Multi-resource theories and applications to quantum thermodynamics

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A thesis submitted to

University College London

for the degree of

Doctor of Philosophy

Department of Physics and Astronomy

University College London

September 17, 2018

I, Carlo Sparaciari confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

Signed ....   Date ....

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Abstract

Resource theories are a set of tools, coming from the field of quantum information theory, that find applications in the study of several physical scenarios. These theories describe the physical world from the perspective of an agent, who acts over a system to modify its quantum state, while having at disposal a limited set of operations. A noticeable example of a physical theory which has recently been described with these tools is quantum thermodynamics, consisting in the study of thermodynamic phenomena at the nano-scale.

In the standard approach to resource theories, it is usually the case that the constraints over the set of available operations single out a unique resource. In this thesis, we extend the resource theoretic framework to include situations where multiple resources can be identified, and we apply our findings to the study of quantum thermodynamics, to gain a better understanding of quantities like work and heat in the microscopic regime.

We introduce a mathematical framework to study resource theories with multiple resources, and we explore under which conditions these multi-resource theories are reversible. Furthermore, we investigate the interconversion of resources, i.e., in which situations it is possible to exchange between two different kinds of resources. We then apply this formalism to quantum thermodynamics, where the two resources under consideration are energy and entropy. This multi-resource theory allows us to explore thermodynamics when the system under examination is closed or coupled with a thermal environment with a finite size. In addition to our work on multi-resource theories, we study the states of equilibrium of closed systems, known as passive states, and analyse under which circumstances it is possible to extract energy from these states. We show that passivity is energetically unstable, and that, even for closed systems, the only stable states are those with a well-defined temperature.

Impact Statement

The results presented in this thesis concern the fields of quantum information theory and quantum thermodynamics. These two branches of physics are both pivotal in the development of quantum technologies, i.e., devices that exploit the quantum features of nature to outperform their classical counterparts. The most renewed representative of these technologies is probably the quantum computer, but other important examples include quantum sensors, microscopic machines, and quantum simulators. Research into these devices has recently started, with giants like Google and IBM working on their own prototypes, with the United Kingdom funding a national network of Quantum Technology Hubs, and with the European Union investing around €1 billion to create a strong community of experts in this area. In order for quantum technologies to become a reality, we need to progress our scientific and engineering knowledge, and in particular we need to fully understand how thermodynamic and information processes work at the quantum scale.

My work focuses on a set of tools, known as resource theories, which have been proven to be particularly suitable for describing thermodynamic processes at the microscopic scale and quantum information-processing protocols. With the help of these tools, I have designed thermodynamic models able to describe physical situations close to those occurring in a laboratory. For example, I considered scenarios where the main system under examination interacts with a surrounding environment which does not have an infinite size, and is not thermal, i.e., is not described by a single parameter, namely, its temperature. These results have the potential to inform new theoretical and experimental work in quantum thermodynamics. Indeed, it should be possible to build upon the ideas introduced in this thesis to develop new non-idealised models to describe thermodynamics of physical systems, for instance where correlations with the envi-

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ronment are considered. Furthermore, some of the protocols developed in the thesis could be realised in the laboratory, for example to increase the efficiency of energy-extracting machines.

Another part of my work concerns the development of a theoretical framework able to describe tasks where several quantities, or resources, are involved. This work widens the scope of applicability of resource theories, and introduces new concepts – such as the notions of banks and resource interconversion – which could help the theoretical research in this field. Furthermore, this framework provides tools that could be used to gain new quantitative insights into the properties of many-body systems, and in the fundamental resources responsible for the computational quantum advantage.

List of Publications and Preprints

The work presented in this thesis contains material from the following publications and preprints:

1. C. Sparaciari, J. Oppenheim, and T. Fritz, "Resource theory for work and heat," *Physical Review A*, vol. 96, no. 5, p. 052112, 2017.

2. C. Sparaciari, D. Jennings, and J. Oppenheim, "Energetic instability of passive states in thermodynamics," *Nature Communications*, vol. 8, no. 1, p. 1895, 2017.

3. C. Sparaciari, L. del Rio, C. M. Scandolo, P. Faist, and J. Oppenheim, "The first law of general quantum resource theories", *arXiv:1806.04937 [quant-ph]*, 2018.

Other publications by the author are:

4. M. Horodecki, J. Oppenheim, and C. Sparaciari, "Extremal distributions under approximate majorization", *Journal of Physics A: Mathematical and Theoretical*, vol. 51, no. 30, p. 305301, 2018.

Acknowledgments

These three years of Ph.D. have been both an exciting and challenging time of my life, and concluding this path would have been impossible without the professional and emotional support I received from many people. First and foremost, I would like to thank Jonathan, my supervisor, for the many chats about physics we had, and for the freedom he gave me in following my interests and doing my research, while always being available for discussion and guidance.

I am grateful to everyone in my group, for creating a great environment to do research and learn new things. In particular, I want to thank Álvaro for the infinite discussions on thermodynamics and resource theories, and for being the best travel companion I could have hoped for, for almost every conference I went to. I am also grateful to all the collaborators I had during these years, from whom I have learnt a lot. Particularly, I want to thank Lídia, who is one of the best researchers I have ever met, from both a scientific and human point of view. I am also extremely grateful to everyone involved in the CDT, and in particular to Andrew, Dan, and Lopa, for sharing with us their research and passion. I thank my examiners, Alessio Serafini and Giulio Chiribella, for their comments, which considerably improved this thesis.

I want to thank all the first cohort of the CDT, for sharing the joy and pain of the first year; from the travel to Whistler to the business case study. The strongest bonds I have in the UK are with some of these people; thanks to Andrew, Cameron, Claudia, Johnnie, Josh, Lorenzo, Sherif, Tim, and Tom. Likewise, I am grateful to Alex, Mike, and Tom, who made working in an overcrowded office a positive and fun experience.

Finally, I would like to thank my family for their constant love and support; without it, I would not have achieved much in life. And my deepest thank you is for Paola. You shared with me the best moments of my life, and your love has helped me overcome the worst ones.

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"Solo dopo aver conosciuto la superficie delle cose, ... ci si puó spingere a cercare quel che c'è sotto. Ma la superficie delle cose è inesauribile."

Il sig. Palomar, *Palomar (Italo Calvino)*

Introduction

The notion of observer is ubiquitous in physics, as it plays a central role in several different theories, from quantum mechanics to special relativity, from information theory to thermodynamics. An observer is an entity – such as a human being, or a device – able to perform measurements on a given system, or on their surrounding environment, and to store the obtained information. While the specific features of this entity might differ from one theory to another (for example, the observer in quantum mechanics is certainly different from the one in special relativity), the main role of the observer, common to all theories, is to provide a "reference frame" from which we can describe the natural world, so that all the physical phenomena of interest are described from this perspective.

Other kinds of theories can be considered, where the subject plays a less passive role than simply observing their surrounding. In these theories the subject is known as an *agent*, i.e., an entity who can perform measurements on the system under examination and record the outcome (as the observer does), but can additionally use the information obtained to perform operations over the system. Theories involving an agent are used to investigate which transformations can be realised over a system, and which ones are instead impossible given the tools the agent has access to. Furthermore, these theories can be used to design efficient schemes and protocols to perform specific physical tasks.

In this thesis, we focus on a particular class of physical theories based on the notion of agency, namely, on *quantum resource theories*. In these theories, whose formalism originated within the field of quantum information theory, the agent acts over a quantum system while having at disposal a (well-defined) constrained set of operations. The constraints over this set depend on the specific theory one is considering, and they can be due, for example, to some

technological restriction, or to some fundamental law of nature. The goal of the agent is to transform the state of the system, from a given initial state into the desired final one. Depending on the specific initial and final states, and on the constraints over the operations the agent can use, the state transformation may or may not be possible. As a result, in resource theories it is possible to classify states according to their resourcefulness; the most resourceful states are those allowing the agent to create many different target states, while the less resourceful ones can be used to prepare only a few other states. Additionally, resource theories are used to quantify the cost, in terms of a specific resource, of the operations which are not apriori available to the agent.

Resource theories find applications in several branches of physics, such as in quantum communication, in thermodynamics of microscopic systems, and in the study of symmetries and conservation laws. Due to the central role of the agent, these theories are particularly suited for describing thermodynamics, where one is often interested in finding protocols that can be used to efficiently perform a given task – a prominent example being the different cycles for extracting work using heat engines. Furthermore, the structure of resource theories, with their well-defined set of operations, provides helpful insights into the processes occurring in thermodynamics. They remove the necessity of considering different types of transformations (like adiabatic, isothermal, and isochoric processes) by grouping them in a more general and consistent set of operations. Finally, the resource theoretic approach allows us to extend thermodynamics to the case where the states under investigation are out of equilibrium, a common situation when we consider quantum systems, whose energy is comparable to the thermal fluctuations. Examples of results obtained with this approach in thermodynamics include, but are not limited to, the derivation of multiple conditions (or second laws) that regulate thermodynamic transformations of microscopic systems, the extension of classical quantities, such as work and heat, to the quantum regime, and the derivation of fully-quantum fluctuations theorems.

While resource theories have already provided interesting results in different branches of quantum physics, they are commonly used to describe scenarios where the constraints over the agent's actions single out a unique kind of resource. For example, when applied to quantum communication processes, resource theories only quantify the amount of quantum correlation

required by a given task. However, several physical phenomena we observe in nature, and many technological tasks we aim to achieve, depend on the interplay of multiple resources, rather than on a single quantity. This is the case, for instance, of many-body physics, where physical systems exhibit quantum correlations while also obeying the laws of thermodynamics. Similarly, in quantum computation the advantage over classical devices cannot be ascribed to a single quantum feature, and quantities like entanglement, coherence, and purity all seem to play a fundamental role.

In this thesis, we study resource theories with multiple resources, and we make use of these theories to gain a better understanding of thermodynamics in the quantum regime. Specifically, we develop a framework for building and studying resource theories able to describe physical scenarios where several different resources are involved. Examples of specific resource theories with multiple resources can be found in the literature, and our work aims to provide a set of tools which can be applied to all such theories. An interesting result of this analysis is the derivation of a first-law relation for multi-resource theories, which generalises the First Law of Thermodynamics. This relation consists of a single equation which regulates state transformations in multi-resource theories, and it links the change in a specific property of the system (in thermodynamics, the Helmholtz free energy) to the weighted sum of the resources provided during the transformation (for our example, the sum of energy and entropy). This formalism can be used to construct new theories, and to gain a better understanding of the physical phenomena occurring, for example, in many-body physics, or inside quantum devices. Furthermore, the study we perform to derive the first-law relation for multi-resource theories provides additional information about the structure of thermodynamics, and about processes such as Landauer's erasure and Maxwell's demons.

We apply these tools to construct a resource theory for thermodynamics, where the main resources are energy and entropy (or, equivalently, information). This theory allows us to describe both scenarios where the system is isolated from its surroundings, and where it is coupled to an environment. Interesting, the theory provides a certain freedom in the choice of the environment, which we can take to be thermal or not, initially uncorrelated from the system or correlated, and whose size (relative to the system) can be taken to be finite. We specialise this theory to the case where the environment is thermal and finite-sized, so that

any interaction with the system changes its properties. As a result, we find that quantities like work, heat, and the efficiency of Carnot engines are different from the standard ones. The versatility of this theory, which allows us to consider different kinds of environment, makes it a good tool to study the processes taking place in, for example, microscopic devices currently realised in the laboratory, such us microscopic motors and heat engines. Indeed, we can expect that any such devices interact with a non-ideal environment, for instance because correlations are present.

The last part of the thesis is devoted to the study of the equilibrium properties of a class of states, known as passive states. These states describe closed thermodynamic systems, and by definition are energetically stable under unitary dynamics, i.e., their energy cannot be reduced any further by a reversible evolution. However, with the help of tools from information theory and resource theories, we show that these states become energetically unstable if a catalyst[1] is added to the picture. When this additional system is available, we show that the sole equilibrium states under unitary dynamics are thermal states, i.e., states with a well-defined temperature. This result provides a new way to understand the process of thermalisation in closed systems, and gives a meaning to the notion of temperature for such systems. In order to prove the energetic instability of passive states, we design an explicit protocol. This protocol might be realised in the laboratory, and could find applications inside microscopic devices, for instance to improve the efficiency of energy-extraction tasks.

Structure of the thesis

The thesis is divided into two main parts, plus an appendix. The first part, composed by Chs. 1 and 2, provides the necessary background material. The results presented in these two chapters are known in the literature. The second part of the thesis – consisting of Chs. 3, 4, and 5 – contains the results of the research work performed during my PhD. This work has been done in collaboration with both internal and external researches, all of whom contributed significantly to the ideas developed in this thesis. Whenever their contribution has extended to the derivation or partial derivation of a result, we clearly specify it inside the text. The last

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[1] An ancillary system which facilitates a given state transformation, and whose state is left unchanged at the end of the evolution.

part of the thesis contains the appendices, where we provide minor results used in the main chapters, together with longer proofs and derivations which, if added to the main body, would complicate the exposition and the presentation of more important results.

In the first chapter of the background, Ch. 1, we provide an introduction to the tools known as quantum resource theories. We start by introducing the formalism, consisting of a set of allowed operations that an agent can apply over a given quantum system. We introduce the notion of free states, i.e., states that the agent can always prepare using the allowed operations, and the notion of monotones, functions from the set of states to the real numbers, whose value never increases under the action of the allowed operations. We then move to the study of resource theories when many copies of the system are considered, and in this setting we introduce reversible theories, i.e., theories in which an agent can perform any cyclic state transformation without losing resources. Finally, we focus on state transformations where the agent is only allowed to act on a single copy of the system. In particular, we specialise the discussion to the class of resource theories where majorization provides necessary and sufficient conditions for state transformations to be realisable. For these resource theories we also introduce the notion of a catalyst, an additional system which facilitates a given state transformation without being modified by it.

The second and last chapter of the background, Ch. 2, is devoted to the analysis of a resource theory for quantum thermodynamics, whose class of allowed operations is referred to as Thermal Operations. We first define this class of operations, which describes situations in which an agent can act over a quantum system while having access to an infinite thermal reservoir at a given temperature. These operations allow the agent to control the microscopic degrees of freedom of system and reservoir, with the sole constraint that the total energy needs to be preserved exactly. We then move to study the physical features of these operations, and their limitations in describing the processes occurring in nature. While presenting these limitations, we also discuss possible modifications to the theory which extend its applicability to more physically relevant scenarios. We study which state transformations can be achieved using Thermal Operations, and provide the necessary and sufficient conditions for realising these transformations – known as thermo-majorization – when the states considered are diagonal in the energy eigenbasis. Furthermore, we show how these necessary and sufficient conditions

are modified when the agent can use a catalyst to facilitate the state transformation. Finally, we analyse the problem of defining work in quantum thermodynamics, and we introduce the notion of battery, an ancillary system used to store and quantify the work exchanged during a transformation.

