NEVANLINNA-PICK INTERPOLATION BY RATIONAL FUNCTIONS WITH A SINGLE POLE INSIDE THE UNIT DISK

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Abstract. We devise an efficient algorithm that, given points $z_1, \dots, z_k$ in the open unit disk $\mathbb{D}$ and a set complex numbers $\{f_{i,0}, f_{i,1}, \dots, f_{i,n_i-1}\}$ assigned to each z_i, produces a rational function f with a single (multiple) pole in $\mathbb{D}$, such that f is bounded on the unit cirlce by a predetermined positive number, and its Taylor expansion at z_i has $f_{i,0}, f_{i,1}, \dots, f_{i,n_i-1}$ as its first n_i coefficients.

In this paper we will be dealing with an interpolation problem whose data set

$$\Delta(\mathbf{f}) = \{z_i,\ n_i,\ f_{i,j}:\ j = 0, \dots, n_i - 1;\ i = 1, \dots, k\} \tag{1}$$

consists of k distinct points $z_1, \dots, z_k$ in the open unit disk $\mathbb{D}$, positive integers $n_1, \dots, n_k$ and a collection $\mathbf{f} = \{f_{i,j}\}_{i=1,\dots,k}^{j=0,\dots,n_i-1}$ of complex numbers $f_{i,j}$. Let us denote by $\mathbf{F}_{\Delta(\mathbf{f})}$ the set of all functions f analytic at $z_1, \dots, z_k$ and satisfying interpolation conditions listed below:

$$\mathbf{F}_{\Delta(\mathbf{f})} = \left\{ f:\ \frac{f^{(j)}(z_i)}{j!} = f_{i,j} \quad \text{for} \quad i = 1, \dots, k;\ j = 0, \dots, n_i - 1 \right\}. \tag{2}$$

In what follows, we denote by N the total number of interpolation conditions in (2): $N = n_1 + \ldots + n_k$.

Let H^∞ denote the space of bounded analytic functions in $\mathbb{D}$ and let L^∞ be the space of essentially bounded measurable functions on the unit circle $\mathbb{T}$. Given an integer $\kappa \geq 0$, we denote by H^∞_κ the set of all functions f of the form

$$f(z) = \frac{s(z)}{b(z)} \tag{3}$$

where the numerator s belongs to H^∞ and the denominator b is a finite Blaschke product of degree κ. Via nontangential boundary limits, H^∞_κ-functions can be identified with L^∞-functions which admit meromorphic continuation inside $\mathbb{D}$ with at most κ poles (counted with multiplicities) in $\mathbb{D}$. Since a finite Blaschke product is unimodular on $\mathbb{T}$, it follows that $\|f\|_{L^\infty} = \|s\|_{H^\infty}$ for an f of the form (3).

The intersection $\mathbf{F}_{\Delta(\mathbf{f})}$ with H^∞ (and therefore, with H^∞_κ for every $\kappa > 0$) contains polynomials and therefore, is not empty. A question of interest is: *find the value of* $\mu_\kappa := \inf_{f \in \mathbf{F}_{\Delta(\mathbf{f})} \cap H^\infty_\kappa} \|f\|_{L^\infty}$ *in terms of interpolation data* (1).

1991 *Mathematics Subject Classification.* 65E05, 30E05.

Key words and phrases. norm-constrained interpolation, Hermite polynomial.

The research was partially supported by National Science Foundation Grant DMS 0901124.

The answer has been obtained in [1] and is recalled in Theorem 1 below. Associated with given $z_i \in \mathbb{D}$ and $n_i \in \mathbb{N}$ from (1) is the block-matrix

$$\mathbf{\Gamma} = [\Gamma(z_i, z_\ell)]_{i,\ell=1}^{k}$$

with $n_i \times n_\ell$ blocks $\Gamma(z_i, z_\ell)$ given entry-wise by the formula

$$[\Gamma(z_i, z_\ell)]_{r,j} = \frac{1}{r!j!}\,\frac{\partial^{r+j}}{\partial z^r \partial \bar{\zeta}^j}\,\frac{1}{1-z\bar{\zeta}}\bigg|_{\substack{z=z_r,\\ \zeta=z_j}} = \sum_{\alpha=0}^{\min\{r,j\}} \frac{(r+j-\alpha)!}{(r-\alpha)!\alpha!(j-\alpha)!}\,\frac{z_i^{r-\alpha}\bar{z}_\ell^{j-\alpha}}{(1-z_i\bar{z}_\ell)^{r+j-\alpha+1}}.$$

