THE JACQUET-LANGLANDS CORRESPONDENCE FOR OVERCONVERGENT HILBERT MODULAR FORMS

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Abstract. We use results by Chenevier to interpolate the classical Jacquet-Langlands correspondence for Hilbert modular forms, which gives us an extension of Chenevier's results to totally real fields. From this we obtain an isomorphisms between eigenvarieties attached Hilbert modular forms and those attached to modular forms on a totally definite quaternion algebra over a totally real field of even degree.

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Introduction

Eigenvarieties are rigid analytic spaces which parametrize systems of hecke eigenvalues of finite slope eigenforms, for this reason they are the subject of much work in number theory. They were first introduced by Colem–Mazur in [13] and Buzzard in [9] and since then many constructions have appeared due to many authors. In particular, we can now construct eigenvarieties associated to any reductive group and one can ask to what extent can we use Langlands functoriality to find maps between different eigenvarieties. Take for example the Jacquet-Langlands correspondence, which tells us (roughly) that the classical spaces of quaternionic modular forms $S_k^D(N)$ of weight k and level N on a quaternion algebra D of discriminant $\mathfrak{d}$ are isomorphic as Hecke modules to the spaces $S_k^{\mathfrak{d}\text{-new}}(N\mathfrak{d})$ of $\mathfrak{d}$-new modular forms for GL_2. One can then ask if this extends to families of modular forms, i.e., if we can use this to relate the eigenvariety $\mathscr{X}_D$ coming from these quaternionic modular forms, to the eigenvariety $\mathscr{X}_{\mathrm{GL}_2}$ coming from the usual spaces of modular forms. Over $\mathbb{Q}$, this was answered by Chenevier in [11], who showed that there is a closed immersion $\mathscr{X}_D \hookrightarrow \mathscr{X}_{\mathrm{GL}_2}$ which interpolates the classical Jacquet-Langlands correspondence[a]. This result is an instance of what is now called p-adic Langlands functoriality. Other examples of this can be found in work of Hansen [16], Ludwig [22], and Newton [23].

Going back to the result of Chenevier, if one picks $D/\mathbb{Q}$ to be a totally definite quaternion algebra, then the spaces of overconvergent quaternionic modular forms are very easy to define and work with since there is much simpler geometry involved. This means that if one is interested in studying or computing the action of the U_p operator on the space of overconvergent modular forms, then it is possible to reduce to computing this on spaces of overconvergent quaternionic modular forms. Our goal here is to extend the results by Chenevier to a totally real field F. In particular, we have the following theorem.

Theorem 1. *Let D/F be a totally definite quaternion algebra of discriminant $\mathfrak{d}$ defined over a totally real field F. Let p be a rational (unramified) prime and $\mathfrak{n}$ an integral ideal of F such that $p \nmid \mathfrak{n}\mathfrak{d}$ and $(\mathfrak{n}, \mathfrak{d}) = 1$. Let $\mathscr{X}_D^{red}(\mathfrak{n}p)$ be the nilreduction (see 2.3.4) of the eigenvariety of level $\mathfrak{n}p$ attached to quaternionic modular forms on D. Similarly, let $\mathscr{X}_{\mathrm{GL}_2}^{red}(\mathfrak{n}\mathfrak{d}p)$ denote nilreduction of the eigenvariety associated to cuspidal Hilbert modular forms of level $\mathfrak{n}\mathfrak{d}p$ (with*

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[a]In general Chenevier proves that one gets a isomorphism onto the $\mathfrak{d}$-new 'part' of the eigenvariety.

the associated moduli problem for this level being representable) as constructed in [2]. *Then there is a closed immersion* $\iota_D : \mathscr{X}_D^{red}(\mathfrak{n}p) \hookrightarrow \mathscr{X}_{\mathrm{GL}_2}^{red}(\mathfrak{n}\mathfrak{d}p)$ *which interpolates the classical Jacquet-Langlands correspondence. Moreover, when* $[F : \mathbb{Q}]$ *is even, one can choose* D *with* $\mathfrak{d} = 1$ *so that the above is an isomorphism between the corresponding eigenvarieties.*

Alongside this we also prove following:

Theorem 2. *The eigenvarieties* $\mathscr{X}_D(\mathfrak{n}p)$ *and* $\mathscr{X}_{\mathrm{GL}_2}(\mathfrak{n}p)$ *as above are reduced, meaing that* $\mathscr{X}_*^{red}(\mathfrak{n}p) \cong \mathscr{X}_*(\mathfrak{n}p)$ *for* $* \in \{D, \mathrm{GL}_2\}$.

To prove the above results, we first recall the construction of the relevant eigenvarieties, making small modifications so that we can apply Chenevier's Interpolation theorem. In the Hilbert case, we construct the spaces of overconvergent cuspidal Hilbert modular forms following [2] and then use Buzzard's eigenmachine to construct the relevant eigenvariety for which we can apply the Interpolation theorem. In the quaternionic case we construct the spaces of overconvergent quaternionic modular forms following [9], but modifying the weight space so that it is equidimensional of dimension $[F : \mathbb{Q}] + 1$ so that it matches with the weight space on the Hilbert eigenvariety.

The importance of the above theorems, is that it allows us to study the geometry of the Hilbert eigenvariety by studying the quaternionic eigenvariety which has a more combinatorial nature. Similar ideas have been used in [20, 28, 21] to understand the geometry of eigenvarieties attached to modular forms over $\mathbb{Q}$. Furthermore, the above theorem is one of the key ingredients in [5], where it is used to compute slopes of Hilbert modular forms, which in turn gives insights into the geometry of the relevant eigenvarieties.

Moreover, by working with totally real fields of even degree we obtain isomorphisms between the relevant eigenvarieties and not just closed immersion, which is a stronger result than what is available over $\mathbb{Q}$. In practice, this means that, in this case, we can study the full (cuspidal) Hilbert eigenvariety by using the quaternionic one.

1. **Notation**

Notation 1.1.1. (1) Let F be a totally real field of degree g, and fix a rational prime p which is unramified with the exception of Sections 3 and 4 where it may be ramified unless otherwise stated.

(2) Let D be a quaternion algebra over F with a fixed maximal order $\mathcal{O}_D$ and let $G_D = \mathrm{Res}_{F/\mathbb{Q}}(D^\times)$ (we will sometimes abuse notation and denote this simply by D). When $D = M_2(F)$ we denote this simply as G. We fix an isomorphism $\mathcal{O}_D \otimes_{\mathcal{O}_F} \mathcal{O}_p \cong M_2(\mathcal{O}_p)$, which induces an isomorphism $D_p := D \otimes_F F_p \cong M_2(F_p)$. Lastly, let T denote a fixed maximal torus of G_D and $\mathbb{T} = \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}} \mathbb{G}_m$.

(3) Let $\mathcal{O}_F$ denote the ring of integers of F and let $\mathfrak{d}_F$ denote the different ideal of F. For each finite place v of F let F_v denote the completion of F with respect to v and $\mathcal{O}_v$ the ring of integers of F_v. For an integral ideal $\mathfrak{n}$, let $F_{\mathfrak{n}} = \prod_{v|\mathfrak{n}} F_v$ and similarly let $\mathcal{O}_{\mathfrak{n}} = \prod_{v|\mathfrak{n}} \mathcal{O}_v$. In particular, if we have $p\mathcal{O}_F = \prod_{i=1}^f \mathfrak{p}_i$, then let $\mathcal{O}_p = \oplus_i \mathcal{O}_{\mathfrak{p}_i} = \mathcal{O}_F \otimes \mathbb{Z}_p$.

(4) Let Σ to be the set of all places of F, Σ_p be the set of all finite places above p and $\Sigma_\infty \subset \Sigma$ the set of all infinite places of F.

(5) Let $\overline{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$ inside $\mathbb{C}$ and we fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Furthermore, we fix embeddings $inc : \overline{\mathbb{Q}} \to \mathbb{C}$ and $inc_p : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$, which allow us to think of the elements of $\overline{\mathbb{Q}}$ as both complex and p-adic numbers.

(6) Let $\pi_{\mathfrak{p}}$ denote the uniformisers of $F_{\mathfrak{p}}$ and $\pi \in \mathcal{O}_p$ be the element whose $\mathfrak{p}$ component of $\mathcal{O}_p$ is $\pi_{\mathfrak{p}}$. By abuse of notation we also let π denote the ideal of $\mathcal{O}_F$ which is the product of all the primes ideals above p, i.e., the radical of $p\mathcal{O}_F$.

(7) For each v $\in \Sigma_\infty$, we have a field embedding ι_v of F into $\mathbb{C}$ given by v. This map extends to a map $F_p \to \overline{\mathbb{Q}}_p$ and then factors through the projection $F_p \to F_{\mathfrak{p}}$ for some $\mathfrak{p}$ above p. This then gives a natural surjection $\Sigma_\infty \to \Sigma_p$ where v $\mapsto \mathfrak{p}_v$. For each prime ideal $\mathfrak{p}_j \in \Sigma_p$ let $\Sigma_{\mathfrak{p}_j}$ be the set of v $\in \Sigma_\infty$ factoring through the projection $F_p \to F_{\mathfrak{p}_j}$.

(8) Let L be a complete extension of $\mathbb{Q}_p$, which contains the compositum of the images of F under $\iota \circ \iota_v$, for $v \in \Sigma_\infty$ where $\iota : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$ such that $\iota \circ inc = inc_p$.

2. Eigenvarieties and the Interpolation Theorem

We begin by briefly reviewing the construction of eigenvarieties and Chenevier's Interpolation Theorem. More details on general constructions can be found in many places, for example [13, 11, 9, 16].

2.1. The weight space.
The weight space is a rigid analytic variety that allows us to make precise the idea of modular forms 'living' in p-adic families. Following [5], we begin with the classical definition of a weight of a Hilbert modular form.

Definition 2.1.1. Let $n \in \mathbb{Z}_{\geq 0}^{\Sigma_\infty}$ and $v \in \mathbb{Z}^{\Sigma_\infty}$ such that $n + 2v = (r, \dots, r)$ for some $r \in \mathbb{Z}$ and set $k = n + 2$ and $w = v + n + 1$. By abuse of notation we denote $(r, \dots, r)$ by r for $r \in \mathbb{Z}$. Note that all the entries of k will have the same parity and $k = 2w - r$. We call the pair $(k, r) \in \mathbb{Z}_{\geq 2}^{\Sigma_\infty} \times \mathbb{Z}$ a *classical algebraic weight*. Note that given k (with all entries paritious and greater than 2) and r we can recover n, v, w. In what follows we will move between both descriptions when convenient. We will call (k, r, n, v, w) satisfying the above a *weight tuple*.

A weight is called *arithmetic* or *classical* if it is the product of an algebraic character (k, r) and a finite character ψ, which we denote by (κ_ψ, r).

Definition 2.1.2. Let $\mathbb{T} = \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}} \mathbb{G}_m$. We define $\mathcal{W}^G$ to be the rigid analytic space over L associated to the completed group algebra $\mathcal{O}_L[\![\mathbb{T}(\mathbb{Z}_p) \times \mathbb{Z}_p^\times]\!]$. We call $\mathcal{W}^G$ the *weight space for G*. Moreover, one has a universal character

$$[-] : \mathbb{T}(\mathbb{Z}_p) \times \mathbb{Z}_p^\times \longrightarrow \mathcal{O}_L[\![\mathbb{T}(\mathbb{Z}_p) \times \mathbb{Z}_p^\times]\!]^\times.$$

It follows from the above that $\mathcal{W}^G(\mathbb{C}_p) = \mathrm{Hom}_{cts}(\mathbb{T}(\mathbb{Z}_p) \times \mathbb{Z}_p^\times, \mathbb{C}_p^\times)$. Moreover, we note that $\mathbb{T}(\mathbb{Z}_p) \times \mathbb{Z}_p^\times \cong H \times \mathbb{Z}_p^{g+1}$, where H is the torsion subgroup of $\mathbb{T}(\mathbb{Z}_p) \times \mathbb{Z}_p^\times$. From this it follows that

$$\mathcal{W}^G \cong H^\vee \times B(1,1)^{g+1} \cong \bigsqcup_{\chi \in H^\vee} \mathcal{W}_\chi$$

as rigid spaces, where $H^\vee$ is the character group of H and $B(1,1)$ is the open ball of radius 1 around 1. It is clear from this that $\mathcal{W}^G$ is equidimensional of dimension $g + 1$.