In Ch. 3, the first chapter containing original research, we derive a formalism to describe multi-resource theories, i.e., theories able to describe physical tasks where several resources are present. An example of such theories, which we extensively study in the following chapter, is thermodynamics, which can be understood as a theory where both energy and information (or entropy) are fundamental resources necessary to transform the state of the system. We provide a framework to build and analyse multi-resource theories, which are obtained by composing the classes of allowed operations of different single resource theories. We then consider reversibility of multi-resource theories, i.e., the ability to transform forward and backward between two states without losing any of the resources. To study reversibility we introduce the notion of batteries, ancillary systems which store a single kind of resource each. Once the batteries are defined, we turn to the problem of interconversion, i.e., how to exchange one resource for another one. We show that, to achieve resource interconversion, the agent needs to have access to an additional system, which we call a bank. With a bank, the agent can exchange between the resources at a given exchange rate. Finally, with the help of batteries and bank, we introduce a first-law for multi-resource theories, i.e., a single relation which regulates state transformations, where the change in some property of the system is linked to the sum of the resources exchanged with it. When the thermodynamic theory is considered, we show that this relation coincides with the First Law of Thermodynamics.

The second chapter with original results, Ch. 4, can be seen as an application of the formalism developed in the previous chapter. Indeed, we defined a multi-resource theory for thermodynamics, where the allowed operations are given by energy-preserving unitary operations. This theory is able to describe situations where the system is isolated from the environment, but also more interesting scenarios where the environment is present, and is not thermal, nor has an infinite size. We show that the sole resources of the theory are energy and information, and that the theory is reversible, i.e., that cyclic state transformations do not consume either of the two resources. We then make use of the tools developed in the previous chapter to study

how energy can be exchanged for information and vice versa, corresponding to the physical scenarios of Landauer's erasure and Maxwell's demon, respectively. Finally, we show how the theory can be modified to consider, instead of energy and information, two other quantities which are central in thermodynamics, namely, work and heat. We study how much of these two resources are exchanged when the state of a system coupled to a finite-sized thermal reservoir is transformed, and we use this result to compute the efficiency of thermal machines exchanging heat between finite-sized reservoirs. We find that the efficiency of such devices is always sub-Carnot, a limitation which follows from the finiteness of the reservoirs.

The last chapter of the main part of the thesis, Ch. 5, is devoted to the study of a particular kind of equilibrium in thermodynamics, known as passivity. We first introduce the notion of passive states, i.e., states whose energy cannot be reduced by means of unitary operations. Then, we consider the case in which the agent is allowed to use both unitary operations and catalysts. In this setting, we show that energy can be extracted from passive states, and we present an explicit protocol that achieves such extraction for any given passive state. At the end of the transformation, we find that the catalyst is returned to its original state, but correlations with the system are created. We show that, in the limit of an infinite-dimensional catalyst, we can keep the correlations infinitesimally small, and we can map any passive state into a thermal state with a specific temperature, depending on the entropy of the initial state. This result shows that passive states are not equilibrium states when the use of a catalyst is allowed, and that the only equilibrium states in thermodynamics (even for microscopic close systems) are thermal states.

Basic definitions and notation

In this section, we introduce some basic notions which are used throughout the text, and we define here for completeness. We start with the big-O and little-o notations, relevant concepts when considering the asymptotic limit. The *big-O notation* is defined as,

Definition 1 (Big-O notation)**.** *Consider a metric space S, a point $x_0 \in S$, and two real-valued functions $f, g : S \to \mathbb{R}$. We assume g to be non-zero for values in the neighbourhood of x_0, i.e., there exists $\varepsilon > 0$ such that $g(x) \neq 0$ for all $x \in B_\varepsilon(x_0)$, the ball of radius ε around x_0. We say*

that

$$f(x) = O\left(g(x)\right) \ \ for \ x \to x_0, \tag{1}$$

if and only if

$$\limsup_{x \to x_0} \left| \frac{f(x)}{g(x)} \right| < \infty. \tag{2}$$

While the *little-o notation* is

Definition 2 (Little-o notation). *Consider a metric space S, a point $x_0 \in S$, and two real-valued functions $f, g : S \to \mathbb{R}$. We assume g to be non-zero for values in the neighbourhood of x_0, i.e., there exists $\varepsilon > 0$ such that $g(x) \neq 0$ for all $x \in B_\varepsilon(x_0)$, the ball of radius ε around x_0. We say that*

$$f(x) = o\left(g(x)\right) \ \ for \ x \to x_0, \tag{3}$$

if and only if

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0. \tag{4}$$

The notion of *completely-positive* maps is useful, instead, for defining the most general quantum channels,

Definition 3 (Positive operators, positive and completely-positive maps). *Consider a Hilbert space $\mathcal{H}$ and the space $\mathcal{B}\left(\mathcal{H}\right)$ of bounded operators acting on $\mathcal{H}$. An operator $A \in \mathcal{B}\left(\mathcal{H}\right)$ is positive, and we write $A \geq 0$, if*

$$\langle\psi| A |\psi\rangle \geq 0, \quad \forall \, |\psi\rangle \in \mathcal{H}. \tag{5}$$

A map $\varepsilon : \mathcal{B}\left(\mathcal{H}\right) \to \mathcal{B}\left(\mathcal{H}\right)$ is positive if it maps any positive operator into a positive operator,

$$\varepsilon(A) \geq 0, \quad \forall \, A \in \mathcal{B}\left(\mathcal{H}\right) \ \ such \ that \ A \geq 0. \tag{6}$$

A map $\varepsilon : \mathcal{B}\left(\mathcal{H}\right) \to \mathcal{B}\left(\mathcal{H}\right)$ is completely-positive if, for any $k \in \mathbb{N}$, the map $\varepsilon \otimes \mathbb{I}_k$ acting on $\mathcal{B}\left(\mathcal{H} \otimes \mathbb{C}^k\right)$ is positive, where $\mathbb{I}_k : \mathcal{B}\left(\mathbb{C}^k\right) \to \mathcal{B}\left(\mathbb{C}^k\right)$ is the identity map, mapping any operator acting on $\mathbb{C}^d$ into itself.

The most general measurement in quantum mechanics, known as *positive-operator valued measure* (POVM), is defined as

Definition 4 (Positive-operator valued measure). *Given a Hilbert space $\mathcal{H}$, a POVM over such space is composed by a set of positive operators $\{E_i \in \mathcal{B}(\mathcal{H})\}_i$ (known as* effects*) such that $\sum_i E_i = \mathbb{I}$. An outcome is associated to each effect, and the probability of obtaining the outcome i when performing the measurement over the state $\rho \in \mathcal{S}(\mathcal{H})$ is given by* $\mathrm{p}(i) = \mathrm{Tr}[\rho E_i]$.

The *total variation distance* quantify maximum distance (in probability) between two classical distributions, and it is defined as

Definition 5 (Total variation distance). *Consider a discrete probability space X, and two distributions $p, q : X \to [0, 1]$. The total variation distance between the two distributions is*

$$\|p - q\|_1 = \sup_{S \in \mathbb{P}(X)} \left| \sum_{x \in S} p(x) - q(x) \right|, \tag{7}$$

where $\mathbb{P}(X)$ is the power set of X.

The ℓ_p-*norms* are the natural norms for finite-dimensional real vector spaces,

Definition 6 (ℓ_p-norms). *Consider a real vector space V of finite dimension d. Given a vector $v \in V$, we define the ℓ_p-norm of that vector, for real $p \geq 1$, as*

$$\|v\|_p = \left(\sum_{i=1}^{d} |v_i|^p \right)^{\frac{1}{p}}, \tag{8}$$

where v_i are the component of the vector.

The *trace norm* is the equivalent of the ℓ_1-norm for quantum states, and the distance induced by this norm quantifies the indistinguishability between two quantum states,

Definition 7 (Trace norm). *Consider a finite dimensional Hilbert space $\mathcal{H}$ and a bound operator $A \in \mathcal{B}(\mathcal{H})$ acting on this space. The trace norm of the operator is defined as*

$$\|A\|_1 = \mathrm{Tr}\left[\sqrt{A^\dagger A} \right]. \tag{9}$$

Finally, we recall the notion of *Shannon entropy*,

Definition 8 (Shannon entropy). *Consider the space of discrete probability distribution associated with a random variable with d outcomes. The Shannon entropy of the probability distribution p is defined as*

$$H(\rho) = -\sum_{i=0}^{d-1} p_i \log p_i. \tag{10}$$

The Shannon entropy has a quantum counterpart, known as the *von Neumann entropy,* which is defined as

Definition 9 (Von Neumann entropy). *Consider a finite dimensional Hilbert space $\mathcal{H}$ and a density operator $\rho \in \mathcal{S}(\mathcal{H})$. The von Neumann entropy of the system described by ρ is defined as,*

$$S(\rho) = -\mathrm{Tr}\left[\rho \log \rho\right]. \tag{11}$$

Part I

Background

Chapter 1

The resource theoretic approach

Resource theories are a set of tools coming from quantum information theory, first introduced in order to study and quantify the entanglement shared between two spatially separated quantum systems [1, 2, 3], and then extended to many more physical scenarios, from asymmetry theory [4, 5, 6] to thermodynamics [7, 8, 9] to the theory of stabiliser states [10, 11, 12]. These theories describe the physical world from the perspective of an agent, i.e., an entity who is able to act over a quantum system and modify its state, and they are used to investigate which state transformation can or cannot be performed by the agent over the system. A crucial feature of resource theories is the fact that the set of operations the agent can use, known as *allowed operations*, is subjected to some constraints. These constraints depend on the physical scenario the theory is describing, and they can arise from either technological restrictions or some fundamental law of nature. For example, in the resource theory of entanglement, it is usually assumed that the quantum system under consideration is divided into multiple parts, each of them spatially separated from the others, and the agent can only act locally over a single part. This limitation reflects the fact that, given our current technological capabilities, it is impractical to send quantum systems from one location to another. Another example concerns the resource theory of thermodynamics, in which it is assumed that the agent has to preserve the energy of the universe while acting over the quantum system. This constraint follows from the principle of energy conservation, and from the fact that the universe is an isolated system.

When a physical theory is recast as a resource theory, and the set of allowed operations is made explicit, investigating which state transformations the agent can or cannot achieve

become possible. For example, we can use resource theories to derive the necessary and sufficient conditions for two states to be converted into one another. Furthermore, we can investigate if there exist states from which all other states can be reached using the allowed operations, or the opposite, that is, whether there exist states which can be always obtained, independently of the initial state of the system. Both classes of states are pivotal in resource theories. The former class contains the most resourceful states of the theory, since when the system is described by one of them, the agent is able to map it into any other state they might need. On the other hand, the states composing the latter class are the least resourceful ones, and they are known as *free states*. If a state describing the system is initially provided, the agent can always transform it into a free state using the allowed operations, and we show that once the system is in one of these states, it cannot be mapped back to a state outside the set of free states, see Prop. 1. It is worth noting that both the set of most resourceful states and that of free states can be empty. For example, the resource theory of tripartite entanglement, equipped with stochastic local operations and classical communication (SLOCC) [13], has two in-equivalent sets of truly tripartite entangled states, and one cannot map states in one set to states in the other [14]. Likewise, one can find situations in which no free states exist, and the absence of free states is a central point of Ch. 3, where resource theories with multiple resources are considered.

We can classify the states describing the quantum system in a more detailed way than just in terms of most/least resourceful states. Indeed, with the resource theory framework one can create a hierarchy of quantum states, ordered in accordance with their resourcefulness. Following the same line of thoughts used in the previous paragraph, we say that a state is more (or equally) *resourceful* than another one if the agent can map the former into the latter with the allowed operations. Notice that, in general, it is possible to find two states such that the agent cannot map one into the other, and vice versa, by means of allowed operations. Therefore, the resourcefulness of quantum states induces a pre-order relation[1] over the state space of the theory, i.e., the set of all states describing the quantum system. This pre-order relation provides us with a hierarchy of states, where at the top we find the most resourceful states, and at the bottom the free states. When the agent performs an allowed operation over

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[1]Given a set S, a pre-order $\leq$ over this set is a relation between pairs of elements of the set which satisfies reflexivity ($\forall\, x \in S$, $x \leq x$), and transitivity ($\forall\, x, y, z \in S$, if $x \leq y$ and $y \leq z$ then $x \leq z$).

the quantum system, they move the state of the system down the hierarchy, toward the free states, and they never move it upward.

The fact that the allowed operations never increase the resourcefulness of a quantum system can be quantified using *monotones*. Monotones assign a value to every state in the state space of the theory, and their value never increases when an allowed operation is performed on the system. Thus, a state which is obtained from another one with an allowed operation always has a value of the monotone which is smaller than (or equal to) the value of the monotone on the initial state. Multiple examples of monotones can be found in physics, and one of the most well-known is certainly the entropy of a closed thermodynamic system. Indeed, this quantity never decreases (and therefore, its negative value never increases) as we act over this system with thermodynamic transformations, as guaranteed by the Second Law of Thermodynamics [15].

In the following, we explore in further details the formalism of resource theories. Specifically, in Sec. 1.1 we introduce the mathematical structure of these theories, describing the state space, the set of allowed operations, and the free states. In Sec. 1.2 we introduce the notion of monotones, and we discuss some examples that will be useful in the following chapters. In Sec. 1.3 we deal with the situation in which many copies of the system are considered, and in Sec. 1.4 we investigate the notion of reversibility in resource theories. Finally, in Sec. 1.5 we study to the case in which a single copy of the system is considered, and we introduce an additional tool, known as a catalyst, which helps to achieve some state transformations that cannot be achieved with the allowed operations. For further information on the mathematical structure of resource theories, we refer the interested reader to the following papers. Refs. [16, 17] study the framework of resource theories through the lens of category theory. In particular, resource theories are here described as *ordered commutative monoids*[2], where the binary operation corresponding to the composition of resources is the tensor product. Refs. [18, 19], instead, analyse resource theories from a different angle and provide a framework able to describe situations in which the systems under investigation do not compose with the tensor product. Finally, for an extensive review on the resource theoretic approach, we refer to Ref. [20].

