On classes in the motivic cohomology of certain Shimura varieties

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A dissertation submitted in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

of

University College London.

Department of Mathematics

University College London

September 13, 2018

Ai miei genitori, ad Eugenio e a Sara.

I, Antonio Cauchi, confirm that the work presented in this thesis is my own, except the contents of Chapter 4, which is in collaboration with Joaquin Jacinto Rodrígues ([CRJ18]). Where information has been derived from other sources, I confirm that this has been indicated in the work.

Abstract

The main theme of this thesis is on the push-forward construction of motivic cohomology classes for Shimura varieties. This strategy was successfully employed in work of Lei-Loeffler-Zerbes *et al* to construct new Euler systems for Galois representations attached to certain cohomological automorphic forms, which have been used to prove new cases of the Bloch-Kato conjecture. In this thesis, we describe two new push-forward constructions for Shimura varieties associated to the symplectic group $\mathbf{GSp}_6$ and the unitary group $\mathbf{GU}(2,2)$, and their distribution relations.

First, we describe the joint work with Joaquín Rodrigues Jacinto on the construction of classes in the seventh cohomology group of the Shimura variety for $\mathbf{GSp}_6$; these classes have coefficients in a local system associated to an irreducible algebraic representation of $\mathbf{GSp}_6$ of arbitrary weight. The classes are defined as push-forward of elements in the cohomology of a triple product of modular curves. We prove a trace compatibility result for these classes and use it to deduce Euler system norm relations in the cyclotomic tower at any rational prime p.

Secondly, we explain the construction of classes in the fifth motivic cohomology group of the Shimura variety for $\mathbf{GU}(2,2)$. They are obtained as the push-forward of $\mathbf{GSp}_4$-Eisenstein classes along the Gysin morphisms of a closed immersion of the Shimura variety for $\mathbf{GSp}_4$ inside the one for $\mathbf{GU}(2,2)$. By perturbing the aforementioned immersion, we construct a two variable family of push-forward classes that satisfies certain norm relations. To derive these, we first prove, more generally, some distribution relations for the $\mathbf{GSp}_{2g}$-Eisenstein classes and then translate them into those for the push-forward classes.

Impact Statement

In 2012, Lei, Loeffler and Zerbes initiated a program devoted to proving new cases of the Birch and Swinnerton-Dyer conjecture, which is listed as one of the Millennium Prize Problems by the Clay Institute, and generalisations, such as the Bloch-Kato conjecture. Their strategy consists in constructing new Euler systems. The main contribution of this thesis to the programme is promising supporting evidence to the construction of a new Euler system, which will lead to the proof of new cases of the Bloch-Kato conjecture. We intend to come back to these points in the near future.

We expect that the results of this thesis will have impact in various areas of Mathematics. Indeed, we have been using tools from Number Theory, Geometry, and Representation theory. Thus, we believe that our work will influence and generate new research in these areas. To achieve this impact, we are planning to publish our work in peer-reviewed journals, and we will make it soon available on the arXiv. As evidence of interest on our work, we have been recently invited to present our research in various occasions around Europe; the list of hosting institutions includes EPFL (Switzerland), University of Regensburg (Germany), and University of Padova (Italy).

Acknowledgements

I would like to express my deepest gratitude to my supervisor Sarah Zerbes for her continued support and guidance. Her ambition, determination, and enthusiasm for Mathematics have been a constant inspiration to me during my studies at UCL. Throughout all stages of my PhD, her encouragement was invaluable for me.

I am indebted to David Loeffler, who generously gave of his time to explain various aspects of the subject to me and to correct several sections of this thesis. He was like a second supervisor for me.

A special thanks goes to my Master's supervisor Matteo Longo for introducing me to the theory of Euler systems and for motivating me to pursue an academic career. I continue to benefit from his encouragement and interest in my work.

I thank Guido Kings and Chris Skinner for fruitful discussions during the last academic year. Parts of this thesis were written while I was visiting the Bernoulli Centre at EPFL. I am very grateful to the Bernoulli Centre and specially to Dimitar Jetchev for their hospitality and invitation to participate in part of the semester "Euler Systems and Special Values of L-functions".

I benefited hugely from countless discussions with Joaquín Rodrigues Jacinto. I would like to thank him for his patience with me and his enthusiasm for Mathematics.

My doctoral studies were funded by the Engineering and Physical Sciences Research Council [EP/L015234/1]. The EPSRC Centre for Doctoral Training in Geometry and Number Theory (The London School of Geometry and Number Theory), University College London. I am honoured to be part of the first cohort of this wonderful programme.

I would like to thank Nicky Townsend and all the members at UCL for making my stay here much more enjoyable.

A special thanks goes to my fellow PhD students at UCL: Gregorio Baldi, (my academic sister) Giada Grossi, Mattia Miglioranza, and Tobias Sodoge. An even more special thanks goes to James Cann, for the innumerable fun and enlightening conversations we had in these four years.

I would like to thank my flatmates/colleagues Momchil Konstantinov and Kwok-Wing Tsoi, aka Ghaleo. It felt like home in the last 3 years.

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Chapter 1

Introduction

The construction of special elements in the cohomology of Shimura varieties constitutes one of the main tools to study the arithmetic of Galois representations appearing in the cohomology of Shimura varieties and their relation to special L-values. It has played a crucial role in proving cases of Beilinson's conjecture (e.g. [Beĭ86], [Kin98], [Lem17] etc.), and in studying the structure of Selmer groups of those Galois representations (e.g. [Kat04], [BDR15b], [LLZ14], [KLZ17], [DR17], [LSZ17] etc.). In the latter situation, the method of bounding Selmer groups often relies on the theory of Euler systems.

1.1 Euler Systems

The theory of Euler systems is a powerful tool to prove cases of the Birch and Swinnerton-Dyer conjecture and generalisations, such as the Bloch-Kato conjecture, and constitutes one of the only few known approaches to tackle those questions down. In the 1960s, Birch and Swinnerton-Dyer conjectured a mysterious relation between the rank of the group of $\mathbf{Q}$-rational points of an elliptic curve E over the field of rational numbers $\mathbf{Q}$ and the behaviour at the central critical value $s = 1$ of its complex Hasse-Weil L-function $L(E,s)$, which was not known to be defined at $s = 1$. (Thanks to the pioneering work of Wiles and successive refinements of Breuil, Conrad, Diamond and Taylor, we now know that the Hasse-Weil L-function of elliptic curves over $\mathbf{Q}$ extends to an entire function on $\mathbf{C}$). Based on work of Gross and Zagier, in the late 1980s Kolyvagin proved special cases of the conjecture for modular elliptic curves over $\mathbf{Q}$ in [Kol90], by giving a bound of their Selmer group. Kolyvagin's method was based on the construction of a system of Galois cohomology classes for the p-adic Galois representation of the elliptic curve, defined over anticyclotomic extensions of an imaginary quadratic field. These classes are constructed from a particular family of points of the elliptic curve, called Heegner points, which are images under the modular parametrisation of CM points in the modular curve.

Inspired by this construction and Thaine's method to bound ideal class groups of real abelian extensions of $\mathbf{Q}$ using cyclotomic units, in [Rub00] Karl Rubin proposed a general machinery to bound Selmer groups associated to p-adic representations V of $Gal(\bar{\mathbf{Q}}/K)$, for K number field.

His method is subject to the existence of an Euler system for V, which is a collection of Galois cohomology classes for V defined over abelian extensions of K satisfying certain relations under the corestriction maps, with non-zero bottom class (i.e. the class defined over K). Rubin showed that the existence of such an Euler system has important Iwasawa-theoretic applications.

Independently, in [Kat04] Kato discovered similar techniques to bound Selmer groups using Euler systems, and he constructed an Euler system for the p-adic representation $V_p(f)$ of $Gal(\bar{\mathbf{Q}}/\mathbf{Q})$, for a normalised cuspidal eigenform f of weight $k \geq 2$ and level N: his starting point are objects of geometric nature, namely cup products of Siegel units in K_2 of modular curves. Siegel units are algebraic realisations of meromorphic functions on complex elliptic curves. The values of their logarithmic derivatives at torsion points are related to Eisenstein series and values of the Hasse-Weil L-function (by the so-called "Kroenecker's second limit formula"). Kato proves an explicit reciprocity law, by relating the non-vanishing of the bottom class to the non-vanishing of the L-function of the modular form f at the point $s = 1$. In the case when this value is non-zero, he obtains bounds on Selmer groups, giving a different proof of some cases of the Birch and Swinnerton-Dyer conjecture proved by Kolyvagin, and he proves partial results on the Iwasawa Main conjecture for modular elliptic curves over $\mathbf{Q}$. The construction of Kato relies on the fact that the p-adic representation attached to f "comes from geometry", in the sense that its dual $(V_p f)^*$ can be realised as a quotient of the first étale cohomology group of the modular curve over $\bar{\mathbf{Q}}$ with coefficients in an opportune étale local system.

Despite the differences between the construction of Kolyvagin and the one of Kato, both families of Galois cohomology classes have the property that they arise in a geometric fashion from objects (points, resp. units) on the modular curve which already satisfy distribution relations in the "geometric world" (as elements of group of rational points, resp. as elements of the étale/motivic cohomology of the modular curve). Thus, these constructions suggest that a good starting point for constructing new Euler systems is the case of p-adic Galois representations which can be realised as subquotients of (a twist of) étale cohomology groups of schemes Y defined over number fields with a rich supply of geometric objects on them.

Because of their incredible applications, it turns out that Euler systems are extremely difficult to construct, and soon after the construction of Kato there was little progress in this direction for about ten years. Building upon the seminal work of Bertolini, Darmon and Rotger ([BDR15a], [BDR15b]) on Beilinson-Flach classes in the étale cohomology of a product of modular curves, Lei, Loeffler and Zerbes constructed in [LLZ14] the Euler system of Beilinson-Flach elements for a certain twist of the tensor product of p-adic representations associated to two modular forms of weights bigger or equal than 2. These p-adic representations are realised as subquotients of a suitable étale cohomology group of a product of modular curves, and the Euler system construction relies again on the existence of Siegel units. The Beilinson-Flach classes are obtained as pushforward along perturbations of the diagonal embedding of a Siegel unit, which can be regarded as an element in the cohomology of one modular curve. Remarkably, in [KLZ17] Kings, Loeffler and Zerbes studied the variation of the Euler system in p-adic families of modular forms and were able to establish an explicity reciprocity law; thus, they deduced certain cases of a refined version of the Birch and Swinnerton-Dyer conjecture ([KLZ17], [KLZ15]). Building on these new ideas, new Euler systems have been recently constructed in the case of Hilbert modular forms (see [LLZ16]) and genus 2 Siegel forms ([LSZ17]), and this strategy has also successfully been applied in various contexts (e.g [LZ16]).

Independently, in [Jet14], [BBJ15], progress has been made in building Euler systems for Galois representations of cohomological automorphic representations of certain products of unitary groups from special cycles, which fit in the context of Gross-Zagier type conjectures (e.g. [Zha17, Conjectures 2.3 and 3.4]) and are the natural higher dimensional analogue of the Euler system of Heegner points.

1.1.1 An orientative definition

Fix a prime number p and let V be a finite dimensional $\mathbf{Q}_p$-vector space with a continuous action of $G_{\mathbf{Q}} = Gal(\bar{\mathbf{Q}}/\mathbf{Q})$, which is unramified outside a finite set of primes Σ containing p. Denote by $V^*(1)$ the $\mathbf{Q}_p$-dual of V twisted by the cyclotomic character, and by $\mathrm{Sel}_{\mathrm{BK}}(V)$ its Bloch-Kato Selmer group, which is a subgroup of $H^1(\mathbf{Q}, V^*(1))$ cut out by local conditions (cf. [BK90]). By $H^i(\mathbf{Q}, V^*(1))$ we denote continuous group cohomology of the absolute Galois group of $\mathbf{Q}$ acting on $V^*(1)$. The (weak) Bloch-Kato conjecture predicts that

$$\mathrm{ord}_{s=0} L(V,s) = \dim_{\mathbf{Q}_p}(\mathrm{Sel}_{\mathrm{BK}}(V)) - \dim_{\mathbf{Q}_p}(H^0(\mathbf{Q}, V^*(1))).$$

Non-trivial Euler systems are used to give bounds of Bloch-Kato Selmer groups which, if in possess of information on the behaviour of $L(V,s)$ around $s = 0$, are used to prove cases of this conjecture. The construction of Euler systems amounts to showing the existence of a family of Galois cohomology classes satisfying precise distribution relations.

Following the definition of Rubin in [Rub00, §1], an Euler system $\mathbf{ES}$ for a $G_{\mathbf{Q}}$-stable $\mathbf{Z}_p$-lattice T of V is a collection of Galois cohomology classes $\{\mathbf{ES}_m\}_{m\geq 1}$, where $\mathbf{ES}_m \in H^1(\mathbf{Q}(\zeta_m), T^*(1))$ and

$$\mathrm{cores}^{\mathbf{Q}(\zeta_{m\ell})}_{\mathbf{Q}(\zeta_m)} \mathbf{ES}_{m\ell} = \begin{cases} \mathbf{ES}_m, & \text{if } \ell \mid m \text{ or } \ell \in \Sigma; \\ P_\ell(Frob_\ell^{-1})\mathbf{ES}_m & \text{otherwise,} \end{cases}$$

where $P_\ell(X) := det(1 - Frob_\ell^{-1} X|V) \in \mathbf{Z}_p[X]$ is the characteristic polynomial of the inverse of the arithmetic Frobenius $Frob_\ell$.

We remark that the Euler system of Heegner points are excluded from this definition. It is also worth mentioning that there are cases (e.g. [DR14], [DR17]) where only the "bottom" class z_1 is used to bound the Selmer group by an argument using Hida Theory.

1.1.2 The push-forward construction

Suppose that a twist of $T^*(1)$ appears as a sub-quotient of

$$H^i_{\text{ét}}(\mathrm{Sh}_G(U)_{\bar{\mathbf{Q}}}, \mathscr{L}(j)), \text{ for some } i, j \geq 0$$

where $\mathrm{Sh}_G(U)$ denotes the canonical model over $\mathbf{Q}$ of a Shimura variety associated to a reductive group G of level $U \subset G(\mathbf{A}_f)$ and $\mathscr{L}$ is an étale $\mathbf{Z}_p$-sheaf. This is not a restrictive condition; such condition will indeed be satisfied in the cases discussed.

In this setting, we can try to build an Euler system as follows. Consider the projection map

$$pr_T : H^i_{\text{ét}}(\mathrm{Sh}_G(U)_{\bar{\mathbf{Q}}}, \mathscr{L}(j)) \longrightarrow T^*(1).$$

We want to construct classes in $H^1(\mathbf{Q}(\zeta_m), H^i_{\text{ét}}(\mathrm{Sh}_G(U)_{\bar{\mathbf{Q}}}, \mathscr{L}(j)))$ satisfying the Euler system norm relations; the Euler system for $T^*(1)$ will be the images of these classes under the projection pr_T. By using the Hochschild-Serre spectral sequence (and eventually killing the Galois invariant classes in $H^{i+1}_{\text{ét}}(\mathrm{Sh}_G(U)_{\bar{\mathbf{Q}}}, \mathscr{L}(j))$), we reduce the problem to the construction of classes $z_m \in H^{i+1}_{\text{ét}}(\mathrm{Sh}_G(U)_{\mathbf{Q}(\zeta_m)}, \mathscr{L}(j))$, satisfying the norm relations under trace maps.

Suppose there exists a sub-variety $\iota : Y \hookrightarrow \mathrm{Sh}_G(U)$ with the following properties:

1. If we let d be the codimension of Y in $\mathrm{Sh}_G(U)$, then we have $k + 2d = i + 1$, so that the pushforward along ι gives

$$\iota_* : H^k_{\text{ét}}(Y, \iota^*\mathscr{L}(j-d)) \longrightarrow H^{i+1}_{\text{ét}}(\mathrm{Sh}_G(U), \mathscr{L}(j));$$

2. There exists a supply of geometric classes in $H^k_{\text{ét}}(Y, \iota^*\mathscr{L}(j-d))$.

We then define the class z_1 as the image under ι_* of a suitable geometric class in $H^k_{\text{ét}}(Y, \iota^*\mathscr{L}(j-d))$. To construct classes over $\mathbf{Q}(\zeta_m)$, we use the right-translation action of $G(\mathbf{A}_f)$ on the Shimura variety Sh_G to perturb the embedding ι. Precisely, we define maps $\{\iota_m\}_{m\geq 1}$, where

$$\iota_m = u_m \circ \iota : Y_{\mathbf{Q}(\zeta_m)} \longrightarrow \mathrm{Sh}_G(U)_{\mathbf{Q}(\zeta_m)},$$

for certain $u_m \in G(\mathbf{A}_f)$ such that $\mathrm{Sh}_G(U)_{\mathbf{Q}(\zeta_m)} \simeq \mathrm{Sh}_G(u_m^{-1}Uu_m)$ as $\mathbf{Q}$-schemes. We then take the push-forward along ι_m of the appropriate class in $H^k_{\text{ét}}(Y_{\mathbf{Q}(\zeta_m)}, \iota^*\mathscr{L}(j-d))$. Notice that the choice of u_m is very delicate and ensures that the class constructed is not a simple base-change to $\mathbf{Q}(\zeta_m)$ of z_1.

In many cases in the literature, one constructs classes

$$z^U_m \in H^{i+1}_{\text{ét}}(\mathrm{Sh}_G(U)_{\mathbf{Q}(\zeta_m)}, \mathscr{L}(j)),$$

which are compatible under push-forwards as U varies in a certain family of level subgroups of

$G(\mathbf{A}_f)$. This is achieved by taking push-forward along $\iota_U : Y_U \to \mathrm{Sh}_G(U)$ of classes, each in $H^k_{\text{ét}}(Y_U, \iota_U^*\mathscr{L}(j-d))$, which are compatible under push-forward of morphisms $\phi^U_{U'} : Y_U \to Y_{U'}$ for U'/U finite étale. The family of subvarieties $\{Y_U\}_U$ is required to satisfy the following: if $\pi^U_{U'} : \mathrm{Sh}_G(U) \to \mathrm{Sh}_G(U')$ is finite étale, we have a commutative diagram

$$\begin{tikzcd} Y_U \arrow[r, "\iota_U"] \arrow[d, "\phi^U_{U'}"'] & \mathrm{Sh}_G(U) \arrow[d, "\pi^U_{U'}"] \\ Y_{U'} \arrow[r, "\iota_{U'}"] & \mathrm{Sh}_G(U'). \end{tikzcd}$$

The push-forward compatibility of z^U_m as U varies follows immediately from the fact that

$$\pi^U_{U',*} \circ \iota_{U,*} = \iota_{U',*} \circ \phi^U_{U',*},$$

and the compatibility of the geometric classes under $\phi^U_{U',*}$.
These push-forward relations are used in the proof of Euler system norm relations and are employed in a crucial way in the study of variation of Euler systems in p-adic families.

1.1.3 Some examples

Finding suitable subvarieties with a rich supply of cohomology classes is quite a hard task to accomplish in general and it is usually dictated by properties of the underlying L-function of the representation. At present, the majority of cases where this strategy has been successful are characterised either by having Y a modular curve or a fibre product of modular curves and as associated cohomology classes étale realisations of Siegel units and Eisenstein classes of the modular curve, pull-back or cup-product of them ([Kat04], [BDR15b], [LLZ14], [LLZ16], [LSZ17] etc.), or by taking the étale realisation of special cycles ([DR17], [Jet14] etc.).

For instance, the motivic class underlying the construction in [LLZ14] of the Euler system of Beilinson-Flach elements (studied in [Beĭ85], [Fla92], [BDR15a], [BDR15b] etc.) is given by the push-forward to $H^3_{\mathrm{mot}}(\mathrm{Sh}_{\mathbf{GL}_2}(K_1(N))^2, \mathbf{Z}(3))$ along the diagonal embedding $\mathrm{Sh}_{\mathbf{GL}_2} \hookrightarrow \mathrm{Sh}^2_{\mathbf{GL}_2}$ of a Siegel unit ${}_cg_N \in \mathscr{O}(\mathrm{Sh}_{\mathbf{GL}_2}(K_1(N)))^*$. The image under the Beilinson regulator r_D to Deligne cohomology of this class is intimately connected to values of the Rankin-Selberg L-function of the convolution of two modular forms ([Beĭ85, §6]).

On the other hand, the use of the étale realisation of cycles coming from subvarieties of appropriate co-dimension has given extraordinary results towards the Bloch-Kato conjecture of the corresponding representation. These cases give Kolyvagin type Euler systems, which consist of a collection of Galois cohomology classes with distribution relations over a tower of ramified extensions of the base field which differs from the cyclotomic one. Examples are given by Heegner points, i.e. CM points of the modular curve $\mathrm{Sh}_{\mathbf{GL}_2}(K_0(N))$, as in [Kol90], [Gro91], or by special CM cycles (e.g. [Jet14], [BBJ15], [Cor09]). In [Jet14], distribution relations over the anti-cyclotomic tower of a CM field E of certain cycles on a unitary Shimura variety for $U(2,1)\times U(1,1)$ of reflex field E are taken under observation. For instance, if $[E:\mathbf{Q}]=2$, the motivic bottom class in $H^4_{\mathrm{mot}}(\mathrm{Sh}_G(K)_{/E},\mathbf{Z}(2))$ is the push-forward of the cycle $\mathbb{1}_Y\in H^0_{\mathrm{mot}}(\mathrm{Sh}_{U(1,1)}(K\cap U(1,1))_{/E},\mathbf{Z})$ corresponding to the identity connected component Y of the Shimura variety $\mathrm{Sh}_{U(1,1)}$ under a *diagonal* embedding $\mathrm{Sh}_{U(1,1)}\hookrightarrow \mathrm{Sh}_{U(2,1)}\times \mathrm{Sh}_{U(1,1)}$. This cycle appears naturally in the setting of the arithmetic Gan-Gross-Prasad conjecture (e.g. [Zha17, §3.1, 3.2]).

Finally, it is worth mentioning the work of Darmon and Rotger, who intensively studied (e.g. in [DR14], [DR17]) the arithmetic significance of diagonal cycles constructed as the push-forward Δ of $\mathbb{1}_{\mathrm{Sh}_{\mathbf{GL}_2}}\in H^0_{\mathrm{mot}}(\mathrm{Sh}_{\mathbf{GL}_2},\mathbf{Z})$ to $H^4_{\mathrm{mot}}(\mathrm{Sh}^3_{\mathbf{GL}_2},\mathbf{Z}(2))$ along the diagonal embedding $\mathrm{Sh}_{\mathbf{GL}_2}\hookrightarrow \mathrm{Sh}^3_{\mathbf{GL}_2}$. These elements do not satisfy distribution relations as in the previous cases, but, by Hida theoretic methods, are used to prove new cases of the equivariant Birch and Swinnerton-Dyer conjecture. If f,g,h are three modular forms of weight 2, the height of the (f,g,h)-isotypic component of Δ can be related to the first derivative of the triple product L-function $L(f\times g\times h,s)$ at $s=2$ ([YZZ], building on the integral formula of [Ich08]).

All these constructions are supported by an intimate connection to special values of the corresponding L-function. This should be seen as a sort of guideline to whether expect interesting arithmetic applications from the construction in exam.

1.1.4 New constructions

The main theme of this thesis is testing the push-forward construction in two new cases:

1. The first construction arises from Siegel units and gives classes in the motivic cohomology of the Shimura variety for the symplectic similitude group $\mathbf{GSp}_6$. Via the étale regulator, we obtain Galois cohomology classes for representations appearing in the middle degree étale cohomology of the Shimura variety. There is evidence that suggests that our motivic class is related to values of the spin L-function for certain cuspidal automorphic representations of $\mathbf{GSp}_6$ ([PS18b]).

2. The second one uses the push-forward of $\mathbf{GSp}_4$ Eisenstein classes ([Kin98], [Wil06], and [Fal05]) to the cohomology of a Shimura variety associated to the unitary group $\mathbf{GU}(2,2)$. The relation of this motivic class with values of the exterior square L-function of certain automorphic representations of $\mathbf{GU}(2,2)$ is yet unknown. We intend to investigate it in a future project.

Before describing the two constructions, we would like to stress the importance of exploring the technique for Eisenstein classes attached to symplectic groups larger than $\mathbf{GL}_2$. Unfortunately, there are cases where either the construction of an Euler system from Eisenstein classes for $\mathbf{GL}_2$ or special cycles might not be possible, or where results from the theory of automorphic forms would suggest that we might expect different and conceptually more suitable constructions. For instance, there are integral representations of automorphic L-functions which use Eisenstein series associated to higher rank groups:

- In [BG92] the integral representation of Bump and Ginzburg of the spin L-function of automorphic forms of $\mathbf{GSp}_{10}$ is obtained by integrating over $\mathbf{GSp}_4 \boxtimes \mathbf{GSp}_6$ a Siegel Eisenstein series for $\mathbf{GSp}_6$;
- The integral in [PSR87] of Piatetski-Shapiro and Rallis gives a representation of the (triple product) L-function associated to the natural 8-dimensional representation of $\mathbf{GL}_2 \boxtimes \mathbf{GL}_2 \boxtimes \mathbf{GL}_2$ by restricting a Siegel Eisenstein series for $\mathbf{GSp}_6$.

We would like to remark that the construction of motivic classes in the cohomology of symplectic Shimura varieties, whose realisation under the Beilinson regulator is related to Siegel Eisenstein series is still an open problem. Moreover, at present little is known on the range of possible applications of the Eisenstein classes for $\mathbf{GSp}_{2g}$ of [Kin99] to the study of the arithmetic of automorphic forms. Nevertheless, Lemma has proved that the motivic Eisenstein classes for $\mathbf{GSp}_{2g}$ are non-zero, by studying their residue to the dimension zero component of the Baily-Borel-Satake compactification of $\mathrm{Sh}_{\mathbf{GSp}_{2g}}$ (see [Lem16]). The p-adic theory of these classes has been extensively explored by Kings in [Kin15c], extending previous results of [Kin15a] in the $\mathbf{GL}_2$-setting.

1.2 Towards an Euler system for $\mathbf{GSp}_6$

Together with Joaquín Rodrigues Jacinto, we construct elements in the cohomology of the Shimura variety of the symplectic similitude group $\mathbf{G} = \mathbf{GSp}_6$. These classes are defined over cyclotomic extensions of $\mathbf{Q}$ and satisfy norm compatibility relations in the cyclotomic tower at p, which differ from the Euler system norm relations because a Hecke operator appears. This phenomenon is present in the constructions of [LLZ14], [LLZ16], and [LSZ17], and it has a conjectural explanation ([LZ17, §5]).

1.2.1 Setting

We consider the subgroup

$$\mathbf{H} = \mathbf{GL}_2 \times_{\det} \mathbf{GL}_2 \times_{\det} \mathbf{GL}_2 = \{(A,B,C) \ : \ A,B,C \in \mathbf{GL}_2, \det A = \det B = \det C\} \subset \mathbf{G},$$

which, after a suitable choice of maps from the Deligne torus to $\mathbf{H}_{\mathbf{R}}$, denoted by $X_{\mathbf{H}}$, induces an embedding $\iota : \mathrm{Sh}_{\mathbf{H}} = \mathrm{Sh}(\mathbf{H}, X_{\mathbf{H}}) \hookrightarrow \mathrm{Sh}(\mathbf{G}, X_{\mathbf{G}}) = \mathrm{Sh}_{\mathbf{G}}$. By pulling back Beilinson's Eisenstein symbol in the motivic cohomology of the modular curve associated to the first $\mathbf{GL}_2$-copy of $\mathbf{H}$, we get elements in the first motivic cohomology group of $\mathrm{Sh}_{\mathbf{H}}$. Their push-forward along ι thus gives elements in the seventh motivic cohomology group of $\mathrm{Sh}_{\mathbf{G}}$. One then uses the natural action of $\mathbf{G}(\mathbf{A}_f)$ on the Shimura variety $\mathrm{Sh}_{\mathbf{G}}$ to perturb these classes and obtain a whole compatible system of cohomology classes defined over ramified extensions of the base field.

1.2.2 Motivation

Let π be a cohomological automorphic cuspidal representation of $\mathbf{G}(\mathbf{A}_f)$. After projecting to the π-isotypic component, the motivic classes that we construct are expected, according to Beilinson's conjectures, to be related to special values of the degree eight spin L-function $L(s,\pi,\mathrm{spin})$ associated to π. This is motivated by recent work of Pollack and Shah ([PS18b]), who have given (under certain hypotheses on π) an integral representation of the (partial) spin L-function of π, by integrating over $\mathbf{H}$ a $\mathbf{GL}_2$-Eisenstein series against a cusp form φ in the space of π.

1.2.3 Main results

By applying the étale regulator map and employing the action of the Hecke algebra of $\mathbf{G}$, we prove the following.

Theorem 1.2.1. *Let $\mathscr{L}_{\mathbf{Z}_p}$ be the $\mathbf{Z}_p$-local system associated to the irreducible algebraic representation of $\mathbf{G}$ of highest weight $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3)$. For each integer k such that $|\lambda_1 - \lambda_2 - \lambda_3| \leq k \leq \lambda_1 - \lambda_2 + \lambda_3$ and $\sum_i \lambda_i \equiv k$ (mod 2), there exists a family of étale cohomology classes*

$$z_{n,m}^{\mathscr{L},k} \in H^7_{ét}(\mathrm{Sh}_{\mathbf{G}}(K_{n,0})_{/\mathbf{Q}(\zeta_{p^m})}, \mathscr{L}_{\mathbf{Z}_p}(4+q)),$$

for $q = \frac{k-\sum_i \lambda_i}{2}$, which satisfies the following norm relations:

1. *For $n \geq 1$, $(\mathrm{pr}^{K_{n+1,0}}_{K_{n,0}})_*(z^{\mathscr{L},k}_{n+1,m}) = z^{\mathscr{L},k}_{n,m}$;*
2. *For $n,m \geq 1$, $\mathrm{norm}^{\mathbf{Q}(\zeta_{p^{m+1}})}_{\mathbf{Q}(\zeta_{p^m})}(z^{\mathscr{L},k}_{n,m+1}) = \frac{\mathscr{U}'_p}{\sigma_p^3} \cdot z^{\mathscr{L},k}_{n,m}$,*

where $\mathscr{U}'_p$ is the Hecke operator associated to the double coset of $diag(p^{-3},p^{-2},p^{-2},p^{-1},p^{-1},1) \in \mathbf{G}(\mathbf{Q}_p) \subset \mathbf{G}(\mathbf{A}_f)$, and σ_p is the image of p^{-1} under the Artin map $\mathbf{Q}_p^* \hookrightarrow \mathbf{A}_f^* \to Gal(\mathbf{Q}(\zeta_{p^m})/\mathbf{Q})$.

By $K_{n,0}$ we mean a tower of sufficiently small level subgroups of $\mathbf{G}(\widehat{\mathbf{Z}})$ defined by certain congruences modulo powers of p (cf. §4.2.3). The proof of the Theorem 1.2.1 is an adaptation of the methods used in [LLZ14].

By using the theory of Λ-adic Eisenstein classes developed in [Kin15b], we also show that the classes vary p-adically in families as the local system $\mathscr{L}_{\mathbf{Z}_p}$ varies. We thus obtain a universal class interpolating them all. Taking specialisations of this universal class, one obtains further étale cohomology classes which do not a priori come from a geometric construction. These aspects will appear in [CRJ18].

Let us briefly mention some immediate applications of our theorem. Using results from [MT02], one can project our classes to the groups

$$H^1(\mathbf{Q}(\zeta_{p^m}), V_\pi(q)),$$

where π is a suitable automorphic representation of $\mathbf{G}(\mathbf{A}_f)$, V_π is the p-adic Galois representation associated to π ([KS16, §8]), and q is as in Theorem 1.2.1. After imposing a $\mathscr{U}'_p$-ordinarity condition on V_π, in Theorem 4.3.3 we modify the classes above constructed to define Galois cohomology classes in the Iwasawa cohomology of V_π. Applying the general machinery of Perrin-Riou, this allows to define a p-adic spin L-function for this automorphic representation.

1.2.4 Towards new Euler systems

We finally mention that this work should be seen as a starting point and that there are still many unsolved questions about these classes. The crux is the proof of the so-called tame norm relations, comparing classes over fields $\mathbf{Q}(\zeta_{m\ell})$ and $\mathbf{Q}(\zeta_m)$ where ℓ does not divide m. At the moment, it seems very hard to adapt the technique introduced in [LSZ17], which relies on the local Gan-Gross-Prasad conjecture for the pair $(\mathrm{SO}_4, \mathrm{SO}_5)$. This is due to the fact that, if π is an unramified principal series representation of $\mathbf{G}(\mathbf{Q}_\ell)$ and ρ is a principal series representation of $\mathbf{GL}_2(\mathbf{Q}_\ell)$, the space of bilinear forms

$$Hom_{\mathbf{H}(\mathbf{Q}_\ell)}((\rho \boxtimes \mathbb{1} \boxtimes \mathbb{1}) \otimes \pi, \mathbf{C})$$

fails to be one dimensional. It also seems to be very difficult to show that these classes are non-zero.

The relation between the special values of the p-adic spin L-function and the complex L-function are still mysterious. We expect an explicit reciprocity law to hold, relating values of Bloch-Kato's dual exponential maps of our Iwasawa class to certain values of the complex spin L-function. One should also be able to calculate the complex regulator of the motivic classes in terms of the complex spin L-function using the techniques of [Kin98] and [Lem17]. We are at the moment working on some of these points and we expect this work to be the first one of a series devoted to the study of the arithmetic of automorphic forms for the group $\mathbf{G}$.

1.3 Norm compatible elements for $\mathbf{GU}(2,2)$

In Chapter 5, we construct a two variable family of cohomology classes in the fifth degree motivic and étale cohomology of the Shimura variety $\mathrm{Sh}_{\mathbf{GU}(2,2)}$ attached to a similitude unitary group $\mathbf{G} = \mathbf{GU}(2,2)$ of signature $(2,2)$, which satisfy certain compatibility relations.

1.3.1 Setting

We consider the subgroup $\mathbf{H} := \mathbf{GSp}_4 \subset \mathbf{G}$ (via the natural map $\mathbf{H} \subset \mathbf{GL}_4$) and an embedding of Shimura data $(\mathbf{H}, X_\mathbf{H}) \hookrightarrow (\mathbf{G}, X_\mathbf{G})$. It induces an embedding of the corresponding Shimura varieties $\iota : \mathrm{Sh}_\mathbf{H} \hookrightarrow \mathrm{Sh}_\mathbf{G}$ of co-dimension 1. The motivic constituents of our family are the push-forward along ι of Eisenstein classes for $\mathbf{H}$, as defined by [Kin99] and [Fal05]. By using the action of $\mathbf{G}(\mathbf{A}_f)$ on $\mathrm{Sh}_\mathbf{G}$, we obtain a compatible system of cohomology classes over a certain two variable tower of level subgroups.

This constitutes the first example in literature where Eisenstein classes for $\mathbf{H}$ are employed in a push-forward construction, and it presents an unexpected behaviour, which we believe to be always present in any push-forward construction of this kind, that obstructs the method employed in [KLZ17], [LLZ16], [CRJ18], [LZ18] to give a norm-compatible system of cohomology classes defined over ramified extensions of the base field.

1.3.2 Main results

Fix a prime p, which is either split or inert in the imaginary quadratic field defining $\mathbf{G}$. By employing the action of the Hecke algebra of $\mathbf{G}$ and the étale regulator, we define a family of cohomology classes

$$z_{n,m} \in H^5_{\text{ét}}(\mathrm{Sh}_{\mathbf{G}}(\tilde{U}_{n,m}), \mathbf{Z}_p(3)).$$

By $\tilde{U}_{n,m}$ we mean a sufficiently small open compact subgroup of $\mathbf{G}(\widehat{\mathbf{Z}})$ formed by elements whose reduction modulo p^n is in a mirabolic subgroup of the Klingen subgroup of $\mathbf{G}$ (Definition 5.1.6), and whose reduction modulo p^m lies in a one dimensional subgroup of the maximal torus of $\mathbf{G}$. We show the following.

Theorem 1.3.1. *There exists a family of cohomology classes* $z_{n,m} \in H^5_{\acute{e}t}(\mathrm{Sh}_{\mathbf{G}}(\tilde{U}_{n,m}), \mathbf{Z}_p(3))$*, which satisfies the norm relations*

$$(\mathrm{pr}^{\tilde{U}_{n+1,m}}_{\tilde{U}_{n,m}})_*(z_{n+1,m}) = z_{n,m};$$
$$(\mathrm{pr}^{\tilde{U}_{n,m+1}}_{\tilde{U}_{n,m}})_*(z_{n,m+1}) = \mathscr{U}'_p \cdot z_{n,m},$$

whenever $m \geq 1$ *and* $n \geq 3m+3$*;* $\mathscr{U}'_p$ *is the Hecke operator associated to the double coset of* $diag(p^{-3}, p^{-2}, p^{-1}, 1) \in \mathbf{G}(\mathbf{Q}_p) \subset \mathbf{G}(\mathbf{A}_f)$.

The compatibility with respect to n follows from the analogous statement for the Eisenstein classes for $\mathbf{H}$. This is the subject of §3.4, where we show, more generally, that the Eisenstein classes for $\mathbf{GSp}_{2g}$ are compatible in the mira-Klingen tower (Proposition 3.4.8) and then deduce compatibility relations for any push-forward of them (Corollaries 3.4.11 and 3.4.12). Proposition 3.4.8 generalises a method used in [Sch98] for the $g=1$ case. As a direct consequence of these compatibility relations, we can construct Λ-adic Eisenstein classes for $\mathbf{GSp}_{2g}$, which arise from the integral construction of [Fal05, §3], and compare them with the Eisenstein-Iwasawa classes introduced in [Kin15c].

The proof of Theorem 1.3.1 for the compatibility with respect to m is more elaborate and it is based on ideas which have been employed in the proof of [KLZ17, Theorem 5.4.1] for the *vertical* Euler system norm relation of the Beilinson-Flach classes.

1.3.3 Cyclotomic norm relations

In §5.4 we discuss the obstruction we face when trying to deduce from Theorem 1.3.1 an Euler system norm compatibility of these classes in the cyclotomic tower at p and discuss a few similar cases where the same obstruction appears.

Finally, we would like to remark that one expects to prove a relation between our classes and values of the exterior square L-function of automorphic representations of $\mathbf{G}$ appearing in the middle degree cohomology of $\mathrm{Sh}_{\mathbf{G}}$.

1.4 Future directions

We intend to generalise the work of this thesis in many different directions; in this section, we briefly describe some of the questions we intend to tackle in the near future.

1.4.1 Tame norm relations for $\mathbf{GSp}_6$

In Chapter 4, we construct a system of cohomology classes compatible in the cyclotomic tower at p for p-adic Galois representations appearing in the middle degree cohomology of $\mathrm{Sh}_{\mathbf{GSp}_6}$. However, in order to apply the Euler system machine of [Rub00] and, thus, obtain bounds (under standard assumptions) on Bloch-Kato Selmer groups of these Galois representations, we need to establish the cyclotomic norm relations of our cohomology classes at tame primes. We are currently working on these relations, by exploring two different directions. On the one hand, we are analysing a very interesting approach, suggested by Dimitar Jetchev, based on adapting techniques used in the case of special cycles on unitary Shimura varieties in [Jet14] and [BBJ15]. On the other hand, we intend to understand how to reduce the problem to a local statement involving the unramified calculations in [PS18b], thus modelling techniques [LSZ17] for a setting which lacks of a local Gan-Gross-Prasad conjecture.

1.4.2 Vertical norm relations for $\mathbf{GSp}_{2g}^{\boxtimes 2}$

Constructing non-trivial elements in the Iwasawa cohomology of Galois representations has highly sophisticated arithmetic consequences, such as the construction of p-adic L-functions (e.g. [Kat04], [Col00]); the method introduced in [KLZ17] and axiomatised in [LZ18] gives a recipe for building classes in the Iwasawa cohomology of Galois representations appearing in the étale cohomology of Shimura varieties.

Recently, we have been investigating a new push-forward construction involving Siegel units for representations appearing in the middle degree étale cohomology of $\mathbf{GSp}_{2g}^{\boxtimes 2}$. At present, we can construct classes defined over cyclotomic fields, and use the method of *loc.cit.* to show the Euler system norm relation in the cyclotomic tower at p for small g cases. We are working on extending the result in general.

1.4.3 Archimedean regulator formula for $\mathbf{GU}(2,2)$

There are various aspects of the theory of Eisenstein classes that are still unexplored. For example, it would be very useful to explicitly calculate the residue at the boundary of the Baily-Borel-Satake compactification of the motivic Eisenstein classes for $\mathbf{GSp}_{2g}$, extending Lemma's recent results in [Lem16]. This would possibly lead to an Archimedean regulator formula for the motivic classes of Chapter 5, by generalising methods used in [Lem17] and [PS18a]. As a first step towards it, we intend to give a representation for the exterior square L-function of certain automorphic representations π of $\mathbf{GU}(2,2)$, integrating a cusp form in the space of π with a $\mathbf{GSp}_4$-Klingen Eisenstein series over the automorphic quotient $Z_{\mathbf{GSp}_4}(\mathbf{A})\backslash[\mathbf{GSp}_4]$.

1.4.4 Euler Systems for $\mathbf{GU}(2,2)$

At present, the classes constructed in Chapter 5 from push-forward of $\mathbf{GSp}_4$-Eisenstein classes do not give norm compatible elements over cyclotomic extensions of $\mathbf{Q}$. Nevertheless, we aim to construct an Euler system for Galois representations appearing in the middle degree étale cohomology group of $\mathrm{Sh}_{\mathbf{GU}(2,2)}$. Similarly to the case of $\mathbf{GSp}_6$ treated in this thesis, the construction is based on taking pull-back and push-forward of Siegel units and Eisenstein classes for modular curves. Precisely, we have an embedding of $\mathbf{H} := \mathbf{GL}_2 \boxtimes \mathbf{GL}_2 \hookrightarrow \mathbf{GSp}_4 \hookrightarrow \mathbf{GU}(2,2) =: \mathbf{G}$ which induces a morphism of Shimura varieties $\iota_U : \mathrm{Sh}_{\mathbf{H}}(U \cap \mathbf{H}) \longrightarrow \mathrm{Sh}_{\mathbf{G}}(U)$ of codimension 2 for sufficiently small $U \subset \mathbf{G}(\mathbf{A}_f)$. Then, we can construct classes in the fifth degree cohomology group of $\mathrm{Sh}_{\mathbf{G}}$ by first pulling back Beilinson's Eisenstein symbol for $\mathrm{Sh}_{\mathbf{GL}_2}$ to $\mathrm{Sh}_{\mathbf{H}}$ and then taking the push-forward under the Gysin morphism of ι. This construction is motivated by the following facts.

- Recent (yet unpublished) work of Aaron Pollack and Shrenik Shah on an Archimedean regulator formula for the corresponding motivic classes by giving a representation of the exterior square L-function of cuspidal automorphic representations π (supporting the appropriate Fourier coefficient) of $\mathbf{G}$ by integrating a $\mathbf{GL}_2$-Eisenstein series against a cusp form $\varphi \in \pi$ over the automorphic quotient $Z_{\mathbf{H}}(\mathbf{A})\backslash[\mathbf{H}]$.
- The technique of [LZ18] applies to this setting and enables us to construct classes defined over cyclotomic extensions, which satisfy the Euler system norm relations in the cyclotomic tower at p.

1.5 General Notation

For the readers' convenience, we collect here some of the notation used in the thesis.

We fix an algebraic closure $\bar{\mathbf{Q}}$ of the field of rational numbers $\mathbf{Q}$. For a rational prime p, we denote by $\mathbf{Q}_p$ the p-adic completion of $\mathbf{Q}$ and by $\mathbf{Z}_p$ the ring of p-adic integers; we also fix an embedding $\bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$. We denote by $\mathbf{Q}(\zeta_m)$ the m-th cyclotomic extension of $\mathbf{Q}$, where ζ_m denotes a primitive m-th root of unity. Let μ_m denote the subgroup of the m-th roots of unity in $\bar{\mathbf{Q}}^*$. We denote the inverse limit $\varprojlim_n \mu_{p^n}$ by $\mathbf{Z}_p(1)$ and for any $\mathbf{Z}_p$-module T and integer j, we denote $T \otimes_{\mathbf{Z}_p} (\mathbf{Z}_p(1)^{\otimes j})$ by $T(j)$. We also fix a basis of $\mathbf{Z}_p(1)$, by choosing a system of compatible p^n-th roots of unity $\zeta = (\zeta_{p^n})_n$, for any rational prime p. For a finite or empty set S of finite primes, we denote $\widehat{\mathbf{Z}}^S = \prod_{p \notin S} \mathbf{Z}_p$ and $\mathbf{Z}_S = \prod_{p \in S} \mathbf{Z}_p$.

For a number field E, we denote by $\mathrm{Gal}_E = \mathrm{Gal}(\bar{\mathbf{Q}}/E)$ its absolute Galois group and by $\mathbf{A}_E$ the ring of adeles of E. Moreover, $\mathrm{Gal}_E^{\mathrm{ab}}$ is the Galois group of the maximal abelian extension of E. In the case of $E = \mathbf{Q}$, we will simply denote by $\mathbf{A}$ and $\mathbf{A}_f$ the adeles and the finite adeles of $\mathbf{Q}$, respectively.

For an integer n, we denote by $\mathbf{GL}_n$ the general linear group over $\mathbf{Z}$; in the case of $n = 1$, we identify $\mathbf{GL}_1$ with the multiplicative group $\mathbf{G}_{\mathrm{m}}$. For a ring homomorphism $R \to S$ and a scheme X over S, we denote by $\mathrm{Res}_{S/R}(X)$ the R-scheme given by the restriction of scalars of X/S. In particular, we denote by $\mathbf{S}$ the Deligne torus $\mathrm{Res}_{\mathbf{C}/\mathbf{R}}(\mathbf{G}_{\mathrm{m}})$. For a reductive group G over $\mathbf{Z}$, we denote by $K_G(N)$ or $K(N)$ the kernel of the reduction modulo an integer N:

$$K_G(N) \hookrightarrow G(\widehat{\mathbf{Z}}) \longrightarrow G(\mathbf{Z}/N\mathbf{Z}) \to 0.$$

For two reductive groups G_1, G_2 with multiplier maps $\nu_i : G_i \to \mathbf{G}_{\mathrm{m}}$, we sometimes denote by $G_1 \boxtimes G_2$ their fibre product $G_1 \times_{\nu_1,\nu_2} G_2$.

By an abelian scheme A over a base S, we mean a proper, smooth S-scheme with geometrically connected fibres. For an abelian scheme $\pi : A \to S$, we denote by $A[N]$ the kernel of the multiplication-by-N morphism and by T_pA or $\mathscr{H}_{\mathbf{Z}_p}$ its p-adic Tate module $(R^1\pi_*\mathbf{Z}_p)^\vee$. We sometimes denote by μ_N/S the S-scheme of primitive N-th roots of unity.

For a commutative ring R and an R-module M, we denote by $\mathrm{Sym}^k(M)$ the module of Σ_k-coinvariants of the k-fold tensor product of M and by $\mathrm{TSym}^k(M)$ the module of Σ_k-invariants of the k-fold tensor product of M, where Σ_k denotes the symmetric group on k elements. There is a natural morphism $\mathrm{Sym}^k(M) \to \mathrm{TSym}^k(M)$, which becomes an isomorphism after inverting $k!$.

Chapter 2

Preliminaries

In this chapter, we recall results on Shimura varieties and on the motivic and étale cohomology theories, which will be extensively used in the later chapters.
In particular, in §2.1.5, we study a certain tower of level structures for symplectic Shimura varieties, which is crucially needed for proving distribution relations of Eisenstein classes in the cohomology of symplectic Shimura varieties in §3.4.

2.1 Shimura varieties

In the following, we give a very brief introduction to some aspects of the theory of Shimura varieties. We mainly follow [Del71], [Mil05], and [Moo98].

2.1.1 Definitions

Let $\mathbf{S}$ denote the Deligne torus $\mathrm{Res}_{\mathbf{C}/\mathbf{R}}(\mathbf{G}_{\mathrm{m},\mathbf{C}})$ and denote by G^{ad} the adjoint group of a group G.

A Shimura datum is a pair (G, X_G), consisting of a reductive group $G/\mathbf{Q}$ and a $G(\mathbf{R})$-orbit X_G in the set of morphisms of $\mathbf{R}$-algebraic groups $\mathrm{Hom}(\mathbf{S}, G_{\mathbf{R}})$, such that, for all $h \in X_G$, we have:

SV1. $\mathrm{Lie}(G)_{\mathbf{R}}$ is of type $\{(-1,1),(0,0),(1,-1)\}$;

SV2. $\mathrm{Inn}(h(i))$ is a Cartan involution of $G^{\mathrm{ad}}_{\mathbf{R}}$;

SV3. for every $\mathbf{Q}$-factor H of G^{ad}, the composition of h with $G_{\mathbf{R}} \to H_{\mathbf{R}}$ is non-trivial.

Remark 2.1.1. These conditions imply that connected components of X_G are Hermitian symmetric domains and that X_G has a unique structure of a complex manifold such that every faithful representation $\rho : G \to \mathbf{GL}(V)$ induces a variation $(V, \rho \circ h)_{h \in X_G}$ of polarisable $\mathbf{Q}$-Hodge structures.

All the Shimura data we work with are of PEL-type.

Definition 2.1.2. A PEL-datum is a tuple $(B, \star, V, \langle\ ,\ \rangle, h)$, where B is a semi-simple $\mathbf{Q}$-algebra, $\star$ is a positive involution on B, $(V, \langle\ ,\ \rangle)$ is a finite dimensional symplectic B-module with a $\mathbf{Q}$-valued pairing $\langle\ ,\ \rangle$ such that

$$\langle bu, v \rangle = \langle u, b^{\star} v \rangle, \text{ for } b \in B \text{ and } u, v \in V,$$

and $h : \mathbf{C} \to \mathrm{End}_{B_{\mathbf{R}}}(V_{\mathbf{R}})$ is an $\mathbf{R}$-algebra homomorphism such that

1. $\langle h(z)u, v\rangle = \langle u, h(\bar{z})v\rangle$, for $z \in \mathbf{C}$, $u, v \in V$;

2. The pairing $(u, v) \mapsto \langle u, h(i)v\rangle$ is positive definite.

Let G denote the subgroup of $\mathbf{GL}(V)$, which preserves the pairing $\langle\ ,\ \rangle$ up to scaling; (G, h) defines a Shimura datum (cf. Proposition [Mil05, Proposition 8.14]). Any Shimura datum arising from a PEL-datum in this fashion is said to be of PEL-type.

Given a Shimura datum (G, X_G) and an open compact subgroup K of $G(\mathbf{A}_f)$, we make $G(\mathbf{Q})$ act on the left on $X \times G(\mathbf{A}_f)$ by left multiplication on both factors and K act on the second one by right multiplication on $G(\mathbf{A}_f)$. Thus, we can consider the double coset space $G(\mathbf{Q})\backslash X \times G(\mathbf{A}_f)/K$.

Recall that an element $g \in G(\mathbf{Q})$ is neat if the subgroup of $\overline{\mathbf{Q}}^*$ generated by the eigenvalues of G with respect to some faithful representation of G is torsion free, while a subgroup of $G(\mathbf{Q})$ is defined to be neat if all its elements are. One can extend this notion to subgroups $G(\mathbf{A}_f)$, as follows (see [Pin90], p.12).

Definition 2.1.3. For each $g = (g_p)_p \in G(\mathbf{A}_f)$, denote by Γ_p the subgroup of $\overline{\mathbf{Q}}_p^*$ generated by the eigenvalues of g_p w.r.t. some faithful representation of G; for each p, fix an embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. Then, $g \in G(\mathbf{A}_f)$ is neat if

$$\bigcap_p (\overline{\mathbf{Q}} \cap \Gamma_p)_{tors} = 1,$$

where $(\bullet)_{tors}$ denotes the torsion part of $\bullet$. A subgroup of $G(\mathbf{A}_f)$ is called neat if all its elements are.