With the rest of the data set (1) (i.e., with the given collection $\mathbf{f} = \{f_{ij}\}_{i=1,\dots,k}^{j=0,\dots,n_i-1}$), we associate the block-diagonal matrix $T_{\mathbf{f}}$ with lower triangular Toeplitz diagonal blocks defined as follows:

$$T_{\mathbf{f}} = \begin{bmatrix} T_{\mathbf{f},1} & & 0 \\ & \ddots & \\ 0 & & T_{\mathbf{f},k} \end{bmatrix}, \quad \text{where} \quad T_{\mathbf{f},i} = \begin{bmatrix} f_{i,0} & 0 & \dots & 0 \\ f_{i,1} & f_{i,0} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ f_{i,n_i-1} & \dots & f_{i,1} & f_{i,0} \end{bmatrix}. \tag{4}$$

We now define the matrix (more precisely, the matrix pencil)

$$P_{\Delta(\mathbf{f})}(\lambda) = \lambda^2 \mathbf{\Gamma} - T_{\mathbf{f}} \mathbf{\Gamma} T_{\mathbf{f}}^*.$$

Theorem 1. *Let $\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_m > 0$ be all positive solutions of the equation $\det P_{\Delta(\mathbf{f})}(\lambda) = 0$. Then for every $\kappa \geq 0$,*

$$\mu_\kappa := \inf_{f \in \mathbf{F}_{\Delta(\mathbf{f})} \cap H_\kappa^\infty} \|f\|_{L^\infty} = \begin{cases} \lambda_\kappa & \text{if} \quad \kappa \leq m, \\ 0 & \text{if} \quad \kappa > m. \end{cases}$$

Theorem 1 tells us that in order to get interpolants f of the form (3) with small L^∞-norm (recall that minimizing the L^∞-norm of an interpolant is of particular importance in model reduction and digital filter design; see e.g., [4]) one must allow f to have up to N poles. Observe that in order to interpolate (2) by an H_κ^∞-function f with relatively small κ, the poles of f must be carefully chosen. We refer to [3] for more information on rational norm-constrained interpolation. The numerical aspects of meromorphic Nevanlinna-Pick interpolations are discussed in [2, 5].

The main result. The objective of this note is to construct a rational function in $\mathbf{F}_{\Delta(\mathbf{f})}$ with a single (multiple) pole at a prescribed point in $\mathbb{D} \setminus \{z_1, \dots, z_k\}$ (by the conformal change of variable, it suffices to construct a solution with a pole at the origin) and with an arbitrarily small L^∞-norm.

Theorem 2. *Given $\varepsilon > 0$ and a data set (1) such that $0 \notin \{z_1, \dots, z_k\}$, there exists a positive integer m and a polynomial p with $\deg p \leq N-1$ and $\|p\|_\infty < \varepsilon$ so that the function*

$$f(z) := \frac{p(z)}{z^m} \tag{5}$$

satisfies interpolation conditions (2).

Proof. Denote $\mathbf{n} = \max\{n_1, \dots, n_k\}$. Assuming that $m \geq \mathbf{n}$, we use the function $b_m(z) := z^m$ to define the numbers

$$b_{m;i,j} = \frac{b_m^{(j)}(z_i)}{j!} = \binom{m}{j} z_i^{m-j} \quad \text{for} \quad i = 1, \dots, k;\ j = 0, \dots, n_i - 1 \tag{6}$$

and the block diagonal matrix $T_{\mathbf{b}_m}$ with the diagonal blocks (see formula (4))

$$T_{\mathbf{b}_m,i} = \begin{bmatrix} b_{m;i,0} & 0 & \cdots & 0 \\ b_{m;i,1} & b_{m;i,0} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ b_{m;i,n_i-1} & \cdots & b_{m;i,1} & b_{m;i,0} \end{bmatrix}, \quad i = 1, \dots, k. \tag{7}$$