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[2]A monoid is a set S together with a binary operation $+ : S \times S \to S$ which satisfies associativity ($\forall x, y, z \in S$, $x + (y + z) = (x + y) + z$), and contains the identity element ($\exists e \in S$ such that $\forall x \in S$, $e + x = x + e = x$). A monoid is called commutative if its binary operation is commutative ($\forall x, y \in S$, $x + y = y + x$).

1.1 The structure of a resource theory

A resource theory R is used to investigate which state transformations can be performed over a quantum system, and their cost. In the following, the quantum system is described by a Hilbert space $\mathcal{H}$, and we consider the case in which the dimension of the system is finite, that is, $d = \dim \mathcal{H} < \infty$. Notice, however, that resource theories can be defined for infinite-dimensional systems, such as the resource theory of non-gaussianity [21, 22, 23, 24]. The set of states describing the quantum system, referred to as the state space of the theory, is composed of *density operators*, i.e., positive operators with unit trace,

$$\mathcal{S}\left(\mathcal{H}\right) = \left\{\rho \in \mathcal{B}\left(\mathcal{H}\right) \mid \rho \geq 0, \ \mathrm{Tr}\left[\rho\right] = 1\right\}, \tag{1.1}$$

where $\mathcal{B}\left(\mathcal{H}\right)$ is the set of bounded operators acting on the Hilbert space $\mathcal{H}$. Let us consider, for example, the resource theory of bipartite entanglement, used to describe quantum systems composed of two parts that are spatially separated from each other. We can label these two parts A and B, and the Hilbert space describing the global system is $\mathcal{H}_A \otimes \mathcal{H}_B$. The state space is then given by the set $\mathcal{S}\left(\mathcal{H}_A \otimes \mathcal{H}_B\right)$ of density operators acting on the global Hilbert space.

A resource theory is defined through its class of allowed operations $\mathcal{A}$. In quantum mechanics, the most general operations that can be applied to a system are known as quantum channels. These channels are linear maps from $\mathcal{S}\left(\mathcal{H}_1\right)$ to $\mathcal{S}\left(\mathcal{H}_2\right)$, where the two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ needs not to be equal. For these maps to transform between quantum states, they need to be completely positive, see Def. 3, and to preserve the trace (CPTP). Within this section we focus on quantum channels which leave the quantum system unchanged, i.e., they have the same initial and final Hilbert space. However, at the end of the section we provide a way to keep the Hilbert space fixed, while still being able to map between states describing different quantum systems. In this way, one is able to extend the following framework to situations where the operations change the system under investigation.

The set of allowed operations $\mathcal{A}$ of the theory is a subset of all CPTP maps acting on the state space $\mathcal{S}\left(\mathcal{H}\right)$. This subset is defined by the constraints that the physical theory is subjected to, and these constraints change depending on the different scenarios we need to model, making it impossible to be more specific about the structure of this subset. It is worth noticing that, in general, the set of allowed operations is closed under composition, a condition that guarantees

the theory to be physically meaningful. For the example we are considering, the theory of bipartite entanglement, one of the most common sets of allowed operations is given by local operations and classical communication (LOCC) [1]. In this setting, each agent is allowed to perform local operations over their part, for example by adding and tracing out an ancillary system, by acting over system and ancilla with a unitary operation, or by measuring the state of their system. Furthermore, the two agents are allowed to transmit classical information to each other, for example to exchange the outcome of the local measurements they have performed.

We are now able to define the set of free states, that is, the set of those states that the agent can always prepare using the allowed operations.

Definition 10 (Free states). *Consider a resource theory with allowed operations $\mathcal{A}$, acting on a system described by the Hilbert space $\mathcal{H}$. A state $\gamma \in \mathcal{S}(\mathcal{H})$ is a* free state *of the theory if*

$$\forall \rho \in \mathcal{S}(\mathcal{H}), \ \exists \varepsilon \in \mathcal{A} \ : \varepsilon(\rho) = \gamma. \tag{1.2}$$

The above definition clarifies the reason why these states are called free. Indeed, the agent is always able to prepare the system in one of these states. In the case of bipartite entanglement, the free states of the theory are known as separable states[3]. These states can always be prepared by the agents using LOCC. For example, the agent A can toss a coin (or roll a dice), and communicate the outcome to the agent B. Conditionally to the outcome, they both prepare an ancillary system in a specific state, swap the state of the main system with that of the ancilla, and trace out the ancilla. In this way, they are able to prepare the system in any separable state.

We now introduce a fundamental property of the allowed operations, which plays a central role in the next section.

Proposition 1. *Consider a resource theory with allowed operations $\mathcal{A}$, acting on a system described by the Hilbert space $\mathcal{H}$. The set of free states of the theory, $\mathcal{F} \subseteq \mathcal{S}(\mathcal{H})$, is closed under the class of allowed operations, that is,*

$$\forall \gamma \in \mathcal{F}, \ \forall \varepsilon \in \mathcal{A} \ : \varepsilon(\gamma) \in \mathcal{F}. \tag{1.3}$$

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[3]A separable state of a bipartite system described by the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ is defined as a convex combination of product states. Mathematically, they corresponds to states of the form $\sum_{i=1}^{m} \mathrm{p}_i \, \rho_A^{(i)} \otimes \rho_B^{(i)}$, where $\rho_A^{(i)} \in \mathcal{S}(\mathcal{H}_A)$ and $\rho_B^{(i)} \in \mathcal{S}(\mathcal{H}_B)$ for each $i = 1, \ldots, m$, and $\mathrm{p}_i \geq 0$, $\sum_{i=1}^{m} \mathrm{p}_i = 1$.

Proof. Suppose that there exists a free state $\gamma \in \mathcal{F}$, and an allowed operation $\tilde{\varepsilon} \in \mathcal{A}$, such that $\tilde{\varepsilon}(\gamma) = \omega \notin \mathcal{F}$. Then, by definition of free state, we have that for all $\rho \in \mathcal{S}(\mathcal{H})$, there exists an operation $\varepsilon \in \mathcal{A}$ such that $\varepsilon(\rho) = \gamma$, see Eq. (1.2). Let us now consider the operation $\varepsilon' = \tilde{\varepsilon} \circ \varepsilon$ which belongs to the set of allowed operations $\mathcal{A}$, since this set is closed under composition. It is now easy to see that

$$\forall \rho \in \mathcal{S}(\mathcal{H}), \ \exists \varepsilon' \in \mathcal{A} \ : \varepsilon'(\rho) = \omega, \tag{1.4}$$

which contradicts our initial assumption that $\omega \notin \mathcal{F}$. $\qquad \square$

The above result implies that the set of free states is invariant[4] under the allowed operations, and we write $\varepsilon(\mathcal{F}) \subseteq \mathcal{F}$. Physically, this means that the agent cannot create resource out of a non-resourceful state. Furthermore, this proposition allows us to extend the set of allowed operations in a way that is less physical (less operational, at least), but that often provides an easier mathematical framework where embedding a problem. Indeed, we can consider a resource theory whose set of allowed operations is given by every CPTP map which leaves $\mathcal{F}$ invariant. This set, that we denote with $\tilde{\mathcal{A}}$, is the biggest one containing operations which do not create the resource, and therefore it contains (sometimes strictly contains) the more operational set $\mathcal{A}$. For instance, in the resource theory of bipartite entanglement, we have that the set $\tilde{\mathcal{A}}$, composed by the operations that map separable states into separable states, also contains the operation which swaps the local states of the two parts. This operation, while keeping any separable state separable, is clearly not a LO.

To conclude the section, let us provide a way to extend the above formalism to describe the situation in which even the quantum system under examination is modified by the agent. Suppose that the agent is mapping the state of the system from an initial state $\rho \in \mathcal{S}(\mathcal{H}_1)$ into the final state $\rho' \in \mathcal{S}(\mathcal{H}_2)$, thus modifying the system itself. The quantum channel could be described by an operation $\varepsilon : \mathcal{S}(\mathcal{H}_1) \to \mathcal{S}(\mathcal{H}_2)$ such that $\varepsilon(\rho) = \rho'$. However, we can also think of the situation in which the agent considers the original system, described by $\mathcal{H}_1$, together with an additional system described by $\mathcal{H}_2$. The first system is in the state ρ, while the second system has to be in a free state, since the agent cannot prepare it in another state with the allowed operation. Then, the operation performed on these two systems maps the first system

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[4]A set X is *invariant* under the map $\varepsilon : X \to X$ if, for all $x \in X$, we have that $\varepsilon(x) \in X$.

into a free state (so that this system is now useless from a resource theoretic point of view) and the second one into the state ρ'. This operation acts on the same global Hilbert space, since it is described by $\bar{\varepsilon} : \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2) \rightarrow \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, and is such that $\bar{\varepsilon}(\rho \otimes \gamma) = \gamma' \otimes \rho'$, where $\gamma \in \mathcal{S}(\mathcal{H}_1)$ and $\gamma' \in \mathcal{S}(\mathcal{H}_2)$ are free states.

1.2 The Second Law of resource theories: monotones

We can now address the problem of how to quantify resources within the resource theory approach. As we have already noticed in the previous sections, we can say that a state is more resourceful than another if there exists an allowed operation in $\mathcal{A}$ mapping the first state into the second one. We can formalise this notion of resourcefulness using monotones, i.e., quantifiers that assign a "price" to each quantum state, such that a state which is obtained from another one with allowed operations never has a bigger price than the other state.

Definition 11 (Monotone). *A monotone is a function $M : \mathcal{S}(\mathcal{H}) \rightarrow \mathbb{R}$ such that*

$$M(\varepsilon(\rho)) \leq M(\rho) \quad \forall \rho \in \mathcal{S}(\mathcal{H}), \forall \varepsilon \in \mathcal{A}. \tag{1.5}$$

It is then clear that, given two states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$, the condition $M(\rho) \geq M(\sigma)$ is necessary for the existence of an operation $\varepsilon \in \mathcal{A}$ able to map ρ into σ. However, this is not a sufficient condition, and much of the work in resource theories is focused on defining necessary and sufficient conditions for a state transformation to occur under allowed operations. For example, in the resource theory of bipartite entanglement, necessary and sufficient conditions for a pure state to be mapped into another pure state under LOCC are known. Indeed, such transformation exists if and only if the Schmidt coefficients of the final state majorize (see Def. 19) the Schmidt coefficients of the initial state [25].

Several monotones can be found for each resource theory, and different theories have different monotones which are only valid for those theories. However, there is a family of monotones that we can build for any resource theory equipped with a set of free states, or more in general with a non-trivial invariant set. These monotones are realised using a *contractive metric*, i.e., a distance between quantum states those value never increases when a quantum channel is applied over the states.

Definition 12 (Contractive distance). *A metric $d_C(\cdot, \cdot) : \mathcal{S}(\mathcal{H}) \times \mathcal{S}(\mathcal{H}) \to \mathbb{R}$ is called contractive if, for all pair of states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$, and for all CPTP map $\varepsilon : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$, the following is satisfied*

$$d_C(\varepsilon(\rho), \varepsilon(\sigma)) \leq d_C(\rho, \sigma). \tag{1.6}$$

It is worth noting that the object d_C does not need to satisfy all axioms of a metric to be suitable for building a monotone. For example, the *quantum relative entropy*, that we introduce below in Eq. (1.9), is not symmetric and does not satisfy the triangle inequality. Nevertheless, the monotone built out of this function plays a central role in reversible resource theories, see Sec. 1.4.

Let us show how a monotone can be built for a resource theory R with state space $\mathcal{S}(\mathcal{H})$, allowed operations $\mathcal{A}$, and free states $\mathcal{F}$. Given a contractive distance d_C over the state space, a monotone $M_C : \mathcal{S}(\mathcal{H}) \to \mathbb{R}$ for the theory is defined as

$$M_C(\cdot) := \inf_{\gamma \in \mathcal{F}} d_C(\cdot, \gamma). \tag{1.7}$$

We now provide a brief proof that the above function is indeed monotonic under the set of allowed operations $\mathcal{A}$.

Proposition 2. *Consider a resource theory R with state space $\mathcal{S}(\mathcal{H})$, allowed operations $\mathcal{A}$, and free states $\mathcal{F}$. Then, the function M_C defined in Eq. (1.7) is a monotone for this resource theory.*

Proof. Given a generic quantum state $\rho \in \mathcal{S}(\mathcal{H})$, we have that

$$M_C(\varepsilon(\rho)) = \inf_{\gamma \in \mathcal{F}} d_C(\varepsilon(\rho), \gamma) \leq \inf_{\gamma \in \mathcal{F}} d_C(\varepsilon(\rho), \varepsilon(\gamma)) \leq \inf_{\gamma \in \mathcal{F}} d_C(\rho, \gamma) = M_C(\rho), \tag{1.8}$$

where the first inequality follows from the fact that $\varepsilon(\mathcal{F}) \subseteq \mathcal{F}$, as showed in Prop. 1, and the fact that we are optimising over the set of free states, while the second inequality follows from the contractivity of the distance d_C, see Eq. (1.6). $\square$

We can now introduce a specific monotone which has the form of Eq. (1.7). This monotone is built out of the quantum relative entropy, a function $D(\cdot \| \cdot) : \mathcal{S}(\mathcal{H}) \times \mathcal{S}(\mathcal{H}) \to \mathbb{R}$ defined as

$$D(\rho \| \sigma) := \text{Tr}\left[\rho(\log \rho - \log \sigma)\right] \quad \forall \rho, \sigma \in \mathcal{S}(\mathcal{H}) : \text{supp}(\rho) \subseteq \text{supp}(\sigma), \tag{1.9}$$

where the support of ρ is the subspace of $\mathcal{H}$ spanned by the eigenvectors of ρ with non-zero eigenvalues. It is easy to show that, when $\mathrm{supp}\,(\rho) \not\subset \mathrm{supp}\,(\sigma)$, the above quantity is ill-defined, as its value is equal to ∞. Although the quantum relative entropy is not a proper distance, it still allows us to define a monotone for a generic resource theory R with free states $\mathcal{F}$. Indeed, since the relative entropy is monotonic under CPTP maps [26], we can introduce the following monotone, known as the *relative entropy distance from the free states*,

$$E_{\mathcal{F}}(\cdot) := \inf_{\gamma \in \mathcal{F}} D(\cdot \,\|\, \gamma). \tag{1.10}$$

For example, when the resource theory of bipartite entanglement is considered, and therefore the set of free states $\mathcal{F}$ is composed of separable states, the above monotone is referred to as the relative entropy of entanglement [27].