Notation 2.1.3. Elements of $\mathcal{W}^G(\mathbb{C}_p)$ will be given by $v : \mathbb{T}(\mathbb{Z}_p) \to \mathbb{C}_p^\times$ and $r : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$. Setting $n = -2v + r$ and $\kappa = n + 2$, we will denote these weights as (κ, r) and call (κ, r, n, v, w) a weight tuple if κ, r, n, v, w satisfy the same relations as in 2.1.1. More generally, if $\mathfrak{U}$ is an affinoid with a morphism of rigid spaces $\mathfrak{U} \to \mathcal{W}^G$, then we will denote by $(\kappa^{\mathfrak{U}}, r^{\mathfrak{U}})$ the restriction of the universal character to $\mathfrak{U}$.

Definition 2.1.4. Let (k, r, n, v, w) be a weight tuple with $(k, r) \in \mathbb{Z}^{\Sigma_\infty} \times \mathbb{Z}$ a classical algebraic weight. This defines an *algebraic weight* by sending $(a, b) \in \mathbb{T}(\mathbb{Z}_p) \times \mathbb{Z}_p^\times$ to $a^v b^r$.

Notation 2.1.5. There is a natural map $\mathbb{T}(\mathbb{Z}_p) \to \mathbb{T}(\mathbb{Z}_p) \times \mathbb{Z}_p^\times$ given by $t \mapsto (t^{-2}, N_{F/\mathbb{Q}}(t))$. In this way we view weights $(\kappa, r) \in \mathcal{W}^G$ with (κ, r, n, v, w) a weight tuple as maps $\mathbb{T}(\mathbb{Z}_p) \to \mathbb{C}_p$ given by $t \mapsto n(t)$.

Remark 2.1.6. In the literature there are slightly more general weight spaces than the one we have introduced. One alternative way of defining the weight space is to let $\mathcal{W}'$ denote the rigid analytic space associated to the completed group algebra $\mathcal{O}_L[\![T(\mathbb{Z}_p)]\!]$, where T is a fixed maximal torus of G. The problem with this weight space is that it contains too many weights for which the associated spaces of modular forms would be empty. For this reason one usually imposes suitable vanishing conditions on these weights. See [9, Part III] and [27, 4.3.2]. The weight spaces one gets this way conjecturally have dimension $g + 1$ (dependent on Leopoldt's conjecture). For this reason we have chosen to work with $\mathcal{W}^G$ which has the correct dimension. Moreover, if Leopoldt's conjecture is true then the resulting eigenvarieties for the different weight spaces will be isomorphic.

Later, when defining the spaces of locally analytic functions it will be convenient for us to extend the definition of the weight space from $\mathbb{T}$ to T, which denotes a fixed maximal torus of G. We do this as follows:

Definition 2.1.7. Let (κ, r, n, v, w) be a weight tuple with $(\kappa, r) \in \mathcal{W}^G$ and set

$$\lambda_{\kappa, r} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \lambda_1(a) \lambda_2(d)$$

where $\lambda_1 = (r + n)/2$, $\lambda_2 = (r - n)/2$.

Remark 2.1.8. Note that if we map $T(\mathbb{Z}_p)$ to $\mathbb{T}(\mathbb{Z}_p) \times \mathbb{Z}_p^\times$ via $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix} \right) \mapsto (a/d, \mathrm{Norm}(a))$ then our weights on T and $\mathbb{T}$ agree.

Using this, we talk about weights λ on T where we implicitly assume that there is some $(\kappa, r) \in \mathcal{W}^G$ such that $\lambda = \lambda_{\kappa, r}$. This construction then lets us take a weight $\mathcal{W}^G$ and get a weight in $\mathcal{W}'$ as in Remark 2.1.6.

2.2. Eigenvarieties.
In order to define an eigenvariety $\mathscr{X}$, we need to specify the eigendata to which it is associated. We begin by recalling some standard definitions that can be found in [9, 16].

Definition 2.2.1. Let U be a compact operator on Banach space M. We define the *Fredholm determinant* as

$$\mathrm{Fred}_M(U) = \det(1 - XU|M) := \lim_i \det(1 - XU_i|M_i),$$

where $(M_i)_i$ is a sequence of projective and finitely generated Banach modules such that $U_i := U|_{M_i}$ converges to U as $i \to \infty$.

For our purposes, we will study the subspace of $\mathcal{W}^G \times \mathbb{A}^1$ cut out by $f := \mathrm{Fred}_{S^\dagger}(U_p)$, denoted $\mathscr{X}(U_p)$ or $\mathscr{X}(f)$, where $\mathcal{W}^G$ is the weight space as in Definition 2.1.2 and U_p is a compact operator on a space $S^\dagger$ of overconvergent Hilbert modular forms. These will give us our spectral varieties.

Definition 2.2.2. Let $\mathfrak{U} \subset \mathcal{W}^G$ be an affinoid and $\mathscr{X}(f)$ a spectral variety. Define $\mathscr{Z}_{\mathfrak{U}, h} = \mathscr{O}(\mathfrak{U})\langle p^h X \rangle/(f(X))$, which we view as an admissible affinoid open subset of $\mathscr{X}(f)$. We have a natural map $\mathscr{Z}_{\mathfrak{U}, h} \to \mathfrak{U}$, which is flat but might not be finite. We say that $\mathscr{Z}_{\mathfrak{U}, h}$ is *slope-adapted* if the above map is finite and flat.

Remark 2.2.3. It is the possible to show that $\mathscr{Z}_{\mathfrak{U}, h}$ is slope adapted if and only if $f|_{\mathfrak{U}}$ admits a slope $\leq h$ factorization $f|_{\mathfrak{U}}(X) = Q(X)R(X)$, from which we have $\mathscr{O}(\mathscr{Z}_{\mathfrak{U}, h}) = \mathscr{O}(\mathfrak{U})[X]/(Q(X))$. Then [9, Theorem 4.6] tells us that the collection slope-adapted affinoids is an admissible cover of $\mathscr{X}(f)$.

Definition 2.2.4. Let A be commutative Noetherian K-Banach algebra, for K a field complete with respect to a non-trivial non-archimedean norm. Following [9], we say a Banach A-module P satisfies property (Pr) if there is a Banach A-module Q, such that $P \oplus Q$ (with its usual norm) is potentially ON-able.

Definition 2.2.5. Let M_1, M_2 be Banach R-modules satisfying (Pr) for R a reduced affinoid and $\boldsymbol{T}$ a commutative R-algebra with maps $\psi_i : \boldsymbol{T} \to \mathrm{End}_R(M_i)$. Let $U \in \boldsymbol{T}$ act compactly on both M_1 and M_2. A continuous R-module and $\boldsymbol{T}$-module homomorphism $\alpha : M_1 \to M_2$ is called a *primitive link* if there is a compact R-linear and $\boldsymbol{T}$-linear map $c : M_2 \to M_1$ such that $\psi_2(U) : M_2 \to M_2$ is $\alpha \circ c$ and $\psi_1(U) : M_1 \to M_1$ is $c \circ \alpha$. More generally a continuous R-module and $\boldsymbol{T}$-module homomorphism $\alpha : M' \to M$ is a *link* if there exists a sequence M_i of Banach R-modules satisfying (Pr) for $i \in \{0, \ldots, n\}$ such that $M' = M_0$, $M = M_n$ and α factors as a compositum of maps $\alpha_i : M_i \to M_{i+1}$ with α_i a primitive link.

Definition 2.2.6. Let $\mathcal{W}$ be a reduced rigid space, R a reduced affinoid and $\boldsymbol{T}$ be a commutative R-algebra with a specified element U. For admissible affinoid open $\mathfrak{U} \subset \mathcal{W}$ let $M(\mathfrak{U})$ a Banach $\mathscr{O}(\mathfrak{U})$-module satisfying (Pr) with an R-module homomorphism $\psi_{\mathfrak{U}} : \boldsymbol{T} \to \mathrm{End}_{\mathscr{O}(\mathfrak{U})}(M_{\mathfrak{U}})$ such that $\psi_{\mathfrak{U}}(U)$ is compact. Finally assume that if $\mathfrak{U} \subset \mathfrak{U}' \in \mathcal{W}$ are two admissible affinoid opens, then there is a continuous $\mathscr{O}(\mathfrak{U})$-module homomorphism $\alpha : M_{\mathfrak{U}} \to M_{\mathfrak{U}'} \hat{\otimes}_{\mathscr{O}(\mathfrak{U}')} \mathscr{O}(\mathfrak{U})$ which is a link and such that if $\mathfrak{U}_1 \subset \mathfrak{U}_2 \subset \mathfrak{U}_2 \subset \mathcal{W}$ are all affinoid subdomains then $\alpha_{13} = \alpha_{23} \circ \alpha_{12}$ for $\alpha_{ij} : M_{\mathfrak{U}_i} \to M_{\mathfrak{U}_i} \hat{\otimes}_{\mathscr{O}(\mathfrak{U}_i)} \mathscr{O}(\mathfrak{U}_j)$.

We give the name of *eigendata* or *eigenvariety data*, to tuple $\mathfrak{E} = (\mathcal{W}, \mathscr{M}, \boldsymbol{T}, U)$ where $\mathscr{M}$ is the coherent sheaf defined by the $M_{\mathfrak{U}}$.

Theorem 2.2.7 (The Eigenmachine). *Attached to $\mathfrak{E} = (\mathcal{W}, \mathscr{M}, \boldsymbol{T}, U)$ there is a canonically associated rigid space $\mathscr{X}(\mathfrak{E})$ with a finite morphism to the spectral variety $\mathscr{Z}(U)$ defined by U and whose points over $z \in \mathscr{Z}(U)$ are in bijection with the generalized eigenspace for the action of $\boldsymbol{T}$ on the fibre $\mathscr{M}_z$. Moreover, if $\mathcal{W}$ is equidimensional of dimension n, then so is $\mathscr{X}(\mathfrak{E})$.*

Proof. This follows from [9, Construction 5.7, Lemmas 5.8-5.9]. $\qquad \square$

2.3. The Interpolation Theorem. In this section we will recall Chenevier's interpolation theorem, which we will use in order to interpolate the Jacquet-Langlands correspondence. For this we will need to find a very Zariski dense subset of the weight space, together with a classical structure on it. As the name suggests, the classical structure will be given by the subspace of classical modular forms inside the space of overconvergent modular forms. We then use this to find closed immersions between different eigenvarieties by relating their classical structures. In our case, it will be the classical Jacquet-Langlands correspondence that will allow us to relate the classical structures. Following [11], we have:

Definition 2.3.1. A subset $X \subset Z$ is Zariski dense in Z if for every analytic subset (see [6, Section 9.5.2]) $Y \subset Z$ such that $X \subset Y$, then $Y = Z$.

Definition 2.3.2. A Zariski dense subset $X \subset \mathcal{W}^G(\mathbb{C}_p)$ is *very Zariski dense* if for each $x \in X$ and each irreducible admissible affinoid open $V \subset \mathcal{W}^G$ containing x, we have that $V(\mathbb{C}_p) \cap X$ is Zariski dense in X.

From this we define the classical structures as follows:

Definition 2.3.3. Let $\mathfrak{E} = (\mathcal{W}, \mathcal{M}, \boldsymbol{T}, U)$ be a set of eigendata as above and let $X \subset \mathcal{W}$ be a very Zariski dense subset. For each $x \in X$, let $\mathcal{M}_x^{cl}$ be a finite dimensional $\boldsymbol{T}$-module contained in $\mathcal{M}_x$ and, for every $h \in \mathbb{R}$, set $X_h = \{x \in X \mid \mathcal{M}_x^{\leq h} \subset \mathcal{M}_x^{cl}\}$. We say that $\mathcal{M}^{cl}$ gives a *classical structure* on X if for every open affinoid neighbourhood $V \subset \mathcal{W}$ and every h, the sets $X \cap V, X_h \cap V$ have the same Zariski closure in V.

Definition 2.3.4. If $\mathscr{X}$ is an eigenvariety, with eigendata $\mathfrak{E} = (\mathcal{W}, \mathcal{M}, \boldsymbol{T}, U)$, we denote the *nilreduction* of $\mathscr{X}$ by $\mathscr{X}^{red}$, and we say that an eigenvariety is *reduced* if $\mathscr{X}^{red} \cong \mathscr{X}$.