Assuming without loss of generality that $|z_1| \geq |z_i|$ for all $i > 1$, we get:

$$\begin{aligned} \|T_{\mathbf{b}_m}\|^2 =& \lambda_{\max}(T^*_{\mathbf{b}_m} T_{\mathbf{b}_m}) \leq \operatorname{trace}(T^*_{\mathbf{b}_m} T_{\mathbf{b}_m}) = \sum_{i=1}^{k} \operatorname{trace}(T^*_{\mathbf{b}_{m,i}} T_{\mathbf{b}_{m,i}}) \\ =& \sum_{i=1}^{k} \sum_{j=0}^{n_i-1} \left| (n_i - j) \binom{m}{j} z_i^{m-j} \right|^2 \leq k\mathbf{n}^2 m^{2\mathbf{n}-2} |z_1|^{2m-2\mathbf{n}+2}, \end{aligned} \tag{8}$$

where the third equality follows from the block diagonal structure of $T_{\mathbf{b}_m}$ and the fourth can be seen from (7). In (8) and what follows, we write $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ for the maximum and the minimum eigenvalues of the matrix A.

It follows from (5) and (6) that f of the form (5) satisfies conditions (2) if and only if p is subject to similar conditions

$$\frac{p^{(j)}(z_i)}{j!} = p_{i,j} \quad \text{for} \quad i = 1, \dots, k;\ j = 0, \dots, n_i - 1, \tag{9}$$

where the numbers $p_{i,j}$ are defined by

$$p_{i,j} = \sum_{\ell=0}^{j} b_{m,i,\ell} f_{i,j-\ell} \quad (i = 1, \dots, k;\ j = 0, \dots, n_i - 1). \tag{10}$$

Although the numbers $p_{i,j}$ depend on m, we drop this dependence from notation. Observe that equalities (10) can be written in the matrix form as

$$C_{\mathbf{p}} = T_{\mathbf{b}_m} C_{\mathbf{f}} \tag{11}$$

where $C_{\mathbf{f}}$ and $C_{\mathbf{p}}$ are the columns associated with the collections $\mathbf{f}$ and $\mathbf{p}$ as follows:

$$\begin{aligned} C_{\mathbf{f}} &= [f_{1,0}, \cdots, f_{1,n_1-1}, \cdots, f_{k,0}, \cdots, f_{k,n_k-1}]^t, \\ C_{\mathbf{p}} &= [p_{1,0}, \cdots, p_{1,n_1-1}, \cdots, p_{k,0}, \cdots, p_{k,n_k-1}]^t. \end{aligned}$$

Let $p(z)$ be the (unique) Hermite osculatory polynomial ([7, 6]) of degree at most $N - 1$ satisfying interpolation conditions (9) and let $C_{\mathbf{a}}$ denote the column of its coefficients:

$$p(z) = a_0 + a_1 z + \ldots + a_{N-1} z^{N-1}, \quad C_{\mathbf{a}} = [a_0,\ a_1, \dots, a_{N-1}]^t. \tag{12}$$

Then

$$\|p\|_\infty := \max_{z\in\mathbb{D}} |p(z)| \le \sum_{i=0}^{N-1} |a_i| \le \left(N \sum_{i=0}^{N-1} |a_i|^2 \right)^{\frac{1}{2}} = \sqrt{N}\, \|C_{\mathbf{a}}\|_2. \tag{13}$$

Furthermore, due to (12), conditions (9) can be written in terms of a_i's as

$$\sum_{\ell=0}^{N-j-1} \binom{j+\ell}{j} a_{\ell+j} z_i^\ell = p_{ij} \qquad (i = 1, \dots, k;\ j = 0, \dots, n_i - 1)$$

and in turn can be written in matrix form as

$$VC_{\mathbf{a}} = C_{\mathbf{p}} \tag{14}$$

where V is the $N \times N$ matrix defined below:

$$V = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_k \end{bmatrix}, \quad V_i = \begin{bmatrix} \binom{0}{0} & \binom{1}{0} z_i & \binom{2}{0} z_i^2 & \binom{3}{0} z_i^3 & \cdots & \binom{N-1}{0} z_i^{N-1} \\ 0 & \binom{1}{1} & \binom{2}{1} z_i & \binom{3}{1} z_i^2 & \cdots & \binom{N-1}{1} z_i^{N-2} \\ \vdots & \ddots & \ddots & & & \\ 0 & \cdots & 0 & \binom{n_i-1}{n_i-1} & \cdots & \binom{N-1}{n_i-1} z_i^{N-n_1} \end{bmatrix}. \tag{15}$$