Definition 2.1.3 does not depend on either the choice of representation of G or the embeddings $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$ (cf. [Pin90]). The kernel $K_G(d)$ of reduction modulo $d \geq 3$, i.e. the subgroup defined by

$$K_G(d) \hookrightarrow G(\widehat{\mathbf{Z}}) \longrightarrow G(\mathbf{Z}/d\mathbf{Z}) \to 0$$

is neat (e.g. [Pin90, p.13]). Indeed, a large family of neat subgroups of $G(\mathbf{A}_f)$ is given by congruent sufficiently small open compact subgroups:

Definition 2.1.4. A compact open subgroup $K \subset G(\mathbf{A}_f)$ is said to be sufficiently small if it acts faithfully on $G(\mathbf{Q})\backslash X_G \times G(\mathbf{A}_f)$.

Whenever K is neat or sufficiently small, $G(\mathbf{Q})\backslash X_G \times G(\mathbf{A}_f)/K$ has a unique structure of a quasi-projective complex algebraic variety, which we denote by $\mathrm{Sh}_G(K)$. As a matter of convention, in the sequel we work with sufficiently small level subgroups rather than neat ones.

Right multiplication by $g \in G(\mathbf{A}_f)$ induces a morphism

$$g : \mathrm{Sh}_G(K) \to \mathrm{Sh}_G(g^{-1}Kg),\ [h, m] \mapsto [h, mg].$$

Thus, we define the Shimura variety Sh_G of infinite level to be the projective system of varieties $(\mathrm{Sh}_G(K))_K$ (under the natural projections), where K runs through sufficiently small compact open subgroups. It has a natural right action of $G(\mathbf{A}_f)$ induced at each finite level K by the morphisms $g : \mathrm{Sh}_G(K) \to \mathrm{Sh}_G(g^{-1}Kg)$, for each $g \in G(\mathbf{A}_f)$.

Definition 2.1.5. A morphism of Shimura data $(H, X_H) \to (G, X_G)$ is a morphism $H \to G$ of groups sending X_H to X_G.

In the following chapters, we will repeatedly use that morphisms of Shimura data induce morphisms of the corresponding Shimura varieties.

Theorem 2.1.6 ([Mil05], Theorem 5.16)**.** *A morphism of Shimura data $(H, X_H) \to (G, X_G)$ defines a morphism $\mathrm{Sh}_H \to \mathrm{Sh}_G$ of Shimura varieties, which is a closed immersion if $H \to G$ is injective.*

2.1.2 Canonical models

The arithmetic significance of Shimura varieties is supported by the existence of canonical models for them over number fields. We refer to [Mil05, §14] and [Moo98, §2].

To a Shimura datum (G, X_G), one can associate a number field $E = E(G, X_G)$, which is called the reflex field. It is defined as follows. For each $h \in X_G$, we define a co-character of $G_{\mathbf{C}}$ by

$$\mu_h : \mathbf{G}_{\mathrm{m}} \to \mathbf{S}_{\mathbf{C}} \to G_{\mathbf{C}},\ z \mapsto h_{\mathbf{C}}(z, 1).$$

Since two different $h, h' \in X_G$ are conjugate, X_G defines an element

$$(\mu_h)_{h \in X_G} \in G(\mathbf{C}) \backslash \mathrm{Hom}(\mathbf{G}_{\mathrm{m}}, G_{\mathbf{C}}).$$

We can regard $(\mu_h)_{h \in X_G}$ as an element of $G(\overline{\mathbf{Q}}) \backslash \mathrm{Hom}(\mathbf{G}_{\mathrm{m}}, G_{\overline{\mathbf{Q}}})$ (c.f. [Mil05, p. 344]).Thus, we define the reflex field E of (G, X_G) to be the fixed field of the subgroup of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ which fixes $(\mu_h)_{h \in X_G}$. In several occasions, we will use the following:

Remark 2.1.7 ([Mil05], Remark 12.3(c)). Let $(H, X_H) \to (G, X_G)$ be a morphism of Shimura data with injective $H \to G$. Then, $E(H, X_H) \supseteq E(G, X_G)$.

2.1.2.1 The case of tori

Before discussing the general case, it is useful to analyse the case of tori. Let $T/\mathbf{Q}$ be a torus and $h : \mathbf{S} \to T_{\mathbf{R}}$ be any morphism of tori, then $(T, \{h\})$ is a Shimura datum. For every compact open subgroup $K \subset T(\mathbf{A}_f)$, $\mathrm{Sh}_T(K)$ consists of finitely many points. To define a model of $\mathrm{Sh}_T(K)$ over $E = E(T, \{h\})$, it suffices to give an action of $\mathrm{Gal}_E = \mathrm{Gal}(\overline{\mathbf{Q}}/E)$. Since T is commutative, it is enough to describe an action of the Galois group $\mathrm{Gal}_E^{\mathrm{ab}}$ of the maximal abelian Galois extension of E. Recall that, by Class field theory, we have a surjective and continuous homomorphism

$$\mathrm{Art}_E : \mathbf{A}_E^* \to \mathrm{Gal}_E^{\mathrm{ab}}$$

such that, for every finite abelian extension L/E, we have a commutative diagram

$$\begin{tikzcd} E^*\backslash \mathbf{A}_E^* \arrow[r, "\mathrm{Art}_E"] \arrow[d] & \mathrm{Gal}_E^{\mathrm{ab}} \arrow[d, "\sigma\mapsto\sigma_{|L}"] \\ E^*\backslash \mathbf{A}_E^*/\mathrm{Norm}_E^L(\mathbf{A}_L^*) \arrow[r, "\sim"] & \mathrm{Gal}(L/E). \end{tikzcd}$$

On the other hand, the co-character μ_h defines

$$r_{(T,h)} : \mathrm{Res}_{E/\mathbf{Q}}(\mathbf{G}_\mathrm{m}) \xrightarrow{\mathrm{Res}(\mu_h)} \mathrm{Res}_{E/\mathbf{Q}}(T_E) \xrightarrow{\mathrm{Norm}_\mathbf{Q}^E} T. \tag{2.1}$$

We define a model of $\mathrm{Sh}_T(K)_E$, by choosing the action of $\sigma \in \mathrm{Gal}_E$, given by sending $[h,g] \in \mathrm{Sh}_T(K)$ to $[h, r_{(T,h)}(a)\cdot g]$, for $a \in \mathbf{A}_E^*$ such that $\mathrm{Art}_E(a) = \sigma_{|_{\mathrm{Gal}_E^{\mathrm{ab}}}}$. Thus, we define the canonical model of $\mathrm{Sh}_{T/E}$ to be the inverse limit of these models as K varies.

2.1.2.2 Canonical models

We are now ready to treat the general case. For a Shimura datum (G, X_G) and for each sufficiently small open compact K, $\mathrm{Sh}_G(K)$ admits a canonical model over E (e.g. [Moo98, Theorem 2.18]), i.e. there exists an E-scheme $\mathrm{Sh}_G(K)_E$ such that

1. $\mathrm{Sh}_G(K) = \mathrm{Sh}_G(K)_E \times_E \mathbf{C}$;
2. the right-multiplication action of $G(\mathbf{A}_f)$ on Sh_G descends to E, i.e. we have E-morphisms
$$g : \mathrm{Sh}_G(K)_E \longrightarrow \mathrm{Sh}_G(g^{-1}Kg)_E, \quad \text{for any } g \in G(\mathbf{A}_f);$$
3. it is canonical, i.e. for every injective morphism $(T,\{h\}) \hookrightarrow (G, X_G)$ of Shimura data, where T is a torus, the induced $\mathbf{C}$-morphism $\mathrm{Sh}_T(K\cap T) \to \mathrm{Sh}_G(K)$ descends to a morphism between the canonical model of $\mathrm{Sh}_T(K\cap T)$ over $E(T,\{h\})$ and $\mathrm{Sh}_G(K)_E \times_E E(T,\{h\})$.

Remark 2.1.8.

- The existence of the (canonical) model is proved by showing that the variety is defined over $\bar{\mathbf{Q}}$ and then by descending it to E, using the (continuous) action of $\mathrm{Gal}(\bar{\mathbf{Q}}/E)$ on the variety determined by 3. above. Indeed, recall that the functor $S \mapsto S \times_E \overline{\mathbf{Q}}$ defines an equivalence of categories (cf. [Moo98, §2.15.1])
$$\{\text{ quasi-projective schemes } S/E\} \to \left\{ \substack{\text{quasi-pr. schemes } S/\overline{\mathbf{Q}} \\ \text{with a continuous semi-linear action of } \mathrm{Gal}(\overline{\mathbf{Q}}/E)} \right\}.$$
- One of the key advantages of working with this Deligne-Shimura formalism of Shimura varieties is that $\mathrm{Sh}_G(K)$ has a canonical model over E, which does not depend on the level group K.

In the sequel, we will denote this canonical model by $\mathrm{Sh}_G(K)$ without explicit reference to the field E. Finally, the Shimura variety Sh_G at infinite level admits a unique canonical model over E, being the projective limit of canonical models at finite levels. In other words, the canonical models of $(\mathrm{Sh}_G(K))_K$, for each sufficiently small open compact K, are constructed in such a way that the Hecke action of $G(\mathbf{A}_f)$ and the Galois action commute, endowing $\mathrm{Sh}_{G,\overline{\mathbf{Q}}}$ with an action of $G(\mathbf{A}_f) \times \mathrm{Gal}(\bar{\mathbf{Q}}/E)$.

2.1.3 Symplectic Shimura varieties

In this thesis, we will mainly work with Shimura varieties associated to similitude symplectic groups, which we now define.

2.1.3.1 Symplectic groups

Denote by I'_g the $g \times g$ anti-diagonal matrix with all entries 1 and $J = \begin{pmatrix} & I'_g \\ -I'_g & \end{pmatrix}$. Let $\mathbf{GSp}_{2g}$ be the group scheme over $\mathbf{Z}$ defined by

$$\mathbf{GSp}_{2g}(R) = \{(h, m_h) \in (\mathbf{GL}_{2g} \times \mathbf{G}_{\mathrm{m}})(R) : h^t J h = m_h J\},$$

for any commutative ring R with 1. Define the symplectic multiplier to be the homomorphism

$$\nu : \mathbf{GSp}_{2g} \longrightarrow \mathbf{G}_{\mathrm{m}}, \quad h \mapsto m_h.$$

It has kernel the symplectic group Sp_{2g}. Denote by $Z_{\mathbf{GSp}_{2g}}$ the center of $\mathbf{GSp}_{2g}$.

2.1.3.2 Shimura datum

Define $h \colon \mathbb{S} \to \mathbf{GSp}_{2g/\mathbf{R}}$ as follows. Let

$$X := \left\{ M \in Sp_{2g}(\mathbf{R}) \ : \ M^2 = -I, \ \langle u, Mv \rangle := u^t JMv \text{ is} \pm\text{-definite} \right\}$$

be the set of positive or negative definite symplectic complex structures on the real vector space given by the standard representation of $\mathbf{GSp}_{2g/\mathbf{R}}$. The set X can be identified with the set of homomorphisms

$$h : \mathbb{S} \longrightarrow \mathbf{GSp}_{2g/\mathbf{R}},$$

by sending $M \in X$ to h such that $h(a+ib) = aI + bM$. Every $\mathbf{GSp}_{2g}(\mathbf{R})$-conjugacy class in X defines a Shimura datum and, since two symplectic complex structures are $\mathbf{GSp}_{2g}(\mathbf{R})$-conjugate, we conclude that X consists of a single $\mathbf{GSp}_{2g}(\mathbf{R})$-conjugacy class. In what follows, we "twist" X by considering the $\mathbf{GSp}_{2g}(\mathbf{R})$-conjugacy class of

$$\tilde{h} : \mathbb{S} \longrightarrow \mathbf{GSp}_{2g/\mathbf{R}}, \ \ a + ib \mapsto \tfrac{1}{a^2+b^2} h(a+ib),$$

for $h \in X$. This has the effect of changing a sign on the Galois action on the connected components (cf. [LSZ17, Remark 5.1.2]).

The pair $(\mathbf{GSp}_{2g}, X)$ defines a Shimura datum with reflex field $\mathbf{Q}$. The attached Shimura variety is a moduli space of polarised abelian schemes of relative dimension g with extra level structures, as we discuss below. Note that this Shimura datum naturally arises from a PEL-datum of type C (e.g. see [Mil05], Definition 8.15 and Example 8.6).

2.1.3.3 Galois action on connected components

A canonical model over $\mathbf{Q}$ of $\mathrm{Sh}_{\mathbf{GSp}_{2g}}$ defines an action of $\mathrm{Gal}_{\mathbf{Q}}^{\mathrm{ab}}$ on the connected components $\pi_0(\mathrm{Sh}_{\mathbf{GSp}_{2g}})$, as described in [Mil05, p. 349]; since the derived subgroup Sp_{2g} is simply connected, the space $\pi_0(\mathrm{Sh}_{\mathbf{GSp}_{2g}})$ admits the following description. Note that the multiplier map $\nu : \mathbf{GSp}_{2g} \to \mathbf{G}_{\mathrm{m}}$ induces a map of Shimura data $(\mathbf{GSp}_{2g}, X) \to (\mathbf{G}_{\mathrm{m}}, y)$, where y is the $\mathbf{G}_{\mathrm{m}}(\mathbf{R})$-conjugacy class of $\nu \circ \tilde{h}$. Denote by $\mathbf{G}_{\mathrm{m}}(\mathbf{Q})^{\dagger}$ the intersection of $\mathbf{G}_{\mathrm{m}}(\mathbf{Q})$ with $\mathrm{Im}(Z_{\mathbf{GSp}_{2g}}(\mathbf{R}) \to \mathbf{G}_{\mathrm{m}}(\mathbf{R}))$, then [Mil05, Theorem 5.17] gives the following

Proposition 2.1.9. *Let U be a sufficiently small level subgroup; the map*

$$\mathbf{GSp}_{2g}(\mathbf{Q}) \backslash X \times \mathbf{GSp}_{2g}(\mathbf{A}_f)/U \to \mathbf{G}_{\mathrm{m}}(\mathbf{Q})^{\dagger} \backslash \mathbf{G}_{\mathrm{m}}(\mathbf{A}_f)/\nu(U)$$

induces an isomorphism

$$\pi_0(\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U))(\mathbf{C}) \simeq \mathbf{G}_{\mathrm{m}}(\mathbf{Q})^{\dagger} \backslash \mathbf{G}_{\mathrm{m}}(\mathbf{A}_f)/\nu(U).$$

It follows that $\pi_0(\mathrm{Sh}_{\mathbf{GSp}_{2g}})(\mathbf{C}) \simeq \mathbf{Q}^*_{>0} \backslash \mathbf{A}_f^* \simeq \widehat{\mathbf{Z}}^*$; thus, if we normalise the Artin reciprocity map

$$\mathrm{Art} : \mathbf{Q}^*_{>0} \backslash \mathbf{A}_f^* \to \mathrm{Gal}_{\mathbf{Q}}^{\mathrm{ab}},$$

such that $\mathrm{Art}(x)$, for $x \in \widehat{\mathbf{Z}}^* \subset \mathbf{A}_f^*$, acts on roots of unity by $\zeta \mapsto \zeta^x$, we have the following (cf. [LSZ17, Proposition 5.4.2]).

Proposition 2.1.10. *The right-multiplication action of $u \in \mathbf{GSp}_{2g}(\mathbf{A}_f)$ on $\pi_0(\mathrm{Sh}_{\mathbf{GSp}_{2g}})(\mathbf{C})$ coincides with the action of* $\mathrm{Art}(\nu(u)^{-1})$.

2.1.3.4 Moduli of abelian schemes for $\mathbf{GSp}_{2g}$

For any open compact subgroup U of $\mathbf{GSp}_{2g}(\widehat{\mathbf{Z}})$, we can associate the set-valued functor F_U from the category $Sch/\mathbf{Q}$ of schemes over $\mathbf{Q}$, which parametrises (isomorphism classes of) abelian schemes of relative dimension g with principal polarisation and U-level structure (whose definition is discussed in §2.1.4). Recall that if U is sufficiently small, then F_U is known to be representable by a smooth quasi-projective scheme $S_g(U)$ over $\mathbf{Q}$ (for instance, see [Lan13] Theorem 1.4.1.11)

Remark 2.1.11.

- The very existence of $S_g(U)$ gives a model of $\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U)$ over $\mathbf{Q}$; by using the *main theorem of complex multiplication of abelian varieties* ([Mil05, Theorem 11.2]), one shows that this is a canonical model, thus $S_g(U)$ is isomorphic to the previously defined $\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U)/\mathbf{Q}$.
- Every Shimura variety of PEL-type admits a similar description as moduli of abelian schemes with extra structure ([Mil05, Theorem 8.17]). For instance, in Chapter 5, we discuss properties of a unitary Shimura variety of PEL-type.

2.1.4 Integral symplectic level structures

In the following, we review the definition of level structures for any open compact subgroup U of $\mathbf{GSp}_{2g}(\widehat{\mathbf{Z}})$. We closely follow [Lan13]. We will denote these structures as symplectic level structures simply to remark that we are dealing with the symplectic group $\mathbf{GSp}_{2g}$. Consider an abelian scheme A of relative dimension g over a locally Noetherian $\mathbf{Q}$-scheme S and fix a principal polarisation λ on it; a "naive" candidate for the definition of a full level N-structure is the following.

Definition 2.1.12. A naive symplectic full level N structure on $(A,\lambda)_{/S}$ is an isomorphism

$$\alpha_N : (\mathbf{Z}/N\mathbf{Z})^{2g}_{/S} \longrightarrow A[N],$$

which respects the symplectic forms defined by J on $(\mathbf{Z}/N\mathbf{Z})^{2g}$ and the one induced by the Weil pairing e_λ and λ on $A[N]$.

Remark 2.1.13. The isomorphism α_N respects the two symplectic forms in the following sense. There exists an isomorphism $\beta_N : (\mathbf{Z}/N\mathbf{Z})_{/S} \longrightarrow \mu_{N/S}$ which makes the diagram

$$\begin{tikzcd}
(\mathbf{Z}/N\mathbf{Z})^{2g}_{/S} \times_S (\mathbf{Z}/N\mathbf{Z})^{2g}_{/S} \arrow[r, "J"] \arrow[d, "\alpha_N \times \alpha_N"'] & (\mathbf{Z}/N\mathbf{Z})_{/S} \arrow[d, "\beta_N"] \\
A[N] \times_S A[N] \arrow[r, "e_\lambda"] & \mu_{N/S}
\end{tikzcd}$$

commutative.

For each geometric point $\bar{s}$ of S, define the Tate module (at $\bar{s}$) of A to be the $\widehat{\mathbf{Z}}$-module

$$T_{\bar{s}}(A) := \varprojlim_N A[N](\bar{s}).$$

Since S is a $\mathbf{Q}$-scheme, $T_{\bar{s}}(A)$ is a free $\widehat{\mathbf{Z}}$-module of rank $2g$. There is another way to define symplectic full level N structures, at the level of Tate modules (after passing to geometric fibres of A/S), which is equivalent to the one of Definition 2.1.12, due to the following classical result.

Lemma 2.1.14. *Let S be a connected locally Noetherian scheme and fix a geometric point $\bar{s} \to S$; there is an equivalence between the category of locally constant constructible étale sheaves of* *abelian groups over S and the one of finite continuous $\pi_1(S,\bar{s})$-modules, given by sending G to its geometric fibre $G_{\bar{s}}$.*

In particular, since $A[N]$ is a locally constant constructible étale sheaf on S, $A[N](\bar{s})$ has the structure of a $\pi_1(S,\bar{s})$-module. The $\widehat{\mathbf{Z}}$-module $T_{\bar{s}}(A)$ acquires an action of $\pi_1(S,\bar{s})$ from the one of each group of N-torsion points.

Let $K(N) := K_{\mathbf{GSp}_{2g}}(N) \subset \mathbf{GSp}_{2g}(\widehat{\mathbf{Z}})$ be the kernel of reduction modulo N. In view of Lemma 2.1.14, we can translate the information given by a naive symplectic full level N structure α_N in terms of a $\pi_1(S,\bar{s})$-equivariant isomorphism at the geometric fibre, say $\alpha_{N\bar{s}}$. Note that $\alpha_{N\bar{s}}$ is the reduction modulo $K(N)$ of a symplectic isomorphism

$$\alpha_{\bar{s}} : \widehat{\mathbf{Z}}^{2g} \longrightarrow T_{\bar{s}}(A),$$

and such a lift is unique up to the action of $K(N)$. We assume that the element $h \in \mathbf{GSp}_{2g}(\widehat{\mathbf{Z}})$ acts on the isomorphism $\alpha_{\bar{s}}$ by $\alpha_{\bar{s}} \circ h$, while $\sigma \in \pi_1(S,\bar{s})$ acts on the left. Hence, we can give the following definition.

Definition 2.1.15. A symplectic full level N-structure on $(A,\lambda)_{/S}$ (at $\bar{s}$) is a $\pi_1(S,\bar{s})$-invariant $K(N)$-orbit of a symplectic isomorphism

$$\alpha_{\bar{s}} : \widehat{\mathbf{Z}}^{2g} \longrightarrow T_{\bar{s}}(A).$$

Thus, a symplectic level N-structure on $(A,\lambda)_{/S}$ is a collection of symplectic full level N-structures at each geometric point $\{\alpha_{\bar{s}}\}_{\bar{s}}$, such that if two geometric points $\bar{s},\bar{r}$ are in the same connected component, then

$$\alpha_{\bar{s}} = \alpha_{\bar{r}}.$$

In Definition 2.1.15, the $\pi_1(S,\bar{s})$-invariance of the $K(N)$-orbit of $\alpha_{\bar{s}}$ is equivalent to asking the symplectic full level N structure in the sense of Definition 2.1.12 to be defined over S and hence it is an essential ingredient to compare the two definitions, as the next proposition shows.

Proposition 2.1.16 ([Lan13] 1.3.6.5-1.3.6.6)**.** *Let $(A,\lambda)_{/S}$ be as above; a symplectic level N-structure on $(A,\lambda)_{/S}$ is equivalent to a tower*

$$(t_M : S_M \longrightarrow S)_{N|M}$$

of finite étale surjective maps such that:

1. *$S_N = S$ and for any $N|M|L$ there are finite étale surjective maps*

$$g_{L,M} : S_L \longrightarrow S_M$$

   such that $t_L = t_M \circ g_{L,M}$;

2. *Over each S_M, we have a symplectic isomorphism*

$$\alpha_{/S_M} : (\mathbf{Z}/M\mathbf{Z})^{2g}_{/S_M} \longrightarrow A[M]_{/S_M},$$

such that, if $N|L|M$, the pullback of $\alpha_{/S_L}$ under $g_{M,L}$ is the reduction modulo L of $\alpha_{/S_M}$.

The proposition above suggests an alternative (and more convenient to us) way to define level structures for general open compact subgroups of $\mathbf{GSp}_{2g}(\widehat{\mathbf{Z}})$.

Definition 2.1.17. Let U be an open compact subgroup of $\mathbf{GSp}_{2g}(\widehat{\mathbf{Z}})$ and for any integer M such that $K(M) \subset U$ denote by U_M the quotient $U/K(M)$. Then, a symplectic level U-structure of $(A,\lambda)_{/S}$ is a collection $\{\alpha_{U_M}\}_M$, where M varies among the integers such that $K(M) \subset U$, of elements α_{U_M} such that

1. α_{U_M} is a locally étale defined U_M-orbit of a naive symplectic full level M-structure;
2. If $L|M$, α_{U_L} corresponds to the reduction modulo L of α_{U_M}.

We finally note that passing to geometric fibres, a symplectic level U-structure gives a $\pi_1(S,\bar{s})$-invariant U-orbit of a symplectic isomorphism

$$\alpha : \widehat{\mathbf{Z}}^{2g} \longrightarrow T_{\bar{s}}(A),$$

at each geometric point.

2.1.5 Tower of symplectic level structures at p

The main result of Chapter 3 involves the computation of distribution relations for Eisenstein classes attached to moduli of abelian schemes with certain level structures at a prime p. Here, we define and study these level structures in order to prepare the territory for proving Lemma 3.4.3.

For representability issues, we work with open compact subgroups U which decompose as $U_p \cdot U^{(p)} \subset \mathbf{GSp}_{2g}(\widehat{\mathbf{Z}})$, where $U_p \subset \mathbf{GSp}_{2g}(\mathbf{Z}_p)$ and $U^{(p)} \subset \mathbf{GSp}_{2g}(\widehat{\mathbf{Z}}^{(p)})$. We suppose that $U = U^{(p)}\mathbf{GSp}_{2g}(\mathbf{Z}_p)$ is sufficiently small, thus $\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U)/\mathbf{Q}$ is a moduli which parametrises (iso. classes of) p.p. abelian schemes of rel. dim. g with U-level structure. Let $\mathscr{A} = \mathscr{A}_g(U)$ denote its universal abelian scheme and consider the following functor $G_1(p^m) : Sch_{/\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U)} \longrightarrow Sets$, defined by

$$S/\,\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U) \mapsto \{\text{points of exact order } p^m \text{ of } \mathscr{A} \times_{\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U)} S/S\}.$$

Remark 2.1.18. Since we are working in characteristic zero, by point of exact order p^m of A/S, we simply mean a section $S \to A$ whose pull-back to each geometric fibre is a point of exact order p^m.

Lemma 2.1.19. *The functor $G_1(p^m)/\,\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U)$ is representable by a finite étale $\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U)$-scheme.*

Proof. The question boils down to show that $G_1(p^m)$ is an étale sheaf on $\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U)$. Indeed, if this is the case, $G_1(p^m)$ is a lcc (i.e. locally constant constructible) sub-sheaf of the lcc étale sheaf $\mathscr{A}[p^m]$ (the natural transformation between the two sheaves is given by the obvious inclusion). Hence, $G_1(p^m)$ is representable by a finite étale $\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U)$-scheme $\mathscr{G}_1(p^m)/\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U)$, by the characterisation of étale lcc sheaves as the ones representable by finite étale schemes. The fact that $G_1(p^m)$ is an étale sheaf follows by observing that, since we are working in characteristic zero, points of exact order p^m of $\mathscr{A}$ can't have "many exact orders". Indeed, $G_1(p^m)$ is an étale sheaf if for any étale $S \to \mathrm{Sh}_{\mathbf{GSp}_{2g}}(U)$ and any étale covering $\{S_i\}_{i\in I}$ of S, the diagram

$$G_1(p^m)(S) \longrightarrow \prod_i G_1(p^m)(S_i) \rightrightarrows \prod_{(i,j)} G_1(p^m)(S_i \times_S S_j)$$

is exact, i.e. the left map is an injection onto the set of I-tuples $(c_i) \in \prod_i G_1(p^m)(S_i)$ such that

$$c_{i|_{S_i\times_S S_j}} = c_{j|_{S_i\times_S S_j}},$$

for all $i, j \in I$. Note that such a I-tuple comes from an element $c \in \mathscr{A}_S[p^m](S)$ such that $c_{|_{S_i}}$ is a point of exact order p^m of $\mathscr{A}_{S_i}$. Therefore, since $\{S_i\}$ is an étale covering of S, all geometric points of s factor through one of the S_i so that c must have exact order p^m, i.e. $c \in G_1(p^m)(S)$. □

We now compare $G_1(p^m)$ with the sheaf induced by the following open compact subgroups of $\mathbf{GSp}_{2g}(\mathbf{Z}_p)$.

Definition 2.1.20. For any integer $m \geq 1$, define the subgroup $U_1(p^m) \subset \mathbf{GSp}_{2g}(\mathbf{Z}_p)$ as follows:

$$U_1(p^m) \quad := \quad \{M \in \mathbf{GSp}_{2g}(\mathbf{Z}_p) | R_{2g}(M) \equiv (0,\cdots,0,1) \bmod p^m\} \tag{2.2}$$

where $R_i(M)$ denotes the i-th row of M. For any integer N, then $U_1(N) \subset \mathbf{GSp}_{2g}(\widehat{\mathbf{Z}})$ is defined to be the subgroup of elements $(g_p)_p$ such that $g_p \in U_1(p^{v_p(N)})$.

Remark 2.1.21. Recall that the Klingen parabolic of $\mathbf{GSp}_{2g}$ is the parabolic associated to the flag variety of lines in the symplectic vector space defining $\mathbf{GSp}_{2g}$. Then, $U_1(p^m)$ is the subgroup of $\mathbf{GSp}_{2g}(\mathbf{Z}_p)$ of elements whose reduction modulo p^m are in the mirabolic subgroup of the Klingen parabolic of the form

$$\begin{pmatrix} * & * & * & * & \cdots & * \\ & * & * & * & \cdots & * \\ & * & * & * & \cdots & * \\ & * & * & * & \cdots & * \\ & * & * & * & \cdots & * \\ & & & & & 1 \end{pmatrix}.$$

Lemma 2.1.22. *Let A/S be an abelian scheme of relative dimension g over a $\mathbf{Q}$-scheme S, with a fixed principal polarisation on it. Then, there is a bijection between points of exact order p^m of A and symplectic level $U_1(p^m)$-structures.*

Proof. Denote by U_{p^m} the image of $U_1(p^m)$ under reduction mod p^m. Since A is of finite presentation over S, we can reduce to work over a locally Noetherian base S ([FC13, I.1.2(a)]); thus, since now S is the disjoint union of its connected components and étale sheaves send co-products into products, it is sufficient to work over a connected locally Noetherian S. Finally, by replacing S by an étale finite surjective cover of it if necessary, we can assume

$$A[p^m] \simeq (\mathbf{Z}/p^m\mathbf{Z})^{2g}_{/S}.$$

A point $t \in A(S)$ of exact order p^m lifts to a symplectic "naive" level p^m-structure and such a lift is unique up to action of U_{p^m}. Indeed, t defines a monomorphism over S

$$t : (\mathbf{Z}/p^m\mathbf{Z})_{/S} \hookrightarrow A[p^m],$$

and it can be completed to a full isomorphism

$$(\mathbf{Z}/p^m\mathbf{Z})^{2g}_{/S} \longrightarrow A[p^m]$$

uniquely up to the action of U_{p^m}.

After passing to a suitable étale cover of S, we can lift t to a point of exact order p^{m+1} of A, which is mapped to t under (the abstract group homomorphism) reduction modulo p^m. Repeating this procedure for any $l \geq m$ uniquely defines a level $U_1(p^m)$-structure of A. The converse is proved similarly. □

This result directly implies the following.

Corollary 2.1.23. *The scheme $\mathscr{G}_1(p^m)$ which represents $G_1(p^m)$ is isomorphic to $\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U^{(p)}U_1(p^m))$ as a covering of $\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U^{(p)}\mathbf{GSp}_{2g}(\mathbf{Z}_p))$.*

This generalises a well-known result for modular curves (e.g. [KLZ17, Theorem 4.3.3]), which plays an important role in the study of the pushforward relations of Eisenstein classes for $\mathbf{GL}_2$. As in *loc. cit.*, we use Corollary 2.1.23 to prove push-forward relations of the Eisenstein classes for $\mathbf{GSp}_{2g}$ in Proposition 3.4.6.

Remark 2.1.24. As a consequence of the Chinese Remainder Theorem, if $N = \prod_{i=1}^r p_i^{e_i}$, $\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U^{(N)}U_1(N))$ parametrises p.p. abelian schemes of relative dimension g with level structure $U^{(N)}$ and r different points each of exact order $p_i^{e_i}$.

2.1.5.1 Integral models

In the following, we recall the existence of integral models for the symplectic Shimura variety $\mathrm{Sh}_{\mathbf{GSp}_{2g}}$ of level $U^{(p)}U_1(p^r)$. We refer to [Moo98, Section 3] or [Hid04, Section 6.4.1] for further details.

By [MFK94, Theorem 7.10], there exists a (fine) moduli space over $\mathbf{Z}[\frac{1}{dp}]$, for an auxiliary integer $d \geq 3$ coprime to p, of isomorphism classes of principally polarised abelian schemes with symplectic level $U(p^r) := U^{(p)} K_{\mathbf{GSp}_{2g}}(p^r)$-structure, which we (again) denote by $\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U(p^r))_{/\mathbf{Z}[\frac{1}{dp}]}$. Let $\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U^{(p)}U_1(p^r))_{/\mathbf{Z}[\frac{1}{dp}]}$ be its quotient by $U_1(p^r)/K_{\mathbf{GSp}_{2g}}(p^r)$.

Similarly to the previous subsection, let $\mathscr{G}_1(p^r)$ be the finite étale sheaf over $\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U^{(p)})_{/\mathbf{Z}[\frac{1}{dp}]}$ associated to points of exact order p^r of the universal abelian scheme of $\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U^{(p)})_{/\mathbf{Z}[\frac{1}{dp}]}$. Then, Corollary 2.1.23 is still true in this setting.

Lemma 2.1.25. *The scheme $\mathscr{G}_1(p^r)$ is isomorphic to $\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U^{(p)}U_1(p^r))_{/\mathbf{Z}[\frac{1}{dp}]}$ as a covering of $\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U^{(p)})_{/\mathbf{Z}[\frac{1}{dp}]}$.*

Proof. Since p is invertible in $\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U^{(p)})_{/\mathbf{Z}[\frac{1}{dp}]}$, the proof is identical to the one of Corollary 2.1.23. □

2.2 Cohomology theories

In this text, we will often work with motivic and étale cohomology groups with non-trivial coefficients of Shimura varieties. In what follows, we quickly list some of the definitions and properties, which we make use of in the upcoming chapters.

2.2.1 Continuous étale cohomology

In this section, we recall the definition of continuous étale cohomology for schemes over a general base, introduced by Jannsen in [Jan88].

Definition 2.2.1. For an inverse system $(\mathscr{F}_n)$ of constructible étale $\mathbf{Z}/p^n$-sheaves over a scheme X, define $H^i_{\text{ét}}(X, (\mathscr{F}_n))$ to be the i-th derived functor of $(\mathscr{F}_n) \mapsto \varprojlim_n H^0_{\text{ét}}(X, \mathscr{F}_n)$. In particular, for p invertible on X and an integer j, we define

$$H^i_{\text{ét}}(X, \mathbf{Z}_p(j)) := H^i_{\text{ét}}(X, (\mathbf{Z}/p^n\mathbf{Z}(j))).$$

Note that if $H^{i-1}_{\text{ét}}(X, \mathscr{F}_n)$ is finite for all n, then

$$H^i_{\text{ét}}(X, (\mathscr{F}_n)) \simeq \varprojlim_n H^i_{\text{ét}}(X, \mathscr{F}_n).$$

Remark 2.2.2. For instance, this last condition is satisfied whenever X is a scheme over S, where the base S is an algebraically closed field or it is a scheme of finite type over $\mathbf{Z}$.

Finally, for $\mathscr{F}_{\mathbf{Z}_p} = (\mathscr{F}_n)_n$ as in Definition 2.2.1, we denote $\mathscr{F}_{\mathbf{Z}_p} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ by $\mathscr{F}_{\mathbf{Q}_p}$ and define

$$H^i_{\text{ét}}(X, \mathscr{F}_{\mathbf{Q}_p}) := H^i_{\text{ét}}(X, \mathscr{F}) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p.$$

2.2.2 Étale coefficient sheaves on Shimura varieties

Let (G, X_G) be a Shimura datum, where G is a reductive group G over $\mathbf{Q}$, and denote by $\mathrm{Rep}_{\mathbf{Q}}(G)$ the category of representations of G over $\mathbf{Q}$.

For any prime p, to $V \in \mathrm{Ob}(\mathrm{Rep}_{\mathbf{Q}}(G))$ we can associate a p-adic étale sheaf $\mathscr{V}_{\mathbf{Z}_p}$ (cf. [Pin90] or [LLZ16]) on $\mathrm{Sh}_G(U)/E$, where E denotes the reflex field of (G, X_G) and U is any sufficiently small level subgroup of $G(\mathbf{A}_f)$. The construction is motivated by the fact that if S is a finite set with a continuous left action of U, we can define a finite étale covering $\mathscr{S}$ of $\mathrm{Sh}_G(U)$ by taking any open normal subgroup V of U which acts trivially on S and defining $\mathscr{S}$ to be

$$(U/V)\backslash(\mathrm{Sh}_G(V) \times S)$$

w.r.t. the left action given by $h \cdot (x, s) = (xh^{-1}, hs)$. This construction extends to representations of U on finite-rank $\mathbf{Z}_p$-modules (cf. [LSZ17, §6.1]) and is functorial in the sense that if we have an injection $\iota : (H, X_H) \hookrightarrow (G, X_G)$ of Shimura data, the pull-back on étale sheaves over the associated Shimura varieties corresponds to restriction of algebraic representation from G to H (cf. loc.cit.).

2.2.3 Relative Chow motives and Ancona's functor

Let us briefly recall the main properties of the functor defined in [Anc15]. For a reductive group G over $\mathbf{Q}$, recall we have denoted by $\mathrm{Rep}_{\mathbf{Q}}(G)$ the category of representations of G over $\mathbf{Q}$.

For a smooth quasi-projective scheme S over a field of characteristic zero, let $\mathrm{CHM}_{\mathbf{Q}}(S)$ denote the $\mathbf{Q}$-linear tensor pseudo-abelian category of relative Chow motives over S. Recall that there is a functor M from the category of smooth projective schemes over S to $\mathrm{CHM}_{\mathbf{Q}}(S)$; let $\mathbb{1}_S := M(S)$, and denote by $\mathbb{L}_S$ the Lefschetz motive appearing in the decomposition of $M(\mathbb{P}^1_S)$ as $\mathbb{1}_S \oplus \mathbb{L}_S$. For any positive integer m and $\mathscr{V} \in \mathrm{Ob}(\mathrm{CHM}_{\mathbf{Q}}(S))$, we denote by $\mathscr{V}(-m)$ and $\mathscr{V}(m)$ the tensor products of $\mathscr{V}$ with $\mathbb{L}_S^{\otimes m}$ and $(\mathbb{L}_S^{\vee})^{\otimes m}$. In order to define Ancona's functor, recall the following.

Proposition 2.2.3 ([DM91])**.** *Let $\pi : A \to S$ be an abelian scheme of relative dimension g; there exists a decomposition in* $\mathrm{CHM}_{\mathbf{Q}}(S)$

$$M(A) = \bigoplus_{i=0}^{2g} h^i(A),$$

where $[n]^*$ acts on $h^i(A)$ as multiplication by n^i and the ℓ-adic realisation of $h^i(A)$ is $R^i\pi_*\mathbf{Q}_\ell$.

Now, consider a Shimura datum (G, X) of PEL-type. For any sufficiently small level subgroup $U \subseteq G(\mathbf{A}_f)$ there is a Shimura variety $\mathrm{Sh}_G(U)$, which admits a model over the reflex field of (G, X), and a universal abelian scheme $\mathscr{A}/\mathrm{Sh}_G(U)$ with PEL structure.

Proposition 2.2.4 ([Anc15])**.** *There is a tensor functor*

$$\mu_U^G : \mathrm{Rep}_{\mathbf{Q}}(G) \longrightarrow \mathrm{CHM}_{\mathbf{Q}}(\mathrm{Sh}_G(U)),$$

which respect duals and satisfies the following:

1. *If V is the standard representation of G, then $\mu_U^G(V) = h^1(\mathscr{A})$;*

2. *If $\nu : G \to \mathbf{G}_{\mathrm{m}}$ is the multiplier, then $\mu_U^G(\nu) = \mathbb{L}_{\mathrm{Sh}_G(U)}$;*

3. *for any prime p, the p-adic étale realisation of $\mu_U^G(V)$ is the étale sheaf associated to $V \otimes \mathbf{Q}_p$, with U acting on the left via $U \hookrightarrow G(\mathbf{A}_f) \to G(\mathbf{Q}_p)$.*

Remark 2.2.5. We have adopted conventions used in [LSZ17]. This is coherent with the fact that, in the case of $\mathbf{GL}_2$, the p-adic Tate module $T_p\mathscr{E}$ of the universal elliptic curve $\mathscr{E}$ corresponds to the dual of the standard representation of $\mathbf{GL}_2(\mathbf{Z}_p)$. Thus, $T_p\mathscr{E}$ gives a lattice in the p-adic étale realisation of $h^1(\mathscr{E})^\vee$.

As explained in [LSZ17], there is a canonical $G(\mathbf{A}_f)$-equivariant structure on $\mu_U^G(V)$ for every V in $\mathrm{Rep}_{\mathbf{Q}}(G)$, which is compatible with the $G(\mathbf{A}_f)$-equivariant structure on the corresponding p-adic étale realisations. Thus, we have a functor

$$\mu^G : \mathrm{Rep}_{\mathbf{Q}}(G) \longrightarrow \mathrm{CHM}_{\mathbf{Q}}(\mathrm{Sh}_G)^{G(\mathbf{A}_f)},$$

where $\mathrm{Sh}_G = \varprojlim_U \mathrm{Sh}_G(U)$. Now, let $\iota : H \subset G$ induce a morphism of Shimura data of PEL type. Then, Ancona's functor satisfies the following:

Proposition 2.2.6. *There is a commutative diagram of functors*

$$\begin{tikzcd}
\mathrm{Rep}_{\mathbf{Q}}(G) \arrow[r, "\mu^G"] \arrow[d, "\bullet|_{\mathbf{H}}"'] & \mathrm{CHM}_{\mathbf{Q}}(\mathrm{Sh}_G)^{G(\mathbf{A}_f)} \arrow[d, "\iota^*"] \\
\mathrm{Rep}_{\mathbf{Q}}(H) \arrow[r, "\mu^H"] & \mathrm{CHM}_{\mathbf{Q}}(\mathrm{Sh}_H)^{H(\mathbf{A}_f)},
\end{tikzcd}$$

where ι^* denotes pull-back.

Proof. This is stated in [LSZ17, Proposition 6.2.5] and a proof will appear in forthcoming work of Alex Torzewski.

□

2.2.4 Motivic cohomology

We now recall the definition of the motivic cohomology group associated to a scheme X. We restrict to the case where X is a smooth quasi-projective scheme over a characteristic zero field. Let $\mathrm{DM}_{B,c}(X)$ be the triangulated category of constructible Beilinson motives over X with $\mathbf{Q}$-coefficients as defined in [CD12, Definition 15.1.1]; then, one can define the following.

Definition 2.2.7.

$$H^\bullet_{\mathrm{mot}}(X, \mathbf{Q}(\star)) := \mathrm{Hom}_{\mathrm{DM}_{B,c}(X)}(\mathbb{1}_X, \mathbb{1}_X(\star)[\bullet]).$$

This is compatible with the definition of motivic cohomology by using Quillen's K-groups (see [Qui73, §7]). Recall that Quillen's K-groups admit a γ-filtration (cf. [Wei13, IV.5])

Proposition 2.2.8 ([CD12], Corollary 14.2.14)**.** *We have*

$$H^{\bullet}_{\mathrm{mot}}(X,\mathbf{Q}(\star)) \simeq \mathrm{Gr}^{\star}_{\gamma}K_{2\star-\bullet}(X)\otimes\mathbf{Q},$$

where $\mathrm{Gr}^{\star}_{\gamma}K_{2\star-\bullet}(X)$ *denotes the* $\star$*-th graded piece of the* γ*-filtration on* $K_{2\star-\bullet}(X)$.

In Chapter 4, we will work with motivic cohomology groups with coefficients given by relative Chow motives. These cohomology groups can be defined similarly to the trivial case. First, notice that an element $\mathscr{V}_{\mathbf{Q}} \in \mathrm{Ob}(\mathrm{CHM}_{\mathbf{Q}}(X))$ is a constructible Beilinson motive; indeed, we have a fully faithful embedding $\mathrm{CHM}_{\mathbf{Q}}(X) \hookrightarrow \mathrm{DM}_{B,c}(X)$ (e.g. [CD12, Corollary 16.1.6] and [CD12, 11.3.8], or [CD12, Proposition 15.2.3]). Thus, it makes sense to define

Definition 2.2.9. Let $\mathscr{V}_{\mathbf{Q}} \in \mathrm{Ob}(\mathrm{CHM}_{\mathbf{Q}}(X))$; define

$$H^{\bullet}_{\mathrm{mot}}(X,\mathscr{V}_{\mathbf{Q}}(\star)) := \mathrm{Hom}_{\mathrm{DM}_{B,c}(X)}(\mathbb{1}_X,\mathscr{V}_{\mathbf{Q}}(\star)[\bullet]).$$

2.2.5 Operations

The triangulated category of constructible Beilinson motives satisfies the Grothendieck 6 functor formalism and duality (in the sense of [CD12, §A.5]), thus, by [CD12, Theorem 7], we have the following operations in cohomology. Let $f: X \to Y$ be a morphism of schemes, with X,Y smooth, quasi-projective over a characteristic zero field, and let $\mathscr{V}_{\mathbf{Q}}$ and $\mathscr{W}_{\mathbf{Q}}$ be relative Chow motives for X and Y respectively.

- **Pullbacks:** $f^*: H^i_{\mathrm{mot}}(Y,\mathscr{V}_{\mathbf{Q}}(j)) \longrightarrow H^i_{\mathrm{mot}}(X,f^*\mathscr{V}_{\mathbf{Q}}(j))$, for any f.
- **Gysin morphisms:** $f_*: H^i_{\mathrm{mot}}(X,f^*\mathscr{W}_{\mathbf{Q}}(j)) \longrightarrow H^{i+2c}_{\mathrm{mot}}(Y,\mathscr{W}_{\mathbf{Q}}(j+c))$, for a closed immersion $f: X \to Y$ of co-dimension c.
- **Traces:**

$$\begin{aligned} f_*: H^i_{\mathrm{mot}}(X,\mathscr{V}_{\mathbf{Q}}(j)) &\xrightarrow{\varphi} H^i_{\mathrm{mot}}(X,f^*\mathscr{W}_{\mathbf{Q}}(j)) \\ &\xrightarrow{\sim} H^i_{\mathrm{mot}}(Y,f_*f^*\mathscr{W}_{\mathbf{Q}}(j)) \\ &\xrightarrow{\mathrm{Tr}_f} H^i_{\mathrm{mot}}(Y,\mathscr{W}_{\mathbf{Q}}(j)), \end{aligned}$$

  for a finite étale f and a morphism $\varphi: \mathscr{V}_{\mathbf{Q}} \to f^*\mathscr{W}_{\mathbf{Q}}$.
- **Cup-products:** $\cup: H^i_{\mathrm{mot}}(X,\mathscr{V}_{\mathbf{Q}}(j)) \times H^{i'}_{\mathrm{mot}}(X,\mathscr{V}'_{\mathbf{Q}}(j')) \to H^{i+i'}_{\mathrm{mot}}(X,\mathscr{V}_{\mathbf{Q}}\otimes\mathscr{V}'_{\mathbf{Q}}(j+j'))$.

- **Projection formula:** Cup-products and traces satisfy the following

$$f_*(a \cup f^*(b)) = f_*(a) \cup b.$$

- **Compatibility in Cartesian diagrams:** Suppose we have a Cartesian diagram

$$\begin{tikzcd} X \arrow[r, "f'"] \arrow[d, "\pi'"'] & Y \arrow[d, "\pi"] \\ X' \arrow[r, "f"] & Y', \end{tikzcd}$$

where f, f' are closed immersions and π, π' are finite étale, then

$$f^* \circ \pi_* = \pi'_* \circ f'^*$$
$$\pi^* \circ f_* = f'_* \circ \pi'^*.$$

Remark 2.2.10.

- We use the same notation for push-forwards and Gysin morphisms since, by construction, they commute with each other.

- The same (opportunely translated) properties are true in étale cohomology.

2.2.6 Gysin morphisms and branching laws

In this section, we consider two Shimura data (H, X_H) and (G, X_G) of PEL-type and the corresponding Shimura varieties Sh_H and Sh_G such that $\iota : H \hookrightarrow G$ induces a closed embedding

$$\iota : \mathrm{Sh}_H \hookrightarrow \mathrm{Sh}_G .$$

By functoriality of Ancona's functor μ (Proposition 2.2.6), we have the following.
Take an irreducible algebraic representation W of G over $\mathbf{Q}$ and consider it as an H-representation $W_{|_H}$ (using ι). The H-representation $W_{|_H}$ might not be irreducible anymore, but it admits a decomposition as sum of its H-irreducible constituents, i.e.

$$W_{|_H} = \oplus_i V_i,$$

where V_i are irreducible algebraic representations for H. Thus, we have morphisms $\varphi_i : V_i \to W_{|_H}$ in $\mathrm{Rep}_{\mathbf{Q}}(H)$. After applying Ancona's functor and Proposition 2.2.6, we get morphisms of relative Chow motives

$$\varphi_i : \mathscr{V}_{i,\mathbf{Q}} \to \iota^* \mathscr{W}_{\mathbf{Q}}.$$

Thus, by composing the corresponding map in cohomology with the Gysin morphism associated to ι_U (for U a sufficiently small open compact of $G(\mathbf{A}_f)$), we have morphisms

$$\begin{aligned} H^r_{\mathrm{mot}}(\mathrm{Sh}_H(U\cap H), \mathscr{V}_{i,\mathbf{Q}}(j)) &\xrightarrow{\varphi_i} H^r_{\mathrm{mot}}(\mathrm{Sh}_H(U\cap H), \iota^*\mathscr{W}_{\mathbf{Q}}(j)) \\ &\xrightarrow{\iota_{U,*}} H^{r+2c}_{\mathrm{mot}}(\mathrm{Sh}_G(U), \mathscr{W}_{\mathbf{Q}}(j+c)). \end{aligned}$$

Remark 2.2.11. Exactly the same story applies in p-adic étale cohomology, so that we have morphisms

$$H^r_{\text{ét}}(\mathrm{Sh}_H(U\cap H), \mathscr{V}_{i,\mathbf{Z}_p}(j)) \to H^{r+2c}_{\text{ét}}(\mathrm{Sh}_G(U), \mathscr{W}_{\mathbf{Z}_p}(j+c)).$$

This observation has been crucially used to construct push-forward classes in the cohomology with non-trivial coefficients in several circumstances ([LLZ16], [Lem17], [LSZ17] etc.).

Chapter 3

Siegel units and Eisenstein classes

The construction of motivic classes in the cohomology of Shimura varieties has played a fundamental role in understanding the arithmetic of zeta values. Siegel units and Eisenstein classes are a fascinating source of such cohomology classes, due to their connection to Eisenstein series and their distribution relations.

In this chapter, we discuss their properties and prove a distribution relation for Eisenstein classes in the cohomology of the Shimura variety for $\mathbf{GSp}_{2g}$, generalising a result known in the case of $g = 1$, which has found various applications in the theory of Euler systems. Notably, these distribution relations are used for proving the norm relations of Kato's Euler system (e.g. [Kat04], [Sch98]).

We proceed as follows.

First, we give a brief account of the properties of Siegel units in §3.1, mainly following [Kat04]; then, we discuss the construction of their higher weight and dimension analogues in motivic cohomology of general abelian schemes, as in [Kin99], [KR17], and [HK15].

In §3.3, we discuss two constructions in the étale cohomology with integral coefficients, appearing in [Fal05] and [Kin15c], and compare them.

Finally, in §3.4, we prove distribution relations for Eisenstein classes in the cohomology of the Shimura variety for $\mathbf{GSp}_{2g}$, by generalising the method adopted in [Sch98], and discuss some immediate consequences.

3.1 Siegel units

Let $\pi : E \to S$ be an elliptic curve over a scheme S of characteristic coprime to $2,3$. For any integer c, consider $\pi_c : E \smallsetminus E[c] \to S$. In [Kat04], Kato defines Siegel units as the evaluation at torsion points of certain canonical functions in $\mathscr{O}(E \smallsetminus E[c])^*$. The motivation behind this construction is of analytic nature and it relies on the more classic study of modular units and their relations to Eisenstein series and values of L-functions.

Siegel units are constructed from Cartier divisors which are invariant under norm maps. Recall we have the following.

Theorem 3.1.1 ([Kat04], Proposition 1.3)**.** *Let E be an elliptic curve over a scheme S and fix an integer c prime to 6. Then, there exists a unique element ${}_c\theta_E \in \mathscr{O}(E \smallsetminus E[c])^*$ such that:*

1. *${}_c\theta_E$ has divisor $c^2(0) - E[c]$ on E, where the zero-section (0) of E and the kernel of the multiplication-by-c morphism $E[c]$ are regarded as Cartier divisors;*

2. *for any integer a coprime to c, ${}_c\theta_E$ is compatible under the norm map $N_a : \mathscr{O}(E \smallsetminus E[ac])^* \to \mathscr{O}(E \smallsetminus E[c])^*$ associated to the pullback of the multiplication-by-a map $E \smallsetminus E[ac] \to E \smallsetminus E[c]$, i.e.*
$$N_a({}_c\theta_E) = {}_c\theta_E.$$

Proof. **Uniqueness**. Suppose that f and g are two distinct elements of $\mathscr{O}(E \smallsetminus E[c])^*$ satisfying (1) and (2), then
$$g = uf, \text{ for } u \in \mathscr{O}(S)^\times.$$

Hence, by (2), for any a coprime to c
$$uf = g = N_a(g) = N_a(uf) = N_a(u)f = u^{a^2} f.$$

This is necessary to force u to be 1. Indeed, for $a = 2,3$ (which is coprime to c by hypothesis) we have that $u^3 - 1 = 0$ and $u^8 - 1 = 0$, conditions that imply $u = 1$.