With these definitions we can now state the Chenevier's *Interpolation theorem*.

Theorem 2.3.5. *(Chenevier) Let $\mathscr{X}_i$ be eigenvarieties associated to the eigendata of $\mathfrak{E}_i = (\mathcal{W}_i, \mathcal{M}_i, \boldsymbol{T}_i, \psi_i)$, for $i = 1, 2$ with $\mathcal{W} = \mathcal{W}_1 = \mathcal{W}_2$ and $\boldsymbol{T} = \boldsymbol{T}_1 = \boldsymbol{T}_2$. Let $X \subset \mathcal{W}$ a very Zariski dense subset such that $\mathcal{M}_i^{cl}$ is a classical structure on X for each $\mathcal{M}_i$. Assume that, for all $t \in \boldsymbol{T}$ and all $x \in X$, we have*

$$\det\left(1 - \psi_1(tU)Y \mid_{\mathcal{M}_{1,x}^{cl}}\right) \ \text{divides} \ \det\left(1 - \psi_2(tU)Y \mid_{\mathcal{M}_{2,x}^{cl}}\right)$$

in $k(x)[Y]$, where $k(x)$ is the residue field at x. Then, there is a canonical closed immersion $\iota : \mathscr{X}_1^{red} \hookrightarrow \mathscr{X}_2^{red}$ such that the following diagrams commute

Proof. See [11, Theorem 1].

$\square$

Corollary 2.3.6. *If $\det\left(1 - \psi_1(tU)Y_{|\mathcal{M}_{1,x}^{cl}}\right) = \det\left(1 - \psi_2(tU)Y_{|\mathcal{M}_{2,x}^{cl}}\right)$ in $k(x)[Y]$ for all $t \in \boldsymbol{T}$ and all $x \in X$, then there is an isomorphism $\mathscr{X}_1^{red} \cong \mathscr{X}_2^{red}$.*

Proof. In this case the above Theorem gives us two closed immersions $\iota_{12} : \mathscr{X}_1^{red} \hookrightarrow \mathscr{X}_2^{red}$ and $\iota_{21} : \mathscr{X}_2^{red} \hookrightarrow \mathscr{X}_1^{red}$, from which the result follows at once by noting that $\iota_{12}\iota_{21} = Id_{\mathscr{X}_2}$ and $\iota_{21}\iota_{12} = Id_{\mathscr{X}_1}$.

$\square$

We now have a result of Chenevier that gives a criterion for an eigenvariety to be reduced. Suppose that $X \subset \mathcal{W}$ is a very Zariski dense subset giving a classical structure. For $h \in \mathbb{R}$, let $X_h^{ss} = \{x \in X \mid \mathcal{M}_x^{cl} \cap \mathcal{M}_x^{\leq h}$ is a semisimple $\boldsymbol{T}$-module$\}$.

Lemma 2.3.7. *If for all $h \in \mathbb{R}$, $x \in X$ and $V \subset \mathcal{W}$ an open affinoid containing x, there exists $W \subset V$ an open affinoid containing x, such that $X_h^{ss} \cap W$ contains an open Zariski dense subset of $X \cap W$, then $\mathscr{X}$ is reduced (here we view $X \cap W$ as a topological subspace of W with the Zariski topology).*

Proof. See [11, Proposition 3.9].

$\square$

3. Eigenvariety of Hilbert modular cusp forms

In this section we define the spaces of Hilbert modular forms, in a way which generalizes to give the spaces of overconvergent Hilbert modular cusp forms. We will only discuss the classical spaces here and give some indication of what is needed to extend this to the overconvergent setting as is done in [2]. Throughout this section our chosen prime p may be ramified unless otherwise stated.

To define the spaces of Hilbert modular forms for G, we first work with the group $G^* = G \times_{\mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m} \mathbb{G}_m$ and define the spaces of modular forms for G^*. Then using a projector one gets the definition for G. The reason for working with G^* is that the relevant moduli problem associated to G^* is representable while the one for G is not.

Following Hida [19], we consider the fibered category $\mathcal{A}_F$ of abelian schemes over the category of schemes.

Definition 3.1.1. An object of $\mathcal{A}_F$ is a triple (A, λ, ι) where:

(1) A is an abelian scheme $A \to S$ of relative dimension g over S.
(2) $\mathcal{O}_F$-multiplication given by an embedding $\iota : \mathcal{O}_F \hookrightarrow \mathrm{End}_S(A)$.
(3) Let $\mathfrak{c}$ be a fractional ideal of F and $\mathfrak{c}^+$ its cone of totally positive elements. If $P \in \mathrm{Hom}_{\mathcal{O}_F}(A, A^\vee)$ is a sheaf for the étale topology on S of symmetric $\mathcal{O}_F$-linear homomorphisms from A to the dual abelian scheme $A^\vee$, and if $P^+ \subset P$ is the subset of polarizations, then

$$\lambda : (P, P^+) \longrightarrow (\mathfrak{c}, \mathfrak{c}^+)$$

is an isomorphism of étale sheaves, as invertible $\mathcal{O}_F$-modules with a notion of positivity. The triple $(A, A^\vee, \mathfrak{c})$ is subject to the Deligne-Pappas conditions, which means the map $A \otimes_{\mathcal{O}_F} \mathfrak{c} \to A^\vee$ is an isomorphism of abelian schemes.

See [19, Section 4.1] and [2] for more details.

We now state the moduli problem for G^* associated to the $\mu_\mathfrak{n}$-level structure on an abelian variety.

Definition 3.1.2. Let (A, ι, λ) in $\mathcal{A}_F$ (so λ is a $\mathfrak{c}$-polarization). Let $\mathfrak{n}$ be a non-zero ideal and let $\mu_\mathfrak{n}$ denote the locally free group scheme of finite rank given by $\mu_\mathfrak{n}(R) = \{x \in \mathbb{G}_m(R) \otimes_\mathbb{Z} \mathfrak{d}_F^{-1} | \mathfrak{n}x = 0\}$. Let $\mathfrak{n} \cap \mathbb{Z} = (N)$ and let $M(\mathfrak{c}, \mu_\mathfrak{n})$ be the Hilbert moduli scheme representing the functor $\mathcal{E}_{\mu_\mathfrak{n}} : \mathrm{Sch}/_{\mathbb{Z}[1/N]} \to \mathrm{Set}$ where $\mathcal{E}_{\mu_\mathfrak{n}}(S)$ is the set of isomorphism classes of $(A_{/S}, \iota, \lambda, \Phi_\mathfrak{n})$. Here $\Phi_\mathfrak{n} : \mu_\mathfrak{n} \hookrightarrow A[N]_{/S}$ is a closed immersion compatible with $\mathcal{O}_F$-actions. We call such a $\Phi_\mathfrak{n}$ a $\mu_\mathfrak{n}$-*level structure* on A.[c]

Remark 3.1.3. For a fractional ideal $\mathfrak{b} \in \mathcal{O}_F$ let $\mathfrak{b}^* = \mathfrak{b}^{-1} \mathfrak{d}_F^{-1}$. Let

$$\Gamma_0(\mathfrak{b}, \mathfrak{n}) = \mathrm{GL}_2(F)^+ \cap \begin{pmatrix} \mathcal{O}_F & \mathfrak{b}^* \\ \mathfrak{b}\mathfrak{n}\mathfrak{d}_F & \mathcal{O}_F \end{pmatrix}$$

$$\Gamma_1(\mathfrak{b}, \mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathfrak{b}, \mathfrak{n}) \mid d \equiv 1 \mod \mathfrak{n} \right\}$$

$$\Gamma_1^1(\mathfrak{b}, \mathfrak{n}) = \Gamma_1^1(\mathfrak{b}, \mathfrak{n}) \cap \mathrm{SL}_2(F).$$

Then one can show that, for $\mathfrak{n}$ with $(N) = \mathfrak{n} \cap \mathbb{Z}$ such that $N \geq 3$, $M(\mathfrak{c}, \mu_\mathfrak{n})(\mathbb{C}) = \Gamma_1^1(\mathfrak{c}, \mathfrak{n}) \backslash \mathcal{H}^g$.

Notation 3.1.4. Let $M(\mathfrak{c}, \mu_\mathfrak{n})$ denote the scheme representing the corresponding moduli problem (for G^*). Denote by $\overline{M}(\mathfrak{c}, \mu_\mathfrak{n})$ (resp. $\overline{M}^*(\mathfrak{c}, \mu_\mathfrak{n})$) a fixed *toroidal* (resp. *the minimal*) compactification of $M(\mathfrak{c}, \mu_\mathfrak{n})$.

Definition 3.1.5. Let A be a versal semiabelian scheme over $\overline{M}(\mathfrak{c}, \mu_\mathfrak{n})$, with real multiplication by $\mathcal{O}_F$. Define ω_A to be the conormal sheaf to the identity of A.

There exist a greatest open subscheme $\overline{M}^R(\mathfrak{c}, \mu_\mathfrak{n}) \subset \overline{M}(\mathfrak{c}, \mu_\mathfrak{n})$ such that ω_A is an invertible $\mathcal{O}_{\overline{M}^R(\mathfrak{c}, \mu_\mathfrak{n})} \otimes_\mathbb{Z} \mathcal{O}_F$-module. We will use ω_A to define *another* invertible sheaf whose sections will be our Hilbert modular forms. But in order to do so, we first need to define the weight space for G^* and showing how it is related to $\mathcal{W}^G$.

Definition 3.1.6. The weight space $\mathcal{W}^{G^*}$ for G^* is defined by setting $\mathcal{W}^{G^*}$ to be the rigid analytic space over L associated to $\mathcal{O}_L[\![\mathbb{T}(\mathbb{Z}_p))]\!]$, where $\mathbb{T}$ and L are as before. We will denote such weights by κ or k.

There is a canonical map $\mathcal{W}^G \to \mathcal{W}^{G^*}$ induced from 2.1.5. We define a classical algebraic weight for G^* as a map from $\mathbb{T}(\mathbb{Z}_p)$ to $\mathbb{C}_p$ defined by an element $k \in \mathbb{Z}_{\geq 0}^g$, as usual.

We now construct an invertible sheaf associated to classical weights in $\mathcal{W}^{G^*}$, from which we can then define the spaces of Hilbert modular forms for G^*:

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[c]If we take $\mu_{(N)}$ with $N \geq 3$, then the associated moduli problem is representable by a scheme $M(\mathfrak{c}, \mu_{(N)})$ (cf. [15, Chapter 3, Theorem 6.9]).

Definition 3.1.7. Recall that L is a splitting field for F, and let $k \in \mathbb{Z}^{\Sigma_\infty}$ be a classical weight for G^*. Then, define the invertible modular sheaf

$$\Omega^k := \bigotimes_{v \in \Sigma_\infty} \omega_{\boldsymbol{A},v}^{\otimes k_v},$$

where $\omega_{\boldsymbol{A},v} := \omega_{\boldsymbol{A}} \otimes_v \mathcal{O}_L$ and here $\otimes_v$ denotes the tensor over $\mathcal{O}_L \otimes \mathcal{O}_F$ via $1 \otimes \iota_v$.

Definition 3.1.8. The L-vector space of $\mathfrak{c}$-polarized, tame level $\Gamma_1^1(\mathfrak{c}, \mathfrak{n})$ and weight k *Hilbert modular forms for G^** is defined by

$$M_k(\Gamma_1^1(\mathfrak{c}, \mathfrak{n})) := H^0(\overline{M}^R(\mathfrak{c}, \mu_{\mathfrak{n}}), \Omega^k).$$

The subspace of cusp forms is defined by

$$S_k(\Gamma_1^1(\mathfrak{c}, \mathfrak{n})) := H^0(\overline{M}^R(\mathfrak{c}, \mu_{\mathfrak{n}}), \Omega^k(-B)),$$

where $B := \overline{M}(\mathfrak{c}, \mu_{\mathfrak{n}}) \backslash M(\mathfrak{c}, \mu_{\mathfrak{n}})$ is the boundary divisor in the toroidal compactification.