It follows from the existence of the osculatory polynomial that the linear system (14) always has a unique solution $C_{\mathbf{a}}$ for any vector $C_{\mathbf{p}}$ and therefore, that the matrix V is invertible. Actually, it can be shown directly that $\det V = \prod_{i<j} (z_j - z_i)^{n_i n_j}$.

We now combine (11) and (14) to get $C_{\mathbf{a}} = V^{-1} T_{\mathbf{b}_m} C_{\mathbf{f}}$. Combining the latter equality with (13) gives

$$\begin{aligned} \|p\|_\infty &\le \sqrt{N}\, \|C_{\mathbf{a}}\|_2 = \sqrt{N}\, \|V^{-1} T_{\mathbf{b}_m} C_{\mathbf{f}}\|_2 \\ &\le \sqrt{N}\, \|V^{-1}\|\, \|T_{\mathbf{b}_m}\|\, \|C_{\mathbf{f}}\|_2 \le \frac{\mathbf{n}\sqrt{Nk}\, m^{\mathbf{n}-1} |z_1|^{m-\mathbf{n}+1} \|C_{\mathbf{f}}\|_2}{\sqrt{\lambda_{\min}(VV^*)}}, \end{aligned}$$

where the operator norm is used for matrices here and in what follows. The last inequality follows from (8) and the fact that the operator norm of V^{-1} is equal to

$$\|V^{-1}\| = \sqrt{\lambda_{\max}(V^{-*}V^{-1})} = \frac{1}{\sqrt{\lambda_{\min}(VV^*)}}.$$

Therefore, given $\varepsilon > 0$, for m satisfying

$$m^{\mathbf{n}-1} |z_1|^{m-\mathbf{n}+1} < \frac{\varepsilon \sqrt{\lambda_{\min}(VV^*)}}{\mathbf{n}\sqrt{Nk} \|C_{\mathbf{f}}\|_2} \tag{16}$$

(which necessarily occurs for large m as $|z_1| < 1$), we obtain $\|p\|_\infty < \varepsilon$. □

Algorithm. The constructive proof of Theorem 2 suggests the following algorithm:

Step 1: Numerically compute $\lambda_{\min}(VV^*)$ for the matrix V given in (15).
Step 2: Numerically find an m satisfying (16).
Step 3: Using this m, compute $p_{i,j}$ as in (10).
Step 4: Solve the (unique) Hermite osculatory polynomial p of degree at most $N-1$ satisfying (9). Then $f(z) = \frac{p(z)}{z^m}$ satisfies the interpolation condition (2).

Example. Take $\varepsilon = 0.2$, $z_1 = 0.4, z_2 = -0.3, f_{1,0} = 2, f_{1,1} = -3, f_{2,0} = i$, and interpolation conditions

$$f(z_1) = f_{1,0}, \quad f'(z_1) = f_{1,1}, \quad f(z_2) = f_{2,0}. \tag{17}$$

By Theorem 1, since the minimum positive λ satisfying $P_{\Delta(\mathbf{f})}(\lambda) = 0$ is 0.92533 (to 5 significant figures), we need a rational function with at least three carefully chosen poles to meet conditions (17). If we are less concerned with the total pole multiplicity, our algorithm allows us to interpolate the data with a single (multiple) pole at a prescribed location.

We compute that $\lambda_{\min}(VV^*) = 0.065537$. The smallest m satisfying (16) is $m = 10$. For this m, we have

$$p_{1,0} = 2.0972 \times 10^{-4} \quad p_{1,1} = 4.9283 \times 10^{-3} \quad p_{2,0} = 5.9049 \times 10^{-6} i$$

The quadratic Hermite osculatory polynomial satisfying these condition is given by

$$\begin{aligned} p(z) = {} & (-7.0362 \times 10^{-4} + 1.9281 \times 10^{-6} i) + (3.6165 \times 10^{-4} + 9.6407 \times 10^{-6} i) z \\ & + (6.6125 \times 10^{-3} + 1.2051 \times 10^{-5} i) z^2 \end{aligned}$$

Hence a solution to the interpolation problem is $f(z) = p(z)/z^{10}$.