Existence. Once uniqueness is proved, we can verify the existence locally and then glue the local pieces to obtain the required unit. The proof essentially relies on Abel's isomorphism, which explains how to give the group structure to E/S ([KM85], Theorem 2.1.2). In particular, we use the isomorphism on the S-rational points
$$Pic^{(0)}(E/S) \xrightarrow{\sim} E(S),$$

which reads as
$$\frac{\{\text{invertible sheaves of degree 0 divisors on } E\}}{\{\text{pullback of ones on } S\}} \xrightarrow{\sim} E(S).$$

Fix now an integer a coprime to c; then, the image of $c^2(0) - E[c]$ under multiplication by a is $c^2(0) - E[c]$ itself. Note that for $a = 2$, this means that
$$\mathscr{L}_{c^2(0)-E[c]} \otimes \mathscr{L}_{c^2(0)-E[c]} = \mathscr{L}_{c^2(0)-E[c]} \text{ in } Pic^{(0)}(E/S),$$

where $\mathscr{L}_\bullet$ denotes the invertible sheaf associated to the divisor $\bullet$. Under Abel's isomorphism, this means that the image of $c^2(0) - E[c]$ in $E(S)$ is 0. In other words, this implies that $c^2(0) - E[c]$ is locally principal on S (note that we have that $c^2(0) - E[c]$ is locally principal on E by definition, but this is a much stronger result), so locally on S there exists $f \in \mathscr{O}(E \smallsetminus E[c])^*$, with divisor $c^2(0) - E[c]$.

Similarly as above, the divisor of $N_a(f)$ is $c^2(0) - E[c]$, hence

$$N_a(f) = u_a f, \text{ for } u_a \in \mathscr{O}(S)^*.$$

In order to get units invariant under the norm maps N_a, we simply take $g := u_2^{-3} u_3 f$. This function has the required property, since $u_a^{b^2-1} = u_b^{a^2-1}$ for a, b coprime to c (the equality comes from the fact that $N_b \circ N_a = N_a \circ N_b$):

$$\begin{aligned} N_a(g) &= u_2^{-3a^2} u_3^{a^2} u_a f \\ &= (u_2^{a^2-1})^{-3} u_3^{a^2-1} u_a (u_2^{-3} u_3 f) \\ &= u_a^{-9} u_a^8 u_a g = g. \end{aligned}$$

The local existence and uniqueness guarantee the global existence of ${}_c\theta_E$. □

Remark 3.1.2. Let E be an elliptic curve over a field K of characteristic 0. Then, the invariance under the norm map $N_a({}_c\theta_E) = {}_c\theta_E$ explicitly tells us that for any point $Q \in E(K)$ which is not $[c]$-torsion,

$${}_c\theta_E(Q) = \prod_{\substack{T \in E(\bar{K}) \\ [a]T = Q}} {}_c\theta_E(T).$$

The following result states some of the fundamental properties of these functions, which will be relevant later.

Proposition 3.1.3 ([Kat04] Proposition 1.3(2)-(4); [Sch98] Theorem 1.2.1 $(iv), (ii), (iii)$)**.** *Let d be an integer prime to 6 and let $E/S, c$ be as in Theorem 3.1.1, then we have the following properties:*

1. *In $\mathscr{O}(E \smallsetminus E[cd])^*$,*
$$({}_d\theta_E)^{c^2} ([c]^*({}_d\theta_E))^{-1} = ({}_c\theta_E)^{d^2} ([d]^*({}_c\theta_E))^{-1}.$$

2. *The functions ${}_c\theta_E$ are invariant under base-change, i.e. for any morphism $S' \to S$ and $g : E' = E \times_S S' \to E$,*
$$g^* {}_c\theta_E = {}_c\theta_{E'}.$$

3. *If $h : E \to E'$ is an isogeny between elliptic curves over S with degree prime to c, then the norm map N_h sends ${}_c\theta_E$ to ${}_c\theta_{E'}$.*

Proof. 1. Note that the divisor of $({}_d\theta_E)^{c^2} ([c]^*({}_d\theta_E))^{-1}$ is

$$c^2(d^2(0) - E[d]) - (d^2 E[c] - E[cd]) = (cd)^2(0) + E[cd] - (c^2 E[d] + d^2 E[c]).$$

Of course, $({}_c\theta_E)^{d^2} ([d]^*({}_c\theta_E))^{-1}$ has the same divisor, hence their ratio is an element $u \in$

$\mathscr{O}(S)^*$. We now conclude the proof using Theorem 3.1.1 (ii), which tells us that

$$N_a(u) = u^{a^2} = u.$$

In particular, for $a = 2,3$, we get

$$u^3 = 1, \quad u^8 = 1 \Rightarrow u = 1.$$

2. In order to prove the result, recall that points of order exact c of E are sent under base-change to points of exact order c in E' ([KM85] (1.4)). Thus, the pullback of the Cartier divisor $c^2(0) - E[c]$ is $c^2(0') - E'[c]$ and $g^*{}_c\theta_E$ satisfies the properties of Theorem 3.1.1.

3. Let us consider any integer a prime to c; then, since h is an isogeny

$$N_a(N_h({}_c\theta_E)) = N_h(N_a({}_c\theta_E)) = N_h({}_c\theta_E),$$

by Theorem 3.1.1(ii). In course of the proof of the existence of Siegel units, we showed that, for any integer a coprime to c, the image of the divisor $c^2(0) - E[c]$ under multiplication by a is $c^2(0) - E[c]$ itself. Since the degree of h is prime to c, the same argument applies in this case, i.e.

$$div(N_h({}_c\theta_E)) = c^2(0') - E'[c].$$

Hence, $N_h({}_c\theta_E)$ satisfies the qualifying properties of ${}_c\theta_{E'}$ and, by uniqueness of such a function, we have

$$N_h({}_c\theta_E) = {}_c\theta_{E'}.$$

□

Kato defines Siegel units as pullback by torsion sections of the units ${}_c\theta_{\mathscr{E}}$, associated to the universal elliptic curve $\mathscr{E}/\operatorname{Sh}_{\mathbf{GL}_2}(K(N))$.

Definition 3.1.4. Fix an integer $N \geq 3$ and an integer c coprime to $6N$ and let $(\alpha,\beta) = (\frac{a}{N}, \frac{b}{N}) \in (\frac{1}{N}\mathbf{Z}/\mathbf{Z})^2 \smallsetminus \{(0,0)\}$, for $a,b \in \mathbf{Z}$. Let $(\mathscr{E}, e_1, e_2)$ be the universal elliptic curve over the modular curve $\operatorname{Sh}_{\mathbf{GL}_2}(K(N))$ with full level N structure. We define the Siegel unit

$${}_c g_{\alpha,\beta} := \iota_{\alpha,\beta}^*({}_c\theta_{\mathscr{E}}) \in \mathscr{O}(\operatorname{Sh}_{\mathbf{GL}_2}(K(N)))^*, \quad \text{for } \iota_{\alpha,\beta} = ae_1 + be_2.$$

Moreover, consider an integer $r > 1$ such that

- $(r,6) = 1$,
- $r \equiv 1 \pmod N$,

and define the element $g_{\alpha,\beta} \in \mathscr{O}(\mathrm{Sh}_{\mathbf{GL}_2}(K(N)))^* \otimes_{\mathbf{Z}} \mathbf{Q}$ as

$$_r g_{\alpha,\beta} \otimes \frac{1}{r^2-1}.$$

Lemma 3.1.5. *Keep the same notation of Definition 3.1.4, then*

1. $g_{\alpha,\beta}$ *is independent of the choice of such an* r*;*

2. *For any integer* c *such that* $(c, 6N) = 1$*, then*

$$_c g_{\alpha,\beta} = (g_{\alpha,\beta})^{c^2} / g_{c\alpha,c\beta} \in \mathscr{O}(\mathrm{Sh}_{\mathbf{GL}_2}(K(N)))^* \otimes_{\mathbf{Z}} \mathbf{Q}.$$

Proof. 1. The proof relies on Proposition 3.1.3(1). Indeed, fix $r, s \neq \pm 1$ integers coprime to 6 and congruent to 1 modulo N. Then,

$$\begin{aligned} {}_r g_{\alpha,\beta} \otimes \frac{1}{r^2-1} &= {}_r g_{\alpha,\beta} \otimes \frac{s^2-1}{(r^2-1)(s^2-1)} = \left({}_r g_{\alpha,\beta}\right)^{s^2-1} \otimes \frac{1}{(r^2-1)(s^2-1)} \\ = \frac{\left({}_s g_{\alpha,\beta}\right)^{r^2} i^*_{\alpha,\beta}([s]^*({}_r\theta_{\mathscr{E}}))}{{}_r g_{\alpha,\beta} i^*_{\alpha,\beta}([r]^*({}_s\theta_{\mathscr{E}}))} &\otimes \frac{1}{(r^2-1)(s^2-1)} \overset{(*)}{=} \frac{\left({}_s g_{\alpha,\beta}\right)^{r^2} {}_r g_{\alpha,\beta}}{{}_r g_{\alpha,\beta} {}_s g_{\alpha,\beta}} \otimes \frac{1}{(r^2-1)(s^2-1)} \\ &= \left({}_s g_{\alpha,\beta}\right)^{r^2-1} \otimes \frac{1}{(r^2-1)(s^2-1)} = {}_s g_{\alpha,\beta} \otimes \frac{1}{s^2-1}. \end{aligned}$$

Note that we crucially use that $r, s \equiv 1 \pmod N$ for $(*)$. Indeed, since the sections e_1, e_2 are killed by N, then $[s] \circ i_{\alpha,\beta} = i_{\alpha,\beta}$ and $[r] \circ i_{\alpha,\beta} = i_{\alpha,\beta}$.

2. In a similar manner, we prove (ii). Let r be as before, then by Proposition 3.1.3(1), we have

$$\begin{aligned} \frac{(g_{\alpha,\beta})^{c^2}}{g_{c\alpha,c\beta}} &= \frac{\left({}_r g_{\alpha,\beta}\right)^{c^2}}{{}_r g_{c\alpha,c\beta}} \otimes \frac{1}{r^2-1} \\ &= \frac{\left({}_c g_{\alpha,\beta}\right)^{r^2} i^*_{\alpha,\beta}([c]^*({}_r\theta_{\mathscr{E}}))}{\left({}_c g_{\alpha,\beta}\right)\left({}_r g_{c\alpha,c\beta}\right)} \otimes \frac{1}{r^2-1} \\ &= \frac{\left({}_c g_{\alpha,\beta}\right)^{r^2}\left({}_r g_{c\alpha,c\beta}\right)}{\left({}_c g_{\alpha,\beta}\right)\left({}_r g_{c\alpha,c\beta}\right)} \otimes \frac{1}{r^2-1} \\ &= \left({}_c g_{\alpha,\beta}\right)^{r^2-1} \otimes \frac{1}{r^2-1} = {}_c g_{\alpha,\beta}. \end{aligned}$$

Implicitly, we used that $[c] \circ i_{\alpha,\beta} = i_{c\alpha,c\beta}$.

□

Before discussing further properties of Siegel units, we wish to make a remark towards the construction of motivic and étale cohomology classes associated to general abelian schemes. The idea behind Theorem 3.1.1 relies on the fact that a Cartier divisor, which by definition is locally principal, that satisfies the rigid condition given by Theorem 3.1.1(2) is globally principal. Once we work with an abelian scheme A of dimension g higher than 1, this strategy does not apply since $c^{2g}(0)-A[c]$ is not a Cartier divisor. We can state Theorem 3.1.1 in a more convenient way for our purposes.

Notation 3.1.6. The motivic residue map for E/S is the map

$$res : H^1_{\mathrm{mot}}(E \smallsetminus E[c], \mathbf{Z}(1)) \longrightarrow H^0_{\mathrm{mot}}(E[c], \mathbf{Z})^{deg=0},$$

which comes from the long exact (Gysin) sequence associated to the triple $E \smallsetminus E[c] \hookrightarrow E \hookleftarrow E[c]$, where deg denotes the map

$$deg : H^0_{\mathrm{mot}}(E[c], \mathbf{Z}) \to H^2_{\mathrm{mot}}(E, \mathbf{Z}(1)).$$

Recall that $H^1_{\mathrm{mot}}(E \smallsetminus E[c], \mathbf{Z}(1)) = \mathscr{O}(E \smallsetminus E[c])^*$ and that $H^0_{\mathrm{mot}}(E[c], \mathbf{Z})^{deg=0}$ is identified with the group of degree 0 divisors supported in $E[c]$, i.e. formal linear combinations of points in $E[c]$ with coefficients in $\mathbf{Z}$.

We can re-interpret Theorem 3.1.1 as follows.

Theorem 3.1.7. *Let E be an elliptic curve over S and let c be an integer prime to 6. The divisor $c^2(0)-E[c]$ lifts canonically under the residue map*

$$res : H^1_{mot}(E \smallsetminus E[c], \mathbf{Z}(1)) \longrightarrow H^0_{mot}(E[c], \mathbf{Z})^{deg=0}$$

to an element ${}_c\theta_E \in H^1_{mot}(E \smallsetminus E[c], \mathbf{Z}(1))$ such that, for any integer a coprime to c,

$$N_a({}_c\theta_E) = {}_c\theta_E.$$

This cohomological interpretation extends to other cohomology theories and, as we will see below, to higher dimension abelian varieties.

3.1.1 Distribution relations

In the following section, we describe some of the compatibility relations that the Siegel units satisfy. For more details, we refer to [Kat04] §2.11-13. These are used essentially in the proof of the norm relations of the Euler Systems constructed from Siegel units. In §3.4, we will give a proof of Proposition 3.1.9, following [Sch98, Lemma 2.3.1], which generalises to the case of Eisenstein classes for arbitrary symplectic Shimura varieties.

Proposition 3.1.8 ([Kat04] 1.7(2))**.** *Let $(\alpha,\beta) \in (\frac{1}{N}\mathbf{Z}/\mathbf{Z})^2 \smallsetminus \{(0,0)\}$ and let a be a non-zero integer.*

Then

$$ {}_c g_{\alpha,\beta} = \prod_{\alpha',\beta'} {}_c g_{\alpha',\beta'}, $$

$$ g_{\alpha,\beta} = \prod_{\alpha',\beta'} g_{\alpha',\beta'}, $$

where c is an integer prime to $6Na$ and α', β' run through all the elements of $\mathbf{Q}/\mathbf{Z}$ such that $a\alpha' = \alpha$ and $a\beta' = \beta$.

Proof. The distribution relations can be seen as a consequence of how the norm map attached to multiplication-by-a morphism is defined. Indeed, consider the universal elliptic curve $(\mathscr{E}, \varepsilon_1, \varepsilon_2)$ of $\mathrm{Sh}_{\mathbf{GL}_2}(K(N))$ and take any geometric point $p : Spec(F) \longrightarrow \mathrm{Sh}_{\mathbf{GL}_2}(K(N))$, for F an algebraically closed field. By universality of the triple $(\mathscr{E}, \varepsilon_1, \varepsilon_2)$, the section $i_{\alpha,\beta} : \mathrm{Sh}_{\mathbf{GL}_2}(K(N)) \to \mathscr{E}$ is base-changed to an N-torsion point $p_{\alpha,\beta}$ of the elliptic curve $E = \mathscr{E} \times_p Spec(F)$ over the field F. More precisely, if we denote by e_1 and e_2 the pull-backs by p of ε_1 and ε_2, then $p_{\alpha,\beta}$ can be written in the form $me_1 + ne_2$, for certain $m, n \in \mathbf{Z}/N\mathbf{Z}$. By Remark 3.1.2, we have

$$ {}_c\theta_E(p_{\alpha,\beta}) = \prod_{\substack{T \in E(F) \\ [a]T = p_{\alpha,\beta}}} {}_c\theta_E(T). \tag{3.1} $$

Note that T is a torsion point which is killed by aN since it is mapped to $p_{\alpha,\beta}$ by $[a]$, i.e. it can be written as

$$ m't_1 + n't_2, $$

where $t_1, t_2 \in E(F)$ form a basis of $E[aN]$, and $m', n' \in \mathbf{Z}/aN\mathbf{Z}$ such that

$$ [a](m't_1 + n't_2) = me_1 + ne_2. $$

Denote the point $m't_1 + n't_2$ by $p_{\alpha',\beta'}$, for $(\alpha', \beta') = (\frac{m'}{aN}, \frac{n'}{aN}) \in (\mathbf{Q}/\mathbf{Z})^2$. Then, (3.1) is

$$ {}_c\theta_E(p_{\alpha,\beta}) = \prod_{\substack{\alpha',\beta' \in \mathbf{Q}/\mathbf{Z} \\ a\alpha' = \alpha \\ a\beta' = \beta}} {}_c\theta_E(p_{\alpha',\beta'}), $$

which is what we are looking for. Since the formula is true for all the $\bar{\mathbf{Q}}$-points of $\mathrm{Sh}_{\mathbf{GL}_2}(K(N))$, it is true globally:

$$ {}_c g_{\alpha,\beta} = i^*_{\alpha,\beta}({}_c\theta_{\mathscr{E}}) = \prod_{\alpha',\beta'} i^*_{\alpha',\beta'}({}_c\theta_{\mathscr{E}}). $$

Similarly, one gets the formula for $g_{\alpha,\beta}$. □

We now describe the norm relations of the elements ${}_c g_{0,\frac{1}{N}} \in \mathscr{O}(\mathrm{Sh}_{\mathbf{GL}_2}(U_1(N)))^*$, where the integer c is chosen to be coprime with $6N$; recall that, for any $N \mid N'$, we have a natural projection

$$\pi_N : \mathrm{Sh}_{\mathbf{GL}_2}(U_1(N')) \to \mathrm{Sh}_{\mathbf{GL}_2}(U_1(N)), \quad (E, e_1) \mapsto (E, \frac{N'}{N} e_1).$$

The morphism π_N induces a norm map $N_{\pi_N} : \mathscr{O}(\mathrm{Sh}_{\mathbf{GL}_2}(U_1(N')))^* \to \mathscr{O}(\mathrm{Sh}_{\mathbf{GL}_2}(U_1(N)))^*$, which can be described as follows. For any $f \in \mathscr{O}(\mathrm{Sh}_{\mathbf{GL}_2}(U_1(N')))^*$,

$$N_{\pi_N}(f) = \prod_{\sigma \in S} \sigma^*(f),$$

where the finite set S consists of a system of coset representatives for $U_1(N)/U_1(N')$. We have the following.

Proposition 3.1.9 ([Kat04], [Sch98])**.** *Let p be prime number. We have*

$$N_{\pi_N}({}_c g_{0, \frac{1}{Np}}) = \begin{cases} {}_c g_{0, \frac{1}{N}} & \text{if } p \mid N; \\ {}_c g_{0, \frac{1}{N}} - d_p^*({}_c g_{0, \frac{1}{N}}) & \text{if } p \nmid N; \end{cases}$$

where $d_p \in \mathbf{GL}_2(\widehat{\mathbf{Z}})$ is any matrix congruent to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$ modulo N.

For a proof, see Proposition 3.4.6 with $g = 1$.

3.1.2 The étale realisation

Fix a prime ℓ which is invertible in S. In the following, we recall how to construct étale cohomology classes from Theorem 3.1.1. There are elements

$${}_c\theta_E^{\text{ét}} \in H^1_{\text{ét}}(E \smallsetminus E[c], \mathbf{Z}_\ell(1)),$$

which satisfy the following:

P1. For any r prime to c, ${}_c\theta_E^{\text{ét}}$ is invariant under trace maps associated to multiplication by r, i.e.

$$[r]_*({}_c\theta_E^{\text{ét}}) = {}_c\theta_E^{\text{ét}};$$

P2. They are invariant under base-change.

Remark 3.1.10. Recall that the trace map associated to multiplication by r prime to c is the composition of

$$[r]_* : H^1_{\text{ét}}(E \smallsetminus E[c], \mathbf{Z}_\ell(1)) \longrightarrow H^1_{\text{ét}}(E \smallsetminus E[rc], \mathbf{Z}_\ell(1)) \longrightarrow H^1_{\text{ét}}(E \smallsetminus E[c], \mathbf{Z}_\ell(1)),$$

where the first map is just restriction to $E \smallsetminus E[rc]$ and the second is the trace map associated to $[r]$.

The étale cohomology classes associated to elliptic curves are the étale realisation of the units of Theorem 3.1.1: using the connecting homomorphism

$$\partial_{\ell^t} : \mathscr{O}(E \smallsetminus E[c])^* \to H^1_{\text{ét}}(E \smallsetminus E[c], \mathbf{Z}/\ell^t\mathbf{Z}(1))$$

of the Kummer exact sequence of étale sheaves

$$0 \longrightarrow \mu_{\ell^t} \longrightarrow \mathbb{G}_m \xrightarrow{(\cdot)^{\ell^t}} \mathbb{G}_m \longrightarrow 0$$

we get the element $\partial_{\ell^t}({}_c\theta_E) \in H^1_{\text{ét}}(E \smallsetminus E[c], \mathbf{Z}/\ell^t\mathbf{Z}(1))$.

Proposition 3.1.11. *The class* ${}_c\theta_E^{ét} := \varprojlim_t \partial_{\ell^t}({}_c\theta_E) \in H^1_{ét}(E \smallsetminus E[c], \mathbf{Z}_\ell(1))$ *satisfies the properties* $P1, P2$ *listed above.*

Proof. It is enough to show the statement at finite levels. The units ${}_c\theta_E$ are invariant under norm maps N_r associated to multiplication by r, for r coprime to c. Hence, property $P1$ follows from the commutativity of the diagram

$$\begin{tikzcd}
\mathscr{O}(E \smallsetminus E[c])^* \arrow[r, "\partial_{\ell^t}"] \arrow[d, "\tilde{N}_r"'] & H^1_{\text{ét}}(E \smallsetminus E[c], \mathbf{Z}/\ell^t\mathbf{Z}(1)) \arrow[d, "{[r]_*}"] \\
\mathscr{O}(E \smallsetminus E[c])^* \arrow[r, "\partial_{\ell^t}"] & H^1_{\text{ét}}(E \smallsetminus E[c], \mathbf{Z}/\ell^t\mathbf{Z}(1)),
\end{tikzcd}$$

where $\tilde{N}_r$ is the obtained by composing N_r on the right by the restriction map $\mathscr{O}(E \smallsetminus E[c])^* \to \mathscr{O}(E \smallsetminus E[rc])^*$. Now, let S'/S be any S-scheme and consider $g : E' = E \times_S S' \to E$. Property $P2$ is a direct consequence of Proposition 3.1.3(2) and the commutative diagram

$$\begin{tikzcd}
\mathscr{O}(E \smallsetminus E[c])^* \arrow[r, "\partial_{\ell^t}"] \arrow[d, "g^*"'] & H^1_{\text{ét}}(E \smallsetminus E[c], \mathbf{Z}/\ell^t\mathbf{Z}(1)) \arrow[d, "g^*"] \\
\mathscr{O}(E' \smallsetminus E'[c])^* \arrow[r, "\partial_{\ell^t}"] & H^1_{\text{ét}}(E' \smallsetminus E'[c], \mathbf{Z}/\ell^t\mathbf{Z}(1)).
\end{tikzcd}$$

□

The étale Eisenstein classes are obtained as pull-back under torsion sections of these étale cohomology classes associated to the universal elliptic curve of $\mathrm{Sh}_{\mathbf{GL}_2}(K(N))$.

3.1.2.1 The étale Gysin sequence

From what we discussed above, the reader might expect that the étale residue of ${}_c\theta_E^{\text{ét}}$ coincides with the étale characteristic class of $c^2(0) - E[c]$, and, indeed, this is correct and follows from the compatibility of the motivic and étale Gysin sequence under the étale regulator map. We explicit this in the case where E/K is an elliptic curve over an algebraically closed field K; in particular, we characterise ${}_c\theta_E^{\text{ét}}$ as the only class which satisfies the property $P1$ and with étale residue $r_{\text{ét}}(c^2(0) - E[c])$. Note that Theorem 3.1.1 allows us to canonically choose a class in $H^1_{\text{ét}}(E \smallsetminus E[c], \mathbf{Z}_\ell(1))$ with the desired properties; however, it is possible to do some reverse engineering and lift canonically a suitable multiple of $r_{\text{ét}}(c^2(0) - E[c])$. This is the approach taken by Faltings in the case of higher dimensional abelian varieties (see Theorem 3.3.6).

Consider the étale Gysin exact sequence for $E[c] \hookrightarrow E \hookleftarrow E \smallsetminus E[c]$. It gives

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & H^0_{\text{ét}}(E, \mathbf{Z}/\ell^t\mathbf{Z}(1)) \arrow[r, "\sim"] & H^0_{\text{ét}}(E \smallsetminus E[c], \mathbf{Z}/\ell^t\mathbf{Z}(1)) \arrow[dll, rounded corners] \\
 & 0 \arrow[r] & H^1_{\text{ét}}(E, \mathbf{Z}/\ell^t\mathbf{Z}(1)) \arrow[r, hook] & H^1_{\text{ét}}(E \smallsetminus E[c], \mathbf{Z}/\ell^t\mathbf{Z}(1)) \arrow[dlll, rounded corners] \\
H^0_{\text{ét}}(E[c], \mathbf{Z}/\ell^t\mathbf{Z}) \arrow[rr, two heads] & & H^2_{\text{ét}}(E, \mathbf{Z}/\ell^t\mathbf{Z}(1)) \arrow[r] & 0
\end{tikzcd}$$

Notation 3.1.12. Let $res_{\text{ét}}$ denote the edge map of the étale Gysin sequence (with $\mathbf{Z}_\ell$-coefficients)

$$H^1_{\text{ét}}(E \smallsetminus E[c], \mathbf{Z}_\ell(1)) \longrightarrow H^0_{\text{ét}}(E[c], \mathbf{Z}_\ell).$$

Lemma 3.1.13. *The étale cohomology class ${}_c\theta_E^{ét}$ is the unique class fixed by $[r]_*$, for r prime to c, such that*

$$res_{ét}({}_c\theta_E^{ét}) = r_{ét}(c^2(0) - E[c]).$$

Proof. The étale residue map $res_{\text{ét}}$ can be explicitly described as follows. Note that the quotient

$$Q_t = H^1_{\text{ét}}(E \smallsetminus E[c], \mathbf{Z}/\ell^t\mathbf{Z}(1))/H^1_{\text{ét}}(E, \mathbf{Z}/\ell^t\mathbf{Z}(1))$$

can be seen as the group of degree 0 divisors with coefficients in $\mathbf{Z}/\ell^t\mathbf{Z}$ and supported on $E[c]$. Indeed, recall that $H^1_{\text{ét}}(E, \mathbf{Z}/\ell^t\mathbf{Z}(1)) \simeq Pic^0(E)[\ell^t]$, while

$$\begin{aligned} H^1_{\text{ét}}(E \smallsetminus E[c], \mathbf{Z}/\ell^t\mathbf{Z}(1)) &\simeq \left\{ (\mathscr{L}, f) | \mathscr{L} \in Pic(E \smallsetminus E[c]), f : \mathscr{L}^{\otimes^{\ell^t}} \xrightarrow{\sim} \mathscr{O}_{E \smallsetminus E[c]} \right\} / \cong \\ &\simeq \left\{ (\mathscr{L}, D, f) | \mathscr{L} \in Pic^0(E), f : \mathscr{L}^{\otimes^{\ell^t}} \xrightarrow{\sim} \mathscr{O}(D) \right\} / \langle (\mathscr{O}(D'), \ell^t D', 1^{\otimes^{\ell^t}}) \rangle. \end{aligned}$$

where D and D' are degree 0 divisors supported on $E[c]$ (e.g. [Sta17, Tag 03RR]). Hence, to each element of Q_t we can associate a divisor with coefficients in $\mathbf{Z}/\ell^t\mathbf{Z}$ supported on the c-torsion points, and the étale residue map is described as

$$res_{\text{ét}} : H^1_{\text{ét}}(E \smallsetminus E[c], \mathbf{Z}/\ell^t\mathbf{Z}(1)) \longrightarrow Q_t \hookrightarrow H^0_{\text{ét}}(E[c], \mathbf{Z}/\ell^t\mathbf{Z}), \quad (\mathscr{L}, D, f) \mapsto D.$$

Passing to the limit, we get a similar description for the quotient with $\mathbf{Z}_\ell$-coefficients. By Theorem

3.1.1, as an element of Q_t the class $\partial_{\ell^t}({}_c\theta_E)$ is the one associated to the degree 0 divisor $c^2(0)-E[c]$ (i.e. the associated class in the kernel of $H^0_{\text{ét}}(E[c],\mathbf{Z}/\ell^t\mathbf{Z}) \to H^2_{\text{ét}}(E,\mathbf{Z}/\ell^t\mathbf{Z}(1))$. Thus, we conclude that

$$res_{\text{ét}}({}_c\theta_E^{\text{ét}}) = r_{\text{ét}}(c^2(0)-E[c]).$$

To prove uniqueness, we use the fact that the Gysin sequence is equivariant for the action of $[r]_*$. Suppose that there is $d_c \in H^1_{\text{ét}}(E\smallsetminus E[c],\mathbf{Z}_\ell(1))^{[r]_*=1}$, for an integer r prime to c, such that $res_{\text{ét}}(d_c) = r_{\text{ét}}(c^2(0)-E[c])$; then, the difference ${}_c\theta_E^{\text{ét}} - d_c$ is fixed by $[r]_*$ and lies in $H^1_{\text{ét}}(E,\mathbf{Z}_\ell(1))$. Since $[r]_*$ acts as multiplication by r on $H^1_{\text{ét}}(E,\mathbf{Z}_\ell(1))$, we conclude that ${}_c\theta_E^{\text{ét}} - d_c = 0$. □

Remark 3.1.14. As briefly mentioned above, the explicit description in Lemma 3.1.13 of the étale residue map

$$res_{\text{ét}} : H^1_{\text{ét}}(E\smallsetminus E[c],\mathbf{Z}_\ell(1)) \longrightarrow H^0_{\text{ét}}(E[c],\mathbf{Z}_\ell)$$

builds a direct analogy with the motivic residue map of Notation 3.1.6. Indeed, we have the commutative diagram

$$\begin{tikzcd}
H^1_{mot}(E\smallsetminus E[c],\mathbf{Z}(1)) \arrow[r, "res"] \arrow[d, "\varprojlim_t \partial_{\ell^t}"'] & H^0_{mot}(E[c],\mathbf{Z})^{deg=0} \arrow[d, "r_{\text{ét}}"] \\
H^1_{\text{ét}}(E\smallsetminus E[c],\mathbf{Z}_\ell(1)) \arrow[r, "res_{\text{ét}}"] & H^0_{\text{ét}}(E[c],\mathbf{Z}_\ell)^{deg=0},
\end{tikzcd}$$

where $H^0_{\text{ét}}(E[c],\mathbf{Z}_\ell)^{deg=0} := \operatorname{Ker}\left(H^0_{\text{ét}}(E[c],\mathbf{Z}_\ell) \to H^2_{\text{ét}}(E,\mathbf{Z}_\ell(1))\right)$, and $r_{\text{ét}}(D)$ is the étale characteristic class of D, which is defined by sending the divisor D to itself (now seen as a formal combination of points in $E[c]$ with coefficients in $\mathbf{Z}_\ell$).

3.1.3 Relation to Eisenstein series

Here, we follow [Kat04, §3]. We explain how to relate the value of the logarithmic derivative of Siegel units to certain Eisenstein series. This should serve as a motivation for the definition of Beilinson's Eisenstein Symbol.

3.1.3.1 Analytic formulae of Siegel units

Let $E = \mathbf{C}/(\mathbf{Z}\tau+\mathbf{Z})$ for $\tau \in \mathfrak{h}$.

Proposition 3.1.15 ([Kat04], Proposition 1.3(3), [Sch98] Theorem 1.2.1 (v))**.** *Let $z \in \mathbf{C} \smallsetminus c^{-1}(\mathbf{Z}\tau+\mathbf{Z})$, $q = e^{2\pi i\tau}$, and $t = e^{2\pi i z}$. Then, the value of ${}_c\theta_\tau$ at z is given by*

$${}_c\theta_\tau(z) = q^{\frac{c^2-1}{12}}(-t)^{\frac{c-c^2}{2}} \prod_{n\geq 0} \frac{(1-q^n t)^{c^2}}{1-q^n t^c} \prod_{n\geq 1} \frac{(1-q^n t^{-1})^{c^2}}{1-q^n t^{-c}}.$$

Proof. The proof consists of checking that our candidate has the required properties described in Theorem 3.1.1, which determine uniquely ${}_c\theta_\tau$. □

Proposition 3.1.15 is used to give analytic formulae for Siegel units $g_{\alpha,\beta}$. Let $\Gamma(N) \subset SL_2(\mathbf{Z})$

be the kernel of reduction modulo N; consider the connected component of $\mathrm{Sh}_{\mathbf{GL}_2}(K(N))$

$$f : \Gamma(N)\backslash\mathfrak{h} \longrightarrow \mathrm{Sh}_{\mathbf{GL}_2}(K(N))(\mathbf{C}),$$

which is given, at the level of moduli, by $\tau \mapsto (\mathbf{C}/(\mathbf{Z}\tau+\mathbf{Z}), \tau/N, 1/N)$; moreover, the action of $SL_2(\mathbf{Z})$ on $\mathfrak{h}$ and the one of $\mathbf{GL}_2(\mathbf{Z}/N\mathbf{Z})$ on $\mathrm{Sh}_{\mathbf{GL}_2}(K(N))(\mathbf{C})$ are compatible under f. We would like to consider Siegel units as holomorphic functions on the upper half-plane $\mathfrak{h}$. In order to do it, we need to fix a N^{th}-root of unity ζ_N and, consequently, the q-expansion of Siegel units has coefficients in $\mathbf{Q}(\zeta_N)$ and not $\mathbf{Q}$. First, note that the pull-back of ${}_cg_{\alpha,\beta}$ under f is ${}_c\theta_{E_\tau}(\alpha\tau+\beta \bmod \mathbf{Z}\tau+\mathbf{Z})$, where $\tau \in \mathfrak{h}$ and $E_\tau = \mathbf{C}/(\mathbf{Z}\tau+\mathbf{Z})$. Using the formula of Proposition 3.1.15, we have the following.

Proposition 3.1.16 ([Kat04] 1.9)**.** *Let $q := e^{2\pi i\tau}$ and $\zeta_N := e^{\frac{2\pi i}{N}}$, then*

$$g_{\alpha,\beta} = q^{\frac{1}{12}-\frac{a}{2N}+\frac{a^2}{2N^2}} \prod_{n\geq 0}(1-q^n q^{\frac{a}{N}}\zeta_N^b)\prod_{n\geq 1}(1-q^n q^{-\frac{a}{N}}\zeta_N^{-b}),$$

where $(\alpha,\beta) = (\frac{a}{N},\frac{b}{N}) \in \left(\frac{1}{N}\mathbf{Z}/\mathbf{Z}\right)^2 \smallsetminus \{0,0\}$.

3.1.3.2 Eisenstein series

In the following two subsections, we give a brief overview of the realisation of Siegel units in de Rham cohomology, under the image of the Chern character dlog. Following Section 3 of [Kat04], one shows that these are Eisenstein series, which can be seen as an additive avatar of the Eisenstein symbols. This section serves as a motivation for the construction of higher weight Eisenstein classes in motivic and étale cohomology.

Denote $Y = \mathrm{Sh}_{\mathbf{GL}_2}(K(N))$ and let $\Omega^1_{\mathscr{E}/Y}$ be the sheaf of relative differentials for the universal elliptic curve $\mathscr{E}/Y$. We can construct modular forms of weight one and two as follows.

1. Consider the logarithmic derivative of ${}_c\theta_{\mathscr{E}}$,

$$\mathrm{dlog}_{\mathscr{E}/Y}({}_c\theta_{\mathscr{E}}) \in \Gamma(\mathscr{E} \smallsetminus \mathscr{E}[c], \Omega^1_{\mathscr{E}/Y}),$$

   and pull it back by $i_{\alpha,\beta}$; we obtain a weight 1 modular form

$$i^*_{\alpha,\beta}(\mathrm{dlog}_{\mathscr{E}/Y}({}_c\theta_{\mathscr{E}})) \in \Gamma(Y, i^*_{\alpha,\beta}\Omega^1_{\mathscr{E}/Y}) = \Gamma(Y, \omega_{\mathscr{E}/Y}),$$

   where the last equality follows from the fact that, since $\Omega^1_{\mathscr{E}/Y}$ is free on the fibres of $\pi : \mathscr{E} \to Y$,

$$\omega_{\mathscr{E}/Y} := \pi_*\Omega^1_{\mathscr{E}/Y} = 0^*\Omega^1_{\mathscr{E}/Y}$$

   is isomorphic to $x^*\Omega^1_{\mathscr{E}/Y}$, for any section $x \in \mathscr{E}(Y)$.

2. We can take the logarithmic derivative

$$\mathrm{dlog}_{Y/\mathbf{Q}}({}_c g_{\alpha,\beta}) \in \Gamma(Y, \Omega^1_{Y/\mathbf{Q}}).$$

Note that the Kodaira-Spencer map, i.e. the $\mathscr{O}_Y$-linear morphism

$$KS : \omega^{\otimes 2}_{\mathscr{E}/Y} \longrightarrow \Omega^1_{Y/\mathbf{Q}}$$

is an isomorphism (see [Sch98] 1.1), hence

$$\mathrm{dlog}_{Y/\mathbf{Q}}({}_c g_{\alpha,\beta}) \in \Gamma(Y, \omega^{\otimes 2}_{\mathscr{E}/Y})$$

gives a weight 2 modular form.

Kato explicitly describes these modular forms, using the analytic formulae for ${}_c\theta_{\mathscr{E}}$ and ${}_c g_{\alpha,\beta}$.

Definition 3.1.17. Fix an integer $k \geq 1$; define the function $E^{(k)}$ on $\mathfrak{h} \times \mathbf{C}$ by

$$E^{(k)}(\tau, z) := (-1)^k (k-1)!(2\pi i)^{-k} E(k, \tau, z, 0),$$

where

$$E(k, \tau, z, s) = \sum_{(m,n)\in\mathbf{Z}^2} \frac{1}{(z + m\tau + n)^k |z + m\tau + n|^s}.$$

Furthermore, if $(\alpha, \beta) = (\frac{a}{N}, \frac{b}{N}) \neq (0,0)$ in $(\mathbf{Q}/\mathbf{Z})^2$, define $E^{(k)}_{\alpha,\beta}$ on $\mathfrak{h}$, by

$$E^{(k)}_{\alpha,\beta}(\tau) := E^{(k)}(\tau, \frac{a}{N}\tau + \frac{b}{N}).$$

In the case of $(\alpha, \beta) = (0,0) \in (\mathbf{Q}/\mathbf{Z})^2$, define $E^{(k)}_{0,0}$ on $\mathfrak{h}$, by

$$E^{(k)}_{0,0}(\tau) := (-1)^k (k-1)!(2\pi i)^{-k} E_{0,0}(k, \tau, 0),$$

where

$$E(k, \tau, s) = \sum_{\substack{(m,n)\in\mathbf{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau + n)^k |m\tau + n|^s}.$$

In particular, note that for $k \geq 3$

$$\begin{aligned} E(k, \tau, \frac{a}{N}\tau + \frac{b}{N}, 0) &= \sum_{(m,n)\in\mathbf{Z}^2} \frac{1}{(\frac{a}{N}\tau + \frac{b}{N} + m\tau + n)^k} = N^k \sum_{(m,n)\in\mathbf{Z}^2} \frac{1}{((a+Nm)\tau + b + Nn)^k} \\ &= N^k \sum_{\substack{(m,n)\in\mathbf{Z}^2 \\ (m,n)\equiv(a,b) \mod N}} \frac{1}{(m\tau + n)^k}, \end{aligned}$$

which is a level N Eisenstein series. An account of their properties and the calculation of their q-expansion can be found in [Kob12], Chapter III, Section 3.

The Eisenstein series defined above are related to the derivatives of $\mathrm{dlog}({}_c\theta_\tau(z))$:

Proposition 3.1.18 ([Kat04] (3.8.1), [Sch98] §1.3). *Let $k \geq 1$. The $(k-1)$-th derivative of* $\mathrm{dlog}({}_c\theta_\tau(z))$ *with respect to z is*

$$(2\pi i)^k(c^2E^{(k)}(\tau,z) - c^kE^{(k)}(\tau,cz)).$$

3.1.3.3 De Rham Eisenstein Classes

We briefly introduce the algebraic avatar of the Eisenstein series defined above, following Section 3 of [Kat04]. We keep the notation adopted in §3.1.3.2. For an integer $k \geq 1$ and $(\alpha,\beta) \in (\frac{1}{N}\mathbf{Z}/\mathbf{Z})^2$, we define elements

1. $E^{(k)}_{\alpha,\beta} \in M_k(\Gamma(N))$, where $k \geq 1$, $k \neq 2$;
2. $\tilde{E}^{(2)}_{\alpha,\beta} \in M_2(\Gamma(N))$;
3. $F^{(k)}_{\alpha,\beta} \in M_k(\Gamma(N))$, where $k \geq 1$ and $(\alpha,\beta) \neq (0,0)$ if $k = 2$,

where $\Gamma(N)$ is the kernel of reduction modulo N of $SL_2(\mathbf{Z})$. One has to keep in mind that their analytic descriptions are given in terms of the functions of Definition 3.1.17. First, we introduce the following operator.

Definition 3.1.19. For the integer $k \geq 1$, define the map

$$D : \omega^{\otimes k}_{\mathscr{E}/Y} \longrightarrow \omega^{\otimes(k+1)}_{\mathscr{E}/Y}$$

which locally is

$$f \otimes \omega^{\otimes k} \mapsto \frac{df}{\omega} \otimes \omega^{\otimes(k+1)},$$

where $f \in \mathscr{O}_{\mathscr{E}}$ and ω is a local basis of $\pi_*\Omega^1_{\mathscr{E}/Y}$, so that $\frac{df}{\omega} \in \mathscr{O}_{\mathscr{E}}$ is the function appearing in $df = \frac{df}{\omega}\cdot\omega$.

Consider the following.

Definition 3.1.20. Let $N \geq 3$ be an integer such that $N\alpha = N\beta = 0$ in $\mathbf{Q}/\mathbf{Z}$ and write $(\alpha,\beta) = (\frac{a}{N},\frac{b}{N}) \in (\frac{1}{N}\mathbf{Z}/\mathbf{Z})^2 \smallsetminus \{(0,0)\}$, for $a,b \in \mathbf{Z}$. Then, for $c > 1$ integer prime to $6N$, define

$${}_cE^{(k)}_{\alpha,\beta} := i^*_{\alpha,\beta}(D^{k-1}\,\mathrm{dlog}_{\mathscr{E}/Y}({}_c\theta_{\mathscr{E}})) \in \Gamma(Y,\omega^{\otimes k}_{\mathscr{E}/Y}).$$

These forms satisfy similar properties to Siegel units, such as distribution and norm relations

([Sch98]). Moreover, Proposition 3.1.3 translates into

$$c^2{}_dE^{(k)}_{\alpha,\beta} - c^k{}_dE^{(k)}_{\alpha,\beta} = d^2{}_cE^{(k)}_{\alpha,\beta} - d^k{}_cE^{(k)}_{\alpha,\beta},$$

for integers c,d coprime to $6N$. Hence, it makes sense to define $E^{(k)}_{\alpha,\beta}$ as follows.

Definition 3.1.21. Let $k \neq 2$, then define $E^{(k)}_{\alpha,\beta} := \frac{1}{r^2-r^k}{}_rE^{(k)}_{\alpha,\beta}$, for an integer $r > 1$ coprime to 6 such that $r \equiv 1 \pmod N$.

As in Lemma 3.1.5, for any integer c coprime to $6N$ and $k \neq 2$, we have

$${}_cE^{(k)}_{\alpha,\beta} = c^2E^{(k)}_{\alpha,\beta} - c^kE^{(k)}_{c\alpha,c\beta}.$$

Thus, in the case of $k = 2$, we can opportunely define $\tilde{E}^{(2)}_{\alpha,\beta}$, with the property that

$${}_cE^{(2)}_{\alpha,\beta} = c^2\tilde{E}^{(2)}_{\alpha,\beta} - c^2\tilde{E}^{(2)}_{c\alpha,c\beta}.$$

Finally, starting from $E^{(k)}_{\alpha,\beta}$ and $\tilde{E}^{(2)}_{\alpha,\beta}$, we can define the elements $F^{(k)}_{\alpha,\beta} \in M_k(\Gamma(N))$, as described in [Kat04] 3.6.

Definition 3.1.22. Let $(\alpha,\beta) = (\frac{a}{N},\frac{b}{N}) \in (\frac{1}{N}\mathbf{Z}/\mathbf{Z})^2$. Then, define

$$\begin{aligned} F^{(k)}_{\alpha,\beta} &= N^{-k}\sum_{x,y\in\mathbf{Z}/N\mathbf{Z}} E^{(k)}_{\frac{x}{N},\frac{y}{N}}\zeta_N^{bx-ay}, \text{ for } k \neq 2; \\ F^{(2)}_{\alpha,\beta} &= N^{-2}\sum_{x,y\in\mathbf{Z}/N\mathbf{Z}} \tilde{E}^{(2)}_{\frac{x}{N},\frac{y}{N}}\zeta_N^{bx-ay}, \text{ for } \left(\frac{a}{N},\frac{b}{N}\right) \neq (0,0). \end{aligned}$$

We now restrict to the case of $(\alpha,\beta) = (0,b/N)$ and give a comparison between $\mathrm{dlog}_{Y/\mathbf{Q}}(g_{0,b/N})$ and $F^{(2)}_{0,b/N}$,, which consists in comparing the two q-expansions. First, recall.

Proposition 3.1.23. *Let $\zeta_N := e^{2\pi i/N}$ and $q := e^{2\pi i\tau}$. For $k \geq -1$ and $b \in \mathbf{Z}/N\mathbf{Z}$ not zero, then*

$$F^{(k+2)}_{0,b/N} = \zeta(-1-k) + \sum_{n>0} q^n \sum_{\substack{dd'=n \\ d,d'>0}} d^{k+1}(\zeta_N^{bd'} + (-1)^k\zeta_N^{-bd'}).$$

Proof. The analytic formulae of $F^{(k+2)}_{0,b/N}$ are extensively studied in [Kat76] Sections 3.2-3.3. $\square$

By explicit calculation, we have the following.

Proposition 3.1.24 ([Kat04] 3.11)**.** *For any $b \in \mathbf{Z}/N\mathbf{Z}$ non-zero, we have*

$$\begin{aligned} \mathrm{dlog}_{Y/\mathbf{Q}}(g_{0,\frac{b}{N}}) &= -F^{(2)}_{0,b/N} \cdot (2\pi i d\tau) \\ \mathrm{dlog}_{Y/\mathbf{Q}}({}_cg_{0,\frac{b}{N}}) &= (-c^2F^{(2)}_{0,b/N} + F^{(2)}_{0,cb/N}) \cdot (2\pi i d\tau). \end{aligned}$$

3.1.4 The Eisenstein Symbol of Beilinson

Let $\mathscr{H}^k_{\mathbf{Q}}$ denote the relative Chow motive over $\mathrm{Sh}_{\mathbf{GL}_2}(U_1(N))$ associated to the $\mathbf{GL}_2$-representation $\mathrm{Sym}^k(\mathrm{Std})\otimes\det^{-k}$, for $k\geq 0$. Given the universal elliptic curve $\pi:\mathscr{E}\to\mathrm{Sh}_{\mathbf{GL}_2}(U_1(N))$, we denote by $\mathscr{H}^k_{\mathrm{dR}}=\mathrm{TSym}^k(\mathscr{H}_{\mathrm{dR}})$ the k-th symmetric tensors of $\mathscr{H}_{\mathrm{dR}}=R^1\pi_*[\mathscr{O}_{\mathscr{E}}\to\Omega^1_{\mathscr{E}/\mathrm{Sh}_{\mathbf{GL}_2}(U_1(N))}]$. In [Beĭ86] Beilinson constructed motivic Eisenstein classes

$$\mathrm{Eis}^k_{0,b/N}\in H^1_{\mathrm{mot}}(\mathrm{Sh}_{\mathbf{GL}_2}(U_1(N)),\mathscr{H}^k_{\mathbf{Q}}(1)),$$

whose pull-back to the upper half plane of the de Rham realisation $r_{dR}(\mathrm{Eis}^k_{0,b/N})$ is the $\mathscr{H}^k_{dR}$-valued 1-form

$$-F^{(k+2)}_{0,b/N}\cdot(2\pi i dz)^k(2\pi i d\tau).$$

We do not discuss the construction of Beilinson; we rather prefer to give an account of the definition of Eisenstein classes as the evaluation at torsion points of the polylogarithm class which generalises to the case of abelian schemes of arbitrary dimension. This has various advantages; one of them is that its norm-compatibility relation is radically built in the structure of the construction.

3.2 Eisenstein classes for abelian schemes

In this section and the following, we give a brief introduction to the theory of Eisenstein classes and we explain how Theorem 3.1.7 generalises to the higher dimensional setting by describing two constructions, one due to Kings and one due to Faltings, comparing them and hopefully explaining advantages of the first against the other. For instance, Faltings constructs only trivial coefficients classes, while Kings' construction is much more general. Our main references are [KR17], [HK15], and [Fal05].

Let us briefly summarise the contents of §3.2 and §3.3. Let A/S be an abelian scheme over a scheme S of relative dimension g and let c be an auxiliary positive integer. In §3.2.1-3, we discuss the construction of Kings of motivic (and étale) Eisenstein classes in the cohomology of S. We start by describing the motivic construction of the class with trivial coefficients in the cohomology of $A\smallsetminus A[c]$, as in [KR17]. This can be achieved directly: by decomposing the motivic cohomology of the abelian scheme into a direct sum of the eigenspaces for the trace $[a]_*$, for a prime to c, one can construct (Definition 3.2.4) an $[a]_*$-invariant class

$${}_cz\in H^{2g-1}_{\mathrm{mot}}(A\smallsetminus A[c],\mathbf{Q}(g)),$$

whose residue is the characteristic class of $c^{2g}(0)-A[c]$. In [KR17], this class is realised as the $\kappa=0$-part of the system of motivic polylogarithm classes

$${}_c\mathrm{pol}^{\kappa}_{\mathrm{mot}}\in H^{2g-1}_{\mathrm{mot}}(A\smallsetminus A[c],\mathrm{Sym}^{\kappa}\mathscr{L}_{\mathbf{Q}}(g)),$$

introduced by [Kin99]. From the polylogarithm, Kings constructs Eisenstein classes (Definition 3.2.14)

$$ {}_{\mathrm{c}}\mathrm{Eis}_x^{\kappa} \in H^{2g-1}_{\mathrm{mot}}(S, \mathrm{Sym}^{\kappa}(h^1(A)^{\vee})(g)), $$

depending on the choice of a torsion section $x : S \to A$, where $h^1(A)^{\vee}$ is the dual of the Chow motive introduced by Proposition 2.2.3.

Fix a prime p invertible in S. In §3.3, we illustrate and compare the two aforementioned p-adic integral constructions. In §3.3.1-2, we discuss Faltings' approach, defining (Theorem 3.3.6) and discussing the properties of

$$ {}_c z_m \in H^{2g-1}_{\text{ét}}(A \smallsetminus A[c], \mathbf{Z}_p(g)). $$

This class is characterised by being $[a]_*$-invariant and having residue the étale characteristic class of $c^{2g}(0) - A[c]$. In §3.3.4, we introduce Kings' construction of the integral p-adic polylogarithm class (Definition 3.3.20)

$$ {}_{\mathrm{c}}\mathrm{pol} \in H^{2g-1}_{\text{ét}}(A \smallsetminus A[c], \mathscr{L}_{\mathbf{Z}_p}(g)), $$

where $\mathscr{L}_{\mathbf{Z}_p}$ is the p-adic integral avatar of the system of motivic sheaves $(\mathrm{Sym}^{\kappa} \mathscr{L}_{\mathbf{Q}})_{\kappa}$. As in the motivic case, from the integral polylogarithm Kings defines (Definition 3.3.24) integral avatars of Eisenstein classes

$$ {}_{\mathrm{c}}\mathrm{Eis}^{\kappa}_{\mathbf{Z}_p, x} \in H^{2g-1}_{\text{ét}}(S, \mathrm{TSym}^{\kappa}(\mathscr{H}_{\mathbf{Z}_p})(g)), $$

where $\mathscr{H}_{\mathbf{Z}_p}$ denotes the p-adic Tate module of A.
Under some hypotheses on S (e.g. of finite type over $\mathbf{Z}$ or equal to an algebraically closed field), in §3.3.3 we construct a class ${}_c\mathscr{Z}_m \in H^{2g-1}_{\text{ét}}(A \smallsetminus A[c], \mathscr{L}_{\mathbf{Z}_p}(g))$ from ${}_c z_m$ (Definition 3.3.15), and compare it with ${}_{\mathrm{c}}\mathrm{pol}$ in §3.3.6: Proposition 3.3.28 asserts that the two integral constructions are equivalent up to an explicit constant.