To define the spaces of Hilbert modular forms associated to G, we need to introduce a certain projector. First, we note that multiplication by $\epsilon \in \mathcal{O}_F^{\times,+}$ gives an isomorphism $(A, \iota, \lambda, \Phi) \cong (A, \iota, \epsilon^2\lambda, \epsilon\Phi)$. Now, let $\mathfrak{S}_{\mathfrak{n}}$ be the elements of $\mathcal{O}_F^{\times,+}$ congruent to 1 modulo $\mathfrak{n}$. Define an action of $\mathfrak{O} := \mathcal{O}_F^{\times,+}/\mathfrak{S}_{\mathfrak{n}}^2$ on $M(\mathfrak{c}, \mu_{\mathfrak{n}})$, by $\epsilon \cdot (A, \iota, \lambda, \Phi) := (A, \iota, \epsilon\lambda, \Phi)$. Since multiplication by ϵ gives an isomorphism $\epsilon : A \to A$ such that $\epsilon^*\lambda = \epsilon^2\lambda$, it follows that, if $\epsilon = \eta^2 \in \mathfrak{S}_{\mathfrak{n}}$, then ϵ acts trivially on $(A, \iota, \lambda, \Phi)$; hence the action factors through $\mathfrak{O}$ as required.

Definition 3.1.9. Let (k, r, n, v, w) be a weight tuple with (k, r) a classical algebraic weight (for G). We define an action of $\mathfrak{O}$ on Ω^k by sending a local section f of Ω^k on $M^R(\mathfrak{c}, \mu_{\mathfrak{n}})$ to

$$(\epsilon \cdot f) : (A, \iota, \lambda, \Phi, \beta) \to w(\epsilon)f(A, \iota, \epsilon^{-1}\lambda, \Phi, \beta)$$

where $\epsilon \in \mathcal{O}_F^{\times,+}$ and β is a local generator for $\omega_{\boldsymbol{A}}$ as a $\mathcal{O}_{M^R(\mathfrak{c}, \mu_{\mathfrak{n}})} \otimes \mathcal{O}_F$-module. If $\epsilon = \eta^2 \in \mathfrak{S}_{\mathfrak{n}}$, then this acts trivially. Hence the action factors through $\mathfrak{O}$. With this we define a *projector* $\mathfrak{e}_{k,r} : M_k(\Gamma_1^1(\mathfrak{c}, \mathfrak{n})) \to M_k(\Gamma_1^1(\mathfrak{c}, \mathfrak{n}))$ by

$$\mathfrak{e}_{k,r} := \frac{1}{|\mathfrak{O}|} \sum_{\epsilon \in \mathfrak{O}} \epsilon.$$

Definition 3.1.10. The L-vector space of *classical Hilbert modular forms for G of level $\Gamma_1(\mathfrak{c}, \mathfrak{n})$, and weight (k, r)* is defined to be the image of $\mathfrak{e}_{k,r}$ and is denoted $M_{k,r}^G(\Gamma_1(\mathfrak{c}, \mathfrak{n}))$. Similarly, we let $S_{k,r}^G(\Gamma_1(\mathfrak{c}, \mathfrak{n}))$ be the image of $S_k(\Gamma_1^1(\mathfrak{c}, \mathfrak{n}))$ under $\mathfrak{e}_{k,r}$.

Remark 3.1.11. We note here that there are other ways of defining Hilbert modular forms for G as sections of a sheaf $\Omega^{(k,r)}$ on $M^G(\mathfrak{c}, \mu_{\mathfrak{n}})$ (cf. [26, Section 2.2]). Working over $\mathbb{C}$ one then recovers the spaces $S_{k,r}(\Gamma_1(\mathfrak{c}, \mathfrak{n}))$ as defined above. To see the relationship with our definition one observes that there is a morphism $m : M^R(\mathfrak{c}, \mu_{\mathfrak{n}}) \to M^G(\mathfrak{c}, \mu_{\mathfrak{n}})$ which is finite and Galois, with Galois group $\mathfrak{O}$ such that $\Omega^{k,r} = (m_*(\Omega^k))^{\mathfrak{O}}$ (cf. [2, Section 1]).

Note that these spaces will not be fixed by the Hecke operators. In fact, note that $F^{\times,+}$ acts on the pairs $(\mathfrak{c}, \mathfrak{c}^+)$ by $\epsilon(\mathfrak{c}, \mathfrak{c}^+) = (\epsilon\mathfrak{c}, \epsilon\mathfrak{c}^+)$, which induces an isomorphism $\alpha_\epsilon : M_{k,r}^G(\Gamma_1(\mathfrak{c}, \mathfrak{n})) \to M_{k,r}^G(\Gamma_1(\epsilon\mathfrak{c}, \mathfrak{n}))$. Moreover, if $\epsilon \in \mathcal{O}_F^{\times,+}$, then $\alpha_\epsilon(f) = f$ for all $f \in M_{k,r}^G(\Gamma_1(\mathfrak{c}, \mathfrak{n}))$. To fix this, we must work adelically. Let $U_1^G(\mathfrak{n}) = \left\{ \gamma \in G(\widehat{\mathbb{Z}}) \mid \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \mod \mathfrak{n} \right\}$. We can now define the space of Hilbert modular forms for G.

Definition 3.1.12. We define the space of *classical Hilbert modular forms for G of level $U_1(\mathfrak{n})$ and weight (k, r)* denoted $M_{k,r}^G(U_1^G(\mathfrak{n}))$ as V/I where

$$V := \bigoplus_{(\mathfrak{c}, \mathfrak{c}^+)} M_{k,r}^G(\Gamma_1(\mathfrak{c}, \mathfrak{n}))$$

and $I = (f - \alpha_\epsilon(f))_{\epsilon \in \left(F^{\times,+}/\mathcal{O}_F^{\times,+}\right)}$. We define $S_{k,r}^G(U_1^G(\mathfrak{n}))$ similarly.

One can define Hecke operators on these spaces, which satisfy the all the usual properties. We denote by $\boldsymbol{T}(U)$ the Hecke algebra consisting of Hecke operators away from the level together with $U_{\mathfrak{p}}$ for $\mathfrak{p} \in \Sigma_p$. Moreover, when we come to the Jacquet-Langlands correspondence, we will assume that $\boldsymbol{T}(U)$ does not contain Hecke operators at the finite places dividing the discriminant of the relevant quaternion algebra (if there are any).

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3.2. **Overconvergent spaces**. Our goal is now to associate to each weight (κ, r) or family of weights $(\kappa^{\mathfrak{U}}, r^{\mathfrak{U}})$, a space of overconvergent Hilbert modular forms. There are several constructions of these spaces but we will be interested in the construction given by [2]. In this case, one defines an overconvergent sheaf which interpolates the sheaf Ω^k. Using this, one can then define the spaces of overconvergent Hilbert modular forms for G^* and then, using a projector, define the spaces for G. The construction of the overconvergent sheaf can be found in [2, 1, 3, 17], so we only give some of its properties.

Let f be the number of primes above p in F and let $t_m \in \mathbb{Q}^f$ be a multi-index with $0 < t_i \leq \frac{1}{p^m}$ for $m \geq 1$. Let $\overline{\mathfrak{M}}(\mathfrak{c}, \mu_{\mathfrak{n}})$ and $\overline{\mathfrak{M}}^*(\mathfrak{c}, \mu_{\mathfrak{n}})$ denote the formal completions of $\overline{M}(\mathfrak{c}, \mu_{\mathfrak{n}}), \overline{M}^*(\mathfrak{c}, \mu_{\mathfrak{n}})$ along their special fibres. Now, let $\overline{\mathcal{M}}(\mathfrak{c}, \mu_{\mathfrak{n}}), \overline{\mathcal{M}}^*(\mathfrak{c}, \mu_{\mathfrak{n}})$ denote the rigid fibres of $\overline{\mathfrak{M}}(\mathfrak{c}, \mu_{\mathfrak{n}}), \overline{\mathfrak{M}}^*(\mathfrak{c}, \mu_{\mathfrak{n}})$ respectively and let $\overline{\mathcal{M}}(\mathfrak{c}, \mu_{\mathfrak{n}}, t_m), \overline{\mathcal{M}}^*(\mathfrak{c}, \mu_{\mathfrak{n}}, t_m)$ denote the neighbourhoods of the respective ordinary locus defined by the condition that $\mathrm{val}_p(h_{\mathfrak{p}_i}) \cap [0, 1] \leq t_i$ (this is the truncated valuation), where $h_{\mathfrak{p}_i}$ are the *partial Hasse invariants* as defined in [2, 3.2.1].

Before continuing let us recall what it means for a weight to be m-analytic, as in [2]:

Definition 3.2.1. Let $\widehat{\mathbb{T}}$ be the formal group obtained by completing $\mathbb{T}$ along the unit section and let $\mathbb{T}_s^0$ be the formal subgroup of $\widehat{\mathbb{T}}$ of elements which are congruent to $1 \mod p^s$. We say a weight $\kappa \in \mathcal{W}^{G^*}$ is m-analytic for some $m \in \mathbb{Z}_{\geq 0}$ if its restriction to $\mathbb{T}_m^0(\mathbb{Z}_p)$ factors as $\exp \circ \psi \circ \log$ where ψ is some $\mathbb{Z}_p$-linear map and $\exp, \log$ are the p-adic exponential and p-adic logarithm. We say a weight tuple is m-analytic if it corresponds to an m-analytic weight under the map $\mathcal{W}^G \to \mathcal{W}^{G^*}$ from 3.1.6.

Now, the overconvergent sheaves are defined over formal models of $\overline{\mathcal{M}}(\mathfrak{c}, \mu_{\mathfrak{n}}, t_m)$ and $\overline{\mathcal{M}}^*(\mathfrak{c}, \mu_{\mathfrak{n}}, t_m)$, which are obtained as follows: let $\overline{\mathfrak{M}}(\mathfrak{c}, \mu_{\mathfrak{n}}, t_m))$ (resp. $\overline{\mathfrak{M}}^*(\mathfrak{c}, \mu_{\mathfrak{n}}, t_m)$) be the normalization (in $\overline{\mathcal{M}}(\mathfrak{c}, \mu_{\mathfrak{n}}, t_m)$ (resp. $\overline{\mathcal{M}}^*(\mathfrak{c}, \mu_{\mathfrak{n}}, t_m)$) of the formal model of $\overline{\mathcal{M}}(\mathfrak{c}, \mu_{\mathfrak{n}}, t_m)$ (resp. $\overline{\mathcal{M}}^*(\mathfrak{c}, \mu_{\mathfrak{n}}, t_m)$) given by taking iterated blow-ups along the ideals $(h_{\mathfrak{p}_i}, p^{t_i})$ of $\overline{\mathfrak{M}}(\mathfrak{c}, \mu_{\mathfrak{n}})$ (resp. $\overline{\mathfrak{M}}^*(\mathfrak{c}, \mu_{\mathfrak{n}})$) and removing all divisors at infinity. Then on $\overline{\mathfrak{M}}(\mathfrak{c}, \mu_{\mathfrak{n}}, t_m))$ we can construct the following sheaves:

Theorem 3.2.2 (Andreatta–Iovita–Pilloni). *For every m-analytic weight $\kappa \in \mathcal{W}^{G^*}(L)$ there exists a coherent sheaf $\Omega^{\dagger, \kappa}$ of $\mathcal{O}_{\overline{\mathfrak{M}}(\mathfrak{c}, \mu_{\mathfrak{n}}, t_m)}$-modules whose restriction to the rigid analytic fibre $\overline{\mathcal{M}}(\mathfrak{c}, \mu_{\mathfrak{n}}, t_m)$ is invertible.*

More generally, to each affinoid $\mathfrak{U}$ with a morphism $\mathfrak{U} \to \mathcal{W}^{G^*}$ and m such that $\kappa^{\mathfrak{U}}$ is locally m-analytic, one can attach a coherent sheaf $\Omega^{\dagger, \kappa^{\mathfrak{U}}}$ of $\mathcal{O}_{\overline{\mathfrak{M}}(\mathfrak{c}, \mu_{\mathfrak{n}}, t_m) \times \hat{\mathfrak{U}}}$-modules where $\hat{\mathfrak{U}} = \mathrm{Spf}(A)$ is the formal model of $\mathfrak{U}$, where A consists of power bounded elements of $\mathfrak{U}$. Moreover, the restriction of $\Omega^{\dagger, \kappa^{\mathfrak{U}}}$ to the rigid fibre is invertible. Lastly, if k is a classical weight (for G^*), then $\Omega^{\dagger, k}$ agrees on $\overline{\mathcal{M}}(\mathfrak{c}, \mu_{\mathfrak{n}}, t_m)$ with the classical Ω^r.