Recall that in the proof of Theorem 2 we assumed that $m \geq \mathbf{n} = \max\{n_1, \ldots, n_k\}$. In fact, the integer m satisfying inequality (16) turns out to be greater than $N = n_1 + \ldots, n_k$ in which case the function f of the form (5) is a polynomial in z^{-1}.

In conclusion, we will discuss the existence of rational solutions f to the problem (2) with $\|f\|_\infty < \varepsilon$ and having the (only) pole at a point $a \in \mathbb{D}$ different from the origin, that is, the functions of the form

$$f(z) = \frac{p(z)}{(z-a)^m}, \qquad a \in \mathbb{D},\ m \in \mathbb{N}, \tag{18}$$

where p is a polynomial. Such solutions are of particular importance in case one of the interpolation nodes falls into the origin.

A conformal change of variable $z \to \frac{z-a}{1-z\overline{a}}$ for a fixed $a \in \mathbb{D}$ enables us to construct a rational solution f to the problem (2) with $\|f\|_\infty < \varepsilon$ and of the form

$$f(z) = q\left(\frac{z-a}{1-z\overline{a}}\right) \Big/ \left(\frac{z-a}{1-z\overline{a}}\right)^m \tag{19}$$

where q is a polynomial of degree at most $N-1$. The function in (19) can be written in the form (18) once $m > N-1$. The order of its unique pole $z = a$ can be decreased by an appropriate choice of a. Indeed, this order m is determined from the inequality (16) with $|z_1|$ replaced by the quantity $\delta = \max_{1\leq i\leq k} \left|\dfrac{z_i - a}{1 - z_i\overline{a}}\right|$. An interesting pseudo-hyperbolic optimization question is: *given $z_1, \ldots, z_k \in \mathbb{D}$, find the value of*

$$\min_{a\in\mathbb{D}} \max_{1\leq i\leq k} \left|\frac{z_i - a}{1 - z_i\overline{a}}\right|$$

and the points $a \in \mathbb{D}$ at which this minimal value is attained.

A slight modification of the proof of Theorem 2 provides another possibility to construct a solution f of the form (18) to the problem (2). First we pick a point $a \in \mathbb{D}$ such that

$$\mu_a := \max_{1 \le i \le k} |z - a| < 1. \tag{20}$$

Then we use the function $b_m(z) := (z-a)^m$ to define the numbers

$$b_{m;i,j} = \frac{b_m^{(j)}(z_i)}{j!} = \binom{m}{j}(z_i - a)^{m-j} \quad \text{for} \quad i = 1, \ldots, k;\ j = 0, \ldots, n_i - 1. \tag{21}$$

The operator norm of the matrix $T_{\mathbf{b}_m}$ defined from the numbers (21) via formula (7) admits the following estimate parallel to (8):

$$\|T_{\mathbf{b}_m}\| \le \sqrt{k}\, \mathbf{n}\, m^{\mathbf{n}-1} \mu^{m-\mathbf{n}+1},$$

where μ is defined in (20). We thus may apply the main algorithm (with μ instead of $|z_1|$ and with the numbers $b_{m;i,j}$ given by (21) rather than by (6)) to construct a solution f of the form (18) to the problem (2). It is hard to say, however, how much a good choice of a may decrease the degree m of the denominator in (18) since moving a away from the origin we may increase the L^∞-norm of f.

Summary. We have shown that given any $\varepsilon > 0$ and data (1), we are able to find a rational function analytic everywhere except at one point in $\mathbb{D}$, satisfying the Lagrange-Hermite interpolation conditions (2), such that the function is bounded by ε on $\mathbb{T}$. The proof is elementary and the algorithm is simple to implement.

Acknolwedgments. The authors are grateful to Professor Vladimir Bolotnikov for calling their attention to the problem and helpful discussions throughout their work on the paper and to the College of William and Mary for hospitality. The authors are also grateful to the anonymous referee for close reading and several useful remarks.

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