3.2.1 Notation

Fix a (connected) smooth and quasi-projective scheme S over a characteristic zero field. Let $\pi : A \to S$ be an abelian scheme of relative dimension g and zero section $e : S \to A$ and let $\pi_c : A \smallsetminus A[c] \to S$. Recall that the trace map associated to multiplication by a prime to c is the composition of

$$ [a]_* : H^{\star}_{\mathrm{mot}}(A \smallsetminus A[c], \mathscr{W}_{\mathbf{Q}}(\bullet)) \longrightarrow H^{\star}_{\mathrm{mot}}(A \smallsetminus A[ac], [a]^* \mathscr{W}_{\mathbf{Q}}(\bullet)) \longrightarrow H^{\star}_{\mathrm{mot}}(A \smallsetminus A[c], \mathscr{W}_{\mathbf{Q}}(\bullet)), $$

where the first map is just restriction to $A \smallsetminus A[ac]$ and the second is the trace map associated to $[a]$. Moreover, for $B \in \{A, A \smallsetminus A[c]\}$ and any integer t, denote by $H^{\star}_{\mathrm{mot}}(B, \mathbf{Q}(\bullet))^{(t)}$ the generalised eigenspace of $[a]_*$

$$ \{z \in H^{\star}_{\mathrm{mot}}(B, \mathbf{Q}(\bullet)) : \quad ([a]_* - a^t)^r z = 0 \text{ for some } r \geq 1\}. $$

As a consequence of Proposition 2.2.3, we have the following.

Proposition 3.2.1 ([KR17], Proposition 2.2.1)**.** *There is a decomposition into $[a]_*$-eigenspaces*

$$H^{\star}_{\mathrm{mot}}(A,\mathbf{Q}(\bullet)) \simeq \bigoplus_{t=0}^{2g} H^{\star}_{\mathrm{mot}}(A,\mathbf{Q}(\bullet))^{(t)},$$

which is independent of a.

This is of fundamental importance in the construction given by Kings of the degree zero part of the polylogarithm class, which is discussed in §3.2.2 and §3.2.3.

3.2.2 The motivic class with trivial coefficients

Generalising Theorem 3.1.7 to this higher dimension setting boils down to constructing an element in $H^{2g-1}_{\mathrm{mot}}(A\smallsetminus A[c],\mathbf{Q}(g))^{(0)}$, whose image in $H^0_{\mathrm{mot}}(A[c],\mathbf{Q})$, under the edge map *res* of the Gysin sequence for the triple $A\smallsetminus A[c]\hookrightarrow A \hookleftarrow A[c]$

$$H^{2g-1}_{\mathrm{mot}}(A,\mathbf{Q}(g)) \longrightarrow H^{2g-1}_{\mathrm{mot}}(A\smallsetminus A[c],\mathbf{Q}(g)) \xrightarrow{res} H^0_{\mathrm{mot}}(A[c],\mathbf{Q}) \xrightarrow{deg} H^{2g}_{\mathrm{mot}}(A,\mathbf{Q}(g))$$

is given by the class of $c^{2g}e(S)-A[c]$.

Remark 3.2.2. Note that one can immediately identify $\mathrm{Ker}(deg)$ with the space $H^0_{\mathrm{mot}}(A[c]\smallsetminus e(S),\mathbf{Q})$, so that the class of $c^{2g}e(S)-A[c]$ corresponds to the fundamental class of $A[c]\smallsetminus e(S)$.

Applying Proposition 3.2.1 and the equivariance of the Gysin sequence for the $[a]_*$-action, we get

Proposition 3.2.3 ([KR17], Corollary 2.2.2)**.** *Let $c\geq 2$, then*

$$H^{2g-1}_{\mathrm{mot}}(A\smallsetminus A[c],\mathbf{Q}(g))^{(0)} \simeq H^0_{\mathrm{mot}}(A[c]\smallsetminus e(S),\mathbf{Q})^{(0)}.$$

Proof. By [KR17, Lemma 2.1.4], the Gysin sequence is equivariant for the $[a]_*$-action, thus the Gysin sequence for $A\smallsetminus A[c]\hookrightarrow A\smallsetminus e(S) \hookleftarrow A[c]\smallsetminus e(S)$, gives the exact sequence

$$H^{2g-1}_{\mathrm{mot}}(A\smallsetminus e(S),\mathbf{Q}(g))^{(0)} \longrightarrow H^{2g-1}_{\mathrm{mot}}(A\smallsetminus A[c],\mathbf{Q}(g))^{(0)} \longrightarrow$$

$$\longrightarrow H^0_{\mathrm{mot}}(A[c]\smallsetminus e(S),\mathbf{Q})^{(0)} \longrightarrow H^{2g}_{\mathrm{mot}}(A\smallsetminus e(S),\mathbf{Q}(g))^{(0)}.$$

The result follows by noticing that

$$H^{\star}_{\mathrm{mot}}(A\smallsetminus e(S),\mathbf{Q}(g))^{(0)} = 0,$$

which is a direct consequence of [KR17, Proposition 2.2.1]. $\square$

As a consequence, we can "lift" the class of $c^{2g}e(S) - A[c]$:

Definition 3.2.4. Let ${}_cz \in H^{2g-1}_{\mathrm{mot}}(A \smallsetminus A[c], \mathbf{Q}(g))^{(0)}$ be the element which maps to the class

$$c^{2g}e_*(1) - \pi^*_{|_{A[c]}}(1) \in H^0_{\mathrm{mot}}(A[c], \mathbf{Q})^{(0)}$$

under the isomorphisms of Proposition 3.2.3 and Remark 3.2.2.

Remark 3.2.5. Notice that similarly we could lift the étale realisation of the class of $c^{2g}e(S) - A[c]$ in p-adic étale cohomology with $\mathbf{Q}_p$-coefficients, where p is a prime which does not divide c. This is not true integrally; we will see that Faltings' idea is based on overcoming the obstruction and construct a section of

$$H^{2g-1}_{\text{ét}}(A \smallsetminus A[c], \mathbf{Z}_p(g))^{(0)} \longrightarrow H^0_{\text{ét}}(A[c], \mathbf{Z}_p)^{(0)},$$

after multiplying by a suitable element of $\mathbf{Z}_p$.

Finally, we mimic Definition 3.1.4:

Definition 3.2.6. Let $x : S \to A$ be an N-torsion section $x : S \to A$, for c, N coprime; define

$${}_cz_x := x^*{}_cz \in H^{2g-1}_{\mathrm{mot}}(S, \mathbf{Q}(g)).$$

How can we describe the class ${}_cz_x$ so obtained? In the one dimensional case, this is described by the image of the Siegel unit of Definition 3.1.4. In the case where $g > 1$, it does not seem possible to have a similar presentation of ${}_cz_x$. Nevertheless, using the logarithmic (motivic) sheaf $\mathscr{L}_{\mathbf{Q}}$ and its symmetric powers, Kings constructs Eisenstein classes

$${}_{\mathrm{c}}\mathrm{Eis}^{\kappa}_x \in H^{2g-1}_{\mathrm{mot}}(S, \mathrm{Sym}^{\kappa}(h^1(A)^{\vee})(g)),$$

such that ${}_{\mathrm{c}}\mathrm{Eis}^0_x = {}_cz_x$. In §3.2.3 below, we give a very brief introduction to Kings' construction of the logarithm sheaf and of ${}_{\mathrm{c}}\mathrm{Eis}^{\kappa}_x$.

3.2.3 The polylogarithm class

We give a very brief account of the construction of the polylogarithm class. Rather than explaining the motivic construction of the logarithm sheaf and Eisenstein classes, we discuss the p-adic case, for a prime p, and refer to [Kin99] and [HK15] for further details on the motivic one, which is morally similar to the p-adic one thanks to the contribution of [HK15]. We keep the notation used above.

Recall we have fixed an abelian scheme $\pi : A \to S$ of relative dimension g. We will work with continuous étale cohomology groups as defined by [Jan88] (see §2.2.1).

Definition 3.2.7. Let $\mathscr{H}_{\mathbf{Q}_p}$ be the dual of the first relative étale cohomology $(R^1\pi_*\mathbf{Q}_p)^\vee$; denote $\mathscr{H}^{\kappa}_{\mathbf{Q}_p} := \mathrm{Sym}^{\kappa}\mathscr{H}_{\mathbf{Q}_p}$.

The Leray spectral sequence induces a short exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_S(\mathbf{Q}_p, \mathscr{H}_{\mathbf{Q}_p}) \xrightarrow{\pi^*} \mathrm{Ext}^1_A(\mathbf{Q}_p, \pi^*\mathscr{H}_{\mathbf{Q}_p}) \longrightarrow \mathrm{Hom}_S(\mathscr{H}_{\mathbf{Q}_p}, \mathscr{H}_{\mathbf{Q}_p}) \longrightarrow 0,$$

which splits since we have a section of π given by the zero section $e : S \to A$.

Definition 3.2.8 ([Kin99], Definition 1.1.2)**.** Let $\mathscr{L}^1_{\mathbf{Q}_p} \in \mathrm{Ext}^1_A(\mathbf{Q}_p, \pi^*\mathscr{H}_{\mathbf{Q}_p})$ be the unique element that maps to $id \in \mathrm{Hom}_S(\mathscr{H}_{\mathbf{Q}_p}, \mathscr{H}_{\mathbf{Q}_p})$ and such that $e^*\mathscr{L}^1_{\mathbf{Q}_p}$ splits. As before denote by $\mathscr{L}^{\kappa}_{\mathbf{Q}_p}$ the κ-th symmetric power $\mathrm{Sym}^{\kappa}\mathscr{L}_{\mathbf{Q}_p}$.

As explained in [KR17], we have maps $\mathscr{L}^{\kappa}_{\mathbf{Q}_p} \to \mathscr{L}^{\kappa-1}_{\mathbf{Q}_p}$, with kernel equal $\pi^*\mathscr{H}^{\kappa}_{\mathbf{Q}_p}$, induced by the morphism $\mathscr{L}^1_{\mathbf{Q}_p} \to \mathbf{Q}_p$ given by the very definition of this extension class. The logarithm sheaf $\mathscr{L}_{\mathbf{Q}_p}$ is defined as the pro-system of p-adic sheaves $(\mathscr{L}^{\kappa}_{\mathbf{Q}_p})_{\kappa}$. Crucially, the logarithm sheaf satisfies the following properties.

Proposition 3.2.9 ([Kin99] Proposition 1.1.3; [KR17] (3.2.1))**.** *We have:*

1. *Let $[a] : A \to A$ be the multiplication-by-$[a]$ morphism. Then, $\mathscr{L}_{\mathbf{Q}_p} \simeq [a]^*\mathscr{L}_{\mathbf{Q}_p}$; if $t \in \mathrm{Ker}[a](S)$, the pullback of $\mathscr{L}^{\kappa}_{\mathbf{Q}_p}$ by t^* is*
$$t^*\mathscr{L}^{\kappa}_{\mathbf{Q}_p} \simeq \prod_{m=0}^{\kappa} \mathscr{H}^m_{\mathbf{Q}_p}.$$
2. *There is a canonical isomorphism*
$$R^i\pi_*\mathscr{L}^{\kappa}_{\mathbf{Q}_p} \simeq \begin{cases} 0 & \text{if } i < 2g \\ \mathbf{Q}_p(-g) & \text{if } i = 2g, \end{cases}$$
for all κ.

Note that Proposition 3.2.9 (with the help of the Leray spectral sequence for π_*) implies that

$$H^i_{\text{ét}}(A, \mathscr{L}_{\mathbf{Q}_p}(g)) := \varprojlim_{\kappa} H^i_{\text{ét}}(A, \mathscr{L}^{\kappa}_{\mathbf{Q}_p}(g)) \simeq \begin{cases} 0 & \text{if } i < 2g \\ H^0_{\text{ét}}(S, \mathbf{Q}_p) & \text{if } i = 2g. \end{cases}$$

This finds an immediate application when we calculate the cohomology group $H^{2g-1}_{\text{ét}}(A \smallsetminus A[c], \mathscr{L}_{\mathbf{Q}_p}(g))$. Indeed, the Gysin sequence for the triangle $A \smallsetminus A[c] \overset{j_c}{\hookrightarrow} A \overset{i_c}{\hookleftarrow} A[c]$ gives

$$0 = H^{2g-1}_{\text{ét}}(A, \mathscr{L}_{\mathbf{Q}_p}(g)) \longrightarrow H^{2g-1}_{\text{ét}}(A \smallsetminus A[c], \mathscr{L}_{\mathbf{Q}_p}(g)) \longrightarrow H^0_{\text{ét}}(A[c], i_c^*\mathscr{L}_{\mathbf{Q}_p}) \longrightarrow H^0_{\text{ét}}(S, \mathbf{Q}_p)\ ,$$

which induces, by [KR17, Corollary 3.3.1], the isomorphism

$$H^{2g-1}_{\text{ét}}(A \smallsetminus A[c], \mathscr{L}_{\mathbf{Q}_p}(g)) \simeq H^0_{\text{ét}}(S, \operatorname{Ker}(\pi_{A[c],*} i_c^* \mathscr{L}_{\mathbf{Q}_p} \to \mathbf{Q}_p)),$$

where $\pi_{A[c]} : A[c] \to S$ is the structure morphism. Notice that Proposition 3.2.9(1) induces the identification

$$\pi_{A[c],*} i_c^* \mathscr{L}_{\mathbf{Q}_p} \simeq \prod_{\kappa} \pi_{A[c],*} \operatorname{Sym}^{\kappa} i_c^* \mathscr{H}_{\mathbf{Q}_p};$$

Thus, the kernel of the residue map $\operatorname{Ker}(H^0_{\text{ét}}(A[c], \mathbf{Q}_p) \to H^0_{\text{ét}}(S, \mathbf{Q}_p))$ lies inside $H^0_{\text{ét}}(S, \operatorname{Ker}(\pi_{A[c],*} i_c^* \mathscr{L}_{\mathbf{Q}_p} \to \mathbf{Q}_p))$ and it makes sense to define the following.

Definition 3.2.10. For each $D \in \operatorname{Ker}(H^0_{\text{ét}}(A[c], \mathbf{Q}_p) \to H^0_{\text{ét}}(S, \mathbf{Q}_p))$, let

$$_D\mathrm{pol}_{\mathbf{Q}_p} \in H^{2g-1}_{\text{ét}}(A \smallsetminus A[c], \mathscr{L}_{\mathbf{Q}_p}(g))$$

be the class of residue D. In particular, denote by $_{\mathrm{c}}\mathrm{pol}_{\mathbf{Q}_p}$ the class whose residue is given by

$$D_c := c^{2g} e_*(1) - \pi^*_{A[c]}(1) \in \operatorname{Ker}(H^0_{\text{ét}}(A[c], \mathbf{Q}_p) \to H^0_{\text{ét}}(S, \mathbf{Q}_p)).$$

We have already anticipated that using the logarithm sheaf has the advantage of having the norm-compatibility built-in; indeed, we have the following.

Proposition 3.2.11 ([KR17], Proposition 3.4.1)**.** *Let a be an integer prime to c; then,*

$$[a]_*(_{\mathrm{c}}\mathrm{pol}_{\mathbf{Q}_p}) = {}_{\mathrm{c}}\mathrm{pol}_{\mathbf{Q}_p}.$$

Proof. It follows from the $[a]_*$-equivariance of the Gysin sequence and the fact that $[a]_*(D_c) = D_c$. □

Remark 3.2.12. The integral avatar of $\mathscr{L}_{\mathbf{Q}_p}$, whose description is the subject of §3.3.4, satisfies the same properties.

In [HK15, Proposition 4.6.1], it is shown that $\mathscr{L}^{\kappa}_{\mathbf{Q}_p}$ is the étale realisation of a motivic object $\mathscr{L}^{\kappa}_{\mathbf{Q}} = \operatorname{Sym}^{\kappa} \mathscr{L}^1_{\mathbf{Q}}$ (in the triangulated category of étale motives over A without transfer and with rational coefficients), which satisfies the equivalent properties of $\mathscr{L}^{\kappa}_{\mathbf{Q}_p}$; thus, one can define ([HK15, Theorem 5.2.3]) a system of motivic polylogarithm classes

$$_{\mathrm{c}}\mathrm{pol}^{\kappa}_{\mathrm{mot}} \in H^{2g-1}_{\mathrm{mot}}(A \smallsetminus A[c], \mathscr{L}^{\kappa}_{\mathbf{Q}}(g)),$$

characterised by having residue equal to $c^{2g} e_*(1) - \pi^*_{A[c]}(1) \in \operatorname{Ker}(H^0_{\mathrm{mot}}(A[c], \mathbf{Q}) \to H^0_{\mathrm{mot}}(S, \mathbf{Q}))$.

As for the étale case, the pullback of $\mathscr{L}_{\mathbf{Q}}^{\kappa}$ by a torsion section $x : S \to A$ splits as

$$x^*\mathscr{L}_{\mathbf{Q}}^{\kappa} = \prod_{m=0}^{\kappa} \mathrm{Sym}^m(h^1(A)^\vee),$$

where $h^1(A)^\vee$, introduced in Proposition 2.2.3, is the underlying motivic object whose p-adic realisation is $\mathscr{H}_{\mathbf{Q}_p}$.

We can finally come back to the question posed at the end of §3.2.2.

Proposition 3.2.13 ([KR17], Corollary 3.4.2)**.** *The degree* 0 *component of the polylogarithm class* ${}_{\mathrm{c}}\mathrm{pol}^0_{\mathrm{mot}} \in H^{2g-1}_{\mathrm{mot}}(A \smallsetminus A[c], \mathbf{Q}(g))$ *is the class* ${}_cz$ *of Definition 3.2.4.*

However, thanks to the splitting of $\mathscr{L}_{\mathbf{Q}}^{\kappa}$ under pullback by torsion section $x : S \to A$ of finite order prime to c, we get maps pr_x^{κ} defined as the composition

$$H^{2g-1}_{\mathrm{mot}}(A \smallsetminus A[c], \mathscr{L}_{\mathbf{Q}}^{\kappa}(g)) \xrightarrow{x^*} H^{2g-1}_{\mathrm{mot}}(S, \textstyle\prod_{m=0}^{\kappa} \mathrm{Sym}^m(h^1(A)^\vee)(g)) \longrightarrow H^{2g-1}_{\mathrm{mot}}(S, \mathrm{Sym}^{\kappa}(h^1(A)^\vee)(g)).$$

Similarly, in the étale case, we have maps $\mathrm{pr}_x^{\kappa} : H^{2g-1}_{\text{ét}}(A \smallsetminus A[c], \mathscr{L}_{\mathbf{Q}_p}(g)) \to H^{2g-1}_{\text{ét}}(S, \mathscr{H}^{\kappa}_{\mathbf{Q}_p}(g))$.

Definition 3.2.14. The motivic and étale Eisenstein classes of weight κ are defined as

$$\begin{aligned} {}_{\mathrm{c}}\mathrm{Eis}_x^{\kappa} &:= \mathrm{pr}_x^{\kappa}({}_{\mathrm{c}}\mathrm{pol}_{\mathrm{mot}}) \in H^{2g-1}_{\mathrm{mot}}(S, \mathrm{Sym}^{\kappa}(h^1(A)^\vee)(g)) \\ {}_{\mathrm{c}}\mathrm{Eis}_{\text{ét},x}^{\kappa} &:= \mathrm{pr}_x^{\kappa}({}_{\mathrm{c}}\mathrm{pol}_{\mathbf{Q}_p}) \in H^{2g-1}_{\text{ét}}(S, \mathscr{H}^{\kappa}_{\mathbf{Q}_p}(g)). \end{aligned}$$

Remark 3.2.15. As a direct consequence of Proposition 3.2.13, we get that

$${}_{\mathrm{c}}\mathrm{Eis}_x^0 = {}_cz_x.$$

3.2.4 Constructing the class in K_1

In [Fal05, §5], Faltings constructs a class ${}_c\tau \in K_1(A \smallsetminus A[c]) \otimes \mathbf{Q}$, dealing directly with Quillen's definition of the K_1-group of a scheme as the second homotopy group $\pi_2(\mathrm{BQM}(A \smallsetminus A[c]), 0)$ of the geometric realisation of the nerve of the Q-category of $M(A \smallsetminus A[c])$, which is the category of coherent sheaves on $A \smallsetminus A[c]$ up to canonical isomorphism. This element in $K_1(A \smallsetminus A[c]) \otimes \mathbf{Q}$, which is shown to have residue a multiple of the fundamental class D_c of $c^{2g}e(S) - A[c]$, comes from determining an explicit and canonical homotopy between the loop of $\mathscr{O}_{e(S)}$ and the one of $\mathscr{O}_t$, for a non-trivial $[c]$-torsion point t of A, in $\mathrm{BQM}(A \smallsetminus A[c])$. This homotopy gives an element of $\pi_1(\Omega\,\mathrm{BQM}(A \smallsetminus A[c]), \tilde{0})$, where $\tilde{0}$ is the constant loop $\tilde{0}(t) = 0$, which is isomorphic to $\pi_2(\mathrm{BQM}(A \smallsetminus A[c]), 0)$.
Pulling back by a torsion point x of order coprime to c, it gives a class in ${}_c\tau_x \in K_1(S) \otimes \mathbf{Q}$.

We do not treat this construction in detail and we will not make use of it at any point in this thesis; however, its component in

$$\mathrm{Gr}^g_\gamma(K_1(S)) \otimes \mathbf{Q} \simeq H^{2g-1}_{\mathrm{mot}}(S, \mathbf{Q}(g))$$

is expected to be a multiple of ${}_c z_x$, since both elements ${}_c\tau$ and ${}_c z$ are fixed by the trace $[a]_*$, for a coprime to c, and have residue one the multiple of the other.

3.3 Integral Eisenstein classes

Fix a prime p. The étale realisation of the motivic Eisenstein classes described above defines an interesting family of étale cohomology classes; however, for arithmetic applications, the integrality of such elements is often needed. For instance, the Euler system machinery requires Galois cohomology classes with values in a lattice of the p-adic Galois representation in question. As the reader might have noticed, Siegel units are naturally integral. However, they form an isolated case and, due to a lack of a theory of integral motivic Eisenstein classes, additional work is needed to construct integral classes.

As in §3.2.2, one can try to directly construct a section of the residue map

$$H^{2g-1}_{\text{ét}}(A \smallsetminus A[c], \mathbf{Z}_p(g)) \longrightarrow \mathrm{Ker}\left(H^0_{\text{ét}}(A[c], \mathbf{Z}_p) \longrightarrow H^{2g}_{\text{ét}}(A, \mathbf{Z}_p(g))\right)$$

of the Gysin sequence for $A[c] \hookrightarrow A \hookleftarrow A \smallsetminus A[c]$. This approach has been explored by Faltings. However, Kings' integral description of the étale logarithm sheaf and of the polylogarithm provides a more conceptual construction of integral étale cohomology classes and immediately sheds some light on their relation with the étale realisation of the motivic Eisenstein classes previously constructed. We will describe both approaches and explore the connection between the two constructions.

3.3.1 A construction of Faltings

We mainly follow [Fal05, Section 3].

Let $\pi : A \longrightarrow S$ be an abelian scheme of relative dimension g and let p be invertible in S. For any integer c not divisible by p, let $\pi_c : A \smallsetminus A[c] \longrightarrow S$. Faltings constructs classes

$${}_c z_m \in H^{2g-1}_{\text{ét}}(A \smallsetminus A[c], \mathbf{Z}_p(g)),$$

depending on the choice of an auxiliary integer m, which satisfy:

P1. For any r prime to c, ${}_c z_m$ is invariant under trace maps associated to multiplication by r, i.e.

$$[r]_*({}_c z_m) = {}_c z_m;$$

P2. The classes ${}_{c_1 \cdot c_2}z_m$, ${}_{c_1}z_m$ and ${}_{c_2}z_m$ are related by

$$ {}_{c_1 \cdot c_2}z_m = [c_1]^*({}_{c_2}z_m) + c_2^{2g}\,{}_{c_1}z_m = [c_2]^*({}_{c_1}z_m) + c_1^{2g}\,{}_{c_2}z_m. $$

Furthermore, pulling them back under torsion sections $x \in A(S)$ of order prime to c, we obtain classes

$$ {}_{c}z_{m,x} \in H^{2g-1}_{\text{ét}}(S, \mathbf{Z}_p(g)). $$

We first recall the construction of these classes and then discuss their properties.

Consider the étale Gysin sequence for $A[c] \hookrightarrow A \hookleftarrow A \smallsetminus A[c]$. It tells us how the direct image étale sheaves (on S) $R^i\pi_{c,*}(\mathbf{Z}/p^t\mathbf{Z}(g))$ are related to $R^i\pi_*(\mathbf{Z}/p^t\mathbf{Z}(g))$. Let $\mathscr{H}^j(A[c], \mathscr{F})$ denote the j-th direct image sheaf of the sheaf $\mathscr{F}$ on $A[c]$.

Proposition 3.3.1. *The sheaf $R^i\pi_{c,*}(\mathbf{Z}/p^t\mathbf{Z}(g))$ coincides with $R^i\pi_*(\mathbf{Z}/p^t\mathbf{Z}(g))$ for $i < 2g-1$; moreover, we have the exact sequence*

$$ 0 \longrightarrow R^{2g-1}\pi_*(\mathbf{Z}/p^t\mathbf{Z}(g)) \longrightarrow R^{2g-1}\pi_{c,*}(\mathbf{Z}/p^t\mathbf{Z}(g)) \longrightarrow \mathscr{H}^0(A[c], \mathbf{Z}/p^t\mathbf{Z}) \longrightarrow \mathbf{Z}/p^t\mathbf{Z}. $$

Proof. The result can be checked at geometric stalks. Then, it follows from Corollary VI.5.3 and Remark VI.5.4,(b) of [Mil80]. □

Remark 3.3.2. By taking inverse limit, this holds for the sheaf $\mathbf{Z}_p$.

Recall that the global sections $\mathscr{H}^0(A[c], \mathbf{Z}_p)(S)$ are by the very definition identified with $\mathbf{Z}_p(A[c])$, since $\pi_{|A[c]} : A[c] \to S$ is a finite étale map. We now define the characteristic class of $c^{2g}(0) - A[c]$.

Definition 3.3.3. Let $e : S \to A[c]$ be the closed immersion defined by the unit section; it induces

$$ e_* : \mathbf{Z}_p(S) \longrightarrow \mathbf{Z}_p(A[c]) = \mathscr{H}^0(A[c], \mathbf{Z}_p)(S). $$

We define the characteristic class of $c^{2g}(0) - A[c]$ to be the global section

$$ D_c := c^{2g}e_*(1) - \pi^*_{|A[c]}(1) \in \mathscr{H}^0(A[c], \mathbf{Z}_p)(S). $$

From Proposition 3.3.1, we have that the cokernel of

$$ R^{2g-1}\pi_*(\mathbf{Z}_p(g)) \hookrightarrow R^{2g-1}\pi_{c,*}(\mathbf{Z}_p(g)) $$

is isomorphic to the kernel of

$$ \phi : \mathscr{H}^0(A[c], \mathbf{Z}_p) \longrightarrow \mathbf{Z}_p. $$

The next lemma shows that the characteristic class D_c defines a global section of the kernel of ϕ, hence of the cokernel of $R^{2g-1}\pi_*(\mathbf{Z}_p(g)) \hookrightarrow R^{2g-1}\pi_{c,*}(\mathbf{Z}_p(g))$.

Lemma 3.3.4. *The global section $D_c \in \mathscr{H}^0(A[c],\mathbf{Z}_p)(S)$ lies in the kernel of $\phi(S)$.*

Proof. Since the kernel of ϕ is a sheaf, we can reduce to show that the restriction of D_c to each geometric fibre is zero. Thus, suppose that A is an abelian scheme over an algebraically closed field, then we want to check that D_c lies in the kernel of the Gysin map

$$\varphi : H^0_{\text{ét}}(A[c],\mathbf{Z}_p) \longrightarrow H^{2g}_{\text{ét}}(A,\mathbf{Z}_p(g)).$$

Note that $H^0_{\text{ét}}(A[c],\mathbf{Z}_p)$ is isomorphic to ${\mathbf{Z}_p}^{c^{2g}}$, while $H^{2g}_{\text{ét}}(A,\mathbf{Z}_p(g))$ is isomorphic to $\mathbf{Z}_p$. By Poincaré duality, the $\mathbf{Z}_p$-dual of $H^{2g}_{\text{ét}}(A,\mathbf{Z}_p(g))$ is $H^0_{\text{ét}}(A,\mathbf{Z}_p)$ which is isomorphic to $\mathbf{Z}_p$ and the pairing is induced by the product structure of $\mathbf{Z}_p$. On the other hand, $H^0_{\text{ét}}(A[c],\mathbf{Z}_p)$ is dual to itself with respect to the pairing induced by multiplication of vectors

$$\langle \bullet,\bullet\rangle : \mathbf{Z}_p^{c^{2g}} = H^0_{\text{ét}}(A[c],\mathbf{Z}_p)\times H^0_{\text{ét}}(A[c],\mathbf{Z}_p) = \mathbf{Z}_p^{c^{2g}} \longrightarrow \mathbf{Z}_p, \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{c^{2g}} \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{c^{2g}} \end{pmatrix} \mapsto a_1b_1 + \cdots a_{c^{2g}}b_{c^{2g}}.$$

Then, the dual map to φ is

$$\mathbf{Z}_p = H^0_{\text{ét}}(A,\mathbf{Z}_p) \longrightarrow H^0_{\text{ét}}(A[c],\mathbf{Z}_p) = \mathbf{Z}_p^{c^{2g}}, a \mapsto (a,a,\cdots,a).$$

Hence, let $v = (a_1,a_2,\cdots,a_{c^{2g}}) \in \mathbf{Z}_p^{c^{2g}}$ and $b \in \mathbf{Z}_p$, then

$$\langle \varphi(v),b\rangle = \langle v, \begin{pmatrix} b, & b, & \cdots, & b \end{pmatrix}\rangle = b\cdot \sum a_i,$$

which gives us the result claimed. $\square$

Unfortunately, we cannot immediately lift D_c to $H^{2g-1}_{\text{ét}}(A \smallsetminus A[c],\mathbf{Z}_p(g))$ as we did in §3.2.2. Faltings overcomes this obstacle, by multiplying for a big enough constant.

Lemma 3.3.5. *Let m be an integer prime to p and let $[m]_*$ denote the trace map associated to multiplication by m on A, then the product*

$$F_m := \prod_{i=0}^{2g-1} ([m]_* - m^{2g-i})$$

annihilates the truncated complex $\tau_{\leq 2g-1}(\mathbf{R}\pi_*(\mathbf{Z}_p(g)))$.

Proof. This is a direct consequence of the action of $[m]_*$ in cohomology, which acts on the i-th degree cohomology as multiplication by m^{2g-i}. Indeed, note that

$$R^i\pi_*(\mathbf{Z}_p(g)) \simeq R^{2g-i}\pi_*(\mathbf{Z}_p) \simeq \bigwedge^{2g-i} R^1\pi_*(\mathbf{Z}_p) \simeq \bigwedge^{2g-i} (T_p(A))^\vee,$$

where $T_p(A)$ denotes the p-adic Tate module of A and $(\bullet)^\vee$ denotes its $\mathbf{Z}_p$-dual. This follows by base change to geometric points and a duality statement for abelian varieties (e.g. [Mil08] Theorem 12.1). Thus, the result follows since $[m]_*$ acts as multiplication by m on $T_p(A)$. □

We now have the following result.

Theorem 3.3.6 ([Fal05], Section 3)**.** *Let m be an integer coprime to c and a generator of $\mathbf{Z}_p^*$ and define $N_m := \prod_{i=0}^{2g-1}(1-m^{2g-i})$. The class of $N_m^2(D_c)$ lifts canonically to $H_{ét}^{2g-1}(A\smallsetminus A[c],\mathbf{Z}_p(g))$.*

Proof. Here we give a sketch of the proof given in [Fal05]. We have a surjective map

$$H_{\text{ét}}^{2g-1}(A\smallsetminus A[c],\mathbf{Z}_p(g)) \longrightarrow \operatorname{Ker}\left(H_{\text{ét}}^0(A[c],\mathbf{Z}_p) \longrightarrow H_{\text{ét}}^{2g}(A,\mathbf{Z}_p(g))\right),$$

which comes from the étale Gysin exact sequence for $(A, A\smallsetminus A[c], A[c])$; denote by ψ the map

$$H_{\text{ét}}^0(A[c],\mathbf{Z}_p) \longrightarrow H_{\text{ét}}^{2g}(A,\mathbf{Z}_p(g)).$$

First, we lift $D_c \in \operatorname{Ker}(\phi(S))$ to the kernel of ψ to construct the desired class. To do so, Faltings constructs a map of complexes

$$\phi_{2g} : R^{2g}\pi_*\mathbf{Z}_p(g)[-2g] \longrightarrow \mathbf{R}\pi_*\mathbf{Z}_p(g)$$

which is multiplication by $(2g)!$ in cohomology. Since, by Lemma 3.3.5, the operator

$$F_m := \prod_{i=0}^{2g-1}([m]_* - m^{2g-i})$$

annihilates the truncated complex $\tau_{\leq 2g-1}(\mathbf{R}\pi_*(\mathbf{Z}_p(g)))$, then

$$F_m \circ \phi_{2g} = F_m \circ (2g)!$$

Let γ be the generator of $R^{2g}\pi_*\mathbf{Z}_p(g)$ defined by the zero section of A; since $[m]_*$ fixes γ, we have that

$$N_m \cdot \phi_{2g}(\gamma) = N_m \cdot (2g)!(\gamma).$$

Thus, $F_m \circ \psi(D_c) = N_m \cdot (2g)!\phi(D_c) = 0$, because $\phi(D_c) = 0$ in $R^{2g}\pi_*\mathbf{Z}_p(g)(S)$.
Since $[m]_*(D_c) = D_c$, F_m acts on D_c as multiplication by N_m. Hence, applying F_m again, we get a

canonical lift of $F_m(N_m D_c) = N_m^2 D_c$ in $H^{2g-1}_{\text{ét}}(A \smallsetminus A[c], \mathbf{Z}_p(g))$. $\square$

Remark 3.3.7. If the prime p is sufficiently large, we can choose m such that N_m is invertible in $\mathbf{Z}_p$; it suffices to assume that $p > 2g+1$ and choose m to be a generator of $(\mathbf{Z}/p\mathbf{Z})^*$.

Remark 3.3.8. The étale Gysin sequence is equivariant for the action of $[r]_*$, thus we have a map

$$H^{2g-1}_{\text{ét}}(A \smallsetminus A[c], \mathbf{Z}_p(g))^{[r]_*=1} \longrightarrow H^0_{\text{ét}}(A[c], \mathbf{Z}_p)^{[r]_*=1}.$$

The dependence of Faltings' classes on the choice of m arises from the obstruction to construct a section of this map if $g > 1$.

Finally, we can define classes over the base of the abelian scheme.

Definition 3.3.9. Denote by ${}_c z_m$ the canonical lift of $N_m^2(D_c)$ to $H^{2g-1}_{\text{ét}}(A \smallsetminus A[c], \mathbf{Z}_p(g))$. Moreover, for any torsion section $x \in A(S)$ of order prime to c, we define the étale Eisenstein classes for A/S as

$${}_c z_{m,x} := x^* {}_c z_m \in H^{2g-1}_{\text{ét}}(S, \mathbf{Z}_p(g)).$$

Let $res_{\text{ét}}$ denote the (residue) map from the étale Gysin sequence

$$res_{\text{ét}} : H^{2g-1}_{\text{ét}}(A \smallsetminus A[c], \mathbf{Z}_p(g)) \longrightarrow H^0_{\text{ét}}(A[c], \mathbf{Z}_p),$$

then

$$res_{\text{ét}}({}_c z_m) = N_m^2(D_c).$$

3.3.2 Properties of Faltings' classes

In the following, we investigate some of the properties that the Eisenstein classes defined above satisfy. In particular, we show that they satisfy the two properties P1,P2 we mentioned above.

Proposition 3.3.10. *Let r be an integer coprime to c and let $[r]_*$ be the trace map associated to the multiplication by $r : A \smallsetminus A[rc] \longrightarrow A \smallsetminus A[c]$, then*

$$[r]_*({}_c z_m) = {}_c z_m.$$

Proof. The trace map $[r]_*$ commutes with the operator induced by multiplication by N_m, hence the result follows from noticing that $[r]_*$ fixes D_c. $\square$

Notice that we can extend this result for trace maps associated to isogenies whose degree is coprime to c. Denote by ${}_c z_m^A$ the class in the cohomology of $A \smallsetminus A[c]$.

Corollary 3.3.11. *Let $h : A \to A'$ be an isogeny over S of degree prime to c, then*

$$h_*({}_c z_m^A) = {}_c z_m^{A'} \in H^{2g-1}_{ét}(A' \smallsetminus A'[c], \mathbf{Z}_p(g)).$$

Now, we prove the compatibility property P2 we stated above.

Proposition 3.3.12. *Let c_1, c_2 be integers, then*

$$ {}_{c_1 \cdot c_2}z = [c_1]^*({}_{c_2}z_m) + c_2^{2g}({}_{c_1}z_m) = [c_2]^*({}_{c_1}z_m) + c_1^{2g}({}_{c_2}z_m) \in H^{2g-1}_{ét}(A \smallsetminus A[c_1c_2], \mathbf{Z}_p(g)). $$

Proof. We are reduced to studying what happens at the level of the characteristic classes of the 0-cycles in $H^0_{\text{ét}}(A[c_1c_2], \mathbf{Z}_p)$ we are lifting. Indeed, note that

$$ [c_1]^*(c_2^{2g}e_*(1) - \pi^*_{|A[c_2]}(1)) + c_2^{2g}(c_1^{2g}e_*(1) - \pi^*_{|A[c_1]}(1)) = D_{c_1c_2} + c_2^{2g}[c_1]^*e_*(1) - c_2^{2g}\pi^*_{|A[c_1]}(1) = $$
$$ = D_{c_1c_2} $$

hence $[c_1]^*({}_{c_2}z) + c_2^{2g}{}_{c_1}z_m$ and ${}_{c_1c_2}z_m$ have same residue and hence are equal. □

In the same fashion of the previous propositions, we can prove that the Eisenstein classes are invariant under base-change. Recall that for Siegel units and Eisenstein classes, this property is well-known.

Proposition 3.3.13. *For any morphism $S' \to S$ and abelian scheme A/S, then*

$$ g^*({}_cz_m^A) = {}_cz_m^{A'}, $$

where $g : A' \to A$ is the base-change of A to S'.

Proof. The result follows from the fact the base change of D_c^A by is $D_c^{A'}$. □

We can now discuss the comparison between the image under the étale regulator of Siegel units and the étale classes defined in the previous section.

Proposition 3.3.14. *Let $\pi : E \to S$ be an elliptic curve and let N_m be as above; fix c to be coprime to $6p$. Then, we have that*

$$ N_m^2({}_c\theta_E^{ét}) = {}_cz_m \in H^1_{ét}(E \smallsetminus E[c], \mathbf{Q}_p(1)), $$

where ${}_c\theta_E^{ét}$ is the class defined in Proposition 3.1.11.

Proof. By the compatibility of étale and motivic Gysin sequences with the étale regulator, we have that the (étale) characteristic class of $res({}_c\theta_E)$ is equal to $res_{\text{ét}}({}_c\theta_E^{\text{ét}})$ (see also Lemma 3.1.13). Thus, $N_m^2({}_c\theta_E^{\text{ét}})$ has residue $N_m^2D_c$. It follows that the difference $N_m^2({}_c\theta_E^{\text{ét}}) - {}_cz_m^E$ lies in the kernel of the residue map and comes from an element

$$ t_m \in H^1_{\text{ét}}(E, \mathbf{Z}_p(1)), $$

which is fixed by $[r]_*$, for any r coprime to c. We are left to show that t_m is torsion. Indeed, after tensoring by $\mathbf{Q}_p$, the Leray spectral sequence gives an isomorphism

$$H^1_{\text{ét}}(E,\mathbf{Q}_p(1)) = H^1_{\text{ét}}(S,R^0\pi_*\mathbf{Q}_p(1)) \oplus H^0_{\text{ét}}(S,R^1\pi_*\mathbf{Q}_p(1)),$$

where $[r]_*$ acts as multiplication by r^2 and by r on each summand. Thus $t_m = 0$ in $H^1_{\text{ét}}(E,\mathbf{Q}_p(1))$. $\square$

3.3.3 Faltings' class as an integral polylogarithm

In the following, we define classes in étale cohomology groups with coefficients in an integral étale logarithm sheaf starting from Faltings' class. This is very close in spirit to the construction of Kings in [Kin15a] and [Kin15c], as we will recall in §3.3.4.

In the remaining part of the section, we fix an abelian scheme A of relative dimension g over a base S of finite type over $\mathbf{Z}$, and a prime p ($p \nmid c$). We consider $A_r := A$ as a cover of A under multiplication for p^r and define the inverse system of étale lisse sheaves

$$\mathscr{L}_{\mathbf{Z}_p} := ([p^r]_*(\mathbf{Z}/p^r\mathbf{Z}))_{r\geq 0},$$

with transition map $[p^r]_*(\mathbf{Z}/p^r\mathbf{Z}) \to [p^{r-1}]_*(\mathbf{Z}/p^{r-1}\mathbf{Z})$ given by composition of trace map and reduction modulo p^{r-1}. Since S is of finite type over $\mathbf{Z}$,

$$H^{2g-1}_{\text{ét}}(A\smallsetminus A[c],\mathscr{L}_{\mathbf{Z}_p}(g)) \simeq \varprojlim_r H^{2g-1}_{\text{ét}}(A_r\smallsetminus A_r[p^rc],\mathbf{Z}/p^r\mathbf{Z}(g)).$$

We can construct an element in this inverse limit by using the étale classes ${}_cz_m$. Indeed, consider the image of the element $i_r({}_cz_m)$ in $H^{2g-1}_{\text{ét}}(A_r\smallsetminus A_r[p^rc],\mathbf{Z}/p^r\mathbf{Z}(g))$ under the natural map

$$i_r : H^{2g-1}_{\text{ét}}(A_r\smallsetminus A_r[c],\mathbf{Z}/p^r\mathbf{Z}(g)) \longrightarrow H^{2g-1}_{\text{ét}}(A_r\smallsetminus A_r[p^rc],\mathbf{Z}/p^r\mathbf{Z}(g)).$$

Since the elements ${}_cz_m$ are trace compatible, i.e. $[p]_*({}_cz_m) = {}_cz_m$, we can define

Definition 3.3.15. Let ${}_c\mathscr{Z}_m := (i_r({}_cz_m))_r \in H^{2g-1}_{\text{ét}}(A\smallsetminus A[c],\mathscr{L}_{\mathbf{Z}_p}(g))$.

3.3.4 The integral polylogarithm of Kings

We now discuss the construction by Kings of the integral étale polylogarithm class

$${}_c\mathrm{pol} \in \varprojlim_r H^{2g-1}_{\text{ét}}(A_r\smallsetminus A_r[p^rc],\mathbf{Z}/p^r\mathbf{Z}(g)),$$

which is characterised by having residue D_c. Recall that A is an abelian scheme over a base S of finite type over $\mathbf{Z}$. Consider a torsion section $x : S \to A$ and let $A[p^r]\langle x\rangle$ be the $A[p^r]$-torsor defined by the Cartesian diagram

$$\begin{tikzcd} A[p^r]\langle x\rangle \arrow[r] \arrow[d, "p_{r,x}"'] & A_r \arrow[d, "{[p^r]}"] \\ S \arrow[r, "x"] & A. \end{tikzcd}$$

Define the sheaf of Iwasawa modules

$$\Lambda(\mathscr{H}_{\mathbf{Z}_p}\langle x\rangle) = (\Lambda_r(A[p^r]\langle x\rangle))_r := (p_{r,x,*}\mathbf{Z}/p^r\mathbf{Z})_r.$$

When x is the unit section $e : S \to A$, denote $\Lambda(\mathscr{H}_{\mathbf{Z}_p}\langle e\rangle)$ by $\Lambda(\mathscr{H}_{\mathbf{Z}_p})$.

Remark 3.3.16. The stalk at a geometric point $\bar{s}$ of $\Lambda_r(A[p^r]\langle x\rangle)$ is isomorphic to the space of $\mathbf{Z}/p^r\mathbf{Z}$-valued measures on $A[p^r]\langle x\rangle_{\bar{s}}$, hence the stalk at $\bar{s}$ of $\Lambda(\mathscr{H}_{\mathbf{Z}_p}\langle x\rangle)_{\bar{s}}$ is the Iwasawa algebra $\varprojlim \Lambda_r(A[p^r]\langle x\rangle_{\bar{s}})$, which motivates the chosen notation.

The sheaves $\Lambda(\mathscr{H}_{\mathbf{Z}_p}\langle x\rangle)$ and $\mathscr{L}_{\mathbf{Z}_p}$ are simply related by the following.

Lemma 3.3.17 ([Kin15c], Lemma 6.1.2)**.** *There is a canonical isomorphism*

$$x^*\mathscr{L}_{\mathbf{Z}_p} \simeq \Lambda(\mathscr{H}_{\mathbf{Z}_p}\langle x\rangle)$$

Proof. It is sufficient to work at finite level. Thus, the result follows from the Cartesian diagram above, since we deduce that

$$x^*[p^r]_*\mathbf{Z}/p^r\mathbf{Z} \simeq p_{r,x,*}\mathbf{Z}/p^r\mathbf{Z}.$$

□

We are now ready to state a result of Kings, which allows to define the integral polylogarithm class in terms of its residue; this is of fundamental importance in the study of integrality of Eisenstein classes and makes explicit the relation between polylogarithm classes and Siegel units in the case of elliptic curves (as in [Kin15a], Theorem 12.4.21), which is of fundamental importance in proving the p-adic interpolation properties of the push-forward classes in [KLZ17], [LSZ17], and [CRJ18].

Proposition 3.3.18 ([Kin15c], Proposition 6.3.1)**.** *We have a short exact sequence*

$$0 \longrightarrow H^{2g-1}_{ét}(A \smallsetminus A[c], \mathscr{L}_{\mathbf{Z}_p}(g)) \xrightarrow{res} H^0_{ét}(S, \pi_{|A[c],*}\pi^*_{|A[c]}\Lambda(\mathscr{H}_{\mathbf{Z}_p})) \longrightarrow H^0_{ét}(S, \mathbf{Z}_p) \longrightarrow 0$$

Remark 3.3.19. The statement differs from the one of [Kin15c, Proposition 6.3.1], since we have used that

$$H^{2g-1}_{\text{ét}}(A \smallsetminus A[c], \mathscr{L}_{\mathbf{Z}_p}(g)) \simeq H^{2g-1}_{\text{ét}}(S, R\pi_* Rj_{c*}j_c^*\mathscr{L}_{\mathbf{Z}_p}(g)),$$

as explained in [Kin15c, Section 6.5].

This result should already shed light on the connection between $\mathscr{L}_{\mathbf{Z}_p}$ and the étale logarithm sheaf $\mathscr{L}_{\mathbf{Q}_p}$, which results in Proposition 3.3.21. Let D_c be the étale characteristic class of $c^{2g}(0) - A[c]$, as defined in Definition 3.3.3. Since

$$D_c \in H^0_{\text{ét}}(S, \pi_{|_{A[c]},*}\pi^*_{|_{A[c]}}\Lambda(\mathscr{H}_{\mathbf{Z}_p})),$$

we can define the integral étale polylogarithm class as follows.

Definition 3.3.20. The integral étale polylogarithm class associated to D_c is

$${}_{\mathrm{c}}\mathrm{pol} \in H^{2g-1}_{\text{ét}}(A \smallsetminus A[c], \mathscr{L}_{\mathbf{Z}_p}(g))$$

such that $res({}_{\mathrm{c}}\mathrm{pol}) = D_c$.

This construction agrees with the one of Definition 3.2.10:

Proposition 3.3.21 ([Kin15c], Corollary 7.2.2)**.** *There exists a map* $\mathrm{comp} : \mathscr{L}_{\mathbf{Z}_p} \to \mathscr{L}_{\mathbf{Q}_p}$, *which induces the comparison*

$$\mathrm{comp}({}_{\mathrm{c}}\mathrm{pol}) = {}_{\mathrm{c}}\mathrm{pol}_{\mathbf{Q}_p} \in H^{2g-1}_{ét}(A \smallsetminus A[c], \mathscr{L}_{\mathbf{Q}_p}(g))$$

3.3.5 The Eisenstein-Iwasawa classes of Kings

We now explain how Kings constructs integral Eisenstein classes from the polylogarithm class of Definition 3.3.20.

Definition 3.3.22. Let $x : S \to A$ be a q-torsion section (for q coprime to c); define the Λ-adic Eisenstein class ${}_c\mathbf{EI}(x)$ to be $x^*({}_{\mathrm{c}}\mathrm{pol})$ in

$$H^{2g-1}_{\text{ét}}(S, x^*\mathscr{L}_{\mathbf{Z}_p}(g)) \simeq \varprojlim_r H^{2g-1}_{\text{ét}}(A[p^r]\langle x\rangle, \mathbf{Z}/p^r\mathbf{Z}(g)) = H^{2g-1}_{\text{ét}}(S, \Lambda(\mathscr{H}_{\mathbf{Z}_p}\langle x\rangle)(g)).$$

Remark 3.3.23. The class ${}_c\mathbf{EI}(x)$ is the Eisenstein-Iwasawa class, which appears in the works [Kin15a], [Kin15c] and [KLZ17].

Now, let $\mathrm{TSym}^k(\mathscr{H}_{\mathbf{Z}_p})$ be the sheaf of k^{th} symmetric tensors of the p-adic Tate module $\mathscr{H}_{\mathbf{Z}_p} = (R^1\pi_*\mathbf{Z}_p)^\vee$ of A. Notice that $\mathrm{TSym}^k(\mathscr{H}_{\mathbf{Z}_p})$ is not isomorphic to the k^{th} symmetric powers $\mathscr{H}^k_{\mathbf{Z}_p}$ (see Definition 3.2.7). The two sheaves become isomorphic after tensoring by $\mathbf{Q}_p$. Indeed, by the universal property of the symmetric algebra, we have a morphism $\mathrm{Sym}^k(\mathscr{H}_{\mathbf{Z}_p}) \to \mathrm{TSym}^k(\mathscr{H}_{\mathbf{Z}_p})$, which becomes an isomorphism after inverting $k!$, thus inducing $\mathrm{Sym}^k(\mathscr{H}_{\mathbf{Q}_p}) \simeq \mathrm{TSym}^k(\mathscr{H}_{\mathbf{Q}_p})$, where $\mathscr{H}_{\mathbf{Q}_p} = (R^1\pi_*\mathbf{Q}_p)^\vee$.

Kings defines moment maps (e.g. [Kin15c, Definition 5.5.2])

$$\mathrm{mom}_k : H^{2g-1}_{\text{ét}}(S, \Lambda(\mathscr{H}_{\mathbf{Z}_p})(g)) \longrightarrow H^{2g-1}_{\text{ét}}(S, \mathrm{TSym}^k(\mathscr{H}_{\mathbf{Z}_p})(g)),$$

for any integer $k \geq 0$, and gives the following:

Definition 3.3.24. Let

$$ {}_{\mathrm{c}}\mathrm{Eis}^{k}_{\mathbf{Z}_p,x} := \mathrm{mom}_k([q]_{*c}\mathbf{EI}(x)) \in H^{2g-1}_{\text{ét}}(S, \mathrm{TSym}^k(\mathscr{H}_{\mathbf{Z}_p})(g)). $$

The definition depends on the integer q, but we prefer to omit it from the notation. Definition 3.3.24 is again motivated by the following comparison of these integral classes with the ones of Definition 3.2.14.