Proof. See [2, Section 3.4-3.5]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remark 3.2.3. In general, the $\Omega^{\dagger, \kappa^{\mathfrak{U}}}$ and $\Omega^{\dagger, \kappa}$ depend on m, but when restricted to the rigid fibres, they are independent of m, for this reason we have suppressed the dependence on m. See [2, Proposition 3.9 and Proposition 3.13].

Using these sheaves, one can then define the spaces of t_m-overconvergent cuspidal Hilbert modular forms for G^* of weight $\kappa^{\mathfrak{U}}$ by setting

$$S_{\mathfrak{U}}^{\dagger}(\Gamma_1^1(\mathfrak{c}, \mathfrak{n}), t_m) = \mathrm{H}^0(\overline{\mathcal{M}}(\mathfrak{c}, \mu_{\mathfrak{n}}, t_m) \times \mathfrak{U}, \Omega^{\dagger, \kappa^{\mathfrak{U}}}(-B))$$

where B is again the boundary divisor. From this, one then uses a projector $\mathfrak{e}_{\kappa^{\mathfrak{U}}, r^{\mathfrak{U}}}$ to define families of t_m-overconvergent cuspidal Hilbert modular forms for G of weight $(\kappa^{\mathfrak{U}}, r^{\mathfrak{U}})$ denoted $S_{\mathfrak{U}}^{G, \dagger}(\Gamma_1(\mathfrak{c}, \mathfrak{n}), t_m)$. Moreover, taking $\mathfrak{U} = \mathrm{Spf}(L)$ gives $S_{\kappa, r}^{G, \dagger}(\Gamma_1(\mathfrak{c}, \mathfrak{n}), t_m)$.

Theorem 3.2.4. *Let $\mathfrak{U}$ be an admissible open affinoid of $\mathcal{W}^G$ and $(\kappa^{\mathfrak{U}}, r^{\mathfrak{U}})$ the restriction of the universal character to $\mathfrak{U}$. Let A be the algebra of power bounded elements of $\mathfrak{U}$. Then for an appropriate[d] choice of m and t_m the spaces $S_{\mathfrak{U}}^{G, \dagger}(\Gamma_1(\mathfrak{c}, \mathfrak{n}), t_m)$ are Banach $(A \otimes_{\mathcal{O}_L} L)$-modules satisfying (Pr). Moreover, for any weight $(\kappa, r) \in \mathfrak{U}(L)$ there is a natural specialization map*

$$S_{\mathfrak{U}}^{G, \dagger}(\Gamma_1(\mathfrak{c}, \mathfrak{n}), t_m) \longrightarrow S_{\kappa, r}^{G, \dagger}(\Gamma_1(\mathfrak{c}, \mathfrak{n}), t_m)$$

which is surjective.

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[d]This means such that $\kappa^{\mathfrak{U}}$ is m-analytic (see Definition 3.2.1).

Proof. This is [2, Theorem 4.4]. □

Now as before these spaces have an action of $F^{\times,+}$ so they will not be fixed under the action of Hecke operators. In particular, we have:

Lemma 3.2.5. *Let $\epsilon \in F^{\times,+}$ and assume that ϵ is also a p-adic unit. Then there is a canonical isomorphism*

$$\mathscr{L}_\epsilon : S_{\mathfrak{U}}^{G,\dagger}(\Gamma_1(\mathfrak{c}, \mathfrak{n}), t) \longrightarrow S_{\mathfrak{U}}^{G,\dagger}(\Gamma_1(\epsilon\mathfrak{c}, \mathfrak{n}), t)$$

which only depends on ϵ modulo totally positive units.

Proof. This is [2, Lemma 4.5]. □

Definition 3.2.6. Let

$$S_{\mathfrak{U}}^{G,\dagger}(U_1(\mathfrak{n}), t) := \left(\bigoplus_{\mathfrak{c} \in Frac(F)^{(p)}} S_{\mathfrak{U}}^{G,\dagger}(\Gamma_1(\mathfrak{c}, \mathfrak{n}), t) \right) / (\mathscr{L}_\epsilon(f) - f)_{\epsilon \in Princ(F)^{+,(p)}}$$

be the Banach module of tame level U, t-*overconvergent cuspidal arithmetic Hilbert modular forms* for G with weights parametrized by $\mathfrak{U}$. Here $Frac(F)^{(p)}$ is the group of fractional ideals prime to p and $Princ(F)^{+,(p)}$ is the group of positive elements which are p-adic units.

Moreover, taking the limit over t we get Frechet spaces $S_{\mathfrak{U}}^{G,\dagger}(U_1(\mathfrak{n}))$ which give a quasi-coherent sheaf of overconvergent cuspidal arithmetic Hilbert modular forms $S^{G,\dagger}(U_1(\mathfrak{n}))$ over $\mathcal{W}^G$, whose value at an open affinoid $\mathfrak{U} \subset \mathcal{W}^G$ is $S_{\mathfrak{U}}^{G,\dagger}(U_1(\mathfrak{n}))$.

Remark 3.2.7. Note that taking $\mathfrak{U} = \mathrm{Sp}(L)$ with image (κ, r) in $\mathcal{W}^G$ will give the spaces of this fixed weight.

Following [2, Section 4.3], for $\mathfrak{q}$ prime to the tame level, one can define commuting Hecke operators $T_\mathfrak{q}, S_\mathfrak{q}$ action on $S^{G,\dagger}(U_1(\mathfrak{n}))$. Moreover, for $\mathfrak{p}|p$ one can define operators $U_\mathfrak{p}$ such that $U_p = \prod_{\mathfrak{p}|p} U_\mathfrak{p}^{e_\mathfrak{p}}$ for $e_\mathfrak{p}$ the ramification degree of $\mathfrak{p}$.

Proposition 3.2.8. *The U_p operator is a compact operator on $S_{\kappa,r}^{G,\dagger}(U_1(\mathfrak{n}))$ for any weight (κ, r).*

Proof. This follows from [2, Lemma 3.27]. □

Definition 3.2.9. Let $h \in \mathbb{Q}_{\geq 0}$. We say an element $f \in S_{\kappa,r}^{G,\dagger}(U_1(\mathfrak{n}))$ has *slope-$\leq h$* for U_p (resp. $U_\mathfrak{p}$ for $\mathfrak{p}|p$) if it is annihilated by a unitary polynomial in U_p (resp. $U_\mathfrak{p}$) whose roots have valuation less than h.[e]

We now wish to use Buzzard's Eigenmachine to construct the eigenvariety of Hilbert modular forms. One of the key ingredients is the existence of links which is checked explicitly in [17, Section 3.3.3].

Theorem 3.2.10. *Associated to the eigendata of $(\mathcal{W}^G, S^{G,\dagger}(U_1(\mathfrak{n})), \boldsymbol{T}, U_p)$ we have an eigenvariety $\mathscr{X}_G(U_1(\mathfrak{n}))$ with the following properties:*

(a) *It is equidimensional of dimension $g + 1$.*
(b) *There is a universal character $\phi : \boldsymbol{T} \to \mathcal{O}_{\mathscr{X}}$.*
(c) *There is a map $\alpha : \mathscr{X} \to \mathcal{W}^G$ that is locally on $\mathscr{X}$ and $\mathcal{W}^G$, finite and surjective.*
(d) *For all $(\kappa, r) \in \mathcal{W}^G$, the points $\alpha^{-1}(\kappa, r)$ are in bijection with the finite slope eigensystems occurring in $S^{G,\dagger}(U_1(\mathfrak{n}))\mid_{\kappa,r} = S_{\kappa,r}^{G,\dagger}(U_1(\mathfrak{n}))$.*
(e) *Let (k, r) be a classical weight in $\mathcal{W}^G$. Let $f \in S_{k,r}^{G,\dagger}(U)$ be a finite slope (for U_p) overconvergent Hilbert modular form whose $U_{\mathfrak{p}_i}$ slope is less than h_i for $\mathfrak{p}_i \in \Sigma_p$. If p is unramified and $h_i < v_{\mathfrak{p}_i}(k, r) + \min_{j \in \Sigma_{\mathfrak{p}_i}}\{k_j - 1\}$ for all i, then f is a classical form.*

Proof. This is essentially [2, Theorem 5.1], the only modification is that using a stronger version of the control theorem due to [26], which matches up with the one we will have on the quaternionic side. □

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[e] Note that if f is in fact an eigenform, then having slope-$\leq h$ for U_p (resp. $U_\mathfrak{p}$) is saying that the p-adic valuation of the U_p (resp. $U_\mathfrak{p}$) eigenvalue is less than h.

4. Eigenvarieties for totally definite quaternion algebras

Following [9, Part III], we will define the eigenvariety attached to a totally definite quaternion algebra D over F and prove the control theorem in this setting. In contrast to [9], we will work with the weight space $\mathcal{W}^G$ which, as we explained above, has the advantage of being equidimensional of dimension $g + 1$ (recall $[F : \mathbb{Q}] = g$). Apart from this small detail, the rest of our construction spaces of overconvergent quaternionic modular forms over F follows [9, Part III]. Throughout this section our chosen prime p may be ramified unless otherwise stated.

4.1. **Classical spaces**. We will define the spaces of classical quaternionic modular forms using a definition that, compared to [18], is more suited to p-adic interpolation. Most of the set-up will be similar to that of *loc. cit.* but with the crucial difference that the actions are 'shifted' from the infinite places to the places above p. We recall here that D/F is a totally definite quaternion algebra split above p.

Notation 4.1.1. (a) Let $G_D/\mathbb{Q}_p := \mathrm{Res}_{F/\mathbb{Q}}(D^\times) \times \mathbb{Q}_p$, which is a connected reductive linear algebraic group over $\mathbb{Q}_p$ (via our choice of splitting). Let T be the standard maximal torus, B the standard Borel subgroup, and the unipotent radical N. We denote by $\overline{B}$ and $\overline{N}$ be the opposite Borel and opposite unipotent radical. Let $I \subset G_D(\mathbb{Z}_p)$ be the standard Iwahori subgroup in good position with respect to B (in good position means that $B, N, T, G_D, \overline{N}$ have fixed compatible integral models over $\mathbb{Z}_p$).

(b) For $m \in \mathbb{Z}_{\geq 1}^{\Sigma_p}$, set

$$I_m = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathbb{Z}_p) \mid c \in \pi^m \mathcal{O}_p \right\},$$

with $I = I_1 = I_{(1,\ldots,1)}$ and let $\overline{I}_m = \overline{N}(\mathbb{Z}_p) \cap I_m$.

Furthermore, we set

$$T^+ = \left\{ t \in T(\mathbb{Q}_p) \mid tN(\mathbb{Z}_p)t^{-1} \subseteq N(\mathbb{Z}_p) \right\} = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in T(\mathbb{Q}_p) \mid ab^{-1} \in \mathcal{O}_p \right\}.$$

With this we define the semigroup $\Delta = IT^+I$. Note that the Iwahori decomposition tells us that $I = \overline{I}_1 T(\mathbb{Z}_p)N(\mathbb{Z}_p)$, and hence any $\delta \in \Delta$ can be written uniquely as $\delta = \overline{n}_\delta t_\delta n_\delta$ with $\overline{n}_\delta \in \overline{I}_1, t_\delta \in T^+, n_\delta \in N(\mathbb{Z}_p)$.