It is worth noting that many other monotones, whose form is different from the one of Eq. (1.7), can be defined for a single resource theory. As an example, we introduce the *global robustness* [28, 29, 30], a monotone that we use in the following when we describe a general model for reversible resource theories, see Thm. 2. For a quantum state $\rho \in \mathcal{S}\,(\mathcal{H})$, this quantity is defined as

$$\mathcal{R}\,(\rho) = \min_{\pi \in \mathcal{S}(\mathcal{H})} \left\{ \lambda \mid \frac{1}{1 + \lambda}\,(\rho + \lambda\,\pi) \in \mathcal{F} \right\}. \tag{1.11}$$

The global robustness quantifies the amount of noise one needs to add to the quantum system under examination in order to destroy the resource contained, and to obtain a free state. Here, the state π is a generic state in the state space $\mathcal{S}\,(\mathcal{H})$, but we can also redefine the robustness by asking that π belongs to the set of free states $\mathcal{F}$ only [28].

1.3 Resource theories in the many-copy regime

In this section, we show how a resource theory can be extended to the case in which the agent has access to many copies of the quantum system, and can act over these copies with global operations. This setting is particularly useful, as we see in the next section, since it allows us to investigate reversibility in the context of resource theories. When the number of copies of the system is big enough, quantum fluctuations can be neglected, and the agent can act over the global system in a reversible way (i.e., without dissipating any amount of resource during

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a state transformation). In the following, we assume the agent to have access to a source of quantum systems (a scenario known as the *i.i.d. limit*). Each use of the source produces a system described by the Hilbert space $\mathcal{H}$, and the source can be used an arbitrary number of times. Furthermore, we assume each use of the source to be *independent* of the others, and the source to produce *identically distributed* states of the system. In this way, if the agent uses the source $n \in \mathbb{N}$ times, and keeps each system produced by the source, the state of the global system obtained is described by n copies of a specific quantum state $\rho \in \mathcal{S}(\mathcal{H})$, i.e., by the tensor product state $\rho^{\otimes n}$.

We now describe how a resource theory R acting on a system described by the Hilbert space $\mathcal{H}$, with allowed operations $\mathcal{A}$ and free states $\mathcal{F}$, can be extended to the scenario in which n copies of the system are considered. In this case, the global system under examination is described by the Hilbert space $\mathcal{H}^{\otimes n}$, and the state space of the theory is $\mathcal{S}(\mathcal{H}^{\otimes n})$. The set of allowed operations $\mathcal{A}^{(n)}$ is a subset of CPTP maps acting on $\mathcal{S}(\mathcal{H}^{\otimes n})$. Intuitively, the same constraints defining the allowed operations for a single copy of the system should be valid for the n-copy case. For example, let us consider the resource theory of bipartite entanglement whose allowed operations are LOCC. When n copies of the system are considered, the operations used by the agent still need to be local. Therefore, even if the agent is able to act collectively on the n local parts they own, they cannot act over the parts owned by the other agent.

The set of free states, when n copies of the system are considered, is trivially obtained by extending Def. 10. Indeed, we say that a state $\gamma_n \in \mathcal{S}(\mathcal{H}^{\otimes n})$ is a free state if we can transform any other state of $\mathcal{S}(\mathcal{H}^{\otimes n})$ into γ_n by means of the allowed operations $\mathcal{A}^{(n)}$. Therefore, the set of free states for n copies of the system is composed of those states that the agent can always prepare with the allowed operations, and we denote it with $\mathcal{F}^{(n)} \subset \mathcal{S}(\mathcal{H}^{\otimes n})$. The states in this set need not to be given by the tensor product of n single-copy free states, and in general we have that $\mathcal{F}^{\otimes n} \subseteq \mathcal{F}^{(n)}$. For example, in the resource theory of bipartite entanglement, when many copies are considered, the free states can be locally entangled, and they only need to be separable between the two spatially-separated parts. Since the definition of free states for n copies coincides with the one for a single system, it is easy to show that Prop. 1 extends to this scenario, i.e., the states in $\mathcal{F}^{(n)}$ are invariant under the allowed operations $\mathcal{A}^{(n)}$.

The monotones of the resource theory R can be extended to the case in which n copies of

the system are considered. Given a monotone $M : \mathcal{S}(\mathcal{H}) \to \mathbb{R}$, we extend it by replacing all the single-copy objects with the corresponding n-copy ones. For example, the relative entropy distance from the free states, given in Eq. (1.10), is extended to n copies as

$$E_{\mathcal{F}}(\rho_n) = \inf_{\gamma_n \in \mathcal{F}^{(n)}} D(\rho_n \| \gamma_n), \tag{1.12}$$

where $\rho_n \in \mathcal{S}(\mathcal{H}^{\otimes n})$. A useful notion, which we use in the context of reversible resource theories, is the *regularisation* of a monotone.

Definition 13 (Regularised monotone). *Consider a resource theory* R *acting over the Hilbert space* $\mathcal{H}$, *with a monotone* $M : \mathcal{S}(\mathcal{H}) \to \mathbb{R}$. *The* regularisation *of this monotone is a function* $M^\infty : \mathcal{S}(\mathcal{H}) \to \mathbb{R}$, *defined as*

$$M^\infty(\rho) = \lim_{n \to \infty} \frac{1}{n} M(\rho^{\otimes n}). \tag{1.13}$$

For example, the regularised relative entropy from the free states is given by

$$E_{\mathcal{F}}^\infty(\rho) = \lim_{n \to \infty} \frac{1}{n} E_{\mathcal{F}}(\rho^{\otimes n}). \tag{1.14}$$

We close this section with a discussion on the properties of the relative entropy distance $E_{\mathcal{F}}$ when many copies of the system are considered. These properties are used in the next section, where we study reversible resource theories and the special role played by $E_{\mathcal{F}}$ in such theories. Consider the sequence of free sets $\{\mathcal{F}^{(n)}\}_{n \in \mathbb{N}}$, each one associated with a different number of copies of the system. We ask the following assumptions over the sets of free states, which are often required in the resource theoretic context [31, 32].

F1 Each set $\mathcal{F}^{(n)}$ is convex. Given $n \in \mathbb{N}$,

$$\lambda \gamma_n + (1 - \lambda) \gamma_n' \in \mathcal{F}^{(n)}, \quad \forall \lambda \in [0, 1], \ \forall \gamma_n, \gamma_n' \in \mathcal{F}^{(n)}. \tag{1.15}$$

F2 The set $\mathcal{F} = \mathcal{F}^{(1)}$ contains at least one full-rank state.

F3 The tensor product of free states is a free state. Given $n, k \in \mathbb{N}$,

$$\gamma_n \otimes \gamma_k \in \mathcal{F}^{(n+k)}, \quad \forall \gamma_n \in \mathcal{F}^{(n)}, \ \forall \gamma_k \in \mathcal{F}^{(k)}. \tag{1.16}$$

F4 The partial trace of a free state is a free state. Given $n, k \in \mathbb{N}$, where $n > k$,

$$\mathrm{Tr}_k [\gamma_n] \in \mathcal{F}^{(n-k)}, \quad \forall \gamma_n \in \mathcal{F}^{(n)}. \tag{1.17}$$

F5 Each set is closed under permutations of the n copies. For all $n \in \mathbb{N}$,

$$P_\pi \, \gamma_n \, P_\pi \in \mathcal{F}^{(n)}, \quad \forall \, \gamma_n \in \mathcal{F}^{(n)}, \; \forall \, \pi \in S_n, \tag{1.18}$$

where P_π is the representation of the symmetric group S_n over the Hilbert space $\mathcal{H}^{\otimes n}$, whose action is

$$P_\pi \, |\psi_1\rangle \otimes \ldots \otimes |\psi_n\rangle = |\psi_{\pi^{-1}(1)}\rangle \otimes \ldots \otimes |\psi_{\pi^{-1}(n)}\rangle, \quad \forall \, |\psi_i\rangle \in \mathcal{H}. \tag{1.19}$$

Notice that the above assumptions over the sets of free states are rather general, and most of the known resource theories satisfy them. Assumption F1, for example, implies that the agent is allowed to use randomness, and therefore they can mix between different free states. Examples of convex sets of free states are the separable states for the resource theory of bipartite entanglement, the incoherent states for the resource theory of coherence [33, 34, 35], and the maximally-mixed state for the resource theory of Noisy Operations [36, 37]. However, it is possible to find sets of free states that are not convex, such as, for instance, the one for the resource theory of non-Gaussianity [23, 24]. Assumption F3 implies that the agent cannot create a resourceful state by combining two or more systems described by free states. Similarly, assumptions F4 tells us that the agent cannot obtain a resourceful state by forgetting about part of a system in a free state. Assumption F5 implies that the agent cannot generate any amount of resource by permuting the copies of a system which is initially in a free state. Assumption F2 is required for the relative entropy distance $E_\mathcal{F}$ to be well-defined, as we show in Lemmas 3 and 4. Finally, notice that it is usually assumed that the set $\left\{\mathcal{F}^{(n)}\right\}_{n \in \mathbb{N}}$ is closed, i.e., any sequence of free states converges to a free state.

Consider the relative entropy distance $E_\mathcal{F}$, where the associated sequence of free sets satisfies the above assumptions. We now derive some properties of this quantity that are useful for the study of reversible theories. The first property concerns the monotonicity of $E_\mathcal{F}$ under partial tracing.

Lemma 1. *Consider a sequence of sets* $\left\{\mathcal{F}^{(n)} \subseteq \mathcal{S}\left(\mathcal{H}^{\otimes n}\right)\right\}_{n \in \mathbb{N}}$ *satisfying the property F4. Then the relative entropy distance from the free states is such that*

$$E_\mathcal{F}(\mathrm{Tr}_k\,[\rho_n]) \leq E_\mathcal{F}(\rho_n), \quad \forall \, \rho_n \in \mathcal{S}\left(\mathcal{H}^{\otimes n}\right), \; \forall \, k < n. \tag{1.20}$$

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Proof. Let us consider the CPTP map $\varepsilon_{\text{Tr}}(\cdot) = \text{Tr}_k[\cdot]$, mapping between the space $\mathcal{S}(\mathcal{H}^{\otimes n})$ and $\mathcal{S}(\mathcal{H}^{\otimes n-k})$. With this map, we can show that

$$
\begin{aligned}
E_{\mathcal{F}}(\rho_n) &= \inf_{\gamma_n \in \mathcal{F}^{(n)}} D(\rho_n \,\|\, \gamma_n) \geq \inf_{\gamma_n \in \mathcal{F}^{(n)}} D(\varepsilon_{\text{Tr}}(\rho_n) \,\|\, \varepsilon_{\text{Tr}}(\gamma_n)) \\
&= \inf_{\gamma_n \in \mathcal{F}^{(n)}} D(\text{Tr}_k[\rho_n] \,\|\, \text{Tr}_k[\gamma_n]) \geq \inf_{\gamma_{n-k} \in \mathcal{F}^{(n-k)}} D(\text{Tr}_k[\gamma_n] \,\|\, \gamma_{n-k}) = E_{\mathcal{F}}(\text{Tr}_k[\gamma_n]),
\end{aligned}
$$

$$(1.21)$$

where the first inequality follows from the monotonicity of the relative entropy distance under CPTP maps, and the second one from the property F4. $\square$

We now move to consider the value of $E_{\mathcal{F}}$ when an ancilla in a free state is added to the main system. Since free states can always be prepared by the agent, we would expect the value of the monotone not to change when the free states are added. This is indeed the case for the relative entropy distance from the free states, as we show in the following lemma.

Lemma 2. *Consider a sequence of sets $\{\mathcal{F}^{(n)} \subseteq \mathcal{S}(\mathcal{H}^{\otimes n})\}_{n \in \mathbb{N}}$ satisfying the properties F3 and F4. Then the relative entropy distance from the free states is such that*

$$
E_{\mathcal{F}}(\rho_n \otimes \gamma_k) = E_{\mathcal{F}}(\rho_n), \quad \forall\, \rho_n \in \mathcal{S}(\mathcal{H}^{\otimes n}), \; \forall\, \gamma_k \in \mathcal{F}^{(k)}. \tag{1.22}
$$

Proof. Let us consider a state $\rho_n \in \mathcal{S}(\mathcal{H}^{\otimes n})$, and a free state $\gamma_k \in \mathcal{F}^{(k)}$. We first introduce the CPTP map $\varepsilon_{\otimes}(\cdot) = \cdot \otimes \gamma_k$, mapping between the space $\mathcal{S}(\mathcal{H}^{\otimes n})$ and $\mathcal{S}(\mathcal{H}^{\otimes n+k})$. Then, the following chain of inequalities holds,

$$
\begin{aligned}
E_{\mathcal{F}}(\rho_n) &= \inf_{\gamma_n \in \mathcal{F}^{(n)}} D(\rho_n \,\|\, \gamma_n) \geq \inf_{\gamma_n \in \mathcal{F}^{(n)}} D(\varepsilon_{\otimes}(\rho_n) \,\|\, \varepsilon_{\otimes}(\gamma_n)) \\
&= \inf_{\gamma_n \in \mathcal{F}^{(n)}} D(\rho_n \otimes \gamma_k \,\|\, \gamma_n \otimes \gamma_k) \\
&\geq \inf_{\gamma_{n+k} \in \mathcal{F}^{(n+k)}} D(\rho_n \otimes \gamma_k \,\|\, \gamma_{n+k}) = E_{\mathcal{F}}(\rho_n \otimes \gamma_k),
\end{aligned} \tag{1.23}
$$

where the first inequality follows from the monotonicity of the relative entropy distance under CPTP maps, and the second one from property F3. Now, using the result of Lem. 1, we can also show that

$$
E_{\mathcal{F}}(\rho_n \otimes \gamma_k) \geq E_{\mathcal{F}}(\rho_n), \tag{1.24}
$$

where $\rho_n = \text{Tr}_k[\rho_n \otimes \gamma_k]$. Eqs. (1.23) and (1.24) prove the lemma. $\square$

The second property we consider concerns the regularisation of the relative entropy distance, see Def. 13. In particular, we show that this quantity is always well-defined, i.e., the limit in Eq. (1.14) never diverges. However, it is worth noting that even if this quantity is always finite, there are resource theories in which $E_{\mathcal{F}}^{\infty}$ is equal to zero for any state in the state space, and therefore this quantity is not a useful monotone. This is the case, for instance, of the resource theory of asymmetry, as first highlighted in Ref. [6, Sec. IV].

