C}^i)$ remain unshifted, thus effectively becoming parameters while the parameters $(\hat{\Upsilon}, \hat{\Gamma})$ and $(\bar{\Upsilon}, \bar{\Gamma})$ are promoted to 'variables'. The equations are

$$\begin{aligned}
\hat{\Upsilon}\bar{\Upsilon} &= \bar{\Upsilon}\hat{\Upsilon},\\
\hat{\Gamma}\bar{\Gamma} &= \bar{\Gamma}\hat{\Gamma},
\end{aligned} \tag{6.51}$$

and

$$\hat{\bar{J}}_{m,n}\left[\hat{O}_{m,n}\bar{J}^{-1}_{m,n}\bar{P}_{m,n}-\bar{O}_{m,n}\hat{J}^{-1}_{m,n}\hat{P}_{m,n}\right]=\left[\bar{P}_{m,n}\hat{J}_{m,n}\hat{O}_{m,n}-\hat{P}_{m,n}\bar{J}_{m,n}\bar{O}_{m,n}\right]J^{-1}_{m,n}. \quad (6.52)$$

That is we see that the variables given by the links and the free-parameters should be considered on the same footing.

6.4 Some reductions of the ASDYM Bianchi system

The idea behind the construction of the ASDYM Bianchi system was the hope that from such system one could recover, by direct reduction, the lattice equations which themselves are born from the permutability of BTs for equations obtained from the ASDYM equations via reduction. In the next two subsections we show that this is indeed the case; inserting the Yang matrix for the SG reduction into the Bianchi system gives the permutability theorem for the SG equation directly. A similar reduction results in the discrete SG equation of Hirota. Furthermore, in the non-autonomous case we have a reduction to the non-autonomous lattice modified KdV (lmKdV) equation. This equation is known to have itself a reduction to the q-difference Painlevé VI equation and therefore, from our framework, one can then re-interpret this as arising from the BTs of the SG equation. Although at the J-matrix reduction level this correspondence is clear, it is not clear how to obtain this from the BTs of the SG equation directly. In fact, attempting this naively from the BTs of the SG equation forces an autonomous reduction of qP_{VI} where q is a root of unity.

6.4.1 From the autonomous ASDYM Bianchi system to the permutability theorem for the SG equation

Consider the autonomous ASDYM Bianchi system

$$J_{m+1,n+1}\left[\tilde{C}^2J^{-1}_{m+1,n}C^1-\tilde{C}^1J^{-1}_{m,n+1}C^2\right]=\left[C^1J_{m,n+1}\tilde{C}^2-C^2J_{m+1,n}\tilde{C}^1\right]J^{-1}_{m,n}, \quad (6.53)$$

together with $[C^{(1)}, C^{(2)}] = [\widetilde{C}^{(1)}, \widetilde{C}^{(2)}] = 0$. Substituting

$$J_{m,n} = U(\tilde{w}) \begin{pmatrix} \mathrm{e}^{i\theta_{m,n}/2} & 0 \\ 0 & \mathrm{e}^{-i\theta_{m,n}/2} \end{pmatrix} V(w),$$

where U and V are given by is given by (4.17) and (4.18), into the permutability equation (6.53) with

$$C^{(j)} = b_j \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \widetilde{C}^{(j)} = \alpha_j \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

gives

$$\begin{aligned} &\sin\left(\frac{\theta_{m+1,n} + \theta_{m,n}}{2}\right) - \sin\left(\frac{\theta_{m,n+1} + \theta_{m+1,n+1}}{2}\right) \\ &+ \kappa\left[\sin\left(\frac{\theta_{m+1,n} + \theta_{m+1,n+1}}{2}\right) - \sin\left(\frac{\theta_{m,n+1} + \theta_{m,n}}{2}\right)\right] = 0, \end{aligned}$$

with $\kappa = b_2\alpha_1/b_1\alpha_2$. This is equivalent to the standard form

$$\tan\left(\frac{\theta_{m+1,n+1} - \theta_{m,n}}{4}\right) = \frac{\kappa+1}{\kappa-1}\tan\left(\frac{\theta_{m,n+1} - \theta_{m+1,n}}{4}\right),$$

of the Bianchi permutability theorem for the sine-Gordon equation.

6.4.2 Reduction of the autonomous ASDYM Bianchi system to the Hirota discrete SG equation

Let

$$C^{(1)}_{m,n} = \frac{1}{\alpha}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C^{(2)}_{m,n} = \frac{1}{\beta}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \widetilde{C}^{(1)}_{m,n} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \widetilde{C}^{(2)}_{m,n} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$J_{m,n} = \begin{pmatrix} u_{m,n} & 0 \\ 0 & 1/u_{m,n} \end{pmatrix}.$$

The autonomous Bianchi system (6.7) then reduces to

$$\frac{1}{\beta}(u_{m+1,n}u_{m,n+1} - u_{m+1,n+1}u_{m,n}) + \frac{1}{\alpha}(u_{m+1,n+1}u_{m+1,n}u_{m,n+1}u_{m,n} - 1) = 0, \quad (6.54)$$

which is Hirota's discrete SG equation, [95].

6.4.3 Reduction of the non-autonomous ASDYM Bianchi system to non-autonomous lattice modified KdV

Let us consider the system (6.7)–(6.8) independently of any connection with the ASDYM equations. Let

$$C^{(1)}_{m,n} = \frac{1}{\alpha_m}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C^{(2)}_{m,n} = \frac{1}{\beta_n}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \widetilde{C}^{(1)}_{m,n} = \widetilde{C}^{(2)}_{m,n} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and

$$J_{m,n} = \begin{pmatrix} u_{m,n} & 0 \\ 0 & 1/u_{m,n} \end{pmatrix}.$$

The nonautonomous Bianchi system (6.7) then reduces to

$$\alpha_m(u_{m,n}u_{m+1,n} - u_{m,n+1}u_{m+1,n+1}) - \beta_n(u_{m,n}u_{m,n+1} - u_{m+1,n}u_{m+1,n+1}) = 0. \quad (6.55)$$

Equation (6.55) is known as the nonautonomous lattice mKdV equation [54].

Note that we may try to interpret this in terms of Bäcklund transformations. For this note that symmetry reductions of the ASDYM equations in the original variables (A_μ) lead to reductions in Yang's form where J has the 'dressed' form $J = AGB$, where G depends on the variables that will be the independent variables of the reduced equations and the 'dressing matrices' A and B depend on some auxiliary combination of variables that do not appear in the reduced equation.