Definition 4.1.2. Let (k, r, n, v, w) be a weight tuple with $(k, r) \in \mathbb{Z}_{\geq 0}^{\Sigma_\infty} \times \mathbb{Z}$. Let V_k be the L-vector space with basis of monomials $\prod_{v \in \Sigma_\infty} Z_v^{m_v}$, with $m \in \mathbb{Z}_{\geq 0}^{\Sigma_\infty}$, $0 \leq m_v \leq k_v - 2$. We define a right action of $\Delta = IT^+I$ on this space as follows: for $\gamma = (\gamma_{\mathfrak{p}})_{\mathfrak{p} \in \Sigma_p} = \begin{pmatrix} a_{\mathfrak{p}} & b_{\mathfrak{p}} \\ c_{\mathfrak{p}} & d_{\mathfrak{p}} \end{pmatrix}_{\mathfrak{p}} \in \Delta$, let

$$\gamma : \prod_{v \in \Sigma_\infty} Z_s^{m_s} \longmapsto \prod_{v \in \Sigma_\infty} (c_v Z_v + d_v)^{n_v} \det(\gamma_v)^{v_v} \left(\frac{a_v Z_v + b_v}{c_v Z_s + d_v} \right)^{m_v}$$

Note that here (following [9]) we have adopted the notation that for $a_{\mathfrak{p}}$ (resp. $b_{\mathfrak{p}}, c_{\mathfrak{p}}, d_{\mathfrak{p}}$) we let a_v (resp. b_v, c_v, d_v) denote the image of $a_{\mathfrak{p}}$ under the corresponding map $\iota \circ \iota_v$ for $v \in \Sigma_{\mathfrak{p}}$. Let $V_{n,v}(L)$ denote the resulting Δ-module.[f]

Definition 4.1.3. Let U be an open compact subgroup of $G_D(\mathbb{A}_f)$, such that its image under the projection $U \to D_p^\times$ lies in I_1 and let (k, r, n, v, w) be a weight tuple with $(k, r) \in \mathcal{W}^G$ a classical weight. The space of *quaternionic modular forms* over D of weight k and level U, denoted $S_{k,r}^D(U)$, is the space of functions

$$f : G_D(\mathbb{A}_f) \longrightarrow V_{n,v}(L)$$

such that:

(a) For $\gamma \in G_D(\mathbb{Q})$, we have $f(\gamma g) = f(g)$ for all $g \in G_D(\mathbb{A})$.
(b) For $u \in U$ we have $f(g) = f(gu^{-1}) \cdot u_p$ for all $g \in G_D(\mathbb{A})$, where u_p denotes the p-part of u.

We will only be interested in the following open compact subgroups of $G_D(\mathbb{A}_f)$:

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[f]Note that since we have chosen weights such that $n + 2v$ is parallel, $\mathcal{O}_F^{\times,+}$ will act trivially, when embedded diagonally into I via $\mathcal{O}_F \to \mathcal{O}_p \to M_2(\mathcal{O}_p)$.

Definition 4.1.4. Let $\mathfrak{n} = \prod_v \mathfrak{q}_v^{e_v}$ be an integral ideal. For D a totally definite quaternion algebra[g] with $(\mathrm{Disc}(D), \mathfrak{n}) = 1$, we fix splitting at all primes dividing $\mathfrak{n}$ of D. Then we define

$$U_1(\mathfrak{n}) := \left\{ \gamma \in (\mathcal{O}_D \otimes \widehat{\mathbb{Z}})^\times \mid \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix} \right) \mod \mathfrak{n} \right\},$$

$$U_0(\mathfrak{n}) := \left\{ \gamma \in (\mathcal{O}_D \otimes \widehat{\mathbb{Z}})^\times \mid \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix} \right) \mod \mathfrak{n} \right\},$$

$$U(\mathfrak{n}) := \left\{ \gamma \in (\mathcal{O}_D \otimes \widehat{\mathbb{Z}})^\times \mid \gamma \equiv 1 \mod \mathfrak{n} \right\}.$$

4.2. Overconvergent spaces.
We are now going to define the spaces of overconvergent modular forms for D. For this we need to find a larger Δ-module containing $V_{n,v}(L)$, so work with the spaces of locally analytic functions.

Let $X \subset \mathbb{Q}_p^s$ be open and compact.

Definition 4.2.1. For a finite extension $L/\mathbb{Q}_p$, we say a function $f : X \to L$ is L-*analytic* if it can be expressed as a converging power series $f(x_1, \ldots, x_s) = \sum_{t_1,\ldots,t_s} \alpha_{t_1,\ldots,t_s}(x_1 - a_1)^{t_1} \cdots (x_s - a_s)^{t_s}$, for $\alpha_{t_1,\ldots t_s} \in L$, and some $(a_1, \ldots, a_s) \in X$. We say it is *algebraic* if almost all α's are zero.

Definition 4.2.2. For $m \in \mathbb{Z}_{\geq 0}^r$, let $\mathcal{A}(X, L, m)$ be the L-vector space of m-*locally analytic functions*, i.e., functions that are analytic on balls of radius p^{-m} covering X. Then $\mathcal{A}_m(X, L)$ is a p-adic Banach space when X is compact. We let $\mathcal{A}(X, L) = \bigcup_{m \geq 0} \mathcal{A}(X, L, m)$. This is the space of functions $f : X \to L$ that are m-locally L-analytic for some n.

We now define the Δ-modules that we will be interested in.

Definition 4.2.3. We begin by identifying $\mathcal{O}_p$ with an open compact subset of $\mathbb{Q}_p^g$ compatible with the identification of I as an open compact of $\mathbb{Q}_p^{4g}$. We then consider $\mathcal{A}(\mathcal{O}_p, L) = \bigcup_{m \geq 0} \mathcal{A}(\mathcal{O}_p, L, m)$. This is a Δ-module with the following action. For (κ, r, n, v, w) a weight tuple with $(\kappa, r) \in \mathcal{W}^{\bar{G}}(L)$, $f \in \mathcal{A}(\mathcal{O}_p, L)$, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Delta$ and $z \in \mathcal{O}_p$, let

$$(f \cdot \gamma)(z) = n(cz + d)v(\det(\gamma))f\left(\frac{az + b}{cz + d} \right).$$

We denote this module by $\mathcal{A}_{n,v}(\mathcal{O}_p, L)$.

Lemma 4.2.4. *For $(\kappa, r) \in \mathcal{W}^{\bar{G}}$ there exists a smallest $m(\kappa, r)$, such that for all $m \geq m(\kappa, r)$, (κ, r) is m-locally analytic.*

Proof. See [27, Lemma 3.2.5]. $\square$

From this it follows that $\mathcal{A}_{n,v}(\mathcal{O}_p, L) = \bigcup_{m \geq m(\kappa,r)} \mathcal{A}_{n,v}(\mathcal{O}_p, L, m)$, where $\mathcal{A}_{n,v}(\mathcal{O}_p, L, m)$ is the Δ-module $\mathcal{A}(\mathcal{O}_p, L, m)$ with the action defined as above. More generally, since we wish to consider families of modular forms, one can extend this definition as follows:

Definition 4.2.5. If $\mathfrak{U}$ is an affinoid subdomain of $\mathcal{W}^{\bar{G}}$ defined over a finite extension $L/\mathbb{Q}_p$ and $(\kappa^{\mathfrak{U}}, r^{\mathfrak{U}})$ is the restriction of the universal character to $\mathfrak{U}$, then we define $\mathcal{A}_{\mathfrak{U}}(\mathcal{O}_p, L) := \mathcal{A}_{n^{\mathfrak{U}}, v^{\mathfrak{U}}}(\mathcal{O}_p, L)$, with the action of Δ defined analogously where $(\kappa^{\mathfrak{U}}, r^{\mathfrak{U}}, n^{\mathfrak{U}}, v^{\mathfrak{U}}, w^{\mathfrak{U}})$ is a weight tuple.

It follows from [27, Lemma 3.4.6], that there exists a smallest integer $m(\mathfrak{U})$ such that $(\kappa^{\mathfrak{U}}, r^{\mathfrak{U}})$ is $m(\mathfrak{U})$-analytic. Moreover, $\mathcal{A}_{\mathfrak{U}}(\mathcal{O}_p, L) = \bigcup_{m \geq m(\mathfrak{U})} \mathcal{A}_{\mathfrak{U}}(\mathcal{O}_p, L, m)$.

Definition 4.2.6. Let (κ, r, n, v, w) be a weight tuple with $(\kappa, r) \in \mathcal{W}^{\bar{G}}(L)$ and U be an open compact subgroup of $G_D(\mathbb{A}_f)$, such that its image under the projection $U \to D_p^\times$ lies in I_m for some $m \geq 1$ and $t \in \mathbb{Z}_{\geq 0}^{\Sigma_p}$ is such that $t + m \geq m(\kappa)$. The space of overconvergent quaternionic modular forms of weight κ, level U and radius of overconvergence p^{-t}, denoted $S_{\kappa,r}^{D,\dagger}(U, t)$ is the space of functions

$$f : G_D(\mathbb{A}_f) \longrightarrow \mathcal{A}_{n,v}(\mathcal{O}_p, L, t)$$

such that:

(a) For $d \in G_D(\mathbb{Q})$, we have $f(dg) = f(g)$ for all $g \in G_D(\mathbb{A})$.
(b) For $\gamma \in U$ we have $f(g) = f(g\gamma^{-1}) \cdot \gamma_p$ for all $g \in G_D(\mathbb{A})$, where γ_p is the p-part of γ.

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[g]Note that if we take $D = M_2(F)$ one obtains the usual level structures for Hilbert modular forms.

If $\mathfrak{U} \subset \mathcal{W}^G$ is an affinoid subdomain defined over L and $t \geq m(\mathfrak{U})$, then define $S_{\mathfrak{U}}^{D,\dagger}(U,t)$ to be the space of functions $f : G_D(\mathbb{A}_f) \longrightarrow \mathcal{A}_{\mathfrak{U}}(\mathcal{O}_p, L, t)$ satisfying $(a),(b)$ above. Lastly, taking the limit over t we obtain Fréchet spaces $S_{\kappa,r}^{D,\dagger}(U)$.

4.3. **Hecke operators and the Control Theorem.** Following Section 12 of [9], we define the Hecke operators on these spaces.

Definition 4.3.1. For $U' = U_*(\mathfrak{n}) \cap U_i(\pi^s)$ with $\mathfrak{n}$ coprime to π, we call $\mathfrak{n}$ the *tame level* and π^s the *wild level*.

Notation 4.3.2. If v is a finite place of F, such that D_v is split, then let $\eta_v \in G_D(\mathbb{A}_f)$ be the element which is the identity at all places different from v and at v it is the matrix $\left(\begin{smallmatrix} \pi_v & 0 \\ 0 & 1 \end{smallmatrix} \right)$, for π_v a uniformiser of F_v. In order to ease notation later on, when $v|p$ we choose the same uniformisers as we had before.

Definition 4.3.3. Let U have tame level $\mathfrak{n}$ and wild level π^s. For each v as above, we define the Hecke operators T_v as the double coset operators given by $[U\eta_v U]$. Moreover, if v is coprime to level, then we can regard π_v as an element of the centre of $D_f^\times$ and we denote by S_v the operator $[U\pi_v U]$. Lastly, for each $\mathfrak{p} \in \Sigma_p$ let $U_\mathfrak{p}$ denote the operator $T_\mathfrak{p}$ and let $U_p = \prod_{\mathfrak{p} \in \Sigma_p} U_\mathfrak{p}^{e_\mathfrak{p}}$. We denote by $\boldsymbol{T} = \boldsymbol{T}^D(U)$, the Hecke algebra generated by the operators[h] $T_\mathfrak{q}, U_\mathfrak{p}$, where $\mathfrak{q} \nmid \mathfrak{n}\mathfrak{d}$ with $\mathfrak{d} = \mathrm{Disc}(D)$ and $\mathfrak{p} \in \Sigma_p$.

We now want to show that the overconvergent quaternionic modular forms of small slope are classical. To do this we will follow the proof of the case $F = \mathbb{Q}$ in [8, Section 7]. We begin with some preliminaries.

Lemma 4.3.4. *The U_p operator acting on $S_{\mathfrak{U}}^{D,\dagger}(U \cap U_0(\pi^s),t)$ for $s + t \geq m(\mathfrak{U})$ is compact. In particular, this holds for the spaces $S_{\kappa,r}^{D,\dagger}(U \cap U_0(\pi^s),t)$ for $s + t \geq m(\kappa,r)$.*

Proof. See [9, Lemma 12.2] or [27, Lemma 3.2.8]. $\qquad\square$

Definition 4.3.5. Let (κ,r) be an algebraic weight and let $\kappa_i = (k_1,\ldots,k_{i-1}, 2 - k_i, k_{i+1},\ldots k_g)$. Note that if $\kappa = 2w - r$ then $\kappa_i = 2w' - r$ where $w'_j = w_j$ for $j \neq i$ and $w'_i = v_i$. For each $i \in \{1,\ldots,g\}$ corresponding to a place in Σ_∞, we define a map $\Theta_i : S_{\kappa,r}^{D,\dagger}(U,0) \longrightarrow S_{\kappa_i,r}^{D,\dagger}(U,0)$ by setting $\Theta_i(f)(h) = \frac{\partial^{k_i-1} f(h)}{\partial z_i^{k_i-1}}$ for $h \in G_D(\mathbb{A}_f)$.