Proposition 3.3.25 ([Kin15c], Theorem 7.3.3)**.** *We have*

$$ {}_{\mathrm{c}}\mathrm{Eis}^{k}_{\mathbf{Z}_p,x} = q^k({}_{\mathrm{c}}\mathrm{Eis}^{k}_{\acute{e}t,x}) \in H^{2g-1}_{\acute{e}t}(S, \mathscr{H}^{k}_{\mathbf{Q}_p}(g)). $$

Remark 3.3.26. This result answers affirmatively to the question on whether Eisenstein classes can be interpolated as the weight varies p-adically, and it is the generalisation of the elliptic curve case treated in [Kin15a]; however, there is a crucial difference between [Kin15a, Theorem 12.4.21] and [Kin15c, Theorem 7.3.3]: in the former, the integral polylogarithm class is seen to be ”motivic”: if $\pi : E \to S$ denotes an elliptic curve over a scheme S of finite type over $\mathbf{Z}$, Kings defines ${}_c\Theta$ as the inverse limit of

$$ \varprojlim_r \partial_{p^r}({}_c\theta_E) \in \varprojlim_r H^1_{\text{ét}}(E \smallsetminus E[p^r c], \mathbf{Z}/p^r\mathbf{Z}(1)) = H^1_{\text{ét}}(E \smallsetminus E[c], \mathscr{L}_{\mathbf{Z}_p}(1)). $$

Then, $\mathrm{comp}({}_{\mathrm{c}}\mathrm{pol})$ is equal to ${}_c\Theta$ as elements of $H^1_{\text{ét}}(E \smallsetminus E[c], \mathscr{L}_{\mathbf{Q}_p}(1))$. A similar description does not seem possible in the higher dimension case.

Remark 3.3.27. The phenomenon of constructing classes in the cohomology with non-trivial coefficients from trivial coefficients ones appears in various occasions. Indeed, the very construction of (higher) étale cyclotomic Soulé elements or the Euler system of Kato for modular forms of weight higher than 2 relies on p-adic deformation techniques (see [Kin15a, Definition 12.3.3], [Kat04, §8.4]).

3.3.6 Comparison between the two polylogarithm classes

The comparison of the construction of Definition 3.3.20 and the one of Definition 3.3.15 relies on the very characterisation of the two classes, as the next proposition shows.

Proposition 3.3.28. *Let c, m be as in Theorem 3.3.6. We have*

$$ N^2_m({}_{\mathrm{c}}\mathrm{pol}) = {}_c\mathscr{Z}_m. $$

Proof. This is a straight-forward consequence of the comparison of the residues at finite levels and the injectivity of the residue map by Proposition 3.3.18. Indeed, both $N_m^2({}_c\mathrm{pol})$ and ${}_c\mathscr{Z}_m$ are inverse system of classes having residues $N_m^2 D_c$ in $H^0(A[cp^r], \mathbf{Z}/p^r\mathbf{Z})$. □

Remark 3.3.29.

- This is the *integral counterpart* of a comparison between the underlying rational motivic classes of Faltings and Kings, which follows from the description of the degree 0 part of the polylogarithm on abelian schemes of [KR17], as it was briefly discussed in §3.2.4.

- There is a clear discrepancy between the construction of Kings and the one of Faltings. The étale polylogarithm class does not depend on the choice of an auxiliary integer m, because of the vanishing of the direct image sheaves of $\mathscr{L}_{\mathbf{Z}_p}$ in degree less than $2g$ ([Kin15c, Theorem 6.2.3]).

- Note that the very construction of the moment maps would allow us to define the class

$$ {}_c z_{m,q}^k := \mathrm{mom}_k([q]_* x^* {}_c\mathscr{Z}_m), $$

  which, after tensoring with $\mathbf{Q}_p$, is in $H^{2g-1}_{\text{ét}}(S, \mathscr{H}^k_{\mathbf{Q}_p}(g))$. However, this is a multiple of ${}_c\mathrm{Eis}^k_{\mathbf{Z}_p,x}$, because, by Proposition 3.3.28, $x^*{}_c\mathscr{Z}_m$ is equal to $N_m^2({}_c\mathbf{EI}(x))$.

3.4 Varying the level

We discuss how the trace compatibilities of $\mathbf{GL}_2$-Eisenstein classes generalise to the general $\mathbf{GSp}_{2g}$-case. We use the method of Scholl in [Sch98], which readily extends to the higher dimensional setting. Crucially, the trace compatibility follows from a detailed study of variation of level structures in towers. These compatibility relations play a key role in proving norm relations and in answering Hida theoretic questions as we will discuss in the next chapters.

3.4.1 Compatibility in the mira-Klingen tower

In the following we study how the Eisenstein classes of symplectic Shimura varieties vary in the tower of levels $U_1(N)$ given in Definition 2.1.20; fix a finite set of primes S coprime to N and p, and consider any sufficiently small open compact subgroup $K_S \subset \mathbf{GSp}_{2g}(\mathbf{Z}_S)$. Consider the congruence subgroup

$$ U_N := U_1(N)K_S \subset \mathbf{GSp}_{2g}(\widehat{\mathbf{Z}}). $$

Since U_N is sufficiently small, the Shimura variety $\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_N)$ is the fine moduli space of isomorphism classes of quadruples (A, λ, α, P), where A is an abelian scheme of relative dimension g, λ is a principal polarisation on A, α is a symplectic level K_S-structure and P is a point of exact order N of A. Its universal abelian scheme $\mathscr{A} = \mathscr{A}_g(U_N) \xrightarrow{\pi} \mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_N)$ comes equipped with the universal

N-torsion section e_N. We show how the Eisenstein classes behave under the trace $\mathrm{pr}_{p,*}$ of

$$\mathrm{pr}_p : \mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_{pN}) \longrightarrow \mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_N), \ (A,\lambda,\alpha,P) \mapsto (A,\lambda,\alpha,[p]P)$$

where $[p]$ denotes multiplication by p. We observe that we prove a trace compatibility in the mira-Klingen tower rather than in the Klingen tower because our Eisenstein classes, which depend on the choice of a torsion section of the universal abelian scheme of $\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_N)$, naturally live in the cohomology of $\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_N)$ (Definition 3.4.4).

First, recall the following.

Lemma 3.4.1. *The universal abelian scheme $\mathscr{A}$ represents the functor*

$$\mathscr{F} : Sch_{/\mathbf{Q}} \longrightarrow Sets, \quad S \mapsto \left\{ \begin{array}{c} \textit{isomorphism classes of } (A,\lambda,\alpha,P,s), \\ \textit{where } (A,\lambda,\alpha,P) \in \mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_N)(S) \\ \textit{and } s \in A(S) \end{array} \right\}.$$

Proof. To ease the notation, denote by Sh_g the Shimura variety $\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_N)$. Define the natural transformation $\eta : \mathscr{A}(\cdot) \longrightarrow \mathscr{F}$, by sending, for any $\mathbf{Q}$-scheme S,

$$\eta_S : \mathscr{A}(S) \to \mathscr{F}(S), \quad x \mapsto ((A,\lambda,\alpha,P),s),$$

where (A,λ,α_D,P) corresponds to $\pi(x) \in \mathrm{Sh}_g(S)$ and $s : S \xrightarrow{(x,id_S)} \mathscr{A} \times_{\mathrm{Sh}_g,\pi(x)} S \xrightarrow{\psi} A$, where the second map ψ is an isomorphism which comes from the universal property of $\mathscr{A}/\mathrm{Sh}_g$.

Note that the natural transformation η has a mutual inverse $\rho : \mathscr{F} \longrightarrow \mathscr{A}(\cdot)$, defined by sending

$$\rho_S : \mathscr{F}(S) \to \mathscr{A}(S), \quad ((A,\lambda,\alpha,P),s) \mapsto t,$$

where $t : S \to \mathscr{A}$ is the composition

$$S \xrightarrow{s} A \xrightarrow{\phi} \mathscr{A} \times_{\mathrm{Sh}_g,y} S \xrightarrow{pr_1} \mathscr{A},$$

where

- $y \in \mathrm{Sh}_g(S)$ corresponds to (A,λ,α,P);
- $\phi : A \longrightarrow \mathscr{A} \times_{\mathrm{Sh}_g,y} S$ is the isomorphism, which arises from the universal property of $\mathscr{A}/\mathrm{Sh}_g$.

Indeed, if $x \in \mathscr{A}(S)$,

$$\rho_S \circ \eta_S(x) = \rho_S(\pi(x), \psi \circ (x,id_S))) = pr_1 \circ \psi^{-1} \circ \psi \circ (x,id_S)) = x.$$

On the other hand, if $((A,\lambda,\alpha,P),s) \in \mathscr{F}(S)$,

$$\eta_S \circ \rho_S((A,\lambda,\alpha,P),s)) = \eta_S(pr_1 \circ \phi \circ s) = (\pi(pr_1 \circ \phi \circ s), \phi^{-1} \circ (pr_1 \circ \phi \circ s, id_S)),$$

which is equal to $((A,\lambda,\alpha,P),s)$ since

- $\pi(pr_1 \circ \phi \circ s) = y := (A,\lambda,\alpha,P)$ thanks to the commutativity of the diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] & \mathscr{A} \times_{\mathrm{Sh}_g, y} S \arrow[r, "pr_1"] \arrow[d, "pr_2"] & \mathscr{A} \arrow[d, "\pi"] \\ S \arrow[u, "s"] \arrow[r, "id"] & S \arrow[r, "y"] & \mathrm{Sh}_g; \end{tikzcd}$$

- $\phi^{-1} \circ (pr_1 \circ \phi \circ s, id_S) = \phi^{-1} \circ \phi \circ s = s.$

□

After much unwinding definitions, we'll see below that the compatibility under the trace of pr_p of the $\mathbf{GSp}_{2g}$-Eisenstein classes basically follows from this simple group theoretic result.

Lemma 3.4.2. *Let $(G,+)$ be an abelian group with identity element 0_G and let $x \in G$ be an element of exact order N, i.e. $Nx = 0_G$ and $Mx \neq 0_G$ for every integer $0 < M < N$. Fix a prime number p and suppose that $y \in G$ satisfies the relation $py = x$, then the following are true.*

1. *The element y has exact order either N or Np;*
2. *Suppose that $p \nmid N$. If y has exact order N and $py = x$, then y is necessarily of the form rx, for r positive integer such that*

$$rp \equiv 1 \ (mod\ N);$$

3. *Suppose that $p|N$. If $py = x$, then y has exact order Np.*

Summing all up, we get the following.

Lemma 3.4.3.

1. *Suppose that $p \nmid N$. We have*

$$\begin{tikzcd} \mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_N) \sqcup \mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_{Np}) \arrow[r, "(re_N, f)"] \arrow[d, "(id, \mathrm{pr}_p)"] & \mathscr{A} \arrow[d, "{[p]}"] \\ \mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_N) \arrow[r, "e_N"] & \mathscr{A} \end{tikzcd} \tag{3.2}$$

is Cartesian, where $f = (id \times \mathrm{pr}_p) \circ e_{Np}$ and r is the inverse of p modulo N.

2. *In the case when $p \mid N$, the diagram*

$$\begin{tikzcd} \mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_{Np}) \arrow[r, "(id\times \mathrm{pr}_p)\circ e_{Np}"] \arrow[d, "\mathrm{pr}_p"] & \mathscr{A} \arrow[d, "{[p]}"] \\ \mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_N) \arrow[r, "e_N"] & \mathscr{A} \end{tikzcd} \tag{3.3}$$

is Cartesian.

Proof. The proof is similar to the one in the case of modular curves, as in [Sch98, Lemma 2.3.1]; note that, by Yoneda lemma, it is enough to check that for every $\mathbf{Q}$-scheme S the diagram evaluated at S-points is Cartesian in the category of Sets.

By Lemma 3.4.1, for any $\mathbf{Q}$-scheme S, $\mathscr{A}(S)$ is the set of pairs (a,P), where $a \in \mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_N)(S)$ and P is a point in the abelian scheme A_a/S defined by a. Denote by $\bullet_a$ the base-change of $\bullet \in \mathscr{A}(\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_N))$ to $A_a(S)$. We treat the two cases in the statement of the lemma separately.

1. Let $p \nmid N$. The morphism e_N sends $a \in \mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_N)(S)$ to $(a, e_{N,a}) \in \mathscr{A}(S)$. By Lemma 3.4.2, the pre-image of $e_{N,a}$ under multiplication by p has either the exact order N or Np. In the first case, the pre-image of $(a, e_{N,a})$ is of the form $(a, re_{N,a})$, with r the inverse of p modulo N, hence it defines a point in $\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_N)(S)$. In the second, it will be of the form $(a, re_{N,a}+y)$, for y point of exact order p of A_a, and, using the universal property of e_{pN}, we get a S-point of $\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_{Np})$.

2. As above, the morphism e_N sends $a \mapsto (a, e_{N,a}) \in \mathscr{A}(S)$. By Lemma 3.4.2, its pre-image under multiplication by p has necessarily exact order Np, thus it defines an S-point of $\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_{Np})$.

□

3.4.2 The distribution relations

We now finally state the distribution relations under the trace $\mathrm{pr}_{p,*}$ of the Eisenstein classes for $\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_N)$:

Definition 3.4.4. Let $\mathscr{H}_{\mathbf{Q}}^{\kappa}$ denote the Chow motive over $\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_N)$ associated to the $\mathbf{GSp}_{2g}$-representation $\mathrm{Sym}^{\kappa}(\mathrm{Std}) \otimes \nu^{-\kappa}$ and denote by

$$_{\mathrm{c}}\mathrm{Eis}^{\kappa}_{g,N} := {}_{\mathrm{c}}\mathrm{Eis}^{\kappa}_{e_N} \in H^{2g-1}_{\mathrm{mot}}(\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_N), \mathscr{H}_{\mathbf{Q}}^{\kappa}(g)),$$

where $_{\mathrm{c}}\mathrm{Eis}^{\kappa}_{e_N}$ is the class introduced in Definition 3.2.14.

Remark 3.4.5. Notice that, by our choice of normalisation of Ancona's functor in Proposition 2.2.4, $\mathscr{H}_{\mathbf{Q}}^{\kappa} = \mathrm{Sym}^{\kappa}(h^1(\mathscr{A})^{\vee})$.

We start by showing the distribution relations for ${}_{\mathrm{c}}\mathrm{Eis}^0_{g,N}$, which, by Proposition 3.2.13, is the class $e_N^*({}_cz)$.

Proposition 3.4.6. *We have*

$$\mathrm{pr}_{p,*}({}_{\mathrm{c}}\mathrm{Eis}^0_{g,Np}) = \begin{cases} {}_{\mathrm{c}}\mathrm{Eis}^0_{g,N} & \text{if } p \mid N; \\ {}_{\mathrm{c}}\mathrm{Eis}^0_{g,N} - d_p^*({}_{\mathrm{c}}\mathrm{Eis}^0_{g,N}) & \text{if } p \nmid N; \end{cases}$$

where $d_p \in \mathbf{GSp}_{2g}(\widehat{\mathbf{Z}})$ is any matrix congruent to $\left(\begin{smallmatrix} I & 0 \\ 0 & pI \end{smallmatrix}\right)$ modulo N.

Proof. The computation follows by using Lemma 3.4.3 and the fact that the Eisenstein classes are invariant under trace maps $[p]_*$ (e.g. Proposition 3.2.11) and under base-change (e.g. [Lem16, Proposition 2.11]). To help the reader follow each passage in the proof, we denote by ${}_cz^{\mathscr{A}_g(U_N)}$ the class of the universal abelian scheme of $\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_N)$. Let $p \mid N$; the Cartesianness of diagram (3.3) gives

$$\begin{aligned} \mathrm{pr}_{p,*}({}_{\mathrm{c}}\mathrm{Eis}^0_{g,pN}) &= \mathrm{pr}_{p,*}\left(e_{pN}^*\left({}_cz^{\mathscr{A}_g(U_{pN})}\right)\right) \\ &= \mathrm{pr}_{p,*}\left(e_{pN}^*\left((id \times \mathrm{pr}_p)^*\left({}_cz^{\mathscr{A}_g(U_N)}\right)\right)\right) \\ &= e_N^*\left([p]_*\left({}_cz^{\mathscr{A}_g(U_N)}\right)\right) && \text{by (3.3)} \\ &= e_N^*\left({}_cz^{\mathscr{A}_g(U_N)}\right) \\ &= {}_{\mathrm{c}}\mathrm{Eis}^0_{g,N}, \end{aligned}$$

where the second equality follows form the compatibility of the class under base-change and the second last from the invariance of ${}_cz^{\mathscr{A}_g(U_N)}$ under $[p]_*$.
In the case of p and N coprime, we use the Cartesianness of the diagram (3.2) to deduce

$$\begin{aligned} (id, \mathrm{pr}_p)_*\left((re_N, f)^*\left({}_cz^{\mathscr{A}_g(U_N)}\right)\right) &= e_N^*\left([p]_*\left({}_cz^{\mathscr{A}_g(U_N)}\right)\right) \\ &= e_N^*\left({}_cz^{\mathscr{A}_g(U_N)}\right) \\ &= {}_{\mathrm{c}}\mathrm{Eis}^0_{g,N}\,. \end{aligned}$$

Moreover, the left hand side of the equation above is just

$$(re_N)^*\left({}_cz^{\mathscr{A}_g(U_N)}\right) + \mathrm{pr}_{p,*}\left(e_{Np}^*(id \times \mathrm{pr}_p)^*\left({}_cz^{\mathscr{A}_g(U_N)}\right)\right) = (d_p)^*\, {}_{\mathrm{c}}\mathrm{Eis}^0_{g,N} + \mathrm{pr}_{p,*}\left({}_{\mathrm{c}}\mathrm{Eis}^0_{g,Np}\right),$$

where d_p is a matrix in $\mathbf{GSp}_{2g}(\widehat{\mathbf{Z}})$ whose reduction modulo N is of the form $\left(\begin{smallmatrix} I & 0 \\ 0 & pI \end{smallmatrix}\right)$. $\square$

Remark 3.4.7. This proposition is the higher dimension analogue of the result of Kato for Siegel units (appearing in the proof of [Kat04, Props 2.3-4]), which was the subject of Proposition 3.1.9.

By the same method, we get compatibility relations for ${}_{\mathrm{c}}\mathrm{Eis}^{\kappa}_{g,N} \in H^{2g-1}_{\mathrm{mot}}(\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_N), \mathscr{H}^{\kappa}_{\mathbf{Q}}(g))$. According to §2.2.5, we define $\mathrm{pr}_{p,*}$ to be the composition of $\mathrm{Tr}_{\mathrm{pr}_p}$ with $[p]_* : \mathscr{H}^{\kappa}_{\mathbf{Q}} \to \mathrm{pr}_p^*(\mathscr{H}^{\kappa}_{\mathbf{Q}})$. Then, we have the following.

Proposition 3.4.8. *We have*

$$\mathrm{pr}_{p,*}({}_{\mathrm{c}}\mathrm{Eis}^{\kappa}_{g,pN}) = \begin{cases} {}_{\mathrm{c}}\mathrm{Eis}^{\kappa}_{g,N} & \text{if } p \mid N; \\ {}_{\mathrm{c}}\mathrm{Eis}^{\kappa}_{g,N} - p^{\kappa} d_p^*({}_{\mathrm{c}}\mathrm{Eis}^{\kappa}_{g,N}) & \text{if } p \nmid N; \end{cases}$$

where $d_p \in \mathbf{GSp}_{2g}(\widehat{\mathbf{Z}})$ is any matrix which reduces modulo N to $\left(\begin{smallmatrix} I & 0 \\ 0 & pI \end{smallmatrix}\right)$ modulo N.

Proof. The proof of the statement goes as before, with the only difference, in the case of $p \nmid N$, being the action of $[p]_*$ on our coefficients, which is given by multiplication by p^{κ}. Indeed, by the Cartesian diagram (3.2), we get

$$(\mathrm{Tr}_{\mathrm{pr}_p} \circ [p]_*) \left({}_{\mathrm{c}}\mathrm{Eis}^{\kappa}_{g,pN}\right) + [p]_*(d_p)^* \, {}_{\mathrm{c}}\mathrm{Eis}^{\kappa}_{g,N} = {}_{\mathrm{c}}\mathrm{Eis}^{\kappa}_{g,N},$$

and thus the desired formula since $[p]_*$ acts by multiplication by p on $\mathscr{H}_{\mathbf{Q}}$. □

Remark 3.4.9. Let p be any prime. Note that the method of [KLZ17, Theorem 4.3.3] readily applies to the p-adic realisation of $\mathrm{Eis}^{\kappa}_{g,N}$ and its integral counterpart: denote by

$${}_{\mathrm{c}}\mathrm{Eis}^{\kappa}_{\mathbf{Z}_p,g,N} := {}_{\mathrm{c}}\mathrm{Eis}^{\kappa}_{\mathbf{Z}_p,e_N} \in H^{2g-1}_{\text{ét}}(\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_N), \mathrm{TSym}^{\kappa}(\mathscr{H}_{\mathbf{Z}_p})(g)),$$

the class of Definition 3.3.24. We have

$$\mathrm{pr}_{p,*}({}_{\mathrm{c}}\mathrm{Eis}^{\kappa}_{\mathbf{Z}_p,g,Np}) = \begin{cases} {}_{\mathrm{c}}\mathrm{Eis}^{\kappa}_{\mathbf{Z}_p,g,N} & \text{if } p \mid N; \\ {}_{\mathrm{c}}\mathrm{Eis}^{\kappa}_{\mathbf{Z}_p,g,N} - p^{\kappa} d_p^*({}_{\mathrm{c}}\mathrm{Eis}^{\kappa}_{\mathbf{Z}_p,g,N}) & \text{if } p \nmid N; \end{cases}$$

where $d_p \in \mathbf{GSp}_{2g}(\widehat{\mathbf{Z}})$ is as in Proposition 3.4.8 and $\mathrm{pr}_{p,*}$ denotes the trace map of pr_p in étale cohomology.

3.4.3 Consequences of the distribution relations

3.4.3.1 Distribution relations for push-forward classes

Proposition 3.4.6 gives immediately a trace-compatibility relation for the push-forward of Eisenstein classes. Let H, G be reductive groups over $\mathbf{Q}$ and fix Shimura data (G,X) and $(H_g := \mathbf{GSp}_{2g} \boxtimes H, Y)$ of PEL-type such that there is an embedding

$$\iota : (H_g, Y) \hookrightarrow (G, X).$$

We allow H to be trivial in the sense that with (H_g, Y) we simply mean the symplectic Shimura datum for $\mathbf{GSp}_{2g}$ introduced in §2.1.3.2. The morphism ι induces morphisms

$$\iota_K : \mathrm{Sh}_{H_g}(K \cap H_g) \longrightarrow \mathrm{Sh}_G(K),$$

for $K \subset G(\mathbf{A}_f)$ sufficiently small open compact subgroup.

Now, choose a family of sufficiently small open compact subgroups $K_{p^n} \subset G(\widehat{\mathbf{Z}})$, for $n \geq 0$, which satisfies the following:

1. The pull-back of each $K_{p^n} \cap H_g = U_1^n \boxtimes U_2$ such that its p-component is

$$U_1(p^n) \boxtimes H(\mathbf{Z}_p).$$

2. The maps

$$\iota_n : \mathrm{Sh}_{H_g}(K_{p^n} \cap H_g) \longrightarrow \mathrm{Sh}_G(K_{p^n})$$

are closed immersions of codimension d.

Remark 3.4.10. These conditions ensure that we have a diagram

$$\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_1^n) \xleftarrow{\pi_{1,n}} \mathrm{Sh}_{H_g}(K_{p^n} \cap H_g) \xhookrightarrow{\iota_n} \mathrm{Sh}_G(K_{p^n}).$$

Moreover, if V is an algebraic representation of G such that we have

$$D^{\kappa,0} := D^{\kappa} \boxtimes W^0 \hookrightarrow V_{|H_g},$$

for W^0 the trivial representation of H_g and D^κ the $\mathbf{GSp}_{2g}$-representation $\mathrm{Sym}^\kappa(\mathrm{Std}) \otimes \nu^{-\kappa}$, then we obtain Gysin morphisms

$$\iota_{n,*}^{\kappa} : H^{2g-1}_{\mathrm{mot}}(\mathrm{Sh}_{H_g}(K_{p^n} \cap H_g), \mathscr{D}^{\kappa,0}_{\mathbf{Q}}(g)) \longrightarrow H^{2(g+d)-1}_{\mathrm{mot}}(\mathrm{Sh}_G(K_{p^n}), \mathscr{V}_{\mathbf{Q}}(g+d)),$$

where $\mathscr{D}^{\kappa,0}_{\mathbf{Q}}$, $\mathscr{V}_{\mathbf{Q}}$ are images under Ancona's functor of $D^{\kappa,0}$ and V.

We can then define elements

$$z^{V,\kappa}_{G,n} := \iota^{\kappa}_{n,*} \circ \pi^*_{1,n}\left({}_{c}\mathrm{Eis}^{\kappa}_{x+e_n}\right) \in H^{2(g+d)-1}_{\mathrm{mot}}(\mathrm{Sh}_G(K_{p^n}), \mathscr{V}_{\mathbf{Q}}(g+d)),$$

where $x \in \mathscr{A}_g(\mathrm{Sh}_{\mathbf{GSp}_{2g}}(pr_1(K_1 \cap H_g)))$ is a suitable torsion point of order an auxiliary integer M coprime to p.

Corollary 3.4.11. *Let $n > 1$ and let τ_n denote the natural degeneracy morphism* $\mathrm{Sh}_G(K_{p^n}) \to$

$\mathrm{Sh}_G(K_{p^{n-1}})$*; then, we have*

$$\tau_{n,*}(z^{V,\kappa}_{G,n}) = z^{V,\kappa}_{G,n-1}.$$

Proof. Note that by the choice of K_{p^n}, we have a Cartesian diagram

$$\begin{tikzcd}
\mathrm{Sh}_{H_g}(U_1^n \boxtimes U_2) \arrow[r, two heads, "\pi_{1,n}"] \arrow[d, "\mathrm{pr}'_n"'] & \mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_1^n) \arrow[d, "\mathrm{pr}_n"] \\
\mathrm{Sh}_{H_g}(U_1^{n-1} \boxtimes U_2) \arrow[r, two heads, "\pi_{1,n-1}"] & \mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_1^{n-1}),
\end{tikzcd}$$

where pr_n and pr'_n are the natural degeneracy maps. This allows us to translate the trace compatibility relation of $\left(\mathrm{Eis}^{\kappa}_{g,p^n}\right)_{n>0}$ of Proposition 3.4.8 to a trace compatibility under $\mathrm{pr}'_{n,*}$ of $\left(\pi^*_{1,n}\left(\mathrm{Eis}^{\kappa}_{g,p^n}\right)\right)_{n>0}$. Thus, the result follows from the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Sh}_{H_g}(K_{p^n} \cap H_g) \arrow[r, "\iota_n"] \arrow[d, "\mathrm{pr}'_n"'] & \mathrm{Sh}_G(K_{p^n}) \arrow[d, "\tau_n"] \\
\mathrm{Sh}_{H_g}(K_{p^{n-1}} \cap H_g) \arrow[r, "\iota_{n-1}"] & \mathrm{Sh}_G(K_{p^{n-1}}).
\end{tikzcd}$$

□

A similar statement holds for the case of $n = 0$.

Corollary 3.4.12. *Let τ_1 denote the natural degeneracy morphism $\mathrm{Sh}_G(K_p) \to \mathrm{Sh}_G(K_1)$; then, we have*

$$\tau_{1,*}(z^{V,\kappa}_{G,1}) = (1 - p^{\kappa} d_p^*) z^{V,\kappa}_{G,0},$$

where $d_p \in H_g(\widehat{\mathbf{Z}})$ is any matrix whose $\mathbf{GSp}_{2g}$-component reduces to $\left(\begin{smallmatrix} I & 0 \\ 0 & pI \end{smallmatrix}\right)$ modulo M.

Proof. The proof is identical to the one of Corollary 3.4.11 and follows from the $p \nmid M$ case of Proposition 3.4.6. □

Remark 3.4.13.

- Corollaries 3.4.11 and 3.4.12 are automatically true in the integral p-adic étale setting;
- Corollaries 3.4.11 and 3.4.12 recover the *first norm relation* of the Beilinson-Flach elements ([LLZ14, Theorem 3.1.1]). One can treat similarly the case of push-forward of cup-products of classes, recovering the level trace compatibility of the $\mathbf{GSp}_4$-Euler system ([LSZ17, Theorem 8.3.2(i)]).

3.4.3.2 Λ-adic Eisenstein classes for $\mathbf{GSp}_{2g}$

As as second direct application, we show how to use the compatibility of Eisenstein classes in the mira-Klingen tower to compare Faltings' classes with Kings' Eisenstein-Iwasawa classes when the base scheme S is an integral model of the $\mathbf{GSp}_{2g}$ Shimura variety. This section should be regarded as a continuation of §3.3.6.

Let U_{p^r} be the sufficiently small open compact subgroup $U_1(p^r)K_S \subset \mathbf{GSp}_{2g}(\widehat{\mathbf{Z}})$; we denote by $\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_{p^r})$ the integral model over $\mathbf{Z}[\frac{1}{dp}]$, for an auxiliary integer d coprime to p, of the corresponding $\mathbf{Q}$-scheme. Denote by $\mathscr{A}(U_{p^r})/\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_{p^r})$ its universal abelian scheme, which is equipped with the universal p^r-torsion section e_{p^r}. Associated to $\mathscr{A}(U_{p^r})$ and e_{p^r}, we have the étale class (Definition 3.3.9)

$$ {}_c z^g_{m,p^r} := e^*_{p^r}\left({}_c z_m\right) \in H^{2g-1}_{\text{ét}}(\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_{p^r}), \mathbf{Z}_p(g)), $$

Lemma 3.4.14. *Let $\pi_{p,*}$ be the trace map of the natural degeneracy map $\pi_p : \mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_{p^{r+1}}) \longrightarrow \mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_{p^r})$. If $r \geq 1$, then*

$$ \pi_{p,*}({}_c z^g_{m,p^{r+1}}) = {}_c z^g_{m,p^r}. $$

Proof. The proof is identical to the one of Proposition 3.4.6. Indeed, it follows from Lemma 2.1.25 and the Cartesianness of

$$\begin{tikzcd} \mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_{p^{r+1}}) \arrow[r, "(id\times\pi_p)\circ e_{p^{r+1}}"] \arrow[d, "\pi_p"'] & \mathscr{A}(U_{p^r}) \arrow[d, "{[p]}"] \\ \mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_{p^r}) \arrow[r, "e_{p^r}"] & \mathscr{A}(U_{p^r}). \end{tikzcd}$$

□

Remark 3.4.15. By the previous lemma, we have

$$ {}_c z^g_{m,p^\infty} := ({}_c z^g_{m,p^r})_{r\geq 1} \in \varprojlim_{r\geq 1} H^{2g-1}_{\text{ét}}(\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_{p^r}), \mathbf{Z}_p(g)), $$

where the inverse limit is taken with respect to the trace maps $\pi_{p,*}$.

The class ${}_c z^g_{m,p^\infty}$ can be directly related to Kings' Eisenstein-Iwasawa class ${}_c\mathbf{EI}(e_p) := e_p^*({}_c\mathrm{pol})$ associated to $\mathscr{A}(U_p)$. Indeed, we get the following generalisation of [KLZ17, Theorem 4.5.1(1)].

Theorem 3.4.16. *There is an isomorphism*

$$ H^{2g-1}_{\acute{e}t}(\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_p), \Lambda(\mathscr{H}_{\mathbf{Z}_p}\langle e_p\rangle)(g)) \simeq \varprojlim_{r\geq 1} H^{2g-1}_{\acute{e}t}(\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_{p^r}), \mathbf{Z}_p(g)), $$

where the inverse limit is with respect to the trace map of the natural degeneracy map

$$ \pi_p : \mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_{p^{r+1}}) \longrightarrow \mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_{p^r}). $$

Moreover, under this isomorphism the Λ-adic class $e^*_{p\,c}\mathscr{Z}_m$ is mapped to ${}_c z^g_{m,p^\infty}$. In particular, we *have*

$$ {}_c z^g_{m,p^\infty} = N^2_m({}_c\mathbf{EI}(e_p)). $$

Proof. The proof is very similar to the one of [KLZ17, Theorem 4.5.1(1)]. Let $\mathscr{A} := \mathscr{A}(U_p)$. Note that there is an isomorphism of schemes $f_r : \mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_{p^{r+1}}) \simeq \mathscr{A}[p^r]\langle e_p\rangle$ such that

$$
\begin{tikzcd}
\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_{p^{r+1}}) \arrow[r, "f_r"] \arrow[d, "\pi_p^r"] & \mathscr{A}[p^r]\langle e_p\rangle \arrow[d, "pr_{r,e_p}"'] \\
\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_p) \arrow[r, equal] & \mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_p).
\end{tikzcd}
$$

This follows from the very definition of $\mathscr{A}[p^r]\langle e_p\rangle$. Indeed, a point of $\mathscr{A}[p^r]\langle e_p\rangle$ over $(A,\lambda,\alpha,P) \in \mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_p)$ is given by a point Q of order p^{r+1} of A such that $[p^r]Q = P$. Hence, f_r is the isomorphism defined by sending

$$ (A,\lambda,\alpha,Q) \mapsto ((A,\lambda,\alpha,[p^r]Q),Q). $$

The isomorphism f_r induces

$$ H^{2g-1}_{\text{ét}}(\mathscr{A}[p^r]\langle e_p\rangle, \mathbf{Z}_p(g)) \simeq H^{2g-1}_{\text{ét}}(\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_{p^{r+1}}), \mathbf{Z}_p(g)), $$

for all $r \geq 0$, and, passing to the limit, we get the desired isomorphism.

Moreover, under f_r, the morphism $\mathscr{A}[p^r]\langle e_p\rangle \longrightarrow \mathscr{A}$ corresponds to the universal p^{r+1} torsion section $e_{p^{r+1}}$ of $\mathrm{Sh}_{\mathbf{GSp}_{2g}}(U_{p^{r+1}})$. This shows that ${}_c z^g_{m,p^{r+1}}$ corresponds to the restriction of ${}_c z_m$ to $\mathscr{A}[p^r]\langle e_p\rangle$. Thus,

$$ e_p^*({}_c\mathscr{Z}_m) = {}_c z^g_{m,p^\infty}. $$

Finally, the equality

$$ {}_c z^g_{m,p^\infty} = N^2_m({}_c\mathbf{EI}(e_p)) $$

follows from Proposition 3.3.28. □

Chapter 4

Towards an Euler system for $\mathbf{GSp}_6$

In what follows, we describe the joint work with Joaquín Rodrigues Jacinto on the construction of global cohomology classes in the middle degree cohomology of the Shimura variety $\mathrm{Sh}_{\mathbf{GSp}_6}$ of the symplectic group $\mathbf{GSp}_6$ compatible when one varies the level at p. They arise as push-forward of elements in the first cohomology group of the Shimura variety for $\mathbf{GL}_2 \times_{\det} \mathbf{GL}_2 \times_{\det} \mathbf{GL}_2$. We show how these classes satisfy Euler system norm relations in the cyclotomic tower at p, and thus provide elements in the Iwasawa cohomology of Galois representations appearing in the middle degree cohomology of the Shimura variety.
The chapter is organised as follows.
In §4.1, we discuss basic properties of the Shimura varieties and establish branching laws for the restriction of algebraic representations of $\mathbf{GSp}_6$ to $\mathbf{GL}_2 \times_{\det} \mathbf{GL}_2 \times_{\det} \mathbf{GL}_2$, which is needed to construct elements in the cohomology with non-trivial coefficients. In §4.2 and §4.4, we define the motivic and étale classes in the cohomology of $\mathrm{Sh}_{\mathbf{GSp}_6}$ and prove their norm compatibility relations. In §4.3, we finally construct the elements in the Iwasawa cohomology of Galois representations appearing in the middle degree cohomology of $\mathrm{Sh}_{\mathbf{GSp}_6}$.

4.1 Preliminaries

4.1.1 Groups

Let

$$\mathbf{H} = \mathbf{GL}_2 \times_{\det} \mathbf{GL}_2 \times_{\det} \mathbf{GL}_2 = \{(A,B,C) \,:\, A,B,C \in \mathbf{GL}_2, \det A = \det B = \det C\}$$

be the group scheme over $\mathbf{Z}$ obtained by taking the product over the determinant of three copies of $\mathbf{GL}_2$, and denote by $\mathbf{G}$ the group scheme $\mathbf{GSp}_6$ over $\mathbf{Z}$. Recall that its R-points are

$$\mathbf{G}(R) = \mathbf{GSp}_6(R) = \{A \in \mathbf{GL}_6(R) \,:\, A^t J A = \nu(A) J,\ \nu(A) \in \mathbf{G}_\mathrm{m}(R)\},$$

for any commutative ring R with 1, where J is the matrix $\left(\begin{smallmatrix} 0 & I_3' \\ -I_3' & 0 \end{smallmatrix}\right)$, for $I_3' = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$. In the following, we will consider $\mathbf{H}$ as a subgroup of $\mathbf{G}$ through the embedding defined by

$$\Delta\colon \left(\left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right), \left(\begin{smallmatrix} a_3 & b_3 \\ c_3 & d_3 \end{smallmatrix}\right)\right) \in \mathbf{H} \mapsto \begin{pmatrix} a_1 & & & & & b_1 \\ & a_2 & & & b_2 & \\ & & a_3 & b_3 & & \\ & & c_3 & d_3 & & \\ & c_2 & & & d_2 & \\ c_1 & & & & & d_1 \end{pmatrix} \in \mathbf{G}$$

We denote by $Z_{\mathbf{H}}$ and $Z_{\mathbf{G}}$ the centers of $\mathbf{H}$ and $\mathbf{G}$ respectively.

4.1.2 Shimura varieties

Recall that the Shimura variety $\mathrm{Sh}_{\mathbf{GL}_2}$ is associated to the Shimura datum $(\mathbf{GL}_2, X_{\mathbf{GL}_2})$, where $X_{\mathbf{GL}_2}$ is the set of $\mathbf{GL}_2(\mathbf{R})$-conjugacy classes of

$$h_{\mathbf{GL}_2} := \mathbb{S} \to \mathbf{GL}_{2/\mathbf{R}}, \quad a+ib \mapsto \tfrac{1}{a^2+b^2}\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right).$$

The diagonal embedding $\mathbf{GL}_2 \to \mathbf{H}$ induces a Shimura datum $(\mathbf{H}, X_{\mathbf{H}})$; denote by $\mathrm{Sh}_{\mathbf{H}}$ the corresponding Shimura variety $\mathrm{Sh}_{\mathbf{H}}$. If $U \subseteq \mathbf{H}(\mathbf{A}_f)$ is a fibre product $U_1 \times_{det} U_2 \times_{det} U_3$ of subgroups of $\mathbf{GL}_2(\mathbf{A}_f)$, we have

$$\mathrm{Sh}_{\mathbf{H}}(U) = \mathrm{Sh}_{\mathbf{GL}_2}(U_1) \times_{\mathbf{G}_{\mathrm{m}}} \mathrm{Sh}_{\mathbf{GL}_2}(U_2) \times_{\mathbf{G}_{\mathrm{m}}} \mathrm{Sh}_{\mathbf{GL}_2}(U_3),$$

where $\times_{\mathbf{G}_{\mathrm{m}}}$ denotes the fibre product over the zero dimensional Shimura variety of level $D = det(U_i)$

$$\pi_0(\mathrm{Sh}_{\mathbf{GL}_2})(D) = \hat{\mathbf{Z}}^*/D$$

given by the connected components of $\mathrm{Sh}_{\mathbf{GL}_2}$. Finally, let $(\mathbf{G}, X_{\mathbf{G}})$ be the Shimura datum defined in §2.1.3.2; the embedding Δ induces an inclusion of Shimura data $(\mathbf{H}, X_{\mathbf{H}}) \hookrightarrow (\mathbf{G}, X_{\mathbf{G}})$ with corresponding embedding $\mathrm{Sh}_{\mathbf{H}} \hookrightarrow \mathrm{Sh}_{\mathbf{G}}$ of codimension 3. For any open compact subgroup U of $\mathbf{G}(\mathbf{A}_f)$, we have

$$\iota_U : \mathrm{Sh}_{\mathbf{H}}(U \cap \mathbf{H}) \longrightarrow \mathrm{Sh}_{\mathbf{G}}(U).$$

For sufficiently small level groups, all the Shimura varieties defined above, as well as the morphisms between them, admit canonical models over $\mathbf{Q}$. The following lemma is an adaptation of [LSZ17, Lemma 5.3.1].

Lemma 4.1.1. *Let U be an open compact subgroup of $\mathbf{G}(\mathbf{A}_f)$ such that there exists a sufficiently small open compact subgroup U' of $\mathbf{G}(\mathbf{A}_f)$ containing U, $w_1 U w_1$ and $w_2 U w_2$, where $w_1 = diag(-1,1,1,1,1,-1)$ and $w_2 = diag(1,-1,1,1,-1,1)$. Then the morphism (of $\mathbf{Q}$-schemes)*

$$\iota_U : \mathrm{Sh}_{\mathbf{H}}(U \cap \mathbf{H}) \longrightarrow \mathrm{Sh}_{\mathbf{G}}(U)$$

is a closed immersion.

Proof. We note that it is enough to show it on the complex points of the Shimura varieties. As it was pointed out before, the map at infinite level $\mathrm{Sh}_{\mathbf{H}}(\mathbf{C}) \longrightarrow \mathrm{Sh}_{\mathbf{G}}(\mathbf{C})$ is an injection, hence we need to show that if $z, z' \in \mathrm{Sh}_{\mathbf{H}}(\mathbf{C})$ have the same image in $\mathrm{Sh}_{\mathbf{G}}(U)(\mathbf{C})$, then $z = z'u$ for $u \in U \cap \mathbf{H}$. This would follow by showing that for any $u \in U \setminus (U \cap \mathbf{H})$, we have $\mathrm{Sh}_{\mathbf{H}}(\mathbf{C}) \cap \mathrm{Sh}_{\mathbf{H}}(\mathbf{C})u = \emptyset$ as subsets of $\mathrm{Sh}_{\mathbf{G}}(\mathbf{C})$.

We show the latter as follows. The quotient $W = Z_{\mathbf{H}}/(\mathbf{H} \cap Z_{\mathbf{G}})$ is generated by the two involutions w_1 and w_2. An easy calculation shows that the centraliser $C_{\mathbf{G}(\mathbf{A}_f)}(\{w_1, w_2\})$ is $\mathbf{H}(\mathbf{A}_f)$. Note that the action of w_1 and w_2 on $\mathrm{Sh}_{\mathbf{G}}(\mathbf{C})$ fixes $\mathrm{Sh}_{\mathbf{H}}(\mathbf{C})$ pointwise. Thus, if $z, zu \in \mathrm{Sh}_{\mathbf{H}}(\mathbf{C})$ for $u \in U$, the elements $v_1 = u(w_1 u^{-1} w_1)$ and $v_2 = u(w_2 u^{-1} w_2)$ fix z. By hypothesis $v_1, v_2 \in U'$, which acts faithfully on $\mathrm{Sh}_{\mathbf{G}}(\mathbf{C})$, thus we conclude that $v_1 = v_2 = 1$. This implies that u centralizes the subgroup generated by w_1 and w_2 and hence $u \in U \cap \mathbf{H}$, which completes the proof. $\square$

Remark 4.1.2. Let $K_G(d)$ denote the kernel of reduction modulo d of $G(\hat{\mathbf{Z}}) \to G(\mathbf{Z}/d\mathbf{Z})$, for $G \in \{\mathbf{GL}_2, \mathbf{G}\}$. If $U \subseteq K_G(d)$ for some $d \geq 3$, then the hypotheses of the lemma are satisfied with $U' = K_G(d)$.

We recall that both $\mathrm{Sh}_{\mathbf{GL}_2}$ and $\mathrm{Sh}_{\mathbf{G}}$ admit a description as moduli spaces of abelian schemes: given sufficiently small open compact subgroups $V \subseteq \mathbf{GL}_2(\mathbf{A}_f)$ and $U \subseteq \mathbf{G}(\mathbf{A}_f)$, $\mathrm{Sh}_{\mathbf{GL}_2}(V)$ is the moduli of (isomorphism classes of) elliptic curves with V-level structure, while $\mathrm{Sh}_{\mathbf{G}}(U)$ parametrises (isomorphism classes of) principally polarised abelian schemes of relative dimension 3 and U-level structure.

Finally, we recall that, for $g \in \mathbf{G}(\mathbf{A}_f)$ and U sufficiently small, we have a map of schemes over $\mathbf{Q}$

$$g : \mathrm{Sh}_{\mathbf{G}}(U) \to \mathrm{Sh}_{\mathbf{G}}(g^{-1}Ug)$$

given by $g \cdot [(z,h)] = [(z,hg)]$. For $g \in \mathbf{G}(\mathbf{A}_f)$, we denote by ι_U^g the composition

$$\mathrm{Sh}_{\mathbf{H}}(gUg^{-1} \cap \mathbf{H}) \xrightarrow{\iota_{gUg^{-1}}} \mathrm{Sh}_{\mathbf{G}}(gUg^{-1}) \xrightarrow{g} \mathrm{Sh}_{\mathbf{G}}(U) .$$

Remark 4.1.3. For U equal to the kernel of reduction modulo d, U-level structures of an abelian scheme A correspond to bases of the d-torsion points of A. Note that the right-translation action of $g \in \mathbf{GL}_2(\widehat{\mathbf{Z}})$ (or $\mathbf{G}(\widehat{\mathbf{Z}})$) on the variety corresponds, at the level of moduli spaces, to the map $g\colon (A, \lambda, \{e_i\}) \to (A, \lambda, \{e_i'\})$, where $(e_i') = g^{-1} \cdot (e_i)$, where $\{e_i\}$ forms a basis of the d-torsion points for A.

4.1.3 Level structures

We introduce next several level structures that we will be using throughout. The reader is urged to skip this section and come back as the situation demands.

Definition 4.1.4. Let $K^{(p)} \subset \mathbf{G}(\widehat{\mathbf{Z}}^{(p)})$ be a compact open subgroup satisfying the hypotheses of Lemma 4.1.1. For any $n \in \mathbf{N}$, let $K_n := K^{(p)} U_{1,\mathbf{G}}(p^n) \subseteq \mathbf{G}(\widehat{\mathbf{Z}})$, for

$$U_{1,\mathbf{G}}(p^n) := \{g \in \mathbf{G}(\mathbf{Z}_p) : \quad R_6(g) \equiv (0,\dots,0,1) \bmod p^n\},$$

where $R_6(g)$ denotes the sixth row of g.

For any $n \in \mathbf{N}$, we let $K_1(n) = \mathrm{pr}_1(K_n \cap \mathbf{H}) \subset \mathbf{GL}_2(\widehat{\mathbf{Z}})$. Observe that its component at p is given by

$$U_{1,\mathbf{GL}_2}(p^n) := \{g \in \mathbf{GL}_2(\mathbf{Z}_p) \mid g \equiv I \bmod \left[\begin{smallmatrix} 1 & 1 \\ p^n & p^n \end{smallmatrix}\right]\}.$$

We will always assume that $K_1(n)$ is a sufficiently small compact open subgroup of $\mathbf{GL}_2(\widehat{\mathbf{Z}})$.

Remark 4.1.5.

- Note that, at p, the level group $K_n \cap \mathbf{H}$ has component

$$U_{1,\mathbf{GL}_2}(p^n) \boxtimes \mathbf{GL}_2(\mathbf{Z}_p) \boxtimes \mathbf{GL}_2(\mathbf{Z}_p).$$

- If $K^{(p)} \times \mathbf{G}(\mathbf{Z}_p) = K_{\mathbf{G}}(d)$ for some integer $d \geq 3$ coprime to p, then K_n and $K_1(n) = (\mathbf{GL}_2(\widehat{\mathbf{Z}}^{(p)}) \times U_{1,\mathbf{GL}_2}(p^n)) \cap K_{\mathbf{GL}_2}(d)$ are sufficiently small.

- By Lemma 4.1.1, ι_{K_n} is a closed immersion and we get

$$\mathrm{Sh}_{\mathbf{GL}_2}(K_1(n)) \xleftarrow{pr_1} \mathrm{Sh}_{\mathbf{H}}(K_n \cap \mathbf{H}) \xrightarrow{\iota_{K_n}} \mathrm{Sh}_{\mathbf{G}}(K_n).$$

  This diagram will be fundamental in the definition of the motivic classes underlying our Euler system construction.

The choice of a dominant co-character η of the maximal torus of $\mathbf{G}$ determines a parabolic subgroup P_η of $\mathbf{G}$. In order to define the tower of auxiliary level subgroups, we will force the reduction of elements of K_n modulo powers of p to lie in P_η, for η as follows. Let η be the co-character of the maximal torus of $\mathbf{G}$ defined by

$$x \mapsto \begin{pmatrix} x^3 &&&&& \\ & x^2 &&&& \\ && x^2 &&& \\ &&& x && \\ &&&& x & \\ &&&&& 1 \end{pmatrix}$$

and let $\eta_p := \eta(p) \in \mathbf{G}(\mathbf{Q}_p) \subseteq \mathbf{G}(\mathbf{A}_f)$. We note that the parabolic subgroup P_η is the intersection of the Siegel parabolic with the Klingen parabolic. The choice of η is motivated by the proof of Lemma 4.2.1.

Definition 4.1.6. Recall that we note $K_{\mathbf{G}}(p^m) \subseteq \mathbf{G}(\widehat{\mathbf{Z}})$ the kernel of the reduction modulo p^m. For $m \in \mathbf{N}$, define subgroups of $\mathbf{G}(\mathbf{A}_f)$

- $K'_{n,m(p)} := K_n \cap \eta_p^{m+1} K_n \eta_p^{-(m+1)} \cap K_{\mathbf{G}}(p^m)$;
- $K'_{n,m+1} := K'_{n,m(p)} \cap K_{\mathbf{G}}(p^{m+1})$.

Remark 4.1.7.

- The group $K'_{n,0(p)}$ is the largest subgroup of K_n such that right multiplication by η_p induces a morphism
$$\eta_p : \mathrm{Sh}_{\mathbf{G}}(K'_{n,0(p)}) \longrightarrow \mathrm{Sh}_{\mathbf{G}}(K_n).$$
- The definition of these last level groups will be justified by Lemma 4.2.1.
- In other words, for $n > m$, these subgroups are defined as follows.
$$K'_{n,m} := \left\{ g \in K_0 \ \middle| \ g \equiv I \bmod \begin{bmatrix} p^n & p^m & p^m & p^{2m} & p^{2m} & p^{3m} \\ p^n & p^m & p^m & p^m & p^m & p^{2m} \\ p^n & p^m & p^m & p^m & p^m & p^{2m} \\ p^n & p^m & p^m & p^m & p^m & p^m \\ p^n & p^m & p^m & p^m & p^m & p^m \\ p^n & p^n & p^n & p^n & p^n & p^n \end{bmatrix} \right\}.$$
$$K'_{n,m(p)} := \left\{ g \in K_0 \ \middle| \ g \equiv I \bmod \begin{bmatrix} p^n & p^{m+1} & p^{m+1} & p^{2(m+1)} & p^{2(m+1)} & p^{3(m+1)} \\ p^n & p^m & p^m & p^{m+1} & p^{m+1} & p^{2(m+1)} \\ p^n & p^m & p^m & p^{m+1} & p^{m+1} & p^{2(m+1)} \\ p^n & p^m & p^m & p^m & p^m & p^{m+1} \\ p^n & p^m & p^m & p^m & p^m & p^{m+1} \\ p^n & p^n & p^n & p^n & p^n & p^n \end{bmatrix} \right\}.$$
- Observe that we have a tower of inclusions
$$K_n = K'_{n,0} \supseteq K'_{n,0(p)} \supseteq K'_{n,1} \supseteq K'_{n,1(p)} \supseteq K'_{n,2} \supseteq \dots$$

4.1.4 Representations of algebraic groups

We study now the branching laws for the restriction of an irreducible algebraic representation of $\mathbf{G}$ to some of its subgroups.