Lemma 3. *Consider a sequence of sets $\left\{\mathcal{F}^{(n)} \subseteq \mathcal{S}\left(\mathcal{H}^{\otimes n}\right)\right\}_{n \in \mathbb{N}}$ satisfying the properties F2 and F3. Then the regularised version of the relative entropy distance from the free states exists, and is well-defined, i.e., $E_{\mathcal{F}}^{\infty} < \infty$.*

Proof. Consider a state $\rho \in \mathcal{S}\left(\mathcal{H}\right)$, and a full-rank free state $\gamma_{\text{full-rank}} \in \mathcal{F}$, which exists due to property F2. Then, for all $n \in \mathbb{N}$, we have

$$\frac{1}{n} E_{\mathcal{F}}(\rho^{\otimes n}) = \frac{1}{n} \inf_{\gamma_n \in \mathcal{F}^{(n)}} D(\rho^{\otimes n} \,\|\, \gamma_n) \leq \frac{1}{n} \inf_{\gamma \in \mathcal{F}} D(\rho^{\otimes n} \,\|\, \gamma^{\otimes n})$$

$$= \inf_{\gamma \in \mathcal{F}} D(\rho \,\|\, \gamma) \leq D(\rho \,\|\, \gamma_{\text{full-rank}}) < \infty, \tag{1.25}$$

where the first inequality follows from property F3, and the last one from the definition of relative entropy, see Eq. (1.9). By sending n to infinity, we prove the lemma. $\qquad\square$

We now consider the continuity properties of the relative entropy distance from the free states. Given a function defined over a family of state spaces, we can introduce the following notion of *asymptotic continuity*.

Definition 14 (Asymptotic continuity). *Consider a family of Hilbert spaces $\mathcal{H}_n$ such that $\dim \mathcal{H}_n \to \infty$ for $n \to \infty$. A sequence of real-valued functions $f : \mathcal{S}\left(\mathcal{H}_n\right) \to \mathbb{R}$ is called asymptotic continuous if, for any two sequences of states $\rho_n, \sigma_n \in \mathcal{S}\left(\mathcal{H}_n\right)$ such that*

$$\lim_{n \to \infty} \|\rho_n - \sigma_n\|_1 = 0, \tag{1.26}$$

we have that

$$\lim_{n \to \infty} \frac{|f\left(\rho_n\right) - f\left(\sigma_n\right)|}{\log \dim \mathcal{H}_n} = 0 \tag{1.27}$$

The following lemma states that the relative entropy distance from the free states, and its regularisation, are asymptotic continuous. The proof of the lemma can be found in Ref. [38,

Lem. C.3], and it is obtained by combining the proofs first derived in Ref. [39, Lem. 2] for $E_{\mathcal{F}}$, and in Ref. [40, Prop. 3.23] for the regularised relative entropy of entanglement.

Lemma 4. *Consider a sequence of sets $\left\{ \mathcal{F}^{(n)} \subseteq \mathcal{S}\left(\mathcal{H}^{\otimes n}\right) \right\}_{n \in \mathbb{N}}$ satisfying the properties F1, F2, F3, and F4. Then, both the relative entropy distance from the free states $E_{\mathcal{F}}$, and its regularisation $E_{\mathcal{F}}^{\infty}$, are asymptotic continuous.*

1.4 Reversibility and resource theories

In this section we study resource theories in the asymptotic limit, i.e., when n copies of the system are considered, $n \to \infty$, and the notion of reversibility. We say that a theory is reversible if, whenever the agent is able to map a state into another one with allowed operations, then there exists another allowed operation which implements the reverse transformation. When this is the case, it is easy to show that during both the forward and backward transformations the amount of resource contained in the system (the value of each monotones) is conserved [41]. In the following, we show that, for a reversible theory, there exists a unique monotone that quantifies the amount of resource contained in the system [42, 31], and this monotone is the regularised version of the relative entropy from the free states of the theory. Furthermore, we provide the conditions under which a generic resource theory is reversible [32], together with an explicit way of designing reversible operations for these theories, based on a generalisation of quantum Stein's lemma [38].

When considering reversibility in the asymptotic setting, one is generally interested in the *rate of conversion* between two quantum states. Suppose the agent is initially given n copies of the state $\rho \in \mathcal{S}\left(\mathcal{H}\right)$, and needs to realise as many copies as possible of the state $\sigma \in \mathcal{S}\left(\mathcal{H}\right)$, using the allowed operations. The rate of conversion is then given by the ratio between the optimum number of final copies k_n of σ, and the initial number of copies n of ρ, when $n \to \infty$. Recall that we are here considering the case in which the allowed operations cannot change the global number of copies of the system. Thus, when $n \neq k_n$, we need to make explicit, in the definition of rate of conversion, that $k_n - n$ copies of the system in a free state are initially added (if $k_n > n$), or that $n - k_n$ copies are traced out after the transformation (if $k_n < n$).

Definition 15 (Rate of conversion)**.** *Consider a resource theory* R *with allowed operations* $\mathcal{A}$

and free states $\mathcal{F}$. Given two states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$, the rate of conversion *of ρ into σ is defined as*

$$R(\rho \to \sigma) = \max \left\{ \frac{k_n}{n} \,\middle|\, \text{either } \exists \varepsilon_n \in \mathcal{A}^{(n)} : \right.$$

$$\lim_{n \to \infty} \left\| \mathrm{Tr}_{n-k_n} \left[\varepsilon_n(\rho^{\otimes n}) \right] - \sigma^{\otimes k_n} \right\|_1 = 0, \tag{1.28}$$

$$\text{or } \exists \varepsilon'_{k_n} \in \mathcal{A}^{(k_n)}, \ \exists \gamma_{k_n - n} \in \mathcal{F}^{(k_n - n)} :$$

$$\left. \lim_{n \to \infty} \left\| \varepsilon'_{k_n}(\rho^{\otimes n} \otimes \gamma_{k_n - n}) - \sigma^{\otimes k_n} \right\|_1 = 0 \right\}. \tag{1.29}$$

It is worth noting that, in the above definition, Eq. (1.28) applies to the case in which $n > k_n$, while Eq. (1.29) applies to the other case, $n < k_n$. Distillation is a particular case of the scenario described above. For example, in the resource theory of bipartite entanglement for pure states with LOCC, one is interested in distilling as many copies of the Bell state $|\Phi\rangle_{AB}$ from n copies of a non-maximally entangled state $|\Psi\rangle_{AB}$. The rate of conversion, for this scenario, is known to be $R(\Psi \to \Phi) = S(\rho_A)$ [2], where S is the Von Neumann entropy, see Def. 9, and $\rho_A = \mathrm{Tr}_B \left[|\Psi\rangle\langle\Psi|_{AB} \right]$ is the reduced density matrix describing one part of the system. Interestingly, this resource theory is reversible, meaning that the agents can reverse the transformation, and create n copies of a non-maximally entangled state $|\Psi\rangle_{AB}$ from $k_n = n S(\rho_A)$ copies of $|\Phi\rangle_{AB}$, for $n \gg 1$. In general, we can define reversibility of resource theories in terms of the relation between the rates of conversion of the forward and backward process,

Definition 16 (Reversible resource theory)**.** *A resource theory* R *with allowed operations $\mathcal{A}$ and free states $\mathcal{F}$ is called* reversible *if*

$$R(\rho \to \sigma) R(\sigma \to \rho) = 1, \tag{1.30}$$

for all non-free states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$.

We can now introduce the following fundamental result for reversible resource theories, proved in Ref. [31, Thm. 1], see also Refs. [42, 36]. This result guarantees that, given a reversible resource theory whose set of free states satisfies the four properties listed in the previous section, the regularised relative entropy distance from the free states is the unique measure of the resource contained in the system. Indeed, this is the sole quantity the rate of

conversion depends on for reversible theories, since we can express the rate as a ratio between the values of $E_{\mathcal{F}}^{\infty}$ evaluated on the initial and final state of the system, respectively.

Theorem 1. *Consider a resource theory with allowed operations $\mathcal{A}$ and free states $\mathcal{F}$. Suppose that*

- *The free set $\mathcal{F}$ satisfies properties F1 – F4.*

- *The theory is reversible, see Def. 16.*

- *There exists a state $\rho_0 \in \mathcal{S}(\mathcal{H})$ such that $E_{\mathcal{F}}^{\infty}(\rho_0) > 0$.*

Then, the regularised version of the relative entropy distance from the free states is the unique quantifier of the resource, that is, we can express the rate of conversion as

$$R(\rho \to \sigma) = \frac{E_{\mathcal{F}}^{\infty}(\rho)}{E_{\mathcal{F}}^{\infty}(\sigma)}, \tag{1.31}$$

for all non-free states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$.

Proof. Let consider two non-free states ρ and σ such that $R(\rho \to \sigma) \leq 1$ (the other case is proved equivalently). Then, according to the definition of rate of conversion, Def. 15, there exists a sequence of allowed operations $\left\{ \varepsilon_n \in \mathcal{A}^{(n)} \right\}_{n \in \mathbb{N}}$ such that

$$\lim_{n \to \infty} \left\| \text{Tr}_{n-k_n} \left[\varepsilon_n(\rho^{\otimes n}) \right] - \sigma^{\otimes k_n} \right\|_1 = 0, \tag{1.32}$$

where $\lim_{n \to \infty} \frac{k_n}{n} = R(\rho \to \sigma)$. Since the relative entropy distance from the free states $E_{\mathcal{F}}$ is asymptotic continuous, see Lem. 4, it follows that

$$E_{\mathcal{F}} \left(\text{Tr}_{n-k_n} \left[\varepsilon_n(\rho^{\otimes n}) \right] \right) = E_{\mathcal{F}} \left(\sigma^{\otimes k_n} \right) + o(n), \tag{1.33}$$

where we are using the little-o notation, see Def. 2.

By monotonicity of $E_{\mathcal{F}}$ under partial tracing, Lem. 1, and under the class of allowed operations $\mathcal{A}$, Prop. 2, we have

$$E_{\mathcal{F}} \left(\rho^{\otimes n} \right) \geq E_{\mathcal{F}} \left(\sigma^{\otimes k_n} \right) + o(n). \tag{1.34}$$

We can now divide the left- and right-hand side of the above equation by n, obtaining

$$\frac{1}{n} E_{\mathcal{F}} \left(\rho^{\otimes n} \right) \geq \frac{k_n}{n} \frac{1}{k_n} E_{\mathcal{F}} \left(\sigma^{\otimes k_n} \right) + o(1). \tag{1.35}$$

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By taking the limit of $n \to \infty$, and using the definition of rate of conversion, see Def. 15, together with the fact that the regularised version of $E_{\mathcal{F}}$ exists, see Lem. 3, we have

$$E_{\mathcal{F}}^{\infty}(\rho) \geq R(\rho \to \sigma) E_{\mathcal{F}}^{\infty}(\sigma). \tag{1.36}$$

Notice that this equation was proved for any asymptotic continuous monotone (not just for $E_{\mathcal{F}}^{\infty}$) whose regularisation exists in Ref. [43, Thm. 4]. We can also consider the reverse transformation, mapping the state σ into the state ρ. Using the fact that free states do not contribute to the value of $E_{\mathcal{F}}$, Lem. 2, and that the resource theory is reversible, Def. 16, we obtain

$$E_{\mathcal{F}}^{\infty}(\rho) \leq R(\rho \to \sigma) E_{\mathcal{F}}^{\infty}(\sigma). \tag{1.37}$$

We now consider the case in which $\sigma = \rho_0$, the state associated with a non-zero value of $E_{\mathcal{F}}^{\infty}$. Then, it is easy to show that Eq. (1.36) implies that $E_{\mathcal{F}}^{\infty}(\rho) > 0$, and since ρ is completely general, any non-free state has to be associated with a non-zero value of the relative entropy distance from the free states. Therefore, we can combine the two Eqs. (1.36) and (1.37), and the result proves the theorem. $\qquad\square$

In the above theorem, the assumption that at least one state $\rho_0 \in \mathcal{S}(\mathcal{H})$ exists such that $E_{\mathcal{F}}^{\infty}(\rho_0) > 0$ is fundamental for the result to hold. Indeed, there are resource theories whose set of free states satisfies properties F1 – F4, but in which the regularised relative entropy distance from this set is equal to zero over all states in $\mathcal{S}(\mathcal{H})$. This is the case of the resource theory of asymmetry, see Ref. [6, Sec. IV].

We have shown that, if the resource theory is reversible, then a unique measure of resource exists. However, showing that a theory is reversible often represents a non-trivial task, and one might be interested in finding some general properties that, when satisfied by a resource theory, imply that the theory is reversible. This is done in Ref. [32], where the authors show that, when the set of free states obeys specific conditions, a reversible resource theory with such set of free states can be built. The class of allowed operations for this reversible theory is given by the most general set of maps that cannot create resource (asymptotically),

Definition 17 (Asymptotically resource-non-generating map)**.** *Consider a resource theory* R *with a set of free states $\mathcal{F}$. The operation $\varepsilon_n : \mathcal{S}(\mathcal{H}^{\otimes n}) \to \mathcal{S}(\mathcal{H}^{\otimes n})$ is* asymptotically resource-

non-generating *if, for all* $\gamma_n \in \mathcal{F}^{(n)}$, *we have that*

$$\lim_{n \to \infty} \mathcal{R} \left(\varepsilon_n(\gamma_n) \right) = 0, \tag{1.38}$$

where $\mathcal{R}$ is the global robustness, introduced in Eq. (1.11).

In order to build a reversible resource theory, we need the set of free states to satisfy the properties F1 – F5 introduced in the previous section. A prominent example of a resource theory whose set of free states satisfies these properties is the one of entanglement [44, 45]. Other examples include the resource theories of purity, thermodynamics, and magic states. For these resource theories, one can introduce an asymptotically resource-non-generating operation able to map between any two states with the optimal rate of conversion. In the following theorem, formally proved in Ref. [32, Thm. 1], we introduce this map and analyse its properties.