Note that $f(h) \in \mathcal{A}_{n,v}(\mathcal{O}_p, L, 0)$ so it can be written as a converging power series in variables $(z_1,\ldots,z_g)$, so $\frac{\partial^{k_i-1} f(h)}{\partial z_i^{k_i-1}}$ makes sense. Moreover, one needs to check that Θ_i is actually well-defined, but this follows at once from the simple check that for any $\gamma \in I$ we have $\Theta_i(f)|_\gamma = \Theta_i(f|_\gamma)$.

Theorem 4.3.6 (Control Theorem). *Let $U' = U_*(\mathfrak{n})$ with $(\mathfrak{n},\pi) = 1$ and $U = U' \cap U_1(\pi^s)$ for $s \geq 1$ and let (k,r) be a classical weight. Let $f \in S_{k,r}^{D,\dagger}(U,t)$ be an eigenform for each $U_{\mathfrak{p}_i}$ with eigenvalue $\alpha_{\mathfrak{p}_i}$. If for each $\mathfrak{p}_i|p$ we have*

$$\mathrm{val}_p(\alpha_{\mathfrak{p}_i}) < \frac{v_{\mathfrak{p}_i}(k,r) + \min_{j \in \Sigma_{\mathfrak{p}_i}}\{k_j - 1\}}{e_{\mathfrak{p}_i}},$$

where $e_{\mathfrak{p}_i}$ is the ramification degree, then $f \in S_{k,r}^D(U)$ (in other words, f is classical).

Proof. We will only sketch the proof, but the full details[i] can be found in [29, Theorem 2.3]. First note that if $\Theta_i(f) = 0$ for all i then f must in fact be classical. The task is now to give a criterion for f to be in this kernel based only on the slope of f. Now, let $U_{\mathfrak{p}_i}^0 = \pi_{\mathfrak{p}_i}^{-v_{\mathfrak{p}_i}(k,r)} U_{\mathfrak{p}_i}$ which has operator norm ≤ 1. Then any eigenform of $U_{\mathfrak{p}_i}^0$ with negative slope must in fact be zero. Now Θ_i sends $U_{\mathfrak{p}_i}^0$-eigenforms of slope h to $U_{\mathfrak{p}_i}^0$-eigenforms of slope $h - \frac{\min_{j \in \Sigma_{\mathfrak{p}_i}}\{k_j - 1\}}{e_{\mathfrak{p}_i}}$, from which one can deduce the result. $\qquad\square$

Using the above and the Eigenmachine we can construct the eigenvariety associated to overconvergent quaternionic modular forms for D/F.

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[h]Note that these operators are independent of choice of uniformiser for v not dividing p.
[i]Up to normalization

Theorem 4.3.7. *Let $U = U' \cap U_0(\pi)$ with U' having level $\mathfrak{n}$ coprime to π. Let $\mathscr{Z}$ be the spectral variety defined as usual and $\boldsymbol{T} = \boldsymbol{T}^D(U)$ as defined in 4.3.3. Lastly, let $S^{D,\dagger}(U)$ be the coherent sheaf given the nuclear Frechet spaces $S^{D,\dagger}_{\mathfrak{U}}(U)$ where $\mathfrak{U}$ is an affinoid with a morphism $\mathfrak{U} \to \mathcal{W}^G$. Associated to the eigendata of $(\mathcal{W}^G, S^{D,\dagger}(U), \boldsymbol{T}, U_p)$ we have an eigenvariety $\mathscr{X}_D(U)$ which is equidimensional of dimension $g + 1$ and satisfies the conditions of Theorem 2.2.7.*

Proof. The existence of such an eigenvariety and the fact that it is equidimensional follows from [9, Section 13]. The fact that is it is equidimensional of dimension $g + 1$ is due to the weight space that we have used. $\qquad\square$

5. p-adic Langlands functoriality

In this section we will relate the different eigenvarieties we have defined in the previous section. In particular, we prove the following:

Theorem 5.1.1. *Let D be a totally definite quaternion algebra with $\mathrm{Disc}\, D = \mathfrak{d}$ and $\mathfrak{n}$ an ideal with $(\mathfrak{n}, \mathfrak{d}) = 1$. Let $\mathscr{X}_G := \mathscr{X}_G(U_1(\mathfrak{nd}))$ and $\mathscr{X}_D := \mathscr{X}_D(U_1(\mathfrak{n}))$ be as in Theorems 3.2.10 and 4.3.7 respectively, with $U_1(\mathfrak{nd})$ a level whose associated moduli problem is representable[j] and let p be unramified. Then these eigenvarieties are reduced and the classical Jacquet-Langlands correspondence can be interpolated to obtain a closed immersion $\mathscr{X}_D \hookrightarrow \mathscr{X}_G$ satisfying the properties of Theorem 2.3.5.*

Corollary 5.1.2. *If $g = [F : \mathbb{Q}]$ is even, then taking D (totally definite) with $\mathfrak{d} = 1$, the closed immersion given by Theorem 5.1.1 becomes an isomorphism.*

We will derive Theorem 5.1.1 from Theorem 2.3.5 (the Interpolation theorem). To this end, we need to exhibit a very Zariski dense set $X \subset \mathcal{W}^G$ on which we can put classical structures for both sets of eigenvariety data. The set of all classical weights (see Definition 2.1.1) is such a candidate. The fact that it is a very Zariski dense subset of $\mathcal{W}^G$ is a well-known fact but we include its proof for the sake of completeness. This requires the following lemma.

Lemma 5.1.3. *If W is a non-empty rigid space. Then W is irreducible if and only if the only analytic subset $Z \subset W$ which set-theoretically contains a non-empty admissible open of W is $Z = W$.*

Proof. See [14, Lemma 2.2.3]. $\qquad\square$

Proposition 5.1.4. *Let X be the set of classical weights, then X is very Zariski dense in $\mathcal{W}^G$.*

Proof. This is a simple generalization of [10, Proposition 6.2.7] or [24, Lemma 4.1]. By 2.1.2 we have

$$\mathcal{W}^G \cong H^\vee \times B(1,1)^{g+1} \cong \bigsqcup_\chi \mathcal{W}_\chi,$$

where we index over the elements of $H^\vee$. Let κ_ψ be a classical weight with $\kappa = 2w - r$ and ψ as Definition 2.1.1. Then under the above isomorphism

$$\kappa_\psi \mapsto \left(\widehat{\kappa}_\psi, \left[\prod_j^g (1+p)^{w_j} \right], (1+p)^r \right),$$

where $\widehat{\kappa}_\psi$ denotes the restriction to $H^\vee$ (note that $\kappa_\psi \in \mathcal{W}^G(E)$, with $E = \mathbb{Q}_p[\psi]$). Assume that $\kappa_\psi \in \mathcal{W}_\chi$ for some χ and take any irreducible admissible affinoid open $V \subset \mathcal{W}^G$ that contains κ_ψ. Then $V \subset \mathcal{W}_\chi$ and moreover, since $V(E)$ is open, there exists $\boldsymbol{s} = (s, s') \in \mathbb{Q}_{>0}^{\Sigma_\infty} \times \mathbb{Q}_{>0}$ such that the closed ball of radius $\boldsymbol{s}$ around κ_ψ is contained in V, i.e.,

$$B[\kappa_\psi, \boldsymbol{s}] := \prod_j^g B[w_j, s_j] \times B[r, s'] \subset V.$$

By Lemma 5.1.3, we see that if $B[\kappa_\psi, \boldsymbol{s}] \cap X$ is Zariski dense in $B[\kappa_\psi, \boldsymbol{s}]$, then $V(E) \cap X$ is Zariski dense in V, which is what we want to prove. So we are reduced to showing that $B[\kappa_\psi, \boldsymbol{s}](E) \cap X$ is Zariski dense in $B[\kappa_\psi, \boldsymbol{s}]$. To see this, let $\kappa, \kappa' \in X$, then

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[j]Meaning that the moduli problem of HBAV with a $\mu_{\mathfrak{nd}}$-level structure is representable.

$$|\kappa - \kappa'| = \max_i \{|(1+p)^{w_i} - (1+p)^{w_i'}|_p, |(1+p)^r - (1+p)^{r'}|_p\}$$
$$= \max_i \{|(1+p)^{w_i - w_i'} - 1|_p, |(1+p)^{r-r'} - 1|_p\}.$$

So taking $\kappa'_\psi \in X$, with $\kappa' = 2w' - r'$ is such that:

- for N large enough, $w_i \equiv w_i' \mod (p-1)p^N$, $r \equiv r' \mod (p-1)p^N$;
- κ_ψ and κ'_ψ lie in the same component of the weight space.

Then we can easily see that $B[\kappa_\psi, \boldsymbol{s}]$ contains infinitely many elements of X, hence we get the result. $\qquad\square$

Definition 5.1.5. Let $\mathscr{Z} \subset \mathcal{W}^G \times \mathbb{A}^1$ be the spectral variety defined by $\mathrm{Fred}_{\mathscr{M}}(U_p)$ for $\mathscr{M}$ (as usual) the coherent sheaf on $\mathcal{W}^G$ of overconvergent Hilbert modular forms on D or $M_2(F)$ with a classical structure $\mathscr{M}^{cl}$. We call a point $z \in \mathscr{Z}$ classical if its projection to $\mathcal{W}^G$ is a classical weight and if $\det(1 - TU_p|_{\mathscr{M}^{cl}})$ vanishes at z. We denote these points by $\mathscr{Z}^{cl}$.

Remark 5.1.6. Note that if $\mathscr{M}$ is given by the 'spreading out' of the spaces of overconvergent Hilbert modular forms and if we write $z = ((\kappa_z, r_z), \alpha) \in \mathcal{W}^G \times \mathbb{A}^1$ then z is classical if (κ_z, r_z) is a classical weight and there exists a classical Hilbert modular form with U_p eigenvalue α^{-1}.

Proposition 5.1.7. *The subset $\mathscr{Z}^{cl}$ of $\mathscr{Z}$ is a very Zariski dense subset.*

Proof. This follows from the proof of [11, Proposition 3.5]. The basic idea is to use the fact that X and X_h are very Zariski dense, together with the fact that the admissible cover of $\mathscr{Z}$ as given by [9, Section 4] is finite flat over its projection to weight space. $\qquad\square$

Let $\mathfrak{n}$ be an ideal of $\mathcal{O}_F$ with $(\mathfrak{n}, \mathfrak{d}) = 1$ and $\pi \nmid \mathfrak{n}\mathfrak{d}$, where $\mathfrak{d} = \mathrm{Disc}(D)$. Let $U^D = U_1(\mathfrak{n}\pi)$ and set $\boldsymbol{T}^D(U^D)$ to be the Hecke algebra.[k] Let U^G be the corresponding level structure when one takes $D = M_2(F)$, which gives the level structure in the Hilbert modular form case. By fixing a splitting at places away from $\mathfrak{d}$, we let $\boldsymbol{T}^D$ act on the spaces of Hilbert modular forms. Therefore, throughout this section we denote $\boldsymbol{T}^D$ simply by $\boldsymbol{T}$.

Theorem 5.1.8 (*The Jacquet–Langlands correspondence*). *Let (k, r) be a classical weight in $\mathcal{W}^G$ and U^D, U^G as above. There is an isomorphism*
$$S_{k,r}^D(U^D) \xrightarrow{\sim} S_{k,r}^{\mathfrak{d}\text{-new}}(U^G \cap U_1^G(\mathfrak{d})).$$

Remark 5.1.9. We note that, for g even, we can pick the quaternion algebra D to be totally definite with $\mathfrak{d} = 1$. Now, by fixing a splitting we can identify U^D and U^G, which we will simply denote by U. In this case the classical Jacquet-Langlands correspondence gives an isomorphism of Hecke modules $S_{k,r}^D(U) \xrightarrow{\sim} S_{k,r}(U)$. However, for g odd, since D is totally definite, we must have $\mathfrak{d} \neq 1$. In this case, we have an isomorphism of Hecke modules
$$S_{k,r}^D(U^D) \xrightarrow{\sim} S_{k,r}^{\mathfrak{d}\text{-new}}(U^G(\mathfrak{d})) \hookrightarrow S_{k,r}(U^G(\mathfrak{d})),$$
where $U^G(\mathfrak{d}) = U^G \cap U_1^G(\mathfrak{d})$.