4.1.4.1 Highest weight representations

Recall that every irreducible algebraic representation of $\mathbf{GL}_2$ is of the form $Sym^p \otimes \det^k$ for some $p \in \mathbf{N}, k \in \mathbf{Z}$, where Sym^p denotes the p-th symmetric power of the standard $\mathbf{GL}_2$-representation. We will next review the highest weight theory for the groups $\mathbf{GSp}_4$ and $\mathbf{GSp}_6$.

Let T be the diagonal torus of $\mathbf{G}$ (which coincides with the diagonal torus of $\mathbf{H}$) and denote by $\chi_i \in X^\bullet(T)$, $1 \leq i \leq 6$, the characters of T given by projection onto the i-th coordinate. We then have $\chi_i \chi_{7-i} = \nu$, $i = 1,2,3$, where ν denotes the similitude factor. We see $\mathbf{GSp}_4$ inside $\mathbf{G}$ and χ_i, $i \in \{1,2,5,6\}$, denote as well the characters of its diagonal torus.

For a, b non-negative integers, let $\mu = (\mu_1 \geq \mu_2)$, $\mu_2 = b, \mu_1 = a + b$ and denote by V^μ the unique (up to isomorphism) irreducible algebraic representation of $\mathbf{GSp}_4$ with highest weight $\chi_1^{\mu_1} \chi_2^{\mu_2}$ with central character $x \mapsto x^{|\mu|}$, where $|\mu| = \mu_1 + \mu_2$, which has dimension $\frac{1}{6}(a+1)(b+$

$1)(a+b+2)(a+2b+3)$. Similarly, given a,b,c positive integers, let $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3)$, $\lambda_3 = c, \lambda_2 = b+c, \lambda_1 = a+b+c$ and denote by V^λ the unique algebraic irreducible representation of $\mathbf{G}$ with highest weight $\chi_1^{\lambda_1}\chi_2^{\lambda_2}\chi_3^{\lambda_3}$ and central character $x \mapsto x^{|\lambda|}$, where $|\lambda| = \lambda_1+\lambda_2+\lambda_3$, which is of dimension $\frac{1}{720}(a+1)(a+2(b+c)+5)(a+b+2)(a+b+2c+4)(b+1)(b+2c+3)(a+b+c+3)(b+c+2)(c+1)$.

4.1.4.2 Branching laws

For a fixed k and λ, we are interested in studying how many $\mathbf{H}$-representations of the form $Sym^{(k,0,0)} := Sym^k \boxtimes Sym^0 \boxtimes Sym^0$ appear in the decomposition of the restriction $V^\lambda_{|\mathbf{H}}$ of V^λ to $\mathbf{H}$. It will be useful to consider the obvious factorisation of our embedding $\mathbf{H} \subseteq \mathbf{G}$ through $\mathbf{H}' := \mathbf{GSp}_4 \boxtimes \mathbf{GL}_2$, this is because any irreducible $\mathbf{H}'$-factor of an irreducible $\mathbf{G}$-representation will have multiplicity one.

We recall the following branching law result, which will be used to answer the question above. For $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3)$ and $\mu = (\mu_1 \geq \mu_2)$ as above, we say that μ doubly interlaces λ if $\lambda_1 \geq \mu_1 \geq \lambda_3$ and $\lambda_2 \geq \mu_2 \geq 0$.

Proposition 4.1.8. *Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0)$ and V^λ be as above. Then*

- *We have a decomposition of $\mathbf{Sp}_4 \boxtimes \mathbf{SL}_2$-representations*

$$V^\lambda = \bigoplus_\mu V^\mu \boxtimes (Sym^{r_1} \otimes Sym^{r_2} \otimes Sym^{r_3}),$$

  where the sum is over all $\mu = (\mu_1 \geq \mu_2 \geq 0)$ doubly interlacing λ and where $r_i = x_i - y_i$ for $\{x_1 \geq y_1 \geq x_2 \geq y_2 \geq x_3 \geq y_3\}$ being the decreasing rearrangement of $\{\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, 0\}$.

- *We have a decomposition of $\mathbf{SL}_2 \boxtimes \mathbf{SL}_2$-representations*

$$V^\mu = \bigoplus_{x=0}^{\mu_1-\mu_2} \bigoplus_{y=0}^{\mu_2} Sym^{\mu_1-x-y} \boxtimes Sym^{\mu_2-y+x}.$$

Proof. The first statement is just [WY09, Theorem 3.3]. We sketch a proof of the second points, which is stated in [LSZ17, Proposition 4.3.1]. For the parametrization of the special case of $\mathbf{GSp}_4$, applying [WY09, Theorem 3.3] we obtain

$$\begin{aligned} V^\mu &= \oplus_{s=0}^{\mu_1} (Sym^{r_1} \otimes Sym^{r_2}) \boxtimes Sym^s \\ &= \oplus_{s=0}^{\mu_1} (Sym^{r_1+r_2} \oplus Sym^{r_1+r_2-2} \oplus \ldots \oplus Sym^{|r_1-r_2|}) \boxtimes Sym^s. \end{aligned}$$

Observe that every factor appears with multiplicity one. Dividing the sum for $0 \leq s \leq \mu_2$ and $\mu_2 < s \leq \mu_1$ we see that $r_1 = \mu_1 - \mu_2$, $r_2 = s$ and $r_1 = \mu_1 - s, r_2 = \mu_2$ respectively. Drawing the points (x,y) such that the representation $Sym^x \boxtimes Sym^y$ appears in the above sum we see that we get every integer pair $(x,y) \in \mathbf{Z}^2$ with $x+y \equiv \mu_1+\mu_2 \pmod 2$ inside the rectangle with vertices $(0,\mu_1-\mu_2)$, $(\mu_1-\mu_2,0)$, (μ_2,μ_1) and (μ_1,μ_2). Choosing the right parametrisation of these points (i.e. taking (μ_2,μ_1) as the origin) we get the desired expression. $\square$

We are ready to prove the following.

Lemma 4.1.9. *The sum of all irreducible sub-$\mathbf{H}'$-representations of V^λ isomorphic (up to a twist) to $V^\mu \boxtimes Sym^0$ for some μ is given by*

$$\bigoplus_{\mu \in \mathscr{A}(\lambda)} (V^\mu \boxtimes Sym^0) \otimes \nu^{\frac{|\lambda|-|\mu|}{2}},$$

where $\mathscr{A}(\lambda) \subseteq \mathbf{Z}^2$ denotes the region of points $(\mu_1,\mu_2) \in \mathbf{Z}^2$ satisfying $|\mu| \equiv |\lambda| \pmod 2$ and lying in the rectangle defined by the inequalities

$$\begin{cases} \mu_1-\mu_2 \le \lambda_1-\lambda_2+\lambda_3, \\ \mu_1-\mu_2 \ge |\lambda_1-\lambda_2-\lambda_3|, \\ \mu_1+\mu_2 \ge \lambda_1-\lambda_2+\lambda_3, \\ \mu_1+\mu_2 \le \lambda_1+\lambda_2-\lambda_3. \end{cases}$$

Proof. Applying Proposition 4.1.8, we obtain a decomposition as $\mathbf{Sp}_4 \boxtimes \mathbf{SL}_2$-representations

$$\begin{aligned} V^\lambda &= \bigoplus_\mu V^\mu \boxtimes (Sym^{r_1} \otimes Sym^{r_2} \otimes Sym^{r_3}) \\ &= \bigoplus_\mu \bigoplus_{i=0}^{\min(r_1,r_2)} V^\mu \boxtimes (Sym^{r_1+r_2-2i} \otimes Sym^{r_3}) \\ &= \bigoplus_\mu \bigoplus_{i=0}^{\min(r_1,r_2)} \bigoplus_{j=0}^{\min(r_3,r_1+r_2-2i)} V^\mu \boxtimes (Sym^{r_1+r_2+r_3-2i-2j}), \end{aligned}$$

where the sum is over all $\mu = (\mu_1 \ge \mu_2 \ge 0)$ doubly interlacing λ and where $r_i = x_i - y_i$ for $\{x_1 \ge y_1 \ge x_2 \ge y_2 \ge x_3 \ge y_3\}$ being the decreasing rearrangement of $\{\lambda_1,\lambda_2,\lambda_3,\mu_1,\mu_2,0\}$.

We deduce that if $V^\mu \boxtimes Sym^0$ appears as a sub-$\mathbf{Sp}_4 \boxtimes \mathbf{SL}_2$-representation then

$$r_1+r_2-2i-j=0, \quad r_3-j=0,$$

which implies $j=r_3$ and $2i = r_1+r_2-r_3$ and hence, since $0 \le i \le \min(r_1,r_2)$,

$$r_1+r_2 \ge r_3 \ge r_1+r_2-2\min(r_1,r_2) = |r_1-r_2| \tag{4.1}$$

and $r_1+r_2+r_3 \equiv 0 \pmod 2$, which is equivalent to saying that $|\mu| \equiv |\lambda| \pmod 2$. The result follows by unfolding the two inequalities of (4.1). □

Lemma 4.1.10. *The sum of all irreducible sub-* **H** *-representations of* V^λ *isomorphic (up to a twist) to* $Sym^{(k,0,0)}$ *for some* $k \geq 0$ *is given by*

$$\bigoplus_{\substack{k=|\lambda_1-\lambda_2-\lambda_3| \\ k\equiv|\lambda|\,(mod\,2)}}^{\lambda_1-\lambda_2+\lambda_3} r \cdot Sym^{(k,0,0)} \otimes \det^{\frac{|\lambda|-k}{2}},$$

for $r = \lambda_2 - \lambda_3 + 1$.

Proof. This follows immediately from Lemma 4.1.9. Indeed observe that, by Proposition 4.1.8, for any μ, the unique sub-$\mathbf{SL}_2 \boxtimes \mathbf{SL}_2$-representation of V^μ of the form $Sym^{(k,0)}$ is $Sym^{(\mu_1-\mu_2,0)}$. The result then follows by analysing the possible values of $\mu_1 - \mu_2$ in the region $\mathscr{A}(\lambda)$ of the above lemma. The value r is the number of $(\mu_1, \mu_2) \in \mathscr{A}(\lambda)$ such that $\mu_1 - \mu_2 = k$, i.e. the length of one of the sides of the rectangle. The twist is there so that the central characters of $Sym^{(k,0,0)}$ and V^λ and the inclusion is **H**-equivariant. □

Remark 4.1.11. The values of k and r can be easily deduced by drawing the region $\mathscr{A}(\lambda)$. For instance, from Figure 4.1 for $\lambda = (9,6,2)$, we have that $Sym^{(k,0,0)}$ appears in the decomposition of the restriction of V^λ to **H** only if $k \in \{1,3,5\}$ with multiplicity $r = 5$.

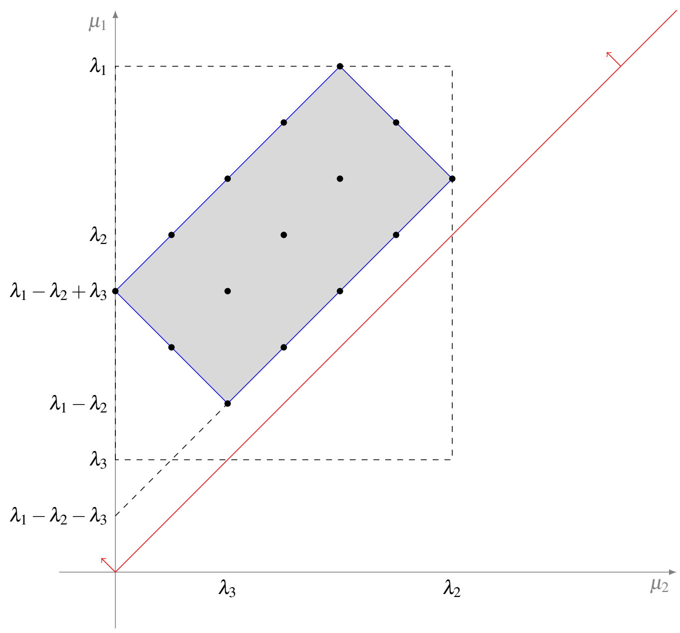

Figure 4.1: The region $\mathscr{A}(\lambda)$ for $\lambda = (9,6,2)$.

4.1.4.3 Integral structures

Denote by $\mathfrak{h}, \mathfrak{g}$ the Lie algebras of **H** and **G** respectively, and write $U(\mathfrak{h}), U(\mathfrak{g})$ for their universal enveloping algebras. For $\mathfrak{a} \in \{\mathfrak{h}, \mathfrak{g}\}$, denote by $U_{\mathbf{Z}}(\mathfrak{a})$ the Kostant **Z**-form in $U(\mathfrak{a})$ ([Ste16, Chapter

2]), which is some subring of $U(\mathfrak{a})$ generated over $\mathbf{Z}$ by an explicit family of ordered monomials given in terms of the choice of a Chevalley basis of $\mathfrak{a}$ (which also forms a *PBW*-basis of $U(\mathfrak{a})$), so that $U(\mathfrak{a})$ is obtained from $U_{\mathbf{Z}}(\mathfrak{a})$ by base-change.

For an $\mathfrak{a}$-module V, an admissible lattice $V_{\mathbf{Z}}$ in V is a $\mathbf{Z}$-lattice which is stable under the action of $U_{\mathbf{Z}}(\mathfrak{a})$. By [Ste16, Corollary 1], we know that admissible lattices exist for any representation of a semi-simple Lie group, and that such a lattice is the direct sum of its weight components. For a weight λ, fix a highest weight vector v^{λ} of weight λ and consider $V_{\mathbf{Z}}^{\lambda}$ the maximal admissible lattice inside V^{λ} whose intersection with the highest weight space is $\mathbf{Z}\cdot v^{\lambda}$. Observe that $V_{\mathbf{Z}}^{\lambda}$ is also an admissible lattice considered as an $\mathbf{H}$-representation (since $U_{\mathbf{Z}}(\mathfrak{h})\subseteq U_{\mathbf{Z}}(\mathfrak{g})$, which can be seen using [Ste16, Theorem 2] and the fact that a set of simple roots for $\mathfrak{h}$ can be extended to a set of simple roots of $\mathfrak{g}$ and that their Cartan subalgebras coincide). Let $\langle e_1,e_2,e_3,f_3,f_2,f_1\rangle$ be a symplectic basis for the standard $\mathbf{G}$-representation $V^{(1\geq 0\geq 0)}$. Denote $Sym_{\mathbf{Z}}^{(k,0,0)}\subseteq (Sym^{(k,0,0)}\cap V_{\mathbf{Z}}^{\lambda})$ the minimal admissible lattice of $Sym^{(k,0,0)}$ such that the intersection $Sym^{(k,0,0)}$ with its highest weight space is $\mathbf{Z}\cdot e_1^k$ (it is isomorphic to the algebra of symmetric tensors $TSym_{\mathbf{Z}}^{k}\boxtimes TSym_{\mathbf{Z}}^{0}\boxtimes TSym_{\mathbf{Z}}^{0}$).

By [Ste16, Corollary 1] (cf. also [Kos66, Corollary 1 to Theorem 1]), the restriction to $\mathbf{H}$ of the lattice $V_{\mathbf{Z}}^{\lambda}$ decomposes as the direct sum of its highest weight components. In particular, for every $\mu=(\mu_1,\mu_2)\in\mathscr{A}(\lambda)$ and $k=\mu_1-\mu_2$, we have that $(Sym^{(k,0,0)}\otimes\det^{\frac{|\lambda|-k}{2}})\cap V_{\mathbf{Z}}^{\lambda}\subseteq V_{\mathbf{Z}}^{\lambda}$ is non empty. By fixing any $\mathbf{H}$-highest weight vector $v^{[\lambda,\mu]}$ in this sub-lattice, we can define a homomorphism of $\mathbf{H}$-representations

$$\mathrm{br}_{\mathbf{Z}}^{[\lambda,\mu]}: Sym_{\mathbf{Z}}^{(k,0,0)}\otimes\det^{\frac{|\lambda|-k}{2}}\to V_{\mathbf{Z}}^{\lambda},$$

by sending e_1^k to $v^{[\lambda,\mu]}$.

4.1.5 Gysin morphisms

In the next section, we will define étale and motivic classes in the cohomology of the $\mathbf{G}$-Shimura variety with coefficients by taking the image under Gysin morphisms of certain classes in the cohomology of the $\mathbf{H}$-Shimura variety. To define these maps, we will translate the branching laws for algebraic representations of $\mathbf{H}$ and $\mathbf{G}$ described above into a statement for the corresponding étale sheaves and relative Chow motives on the Shimura varieties, by using Ancona's construction (§2.2.3). In particular, recall from Proposition 2.2.6 that we have a commutative diagram of functors

$$\begin{tikzcd}
\mathrm{Rep}_{\mathbf{Q}}(\mathbf{G}) \arrow[r, "\mu^{\mathbf{G}}"] \arrow[d, "\bullet|_{\mathbf{H}}"'] & \mathrm{CHM}_{\mathbf{Q}}(\mathrm{Sh}_{\mathbf{G}})^{\mathbf{G}(\mathbf{A}_f)} \arrow[d, "\Delta^*"] \\
\mathrm{Rep}_{\mathbf{Q}}(\mathbf{H}) \arrow[r, "\mu^{\mathbf{H}}"] & \mathrm{CHM}_{\mathbf{Q}}(\mathrm{Sh}_{\mathbf{H}})^{\mathbf{H}(\mathbf{A}_f)},
\end{tikzcd}$$

where Δ^* denotes pull-back.

Let $U\subseteq\mathbf{G}(\mathbf{A}_f)$ be a sufficiently small open compact subgroup so that ι_U is a closed immersion (e.g. Lemma 4.1.1), then we have Gysin morphisms

$$\iota_{U,*} : H^i_{\mathrm{mot}}(\mathrm{Sh}_{\mathbf{H}}(U \cap \mathbf{H}), \Delta^* \mathscr{V}_{\mathbf{Q}}(m)) \longrightarrow H^{i+6}_{\mathrm{mot}}(\mathrm{Sh}_{\mathbf{G}}(U), \mathscr{V}_{\mathbf{Q}}(3+m)).$$

We want to compose these maps $\iota_{U,*}$ with the maps in cohomology coming from the branching laws in $\mathrm{Rep}_{\mathbf{Q}}(\mathbf{H})$ described above. To do this, we let $\mathscr{W}^{\lambda}_{\mathbf{Q}}$ be the relative Chow motive $\mu^{\mathbf{G}}(W^{\lambda})$ over $\mathrm{Sh}_{\mathbf{G}}$, where W^{λ} is the algebraic representation of $\mathbf{G}$ given by $V^{\lambda} \otimes \nu^{-|\lambda|}$; we also let $\mathscr{H}^{(k,0,0)}_{\mathbf{Q}}$ be the relative Chow motive $\mu^{\mathbf{H}}(Sym^{(k,0,0)} \otimes \det^{-k})$ over $\mathrm{Sh}_{\mathbf{H}}$.

Proposition 4.1.12. *Let $\mu = (\mu_1 \geq \mu_2) \in \mathscr{A}(\lambda)$ and let $k = \mu_1 - \mu_2$; we have*

$$\iota^{[\lambda,\mu]}_{U,*} : H^{\bullet}_{\mathrm{mot}}(\mathrm{Sh}_{\mathbf{H}}(U \cap \mathbf{H}), \mathscr{H}^{(k,0,0)}_{\mathbf{Q}}(\star)) \longrightarrow H^{\bullet+6}_{\mathrm{mot}}(\mathrm{Sh}_{\mathbf{G}}(U), \mathscr{W}^{\lambda}_{\mathbf{Q}}(\star + 3 + \tfrac{k-|\lambda|}{2})).$$

Proof. Note that by Lemma 4.1.10, we have

$$Sym^{(k,0,0)} \otimes \det^{\frac{|\lambda|-k}{2}} \hookrightarrow V^{\lambda}.$$

After twisting it, this gives a map

$$Sym^{(k,0,0)} \otimes \det^{-k} \hookrightarrow V^{\lambda} \otimes \nu^{-|\lambda| + \frac{|\lambda|-k}{2}} = W^{\lambda} \otimes \nu^{\frac{|\lambda|-k}{2}}.$$

By Proposition 2.2.6 and Proposition 2.2.4(2), we get a morphism

$$\mathrm{br}^{[\lambda,\mu]} : \mathscr{H}^{(k,0,0)}_{\mathbf{Q}} \longrightarrow \Delta^* \mathscr{W}^{\lambda}_{\mathbf{Q}}(-\tfrac{|\lambda|-k}{2}).$$

The composition of the corresponding map in cohomology $\mathrm{br}^{[\lambda,\mu]}$ with $\iota_{U,*}$ defines the desired map $\iota^{[\lambda,\mu]}_{U,*}$. ◻

Remark 4.1.13. By §4.1.4.3, we have "integral" Gysin morphisms in étale cohomology. Let $\mathscr{H}^{(k,0,0)}_{\mathbf{Z}_p}$ (resp. $\mathscr{W}^{\lambda}_{\mathbf{Z}_p}$) denote the $\mathbf{Z}_p$-sheaf associated to the lattice $Sym^{(k,0,0)}_{\mathbf{Z}} \otimes \det^{-k}$ (resp. $V^{\lambda}_{\mathbf{Z}} \otimes \nu^{-|\lambda|}$), then we have

$$\iota^{[\lambda,\mu]}_{U,*} : H^{\bullet}_{\mathrm{\acute{e}t}}(\mathrm{Sh}_{\mathbf{H}}(U \cap \mathbf{H}), \mathscr{H}^{(k,0,0)}_{\mathbf{Z}_p}(\star)) \longrightarrow H^{\bullet+6}_{\mathrm{\acute{e}t}}(\mathrm{Sh}_{\mathbf{G}}(U), \mathscr{W}^{\lambda}_{\mathbf{Z}_p}(\star + 3 + \tfrac{k-|\lambda|}{2})).$$

4.2 Definition of the classes

We give the definition of the zeta classes and we study their norm compatibility as we vary the level of the Shimura variety.

4.2.1 Notation

Recall that $\mathrm{Sh}_{\mathbf{GL}_2}(K_1(n))$ is the moduli of isomomorphism classes of (E, P_n, α), where P_n is an p^n-torsion point of the elliptic curve E and α is a level $\mathrm{pr}_1(K^{(p)} \cap \mathbf{H})$-structure on E. Denote by $(\mathscr{E}, e_n, \alpha)/\,\mathrm{Sh}_{\mathbf{GL}_2}(K_1(n))$ the universal object of $\mathrm{Sh}_{\mathbf{GL}_2}(K_1(n))$. Moreover, fix a torsion section $x : \mathrm{Sh}_{\mathbf{GL}_2}(K_1(0)) \to \mathscr{E}$ of order an auxiliary integer N coprime to p.
For an auxiliary positive integer c coprime to 6, denote by ${}_c g_n := (x+e_n)^*({}_c\theta_{\mathscr{E}}) \in \mathscr{O}(\mathrm{Sh}_{\mathbf{GL}_2}(K_1(n)))^*$ the unit of Definition 3.1.4. Similarly, let $\mathscr{H}^k_{\mathbf{Q}}$ denote the relative Chow motive over $\mathrm{Sh}_{\mathbf{GL}_2}(K_1(n))$ associated to the $\mathbf{GL}_2$-representation $\mathrm{Sym}^k \otimes \det^{-k}$, for $k \geq 0$ and consider the motivic Eisenstein classes

$$ {}_{\mathrm{c}}\mathrm{Eis}^k_n := {}_{\mathrm{c}}\mathrm{Eis}^k_{x+e_n} \in H^1_{\mathrm{mot}}(\mathrm{Sh}_{\mathbf{GL}_2}(K_1(n)), \mathscr{H}^k_{\mathbf{Q}}(1)). $$

We denote by ${}_{\mathrm{c}}\mathrm{Eis}^k_{\text{ét},n}$ the image of the (motivic) Eisenstein class under the étale regulator. Recall that Kings has constructed an underlying integral étale Eisenstein class (Definition 3.3.24):

$$ {}_{\mathrm{c}}\mathrm{Eis}^k_{\mathbf{Z}_p,n} \in H^1_{\text{ét}}(\mathrm{Sh}_{\mathbf{GL}_2}(K_1(n)), \mathscr{H}^k_{\mathbf{Z}_p}(1)), $$

where $\mathscr{H}^k_{\mathbf{Z}_p} := \mathrm{TSym}^k(\mathscr{H}_{\mathbf{Z}_p})$ is the $\mathbf{Z}_p$-sheaf associated to the lattice $\mathrm{TSym}^k_{\mathbf{Z}} \otimes \det^{-k}$.

4.2.2 The classes at level $K'_{n,m}$

We first construct classes in the cohomology of the Shimura variety of level $K'_{n,m}$.

Lemma 4.2.1 below is the key ingredient for proving the *vertical* norm relations of our classes (Theorem 4.2.17) and, indeed, it constitutes the main motivation for working with the level $K'_{n,m}$.

Lemma 4.2.1. *Let $n, m \geq 1$ be such that $n \geq 3m+3$. There exists an element $u \in \mathbf{G}(\mathbf{A}_f)$ such that the commutative diagram*

$$ \begin{tikzcd} & \mathrm{Sh}_{\mathbf{G}}(K'_{n,m+1}) \arrow[d, "pr"] \\ \mathrm{Sh}_{\mathbf{H}}(uK'_{n,m+1}u^{-1} \cap \mathbf{H}) \arrow[ur, "\iota^u_{K'_{n,m+1}}"] \arrow[r, "pr \circ \iota^u_{K'_{n,m+1}}"] \arrow[d, "\pi_p"'] & \mathrm{Sh}_{\mathbf{G}}(K'_{n,m(p)}) \arrow[d, "\pi'_p"] \\ \mathrm{Sh}_{\mathbf{H}}(uK'_{n,m}u^{-1} \cap \mathbf{H}) \arrow[r, "\iota^u_{K'_{n,m}}"] & \mathrm{Sh}_{\mathbf{G}}(K'_{n,m}) \end{tikzcd} \tag{4.2} $$

has Cartesian bottom square, where π_p, π'_p, and pr denote the natural projections.

Remark 4.2.2.

1. A proof of Lemma 4.2.1 is a direct and not very pleasant calculation and it is given in §4.4.

2. The choice of u does not depend on either m or n and it is not unique. More precisely, u has to be the representative of an open $\mathbf{H}$-orbit of the flag variety $\mathbf{G}/P_\eta$ over $\mathbf{Z}_p$ with trivial

$\mathbf{H}$-stabiliser. In what follows, we take $u \in \mathbf{G}(\widehat{\mathbf{Z}})$, whose component at p equals to $\left(\begin{smallmatrix} I & T \\ 0 & I \end{smallmatrix}\right)$, with

$$T = \left(\begin{smallmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{smallmatrix}\right),$$

and having trivial components elsewhere.

We now define the push-forward classes in the cohomology of the $\mathbf{G}$-Shimura variety of level $K'_{n,m}$. Recall that we have étale regulator maps

$$r_{\text{ét}} : H^j_{\text{mot}}(\mathrm{Sh}_{\mathbf{G}}(U), \mathscr{W}^{\lambda}_{\mathbf{Q}}(\star)) \longrightarrow H^j_{\text{ét}}(\mathrm{Sh}_{\mathbf{G}}(U), \mathscr{W}^{\lambda}_{\mathbf{Q}_p}(\star)),$$

where $\mathscr{W}^{\lambda}_{\mathbf{Q}_p}$ is the the p-adic étale sheaf associated to $W^{\lambda}_{\mathbf{Q}_p}$. Moreover, notice that we have a projection $\mathrm{pr}_{1,n,m} : \mathrm{Sh}_{\mathbf{H}}(uK'_{n,m}u^{-1} \cap \mathbf{H}) \longrightarrow \mathrm{Sh}_{\mathbf{GL}_2}(K_1(n))$. To slightly ease the notation, for any $g \in \mathbf{G}(\mathbf{A}_f)$, we denote by $\iota^{[\lambda,\mu]}_{K'_{n,m},g,*}$ the composition $g_* \circ \iota^{[\lambda,\mu]}_{gK'_{n,m}g^{-1},*}$.

Definition 4.2.3. Let V^{λ} be the irreducible representation of $\mathbf{G}$ of highest weight $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3)$, $\mu = (k + j \geq j) \in \mathscr{A}(\lambda)$ and let $n, m \in \mathbf{N}$.

- Let $\tilde{\mathscr{Z}}^{[\lambda,\mu]}_{n,m}$ be the class given by

$$\iota^{[\lambda,\mu]}_{K'_{n,m},u,*} \circ \mathrm{pr}^*_{1,n,m}({}_{\mathrm{c}}\mathrm{Eis}^k_n) \in H^7_{\text{mot}}(\mathrm{Sh}_{\mathbf{G}}(K'_{n,m}), \mathscr{W}^{\lambda}_{\mathbf{Q}}(4 + \tfrac{k-|\lambda|}{2})).$$

- Let $\tilde{z}^{[\lambda,\mu]}_{n,m}$ be the class

$$r_{\text{ét}}({}_{c}\tilde{\mathscr{Z}}^{[\lambda,\mu]}_{n,m}) \in H^7_{\text{ét}}(\mathrm{Sh}_{\mathbf{G}}(K'_{n,m}), \mathscr{W}^{\lambda}_{\mathbf{Q}_p}(4 + \tfrac{k-|\lambda|}{2})).$$

The motivic classes defined above are not a priori integral, which is due to a lack of theory of integral motivic Eisenstein classes. Building on the work of Kings (e.g. Proposition 3.3.25), we give an integral construction of the p-adic étale classes as follows. This is better suited for studying p-adic interpolation properties.

Definition 4.2.4. Let ${}_{c}\tilde{z}^{[\lambda,\mu]}_{n,m}$ be the class given by

$$\iota^{[\lambda,\mu]}_{K'_{n,m},u,*} \circ \mathrm{pr}^*_{1,n,m}({}_{c}\,\mathrm{Eis}^k_{\mathbf{Z}_p,n}) \in H^7_{\text{ét}}(\mathrm{Sh}_{\mathbf{G}}(K'_{n,m}), \mathscr{W}^{\lambda}_{\mathbf{Z}_p}(4 + \tfrac{k-|\lambda|}{2})).$$

Thanks to Lemma 4.2.1, these classes are proved to be compatible as m varies (cf. Theorem 4.2.17).

4.2.3 The level groups $K_{n,m}$

Let $n \in \mathbf{N}$ and denote by $K_{n,0} \subseteq \mathbf{G}(\widehat{\mathbf{Z}})$ the subgroup of K_n defined by

$$K_{n,0} := K_n \cap \left\{ g \in \mathbf{G}(\mathbf{Z}_p) \;\middle|\; g \equiv I \bmod \begin{bmatrix} 1 & p & p & p & p & p \\ p & 1 & p & p & p & p \\ p & p & 1 & p & p & p \\ 1 & 1 & 1 & p & p & p \\ 1 & 1 & 1 & p & p & p \\ 1 & 1 & 1 & p & p & p \end{bmatrix} \right\}.$$

Remark 4.2.5. The definition of $K_{n,0}$ is motivated by the proof of Theorem 4.2.17.

For $m, n \in \mathbf{N}$, we aim to define classes in the cohomology of

$$\mathrm{Sh}_{\mathbf{G}}(K_{n,0}) \times_{\mathrm{Spec}(\mathbf{Q})} \mathrm{Spec}(\mathbf{Q}(\zeta_{p^m})).$$

Definition 4.2.6. Let $n, m \in \mathbf{N}$. Define subgroups $K_{n,m} \subseteq K_{n,0}$ by

$$K_{n,m} := K_{n,0} \cap \nu^{-1}(1 + p^m \widehat{\mathbf{Z}}) = \{ g \in K_{n,0} : \nu(g) \equiv 1 \ (\mathrm{mod}\ p^m \widehat{\mathbf{Z}}) \}.$$

Remark 4.2.7. As explained in [LSZ17, 5.4], if $U \subseteq \mathbf{G}(\mathbf{A}_f)$ is an open compact subgroup such that

$$\nu(U) \cdot (1 + p^m \widehat{\mathbf{Z}}) = \widehat{\mathbf{Z}}^\times,$$

then there is an isomorphism of $\mathbf{Q}$-schemes

$$\mathrm{Sh}_{\mathbf{G}}(U \cap \nu^{-1}(1 + p^m \widehat{\mathbf{Z}})) \simeq \mathrm{Sh}_{\mathbf{G}}(U) \times_{\mathrm{Spec}(\mathbf{Q})} \mathrm{Spec}(\mathbf{Q}(\zeta_{p^m})),$$

which intertwines the action of $g \in \mathbf{G}(\mathbf{A}_f)$ on the left-hand side with the one of (g, σ_g) on the right-hand side, where $\sigma_g = \mathrm{Art}(\nu(g)^{-1})|_{\mathbf{Q}(\zeta_{p^m})}$. In particular, we have

$$\mathrm{Sh}_{\mathbf{G}}(K_{n,m}) \simeq \mathrm{Sh}_{\mathbf{G}}(K_{n,0}) \times_{\mathrm{Spec}(\mathbf{Q})} \mathrm{Spec}(\mathbf{Q}(\zeta_{p^m})).$$

4.2.4 The classes at level $K_{n,m}$

For two given integers $n, m \geq 1$, take $n' = n + 3m$ and define the projection

$$t_m : \mathrm{Sh}_{\mathbf{G}}(K'_{n',m}) \to \mathrm{Sh}_{\mathbf{G}}(K_{n,m}),$$

induced by right multiplication by the element $\eta_p^m = diag(p^{3m}, p^{2m}, p^{2m}, p^m, p^m, 1) \in \mathbf{G}(\mathbf{Q}_p)$ defined in §4.1.3.

Remark 4.2.8. The map t_m is well-defined. Indeed, we need to check that $\eta_p^{-m} K'_{n',m} \eta_p^m \subseteq K_{n,m}$. Recall that $K'_{n',m} = K_{n'} \cap \eta_p^m K_{n'} \eta_p^{-m} \cap K_{\mathbf{G}}(p^m)$ and $K_{n,m} = K_{n,0} \cap \nu^{-1}(1 + p^m \widehat{\mathbf{Z}})$, so we have $\eta_p^{-m} K'_{n',m} \eta_p^m = \eta_p^{-m} K_{n'} \eta_p^m \cap K_{n'} \cap \eta_p^{-m} K_{\mathbf{G}}(p^m) \eta_p^m$. This is obviously contained in K_n and in

$\nu^{-1}(1+p^m\widehat{\mathbf{Z}})$. Finally, if $g \in K_{n'} \cap \eta_p^{-m} K_{\mathbf{G}}(p^m)\eta_p^m$, it satisfies the extra conditions modulo p imposed in the definition of $K_{n,0}$.

Before defining the classes we note that the push-forward by $t_{m,*}$ makes sense with our p-adic integral coefficients.

Lemma 4.2.9. *There is a well defined action of $\eta_p^{-1}/p^{\lambda_2+\lambda_3}$ on $W^\lambda_{\mathbf{Z}_p}$ defining a morphism of sheaves*

$$t^\lambda_{m,\flat} : \mathscr{W}^\lambda_{\mathbf{Z}_p} \to t_m^*(\mathscr{W}^\lambda_{\mathbf{Z}_p}).$$

In particular, we have a map

$$t^\lambda_{m,*} : H^7_{ét}(\mathrm{Sh}_{\mathbf{G}}(K'_{n',m}), \mathscr{W}^\lambda_{\mathbf{Z}_p}(4+\tfrac{k-|\lambda|}{2})) \to H^7_{ét}(\mathrm{Sh}_{\mathbf{G}}(K_{n,m}), \mathscr{W}^\lambda_{\mathbf{Z}_p}(4+\tfrac{k-|\lambda|}{2})),$$

defined by composing the map in cohomology induced by $t^\lambda_{m,\flat}$ with the trace of t_m in étale cohomology.

Proof. We need to show that the matrix $\eta_p^{-1} = diag(p^{-3}, p^{-2}, p^{-2}, p^{-1}, p^{-1}, 1)$ acts on $W^\lambda_{\mathbf{Z}_p}$ and that its image is contained in $p^{\lambda_2+\lambda_3} W^\lambda_{\mathbf{Z}_p}$. Let S be the one dimensional split torus $diag(x^3, x^2, x^2, x, x, 1)$ of $\mathbf{G}$. Then V^λ decomposes as the direct sum of its weight spaces relative to S, with weights between 0 and $3\lambda_1 + 2\lambda_2 + 2\lambda_3$. We deduce that S acts on the highest weight subspace of $W^\lambda = V^\lambda \otimes \nu^{-|\lambda|}$ through the character $diag(x^3, x^2, x^2, x, x, 1) \mapsto x^{-(\lambda_2+\lambda_3)}$ and, in particular, the action of η_p^{-1} on every S-weight space (and hence on all W^λ) will be divisible by $p^{\lambda_2+\lambda_3}$, thus showing the claim. □

Remark 4.2.10. Observe that the normalisation by $p^{-(\lambda_2+\lambda_3)}$ is such that the action of $p^{-(\lambda_2+\lambda_3)}\eta_p^{-1}$ on the S-highest weight subspace of W^λ is trivial and divisible by p elsewhere. This optimal normalisation of the map $t^\lambda_{m,*}$ will be very helpful (in a rather subtle way) when defining our cohomology classes at integral level and proving their norm relations (cf. Theorem 4.2.17).

We are now ready to define the following.

Definition 4.2.11.

- Let $\mathscr{Z}^{[\lambda,\mu]}_{n,m} := t^\lambda_{m,*}(\tilde{\mathscr{Z}}^{[\lambda,\mu]}_{n',m}) \in H^7_{\mathrm{mot}}(\mathrm{Sh}_{\mathbf{G}}(K_{n,m}), \mathscr{W}^\lambda_{\mathbf{Q}}(4+\frac{k-|\lambda|}{2}))$.

- Let ${}_cz^{[\lambda,\mu]}_{n,m}$ be the class

$$t^\lambda_{m,*}({}_c\tilde{z}^{[\lambda,\mu]}_{n',m}) \in H^7_{\text{ét}}(\mathrm{Sh}_{\mathbf{G}}(K_{n,m}), \mathscr{W}^\lambda_{\mathbf{Z}_p}(4+\tfrac{k-|\lambda|}{2})).$$

4.2.5 Norm relations at p: varying the level

We now show that the various classes that we constructed are compatible when we vary the variable n. Denote by

$$\varphi_{1,n} : \mathrm{Sh}_{\mathbf{GL}_2}(K_1(n+1)) \to \mathrm{Sh}_{\mathbf{GL}_2}(K_1(n))$$
$$\phi'_n : \mathrm{Sh}_{\mathbf{G}}(K'_{n+1,m}) \to \mathrm{Sh}_{\mathbf{G}}(K'_{n,m})$$

the natural projection maps. We have the following.

Proposition 4.2.12. *We have*

$$\phi'_{n,*}({}_c\tilde{z}^{[\lambda,\mu]}_{n+1,m}) = \begin{cases} {}_c\tilde{z}^{[\lambda,\mu]}_{n,m} & \text{if } n \geq 1, \\ (1-p^k D_p^*)\,{}_c\tilde{z}^{[\lambda,\mu]}_{n,m} & \text{if } n = 0, \end{cases}$$

where $D_p \in \mathbf{H}(\widehat{\mathbf{Z}}) \subseteq \mathbf{G}(\widehat{\mathbf{Z}})$ is any matrix whose first $\mathbf{GL}_2$-component is congruent to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$ modulo N.

Proof. This is a particular case of Corollaries 3.4.11, 3.4.12. □

This immediately translates into the identical norm relations for the level $K_{n,m}$ classes.

Corollary 4.2.13. *Let $\phi_n : \mathrm{Sh}_{\mathbf{G}}(K_{n+1,m}) \to \mathrm{Sh}_{\mathbf{G}}(K_{n,m})$ be the natural projection map. We have*

$$\phi_{n,*}({}_c z^{[\lambda,\mu]}_{n+1,m}) = \begin{cases} {}_c z^{[\lambda,\mu]}_{n,m} & \text{if } n \geq 1, \\ (1-p^k D_p^*)\,{}_c z^{[\lambda,\mu]}_{n,m} & \text{if } n = 0, \end{cases}$$

where $D_p \in \mathbf{H}(\widehat{\mathbf{Z}}) \subseteq \mathbf{G}(\widehat{\mathbf{Z}})$ is any matrix whose first $\mathbf{GL}_2$-component is congruent to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$ modulo N.

Proof. Apply $t^{\lambda}_{m,*}$ to both sides of Proposition 4.2.12. □

4.2.6 Norm relations at p: cyclotomic variation

In this section, we prove our main result stating that our cohomology classes satisfy the Euler system relations at powers of p.

4.2.6.1 Hecke operators

We now define the Hecke operator which is going to show up in the norm compatibility relations of our cohomology classes.

Definition 4.2.14. We define the Hecke operator $\mathscr{U}'_p$ acting on $H^7_{\text{ét}}(\mathrm{Sh}_{\mathbf{G}}(K'_{n,m}), \mathscr{W}^{\lambda}_{\mathbf{Z}_p}(4+\frac{k-|\lambda|}{2}))$ to be the action of $p^{-(\lambda_2+\lambda_3)} \cdot K'_{n,m}\eta_p^{-1}K'_{n,m}$, where $K'_{n,m}\eta_p^{-1}K'_{n,m}$ is seen as an element of the Hecke algebra $\mathscr{H}(K'_{n,m}\backslash G(\mathbf{A}_f)/K'_{n,m})_{\mathbf{Z}_p}$ of $K'_{n,m}$-bi-invariant smooth compactly supported $\mathbf{Z}_p$-valued functions on

$\mathbf{G}(\mathbf{A}_f)$. In other words, the action of $K'_{n,m}\eta_p^{-1}K'_{n,m}$ on cohomology is the one induced from the following correspondence on $\mathrm{Sh}_{\mathbf{G}}$:

$$\begin{tikzcd} \mathrm{Sh}_{\mathbf{G}}(K'_{n,m(p)}) \arrow[d, "\pi'_p"'] \arrow[dr, "\eta_p"] & \\ \mathrm{Sh}_{\mathbf{G}}(K'_{n,m}) \arrow[r, dotted] & \mathrm{Sh}_{\mathbf{G}}(K'_{n,m}), \end{tikzcd}$$

where the vertical arrow is the natural projection π'_p and the diagonal one is induced by right multiplication by η_p, and hence $\mathscr{U}'_p$ is given by the composition

$$H^7_{\text{ét}}(\mathrm{Sh}_{\mathbf{G}}(K'_{n,m}),\mathscr{W}^{\lambda}_{\mathbf{Z}_p}(4+\tfrac{k-|\lambda|}{2}))\xrightarrow{(\pi'_p)^*}H^7_{\text{ét}}(\mathrm{Sh}_{\mathbf{G}}(K'_{n,m(p)}),\mathscr{W}^{\lambda}_{\mathbf{Z}_p}(4+\tfrac{k-|\lambda|}{2}))\xrightarrow{\eta^{\lambda}_{p,*}}H^7_{\text{ét}}(\mathrm{Sh}_{\mathbf{G}}(K'_{n,m}),\mathscr{W}^{\lambda}_{\mathbf{Z}_p}(4+\tfrac{k-|\lambda|}{2})),$$

where $\eta^{\lambda}_{p,*}$ is the normalised map defined exactly in the same way as the map $t^{\lambda}_{m,*}$ of Lemma 4.2.9.

Remark 4.2.15. The notation chosen for the Hecke operator is motivated by the fact that $\mathscr{U}'_p$ is dual to the Hecke operator associated to η_p.

4.2.6.2 Norm relation for the classes ${}_c\tilde{z}^{[\lambda,\mu]}_{n,m}$

Recall that the diagonal matrix $\eta_p := (p^3,p^2,p^2,p,p,1)\in\mathbf{G}(\mathbf{Q}_p)$ induces a morphism of Shimura varieties $\eta_p:\mathrm{Sh}_{\mathbf{G}}(K'_{n,m(p)})\to\mathrm{Sh}_{\mathbf{G}}(K'_{n,m})$ and, by Lemma 4.2.9, a map

$$\eta^{\lambda}_{p,*}:H^7_{\text{ét}}(\mathrm{Sh}_{\mathbf{G}}(K'_{n,m(p)}),\mathscr{W}^{\lambda}_{\mathbf{Z}_p}(4+\tfrac{k-|\lambda|}{2}))\to H^7_{\text{ét}}(\mathrm{Sh}_{\mathbf{G}}(K'_{n,m}),\mathscr{W}^{\lambda}_{\mathbf{Z}_p}(4+\tfrac{k-|\lambda|}{2})).$$

Let $m\geq 1$, $n\geq 3(m+1)$, and denote by $\tilde{\eta}_p$ the composition of the natural projection map $pr:\mathrm{Sh}_{\mathbf{G}}(K'_{n,m+1})\to\mathrm{Sh}_{\mathbf{G}}(K'_{n,m(p)})$ with the map $\eta_p:\mathrm{Sh}_{\mathbf{G}}(K'_{n,m(p)})\to\mathrm{Sh}_{\mathbf{G}}(K'_{n,m})$. By the same arguments as in Lemma 4.2.9, we can once more define a normalised trace

$$\tilde{\eta}^{\lambda}_{p,*}:H^7_{\text{ét}}(\mathrm{Sh}_{\mathbf{G}}(K'_{n,m+1}),\mathscr{W}^{\lambda}_{\mathbf{Z}_p}(4+\tfrac{k-|\lambda|}{2}))\to H^7_{\text{ét}}(\mathrm{Sh}_{\mathbf{G}}(K'_{n,m}),\mathscr{W}^{\lambda}_{\mathbf{Z}_p}(4+\tfrac{k-|\lambda|}{2})),$$

as the composition the trace of pr with $\eta^{\lambda}_{p,*}$.

We have the following push-forward compatibility relation.

Theorem 4.2.16. *For $m\geq 1$, $n\geq 3(m+1)$, we have*

$$\tilde{\eta}^{\lambda}_{p,*}({}_c\tilde{z}^{[\lambda,\mu]}_{n,m+1})=\mathscr{U}'_p\cdot{}_c\tilde{z}^{[\lambda,\mu]}_{n,m},$$

where $\mathscr{U}'_p$ is the Hecke operator defined in Definition 4.2.14.

Proof. Denote by ${}_cz^k_{\mathbf{H},n,m}$ the class $\mathrm{pr}^*_{1,n,m}({}_c\mathrm{Eis}^k_{\text{ét},n})$. The result follows from Lemma 4.2.1. Indeed, by the definition of the class ${}_c\tilde{z}_{n,m+1}^{[\lambda,\mu]}$ we have

$$\begin{aligned} pr_*({}_c\tilde{z}_{n,m+1}^{[\lambda,\mu]}) &= pr_* \circ \iota^{[\lambda,\mu]}_{K'_{n,m+1},u,*}({}_c z^k_{\mathbf{H},n,m+1}) \\ &= pr_* \circ \iota^{[\lambda,\mu]}_{K'_{n,m+1},u,*} \circ \pi_p^*({}_c z^k_{\mathbf{H},n,m}), \end{aligned}$$

where π_p is as in Lemma 4.2.1. By the Cartesianness of the square of the diagram of Lemma 4.2.1, we have that

$$pr_* \circ \iota^{[\lambda,\mu]}_{K'_{n,m+1},u,*} \circ \pi_p^* = (\pi'_p)^* \circ \iota^{[\lambda,\mu]}_{K'_{n,m},u,*},$$

so we deduce

$$pr_*({}_c\tilde{z}_{n,m+1}^{[\lambda,\mu]}) = (\pi'_p)^* \circ \iota^{[\lambda,\mu]}_{K'_{n,m},u,*}({}_c z^k_{\mathbf{H},n,m}) = (\pi'_p)^*({}_c\tilde{z}_{n,m}^{[\lambda,\mu]}),$$

where the last equality follows by definition. Hence, by applying $\eta^\lambda_{p,*}$ to both sides, we get

$$\tilde{\eta}^\lambda_{p,*}({}_c\tilde{z}_{n,m+1}^{[\lambda,\mu]}) = \eta^\lambda_{p,*} \circ (\pi'_p)^*({}_c\tilde{z}_{n,m}^{[\lambda,\mu]}) = \mathscr{U}'_p \cdot {}_c\tilde{z}_{n,m}^{[\lambda,\mu]}$$

as desired. □

4.2.6.3 Norm relation for the classes ${}_c z_{n,m}^{[\lambda,\mu]}$

Call $\mathrm{norm}^{\mathbf{Q}(\zeta_{p^{m+1}})}_{\mathbf{Q}(\zeta_{p^m})}$ the norm map of the natural projection $\mathrm{Sh}_{\mathbf{G}}(K_{n,0})_{/\mathbf{Q}(\zeta_{p^{m+1}})} \to \mathrm{Sh}_{\mathbf{G}}(K_{n,0})_{/\mathbf{Q}(\zeta_{p^m})}$. Moreover, let σ_p denotes the image of $\frac{1}{p} \in \mathbf{Q}_p^*$ under the Artin reciprocity map.

Theorem 4.2.17. *For $n,m \geq 1$, we have*

$$\mathrm{norm}^{\mathbf{Q}(\zeta_{p^{m+1}})}_{\mathbf{Q}(\zeta_{p^m})}({}_c z_{n,m+1}^{[\lambda,\mu]}) = \frac{\mathscr{U}'_p}{\sigma_p^3} \cdot {}_c z_{n,m}^{[\lambda,\mu]},$$

where $\mathscr{U}'_p$ is the Hecke operator associated to $p^{-(\lambda_2+\lambda_3)} \cdot K_{n,m}\eta_p^{-1}K_{n,m}$.

Proof. We first deduce the norm relation at levels $K_{n,m}$. By Theorem 4.2.16 and the commutative diagram

$$\begin{tikzcd} \mathrm{Sh}_{\mathbf{G}}(K'_{n,m+1}) \arrow[r, "\eta_p^{m+1}"] \arrow[d, "\tilde{\eta}_p"'] & \mathrm{Sh}_{\mathbf{G}}(K_{n,m+1}) \arrow[d] \\ \mathrm{Sh}_{\mathbf{G}}(K'_{n,m}) \arrow[r, "\eta_p^m"] & \mathrm{Sh}_{\mathbf{G}}(K_{n,m}), \end{tikzcd}$$

where the right vertical arrow is the natural projection map, it suffices to show that the Hecke operator

$\mathscr{U}_p'$ commutes with $t^{\lambda}_{m,*}$, i.e. that we have a commutative diagram

$$\begin{tikzcd} H^7(\mathrm{Sh}_{\mathbf{G}}(K'_{n',m}), \mathscr{W}^{\lambda}_{\mathbf{Z}_p}(4+\frac{k-|\lambda|}{2})) \arrow[r, "t^{\lambda}_{m,*}"] \arrow[d, "\mathscr{U}_p'"'] & H^7(\mathrm{Sh}_{\mathbf{G}}(K_{n,m}), \mathscr{W}^{\lambda}_{\mathbf{Z}_p}(4+\frac{k-|\lambda|}{2})) \arrow[d, "\mathscr{U}_p'"] \\ H^7(\mathrm{Sh}_{\mathbf{G}}(K'_{n',m}), \mathscr{W}^{\lambda}_{\mathbf{Z}_p}(4+\frac{k-|\lambda|}{2})) \arrow[r, "t^{\lambda}_{m,*}"] & H^7(\mathrm{Sh}_{\mathbf{G}}(K_{n,m}), \mathscr{W}^{\lambda}_{\mathbf{Z}_p}(4+\frac{k-|\lambda|}{2})). \end{tikzcd}$$

Recall that the Hecke operator $\mathscr{U}_p'$ at level $K_{n,m}$ is defined as the correspondence $pr_{2,*} \circ \eta^{\lambda}_{p,*} \circ pr_1^*$, where pr_1, pr_2 are natural projections sitting in the diagram

$$\mathrm{Sh}_{\mathbf{G}}(K_{n,m}) \xleftarrow{pr_1} \mathrm{Sh}_{\mathbf{G}}(K_{n,m} \cap \eta_p K_{n,m} \eta_p^{-1}) \xrightarrow{\eta_p} \mathrm{Sh}_{\mathbf{G}}(\eta_p^{-1} K_{n,m} \eta_p \cap K_{n,m}) \xrightarrow{pr_2} \mathrm{Sh}_{\mathbf{G}}(K_{n,m}).$$

Then, the two Hecke operators commute if $|K_{n,m} \cap \eta_p^{-1} K_{n,m} \eta_p \backslash K_{n,m}| = |K'_{n',m} \cap \eta_p^{-1} K'_{n',m} \eta_p \backslash K'_{n',m}|$. This is indeed the case, since both sizes can be checked to be p^{12}. We are making an essential use of the extra congruences modulo p satisfied by the elements in $K_{n,0}$. Finally, the result follows after using the isomorphism

$$\mathrm{Sh}_{\mathbf{G}}(K_{n,m}) \simeq \mathrm{Sh}_{\mathbf{G}}(K_{n,0}) \times_{\mathrm{Spec}(\mathbf{Q})} \mathrm{Spec}(\mathbf{Q}(\zeta_{p^m})),$$

which intertwines the Hecke operator $\mathscr{U}_p'$ on the cohomology $\mathrm{Sh}_{\mathbf{G}}(K_{n,m})$ with $\mathrm{Art}(\nu(\eta_p))|_{\mathbf{Q}(\zeta_{p^m})} \mathscr{U}_p' = \sigma_p^{-3} \mathscr{U}_p'$.