Substituting the form $J_{m,n} = AG_{m,n}B$ into the nonautonomous Bianchi permutability equation (6.7) gives

$$G^{-1}_{m+1,n+1}\left(D^{(2)}_{m+1,n}G_{m+1,n}\widetilde{D}^{(1)}_{m,n} - D^{(1)}_{m,n+1}G_{m,n+1}\widetilde{D}^{(2)}_{m,n}\right)$$

$$+\left(\widetilde{D}^{(2)}_{m+1,n}G^{-1}_{m+1,n}D^{(1)}_{m,n}-\widetilde{D}^{(1)}_{m,n+1}G^{-1}_{m,n+1}D^{(2)}_{m,n}\right)G_{m,n}=0,$$

where $D^{(j)}=A^{-1}C^{(j)}A$ and $\widetilde{D}^{(j)}=B\widetilde{C}^{(j)}B^{-1}$.

Furthermore, the conditions

$$C^{(1)}_{m,n+1}C^{(2)}_{m,n}=C^{(2)}_{m+1,n}C^{(1)}_{m,n}\quad\text{and}\quad\widetilde{C}^{(1)}_{m,n+1}\widetilde{C}^{(2)}_{m,n}=\widetilde{C}^{(2)}_{m+1,n}\widetilde{C}^{(1)}_{m,n}$$

become

$$D^{(1)}_{m,n+1}D^{(2)}_{m,n}=D^{(2)}_{m+1,n}D^{(1)}_{m,n}\quad\text{and}\quad\widetilde{D}^{(1)}_{m,n+1}\widetilde{D}^{(2)}_{m,n}=\widetilde{D}^{(2)}_{m+1,n}\widetilde{D}^{(1)}_{m,n}.$$

Now we return to the problem of interpreting the nonautonomous lattice mKdV (6.55) in terms of Bäcklund transformations. The ASDYM Bianchi system (6.7) with

$$J_{m,n}=F(\tilde{w})\begin{pmatrix}u_{m,n}(z,\tilde{z}) & 0\\ 0 & \frac{1}{u_{m,n}(z,\tilde{z})}\end{pmatrix}F(w)^{-1}$$

and

$$C^{(1)}_{m,n}=\frac{1}{\alpha_m}\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad C^{(2)}_{m,n}=\frac{1}{\beta_n}\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \widetilde{C}^{(1)}_{m,n}=\widetilde{C}^{(2)}_{m,n}=\begin{pmatrix}0&1\\1&0\end{pmatrix},$$

again gives the nonautonomous lattice mKdV (6.55). However this is now a statement about BTs of a reduction of the ASDYM equations (specifically the sine-Gordon equation with $u_{m,n}(z,\tilde{z})=\mathrm{e}^{i\theta_{m,n}(z,\tilde{z})/2}$). What is noteworthy here is that Ormerod [96] has shown that the non-autonomous lattice modified KdV has a reduction to $\mathrm{qP_{VI}}$. This leads to the natural question - can we reinterpret the $\mathrm{qP_{VI}}$ as governing the evolution of solutions to the SG equation related by BTs under some constraint?

Chapter 7

Painlevé test for the ASDYM equations

7.1 The Painlevé property and Painlevé tests

The Painlevé property (PP) describes a particular singularity structure of solutions to certain differential equations (DE) in the complex plane. In this respect there is a stark difference between linear equations and nonlinear ones in their possible singularity structure. Solutions of linear DEs have singularities (that is poles, branch points, essential singularities) only where the coefficients of the equation have singularities and these singularities are termed ‘fixed’ since their positions do not move as the initial conditions are varied. In contrast, the solutions of nonlinear equations can develop singularities at arbitrary points depending on the initial conditions. For example, the equation $dw/dz = 1/2w$ has solution $w = \sqrt{z-c}$ and we notice that the solution has a branch point at $z = c$, thus the singularity’s location will depend on the initial conditions. We call these singularities, which depend on the initial conditions, ‘movable’ singularities. It is a fact that, generally, one cannot avoid a solution from developing movable poles, however, one might attempt to classify all equations whose solutions are single-valued about all movable singularities, i.e. which have no movable singularities *other* than poles. Stated differently, look to classify equations whose branch points and essential singularities of all solutions are fixed (see 3.2.3). This is what is known as the Painlevé property. The classification for differential equations of the second order of the form

$$\frac{d^2w}{dz^2} = F(\frac{dw}{dz}, w, z), \tag{7.1}$$

where F is rational in dw/dz, algebraic in w and locally analytic in z with such property was done by Painlevé and Gambier. They found 50 such equations, all of which could be integrated in terms of previously knows functions with the exception of six. These six *Painlevé equations* we met in earlier chapters and their solutions are what we call the *Painlevé transcendents*, [7] and [29].

Interest in systems with the Painlevé property in the framework of integrable systems dates back to the work of S. Kovalevskaya on the motion of a heavy rigid body spinning about a fixed point, [36]. The same problem was studied by both Euler and Lagrange and each determined conditions on the parameters in the problem such that the required number of integrals of motion could be found and thus the equations be integrated. With the exception of these two specific cases (Euler's case and Lagrange's case) one other integrable case was known (what is called the case of complete kinetic symmetry). Kovalevskaya, in search of novel integrable cases, reasoned on what these integrable cases shared in common. In both the Euler and Lagrange case the solutions, obtained with the help of elliptic functions, turned out to be single valued functions of t and had the additional property that, for finite values of t, no singular points other than poles were present. She thus looked for Laurent series solutions of the systems, i.e. solutions of the form

$$w(z) = (z - z_0)^m \sum_{j=0}^{\infty} a_j (z - z_0)^j, \tag{7.2}$$

with m a negative integer, and indeed was able to show that solutions of that form are only possible in four cases. Three of these were precisely the ones already known, the fourth is what is now called the Kovalevskaya case. She was therefore able to determine an additional integrable case via analysis of the singularity structure of solutions. This might possibly be, in this respect, her greatest contribution to the study of integrable systems - the development of a test for integrability which may be applied directly to the equations of motion without need to explicitly determine the integrals of motion and, further, the idea of projecting the problem on the complex plane. That is, making the time parameter a complex variable. It is worth pointing out that this method does not use information of the explicit solution. Rather, it probes the structural properties of the equation to determine how complex the solutions might be.