Theorem 5.1.1 then follows from Theorem 5.1.10 (below) together with Lemma 5.1.13:

Theorem 5.1.10. *Let $\mathscr{X}_G$ and $\mathscr{X}_D$ be the eigenvarieties associated to the eigendata*
$$\mathfrak{D}_1 = (\mathcal{W}^G, S^{G,\dagger}(U^G(\mathfrak{d})), \boldsymbol{T}, U_p) \text{ and } \mathfrak{D}_2 = (\mathcal{W}^G, S^{D,\dagger}(U^D), \boldsymbol{T}, U_p)$$
respectively and let p be unramified. Then we can interpolate the classical Jacquet-Langlands correspondence and obtain a closed immersion $\iota_D : \mathscr{X}_D^{red} \hookrightarrow \mathscr{X}_G^{red}$.

Proof. We will prove this using Theorem 2.3.5. Let X be the set of classical weights, whose elements we will denote by k. We now define classical structure on X. For each $(k, r) \in X$, let $\mathscr{M}_{G,k,r}^{cl}$ and $\mathscr{M}_{D,k,r}^{cl}$ be the $\boldsymbol{T}$-modules $S_{k,r}(U^G(\mathfrak{d}))$ and $S_{k,r}^D(U^D)$ respectively of classical cusp forms of weight k and level $U^G(\mathfrak{d}), U^D$ respectively. We need to check that this is indeed a classical structure. Pick $h \in \mathbb{R}_{\geq 0}$. Then the set of $(k, r) \in X$ such that $S_{k,r}^{G,\dagger}(U^G(\mathfrak{d}))^{\leq h} \subset \mathscr{M}_{G,k,r}^{cl}$ contains all $(k, r) \in \mathcal{W}^G$, such that $h < v_p(k, r) + \min_{i \in \Sigma_\infty}\{k_i - 1\}$ by the Control

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[k]Note that the Hecke algebra consists of all Hecke operators away from $\mathfrak{d}$.

Theorem and hence satisfies the properties of Definition 2.3.3. Recall that the superscript $\leq h$ denotes slope decomposition with respect to U_p.

Similarly, if $(k, r) \in X$ is such that[l] $h < v_p(k, r) + \min_{i \in \Sigma_\infty}\{k_i - 1\}$, then $S_{k,r}^{D,\dagger}(U^D)^{\leq h} \subset \mathscr{M}_{D,k,r}^{cl}$. It follows that we again have a classical structure. Now, as a consequence of the classical Jacquet-Langlands correspondence we have that $\det\left(1 - U_p X_{|\mathscr{M}_{D,k,r}^{cl}}\right)$ divides $\det\left(1 - U_p X_{|\mathscr{M}_{G,k,r}^{cl}}\right)$. Hence we can apply Theorem 2.3.5 to obtain the closed immersion $\iota_D : \mathscr{X}_D^{red} \hookrightarrow \mathscr{X}_G^{red}$.

$\square$

Now observe that if g is even, then we can pick D to be totally definite *and* have $\mathfrak{d} = 1$ (i.e. trivial discriminant). Then the classical Jacquet-Langlands correspondence gives that $\mathscr{M}_{G,k,r}^{cl} \cong \mathscr{M}_{D,k,r}^{cl}$ at classical weights, and thus

$$\det(1 - U_p X \mid \mathscr{M}_{D,k,r}^{cl}) = \det(1 - U_p X \mid \mathscr{M}_{G,k,r}^{cl})$$

therefore Corollary 2.3.6 gives us an isomorphism $\mathscr{X}_D^{red} \cong \mathscr{X}_G^{red}$. This proves most of Corollary 5.1.2, it only remains to show that the eigenvarieties are reduced which we will follow from Lemma 2.3.7.

Proposition 5.1.11. *Fix $h \in \mathbb{R}_{\geq 0}$. There is a Zariski dense subset $X' \subset X$ (of $\mathcal{W}^G$) such that for all $k \in X'$, the $\boldsymbol{T}$-module $\mathscr{M}_{G,k,r}^{cl, \leq h}$ is semisimple.*

This result will be consequence of the two lemmas below. We begin by noting that the classical Jacquet-Langlands correspondence gives us that if $\mathscr{M}_{G,k,r}^{cl, \leq h}$ is a semisimple $\boldsymbol{T}$-module, then so is $\mathscr{M}_{D,k,r}^{cl, \leq h}$. To ease notation, we let

$$R_{k,r}^h := \mathscr{M}_{G,k,r}^{cl, \leq h} = S_{k,r}(U^G(\mathfrak{d}))^{\leq h}.$$

Now, since we are working with classical Hilbert modular forms, the action of the Hecke operators can be described by their action on q-expansions. Next we note that the only Hecke operators that might not be semisimple are the $U_{\mathfrak{p}_i}$, for $p\mathcal{O}_F = \prod_i \mathfrak{p}_i$. This is because all the other operators are normal (commute with their adjoints), so they are semisimple. Hence we must show that for each i, the operators $U_{\mathfrak{p}_i}$ act semisimply on the space of cusp forms of slope-$\leq h$. In fact we shall show that $U_{\mathfrak{p}_i}$ acts semisimply on $R_{k',r'}^h$ for a Zariski dense subset of $X' \subset X$. Lastly, we need to relate slope decomposition of $R_{k,r}^h$ with respect to U_p, to the slope decompositions with respect to the $U_{\mathfrak{p}_i}$. To do this we have the following:

Lemma 5.1.12. *Let S be a Banach space on which we have pairwise commuting operators U_i for $i = 1, \ldots, n$, all of which have operator norm ≤ 1 (which means they have positive slopes) and such that $U = \prod_i U_i$ is a compact operator. Then the slopes of the U_i operators acting on the space $S^{\leq h}$ (this is the slope decomposition with respect to U) are all $\leq h$.*

Proof. By definition we have that $S^{\leq h}$ is a finite dimensional subspace of S. Therefore by choosing a basis we can view the U_i operators as matrices. Now since the U_i are pairwise commuting operators, we can simultaneously upper triangularize them (after possibly extending the base field). From this it follows that the eigenvalues of U acting on $S^{\leq h}$ are the product of the eigenvalues of the U_i.

Now since the slopes of an operator are simply the p-adic valuation of its eigenvalues, we have that on $S^{\leq h}$ the slopes of U are the sum of the slopes of the U_i operators and therefore, since they all have positive slopes, it follows that the slopes of the U_i acting on $S^{\leq h}$ are all $\leq h$ as required.

$\square$

After renormalizing our operators, we can apply this to our situation to see that since $U_p = \prod_i U_{\mathfrak{p}_i}$ is compact, then we have a slope decomposition for any h. Moreover, for each h we have that the slope of each $U_{\mathfrak{p}_i}$ acting on $R_{k,r}^h$ is less than or equal to h. With this we can prove the following Lemma:

Lemma 5.1.13. *There is a Zariski dense subset $X' \subset \mathcal{W}^G$, such that for each i, $U_{\mathfrak{p}_i}$ acts semisimply on $R_{k,r}^h$ for $k, r \in X'$.*

Proof. Using Lemma 5.1.12 the result is a simple generalization of the classical situation, as is done in [4, Theorem 3.30], or from the proof of [12, Theorem 4.2]. But for completeness we prove it here.

First note that we can decompose $R_{k,r}^h$ into its $\mathfrak{p}_i$-new and $\mathfrak{p}_i$-old parts. The action of $U_{\mathfrak{p}_i}$ on $R_{k,r}^{h, \mathfrak{p}_i\text{-new}}$ is normal and hence diagonalizable. With this we are reduced to showing that this operator acts semisimply on $R_{k,r}^{h, \mathfrak{p}_i\text{-old}}$.

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[l]Note that we are in the case where p is unramified.

In order to prove this, it is enough to show that on each generalized $\boldsymbol{T}$-eigenspace of $R_{k,r}^{h,\mathfrak{p}_i\text{-old}}$ it acts semisimply. Each of these spaces will correspond to a newform f of (lower) level not divisible by $\mathfrak{p}_i$. Now, let $a_{\mathfrak{p}_i} = \boldsymbol{a}(\mathfrak{p}_i, f|T_{\mathfrak{p}_i})$ be the $T_{\mathfrak{p}_i}$ eigenvalue of f and Ψ its nebentypus. Since we are assuming that for each i, we have $(\mathfrak{n}\mathfrak{d}, \mathfrak{p}_i) = 1$, then it follows from Atkin-Lehner Theory that each of these $\mathfrak{p}_i$-old subspaces is 2-dimensional, and generated by f and $f|B_{\mathfrak{p}_i}$ (see [25], for the definition of $B_{\mathfrak{p}_i}$). A simple calculation then shows that on this subspace the $U_{\mathfrak{p}_i}$ operator has minimal polynomial given by

$$X^2 - a_{\mathfrak{p}_i}X + N_{F/\mathbb{Q}}(\mathfrak{p}_i)^{r+1}\Psi(\mathfrak{p}_i).$$

Therefore, since $N_{F/\mathbb{Q}}(\mathfrak{p}_i) = p^{l_{\mathfrak{p}_i}}$ (here $l_{\mathfrak{p}_i}$ is the residue degree), we see that if we pick (k, r), such that $r > (2h - l_{\mathfrak{p}_i})/l_{\mathfrak{p}_i}$, then $h < l_{\mathfrak{p}_i}(r+1)/2$. Therefore, the polynomial must have a unique root α with valuation $\leq h$, from which it follows that on the generalized $\boldsymbol{T}$-eigenspace of $R_{k,r}^{h,\mathfrak{p}_i\text{-old}}$ corresponding to f, we have that $U_{\mathfrak{p}_i}$ acts as the scalar α. Hence it is diagonalizable. This then shows that on $R_{k,r}^h$ the $U_{\mathfrak{p}_i}$ operators act semisimply for (k, r) large enough as required.

$\square$

From this it follows that for any $h \geq 0$, the operators $U_{\mathfrak{p}_i}$ act semisimply on $S_{k,r}(U')^{\leq h}$ for (k, r) in a Zariski dense subset of $\mathcal{W}^G$, proving Proposition 5.1.11. Then by Lemma 2.3.7, we have at once that $\mathscr{X}_D^{red} \cong \mathscr{X}_D$ and $\mathscr{X}_G^{red} \cong \mathscr{X}_G$, which proves Corollary 5.1.2.

Remark 5.1.14. In light of Theorem 5.1.10 and Remark 5.1.9, we see that for g even and D totally definite with $\mathfrak{d} = 1$, we have an isomorphism of eigenvarieties $\iota_D : \mathscr{X}_D^{red} \overset{\sim}{\to} \mathscr{X}_G^{red}$. However, for g odd and D totally definite, the closed immersion $\iota_D : \mathscr{X}_D^{red} \hookrightarrow \mathscr{X}_G^{red}$ is never an isomorphism since $\mathfrak{d} \neq 1$. At best, we can say that its image is the $\mathfrak{d}$-new part of of $\mathscr{X}_G^{red}$ as in the case of modular forms over $\mathbb{Q}$ (cf. [11]).

Remark 5.1.15. We note that, for g odd, there are alternative constructions for the eigenvariety $\mathscr{X}_D^{red}$. Let D be the quaternion algebra ramified at all infinite places but one, with $\mathfrak{d} = 1$. Then Brasca [7] constructs an eigenvariety associated to $\mathscr{X}_D^{red}$ from which one can use the above to obtain a closed immersion $\iota_D : \mathscr{X}_D^{red} \hookrightarrow \mathscr{X}_G^{red}$, which is an isomorphism on the closed subvariety given by parallel weights. His construction combines the theory of Shimura curves with work of Andreatta-Iovita-Pilloni to construct the relevant eigenvarieties.

Remark 5.1.16. Using the work of [16], one can construct eigenvarieties coming from overconvergent cohomology groups associated to any quaternion algebra. In this case, the resulting eigenvarieties are larger than the ones we have constructed, in particular, they may not be equidimensional of dimension $g + 1$. Using a more general version of the Interpolation theorem as in *loc.cit.* one can obtain closed immersions between the 'cores' (as defined in *loc.cit.*) of the relevant eigenvarieties.

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