Theorem 2. *Consider a resource theory R with a set of free states $\mathcal{F}$ satisfying properties F1 – F5. Then, the theory R equipped with a set of allowed operations given by all the asymptotically resource-non-generating maps, see Def. 17, is reversible.*

Sketch of proof. In the following, we introduce a map able to transform between any two quantum states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$, and we provide an intuition why this map is asymptotically resource-non-generating. For a given $n \in \mathbb{N}$, we introduce a map acting over the state space $\mathcal{S}(\mathcal{H}^{\otimes n})$, defined as

$$\varepsilon_n(\cdot) = \mathrm{Tr}\left[E_n \cdot \right] \sigma_{k_n} \otimes \gamma_{n-k_n} + \mathrm{Tr}\left[(\mathbb{I}_n - E_n) \cdot \right] \pi_{k_n} \otimes \gamma_{n-k_n}. \tag{1.39}$$

In particular, the map is completely defined by the following conditions,

1. For a given $n \in \mathbb{N}$, we define $k_n = n \frac{E_{\mathcal{F}}^\infty(\rho)}{E_{\mathcal{F}}^\infty(\sigma)}$. Notice that we are here assuming $E_{\mathcal{F}}^\infty(\rho) < E_{\mathcal{F}}^\infty(\sigma)$. The map ε_n can be easily modified to describe the opposite case.

2. The sequence of states $\sigma_{k_n} \in \mathcal{S}\left(\mathcal{H}^{\otimes k_n} \right)$ is such that,

$$\lim_{n \to \infty} \left\| \sigma_{k_n} - \sigma^{\otimes k_n} \right\|_1 = 0. \tag{1.40}$$

3. The sequence of states $\pi_{k_n} \in \mathcal{S}\left(\mathcal{H}^{\otimes k_n} \right)$ is chosen to be such that

$$\frac{1}{1 + \mathcal{R}\left(\sigma_{k_n} \right)} \left(\sigma_{k_n} + \mathcal{R}\left(\sigma_{k_n} \right) \pi_{k_n} \right) \in \mathcal{F}^{(k_n)}, \quad \forall n \in \mathbb{N}. \tag{1.41}$$

where $\mathcal{R}\left(\sigma_{k_n} \right)$ is the global robustness of the state σ_{k_n}, see Eq. (1.11).

4. The POVM $\{E_n, \mathbb{I}_n - E_n\}$, where $E_n \in \mathcal{B}\left(\mathcal{H}^{\otimes n}\right)$, is such that

$$\mathrm{Tr}\left[(\mathbb{I}_n - E_n)\rho^{\otimes n}\right] \to 0, \quad \text{for } n \to \infty, \tag{1.42}$$

and

$$\max_{\gamma_n \in \mathcal{F}^{(n)}} \mathrm{Tr}\left[E_n \gamma_n\right] = e^{-nE_{\mathcal{F}}^{\infty}(\rho)+o(n)}. \tag{1.43}$$

The existence of such POVM follows from a generalisation of quantum Stein's lemma, see Thm. 13 in appendix A.

In the limit of many copies $n \gg 1$, the operation ε_n described in Eq. (1.39) maps n copies of ρ into k_n copies of σ, with a rate of conversion $R(\rho \to \sigma) = \frac{E_{\mathcal{F}}^{\infty}(\rho)}{E_{\mathcal{F}}^{\infty}(\sigma)}$. This directly follows from the definition of rate of conversion, see Def. 15, since we have that

$$\lim_{n\to\infty} \left\| \varepsilon_n(\rho^{\otimes n}) - \sigma^{\otimes k_n} \otimes \gamma_{n-k_n} \right\|_1 = 0, \tag{1.44}$$

and the trace distance is monotonic under partial tracing. The above equation is a consequence of Eq. (1.42), and condition 2.

We now provide an intuition why the map ε_n is asymptotically resource-non-generating. To do so, we use the following relation between the global robustness of a sequence of states and the regularised relative entropy distance of the state the sequence converges to, see Ref. [38, Prop. II.1],

$$\lim_{n\to\infty} \frac{1}{k_n} \log 1 + \mathcal{R}\left(\sigma_{k_n}\right) = E_{\mathcal{F}}^{\infty}(\sigma), \tag{1.45}$$

where the fact that $\{\sigma_{k_n}\}_n$ converges to the i.i.d. limit of σ follows from condition 2. For $n \gg 1$, and if we only consider the leading order in both Eqs. (1.43) and (1.45), we have that

$$\max_{\gamma_n \in \mathcal{F}^{(n)}} \mathrm{Tr}\left[E_n \gamma_n\right] \approx \frac{1}{1 + \mathcal{R}\left(\sigma_{k_n}\right)}, \tag{1.46}$$

where we made use of condition 1, and we use the symbol $\approx$ to highlight that the relation is only valid when considering the leading order. Let us now compute the state $\varepsilon_n(\gamma_n)$ for $\gamma_n \in \mathcal{F}^{(n)}$. When $n \gg 1$, and we only consider the leading order in n, this state can be expressed as

$$\varepsilon_n(\gamma_n) \approx \frac{1}{1 + \mathcal{R}\left(\sigma_{k_n}\right)}\left(\sigma_{k_n} \otimes \gamma_{n-k_n} + \mathcal{R}\left(\sigma_{k_n}\right)\pi_{k_n} \otimes \gamma_{n-k_n}\right), \tag{1.47}$$

that belongs to $\mathcal{F}^{(n)}$, as it follows from condition 3. Thus, this operation maps free states into approximately free states, and it is possible to formally show that the global robustness of the final state tends to zero for $n \to \infty$. $\qquad\square$

The above theorem provides a way to build a reversible resource theory when a family of free sets is given. However, the class of allowed operations associated with this theory does not (in general) follow from any operational constraints. Indeed, the allowed operation introduced in Thm. 2 only needs to be asymptotically resource-non-generating. Notice that this set of allowed operations contains the operational set of the theory. For example, in entanglement theory one can consider the resource theory whose allowed operations are LOCC, or the one whose operations preserve the separable states, and the former set is contained into the latter.

1.5 Single-copy regime and catalytic transformations

In this last section, we consider situations in which the agent can only act over a single copy of the system under examination [46]. In this scenario, quantum fluctuations are important, and whether or not a state transformation is possible depends on the value of multiple monotones. In particular, we here provide necessary and sufficient conditions for a single-copy state transformation to be possible, for a specific class of resource theories. Then, we introduce the notion of *catalysts*, ancillary systems that can be added to the main system in order to facilitate a state transformation that would otherwise be impossible to achieve. Some of the notions we introduce in this section are used in Sec. 2.3, where we discuss the conditions for a thermodynamic transformation to be realisable, and in Ch. 5, where a catalyst is used to realise a work extraction protocol that would be otherwise impossible.

1.5.1 Necessary and sufficient conditions for state transformations

We are interested in finding necessary and sufficient conditions under which the agent is able to map a state ρ into a different state σ using the allowed operations of the theory $\mathcal{A}$. Clearly, these conditions depend on the type of allowed operations we are considering, and in the following we focus on a specific case that nevertheless applies to different resource theories, such as the one of pure bipartite entanglement [25, 47], purity [37], and thermodynamics [8]. These results are valid in the so called "single-shot regime", where the agent has access to a single copy of the state (in contrast with the situation we have considered in the previous two sections).

Before we introduce the features of the resource theories we are studying in this section, let

us define the notion of doubly-stochastic matrix.

Definition 18 (Doubly-stochastic matrix). *Consider the set of $d \times d$ real square matrices $M_{d \times d}(\mathbb{R})$. A matrix $D \in M_{d \times d}(\mathbb{R})$ with non-negative elements is called doubly-stochastic iff*

1. $\sum_{i=0}^{d-1} D_{i,j} = 1, \quad \forall j \in \{0, \ldots, d-1\}$ *(stochastic matrix)*.

2. $\sum_{j=0}^{d-1} D_{i,j} = 1, \quad \forall i \in \{0, \ldots, d-1\}$ *(identity-preserving matrix)*.

The necessary and sufficient conditions for state transformations that we are going to obtain apply to any resource theory in which,

R1 The quantum states we are considering can be represented by vectors of dimension d (where d is the dimension of the Hilbert space $\mathcal{H}$ describing the system), whose elements are positive, and whose ℓ_1-norm (see Def. 6) is equal to 1.

R2 The allowed operations in $\mathcal{A}$ act over these unit vectors as doubly-stochastic matrices in $M_{d \times d}(\mathbb{R})$, see Def. 18. Formally, this means that an allowed operation $\varepsilon \in \mathcal{A}$ maps a state ρ into a state σ if and only if there exists a doubly-stochastic matrix D mapping the vector v_ρ into the vector v_σ.

An example of a resource theory where these two conditions are satisfied is, for instance, the resource theory of pure bipartite entanglement. Indeed, any pure entangled state $|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ can be written, using the Schmidt decomposition [48], as

$$|\psi\rangle_{AB} = \sum_{i=0}^{d} \sqrt{\lambda_\psi^i} \, |\phi_i\rangle_A \otimes |\theta_i\rangle_B \,, \tag{1.48}$$

where we assume for simplicity that $d = \dim(\mathcal{H}_A) = \dim(\mathcal{H}_B)$, and we have that the sets $\{|\phi_i\rangle\}$ and $\{|\theta_i\rangle\}$ form a basis for $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. The d-dimensional vector of Schmidt coefficients, $v_\psi = \left(\lambda_\psi^0, \lambda_\psi^1, \ldots, \lambda_\psi^{d-1}\right)^T$, whose length is equal to 1 with respect to the ℓ_1-norm, represents the state $|\psi\rangle_{AB}$ (and any other state obtained through $|\psi\rangle_{AB}$ with local unitary operations). Furthermore, one can show that a state transformation from the state $|\psi\rangle_{AB}$ to the state $|\varphi\rangle_{AB}$ is possible, using LOCC, if and only if there exists a doubly-stochastic matrix mapping the vector v_φ into the vector v_ψ [25]. It is worth noting that the resource theory of pure entanglement does not satisfy condition R2, since the doubly-stochastic matrix is acting

on the vector of the final state, rather than on the vector of the initial one. However, the necessary and sufficient conditions we find in Thm. 3 do apply to this theory as well, although the position of the vectors in these conditions need to be inverted.

Another example of a resource theory where the above two conditions apply is the one of purity, whose allowed operations are Noisy Operations, given by the possibility of adding an arbitrary number of ancillary systems in the maximally-mixed state, of acting over system and ancilla with any unitary operation, and of discarding any subsystem. As we show in the next chapter, this set of operations is very similar to the one used in thermodynamics, and coincides with it when the Hamiltonian of the system is fully-degenerate. Without loss in generality, in this theory we restrict the attention to states diagonal in a given basis (since we can always use a unitary operation to diagonalise the state), and therefore the element of the state space are represented by d-dimensional unit vectors, whose components are the diagonal elements of the density operator. It can be shown that, also in this case, the allowed operations act over these vectors as doubly-stochastic matrices [37, Sec. 3].

We can now provide necessary and sufficient conditions for a state transformation to be possible in this class of resource theories. Before we do so, we need to introduce the notion of *majorization,*

Definition 19 (Majorization). *Consider the set V of d-dimensional vectors with positive elements, and with ℓ_1-norm equal to 1. Given two vectors $x, y \in V$, we say that x majorizes y, in symbols $x \succ y$, if and only if*

$$\sum_{i=0}^{k} x_i^{\downarrow} \geq \sum_{i=0}^{k} y_i^{\downarrow} \quad \forall\, k \in \{0, \ldots, d-1\}\,, \tag{1.49}$$

where $x^{\downarrow}$ ($y^{\downarrow}$) is the vector obtained by ordering the element of x (y) in decreasing order.

When a resource theory satisfies the two conditions above, we find that majorization provides the necessary and sufficient conditions for a state transformation to be realisable with the allowed operations.

Theorem 3. *Consider a resource theory* R *acting on a system described by the Hilbert space $\mathcal{H}$, with allowed operations $\mathcal{A}$. If the resource theory satisfies the conditions R1 and R2, then a state transformation from ρ to σ is possible if and only if the vectors associated to these two states, v_ρ and v_σ respectively, are such that $v_\rho \succ v_\sigma$.*

Proof. From condition R1, it follows that for any state ρ in the state space of the theory, there exists an associated unit vector v_ρ. Furthermore, given any two states ρ and σ, condition R2 tells us that there exists an allowed operation $\varepsilon \in \mathcal{A}$ such that $\varepsilon(\rho) = \sigma$ if and only if there exists a doubly-stochastic matrix D such that $D\,v_\rho = v_\sigma$. Using the result of Ref. [48, Prop. 12.11] together with Birkhoff's theorem, see Ref. [48, Thm. 12.12], one can show that a doubly-stochastic matrix D mapping v_ρ into v_σ exists if and only if $v_\rho \succ v_\sigma$. This proves the theorem. □

1.5.2 Catalytic transformations

Catalysts were first studied in the context of the resource theory of pure bipartite entanglement [49, 50, 51, 52], but due to their relation with majorization, they are used in other resource theories, such as the one of thermodynamics. A catalyst is an ancillary system that makes possible a state transformation which otherwise would not be realisable with the allowed operations, and whose state remains unchanged after the transformation.

Definition 20 (Catalyst)**.** *Consider the sets V and W, containing unit vectors (with respect to the ℓ_1-norm) whose elements are positive. Suppose the vectors in V are d-dimensional, and the ones in W are d'-dimensional. If the vectors $x, y \in V$ and $z \in W$ are such that $x \nsucc y$, but $x \otimes z \succ y \otimes z$, then we say that z is a* catalyst *for the transformation of x into y.*

Not every two vectors admit a catalyst. Nevertheless, there are examples in which a catalyst helps one vector majorizing another. For example, consider the two vectors

$$x = (0.5, 0.25, 0.25, 0)^T \,,$$
$$y = (0.4, 0.4, 0.1, 0.1)^T \,,$$

and the vector

$$z = (0.6, 0.4)^T \,.$$

It is easy to show that $x \nsucc y$ (nor the opposite), but if the vector z is added we find that $x \otimes z \succ y \otimes z$. Thus, z is a catalyst for the transformation mapping x into y.

Catalysts play a fundamental role in resource theories. If we enlarge the set of allowed operations to include the possibility of adding catalysts, the necessary and sufficient conditions

we found in Thm. 3 can be re-expressed using a one-parameter family of monotones. This was first shown in Refs. [51, 52], where a family of monotones was found for exact catalytic transformation, i.e., transformations which map the initial state into the final state exactly. However, in this thesis we are more interested in approximate transformations, where the initial state is mapped into a final state which is close[5] to the target state of the transformation. In fact, the operations that an experimentalist is able to realise in the laboratory are always of this latter kind, and therefore it seems reasonable to consider them. If approximate catalytic transformations are considered, the necessary and sufficient conditions are given by the following family of monotones, known as *Rényi entropies* [53].