The singularity analysis performed by Kovalevskaya, now called the Painlevé test (PT), has been used successfully to predict integrable cases of a variety of systems and the idea, outlined implicitly above, is as follows: expand every solution of the differential equation in an infinite series near a movable singularity of the equation, i.e. expand as

$$w(z) = (z - z_0)^m \sum_{j=0}^{\infty} a_j (z - z_0)^j,$$

with z_0 the arbitrary location of the singularity and m the leading power which has to be found. The equation is then assumed to have the PP if the series is self-consistent, single-valued and contains a sufficient number of degrees of freedom to describe all possible solutions or the general solution. These only give necessary conditions for the PP property to hold and no mention has been made on the issue of convergence of the series. Note also that the series expansion and the techniques implemented are analogues of the Frobenius expansion for linear ODEs. This test is certainly not without complications nor defects, for instance the PP is not invariant under change of variables. Indeed, this property is easily destroyed by even simple transformations of the dependent variables. For example ([38]) solutions $u(z)$ of some ODE with movable simple poles are transformed to solutions, $w(z)$, with movable branch points under $u(z) \mapsto w^2(z)$. What is more, the form in which the equation is presented will also be important in order to investigate the PP. For instance, to implement the test on the scaling reduction of the sine-Gordon equation one must first transform the equation to one in rational form.

Ablowitz et al., [37], conjectured that when one can reduce an integrable PDE to a system of ODEs by exact similarity reduction, then the solution (of the ODE) will possess the Painlevé property. Furthermore, they conjectured that when *all* the ODEs obtained by exact similarity transformations from a given PDE have the Painlevé property, then the PDE will be 'integrable'. A complication here is that most PDEs do not have any symmetries and therefore cannot be reduced to ODEs. We shall, in the next subsection, show how a more direct test can be applied to PDEs directly. The standard test as applied to ODEs can be found in a variety of references see, for example, [12, 38]. We then apply such test to the ASDYM equations. In performing the PT on the ASDYM equations we shall describe how the test had already been implemented by Kruskal et al. in [40] but with a chosen expansion which was trivial in the sense that it was taken as a combination resulting in an analytic expansion. Thus their test never probed the issue of the singular structure of solutions as the chosen expansion meant that the solutions searched were essentially analytic. We apply the test with an alternative expansion and show that the ASDYM equation does indeed pass the PT for PDEs. The PDE PT has also been a successful method for the construction of the BTs of the PDE under analysis. Our work is principally motivated by the hope to reproduce the BT for the ASDYM equation from such singularity analysis.

7.1.1 The Painlevé test for PDEs

Carnevale, Tabor and Weiss, [39], proposed a method to apply the Painlevé test to PDEs directly. One of the main differences between the test as applied to PDEs is that, in general, for PDEs, the singularities of the solution cannot be isolated. Thus, if $f(z_1,...,z_n)$ is a meromorphic function of n complex variables, the singularities of f develop along analytic manifolds. We can specify these manifolds (locally) by conditions of the form

$$\phi(z_1,...,z_n) = 0, \tag{7.3}$$

where ϕ is an analytic function of $(z_1,..,z_n)$ in a neighbourhood of the manifold. Hence we say that a PDE has the Painlevé property when its solutions are 'single-valued' about all movable singularity manifolds. We also require that the singularity manifold be non-characteristic (this is a manifold on which one may freely specify the Cauchy initial data) such as to avoid the singularities from propagating during the evolution of the system, [10]. Consider the simple case of the wave equation

$$\phi_{tt} - \phi_{xx} = 0$$

with general solution $\phi(x,t) = f(x-t) + g(x+t)$. We see that f and g may have any type of singularity on $t - x = \kappa_1$ and $t + x = \kappa_2$ and this is why the above statement says nothing about the singular behaviour of solutions on the characteristic manifold. Let $u = u(z_1, ..., z_n)$ be a solution of the PDE in question and let the singularity manifold be determined by (7.3). Assume that

$$u(z_1, ..., z_n) = \phi^{\alpha} \sum_{j=0}^{\infty} u_j \phi^j, \tag{7.4}$$

is a solution of the PDE where $u_j = u_j(z_1, ..., z_n)$ and $\phi = \phi(z_1, ..., z_n)$ are analytic functions of $(z_1, ..., z_n)$ in a neighbourhood of the singularity manifold (7.3), and α is an integer. Substituting (7.4) into the PDE in question will determine, by the principle of dominant balance (see [12] and references therein) the possible values of α and also defines the recursion relations for u_j, $j = 0, 1, 2, 3, ...$ Again, by the Cauchy-Kovalevskaya theorem, such an expansion must have as many arbitrary functions as the order of the system. The procedure is in close analogy to that for ODEs.

This method has been applied successfully to a great number of well known PDEs and has been successful in determining those equations which reduce, via symmetry reduction, to ODEs which themselves possess the Painlevé property. In addition, this method often allows for Bäcklund transformations and Lax pairs to be constructed by truncation of the series expansion and we intend to use the test applied to the ASDYM equations to recover both the BT constructed in chapter 4 and the ASDYM Lax pair.

7.1.1.1 An example: Burgers equation

Let us assume that

$$u = \phi^{\alpha} \sum_{j=0}^{\infty} u_j \phi^j, \tag{7.5}$$

is a solution to Burgers equation

$$u_t + u u_x = \sigma u_{xx}, \tag{7.6}$$

with $\phi = \phi(x,t)$, $u_j = u_j(x,t)$ analytic functions of (x,t) in the neighbourhood of $M = \{(x,t) : \ \phi(x,t) = 0\}$, [39, 97]. Leading order behaviour, i.e. $u \sim u_0 \phi^{\alpha}$ with $\alpha < 0$, tells us that

$$\alpha = -1. \tag{7.7}$$

One may then obtain, by equating coefficients of powers of ϕ, a recursion relation for u_j by the requirement that terms be balanced. Indeed, collecting terms, one finds that

$$\sigma \phi_x^2 (j-2)(j+1) u_j = F(u_{j-1}, ..., u_0, \phi_t, \phi_x, \phi_{xx}, ...) \tag{7.8}$$

for $j = 0, 1, 2, 3, ...$

We see that the recursion relations (7.8) are undefined when $j = -1, 2$. These values of j are what are called the *resonances*. The *resonances* are the values of j at which arbitrary functions appear in the expansion and it is evident from the above example that for every *positive* resonance there is a compatibility condition which must be identically satisfied in order that (7.6) has a solution of the form (7.5). The resonance at $j = -1$ corresponds to the arbitrary singularity manifold, ϕ (i.e. define $u_j = 0$ for $j < 0$)[1]. That at $j = 2$, on the other hand, introduces the arbitrary function u_2 and gives us a form of compatibility condition on the functions (ϕ, u_0, u_1) that requires the right-hand side of (7.8) to vanish identically if the expansion is to work. Expressing the terms in the sequence explicitly we find that the term at $j = 1$ induces the compatibility condition to be satisfied and thus the equation passes the test.