□

Remark 4.2.18. For calculating the size of the quotient for $K_{n,m}$, one actually crucially uses the congruences modulo p appearing in the definition of the level group $K_{n,0}$, and the result would not hold if we didn't impose those congruences.

4.3 Mapping to Galois cohomology

Let $\pi = \pi_f \otimes \pi_\infty$ be a cuspidal automorphic representation of $\mathbf{G}(\mathbf{A})$ of level U such that U is sufficiently small and satisfies $\nu(U) = \widehat{\mathbf{Z}}^\times$, π_∞ is in the discrete series and π appears in $H^6_{\text{ét}}(\mathrm{Sh}_{G,\overline{\mathbf{Q}}}(U), \mathscr{W}^{\lambda}_L(4+q))$ for some weight λ and finite extension L of $\mathbf{Q}_p$, and $q = \frac{k-|\lambda|}{2}$, for some $k \geq 0$ as in Lemma 4.1.10.

Let $N \in \mathbf{N}$ be the smallest number such that $K_{\mathbf{G}}(N) \subseteq U$ (recall that $K_{\mathbf{G}}(N)$ denotes the the principal congruence subgroup of level N), let $\mathscr{H}$ denote the Hecke algebra generated over $\mathbf{Z}$ by the standard Hecke operators for primes ℓ not diving N. Let L be the p-adic completion of the smallest number field containing the Hecke eigenvalues of π and note $\mathscr{O}_L$ its ring of integers, k_L its residue field and note $\mathscr{H}_{\mathscr{O}_L} = \mathscr{H} \otimes_{\mathbf{Z}} \mathscr{O}_L$. Finally, we denote $\mathfrak{m} \subseteq \mathscr{H}_{\mathscr{O}_L}$ to be the kernel of the character $\mathscr{H}_{\mathscr{O}_L} \to k_L$ defined by π.

The study of the localisation at the Hecke ideal $\mathfrak{m}$ of the cohomology of Siegel varieties with integral coefficients has been carried in [MT02]. Their study relies on the existence of a Galois representation associated to π, which is now known to exists thanks to the recent work [KS16]. We will assume throughout the hypotheses **(GO)** (Galois ordinary) and **(RLI)** (residually large image, i.e. *non-Eisenstein-ness*) made in [MT02, §1] (where the reader is referred for the appropriate definitions).

Proposition 4.3.1 ([MT02, Theorem 1])**.** *If $p > 5$ and $p-1 > |\lambda| + 6$ then*

$$H^{\bullet}_{\acute{e}t}(\mathrm{Sh}_{\mathbf{G},\overline{\mathbf{Q}}}(U), \mathscr{W}^{\lambda}_{\mathscr{O}_L}(\star))_{\mathfrak{m}} = H^{6}_{\acute{e}t}(\mathrm{Sh}_{\mathbf{G},\overline{\mathbf{Q}}}(U), \mathscr{W}^{\lambda}_{\mathscr{O}_L}(\star))_{\mathfrak{m}}$$

is a free $\mathscr{O}_L$-module of finite rank.

Let now V_π be the Galois representation associated to π (up to the twist for $\mathbf{Q}_p(4+q)$), where π satisfies the hypotheses **(St)** and **(spin-reg)** made in [KS16]. When $U = K_{n,0}$ for some $n \in \mathbf{N}$, Proposition 4.3.1 and the Hochschild-Serre spectral sequence give a map

$$\begin{aligned} H^7_{\text{ét}}(\mathrm{Sh}_{\mathbf{G}}(K_{n,m}), \mathscr{W}^{\lambda}_{\mathscr{O}_L}(4+q)) \to & H^1(\mathbf{Q}(\zeta_{p^m}), H^6_{\text{ét}}(\mathrm{Sh}_{\mathbf{G},\overline{\mathbf{Q}}}(K_{n,0}), \mathscr{W}^{\lambda}_{\mathscr{O}_L}(4+q))_{\mathfrak{m}}) \\ \to & H^1(\mathbf{Q}(\zeta_{p^m}), T_\pi) \\ \to & H^1(\mathbf{Q}_p(\zeta_{p^m}), T_\pi), \end{aligned}$$

where T_π denotes the $\mathscr{O}_L$-stable lattice in V_π given by the π_f-isotypic component of the étale cohomology with $\mathscr{W}^{\lambda}_{\mathscr{O}_L}(4+q)$-coefficients. In the above composition, the last arrow is the restriction to the decomposition group.

Definition 4.3.2. We let ${}_c z^{\pi}_m$ be the image of ${}_c z^{[\lambda,\mu]}_{n,m}$, for a $\mu = (k+j \geq j) \in \mathscr{A}(\lambda)$, in any of the groups appearing in the above composition.

In addition to what previously asked, suppose that π is $\mathscr{U}'_p$-ordinary, in the sense that $\mathscr{U}'_p$ acts on T_π as multiplication by a p-adic unit α. Then, Theorem 4.2.17 immediately gives the following.

Theorem 4.3.3. *Let ${}_c z^{\pi}_{m,\alpha} \in H^1(\mathbf{Q}(\zeta_{p^m}), T_\pi)$ be the class defined as $\left(\frac{\sigma_p^3}{\alpha}\right)^m \cdot {}_c z^{\pi}_m$. For $m \geq 1$, we have*

$$\mathrm{cores}^{\mathbf{Q}(\zeta_{p^{m+1}})}_{\mathbf{Q}(\zeta_{p^m})}({}_c z^{\pi}_{m+1,\alpha}) = {}_c z^{\pi}_{m,\alpha}.$$

As a consequence, after applying the restriction maps $H^1(\mathbf{Q}(\zeta_{p^m}), T_\pi) \to H^1(\mathbf{Q}_p(\zeta_{p^m}), T_\pi)$, the system of elements $({}_c z^{\pi}_{m,\alpha})_{m \geq 1}$ gives a class

$${}_c z^{\pi}_{\mathrm{Iw},\alpha} \in H^1_{\mathrm{Iw}}(\mathbf{Q}_p, V_\pi) := \varprojlim_m H^1(\mathbf{Q}_p(\zeta_{p^m}), T_\pi) \otimes_{\mathscr{O}_L} L.$$

Remark 4.3.4. Applying Perrin-Riou's machine to ${}_c z^{\pi}_{\mathrm{Iw},\alpha}$ (e.g. [Col00] for references), we construct a p-adic L-function for V_π. Some of these aspects are discussed in [CRJ18].

4.4 Proof of Lemma 4.2.1

We finally give a proof of Lemma 4.2.1. Let $u \in \mathbf{G}(\mathbf{A}_f)$ be the element whose component at p is $\left(\begin{smallmatrix} I & T \\ 0 & I \end{smallmatrix}\right)$, for $T = \left(\begin{smallmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{smallmatrix}\right)$, and let $n, m \geq 1$ be such that $n \geq 3m+3$. Then, the commutative diagram

$$\begin{tikzcd}
 & \mathrm{Sh}_{\mathbf{G}}(K'_{n,m+1}) \arrow[d, "pr"] \\
\mathrm{Sh}_{\mathbf{H}}(uK'_{n,m+1}u^{-1} \cap \mathbf{H}) \arrow[ur, "\iota^u_{K'_{n,m+1}}"] \arrow[r, "pr \circ \iota^u_{K'_{n,m+1}}"] \arrow[d, "\pi_p"'] & \mathrm{Sh}_{\mathbf{G}}(K'_{n,m(p)}) \arrow[d, "\pi'_p"] \\
\mathrm{Sh}_{\mathbf{H}}(uK'_{n,m}u^{-1} \cap \mathbf{H}) \arrow[r, "\iota^u_{K'_{n,m}}"] & \mathrm{Sh}_{\mathbf{G}}(K'_{n,m})
\end{tikzcd} \tag{4.3}$$

has Cartesian bottom square.

In order to show the Cartesianness of diagram 4.3, it is enough to check that

1. The map $pr \circ \iota^u_{K'_{n,m+1}}$ is a closed immersion or, equivalently,

$$uK'_{n,m(p)}u^{-1} \cap \mathbf{H} = uK'_{n,m+1}u^{-1} \cap \mathbf{H};$$

2. $[uK'_{n,m}u^{-1} \cap \mathbf{H} : uK'_{n,m+1}u^{-1} \cap \mathbf{H}] = [K'_{n,m} : K'_{n,m(p)}]$.

These two facts are shown in the next two lemmas.

Lemma 4.4.1. *We have the equality of subgroups of* $\mathbf{H}(\mathbf{A}_f)$

$$uK'_{n,m(p)}u^{-1} \cap \mathbf{H} = uK'_{n,m+1}u^{-1} \cap \mathbf{H}.$$

Proof. It suffices to show that if $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in K'_{n,m(p)}$ is such that $ugu^{-1} \in \mathbf{H}$ then $g \in K'_{n,m+1}$, i.e that $g \equiv I \bmod p^{m+1}$. Writing down the condition $ugu^{-1} \in \mathbf{H}$, we get that g is of the form

$$g = \begin{pmatrix} a_1 & -c_2 & -c_3 & (a_1-d_3)-c_2 & (a_1-d_2)-c_3 & b_1 \\ -c_1 & a_2 & -c_3 & (a_2-d_3)-c_1 & b_2 & (a_2-d_1)-c_3 \\ -c_1 & -c_2 & a_3 & b_3 & (a_3-d_2)-c_1 & (a_3-d_1)-c_2 \\ & & c_3 & d_3 & c_3 & c_3 \\ & c_2 & & c_2 & d_2 & c_2 \\ c_1 & & & c_1 & c_1 & d_1 \end{pmatrix}.$$

The congruences of the (1,2) and (1,3) entries give $c_2 \equiv c_3 \equiv 0 \bmod p^{m+1}$. Moreover, taking a look at the elements off the anti-diagonal of B, we easily deduce that $a_1 \equiv a_2 \equiv a_3 \equiv d_3 \equiv d_2 \equiv d_1 \equiv 1 \bmod p^{m+1}$. $\square$

We are left with showing that the degrees of the two vertical maps of the bottom square of (4.3) are equal.

Lemma 4.4.2. *We have*

$$[uK'_{n,m}u^{-1} \cap \mathbf{H} : uK'_{n,m+1}u^{-1} \cap \mathbf{H}] = [K'_{n,m} : K'_{n,m(p)}].$$

Proof. Since a system of coset representatives of $Q = K'_{n,m}/K'_{n,m(p)}$ determines one for

$$uK'_{n,m}u^{-1}/uK'_{n,m(p)}u^{-1},$$

it suffices to prove that we can find $\{\sigma_i\}_{i\in I}$ system of coset representatives for Q such that $u\sigma_i u^{-1} \in \mathbf{H}$ for all $i \in I$. Consider the following set of elements of $K'_{n,m}$ whose conjugation by u is in $\mathbf{H}$:

$$\sigma_{\underline{v}} = \begin{pmatrix} 1+p^m a & -p^m r' & -p^m r & p^{2m}t''' & p^{2m}t'' & p^{3m}k \\ & 1+p^m b & -p^m r & p^m c & p^m k' & p^{2m}t \\ & -p^m r' & 1+p^m d & p^m k'' & p^m e & p^{2m}t' \\ & & p^m r & 1+p^m s & p^m r & p^m r \\ & p^m r' & & p^m r' & 1+p^m f & p^m r' \\ & & & & & 1 \end{pmatrix},$$

where for each vector $\underline{v} \in \mathbf{Z}/p^3\mathbf{Z} \times (\mathbf{Z}/p^2\mathbf{Z})^4 \times (\mathbf{Z}/p\mathbf{Z})^5 =: V$ we consider one (and only one) lift

$$(k,t,t',t'',t''',k',k'',r,r',s) \in \hat{\mathbf{Z}}^{10}$$

so that $\sigma_{\underline{v}} \in \mathbf{G}(\mathbf{A}_f)$, where we have set

$$a = r' + s + p^m t'', \quad b = r + p^m t,$$

$$c = r - s + p^m t, \quad d = r' + p^m t',$$

$$e = r - s + p^m(t' + t'' - t'''), \quad f = r' - r + s + p^m(t''' - t'').$$

We claim that $\{\sigma_{\underline{v}}^{-1}\}_{\underline{v}\in V}$ (or a subset of it) is a system of coset representatives for the quotient Q. We only sketch the proof of this, which consists of a very long but straightforward calculation. Given $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in K'_{n,m}$, we wish to prove that there exists $\underline{v} \in V$ such that $\sigma_{\underline{v}} g = \left(\begin{smallmatrix} E & F \\ G & H \end{smallmatrix}\right) \in K'_{n,m(p)}$.

Writing down carefully the eight equations modulo p^{m+1}, the four modulo $p^{2(m+1)}$ and the remaining one modulo $p^{3(m+1)}$, we determine $\underline{v}$ by choosing ten of those equations and showing, by the use of the symplectic equations, that the other three equations are redundant. Slightly more precisely, we have, after reducing the equations modulo p^{m+1}

$$\begin{cases} a_{12} - p^m r' a_{22} - p^m r a_{32} \equiv 0 \ [p^{m+1}] \\ a_{13} - p^m r' a_{23} - p^m r a_{33} \equiv 0 \ [p^{m+1}] \\ d_{13} + p^m r \equiv 0 \ [p^{m+1}] \\ d_{23} + p^m r' \equiv 0 \ [p^{m+1}] \\ b_{21} + p^m (r-s) d_{11} + p^m k' d_{21} \equiv 0 \ [p^{m+1}] \\ b_{22} + p^m (r-s) d_{12} + p^m k' d_{22} \equiv 0 \ [p^{m+1}] \\ b_{31} + p^m k'' d_{11} + p^m (r-s) d_{21} \equiv 0 \ [p^{m+1}] \\ b_{32} + p^m k'' d_{12} + p^m (r-s) d_{22} \equiv 0 \ [p^{m+1}] \end{cases}$$

From the second pair of equations we get r and r' and, after replacing $p^m r$ and $p^m r'$, the first pair becomes redundant by the use of the symplectic equations of g

$$A^t I_3' D - C^t I_3' B = \nu(g) I_3'.$$

Indeed, comparing the entries $(2,3)$ gives

$$a_{12}d_{33} + a_{22}d_{23} + a_{32}d_{13} - c_{12}b_{33} + c_{22}b_{23} + c_{32}b_{13} = 0,$$

which reduces modulo p^{m+1} to

$$a_{12} + a_{22}d_{23} + a_{32}d_{13} \equiv 0 \ [p^{m+1}],$$

which coincides with the first equation after substituting d_{12} and d_{23} with $-p^m r$ and $-p^m r'$. Similarly, we get the redundancy of the second equation by comparing the entries $(3,3)$ modulo p^{m+1}.

To solve s, k and k'' from the third series of equations, one has to show that the rank of the matrix

$$\begin{pmatrix} d_{11} & d_{21} & 0 & -b_{21} \\ d_{12} & d_{22} & 0 & -b_{22} \\ d_{21} & 0 & d_{11} & -b_{31} \\ d_{22} & 0 & d_{12} & -b_{32} \end{pmatrix}$$

is three. The fact that its rank is at least three follows by the fact that the determinant of $A^t I_3' D$ is invertible modulo p^{m+1} (all entries of B are divisible by p) and so

$$det(D) \equiv d_{11}d_{22} - d_{21}d_{12} \equiv d_{11}d_{22} \ [p^{m+1}]$$

is invertible as well. Hence, we can find a 3×3 minor with invertible determinant. Finally, the fact that the big determinant is zero follows from an application of the relation

$$d_{12}b_{31} + d_{22}b_{21} \equiv d_{11}b_{32} + d_{21}b_{22} \ [p^{m+1}],$$

from the symplectic equations of g

$$B^t I_3' D - D^t I_3' B = 0.$$

Indeed, unfolding the calculation of the determinant we get

$$d_{11}\left[d_{22}(d_{22}b_{21} - d_{11}b_{32}) - d_{21}(d_{22}b_{22} - d_{12}b_{32})\right] - d_{12}\left[d_{22}(d_{21}b_{21} - d_{11}b_{31}) - d_{21}(d_{21}b_{22} - d_{12}b_{31})\right] \equiv$$

$$\equiv d_{11}d_{22}(d_{22}b_{21} + d_{12}b_{31} - d_{11}b_{32} - d_{21}b_{22}) + d_{12}d_{21}(d_{12}b_{32} + d_{21}b_{22} - d_{22}b_{21} - d_{12}b_{31}) \equiv$$

$$\equiv (d_{11}d_{22} - d_{12}d_{21})(d_{22}b_{21} + d_{12}b_{31} - d_{11}b_{32} - d_{21}b_{22}) \equiv 0 \ [p^{m+1}]$$

The rest of the equations follow more easily. $\square$

Chapter 5

Norm compatible elements for $\mathbf{GU}(2,2)$

In what follows, we describe the construction of "push-forward" cohomology classes in the fifth cohomology group of a unitary Shimura variety associated to the unitary group $\mathbf{GU}(2,2)$. We prove that the resulting classes are trace compatible with respect to a two variable family of level subgroups of $\mathbf{GU}(2,2)(\widehat{\mathbf{Z}})$.

The chapter is organised as follows. In §5.1, we discuss general properties of the $\mathbf{GU}(2,2)$ Shimura variety $\mathrm{Sh}_{\mathbf{GU}(2,2)}$. In §5.2 and §5.3, which are the main core of the chapter, we explain how to construct a two-variable family of norm compatible elements in the cohomology of $\mathrm{Sh}_{\mathbf{GU}(2,2)}$. Finally, in §5.4, we describe why we do not get a family of classes which are norm compatible over the cyclotomic tower at p, and we discuss a few similar cases where the same obstruction occurs.

5.1 The Shimura variety for $\mathbf{GU}(2,2)$

5.1.1 The groups

Let $\mathbf{H} := \mathbf{GSp_4}$ be the group scheme over $\mathbf{Z}$, which was previously defined as

$$\mathbf{H}(R) = \{(g,m_g) \in (\mathbf{GL}_4 \times \mathbf{GL}_1)(R) : g^t J g = m_g J\}.$$

After an auxiliary choice of imaginary quadratic field K with ring of integers $\mathscr{O}_K$, define $\mathbf{G} := \mathbf{GU}(2,2)$ the group scheme over $\mathbf{Z}$ given by

$$\mathbf{G}(R) := \mathbf{GU}(2,2)(R) = \{(g,m'_g) \in \mathbf{GL}_4(R \otimes_{\mathbf{Z}} \mathscr{O}_K) \times \mathbf{GL}_1(R) : \bar{g}^t J g = m'_g J\},$$

where $\bar{\bullet}$ denotes the non-trivial automorphism of order 2 of $K/\mathbf{Q}$. We denote by $\nu : \mathbf{G} \longrightarrow \mathbf{GL}_1$ the unitary multiplier map given by $g \mapsto m'_g$ with kernel $\mathrm{U}(2,2)$. The derived subgroup of $\mathbf{G}$ (and $\mathrm{U}(2,2)$) is $\mathrm{SU}(2,2)$ which is defined as the intersection of $\mathrm{U}(2,2)$ and the kernel of the determinant map $\det : \mathbf{G} \to Res_{\mathscr{O}_K/\mathbf{Z}}(\mathbf{GL}_1)$. As easily depicted from the definition, the description of the $\mathbf{Q}_p$-points of $\mathbf{G}$ for p finite prime depends on whether p splits, ramifies or is inert in K. Denote $\mathbf{Q}_p \otimes_{\mathbf{Q}} K$ by K_p and $\mathbf{Z}_p \otimes_{\mathbf{Z}} \mathscr{O}_K$ by $\mathscr{O}_{K,p}$. Recall that K_p is a product of finite extensions of $\mathbf{Q}_p$ in $1-1$

correspondence with completions of K under extensions of the p-adic valuation on $\mathbf{Q}$. Hence,

1. If p splits in K there is a $\mathbf{Q}_p$-algebra isomorphism $i_p : K_p \longrightarrow \mathbf{Q}_p \times \mathbf{Q}_p$. Explicitly, let i_1, i_2 be the two distinct embeddings of K into $\mathbf{Q}_p$, then

$$a \otimes b \mapsto (i_1(b)a, i_2(b)a).$$

   Under this isomorphism,

$$i_p(a \otimes \bar{b}) = (i_2(b)a, i_1(b)a);$$

   hence, we can identify $\mathbf{G}(\mathbf{Q}_p)$ with

$$\left\{(M,N) \in \mathbf{GL}_4(\mathbf{Q}_p) \times \mathbf{GL}_4(\mathbf{Q}_p) : N^t JM = aJ, \text{ for } a \in \mathbf{Q}_p^*\right\}.$$

   Moreover, the map $(M,N) \mapsto (M,a)$ defines an isomorphism between $\mathbf{G}(\mathbf{Q}_p)$ and $\mathbf{GL}_4(\mathbf{Q}_p) \times \mathbf{GL}_1(\mathbf{Q}_p)$.

2. If p is inert or ramified, K_p is an extension of degree 2 over $\mathbf{Q}_p$; denoting by $a \mapsto \bar{a}$ the non-trivial automorphism of $K_p/\mathbf{Q}_p$, we have

$$\mathbf{G}(\mathbf{Q}_p) = \left\{M \in \mathbf{GL}_4(K_p) : \bar{M}^t JM = a_M J, \text{ for } a_M \in \mathbf{Q}_p^*\right\}.$$

Of fundamental importance in this chapter is the embedding $\varphi : \mathbf{H} \hookrightarrow \mathbf{G}$, given by the natural embedding of $\mathbf{GL}_4$ inside $Res_{\mathcal{O}_K/\mathbf{Z}}\mathbf{GL}_4$.

5.1.2 The Shimura variety

We recall the definition of the Shimura variety attached to $\mathbf{G}$ and its moduli interpretation. Let K be the imaginary quadratic field used to define $\mathbf{G}$ and fix an integral basis $\{1, y\}$ for K. The Shimura datum for $\mathbf{H}$ defines a Shimura datum $(\mathbf{G}, X_{\mathbf{G}})$ via the embedding $\mathbf{H} \hookrightarrow \mathbf{G}$.

Remark 5.1.1. $X_{\mathbf{G}}$ is isomorphic to the Hermitian half-space

$$\mathbb{H} = \{M \in M_{2\times 2}(\mathbf{C}) : \quad -i(M - \bar{M}^t) > 0\}.$$

For any open compact subgroup U of $\mathbf{G}(\mathbf{A}_f)$, we can consider the double quotient space

$$\mathrm{Sh}_{\mathbf{G}}(U)(\mathbf{C}) = \mathbf{G}(\mathbf{Q}) \backslash X_{\mathbf{G}} \times \mathbf{G}(\mathbf{A}_f)/U.$$

As discussed in §2.1, if U is sufficiently small, the quotient $\mathrm{Sh}_{\mathbf{G}}(U)(\mathbf{C})$ is the set of complex points of a smooth quasi-projective variety $\mathrm{Sh}_{\mathbf{G}}(U)$ over the reflex field $E(\mathbf{G}, X_{\mathbf{G}})$.

Remark 5.1.2. Note that the reflex field of $(\mathbf{G}, X_{\mathbf{G}})$ is $\mathbf{Q}$. This easily follows from the existence of the embedding $(\mathbf{H}, X_{\mathbf{H}}) \hookrightarrow (\mathbf{G}, X_{\mathbf{G}})$, which implies that $\mathbf{Q} \subseteq E(\mathbf{G}, X_{\mathbf{G}}) \subseteq E(\mathbf{H}, X_{\mathbf{H}}) = \mathbf{Q}$.

We have a description of $\mathrm{Sh}_{\mathbf{G}}(U)$ as a moduli space of abelian schemes of relative dimension 4 with principal polarisation, level U structure with compatible action of $\mathscr{O}_K$. Consider the functor $\mathscr{L}_U$ from the category of locally Noetherian K-schemes to *Sets*, which to S assigns isomorphism classes of $(A, \lambda, \iota, \alpha_U)$, where

- A/S is an abelian scheme of relative dimension 4;
- λ is a principal polarisation of A;
- $\iota : \mathscr{O}_K \to End_S(A)$ is a homomorphism which is compatible with λ, i.e.

  $$\iota(a)^\vee \circ \lambda = \lambda \circ \iota(\bar{a}),$$

  for all $a \in \mathscr{O}_K$, such that there is a splitting

  $$Lie_S(A) = Lie_S(A)^+ \oplus Lie_S(A)^-,$$

  where the direct summands are locally free $\mathscr{O}_S$-sheaves of ranks 2 and $z \in \mathscr{O}_K$ acts on $Lie_S(A)^+$ by $f(z)$ and on $Lie_S(A)^-$ by $f(\bar{z})$, where $f : K \to \mathscr{O}_S$ is the structure homomorphism;
- α_U is a unitary U-level structure.

Remark 5.1.3.

1. We call the level structure unitary to emphasise the difference with the $\mathbf{GSp}_{2g}$-case and we recall its definition below;
2. The splitting condition on $Lie_S(A)$ is equivalent to requiring that

   $$det(T - \iota(z)|Lie_S(A)) = (T - f(z))^2 (T - f(\bar{z}))^2 \in \mathscr{O}_S[T],$$

   where $f : K \to \mathscr{O}_S$ is the structure homomorphism coming from the K-structure of S;
3. In the case of sufficiently small level U of $\mathbf{G}(\widehat{\mathbf{Z}})$, the functor $\mathscr{L}_U$ is representable by a quasi-projective K-scheme (e.g. [Lan13] Theorem 1.4.1.11) which is identified with the base-change over K of the canonical model $\mathrm{Sh}_{\mathbf{G}}(U)/\mathbf{Q}$.
4. Following (3), the reader might wonder why we do not define an analogous functor, say $\mathscr{L}_U^{\mathbf{Q}}$, over the category of $\mathbf{Q}$-schemes. Note that the condition on $Lie_S(A)$ does not make sense anymore. Namely, if we have a $\mathbf{Q}$-vector space V which does not have a K-structure but an action of $\mathscr{O}_K$ by $\mathbf{Q}$-linear maps, then the endomorphism corresponding to y (where $\{1, y\}$ is the fixed integral basis for K) is only diagonalisable after tensoring by $\mathscr{O}_K$ and the

required decomposition would work for the K-vector space $V \otimes_{\mathbf{Z}} \mathscr{O}_K$. Hence the $\mathbf{Q}$-points of $\mathscr{L}_U^{\mathbf{Q}}$ are the empty set. Note that the same idea applies to any $\mathbf{Q}$-scheme S and locally free $\mathscr{O}_S$-sheaf without K-structure. Since $\mathscr{L}_U$ is representable by $\mathrm{Sh}_{\mathbf{G}}(U) \times_{\mathbf{Q}} \mathrm{Spec}(K)$, then $\mathscr{L}_U^{\mathbf{Q}}$ is representable by the scheme $\mathrm{Sh}_{\mathbf{G}}(U) \times_{\mathbf{Q}} \mathrm{Spec}(K) \to Spec(K) \to Spec(\mathbf{Q})$, obtained by composing with the inclusion $\mathbf{Q} \hookrightarrow K$, which is not isomorphic to the canonical model $\mathrm{Sh}_{\mathbf{G}}(U)$.

5.1.3 Unitary level structures

We now describe what a unitary level structure for an abelian scheme A/S with $\mathscr{O}_K$-action is. Denote by $K_{\mathbf{G}}(p)$ the kernel of reduction modulo p homomorphism $\mathbf{G}(\widehat{\mathbf{Z}}) \longrightarrow \mathbf{G}(\mathbf{Z}/p\mathbf{Z})$. Consider the sesquilinear pairing H on $\mathscr{O}_K^4$ given by $y \cdot J$ and let $\langle \bullet, \bullet \rangle : \mathscr{O}_K^4 \times \mathscr{O}_K^4 \to \mathbf{Z}$ be the skew-symmetric pairing defined by

$$H(x_1, x_2) = \langle x_1, y \cdot x_2 \rangle + y \langle x_1, x_2 \rangle.$$

Note that $\langle \bullet, \bullet \rangle$ restricted to $\mathbf{Z}^4$ is the pairing defined by the matrix J.

As we previously did for symplectic structures, we can define level U-structures for general open compact subgroups of $U \subset \mathbf{G}(\widehat{\mathbf{Z}})$, as follows. Let $K_{\mathbf{G}}(M) \subset \mathbf{G}(\widehat{\mathbf{Z}})$ denote the kernel of reduction modulo M.

Definition 5.1.4. Let U be an open compact subgroup of $\mathbf{G}(\widehat{\mathbf{Z}})$ and for any integer M such that $K_{\mathbf{G}}(M) \subset U$ denote by U_M the quotient $U/K_{\mathbf{G}}(M)$. Then, a unitary level U structure of $(A, \lambda, \iota)_{/S}$ is a collection $\{\alpha_{U_M}\}_M$, where M varies among the integers such that $K_{\mathbf{G}}(M) \subset U$, of elements α_{U_M} such that

1. α_{U_M} is a locally étale defined U_M-orbit of an $\mathscr{O}_K$-equivariant isomorphism

$$\alpha_M : (\mathscr{O}_K/M\mathscr{O}_K)^4 \longrightarrow A[M],$$

with the property that there is an isomorphism $\beta_M : (\mathbf{Z}/M\mathbf{Z})_{/S} \longrightarrow \mu_{M/S}$ which makes the diagram

$$\begin{tikzcd}
(\mathscr{O}_K/M\mathscr{O}_K)^4_{/S} \times_S (\mathscr{O}_K/M\mathscr{O}_K)^4_{/S} \arrow[r, "\langle \bullet, \bullet \rangle"] \arrow[d, "\alpha_M \times \alpha_M"'] & (\mathbf{Z}/M\mathbf{Z})_{/S} \arrow[d, "\beta_M"] \\
A[M] \times_S A[M] \arrow[r, "e_\lambda"] & \mu_{M/S}
\end{tikzcd}$$

commutative.

2. If $L|M$, α_{U_L} correspond to the reduction modulo L of α_{U_M}.

Remark 5.1.5. One could define unitary level structure on the Tate module of the abelian scheme. For instance, a unitary full level p structure on $(A, \lambda, \iota)_{/S}$ corresponds to a collection $\{\alpha_{\bar{s}}\}_{\bar{s}}$ of $\pi_1(S, \bar{s})$-

invariant $K_{\mathbf{G}}(p)$-orbit of an $\mathscr{O}_K$-equivariant isomorphism

$$\alpha_{\bar{s}} : \hat{\mathscr{O}}_K^4 \longrightarrow T_{\bar{s}}(A)$$

which respects the two forms e_λ and $\langle \bullet, \bullet \rangle$ as in Definition 5.1.4, and such that $\alpha_{\bar{s}}$ and $\alpha_{\bar{s}'}$ are canonically identified for any two geometric points $\bar{s}, \bar{s}'$ in the same connected component. As in the symplectic case, $g \in \mathbf{G}(\widehat{\mathbf{Z}})$ acts on the isomorphism $\alpha_{\bar{s}}$ by $\alpha_{\bar{s}} \circ g$.

Consider the following subgroups of $\mathbf{G}(\mathbf{Z}_p)$.

Definition 5.1.6. For any integer $r \geq 1$, define the subgroup $\tilde{U}_1(p^r) \subset \mathbf{G}(\mathbf{Z}_p)$ as follows:

$$\tilde{U}_1(p^r) \quad := \quad \{M \in \mathbf{G}(\mathbf{Z}_p) | R_4(M) \equiv (0, \cdots, 0, 1) \bmod p^r \mathscr{O}_K\} \tag{5.1}$$

where $R_i(M)$ denotes the i-th row of M. If $N = \prod p_i^{e_i}$, then $\tilde{U}_1(N) \subset \mathbf{G}(\widehat{\mathbf{Z}})$ is defined to be the subgroup of elements $(g_p)_p$ such that $g_{p_i} \in \tilde{U}_1(p_i^{e_i})$.

Note that $\tilde{U}_1(p^r) \cap \mathbf{H}(\mathbf{Q}_p) = U_1(p^r)$, which was defined in Definition 2.1.20. As in Remark 2.1.21, $\tilde{U}_1(p^r)$ is given by matrices whose reduction modulo $p^r\mathscr{O}_K$ is in the mirabolic subgroup of the Klingen parabolic of the form

$$\begin{pmatrix} * & * & * & * \\ & * & * & * \\ & * & * & * \\ & & & 1 \end{pmatrix}.$$

Remark 5.1.7. Similarly to the symplectic case, unitary $\tilde{U}_1(p^r)$-level structures correspond to $p^r\mathscr{O}_K$-points. More precisely, let $\mathfrak{P}$ be an ideal of $\mathscr{O}_K$ lying above p and define a $\mathfrak{P}^r$-point of an abelian scheme A/S with $\mathscr{O}_K$-action $\iota : \mathscr{O}_K \longrightarrow End_S(A)$ to be an $\mathscr{O}_K$-linear monomorphism

$$(\mathscr{O}_K/\mathfrak{P}^r)_{/S} \hookrightarrow A[p^r].$$

If p is inert in $\mathscr{O}_K$, a $p^r\mathscr{O}_K$-point is just a point of exact order p^r. Indeed, if $P \in A(S)$ is a point of exact order p^r, it is killed by $\iota(a)$ for all $a \in p^r\mathscr{O}_K$ because $p^r\mathscr{O}_K = (p)^r$ and $\iota(a)P = \iota(a'p^r)P = \iota(a')[p^r]P = 0$, for all $a = a'p^r \in p^r\mathscr{O}_K$. Thus, P defines a monomorphism

$$(\mathscr{O}_K/p^r\mathscr{O}_K)_{/S} \hookrightarrow A[p^r],$$

defined on each geometric fibres (corresponding to $s \to S$) by sending 1 to P_s and identifying $\mathscr{O}_K/p^r\mathscr{O}_K$ with $\langle i(a)P_s \rangle_{a \in \mathscr{O}_K}$. On the other hand, any $p^r\mathscr{O}_K$-point is determined by the point of exact order p^r obtained by the image of 1. In the case where p splits in K, say $p = \mathfrak{P}\bar{\mathfrak{P}}$, then a $p^r\mathscr{O}_K$-point P corresponds to P_1 and P_2 where

- P_1 is a point of exact order p^r which is killed by $\iota(a)$ for all $a \in \mathfrak{P}^r$,
- P_2 is a point of exact order p^r which is killed by $\iota(a)$ for all $a \in \bar{\mathfrak{P}}^r$.

We can now state the unitary analogue of Lemma 2.1.22.

Lemma 5.1.8. *Let (A,λ,ι) be an abelian scheme of relative dimension 4 over a K-scheme S with $\mathscr{O}_K$-module structure and principal polarisation. There is a bijection between unitary level $\tilde{U}_1(N)$-structures and $N\mathscr{O}_K$-points.*

Proof. It is a straightforward modification of the proof of Lemma 2.1.22. □

5.1.4 A remark on the embedding at the level of moduli

We note that $\varphi : \mathbf{H} \hookrightarrow \mathbf{G}$ induces closed immersions $\phi_U : \mathrm{Sh}_{\mathbf{H}}(U \cap \mathbf{H}) \longrightarrow \mathrm{Sh}_{\mathbf{G}}(U)$, for certain open compact subgroups U of $\mathbf{G}(\mathbf{A}_f)$. Due to the moduli space description of these spaces, it is reasonable to ask whether we have a nice explicit description of the pull-back of the universal element of $\mathrm{Sh}_{\mathbf{G}}(U)_K := \mathrm{Sh}_{\mathbf{G}}(U) \times_{\mathbf{Q}} \mathrm{Spec}(K)$ in terms of the (base-change to K) of the universal element of $\mathrm{Sh}_{\mathbf{H}}(U \cap \mathbf{H})$.

First, recall the main properties of Serre's tensor construction for abelian schemes.

Lemma 5.1.9 (Serre's Tensor Construction)**.** *Let R be a ring and M be a finite projective R-module; for any group scheme A with R-module structure over S, the functor from S-schemes to Sets*

$$\mathscr{F} : T \to A(T) \otimes_R M$$

is representable by a group scheme, which is denoted by

$$A \otimes_R M.$$

Moreover, in the case where A/S is an abelian scheme, then we have:

1. *$A \otimes_R M$ is an abelian scheme over S;*
2. *There is a canonical isomorphism*
$$T_\ell(A \otimes_R M) \simeq T_\ell(A) \otimes_R M;$$
3. *There is a canonical isomorphism of $\mathscr{O}_S$-modules*
$$Lie_S(A \otimes_R M) \simeq Lie_S(A) \otimes_R M.$$
4. *We have*
$$(A \otimes_R M)^\vee \simeq A^\vee \otimes_R M^\vee,$$
where $A^\vee$ denotes the dual abelian scheme of A and $M^\vee = Hom_R(M,R)$.

Proof. We only give a sketch of the proof. Note that if M is a free R-module of rank n, $\mathscr{F}$ is representable by A^n; more generally, take a presentation

$$R^m \xrightarrow{g} R^n \longrightarrow M^\vee \longrightarrow 0,$$

and apply $Hom_R(-, A(T))$ to get

$$0 \longrightarrow Hom_R(M^\vee, A(T)) \longrightarrow A(T)^n \xrightarrow{g_T} A(T)^m.$$

Since $A(T) \otimes_R M \simeq Hom_R(M^\vee, A(T))$, we conclude that $\mathscr{F}$ is representable by the kernel of $(g_T)_T$. By [Con04] Theorem 7.2 and Theorem 7.5, the tensor construction preserves smoothness, properness and geometric connectedness of fibres, hence if A is an abelian scheme so is $A \otimes_R M$. Property (2) is a direct consequence of the fact that tensoring by M is left exact (since M is projective), hence $A[\ell^n] \otimes_R M \simeq (A \otimes_R M)[\ell^n]$. (3) is proved similarly (see [AK15], Lemma 3); for a proof of (4), we refer to Proposition 5 of [AK15]. □

We can now prove the main result of this section. Denote by $\mathrm{Sh}_{\mathbf{H}}(U \cap \mathbf{H})_K$ the base-change to K of $\mathrm{Sh}_{\mathbf{H}}(U \cap \mathbf{H})$ and by $(\mathscr{A}_{\mathbf{H},K}, \lambda_{\mathbf{H},K}, \alpha_{\mathbf{H},U\cap\mathbf{H},K})$ the base-change to K of the universal object of $\mathrm{Sh}_{\mathbf{H}}(U \cap \mathbf{H})$.

Proposition 5.1.10. *The abelian scheme $\mathscr{A}_{\mathbf{H},K} \otimes_{\mathbf{Z}} \mathscr{O}_K / \mathrm{Sh}_{\mathbf{H}}(U \cap \mathbf{H})_K$ (and the extra structure) is identified with the pull-back by ϕ_U of the universal object $(\mathscr{A}_{\mathbf{G},K}, \lambda_{\mathbf{G},K}, \iota_K, \alpha_{\mathbf{G},U,K}) / \mathrm{Sh}_{\mathbf{G}}(U)_K$.*

Proof. Consider the pull-back by ϕ_U of $\mathscr{A}_{\mathbf{G},K}$:

$$\begin{tikzcd} ? \arrow[r] \arrow[d] & \mathscr{A}_{\mathbf{G},K} \arrow[d] \\ \mathrm{Sh}_{\mathbf{H}}(U \cap \mathbf{H})_K \arrow[r, "\phi_U"] & \mathrm{Sh}_{\mathbf{G}}(U)_K. \end{tikzcd}$$

We would like to prove it is isomorphic to $\mathscr{A}_{\mathbf{H},K} \otimes_{\mathbf{Z}} \mathscr{O}_K$. By Lemma 5.1.9, we have that $\mathscr{A}_{\mathbf{H},K} \otimes_{\mathbf{Z}} \mathscr{O}_K$ is an abelian scheme over $\mathrm{Sh}_{\mathbf{H}}(U \cap \mathbf{H})_K$ of dimension 4; consider the action

$$\iota : \mathscr{O}_K \longrightarrow End_{\mathrm{Sh}_{\mathbf{H}}(U\cap\mathbf{H})_K}(\mathscr{A}_{\mathbf{H},K} \otimes_{\mathbf{Z}} \mathscr{O}_K),$$

where $\iota(z)$ is given by multiplication by z in the second factor (which is a well-defined endomorphism by [AK15] Proposition 2 (c)). Moreover, ι induces the required decomposition of

$$\mathscr{V} := Lie_{\mathrm{Sh}_{\mathbf{H}}(U\cap\mathbf{H})_K}(\mathscr{A}_{\mathbf{H},K} \otimes_{\mathbf{Z}} \mathscr{O}_K).$$

Indeed, $\mathscr{V}$ is a $\mathscr{O}_S \otimes_{\mathbf{Z}} \mathscr{O}_K$-module and one can consider the elements

$$x_1 = f(y)\otimes 1 - 1\otimes \iota(y),$$
$$x_2 = f(\bar{y})\otimes 1 - 1\otimes \iota(y),$$

where $f : K \to \mathscr{O}_S$ is the structure homomorphism and $\{1, y\}$ is an integral basis for K. The quotients $\mathscr{V}/(x_1)$ and $\mathscr{V}/(x_2)$ are the maximal quotients on which $\mathscr{O}_K$ acts respectively by the structure monomorphism and by its conjugate and since the discriminant D_K is invertible in $\mathscr{O}_S$ we have

$$\mathscr{V} = \mathscr{V}/(x_1) \oplus \mathscr{V}/(x_2).$$

We would like to show that $\lambda_{\mathbf{H},K}$ and the choice of ι give a $\mathscr{O}_K$-linear polarisation on $\mathscr{A}_{\mathbf{H},K}\otimes_{\mathbf{Z}}\mathscr{O}_K$. This amounts to choose a $\mathbf{Z}$-linear isomorphism $g : \mathscr{O}_K \longrightarrow \mathscr{O}_K^\vee$, such that

$$\lambda_{\mathbf{H},K}\otimes g : \mathscr{A}_{\mathbf{H},K}\otimes_{\mathbf{Z}}\mathscr{O}_K \longrightarrow \mathscr{A}_{\mathbf{H},K}^\vee\otimes_{\mathbf{Z}}\mathscr{O}_K^\vee \simeq (\mathscr{A}_{\mathbf{H},K}\otimes_{\mathbf{Z}}\mathscr{O}_K)^\vee$$

is compatible with ι, in the sense that

$$(\lambda_{\mathbf{H},K}\otimes g)\circ\iota(\bar{z}) = \iota(z)^\vee\circ\lambda_{\mathbf{H},K}\otimes g.$$

The isomorphism g is defined to be the composition on the right of $\bar{\bullet} : \mathscr{O}_K \to \mathscr{O}_K$ with the isomorphism $\mathscr{O}_K \to \mathscr{O}_K^\vee$ determined by the choice of our integral basis of K. Hence, $\lambda_{\mathbf{H},K}\otimes g$ defines a polarisation ([AK15], Theorem 17) compatible with the $\mathscr{O}_K$-action. Moreover, $\lambda_{\mathbf{H},K}\otimes g$ is an isomorphism. We are left to show that there is a unitary level structure induced from the symplectic level structure $\alpha_{\mathbf{H},U\cap\mathbf{H},K}$. By part (2) of Lemma 5.1.9, there is an isomorphism

$$(\mathscr{A}_{\mathbf{H},K}\otimes_{\mathbf{Z}}\mathscr{O}_K)[\ell^m] \simeq \mathscr{A}_{\mathbf{H},K}[\ell^m]\otimes_{\mathbf{Z}}\mathscr{O}_K,$$

hence a symplectic full ℓ^m-level structure on $\mathscr{A}_{\mathbf{H},K}$ induces a unitary full ℓ^m-level structure on

$$\mathscr{A}_{\mathbf{H},K}\otimes_{\mathbf{Z}}\mathscr{O}_K.$$

Indeed, take a geometric point s of $\mathrm{Sh}_{\mathbf{H}}(U\cap\mathbf{H})_K$; we consider the base-change to s of the symplectic level structure, so that we get a (symplectic) isomorphism

$$\alpha_s : (\mathbf{Z}/\ell^m\mathbf{Z})^4 \longrightarrow \mathscr{A}_{\mathbf{H},K,s}[\ell^m].$$

Since the tensor product construction commutes with base-change, we have an isomorphism

$$\alpha_s \otimes \mathscr{O}_K : (\mathscr{O}_K/\ell^m \mathscr{O}_K)^4 \longrightarrow (\mathscr{A}_{\mathbf{H},K} \otimes \mathscr{O}_K)_s[\ell^m],$$

which is $\mathscr{O}_K$-equivariant and respects the form given by $\langle \bullet, \bullet \rangle$ and the form given by composition of $\lambda_{\mathbf{H},K} \otimes g$ and Weil pairing, since ι coincide with the $\mathscr{O}_K$-module structure of $(\mathscr{O}_K/\ell^m\mathscr{O}_K)^4$. Thus, it follows that a $U\cap\mathbf{H}$-level symplectic structure induces a U-level unitary structure. Summing all, we constructed a point $\psi \in \mathrm{Sh}_{\mathbf{G}}(U)_K(\mathrm{Sh}_{\mathbf{H}}(U\cap\mathbf{H})_K)$. In particular, for any locally Noetherian K-scheme S, $\psi(S)$ is described by sending the isomorphism class of $(A,\lambda,\eta_{U\cap H})/S$ to the isomorphism class of $(A\otimes_{\mathbf{Z}}\mathscr{O}_K, \lambda\otimes g, \iota, \eta_U)/S$.

By Remark 2.1.8, this morphism corresponds uniquely to a $Gal(\bar{\mathbf{Q}}/K)$-equivariant morphism

$$\psi_{\bar{\mathbf{Q}}} : \mathrm{Sh}_{\mathbf{H}}(U\cap\mathbf{H}) \times_K \mathrm{Spec}(\bar{\mathbf{Q}}) \to \mathrm{Sh}_{\mathbf{G}}(U) \times_K \mathrm{Spec}(\bar{\mathbf{Q}}),$$

which is equal to the descent to $\bar{\mathbf{Q}}$ of ϕ_U. This can be checked by evaluating the two morphisms on the set of special points, which forms a Zariski dense set of $\mathrm{Sh}_{\mathbf{H}}(U\cap\mathbf{H})(\mathbf{C})$. Recall that each special point of $(h,g) \in \mathrm{Sh}_{\mathbf{H}}(U\cap\mathbf{H})(\mathbf{C})$ is associated to a $\mathbf{Q}$-torus T in $\mathbf{H}$ with the property that $h \in X_{\mathbf{H}}$ factors through $T_{/\mathbf{R}}$ (recall that $(\mathbf{H}, X_{\mathbf{H}})$ is the Shimura datum of $\mathrm{Sh}_{\mathbf{H}}$). We denote it by s_T. On the one hand, $\phi(s_T)$ is the special point of $\mathrm{Sh}_{\mathbf{G}}(U)(\mathbf{C})$, associated to T, where T is seen inside $\mathbf{G}$ via $\varphi : \mathbf{H} \to \mathbf{G}$. On the other hand, s_T corresponds to (an isomorphism class of) a CM abelian variety $A = A_T$ with polarisation λ and symplectic level $U\cap\mathbf{H}$ structure $\eta_{U\cap\mathbf{H}}$. Thus,

$$\psi(s_T) = (A\otimes_{\mathbf{Z}}\mathscr{O}_K, \iota, \lambda\otimes g, \eta_U).$$

This coincides with $\phi(s_T)$, which corresponds to the $\mathscr{O}_K$-ification of the polarised Hodge structure corresponding to A. This completes the proof, since the equality $\psi_{\bar{\mathbf{Q}}} = \phi_{U,\bar{\mathbf{Q}}}$ implies that ϕ_U and ψ coincide as morphisms of the models of the Shimura varieties over K. □

Corollary 5.1.11. *The morphism $\phi_U : \mathrm{Sh}_{\mathbf{H}}(U\cap\mathbf{H})_{/K} \longrightarrow \mathrm{Sh}_{\mathbf{G}}(U)_K$ is given by sending the S-point $(A,\lambda,\eta) \in \mathrm{Sh}_{\mathbf{H}}(U\cap\mathbf{H})_{/K}(S)$ to the S-point $(A\otimes_{\mathbf{Z}}\mathscr{O}_K, \lambda\otimes g, \iota, \eta') \in \mathrm{Sh}_{\mathbf{G}}(U)_K(S)$, where $\iota : \mathscr{O}_K \longrightarrow End_S(A\otimes_{\mathbf{Z}}\mathscr{O}_K)$, $\lambda\otimes g$ and η' are defined as in the proof of Proposition 5.1.10.*

Remark 5.1.12. Proposition 5.1.10 and its proof readily generalise to the case where ϕ is the morphism of Shimura data $(\mathbf{GSp}_{2n}, X_{\mathbf{GSp}_{2n}}) \hookrightarrow (\mathbf{GU}(n,n), X_{\mathbf{GU}(n,n)})$.

5.2 A family of trace compatible classes

In this section, we explain how to construct cohomology classes for the Shimura variety $\mathrm{Sh}_{\mathbf{G}}$ starting from Eisenstein classes for $\mathbf{H}$. We prove that the resulting classes are trace compatible with respect to a two variable family of level subgroups of $\mathbf{G}(\widehat{\mathbf{Z}})$.

5.2.1 Definitions

Associated to the closed immersion $\phi_U : \mathrm{Sh}_{\mathbf{H}}(U \cap \mathbf{H}) \longrightarrow \mathrm{Sh}_{\mathbf{G}}(U)$, (for good U) there is the Gysin map

$$\phi_{U,*} : H^3_{\mathrm{mot}}(\mathrm{Sh}_{\mathbf{H}}(U \cap \mathbf{H}), \mathbf{Q}(2)) \longrightarrow H^5_{\mathrm{mot}}(\mathrm{Sh}_{\mathbf{G}}(U), \mathbf{Q}(3)).$$

What are those good U? For our purposes, the following lemma is enough.

Lemma 5.2.1. *Let $U_q \subset \mathbf{G}(\mathbf{Q}_q)$ be a sufficiently small open compact subgroup and let $T = T_q T^{(q)}$ be an open compact subgroup of $\mathbf{H}(\mathbf{A}_f)$ such that $T_q = U_q \cap \mathbf{H}(\mathbf{Q}_q)$; then*

1. *there exists a compact open subgroup $U = U_q U^{(q)}$ of $\mathbf{G}(\mathbf{A}_f)$ with $T \subset U$ and such that φ induces a closed immersion*

$$\phi_U : \mathrm{Sh}_{\mathbf{H}}(T) \hookrightarrow \mathrm{Sh}_{\mathbf{G}}(U);$$

2. *let $p \neq q$ be any prime such that $U^{(q)}$ has trivial component at p; for any open compact $U_p \subset \mathbf{G}(\mathbf{Q}_p)$ with $T_p = U_p \cap \mathbf{H}(\mathbf{Q}_p)$, the morphism*

$$\mathrm{Sh}_{\mathbf{H}}(T_q T_p T^{(q)}) \longrightarrow \mathrm{Sh}_{\mathbf{G}}(U_q U_p U^{(q)})$$

is still a closed immersion.

Proof. (1) is a particular case of [Kis10], Lemma 2.1.2. Now, suppose that $z, z' \in \mathrm{Sh}_{\mathbf{H}}$ have same image in $\mathrm{Sh}_{\mathbf{G}}(U_q U_p U^{(q)})$, i.e. there is $u \in U_q U_p U^{(q)}$ such that $z = z' \cdot u$. We claim that u lies in $T_q T_p T^{(q)}$.
Indeed, since z, z' have same image in $\mathrm{Sh}_{\mathbf{G}}(U_q U_p U^{(q)})$, they map to the same element in $\mathrm{Sh}_{\mathbf{G}}(U)$. Thus, by (1), there exists $t \in T$ such that $z = z' \cdot t$. Hence,

$$z = z' \cdot t = z' \cdot u$$

implies that ut^{-1} fixes z'. Since U acts without fixed points, we have that $ut^{-1} = 1$. In particular, we conclude that $u \in \mathbf{H}$ and, consequently, $u \in T_q T_p T^{(q)}$, which completes the proof of (2). $\square$

Let $\tilde{U}_N$ be the subgroup $\tilde{U}_1(N)\tilde{U}^{(N)} \subset \mathbf{G}(\widehat{\mathbf{Z}})$, where $\tilde{U}^{(N)} \subset \mathbf{G}(\widehat{\mathbf{Z}}^{(N)})$ is a sufficiently small open compact subgroup which satisfies the hypotheses of Lemma 5.2.1 (for a suitable prime $q \nmid N$).