Definition 21 (Rényi entropies)**.** *Consider the set V of d-dimensional vectors with positive elements, and with ℓ_1-norm equal to 1. We define the family of α-Rényi entropies, for $\alpha \geq 0$, and $\alpha \neq 1$, as*

$$H_\alpha(x) = \frac{1}{1-\alpha} \log \sum_{i=0}^{d-1} x_i^\alpha, \quad x \in V. \tag{1.50}$$

The $\alpha = 1$ Rényi entropy is the Shannon entropy $H(x)$, see Def. 8.

We now present the necessary and sufficient conditions for state transformations in a resource theory that satisfies the two properties introduced in the previous section, whose allowed operations include the possibility to use catalysts. This result is proven in Ref. [37, Lem. 46].

Theorem 4. *Consider a resource theory R acting on a system described by the Hilbert space $\mathcal{H}$, and the set of catalytic allowed operations $\mathcal{A}$, which extends the original set of allowed operations with the possibility of adding a catalyst to facilitate the transformation. If the resource theory satisfies the conditions R1 and R2, then a state transformation from ρ to σ is possible if and only if the vectors associated to these two states, v_ρ and v_σ respectively, are such that*

$$I_\alpha(v_\rho) \geq I_\alpha(v_\sigma), \quad \forall \alpha \geq 0, \tag{1.51}$$

where $I_\alpha(x) = \log d - H_\alpha(x)$, and d is the dimension of the Hilbert space $\mathcal{H}$.

Notice that in Eq. (1.51) we use I_α in place of H_α since this result holds even in the case in which the allowed operations maps between two Hilbert spaces with different dimensions. In

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[5]Here, a state is "close" to another one with respect to a given mathematical distance, for example the one induced by the ℓ_1-norm.

that situation, the constant $\log d$ in I_α would be different for v_ρ and v_σ, and it would not cancel out. In Sec. 2.3 we see how a similar family of monotones provides the necessary and sufficient conditions for state transformations in thermodynamics.

It is worth noting that the conditions we have imposed over the catalyst state in Def. 20 are quite strong. The catalyst needs to be returned exactly in its original state, and no correlation between the main system and this ancilla can be created during the transformation. These conditions are important in order not to trivialise the theory, since no constraint over the size of the catalysts is made, which therefore can even be of infinite size. However, if we ask for the catalysts to be returned approximately in its initial state, one can show that any state transformation becomes possible, and the resource theory becomes trivial [54]. A detailed study of the different ways in which the final state of the catalyst can be approximated, and the consequences for the resource theory of thermodynamics, can be found in Ref. [55].

Another way in which the constraints over the catalyst can be weakened concerns the possibility of creating correlations within the catalysts [56], or between the catalyst and the system [57, 58]. One can imagine, for example, that the catalyst is composed of several subsystems, and the initial state of the catalyst is factorised. After the transformation, the local states of the catalyst are unchanged, but correlations have been created. Similarly, we can think of a catalyst initially uncorrelated from the main system, which later becomes correlated with it, even if the local state is preserved. In these situations, the necessary and sufficient conditions given in Thm. 4 collapse into a single condition, given by the Shannon entropy H [58]. Finally, in Ch. 5 we show that correlating catalysts are helpful even in resource theories where majorization does not play a role.

Chapter 2

Thermodynamics as the resource theory of athermality

Quantum thermodynamics studies the energetic and entropic flows which occur between a microscopic system and its environment, as well as the interplay between these flows and the quantum properties of the system, such as coherence and entanglement. Although thermodynamics, and statistical mechanics, historically focused on the properties of systems at equilibrium, the field of quantum thermodynamics can describe processes in which the system is driven far away from equilibrium. Quantum thermodynamics addresses both fundamental and applied questions. For example, its results find applications in the upcoming field of quantum technologies, devices that exploit quantum phenomena to outperform their classical counterpart. Indeed, the processes taking place inside these devices require some of their components to exhibit a quantum behaviour, and therefore to be described by states out of thermal equilibrium. Examples of such devices, some of which currently realisable in the laboratory, are quantum sensors [59, 60], microscopic heat engines [61, 62], many-body simulators [63, 64], and prototypes for quantum computation [65, 66].

There exist different approaches to the study of thermodynamics in the microscopic regime, for example coming from the fields of statistical mechanics [67], of open quantum systems [68], and of quantum information theory [69, 70]. In this chapter we focus on the latter approach, and precisely on the formulation of thermodynamics as a resource theory, a framework that

we have introduced in the previous background chapter. By re-casting thermodynamics as a resource theory, one can systematically investigate the conditions under which a thermodynamic transformation is realisable. It is worth noting that other, more abstract approaches to the study of thermodynamics exist, within the framework of General Probabilistic Theories [71, 72, 73, 74, 75].

Thermodynamics and information theory are two fields that are deeply interconnected [69]. The erasure of a bit of information has a fundamental thermodynamic cost, in terms of heat dissipated, as stated by Landauer's principle [76]. When the physical system storing the bit is in contact with an environment at temperature T, one has to dissipate $k_B T \log 2$ of energy in order to reset the state of a bit, from an unknown state to the state '0'. Likewise, having information about a system allows us to extract work from it. This is the case of the Szilárd engine [77], consisting of a box divided into two partitions and containing a particle of gas, see Fig. 2.1. When this engine is in contact with an environment at temperature T, the knowledge on the position of the particle (whether it is in one partition or another) allows us to extract $k_B T \log 2$ of work.

Within the field of quantum thermodynamics there exist several distinct lines of research. Below, we provide a (non-exhaustive) list of the main theoretical research directions, which are investigated with tools from both statistical mechanics and information theory.

- Studying the properties and the efficiency of heat engines at the quantum scale. These are microscopic devices able to extract work from the heat flow generated between two thermal reservoirs at different temperature. Different topics are investigated, such as which limitations are imposed on the efficiency of these machines by the fact that they operate in the quantum regime [78, 79, 80, 81], what role is played by quantum features (like coherence and entanglement) during the work extraction process [82, 83, 84, 85], and which new cycles can be designed for machines operating in the quantum regime [86, 87].

- Extending the fluctuation theorems [88, 89] to the quantum case. These theorems relate the equilibrium properties of a system to its out-of-equilibrium properties, and offer a powerful tool for experimentalists working in the field. Results on the quantum version of the fluctuation theorems can be found from both a statistical mechanical [90] and a resource theoretic [91, 92] perspective. A key ingredient of these theorems is the proba-
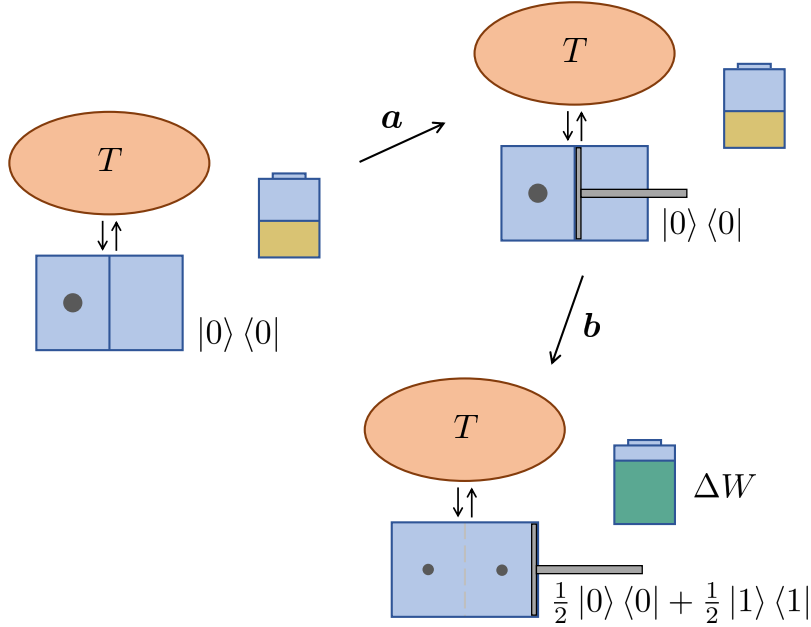
Figure 2.1: The work extraction process in a Szilárd engine. The engine consists in a box divided into two partitions, where a piston can be inserted from each side, and the wall separating the partitions can be removed. This box contains a gas, and is in contact with a thermal bath at temperature T. The piston can be attached to a battery to exchange work with the box. At the beginning (top-left) the gas in a single partition, so that the state describing the box is $|0\rangle$. **a.** Given this knowledge, the agent can insert a piston in the partition where no gas is present, and push it until it reaches the separating wall. If the piston is friction-less, no work is used to perform this passage. The piston is then connected to the battery, so as to be able to exchange work with it. **b.** The separating wall is removed at no expense of work, and the gas expands in the box, while the box is in thermal contact with the reservoir. The expansion pushes the piston, and allows us to extract an amount of work ΔW which is stored in the battery. After the expansion, the agent has no remaining information on where the gas is located, so that the final state of the box is the maximally-mixed state $\frac{1}{2}|0\rangle\langle0| + \frac{1}{2}|1\rangle\langle1|$. The amount of work extracted is $\Delta W = k_B T \log 2$.

bility distribution of the work exchanged during a thermodynamic process, and therefore this line of research is linked to the definition of work in the microscopic regime, another

important topic in quantum thermodynamics.

- Studying thermodynamics of systems with multiple conserved quantities, both in the case in which these quantities commute or do not commute with each other. Research lines include the definition of resource theories to describe this scenario [93, 94, 95], and the development of protocols for extracting and trading these quantities [96, 97].

- Understanding the phenomena of equilibration, i.e., the process in which thermodynamic systems reach a state whose properties are left unchanged by the dynamics, and thermalisation, i.e., the process where systems equilibrate to a thermal state, defined by a given temperature. The study of these phenomena from a quantum mechanical point of view is particularly interesting and challenging, given the reversible nature of this theory [98, 99]. A detailed review on this subject is Ref. [100].

- Designing autonomous machines and clocks within the quantum mechanical framework. Questions that are investigated include how clocks can be realised as quantum mechanical systems [101, 102], and which limitations affects these devices [103, 104, 105]. This topic is relevant in thermodynamics, since it is connected to the realisations of those processes in which the system's Hamiltonian is allowed to change, and to those scenarios where the agent needs to manipulate the coherence, in the energy eigenbasis, of a quantum system. Additionally, research into autonomous machines can inform the design of heat engines [106, 107].

In the following, we introduce a resource theoretic framework for studying quantum thermodynamics. In particular, in Sec. 2.1 we present the resource theory whose allowed operations are Thermal Operations. This resource theory describes the scenario in which a system is coupled to an infinite thermal reservoir at a given temperature, and system and reservoir are treated as a global isolated system. In Sec. 2.2, we describe the main features of this resource theory, together with its limitations. For example, we comment on the fine-grained control the agent is given over system and environment, and on the impossibility, using the allowed operations of the theory, to create coherence in the energy eigenbasis. In Sec. 2.3, we study state transformations in this setting, and we find necessary and sufficient conditions for a class of state transformations to be realisable. We additionally study the case in which catalysts,

introduced in the previous chapter, are allowed. In Sec. 2.4, we introduce one notion of work for the quantum case, and we show that in the single-copy case, quantum thermodynamics is an irreversible theory, since work is lost during a cyclic state transformation, while in the many-copy case, reversibility is restored.

2.1   Thermodynamic setup and Thermal Operations

Quantum thermodynamics studies the energetic properties of a microscopic system in contact with an environment, and it focuses on processes outside equilibrium. The system under consideration can have quantum features, such as coherence over the energy eigenbasis, or being entangled with another system. We can study the thermodynamic properties of systems which are isolated from the outside world, or that interact with an environment and exchange energy, particles, or other quantities with it. For example, a common choice for the environment in thermodynamics is a system of infinite size (or infinite heat capacity), described by a single parameter, its temperature. This system, known as *thermal reservoir*, is in equilibrium with respect to its own dynamics, and is able to exchange an infinite amount of heat with an external system without changing its temperature. Other examples of environments, some of which are considered in the next chapters, are finite-sized thermal reservoirs, or environments that are not in thermal equilibrium, possibly because their dynamics admits multiple conserved quantities.

When the system is interacting with an environment, we can consider these two systems as a single, isolated one, which we refer to as the *universe*. The evolution of this global system obeys the laws of quantum mechanics, and it is therefore represented with unitary operations. Thus, the global evolution is reversible, but if we restrict our investigation to the sole system, and we forget the environment, we obtain an irreversible evolution. Furthermore, since the universe is an isolated system, its global evolution needs to preserve the total energy, in accordance with the First Law of Thermodynamics.

We can now introduce a well-studied resource theory for thermodynamics, whose allowed operations are referred to as *Thermal Operations* [108, 7, 8, 109], see also the review in Ref. [110]. The system under investigation is generally taken to be finite-dimensional, and it is described by a Hilbert space $\mathcal{H}_S$. In this theory, the system is in contact with an infinite thermal reservoir at

a given temperature, and the evolution of system and environment preserves the global energy.

Definition 22 (Thermal Operations)**.** *The class of allowed operations $\mathcal{A}_{TO}$, known as* Thermal Operations*, describes a thermodynamic system S in contact with a thermal reservoir at temperature β^{-1}. This set is composed by three fundamental operations,*

1. *The agent can add any (finite-dimensional) ancillary system B to the main one, provided that the ancilla's state is the Gibbs state of its Hamiltonian H_B at the background temperature,*

$$\rho_S \mapsto \rho_S \otimes \tau_\beta, \quad \rho_S \in \mathcal{S}(\mathcal{H}_S), \tag{2.1}$$

   where

$$\tau_\beta = \frac{e^{-\beta H_B}}{Z} \in \mathcal{S}(\mathcal{H}_B), \tag{2.2}$$

   with H_B the Hamiltonian of the ancilla, and $Z = \mathrm{Tr}\left[e^{-\beta H_B}\right]$ its partition function.