[1] In fact, the one at -1 is obviously outside the range of meaningful values of n.

7.1.2 Painlevé test for the ASDYM equations

Recall that when the gauge group is $SU(2)$ we can parametrise J as

$$J = \frac{1}{f}\begin{pmatrix} 1 & g \\ e & f^2+ge \end{pmatrix}, \tag{7.9}$$

and the ASD equations are given by

$$f\Box f = \nabla f \cdot \bar{\nabla} f - \nabla g \cdot \bar{\nabla} e, \tag{7.10}$$

$$f\Box e = 2\nabla f \cdot \bar{\nabla} e, \tag{7.11}$$

$$f\Box g = 2\nabla g \cdot \bar{\nabla} f, \tag{7.12}$$

where $\nabla = (\partial_w, \partial_z)$, $\bar{\nabla} = (\partial_{\tilde{w}}, -\partial_{\tilde{z}})$ and $\Box = (\partial_{w\tilde{w}} - \partial_{z\tilde{z}})$.

Let us look for solutions with polar singularity

$$f = \sum_{i=0}^{\infty} f_i \phi^{i+1}, \;\; g = \sum_{i=0}^{\infty} g_i \phi^i, \;\; e = \sum_{i=0}^{\infty} e_i \phi^i, \tag{7.13}$$

where $\phi = \phi(z,w,\tilde{z},\tilde{w})$ is the singular set called the singularity manifold. More specifically, the singularities occur along an analytic manifold of real dimension $2(4)-2=6$ and this manifold is determined by the condition that $\phi(z,w,\tilde{z},\tilde{w}) = 0$, where ϕ is an analytic function of $(z,w,\tilde{z},\tilde{w})$ in a neighbourhood of the manifold. By the Cauchy-Kovalevski theorem, we require the above solutions to contain 6 arbitrary functions (including ϕ).

Upon substitution of (7.13) into (7.10)–(7.12), the leading terms ($O(1)$) produce

$$\begin{aligned} f_0^2\eta - (\nabla e_0 + e_1\bar{\nabla}\phi)(\nabla g_0 + g_1\nabla\phi) &= 0, \\ 2f_0\nabla\phi(\bar{\nabla} e_0 + e_1\bar{\nabla}\phi) &= 0, \\ 2f_0\bar{\nabla}\phi(\nabla g_0 + g_1\nabla\phi) &= 0, \end{aligned} \tag{7.14}$$

where $\eta = \nabla\phi \cdot \bar{\nabla}\phi$ and we assume that $\eta \neq 0$ such that the variety $\{\phi = 0\}$ is non-characteristic. From these we then have

$$e_1\eta = -\bar{\nabla}e_0 \cdot \nabla\phi, \tag{7.15}$$

$$g_1\eta = -\nabla g_0 \cdot \bar{\nabla}\phi, \tag{7.16}$$

$$(f_0\eta)^2 = (\bar{\nabla}e_0 \cdot \nabla g_0)\eta - (\bar{\nabla}\phi\bar{\nabla}g_0)(\bar{\nabla}e_0 \cdot \nabla\phi), \tag{7.17}$$

from which we see that e_0, g_0 and ϕ are arbitrary functions which will determine f_0, e_1 and g_1.

The first order terms result in

$$\begin{aligned}(f_0f_1 - e_1g_2 - g_1e_2)\eta - g_2\bar{\nabla}e_0 - e_2\nabla g_0\bar{\nabla}\phi \\ = \frac{1}{2}\left(f_0^2\bar{\nabla}\cdot\nabla\phi + (\bar{\nabla}e_0 + e_1\bar{\nabla}\phi)\nabla g_1 + (\nabla g_0 + g_1\nabla\phi)\bar{\nabla}e_1\right),\end{aligned} \tag{7.18}$$

$$\begin{aligned}2f_0\eta e_2 = f_0\left(\nabla\cdot\bar{\nabla}e_0 + \nabla e_1\cdot\bar{\nabla}\phi - \nabla\phi\cdot\bar{\nabla}e_1 + e_1\nabla\cdot\bar{\nabla}\phi\right) \\ -2\left(\bar{\nabla}e_0\cdot\nabla f_0 + (\bar{\nabla}\phi\cdot\nabla f_0)e_1\right),\end{aligned} \tag{7.19}$$

$$\begin{aligned}2f_0\eta g_2 = f_0\left(\nabla\cdot\bar{\nabla}g_0 + \bar{\nabla}g_1\cdot\nabla\phi - \bar{\nabla}\phi\cdot\nabla g_1 + g_1\nabla\cdot\bar{\nabla}\phi\right) \\ -2\left(\nabla g_0\cdot\bar{\nabla}f_0 + (\nabla\phi\cdot\bar{\nabla}f_0)g_1\right),\end{aligned} \tag{7.20}$$

from which we see that f_1, e_2 and g_2 are also determined.

The second order terms, after simplification through (7.15) – (7.17) give equations relating ϕ, f_0, f_1, e_0, e_1, e_2, g_0, g_1 and g_2 and where the variables f_2, e_3 and g_3 vanish. These equations are satisfied given the use of (7.15) – (7.20) and therefore the resonance conditions are indeed satisfied.

In conclusion, we have shown that the ASDYM equations do indeed allow solutions of the form (7.13) with the 6 required arbitrary functions $\{\phi, e_0, g_0, f_2, e_3, g_3\}$ which then fully determine all future coefficients. Jimbo, Kruskal and Miwa in [40] performed a similar analysis concluding that the ASDYM equations pass the PT. However in that work the authors take the expansion to be

$$f=\sum_{i=0}^{\infty} f_i\phi^{i-m},\ \ g=\sum_{i=0}^{\infty} g_i\phi^{i-m},\ \ e=\sum_{i=0}^{\infty} e_i\phi^{i-m}, \tag{7.21}$$

which, referring back to the form of the J-matrix in (7.9), we see yields components which are analytic. In fact for the above expansion the terms $1/f$, g/f and e/f furnish combinations resulting in analytic expansions and therefore the Cauchy-Kovalevskaya requirements are satisfied automatically. This is a good example of one of the issues intrinsic in the PT; it is dependent on the form in which the equations analysed are presented. Looking at (7.10)–(7.12) it is in fact not obviously clear (without considering different forms of the equations) that such expansion, (7.21), should result in testing the system for analytic solutions rather than meromorphic.

We have already remarked that knowledge of the relevant expansions has been successfully used to recover the Bäcklund transforms, the linearising transforms and the Lax pairs of the PDEs being tested. In future work we will attempt to recover the above properties from the expansions found in this chapter. Reproducing the BT for the ASDYM equations found in chapter 4 from such analysis would link up the work in the rest of the thesis in a harmonious way in addition to giving it closure.