Remark 5.2.2. Note that the level subgroup $\tilde{U}_N$ is trivial outside a finite set of primes $\Sigma_{\tilde{U}_N} \ni q$.

Denote by ϕ_N the closed immersion

$$\mathrm{Sh}_{\mathbf{H}}(\mathbf{H} \cap \tilde{U}_N) \longrightarrow \mathrm{Sh}_{\mathbf{G}}(\tilde{U}_N).$$

In the next section, we show a trace compatibility relation of the push-forward of the Eisenstein classes for $\mathrm{Sh}_{\mathbf{H}}$ in the p-direction.

We will work with the Eisenstein class of trivial weight ${}_c\mathrm{Eis}^0_{2,N} \in H^3_{\mathrm{mot}}(\mathrm{Sh}_{\mathbf{H}}(\tilde{U}_N \cap \mathbf{H}), \mathbf{Q}(g))$, introduced in Definition 3.4.4, and its integral p-adic counterpart ${}_c\mathrm{Eis}^0_{\mathbf{Z}_p,2,N} \in H^3_{\text{ét}}(\mathrm{Sh}_{\mathbf{H}}(\tilde{U}_N \cap \mathbf{H}), \mathbf{Z}_p(g))$, where c is an auxiliary integer coprime to $6N$.

Definition 5.2.3.

1. Let $Z_{\mathbf{G},N}$ be the motivic cohomology class defined by

$$\phi_{N,*}({}_c\mathrm{Eis}^0_{2,N}) \in H^5_{\mathrm{mot}}(\mathrm{Sh}_{\mathbf{G}}(\tilde{U}_N), \mathbf{Q}(3)).$$

2. Let $z_{\mathbf{G},N}$ be the étale class defined by

$$\phi_{N,*}({}_c\mathrm{Eis}^0_{\mathbf{Z}_p,2,N}) \in H^5_{\text{ét}}(\mathrm{Sh}_{\mathbf{G}}(\tilde{U}_N), \mathbf{Z}_p(3)).$$

Remark 5.2.4.

1. As explained in §2.2.6, we can construct classes for $\mathbf{G}$ with non-trivial coefficients by studying branching formulas for the pair of groups $\mathbf{H}, \mathbf{G}$. Branching formulas for these groups can be deduced as follows. Recall that we have (e.g. [FH13, 20.39-20.40])

$$\mathrm{Sp}_4\,\mathbf{C} \simeq \mathrm{Spin}_5\,\mathbf{C} \hookrightarrow \mathrm{Spin}_6\,\mathbf{C} \simeq \mathrm{SU}(2,2)\mathbf{C},$$

where the first and last maps are *exceptional isomorphisms*. Thus, one can use branching formulas [FH13, (25.34)] for $\mathrm{SO}_5\,\mathbf{C} \hookrightarrow \mathrm{SO}_6\,\mathbf{C}$ to deduce formulas for $\mathrm{Sp}_4 \hookrightarrow \mathrm{Spin}_6$ and lift them to formulas for $\mathbf{GSp}_4 \hookrightarrow \mathbf{H}' \hookrightarrow \mathbf{G}$, where $\mathbf{H}'$ is the quasi-split form of GSpin_6 determined by K, which sits in the short exact sequence

$$1 \longrightarrow \mathbf{H}' \longrightarrow \mathbf{G} \xrightarrow{g \mapsto det(g)/\nu(g)^2} ker(N_{K/\mathbf{Q}}) \longrightarrow 1.$$

2. For arithmetic applications, the previous point (together with multiplicity one in branching formulas [FH13, (25.34)]) suggests it might be better to work with the four dimensional sub-Shimura variety $\mathrm{Sh}_{\mathbf{H}'}$ of $\mathrm{Sh}_{\mathbf{G}}$ given by the Shimura datum $(\mathbf{H}', X_{\mathbf{H}'})$, where $X_{\mathbf{H}'}$ is the $\mathbf{H}'(\mathbf{R})$-conjugacy class of

$$h' : \mathbf{S} \longrightarrow \mathbf{H}'_{/\mathbf{R}}, \ a+ib \mapsto \frac{1}{a^2+b^2}\begin{pmatrix} a & & & b \\ & a & b & \\ & -b & a & \\ -b & & & a \end{pmatrix}.$$

5.2.2 Compatibility in the mira-Klingen tower

In the following, we use Corollary 3.4.11 to deduce compatibility relations with respect to traces of natural projection maps $\tau_p : \mathrm{Sh}_{\mathbf{G}}(\tilde{U}_{Np}) \to \mathrm{Sh}_{\mathbf{G}}(\tilde{U}_N)$.

Proposition 5.2.5. *Let $p \notin \Sigma_{\tilde{U}_N}$. Then,*

$$\tau_{p,*}(Z_{\mathbf{G},Np}) = \begin{cases} Z_{\mathbf{G},N} & \text{if } p \mid N; \\ \left(id - d_p^*\right) Z_{\mathbf{G},N} & \text{if } p \nmid N; \end{cases}$$

where $d_p \in \mathbf{H}(\widehat{\mathbf{Z}})$ is any matrix which reduces to $\left(\begin{smallmatrix} I & 0 \\ 0 & pI \end{smallmatrix}\right)$ modulo N.

Proof. It follows from Corollaries 3.4.11 and 3.4.12 . □

Remark 5.2.6. The étale classes $\{z_{\mathbf{G},N}\}_N$ satisfy identical relations.

While the compatibility of these classes in the tower of level subgroups $\{\tilde{U}_N\}_N$ is a natural consequence of the trace compatibility relations of the Eisenstein classes, by using a more sophisticated method, we show in the next sections how to obtain trace compatibility relations in a "two variable" tower of level subgroups $\{\tilde{U}_{N,M}\}$.

5.2.3 Perturbing the embedding I: definitions

When $p \mid N$, we showed that the push-forward of Eisenstein classes defines an element

$$Z_{Np^\infty} \in \varprojlim_i H^5_{\text{mot}}(\mathrm{Sh}_{\mathbf{G}}(\tilde{U}_{Np^i}), \mathbf{Q}(3)),$$

where the limit is taken with respect to traces of the natural projection maps $\mathrm{Sh}_{\mathbf{G}}(\tilde{U}_{Np^{i+1}}) \to \mathrm{Sh}_{\mathbf{G}}(\tilde{U}_{Np^i})$. In order to improve this result, it is necessary to enrich our push-forward classes with extra structure. This is done by employing the action of $G(\mathbf{A}_f)$ on $\mathrm{Sh}_{\mathbf{G}}$, as we see below in Definition 5.2.9. This idea has already been successfully used in the constructions of the Euler systems in [LLZ14], [LLZ16], [LSZ17], and [CRJ18] (or Chapter 4). The method to prove extra trace compatibility relations used here is an adaptation to this setting of the method used in the proof of the vertical norm relation of the Beilinson-Flach Euler system as in [KLZ17, Theorem 5.4.1], and its generalisation in [LZ18].

In the rest of the section we suppose that $p \notin \Sigma_{\tilde{U}_N}$ is inert or split in the imaginary quadratic field K and that N is an integer coprime with p. We now define the tower of level subgroups we are interested in. Let η be the cocharacter of the maximal torus of $\mathbf{G}$ defined by

$$x \mapsto \begin{pmatrix} x^3 & & & \\ & x^2 & & \\ & & x & \\ & & & 1 \end{pmatrix}$$

Let $\eta_p = \eta(p) \in \mathbf{G}(\mathbf{Q}_p)$. It defines an open compact subgroup of $\tilde{U}'_{n,0} \subset \tilde{U}_{Np^n}$ by requiring that $\tilde{U}'_{n,0}$ is the largest subgroup such that we have

$$\eta_p : \mathrm{Sh}_{\mathbf{G}}(\tilde{U}'_{n,0}) \longrightarrow \mathrm{Sh}_{\mathbf{G}}(\tilde{U}_{Np^n}).$$

Moreover, let $\tilde{U}_{n,1} \subset \tilde{U}'_{n,0}$ be the subgroup given by the intersection of $\tilde{U}'_{n,0}$ with

$$\{g \in \mathbf{G}(\mathbf{Z}_p): \quad g \equiv \begin{pmatrix} * & & & \\ & 1 & & \\ & & * & \\ & & & 1 \end{pmatrix} \bmod p\}$$

Reiterating the procedure, we define subgroups of $\mathbf{G}(\mathbf{A}_f)$

- $\tilde{U}'_{n,m} := \tilde{U}_{Np^n} \cap \eta_p^{m+1} \tilde{U}_{Np^n} \eta_p^{-(m+1)} \cap \{g \in \mathbf{G}(\mathbf{Z}_p): \quad g \equiv \begin{pmatrix} * & & & \\ & 1 & & \\ & & * & \\ & & & 1 \end{pmatrix} \bmod p^m\}$;

- $\tilde{U}_{n,m+1} := \tilde{U}'_{n,m} \cap \{g \in \mathbf{G}(\mathbf{Z}_p): \quad g \equiv \begin{pmatrix} * & & & \\ & 1 & & \\ & & * & \\ & & & 1 \end{pmatrix} \bmod p^{m+1}\}$.

Remark 5.2.7. Concretely, for $n > 3m$, $\tilde{U}_{n,m}$ has component at p

$$\left\{ g \in \mathbf{G}(\mathbf{Z}_p) : g \equiv I \bmod \begin{bmatrix} 1 & p^m & p^{2m} & p^{3m} \\ p^n & p^m & p^m & p^{2m} \\ p^n & p^m & 1 & p^m \\ p^n & p^n & p^n & p^n \end{bmatrix} \mathscr{O}_K \right\}.$$

For $u \in \mathbf{G}(\mathbf{A}_f)$, let ϕ_U^u be the composition

$$u \circ \phi_{uUu^{-1}} : \mathrm{Sh}_{\mathbf{H}}(\mathbf{H} \cap uUu^{-1}) \longrightarrow \mathrm{Sh}_{\mathbf{G}}(uUu^{-1}) \longrightarrow \mathrm{Sh}_{\mathbf{G}}(U),$$

where the second arrow is given by right multiplication by u. In particular, consider $u \in \mathbf{G}(\mathbf{A}_f)$ such that

1. $\mathbf{H} \cap u\tilde{U}_{n,m}u^{-1} \subset U_{Np^n}$,

2. $\phi_{u\tilde{U}_{n,m}u^{-1}}$ is a closed immersion.

Remark 5.2.8. The two conditions are satisfied by $u \in \tilde{U}_{Np^n}$ with trivial components at places in $\Sigma_{\tilde{U}_N}$.

For such u, we denote by $\phi^u_{n,m}$ the map $\phi^u_{\tilde{U}_{n,m}}$; moreover, consider pull-backs, which we denote the same way, of ${}_c\mathrm{Eis}^0_{2,Np^n}$ and ${}_c\mathrm{Eis}^0_{\mathbf{Z}_p,2,Np^n}$ to the cohomology groups of $\mathrm{Sh}_{\mathbf{H}}(\mathbf{H} \cap u\tilde{U}_{n,m}u^{-1})$.

Definition 5.2.9. For $u \in G(\mathbf{A}_f)$ satisfying the above conditions, define

$$Z_{n,m,u} := \phi^u_{n,m,*}({}_c\mathrm{Eis}^0_{2,Np^n}) \in H^5_{\mathrm{mot}}(\mathrm{Sh}_{\mathbf{G}}(\tilde{U}_{n,m}), \mathbf{Q}(3));$$

$$z_{n,m,u} := \phi^u_{n,m,*}({}_c\mathrm{Eis}^0_{\mathbf{Z}_p,2,Np^n}) \in H^5_{\mathrm{\acute{e}t}}(\mathrm{Sh}_{\mathbf{G}}(\tilde{U}_{n,m}), \mathbf{Z}_p(3)).$$

5.2.4 Perturbing the embedding II: a two variable compatible family

Consider the following.

Definition 5.2.10. Let $\mathscr{U}_p$ be the Hecke operator defined as the correspondence $\eta_{p,*}\circ \bar{\mathrm{pr}}^*$, where

$$\begin{tikzcd}
\mathrm{Sh}_{\mathbf{G}}(\tilde{U}'_{n,m}) \arrow[d, "\bar{\mathrm{pr}}"'] \arrow[dr, "\eta_p"] & \\
\mathrm{Sh}_{\mathbf{G}}(\tilde{U}_{n,m}) \arrow[r, dotted, "\mathscr{U}_p"] & \mathrm{Sh}_{\mathbf{G}}(\tilde{U}_{n,m}),
\end{tikzcd}$$

with the vertical arrow $\bar{\mathrm{pr}}$ is equal to the natural projection.

We can finally state the main theorem of the chapter.

Theorem 5.2.11. *Suppose $n \geq 3m+3$ and let $m \geq 1$.*

1. *Let $\tau_n : \mathrm{Sh}_{\mathbf{G}}(\tilde{U}_{n+1,m}) \longrightarrow \mathrm{Sh}_{\mathbf{G}}(\tilde{U}_{n,m})$ be the natural projection, then*

$$\tau_{n,*}(Z_{n+1,m,u}) = Z_{n,m,u}.$$

2. *Let $f_{m,*}$ be the trace map associated to $f_m : \mathrm{Sh}_{\mathbf{G}}(\tilde{U}_{n,m+1}) \longrightarrow \mathrm{Sh}_{\mathbf{G}}(\tilde{U}'_{n,m}) \longrightarrow \mathrm{Sh}_{\mathbf{G}}(\tilde{U}_{n,m})$, where the first arrow is the natural projection and the second is right multiplication by η_p. There exists $u \in G(\mathbf{A}_f)$ such that*

$$f_{m,*}(Z_{n,m+1,u}) = \mathscr{U}_p \cdot Z_{n,m,u}.$$

Remark 5.2.12.

1. The analogous statement for the étale classes $\{z_{n,m,u}\}_{n,m}$ holds (and its proof is identical to the one of Theorem 5.2.11). Moreover, after applying the ordinary idempotent $e_{\eta_p} := \lim_{k\to\infty} \mathscr{U}_p^{k!}$ acting on $H^5_{\text{ét}}(\mathrm{Sh}_{\mathbf{G}}(\tilde{U}_{n,m}), \mathbf{Z}_p(3))$, we get a neat compatibility for the étale classes in the second variable (Definition 5.2.14).
2. The proof of Theorem 5.2.11(1) is Proposition 5.2.5.
3. The choice of u of Theorem 5.2.11(2) does not depend on either n or m and it is not unique (see §5.4 and the proof of 5.2.13 for further explanations).

Theorem 5.2.11(2) follows from the following.

Proposition 5.2.13. *There exists an element $u \in G(\mathbf{A}_f)$ such that the commutative diagram*

$$\begin{tikzcd}
 & \mathrm{Sh}_{\mathbf{G}}(\tilde{U}_{n,m+1}) \arrow[d, "\mathrm{pr}"] \\
\mathrm{Sh}_{\mathbf{H}}(u\tilde{U}_{n,m+1}u^{-1}\cap \mathbf{H}) \arrow[ur, "\phi^u_{n,m+1}"] \arrow[r, "pr\circ\phi^u_{n,m+1}"] \arrow[d, "\pi"'] & \mathrm{Sh}_{\mathbf{G}}(\tilde{U}'_{n,m}) \arrow[d, "\bar{\mathrm{pr}}"] \\
\mathrm{Sh}_{\mathbf{H}}(u\tilde{U}_{n,m}u^{-1}\cap \mathbf{H}) \arrow[r, "\phi^u_{n,m}"] & \mathrm{Sh}_{\mathbf{G}}(\tilde{U}_{n,m})
\end{tikzcd}$$

has Cartesian bottom square.

Proposition 5.2.13 is proved in §5.3.

Proof of Theorem 5.2.11(2). Let u be the matrix which appears in Proposition 5.2.13. From the compatibility of pull-backs and push-forwards in Cartesian diagrams, we get

$$\mathrm{pr}_*(Z_{n,m+1,u}) = \bar{\mathrm{pr}}^*(Z_{n,m,u}).$$

Applying the trace of $\eta_p : \mathrm{Sh}_{\mathbf{G}}(\tilde{U}'_{n,m}) \longrightarrow \mathrm{Sh}_{\mathbf{G}}(\tilde{U}_{n,m})$, we have

$$\begin{aligned}\eta_{p,*}(\mathrm{pr}_*(Z_{n,m+1,u})) &= \eta_{p,*}(\bar{\mathrm{pr}}^*(Z_{n,m,u})) \\ &= \mathscr{U}_p \cdot Z_{n,m,u},\end{aligned}$$

where the last equality follows from the very definition of $\mathscr{U}_p$ as the correspondence $\eta_{p,*} \circ \bar{\mathrm{pr}}^*$; this is the desired formula since $\eta_{p,*} \circ \mathrm{pr}_* = f_{m,*}$. □

5.2.5 Projection to the ordinary part

Using the analogous of Theorem 5.2.11 for $z_{n,m,u}$, we define a limiting element where both n and m go to infinity. If we substitute the tower of level subgroups $V_{n,m} := \eta_p^{-m}\tilde{U}_{n,m}\eta_p^m$ for $\tilde{U}_{n,m}$ and the class

$$\mathfrak{z}_{n,m,u} := (\eta_p^m)_*(z_{n,m,u}) \in H^5_{\text{ét}}(\mathrm{Sh}_{\mathbf{G}}(V_{n,m}), \mathbf{Z}_p(3))$$

for $z_{n,m,u}$, Theorem 5.2.11 gives a trace compatibility relation with respect to the natural projections for both n and m varying. This follows from having the commutative diagram

$$\begin{array}{ccc} \mathrm{Sh}_{\mathbf{G}}(\tilde{U}_{n,m+1}) & \xrightarrow{\eta_p^{m+1}} & \mathrm{Sh}_{\mathbf{G}}(V_{n,m+1}) \\ \downarrow{\scriptstyle f_p} & & \downarrow \\ \mathrm{Sh}_{\mathbf{G}}(\tilde{U}_{n,m}) & \xrightarrow{\eta_p^{m}} & \mathrm{Sh}_{\mathbf{G}}(V_{n,m}), \end{array}$$

where right vertical map is the natural projection. Set

$$H^5_{\text{ét}}(\mathrm{Sh}_{\mathbf{G}}(V_\infty), \mathbf{Z}_p(3)) := \varprojlim_{n,m} H^5_{\text{ét}}(\mathrm{Sh}_{\mathbf{G}}(V_{n,m}), \mathbf{Z}_p(3)).$$

The ordinary idempotent $e_{\eta_p} := \lim_{k\to\infty} \mathscr{U}_p^{k!}$ acts on it.

Proposition 5.2.14. *We define*

$$\mathfrak{z}_u^{ord} := (\mathscr{U}_p^{-m} \cdot e_{\eta_p}(\mathfrak{z}_{n,m,u}))_{n,m\geq 1} \in e_{\eta_p} \cdot H^5_{\acute{e}t}(\mathrm{Sh}_{\mathbf{G}}(V_\infty), \mathbf{Z}_p(3)).$$

Proof. The class $\mathfrak{z}_u^{\mathrm{ord}}$ is well-defined, thanks to the trace compatibility relations of Theorem 5.2.11. $\square$

5.3 Proof of Proposition 5.2.13

In the following, we show that there exists $u \in \mathbf{G}(\mathbf{A}_f)$ such that the diagram of Proposition 5.2.13 has Cartesian bottom square. This follows from showing that there exists $u \in \mathbf{G}(\mathbf{A}_f)$ such that

1. $\mathrm{pr} \circ \phi^u_{n,m+1}$ is a closed immersion, i.e.

$$u\tilde{U}_{n,m+1}u^{-1} \cap \mathbf{H} = u\tilde{U}'_{n,m}u^{-1} \cap \mathbf{H}.$$

2. the degrees of π and $\bar{\mathrm{pr}}$ agree, i.e.

$$[\tilde{U}_{n,m} : \tilde{U}'_{n,m}] = [u\tilde{U}_{n,m}u^{-1} \cap \mathbf{H} : u\tilde{U}_{n,m+1}u^{-1} \cap \mathbf{H}].$$

Here n is always assumed to be bigger than $3m+3$. We treat the cases of split and inert p in K separately.

5.3.1 The split case

Let p be split in K and denote $\mathbf{Z}_p \otimes_{\mathbf{Z}} \mathscr{O}_K$ by $\mathscr{O}_{K,p}$. As we discussed in §5.1.1, there is an isomorphism between $\mathbf{G}(\mathbf{Z}_p)$ and $\mathbf{GL}_4(\mathbf{Z}_p) \times \mathbf{G}_{\mathrm{m}}(\mathbf{Z}_p)$ and $\mathbf{H}(\mathbf{Z}_p)$ embeds into $\mathbf{GL}_4(\mathbf{Z}_p) \times \mathbf{G}_{\mathrm{m}}(\mathbf{Z}_p)$ via $M \mapsto (M, \mu(M))$, where μ denotes the symplectic multiplier.

We claim that

$$u := \left(\begin{pmatrix} 1 & & -1 & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, 1 \right) \in \mathbf{GL}_4(\mathbf{Z}_p) \times \mathbf{G}_{\mathrm{m}}(\mathbf{Z}_p)$$

satisfies the properties (1),(2) listed above.

Lemma 5.3.1. *We have $u\tilde{U}_{n,m+1}u^{-1} \cap \mathbf{H} = u\tilde{U}'_{n,m}u^{-1} \cap \mathbf{H}$.*

Proof. We want to show that if $g = (g_p)_p \in \tilde{U}'_{n,m}$ is such that $ugu^{-1} \in \mathbf{H}$, then $g \in \tilde{U}_{n,m+1}$. Since this is a local statement at p, it is enough to verify that if $g_p = \left(\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right), \alpha \right)$ satisfies $ug_pu^{-1} \in \mathbf{H}(\mathbf{Z}_p)$, then

$$g_p \equiv \left(\begin{pmatrix} x & & & \\ & 1 & & \\ & & x & \\ & & & 1 \end{pmatrix}, x \right) \pmod{p^{m+1}}.$$

Reducing modulo p^{m+1}, we get $ug_pu^{-1} \equiv \left(\begin{pmatrix} a_1 & -c & a_1-d_1 & 0 \\ 0 & a_4 & 0 & 0 \\ 0 & c & d_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \alpha \right)$, which is symplectic if

$$\begin{pmatrix} 0 & a_1 \\ a_4d_1 & -c \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & a_1-d_1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a_1-d_1 & 0 \end{pmatrix}.$$

Thus, $a_1 - d_1 \equiv a_4 - 1 \equiv c \equiv 0 \pmod{p^{m+1}}$, which implies that

$$g_p \equiv \left(\begin{pmatrix} a_1 & & & \\ & 1 & & \\ & & a_1 & \\ & & & 1 \end{pmatrix}, a_1 \right) \pmod{p^{m+1}}.$$

□

We are left to check that

Lemma 5.3.2. *We have*

$$[\tilde{U}_{n,m} : \tilde{U}'_{n,m}] = [u\tilde{U}_{n,m}u^{-1} \cap \mathbf{H} : u\tilde{U}'_{n,m}u^{-1} \cap \mathbf{H}].$$

Proof. Note that $[\tilde{U}_{n,m} : \tilde{U}'_{n,m}] = p^{10}$, since a left coset of representatives is given by

$$\sigma_{\underline{v}} = \left(\begin{pmatrix} 1 & p^m k_1 & p^{2m} r_1 & p^{3m} r_2 \\ & 1 & p^m r_3 & p^{2m} r_4 \\ & & 1 & p^m k_2 \\ & & & 1 \end{pmatrix}, 1 \right),$$

where for each vector $\underline{v} \in \mathbf{Z}/p^3\mathbf{Z} \times (\mathbf{Z}/p^2\mathbf{Z})^2 \times (\mathbf{Z}/p\mathbf{Z})^3$ we consider one and only one lift

$$(r_2, r_1, r_4, k_1, r_3, k_2) \in \mathbf{Z}_p^{10}.$$

Now, recall that the p-component of $u\tilde{U}_{n,m}u^{-1} \cap \mathbf{H}$ is isomorphic to the subgroup of $\mathbf{G}(\mathbf{Z}_p)$ given by elements (g, α) such that $u(g, \alpha)u^{-1} \in \mathbf{H}$ and

$$g \equiv \begin{pmatrix} \alpha & & & \\ & 1 & & \\ & & \alpha & \\ & & & 1 \end{pmatrix} \bmod \begin{bmatrix} p^m & p^m & p^{2m} & p^{3m} \\ p^n & p^m & p^m & p^{2m} \\ p^n & p^m & p^m & p^m \\ p^n & p^n & p^n & p^n \end{bmatrix}.$$

Moreover, from Lemma 5.3.1 we have $u\tilde{U}'_{n,m}u^{-1} \cap \mathbf{H} = u\tilde{U}_{n,m+1}u^{-1} \cap \mathbf{H}$, hence

$$[u\tilde{U}_{n,m}u^{-1} \cap \mathbf{H} : u\tilde{U}'_{n,m}u^{-1} \cap \mathbf{H}] = [u\tilde{U}_{n,m}u^{-1} \cap \mathbf{H} : u\tilde{U}_{n,m+1}u^{-1} \cap \mathbf{H}].$$

We claim that a system of coset representatives is given by a subset of the set of elements

$$\sigma'_{\underline{v}} = \left(\begin{pmatrix} 1+p^m s_1 & p^m k_1 & p^{2m} r_1 & p^{3m} r_2 \\ & 1+p^m s_2 & p^m r_3 & p^{2m} r_4 \\ & p^m t & 1 & p^m k_2 \\ & & & 1 \end{pmatrix}, 1 + p^m s_1 \right) \in \tilde{U}_{n,m}$$

where for each vector $\underline{v} \in \mathbf{Z}/p^3\mathbf{Z} \times (\mathbf{Z}/p^2\mathbf{Z})^2 \times (\mathbf{Z}/p\mathbf{Z})^6 =: S$ we consider one and only one lift

$$(r_2, r_1, r_4, k_1, r_3, k_2, s_1, s_2, t) \in \mathbf{Z}_p^{13}.$$

More precisely, a system of left coset representatives is $\{u(\sigma'_{\underline{w}})^{-1}u^{-1}\}_{\underline{w} \in W}$, where $W \subset S$, which is determined by the symplectic conditions for $u\sigma'_{\underline{w}}u^{-1}$ *modulo (powers of) p*, is defined to be the subset of cardinality p^{10} of elements of the form

$$(r_2, r_1, r_4, k_1, r_3, k_2, 0, 0, k_1 + k_2) \in S.$$

To prove our claim we need to show that for any $g = \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ 0 & a_4 & b_3 & b_4 \\ 0 & c_2 & d_1 & d_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \tilde{U}_{n,m}$ such that $ugu^{-1} \in \mathbf{H}$, there exists $\sigma'_{\underline{v}}$ with $\underline{v} \in W$ such that

$$u\sigma'_{\underline{v}} g u^{-1} \in u\tilde{U}_{n,m+1}u^{-1} \cap \mathbf{H}.$$

This boils down to solving the system of equations

$$\begin{cases} b_2 + p^m k_1 b_4 + p^{2m} r_1 d_2 + p^{3m} r_2 \equiv 0 \ [p^{3m+3}] \\ b_1 + p^m k_1 b_3 + p^{2m} r_1 d_1 \equiv 0 \ [p^{2m+2}] \\ b_4 + p^m r_3 d_2 + p^{2m} r_4 \equiv 0 \ [p^{2m+2}] \\ a_2 + p^m k_1 a_4 \equiv 0 \ [p^{m+1}] \\ b_3 + p^m r_3 \equiv 0 \ [p^{m+1}] \\ p^m(k_1 + k_2) b_4 + d_2 + p^m k_2 \equiv 0 \ [p^{m+1}] \end{cases}$$
$$\begin{cases} a_4 + p^m r_3 c_2 \equiv 1 \ [p^{m+1}] \\ p^m(k_1 + k_2) a_4 + c_2 \equiv 0 \ [p^{m+1}] \\ p^m(k_1 + k_2) b_3 + d_1 \equiv a_1 \ [p^{m+1}] \end{cases}$$

From the first system of equations, we determine the values of r_2, r_1, r_4, k_1, r_3, and k_2 since d_1 and a_4 are invertible modulo p. In particular, the fourth and sixth equations give

$$p^m(k_1 + k_2) \equiv -a_4^{-1} a_2 - d_2 \ [p^{m+1}].$$

Thus, the second system of equations reduces to

$$\begin{cases} a_4 \equiv 1 \ [p^{m+1}] \\ -a_2 - d_2 + c_2 \equiv 0 \ [p^{m+1}] \\ d_1 \equiv a_1 \ [p^{m+1}] \end{cases}$$

and it is redundant. Indeed, unfolding the symplectic conditions of ugu^{-1} modulo p^{m+1}, we get

$$\begin{cases} a_1 \equiv a_4 d_1 \ [p^{m+1}] \\ a_2 - c_2 + a_4 d_2 \equiv 0 \ [p^{m+1}] \\ d_1 b_4 \equiv b_2 + a_1 - d_1 + d_2 b_3 \ [p^{m+1}] \end{cases}$$

which gives

$$\begin{cases} a_4 \equiv 1 \ \ [p^{m+1}] \\ c_2 \equiv a_2 + d_2 \ \ [p^{m+1}] \\ d_1 \equiv a_1 \ \ [p^{m+1}] \end{cases}$$

since $b_4, b_2, b_3 d_2 \equiv 0$ modulo p^{m+1}. Thus, we conclude that

$$[\tilde{U}_{n,m} : \tilde{U}'_{n,m}] = [u\tilde{U}_{n,m}u^{-1} \cap \mathbf{H} : u\tilde{U}'_{n,m}u^{-1} \cap \mathbf{H}] = p^{10}.$$

□

5.3.2 The inert case

Let K_p be the p-adic completion of K at p; it is an extension of degree 2 over $\mathbf{Q}_p$ and denote by $\bar{\bullet}$ the non-trivial automorphism in the Galois group. Let $e \in \mathscr{O}_{K_p}$ be a generator of $\mathscr{O}_K/(p\mathscr{O}_K + \mathbf{Z})$ and consider

$$u = \begin{pmatrix} 1 & & e & \\ & 1 & & \bar{e} \\ & & 1 & \\ & & & 1 \end{pmatrix} \in \mathbf{G}(\mathbf{Z}_p).$$

We claim that u satisfies (1),(2) listed above.

Lemma 5.3.3. *We have $u\tilde{U}_{n,m+1}u^{-1} \cap \mathbf{H} = u\tilde{U}'_{n,m}u^{-1} \cap \mathbf{H}$.*

Proof. As for Lemma 5.3.1, it suffices to show that if $g = (g_p)_p \in \tilde{U}'_{n,m}$ is such that $ug_pu^{-1} \in \mathbf{H}$, then $g \in \tilde{U}_{n,m+1}$. Note that the condition $ug_pu^{-1} \in \mathbf{H}(\mathbf{Z}_p)$ is equivalent to asking that ug_pu^{-1} has entries in $\mathbf{Z}_p$. Modulo $p^{m+1}\mathscr{O}_K$, we have

$$ug_pu^{-1} \equiv \begin{pmatrix} a_1 & ec_2 & e(d_1-a_1) & e\bar{e}c_2 \\ 0 & a_4 & 0 & \bar{e}(1-a_4) \\ 0 & c_2 & d_1 & \bar{e}c_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus, $ug_pu^{-1} \in \mathbf{H}(\mathbf{Z}_p)$ implies

$$a_1 - d_1 \equiv a_4 - 1 \equiv c_2 \equiv 0 \ (\mathrm{mod}\ p^{m+1}),$$

hence $g \in \tilde{U}_{n,m+1}$. □

Lemma 5.3.4. *We have*

$$[\tilde{U}_{n,m} : \tilde{U}'_{n,m}] = [u\tilde{U}_{n,m}u^{-1} \cap \mathbf{H} : u\tilde{U}'_{n,m}u^{-1} \cap \mathbf{H}].$$

Proof. As in the split case, we have

$$[\tilde{U}_{n,m} : \tilde{U}'_{n,m}] = p^{10}.$$

Indeed, a system of coset representatives is given by

$$\sigma_{\underline{v}} = \begin{pmatrix} 1 & p^m k_1 & p^{2m} r_1 & p^{3m} r_2 \\ & 1 & p^m r_3 & p^{2m} r_4 \\ & & 1 & -p^m \bar{k}_1 \\ & & & 1 \end{pmatrix} \; : \; \begin{cases} r_2 - k_1 r_4 \in \mathbf{Z}_p, \\ r_4 = \bar{r}_1 - \bar{k}_1 r_3, \end{cases}$$

where for each vector $\underline{v} \in \mathbf{Z}/p^3\mathbf{Z} \times \mathscr{O}_K/p^2\mathscr{O}_K \times \mathscr{O}_K/p\mathscr{O}_K \times \mathbf{Z}/p\mathbf{Z}$ we consider one and only one lift

$$(\tilde{r}_2, r_1, k_1, r_3) \in \mathbf{Z}_p \times \mathscr{O}_{K_p}^2 \times \mathbf{Z}_p,$$

so that $r_2 = \tilde{r}_2 + k_1(\bar{r}_1 - \bar{k}_1 r_3)$.
The calculation of $[u\tilde{U}_{n,m}u^{-1} \cap \mathbf{H} : u\tilde{U}'_{n,m}u^{-1} \cap \mathbf{H}] = p^{10}$ is very similar to the one in Lemma 5.3.2. Here, a system of left coset representatives is formed by elements $u\sigma'_{\underline{w}}u^{-1} \in \mathbf{H}$, where

$$\sigma'_{\underline{w}} := \begin{pmatrix} 1+p^m s_1 & p^m k_1 & p^{2m} r_1 & p^{3m} r_2 \\ & 1+p^m s_2 & p^m r_3 & p^{2m} r_4 \\ & p^m t & 1+p^m s_3 & p^m k_2 \\ & & & 1 \end{pmatrix} \in \tilde{U}_{n,m}$$

where for each vector

$$\underline{w} \in (\mathbf{Z}/p\mathbf{Z})^5 \times \mathbf{Z}/p^3\mathbf{Z} \times (\mathscr{O}_K/p\mathscr{O}_K)^2 \times (\mathscr{O}_K/p^2\mathscr{O}_K)^2$$

we consider one and only one lift

$$(s_1, s_2, s_3, r_3, t, r_2, k_1, k_2, r_1, r_4) \in \mathbf{Z}_p^6 \times \mathscr{O}_{K_p}^4$$

subject to conditions

$$\begin{cases} s_1 = s_2 + s_3 + p^m(s_2 s_3 - t r_3), \\ \bar{k}_1 = -k_2 - p^m(k_2 s_2 + p^m t r_4), \\ \bar{k}_2 r_4 \in \mathbf{Z}_p, \\ \bar{r}_1 = (1 + p^m s_3) - k_2 r_3, \\ k_2 - \bar{e}t \in \mathbf{Z}_p, \\ p^m r_4 + \bar{e} s_2 \in \mathbf{Z}_p, \end{cases}$$

Looking carefully at the system, we can recover s_1, s_2, k_1, k_2, and r_1 in terms of s_3, r_3, t, r_2, and r_4. Indeed, k_2 is obtained from equations 3 and 5, while the values s_1, s_2, k_1, and r_1 are determined respectively by equations 1, 6 , 2, and 4. Thus, a system of left coset representatives is given by $\{u\sigma'_{\underline{w}}u^{-1}\}_{\underline{w}}$, where $\underline{w}$ ranges through all the vectors in $(\mathbf{Z}/p\mathbf{Z})^3 \times \mathbf{Z}/p^3\mathbf{Z} \times (\mathscr{O}_K/p^2\mathscr{O}_K)$.

□

5.4 A few remarks

5.4.1 Cyclotomic Norm relations

In [LLZ14], [LLZ16], [LSZ17], and [CRJ18] (or Chapter 4), the trace compatibility relations obtained by varying level subgroups $\tilde{U}_{n,m}$ with respect to m has a primary role in proving the vertical Euler system norm relations in the p-cyclotomic tower. Theorem 5.2.11 does not give any result in this direction. In this section, we describe the nature of the obstruction that we encounter when trying to prove cyclotomic norm relations using this method. This informal discussion is much inspired by the work [LZ18], which treats an axiomatisation of the technique used in *op.cit.*

Ideally we would have projections

$$\mathrm{Sh}_{\mathbf{G}}(\tilde{U}_{n,m}) \longrightarrow \mathrm{Sh}_{\mathbf{G}}(\tilde{U}_{Np^n}) \times_{\mathbf{Q}} \operatorname{Spec} \mathbf{Q}(\zeta_{p^m})$$

and we would read the compatibility under $f_{p,*}$ of Theorem 5.2.11 as one under the trace map associated to the natural projection $\operatorname{Spec} \mathbf{Q}(\zeta_{p^{m+1}}) \to \operatorname{Spec} \mathbf{Q}(\zeta_{p^m})$. Unfortunately,

$$\nu(\tilde{U}_{n,m}) \subset \widehat{\mathbf{Z}}^*$$

has p-part equal to $\mathbf{Z}_p^*$, thus $\mathrm{Sh}_{\mathbf{G}}(\tilde{U}_{n,m})$ does not surjects onto $\mathrm{Sh}_{\mathbf{G}}(\tilde{U}_{Np^n}) \times_{\mathbf{Q}} \operatorname{Spec} \mathbf{Q}(\zeta_{p^m})$ since it does not have enough connected components at p.
Motivated by the Euler system constructions mentioned above, we could try to modify the tower of subgroups $\{\tilde{U}_{n,m}\}$ by defining

$$\tilde{V}_{n,m} := \tilde{U}_{Np^n} \cap \eta_p^m \tilde{U}_{Np^n} \eta_p^{-m} \cap \{g \in \mathbf{G}(\mathbf{Z}_p) : \quad \nu(g) \equiv 1 \ (\mathrm{mod}\ p^m)\},$$

instead of intersecting $\tilde{U}_{Np^n} \cap \eta_p^m \tilde{U}_{Np^n} \eta_p^{-m}$ with

$$\{g \in \mathbf{G}(\mathbf{Z}_p) : \quad g \equiv \begin{pmatrix} * & & & \\ & 1 & & \\ & & * & \\ & & & 1 \end{pmatrix} \bmod p^m\}.$$

Then, the corresponding Shimura variety $\mathrm{Sh}_{\mathbf{G}}(\tilde{V}_{n,m})$ has enough connected components and it is reasonable to ask if it is possible to find $u \in \mathbf{G}(\mathbf{A}_f)$ such that the diagram

$$\begin{tikzcd}
& & \mathrm{Sh}_{\mathbf{G}}(\tilde{V}_{n,m+1}) \arrow[d, "\mathrm{pr}"] \\
\mathrm{Sh}_{\mathbf{H}}(u\tilde{V}_{n,m+1}u^{-1} \cap \mathbf{H}) \arrow[urr, "\phi^u_{\tilde{V}_{n,m+1}}"] \arrow[rr, "pr\circ\phi^u_{\tilde{V}_{n,m+1}}"] \arrow[d] & & \mathrm{Sh}_{\mathbf{G}}(\tilde{V}'_{n,m}) \arrow[d] \\
\mathrm{Sh}_{\mathbf{H}}(u\tilde{V}_{n,m}u^{-1} \cap \mathbf{H}) \arrow[rr, "\phi_{\tilde{V}^u_{n,m}}"] & & \mathrm{Sh}_{\mathbf{G}}(\tilde{V}_{n,m})
\end{tikzcd}$$

has Cartesian bottom square. Crucially, $u \in G(\mathbf{A}_f)$ has to be chosen so that the morphism $\mathrm{pr} \circ \phi^u_{\tilde{V}_{n,m+1}}$ is a closed immersion. This boils down to showing that, for any $m \geq 1$,

$$u\tilde{V}_{n,m+1}u^{-1} \cap \mathbf{H} = u\tilde{V}'_{n,m}u^{-1} \cap \mathbf{H}. \tag{5.2}$$

How does one determine u such that equality (5.2) is satisfied? Our choice of the co-character η determines a parabolic and its opposite of $\mathbf{G}$, which are respectively the upper and lower-triangular Borels $B_{\mathbf{G}}$ and $\bar{B}_{\mathbf{G}}$ (indeed, conjugation by powers of η_p induces congruences modulo powers of p for upper triangular entries of the elements in $\tilde{U}_{n,m}$). The equality (5.2) follows (by reducing modulo p^{m+1}) from the condition

$$\mathrm{Klin}^o_{\mathbf{H}} \cap u\bar{B}_{\mathbf{G}}u^{-1} \subset \mathrm{Sp}_4, \tag{5.3}$$

where $\mathrm{Klin}^o_{\mathbf{H}}$ is the p-part of the stabiliser of the Eisenstein class for $\mathbf{H}$

$${}_c\mathrm{Eis}^0_{\mathbf{Z}_p,2,Np^\infty} \in H^3_{\text{ét}}(\mathrm{Sh}_{\mathbf{H}}(U_{Np^\infty}), \mathbf{Z}_p(2)).$$

In other words, it denotes the subgroup over $\mathbf{Z}_p$ of the Klingen parabolic of $\mathbf{H}$ of matrices of the form

$$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Remark 5.4.1. Note that $\mathrm{Klin}^o_{\mathbf{H}} \cap u\bar{B}_{\mathbf{G}}u^{-1}$ is the stabiliser of the $\mathrm{Klin}^o_{\mathbf{H}}(\mathbf{Z}_p)$-orbit $u\bar{B}_{\mathbf{G}}$ in the flag variety $\mathbf{G}/\bar{B}_{\mathbf{G}}$.

We cannot find u such that its stabiliser is contained in Sp_4. For instance, let p be split in K, so that $\mathbf{G}(\mathbf{Q}_p)$ is isomorphic to $\mathbf{GL}_4(\mathbf{Q}_p) \times \mathbf{G}_{\mathrm{m}}(\mathbf{Q}_p)$. The lower-triangular Borel $\bar{B}_{\mathbf{G}}$ has co-dimension 6 in $\mathbf{G}$, while $\mathrm{Klin}^o_{\mathbf{H}}$ has dimension 7, thus u satisfies (5.3) if the image under the symplectic multiplier of a space of dimension bigger or equal than 1 is trivial. For sufficiently generic u, conjugation by u rearranges the entries of matrices in $\bar{B}_{\mathbf{G}}$, but the condition that they need to lie in $\mathrm{Klin}^o_{\mathbf{H}}$ does not give enough equations to force these matrices to have multiplier one.

Remark 5.4.2. This is the reason why we define the tower of subgroups $\tilde{U}_{n,m}$ by intersecting it with

$$\{g \in \mathbf{G}(\mathbf{Z}_p) : \quad g \equiv \begin{pmatrix} * & & & \\ & 1 & & \\ & & * & \\ & & & 1 \end{pmatrix} \bmod p^m\}.$$

Indeed, the calculation underlying Proposition 5.2.13 shows that the stabiliser of the open $\mathrm{Klin}^o_{\mathbf{H}}$-orbit $u\bar{B}_{\mathbf{G}}$ is one dimensional and it is isomorphic to the one dimensional subgroup

$$\left\{ \begin{pmatrix} x & & & \\ & 1 & & \\ & & x & \\ & & & 1 \end{pmatrix} \right\}$$

of the maximal torus of $\mathbf{H}$.

5.4.2 Similar Constructions

The natural question arising at this point is on whether there is any push-forward construction of Eisenstein classes for $\mathbf{H}$, where the obstruction described above is not present. Unfortunately, this phenomenon occurs in several similar push-forward constructions. For instance, an analogous of Theorem 5.2.11 works in the case where we consider

1. $\mathbf{H}'_1 := \mathbf{H} \boxtimes \mathbf{GL}_2 \hookrightarrow \mathbf{GSp}_6, \quad \left(\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) \mapsto \begin{pmatrix} A & & & B \\ & a & b & \\ & c & d & \\ C & & & D \end{pmatrix}$
2. $\mathbf{H}'_2 := \mathbf{H} \boxtimes \mathbf{H} \hookrightarrow \mathbf{GSp}_8, \quad \left(\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), \left(\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right)\right) \mapsto \begin{pmatrix} A & & & B \\ & A' & B' & \\ & C' & D' & \\ C & & & D \end{pmatrix}$

and the push-forward of the pull-back of ${}_c\mathrm{Eis}^0_{2,Np^n}$ along the diagram

$$\mathrm{Sh}_{\mathbf{H}}(U_{Np^n}) \xleftarrow{\mathrm{pr}_1} \mathrm{Sh}_{\mathbf{H}'_i}(L_{Np^n} \cap \mathbf{H}'_i) \longrightarrow \mathrm{Sh}_G(L_{Np^n}) \ ,$$

for sufficiently nice level subgroup $L_{Np^n} \subset G(\widehat{\mathbf{Z}})$, where $G \in \{\mathbf{GSp}_6, \mathbf{GSp}_8\}$. In these cases, there is $u \in G$ such that the analogous diagram of Proposition 5.2.13 has Cartesian bottom square. Notice that

1. the stabiliser of the $\mathrm{Klin}^o_{\mathbf{H}} \boxtimes \mathbf{GL}_2$-orbit $u\bar{B}_{\mathbf{GSp}_6}$ in the flag variety $\mathbf{GSp}_6/\bar{B}_{\mathbf{GSp}_6}$ is isomorphic to the one dimensional subgroup $\{\mathrm{diag}(x,1,x,1,x,1)\}$. For instance, u can be taken of the form
$$u = \begin{pmatrix} 1 & 2 & 1 & -1 & 1 & 1 \\ 0 & 3 & 2 & 0 & 1 & -1 \\ 0 & 2 & 2 & 1 & 0 & -1 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$
2. the stabiliser of the $\mathrm{Klin}^o_{\mathbf{H}} \boxtimes \mathbf{H}$-orbit $u\bar{B}_{\mathbf{GSp}_8}$ in the flag variety $\mathbf{GSp}_8/\bar{B}_{\mathbf{GSp}_8}$ is isomorphic to $\{\mathrm{diag}(x,1,x,1,x,1,x,1)\}$.

5.4.2.1 Numerology

The reader might wonder which is the connection between the three cases listed above. They obey the following. Let $G \in \{\mathbf{GSp}_6, \mathbf{GSp}_8, \mathbf{GU}(2,2)\}$, then we have an embedding, for some reductive group H,

$$\mathbf{GL}_2^{\boxtimes 2} \boxtimes H \hookrightarrow \mathbf{H} \boxtimes H \hookrightarrow G \ \rightsquigarrow \mathrm{Sh}_{\mathbf{GL}_2^{\boxtimes 2} \boxtimes H} \hookrightarrow \mathrm{Sh}_{\mathbf{H} \boxtimes H} \hookrightarrow \mathrm{Sh}_G,$$

such that the corresponding push-forward construction of the pull-back of an Eisenstein class for $\mathbf{GL}_2$ to the cohomology of the Shimura variety for $\mathbf{GL}_2^{\boxtimes 2} \boxtimes H$ gives a class in the middle degree plus 1 cohomology group of the variety for G and the aforementioned method of [LZ18] applies, giving a family of norm compatible classes in the cyclotomic tower at p. One of the hypotheses required to apply the method of *loc.cit.* is that the dimension, as a $\mathbf{Z}_p$-group, of the stabiliser $\mathrm{Klin}^o_{\mathbf{GL}_2} \boxtimes \mathbf{GL}_2 \boxtimes H$

of the pull-back of the Eisenstein class ${}_c\mathrm{Eis}^0_{\mathbf{Z}_p,1,p^\infty} \in H^1_{\text{ét}}(\mathrm{Sh}_{\mathbf{GL}_2}(U_{p^\infty}),\mathbf{Z}_p(1))$ is smaller or equal to the dimension of the unipotent radical of the Borel of G. In these cases, we have that

$$\dim(\mathrm{Klin}^o_{\mathbf{GL}_2} \boxtimes \mathbf{GL}_2 \boxtimes H) = \dim U_{B_G} - 1,$$

thus

$$\dim(\mathrm{Klin}^o_{\mathbf{H}} \boxtimes H) = \dim(\mathrm{Klin}^o_{\mathbf{GL}_2} \boxtimes \mathbf{GL}_2 \boxtimes H) + 2 = \dim U_{B_G} + 1,$$

contradicting the same numerology for the push-forward of the pullback of the Eisenstein class ${}_c\mathrm{Eis}^0_{\mathbf{Z}_p,2,p^\infty} \in H^3_{\text{ét}}(\mathrm{Sh}_{\mathbf{H}}(U_{p^\infty}),\mathbf{Z}_p(2))$ to the cohomology of the Shimura variety for $\mathbf{H} \boxtimes H$. This simple heuristic suggests that, when the method of proving norm relations at p holds for the push-forward of a Eisenstein class for $\mathbf{GL}_2$, it seems not possible to apply it to the push-forward of an Eisenstein class for $\mathbf{H}$. The underlying defect of $\dim(\mathrm{Klin}^o_{\mathbf{H}} \boxtimes H) - \dim U_{B_G}$ is interpreted as the dimension of the stabiliser of the open $\mathrm{Klin}^o_{\mathbf{H}} \boxtimes H(\mathbf{Z}_p)$-orbit $uG/\bar{B}_G$ used to perturb the embedding of the Shimura variety for $\mathbf{H} \boxtimes H$ into the one for G.

Remark 5.4.3. We could speculate further and ask whether there might exist a case where the numerology for both cases is satisfied. For example, consider an embedding $\mathbf{H} \boxtimes H \hookrightarrow G$ for G either $\mathbf{GSp}_{2n}$ or $\mathbf{GU}(n,n)$ such that the push-forward of (the pullback of) an Eisenstein class for $\mathbf{H}$ lands in the middle degree + 1 cohomology group. Then, the push-forward of (the pullback of) a $\mathbf{GL}_2$-Eisenstein class through

$$\mathbf{GL}_2 \boxtimes \mathbf{GL}_2 \boxtimes H \hookrightarrow \mathbf{H} \boxtimes H \hookrightarrow G$$

gives another class in the same cohomology group. We can perturb both two embeddings to get classes defined over cyclotomic extensions only when

$$\dim(\mathrm{Klin}^o_{\mathbf{GL}_2} \boxtimes \mathbf{GL}_2 \boxtimes H) \leq \dim U_{B_G} - 2 \quad \rightsquigarrow \quad \dim(H) - 1 \leq \dim U_{B_G} - 7.$$

Now, suppose that $G = \mathbf{GSp}_{2n}$ and that H is a product of symplectic groups, i.e. $H \simeq \boxtimes_{j=1}^{i} \mathbf{GSp}_{2r_j}$ such that $\sum r_j = n-2$; the inequality above becomes

$$\dim(H) - 1 = \sum_j r_j(2r_j+1) \leq n^2 - 7.$$

Since

$$\dim(\mathrm{Sh}_G) = 2\,\mathrm{cod}(\mathrm{Sh}_{\mathbf{GL}_2^{\boxtimes 2} \boxtimes H})$$

we get

$$2\sum r_j^2 = n^2 - n - 4.$$

Thus, we are left to study if there exists $(r_1,\ldots,r_i,n) \in \mathbf{N}^{i+1}$ with $\sum r_j = n-2$ and $2\sum r_j^2 = n^2 - n - 4$

such that

$$\sum_j r_j(2r_j+1) - n^2 + 7 \leq 0 \longleftrightarrow n^2 - n - 4 + n - 2 - n^2 + 7 \leq 0,$$

i.e. $1 \leq 0$, which is false.

Notice that having imposed the fact that our push-forward class lands in the middle degree plus 1 cohomology group is fundamental for arithmetic applications. This is because a large family of Galois representations associated to (cohomological) cuspidal automorphic representations for G tends to appear in the middle degree geometric étale cohomology group of the Shimura variety (e.g. [MT02, Theorem 1]).

At present, it seems reasonable to expect that the technique for proving Euler system norm relations in the tower at p of [LZ18] is suited for infinitely many cases of push-forward constructions involving $\mathbf{GL}_2$-Eisenstein classes, in opposition to what seems to happen for constructions which use Eisenstein classes for symplectic groups greater than $\mathbf{GL}_2$. We will come back to this analysis in future projects.

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