Chapter 8

Summary and Outlook

The work in this thesis presents the construction of a transformation for the ASDYM equations of considerable generality allowing it to inherit rich features induced by the ASDYM equations. The transformation exhibits properties analogous to the ASDYM equations in that it may be considered a 'master' Bäcklund transformation. Implications of such construction are studied and presented in two main parts.

After introducing the relevant background, the first part is devoted to the construction of a Bäcklund transformation for the ASDYM equations arising from a Darboux matrix affine in the spectral parameter acting on solutions of the associated linear problem, [1, 2]. It is then shown that symmetry reduction of this transformation leads to the relevant transformations of the reduced equations such that the following diagram commutes:

$$
\begin{tikzcd}
\text{ASD Yang-Mills} \arrow[r, "\mathbb{D}"] \arrow[d, dashed, "\text{reduction}"'] & \text{ASD Yang-Mills BT} \arrow[d, dashed, "\text{reduction}"] \\
\text{reduced equation} \arrow[r] & \text{BT for reduced equation.}
\end{tikzcd}
$$

Figure 8.1

Two 'transporters', $C(z,w)$ and $C(\tilde{z},\tilde{w})$, appear in the Darboux matrix and the Bäcklund transformation which encode information associated to the specific transformation. These are responsible for injecting the Bäcklund parameters when the

BT is *parametric*, determining the relevant Schlesinger action in the transformations for the Painlevé equations in addition to 'absorbing' gauge information in such a way to make the transformation gauge invariant. Analysis of the geometry of these matrices has not been performed and is intended as future work, however it seems to be the case that the functional dependence of each on half the independent variables allows for the transformed solution to the ASDYM equation to inherit factorizability from that of the 'seed' solution, $J = K^{-1}H$. The first part ends with a presentation of how the coalescence cascade property of the Painlevé equations may be interpreted as arising, if one exploits the geometry of the ASDYM equations, from the confluence of the relevant elements of the 'Painlevé groups' associated with each reduction. This process allows to construct explicitly the limit process by which all Painlevé equations for the partitions $\lambda \neq (1,1,1,1)$ can be obtained from the system, $\mathrm{P_{VI}}$, of type $\lambda = (1,1,1,1)$. Such confluence may be lifted to the Darboux matrix representing the relevant Schlesinger transformation and therefore a limit of Schlesinger transformations from one Painlevé equation to another is recovered, that is that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{P_A} \arrow[r] \arrow[d, dashed, "\text{coalescence limit}"'] & \mathscr{S}_{\mathrm{P_A}} \arrow[d, dashed, "\text{coalescence limit}"] \\
\mathrm{P_B} \arrow[r] & \mathscr{S}_{\mathrm{P_B}}.
\end{tikzcd}$$

Figure 8.2: Here $\mathrm{P_A}$ refers to some Painlevé equation and $\mathrm{P_B}$ another Painlevé equation adjacent in confluence to $\mathrm{P_A}$. $\mathscr{S}_{\mathrm{P_A}}$ refers to some Schlesinger transformation for $\mathrm{P_A}$ and similar for $\mathscr{S}_{\mathrm{P_B}}$. In our framework the operation of taking the limit and obtaining the relevant Schlesinger transformation commute.

We stress that within our framework the geometric properties of the ASDYM equations are the crucial aspects allowing the unification of a variety of properties of the integrable sub-systems contained within the self-duality equations. In particular the extension of the coalescence cascade to the Schlesinger transformations for the

Painlevé is natural and likely able to shed more light onto the relationship between the continuous and discrete versions of the equations.

Motivated by the result of Bianchi on the permutability of Bäcklund transformations for the SG equation and by the more recent reinterpretation of such algebraic relation as a lattice equation, in the second part we make use of the Bäcklund-Darboux transformation for the construction of a general, $N \times N$ matrix lattice equation over $\mathbb{Z}^2$. This system is best understood in terms of lattice gauge theory and has continuous limits to important two-dimensional reductions of the ASDYM equations. The transformation used for this construction, as shown in the first part of the thesis, encodes the Bäcklund transformations of the reduced equations and, therefore, the recovered discrete system encodes within it the lattice equations which arise from permutability of the transformations for the reduced equations. By reduction then one may recover the relevant lattice equations, that is the following diagram commutes:

$$\begin{tikzcd}
\text{ASDYM BT} \arrow[r, "\text{permutability}"] \arrow[d, dashed, "\text{reduction}"'] & \text{ASDYM Bianchi system} \arrow[d, dashed, "\text{reduction}"] \\
\text{reduced BT} \arrow[r, "\text{permutability}"'] & \text{lattice equation.}
\end{tikzcd}$$

Figure 8.3: Reduction of the ASDYM Bianchi system gives lattice equations arising from permutability of the BTs for the reduced equations.

After completion of this work we were made aware of interesting work by Fordy and Xenitidis ([98]) in which the authors study a class of $\mathbb{Z}_N$ graded discrete autonomous Lax pairs with $N \times N$ matrices which may be considered a special case of our system given that we do not restrict the matrices in the Lax pair to be cyclic of level k (see [98]) and that our system may be generalised to be non-autonomous. In this work the authors give a partial classification of the reductions of the graded Lax pair. The potential forms of our system, i.e. (6.12) and (6.19), are used in their work for certain reductions, e.g. reduction to the discrete potential KdV. Furthermore the authors present a rich list of equations which are contained in the ASDYM Bianchi system such as the discrete modified Boussinesq ([99]), the discrete Boussinesq ([99]), the N-component nonlinear superposition formula for the 2 dimensional Toda lattice, the discrete potential KdV ([59, 100]), the discrete Scwarzian KdV equation ([59, 100]), Hirota's discrete KdV equation, the modified Hirota-Satsuma equation ([101]) to mention a few. Furthermore, there are many examples of new discrete equations that arise as reductions. Particularly interesting is their study concerning the non-local symmetries of the discrete system which from our perspective seem to be yielding special cases of the ASDYM Lax pair. We intend to develop work in this direction.

Appendix A

Other confluences

A.0.1 $P_{\mathrm{V}} \to P_{\mathrm{IV}}$

The element corresponding to $\mathrm{P_V}$ is associated to the partition $\lambda = (\lambda_0, \lambda_1, \lambda_2) = (2,1,1)$, thus

$$\mathfrak{P}_{\mathrm{V}} = \begin{pmatrix} a & b & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} = (aI_2 + b\Lambda_2^1) \oplus (cI_1) \oplus (dI_1), \tag{A.1}$$

while that for $\mathrm{P_{IV}}$ has partition $\mu = (\mu_0, \mu_1) = (3,1)$, i.e. $\mu_0 = \lambda_0 + \lambda_1 = 2+1$ and $p = \lambda_0 = 2, q = \lambda_1 = 1$,

$$\mathfrak{P}_{\mathrm{IV}} = \begin{pmatrix} h_0^{(0)} & h_1^{(0)} & h_2^{(0)} & 0 \\ 0 & h_0^{(0)} & h_1^{(0)} & 0 \\ 0 & 0 & h_0^{(0)} & 0 \\ 0 & 0 & 0 & h_0^{(1)} \end{pmatrix} = (h^{(0)} I_3 + h_1^{(0)} \Lambda_3^1 + h_2^{(0)} \Lambda_3^2) \oplus (h_0^{(1)} I_1). \tag{A.2}$$

From here we have (5.35) giving us

$$g_{\lambda\to\mu}(\varepsilon) = (g^{(0)}(\varepsilon) \oplus I_{\mu_1=1}), \tag{A.3}$$

where $g^{(0)}(\varepsilon) \in GL(\mu_0 = 3;\mathbb{C})$ is

$$g^{(0)}(\varepsilon) = \begin{pmatrix} I_2 & G_{12} \\ 0 & G_{22} \end{pmatrix}, \tag{A.4}$$

and (5.32) yields

$$\begin{pmatrix} G_{12} \\ G_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon^2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \varepsilon \\ \varepsilon^2 \end{pmatrix}. \tag{A.5}$$

Therefore $G_{12} = \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix}$ and $G_{22} = \varepsilon^2$ from which we finally have that

$$g_{\lambda\to\mu}(\varepsilon) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & \varepsilon & 0 \\ 0 & 0 & \varepsilon^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{A.6}$$

From this matrix, which gives us the map at the vector space level, we are able to construct the relevant map for the convergence of the Lie algebras. Start by mapping this element to $\mathbb{C}^4$ via $\iota_\mu[\mathfrak{P}_{\mathrm{IV}}] = (h_0^{(0)}, h_1^{(0)}, h_2^{(0)}, h_0^{(1)})$ and from this

$$\begin{aligned} R_{g_{\lambda\to\mu}(\varepsilon)} \circ \iota_\mu[\mathfrak{P}_{\mathrm{IV}}] &= (h_0^{(0)}, h_1^{(0)}, h_2^{(0)}, h_0^{(1)}) \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & \varepsilon & 0 \\ 0 & 0 & \varepsilon^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= (h_0^{(0)}, h_1^{(0)}, h_0^{(0)} + \varepsilon h_1^{(0)} + \varepsilon^2 h_2^{(0)}, h_0^{(1)}). \end{aligned}$$

We can now map back to the maximal Abelian subgroup via ι^{-1}

$$b_\varepsilon = \iota_\lambda^{-1} \circ R_{g_{\lambda\to\mu}(\varepsilon)} \circ \iota_\mu[\mathfrak{P}_{\mathrm{IV}}] = \begin{pmatrix} h_0^{(0)} & h_1^{(0)} & 0 & 0 \\ 0 & h_0^{(0)} & 0 & 0 \\ 0 & 0 & h_0^{(1)} + \varepsilon h_1^{(0)} + \varepsilon^2 h_2^{(0)} & 0 \\ 0 & 0 & 0 & h_0^{(1)} \end{pmatrix},$$

yielding

$$\left(\mathrm{Ad}_{g_{\lambda\to\mu}(\varepsilon)}\right)[b_\varepsilon] = \begin{pmatrix} h_0^{(0)} & h_1^{(0)} & h_2^{(0)} & 0 \\ 0 & h_0^{(0)} & h_1^{(0)} + \varepsilon h_2^{(0)} & 0 \\ 0 & 0 & h_0^{(0)} + \varepsilon h_1^{(0)} + \varepsilon^2 h_2^{(0)} & 0 \\ 0 & 0 & 0 & h_0^{(1)} \end{pmatrix}, \tag{A.7}$$

which in the limit $\varepsilon \to 0$ gives precisely $\mathfrak{P}_{\mathrm{IV}}$. The relevant transformations of the independent variables are obtained from (see (5.50))

$$\hat{x} = (\mathrm{Ad}_{g_{\lambda\to\mu}^{-1}(\varepsilon)})x, \tag{A.8}$$

implying the relevant transformations are

$$\begin{aligned} Z &= \frac{z}{\varepsilon^2}, \quad \tilde{Z} = \tilde{z} - \frac{1}{\varepsilon}, \\ W &= \frac{w}{\varepsilon^2}, \quad \tilde{W} = \tilde{w} - \frac{1}{\varepsilon}. \end{aligned} \tag{A.9}$$

Finally the transformation of the independent variable transforms as

$$t_{\mathrm{V}} = \frac{\tilde{Z}}{\tilde{W}} - \frac{W}{Z} = \frac{1}{\varepsilon} + \left(\tilde{w} - \frac{w}{z}\right) + O(\varepsilon) = \frac{1}{\varepsilon} + t_{\mathrm{IV}} + O(\varepsilon). \tag{A.10}$$

A.0.2 $P_{\mathrm{III}} \to P_{\mathrm{II}}$

The element corresponding to $\mathrm{P_{III}}$ is associated to the partition $\lambda = (\lambda_0, \lambda_1) = (2,2)$, thus

$$\mathfrak{P}_{\mathrm{III}} = \begin{pmatrix} a & b & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & 0 & c \end{pmatrix} = (aI_2 + b\Lambda_2^1) \oplus (cI_2 + c\Lambda_2^1), \tag{A.11}$$

while that for $\mathrm{P_{II}}$ has partition $\mu = (\mu_0) = (4)$, i.e. $\mu_0 = \lambda_0 + \lambda_1 = 2+2$ and $p = \lambda_0 = 2, q = \lambda_1 = 2$,

$$\mathfrak{P}_{\mathrm{II}} = \begin{pmatrix} h_0^{(0)} & h_1^{(0)} & h_2^{(0)} & h_3^{(0)} \\ 0 & h_0^{(0)} & h_1^{(0)} & h_2^{(0)} \\ 0 & 0 & h_0^{(0)} & h_1^{(0)} \\ 0 & 0 & 0 & h_0^{(0)} \end{pmatrix} = (h^{(0)} I_4 + h_1^{(0)} \Lambda_4^1 + h_2^{(0)} \Lambda_4^2 + h_3^{(0)} \Lambda_4^3). \tag{A.12}$$

From here we have (5.35) giving us

$$g_{\lambda \to \mu}(\varepsilon) = g^{(0)}(\varepsilon), \tag{A.13}$$

where $g^{(0)}(\varepsilon) \in GL(\mu_0 = 4; \mathbb{C})$ is

$$g^{(0)}(\varepsilon) = \begin{pmatrix} I_2 & G_{12} \\ 0 & G_{22} \end{pmatrix}, \tag{A.14}$$

and (5.32) yields

$$\begin{pmatrix} G_{12} \\ G_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \varepsilon & 0 & 0 \\ 0 & 0 & \varepsilon^2 & 0 \\ 0 & 0 & 0 & \varepsilon^3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \\ \varepsilon^2 & 2\varepsilon \\ \varepsilon^3 & 3\varepsilon^2 \end{pmatrix}. \tag{A.15}$$

Therefore $G_{12} = \begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix}$ and $G_{22} = \begin{pmatrix} \varepsilon^2 & 2\varepsilon \\ \varepsilon^3 & 3\varepsilon^2 \end{pmatrix}$ from which we finally have that

$$g_{\lambda\to\mu}(\varepsilon) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & \varepsilon & 1 \\ 0 & 0 & \varepsilon^2 & 2\varepsilon \\ 0 & 0 & \varepsilon^3 & 3\varepsilon^2 \end{pmatrix}. \tag{A.16}$$

This matrix again gives us the map at the vector space level and we are able to construct the relevant map for the convergence of the Lie algebras. Start by mapping this element to $\mathbb{C}^4$ via $\iota_\mu[\mathfrak{P}_{\mathrm{II}}] = (h_0^{(0)}, h_1^{(0)}, h_2^{(0)}, h_3^{(0)})$ and from this

$$\begin{aligned} R_{g_{\lambda\to\mu}(\varepsilon)} \circ \iota_\mu[\mathfrak{P}_{\mathrm{II}}] &= (h_0^{(0)}, h_1^{(0)}, h_2^{(0)}, h_3^{(0)}) \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & \varepsilon & 1 \\ 0 & 0 & \varepsilon^2 & 2\varepsilon \\ 0 & 0 & \varepsilon^3 & 3\varepsilon^2 \end{pmatrix} \\ &= (h_0^{(0)}, h_1^{(0)}, h_0^{(0)} + \varepsilon h_1^{(0)} + \varepsilon^2 h_2^{(0)} + \varepsilon^3 h_3^{(0)}, \varepsilon h_1^{(0)} + 2\varepsilon h_2^{(0)} + 3\varepsilon^2 h_3^{(0)}). \end{aligned}$$

We can now map back to the maximal Abelian subgroup via ι^{-1}

$$\begin{aligned} b_\varepsilon &= \iota_\lambda^{-1} \circ R_{g_{\lambda\to\mu}(\varepsilon)} \circ \iota_\mu[\mathfrak{P}_{\mathrm{II}}] \\ &= \begin{pmatrix} h_0^{(0)} & h_1^{(0)} & 0 & 0 \\ 0 & h_0^{(0)} & 0 & 0 \\ 0 & 0 & h_0^{(0)} + \varepsilon h_1^{(0)} + \varepsilon^2 h_2^{(0)} + \varepsilon^3 h_3^{(0)} & h_1^{(0)} + 2\varepsilon h_2^{(0)} + 3\varepsilon^2 h_3^{(0)} \\ 0 & 0 & 0 & h_0^{(0)} + \varepsilon h_1^{(0)} + \varepsilon^2 h_2^{(0)} + \varepsilon^3 h_3^{(0)} \end{pmatrix}, \end{aligned}$$

and then

$$\left(\mathrm{Ad}_{g_{\lambda\to\mu}(\varepsilon)}\right)[b_\varepsilon] =$$
$$\begin{pmatrix} h_0^{(0)} & h_1^{(0)} & h_2^{(0)} & h_3^{(0)} \\ 0 & h_0^{(0)} & h_1^{(0)} - \varepsilon^2 h_3^{(0)} & h_2^{(0)} + 2\varepsilon h_3^{(0)} \\ 0 & 0 & h_0^{(0)} - \varepsilon^2 h_2^{(0)} - 2\varepsilon^3 h_3^{(0)} & h_1^{(0)} + \varepsilon\left(2h_2^{(0)} + 3\varepsilon h_3^{(0)}\right) \\ 0 & 0 & -\varepsilon^2\left(h_1^{(0)} + \varepsilon\left[2h_2^{(0)} + 3\varepsilon h_3^{(0)}\right]\right) & h_0^{(0)} + 2\varepsilon h_1^{(0)} + 3\varepsilon^2 h_2^{(0)} + 4\varepsilon^2 h_3^{(0)} \end{pmatrix}, \tag{A.17}$$

which in the limit $\varepsilon \to 0$ gives precisely $\mathfrak{P}_{\mathrm{II}}$. The relevant transformations of the independent variables are obtained from (see (5.50))

$$\hat{x} = (\mathrm{Ad}_{g^{-1}_{\lambda\to\mu}(\varepsilon)})x, \tag{A.18}$$

and then the relevant transformations are

$$\begin{aligned} Z &= -\frac{1}{\varepsilon^4} + 2\frac{\tilde{w}}{\varepsilon^3} + 3\frac{z}{\varepsilon^2}, \quad \tilde{Z} = -\frac{1}{\varepsilon^4} + \frac{w}{\varepsilon} + \frac{\tilde{z}}{\varepsilon^4}, \\ W &= 2\frac{\tilde{z}}{\varepsilon^3} + 3\frac{w}{\varepsilon^2}, \quad \tilde{W} = -\frac{1}{\varepsilon^3} + \frac{\tilde{w}}{\varepsilon^2} + \frac{z}{\varepsilon}. \end{aligned} \tag{A.19}$$

Finally the transformation of the independent variable transforms as

$$\begin{aligned} t_{\mathrm{III}} = \frac{1}{\tilde{W}}\left(Z\tilde{Z} - W\tilde{W}\right)^{1/2} &= -\frac{1}{\varepsilon} - \frac{\varepsilon}{2}\left(\tilde{z} - z - \tilde{w}^2\right) + O(\varepsilon^2) \\ &= -\frac{1}{\varepsilon} - \frac{\varepsilon}{2}t_{\mathrm{II}} + O(\varepsilon^2). \end{aligned} \tag{A.20}$$

A.0.3 $P_{\mathrm{IV}} \to P_{\mathrm{II}}$

The element corresponding to $\mathrm{P_{IV}}$ is associated to the partition $\lambda = (\lambda_0, \lambda_1) = (3,1)$, thus

$$\mathfrak{P}_{\mathrm{IV}} = \begin{pmatrix} a & b & c & 0 \\ 0 & a & b & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & d \end{pmatrix} = (aI_3 + b\Lambda_3^1 + c\Lambda_3^2) \oplus (dI_1), \tag{A.21}$$

while that for P_{II} has partition $\mu = (\mu_0) = (4)$, i.e. $\mu_0 = \lambda_0 + \lambda_1 = 3+1$ and $p = \lambda_0 = 3, q = \lambda_1 = 1$,

$$\mathfrak{P}_{\mathrm{II}} = \begin{pmatrix} h_0^{(0)} & h_1^{(0)} & h_2^{(0)} & h_3^{(0)} \\ 0 & h_0^{(0)} & h_1^{(0)} & h_2^{(0)} \\ 0 & 0 & h_0^{(0)} & h_1^{(0)} \\ 0 & 0 & 0 & h_0^{(0)} \end{pmatrix} = (h^{(0)} I_4 + h_1^{(0)} \Lambda_4^1 + h_2^{(0)} \Lambda_4^2 + h_3^{(0)} \Lambda_4^3). \quad \text{(A.22)}$$

From here we have (5.35) giving us

$$g_{\lambda \to \mu}(\varepsilon) = g^{(0)}(\varepsilon), \qquad \text{(A.23)}$$

where $g^{(0)}(\varepsilon) \in GL(\mu_0 = 4; \mathbb{C})$ is

$$g^{(0)}(\varepsilon) = \begin{pmatrix} I_3 & G_{12} \\ 0 & G_{22} \end{pmatrix}, \qquad \text{(A.24)}$$

and (5.32) yields

$$\begin{pmatrix} G_{12} \\ G_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \varepsilon & 0 & 0 \\ 0 & 0 & \varepsilon^2 & 0 \\ 0 & 0 & 0 & \varepsilon^3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \varepsilon \\ \varepsilon^2 \\ \varepsilon^3 \end{pmatrix}. \qquad \text{(A.25)}$$

Therefore $G_{12} = \begin{pmatrix} 1 \\ \varepsilon \\ \varepsilon^2 \end{pmatrix}$ and $G_{22} = \varepsilon^3$ from which we finally have that

$$g_{\lambda \to \mu}(\varepsilon) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & \varepsilon \\ 0 & 0 & 1 & \varepsilon^2 \\ 0 & 0 & 0 & \varepsilon^3 \end{pmatrix}, \qquad \text{(A.26)}$$

presenting us with the map at the vector space level which we employ to construct the relevant map for the convergence of the Lie algebras. Map this element to $\mathbb{C}^4$ via $\iota_\mu[\mathfrak{P}_{\mathrm{II}}] = (h_0^{(0)}, h_1^{(0)}, h_2^{(0)}, h_3^{(0)})$ and from this

$$\begin{aligned}
R_{g_{\lambda\to\mu}(\varepsilon)} \circ \iota_\mu[\mathfrak{P}_{\mathrm{II}}] &= (h_0^{(0)}, h_1^{(0)}, h_2^{(0)}, h_3^{(0)}) \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & \varepsilon \\ 0 & 0 & 1 & \varepsilon^2 \\ 0 & 0 & 0 & \varepsilon^3 \end{pmatrix} \\
&= (h_0^{(0)}, h_1^{(0)}, h_2^{(0)}, h_0^{(0)} + \varepsilon h_1^{(0)} + \varepsilon^2 h_2^{(0)} + \varepsilon^3 h_3^{(0)}).
\end{aligned}$$

We can now map back to the maximal Abelian subgroup via ι^{-1}

$$b_\varepsilon = \iota_\lambda^{-1} \circ R_{g_{\lambda\to\mu}(\varepsilon)} \circ \iota_\mu[\mathfrak{P}_{\mathrm{II}}] = \begin{pmatrix} h_0^{(0)} & h_1^{(0)} & h_2^{(0)} & 0 \\ 0 & h_0^{(0)} & h_1^{(0)} & 0 \\ 0 & 0 & h_0^{(0)} & 0 \\ 0 & 0 & 0 & h_0^{(0)} + \varepsilon h_1^{(0)} + \varepsilon^2 h_2^{(0)} + \varepsilon^3 h_3^{(0)} \end{pmatrix},$$

therefore

$$\left(\mathrm{Ad}_{g_{\lambda\to\mu}(\varepsilon)}\right)[b_\varepsilon] = \begin{pmatrix} h_0^{(0)} & h_1^{(0)} & h_2^{(0)} & h_3^{(0)} \\ 0 & h_0^{(0)} & h_1^{(0)} & h_2^{(0)} + \varepsilon h_3^{(0)} \\ 0 & 0 & h_0^{(0)} & h_1^{(0)} + \varepsilon\left(h_2^{(0)} + \varepsilon h_3^{(0)}\right) \\ 0 & 0 & 0 & h_0^{(0)} + \varepsilon h_1^{(0)} + \varepsilon^2 h_2^{(0)} + \varepsilon^3 h_3^{(0)} \end{pmatrix}, \tag{A.27}$$

which in the limit $\varepsilon \to 0$ gives precisely $\mathfrak{P}_{\mathrm{II}}$. The transformations of the independent variables are obtained from (see (5.50))

$$\hat{x} = (\mathrm{Ad}_{g^{-1}_{\lambda\to\mu}(\varepsilon)})x, \tag{A.28}$$

yielding

$$\begin{aligned}
Z &= z + \frac{\varepsilon\tilde{w} - 1}{\varepsilon^2}, \quad & \tilde{Z} &= \frac{\tilde{z}}{\varepsilon^3}, \\
W &= w + \frac{\varepsilon^2\tilde{z} - 1}{\varepsilon^3}, \quad & \tilde{W} &= \frac{\tilde{w}}{\varepsilon^3}.
\end{aligned} \tag{A.29}$$

Finally the transformation of the independent variable transforms as

$$
\begin{aligned}
t_{\mathrm{IV}} = \tilde{W} - \frac{W}{Z} = \frac{\tilde{w}}{\varepsilon^3} - \frac{1}{\varepsilon} - \tilde{w} + \varepsilon\left(\tilde{z} - z - \tilde{w}^2\right) + O(\varepsilon^2) \\
= \frac{\tilde{w}}{\varepsilon^3} - \frac{1}{\varepsilon} - \tilde{w} + \varepsilon t_{II} + O(\varepsilon^2).
\end{aligned}
\tag{A.30}
$$

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