2. *The agent can apply any energy-preserving unitary operation over system and ancilla,*

$$\rho_{SB} \mapsto U \rho_{SB} U^\dagger, \quad \rho_{SB} \in \mathcal{S}(\mathcal{H}_S \otimes \mathcal{H}_B), \; U \in \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_B), \tag{2.3}$$

   If the total Hamiltonian of system and ancilla is $H = H_S \otimes \mathbb{I}_B + \mathbb{I}_S \otimes H_B = H_S + H_B$, then the global unitary operation U is such that,

$$[H_S + H_B, U] = 0. \tag{2.4}$$

3. *The agent can discard the state of part of the global system,*

$$\rho_{SB} \mapsto \mathrm{Tr}_{B'}[\rho_{SB}] = \rho_{S'}, \quad \rho_{SB} \in \mathcal{S}(\mathcal{H}_S \otimes \mathcal{H}_B), \; \rho_{S'} \in \mathcal{S}(\mathcal{H}_{S'}), \tag{2.5}$$

   where we have that $\mathcal{H}_S \otimes \mathcal{H}_B = \mathcal{H}_{S'} \otimes \mathcal{H}_{B'}$.

The most general allowed operation the agent can apply to the system is then obtain by composing the three fundamental maps shown above. The form of this operation is

$$\varepsilon_{TO} : \mathcal{S}(\mathcal{H}_S) \to \mathcal{S}(\mathcal{H}_{S'}), \tag{2.6}$$

$$\rho_S \mapsto \varepsilon_{TO}(\rho_S) = \mathrm{Tr}_{B'}\left[U\rho_S \otimes \tau_\beta U^\dagger\right]$$

where τ_β is the thermal state with Hamiltonian H_B and temperature β^{-1}, see Eq. (2.2), and the unitary operations U is energy-preserving in the sense of Eq. (2.4). It is worth noting that the above operation is changing the quantum system under examination, since the agent is allowed to forget the state of part of the global system. However, we can use the method shown in the last paragraph of Sec. 1.1 to make these operations endomorphisms of the state space $\mathcal{S}(\mathcal{H}_S \otimes \mathcal{H}_{S'})$. For simplicity, in the following we consider the case in which the initial and final state spaces are the same.

Thermal Operations can be physically understood as follow. The agent is in the presence of a quantum system and an infinite thermal reservoir which are initially isolated from each other. According to the fundamental operation 1, the agent can take any ancillary system B with Hamiltonian H_B, and they can put it in contact with the thermal bath until it thermalises. Equivalently, we can assume the bath to be composed by an infinite number of finite-dimensional systems, each one with different dimension and different Hamiltonian, but all described by a Gibbs state at the background temperature. The fundamental operation 2 then implies that the agent can put the system in contact with the chosen ancilla in the thermal state, and make them interact using any reversible evolution which preserves the global energy exactly. After the interaction, according to the fundamental operation 3, the agent is free of considering only a part of the global system. For example, they can forget the state of ancilla and solely consider the state of the initial system, or they can decide to focus on the state of a completely different partition. In Sec. 2.2, we examine in more details the physical consequences of using these allowed operations to describe thermodynamics.

Let us now turn to the free states of the theory. These are the states that can be always prepared using the allowed operations, see Def. 10. For Thermal Operations, the set of free states contains a single state (when we consider the case in which the operations map the state space into itself), namely, the Gibbs state of the system Hamiltonian at the background temperature.

Proposition 3. *Consider the resource theory of thermodynamics with Thermal Operations acting on the finite-dimensional system S with Hamiltonian H_S. Then, the set of free states of this theory is composed by a single state, the thermal state $\tau_\beta = e^{-\beta H_S}/Z$, where β is the inverse temperature of the thermal reservoir.*

Proof. Let us first show that we can always map the state of the system S into the thermal state of the Hamiltonian H_S with temperature β. This can be achieved by the following map

$$\varepsilon_{\text{thermal}}(\rho_S) = \text{Tr}_B\left[U_{\text{swap}}\, \rho_S \otimes \tau_\beta\, U_{\text{swap}}^\dagger\right] = \tau_\beta, \quad \forall\, \rho_S \in \mathcal{S}\left(\mathcal{H}_S\right), \tag{2.7}$$

where we chose an ancillary system with the same Hilbert space of the main system, $\mathcal{H}_B = \mathcal{H}_S$, and the same Hamiltonian, $H_B = H_S$. The operation U_{swap} swaps the state of the main system with the one of the ancilla, and preserves the global energy since the Hamiltonian of these two systems is the same. Thus, according to Eq. (1.2), τ_β is a free state of the theory.

We now show that τ_β is a fixed point for the set of Thermal Operations. This implies that τ_β is the sole free state of the theory, since no other state could be reached once we map the system into this state. Consider an arbitrary allowed operation ε_{TO} in the form of Eq. (2.6), mapping the state space $\mathcal{S}\left(\mathcal{H}_S\right)$ in itself. Then,

$$\begin{aligned}
\varepsilon_{TO}(\tau_\beta^{(S)}) &= \text{Tr}_B\left[U\tau_\beta^{(S)} \otimes \tau_\beta^{(B)}U^\dagger\right] = \frac{1}{Z_S\, Z_B}\text{Tr}_B\left[U\, e^{-\beta H_S} \otimes e^{-\beta H_B}\, U^\dagger\right] \\
&= \frac{1}{Z_S\, Z_B}\text{Tr}_B\left[U\, e^{-\beta(H_S+H_B)}\, U^\dagger\right] = \frac{1}{Z_S\, Z_B}\text{Tr}_B\left[e^{-\beta(H_S+H_B)}\right] \\
&= \frac{e^{-\beta H_S}}{Z_S} = \tau_\beta^{(S)},
\end{aligned} \tag{2.8}$$

where the forth equality follows from the fact that $U \in \mathcal{B}\left(\mathcal{H}_S \otimes \mathcal{H}_B\right)$ commutes with the total Hamiltonian $H_S + H_B$, see Eq. (2.4). This closes the proof of the proposition. $\qquad\square$

The resource theory of Thermal Operations is not the only theory able to describe thermodynamic phenomena at the quantum scale. Other sets of allowed operations can be considered. For example, one can modify $\mathcal{A}_{TO}$ to include any unitary operation that preserves the average energy of the system, rather than preserving the energy exactly [111]. Or we can rephrase thermodynamics as a multi-resource theory, whose allowed operations are given by noisy maps which also preserve the energy of the system, see Ch. 4 for a detailed study of this theory. Otherwise, we can consider all those operations for which the thermal state is a fixed point, known as *Gibbs-preserving maps* [112, 113],

Definition 23 (Gibbs-preserving map)**.** *Consider a finite-dimensional system associated with the Hilbert space $\mathcal{H}$, whose Hamiltonian is H. The set of* Gibbs-preserving maps *is defined as*

$$\mathcal{A}_{GP} = \{\varepsilon : \mathcal{S}\left(\mathcal{H}\right) \to \mathcal{S}\left(\mathcal{H}\right) \mid \varepsilon(\tau_\beta) = \tau_\beta\}, \tag{2.9}$$

where $\tau_\beta = e^{-\beta H}/Z$.

It is worth noting that Thermal Operations are a strict subset of Gibbs-preserving maps. Indeed, from Prop. 3 it follows that Thermal Operations are also Gibbs-preserving maps, since they map the thermal state in itself. However, in order to show that $\mathcal{A}_{TO} \subsetneq \mathcal{A}_{GP}$, one needs to find an operation that preserves τ_β but is not a Thermal Operation. In Ref. [112], the authors provide one such operation, exploiting the fact that Thermal Operations cannot create coherence over the energy eigenbasis (see Sec. 2.2.5 for a discussion on this topic). They construct an operation that is able to create a superposition over the eigenstates of the Hamiltonian operator, while also keeping the thermal state fixed.

2.2 Physical features of Thermal Operations

In this section, we study the physical implications of the mathematical structure of Thermal Operations. Specifically, we are here interested in understanding which thermodynamic scenarios can be described with these operations, and which ones cannot be described. Furthermore, we focus on the peculiarities of this set of operations, for instance on the fact that the global energy needs to be conserved exactly, or that coherence over the energy basis cannot be created. Finally, we investigate which of the assumptions made while defining the set of Thermal Operations might be considered too strong, and we describe the steps that have been done (or need to be done) in order to make this class of operations closer to experimentally realisable processes [114].

2.2.1 No correlations between system and environment

The first of the fundamental operations composing the set of Thermal Operations tells us that the agent can take any subsystem of the thermal reservoirs, and this subsystem is not correlated with the main system. The assumption of an initially-uncorrelated system and thermal reservoir restricts the range of physical situations which can be described by Thermal Operations. For example, this set of operations is suitable to describe the case in which the system is initially isolated, and is subsequently put in contact with the thermal bath. When instead the system has been in the contact with the environment before, it is reasonable to expect that correlations

have been created, and the framework does not apply.

One can nevertheless modify the current theory to include situations where system and thermal reservoir are correlated [115]. In this case, the global initial state is given by a generic $\rho_{SB} \in \mathcal{S}(\mathcal{H}_S \otimes \mathcal{H}_B)$, such that, if we only consider the reservoir, we have that $\text{Tr}_S[\rho_{SB}] = \tau_\beta$, the Gibbs state of the reservoir Hamiltonian at temperature β^{-1}. Then, the agent can act on this global system with energy-preserving unitary operations. Within this framework, we can investigate how the cost of different thermodynamic processes changes due to the initial correlations between system and environment. For example, one can study the optimal amount of work that can be extracted from correlations [116, 117], i.e., from a process mapping a state ρ_{SB} into $\rho_S \otimes \rho_B$, where $\rho_S = \text{Tr}_B[\rho_{SB}]$, and $\rho_B = \text{Tr}_S[\rho_{SB}]$. Similarly, one can study the energy cost of the opposite process, where correlations between two initially uncorrelated thermodynamic systems are created [118]. We can also investigate whether the onset of correlation during a thermodynamic process facilitate energy extraction from a collection of systems [119].

Interesting thermodynamic effects arise when correlations are present between two thermodynamic systems. For instance, anomalous heat flows can be observed [120, 121, 122], as well as violations of Landauer's principle [76]. Indeed, when two thermal reservoirs with different temperatures are correlated, one can observe heat flowing from the colder reservoir to the hotter one, in apparent violation of the Second Law. A similar violation occurs when we erase the state of a memory which is correlated to another system. In particular, if the memory is entangled with the other system, one can erase its state while extracting energy during the process [123], in apparent violation of Landauer's principle.

2.2.2 High degree of control over the environment

With Thermal Operations, the agent is allowed to address any subsystem within the thermal bath, operation 1, and they can perform any (energy-preserving) reversible transformation on this subsystem, see operation 2. In practice, an experimentalist in their laboratory does not have this degree of control over the environment, and they can solely address the degrees of freedom of the system under examination. Thus, Thermal Operations describe a situation in which the agent is able to perform more powerful operations than the one we can realise in practice. Consequently, the results obtained within this framework provide lower bounds to

the amount of resource needed to perform a thermodynamic process. It is worth noting that the existence of these lower bounds is not a consequence of the imperfect control over the thermodynamic processes, but rather a fundamental limitation arising from the thermal nature of the environment, and from the principle of energy conservation.

Thermal Operations are not the sole set of operations that can be used for describing thermodynamics from a resource theoretic perspective. More "experimental friendly" sets of operations have been defined, which only require a coarse-grained control over the system and the thermal reservoir [124, 125]. When equipped with these operations, the agent can act on any two levels of the system, and they can make these two levels thermalise, or they can change the energy gap between these two levels. Interestingly, one can show that any process which is realised with Thermal Operations can also be realised with this experimentally less demanding set of operations [124].

2.2.3   Exact energy conservation

When the agent is equipped with Thermal Operations, they can apply any unitary operation which preserves the energy of system and environment exactly, Eq. (2.4). The idea behind this requirement is that system and environment can be considered as a global, isolated system, and, according to the principle of energy conservation, the energy of such isolated system needs to be preserved during its evolution. Furthermore, the fact that energy is conserved allows us to precisely quantify the transfers occurring between system and environment, or between any other partition we might additionally consider. Notice that if energy were not conserved, we could also interpret any change in this quantity as an exchange with an additional system that we have not included yet into our description, that would act as a sink/source of energy.

One might question whether considering system and environment as an isolated system is a physically motivated assumption. For it to be a reasonable assumption, we need to include in our description a big enough portion of the environment surrounding the system, so that the interactions with the remaining environment are negligible compared to the energy scale of the global system under examination. This is the case, for example, of any system with local interactions, since the energy of the bulk scales like the volume of the system, whereas the energy on the boundary scales like its area. Otherwise, we can simply consider the entirety of

the surrounding environment, up to the point in which there is nothing else the global system can interact with, since we are essentially considering the whole universe.

In order to describe the interactions between system and environment, the formalism of Thermal Operations makes use of the unitary representation. An alternative description is given in terms of interaction Hamiltonians, which can be either time-dependent or -independent. Since this latter description is commonly used to describe processes occurring in a laboratory, it is worth investigating its connection with Thermal Operations, and understanding in which situations an interacting Hamiltonian can be linked to an energy-preserving unitary operation. A comparison of these two approaches can be found in Ref. [7, Supplemental Material]. The easiest example consists in the one in which the interaction Hamiltonian $H_{\text{int}} \in \mathcal{B}\left(\mathcal{H}_S \otimes \mathcal{H}_B\right)$ commutes with the total Hamiltonian of system and environment, that is $[H_{\text{int}}, H_S + H_B] = 0$. In this scenario, the strength of the interaction can be arbitrary, and the coupling can be time-dependent or -independent, but the resulting unitary evolution still commutes with the total Hamiltonian. An example of such interaction Hamiltonian can be found in the (perfectly resonant) Jaynes-Cummings model [125, 126]. This model describes the interaction between a two-level system inside a cavity, and a single mode of the electromagnetic field in that cavity. In this picture, the system absorbs a photon of the field to get excited, and emits a photon while decaying. If the energy gap of the system is equal to the energy of the absorbed/emitted photons (that is, when the field is perfectly resonant), the interaction Hamiltonian commutes with the Hamiltonian of system and radiation.

Another situation that can be approximatively described with Thermal Operations is the one in which system and environment are weakly coupled. In this case, the energy scale of the interaction Hamiltonian is negligible compared to the energy scale of the Hamiltonian of system and environment, and therefore these two operators (almost) commute. In classical thermodynamics, where the main system is macroscopic, the weak coupling assumption is often satisfied, and Thermal Operations would therefore apply to this scenario. However, when microscopic systems are considered, they can be strongly coupled with the environment. Our formalism is still able to describe this situation, if for example we slowly bring system and environment in contact, we make them interact (even strongly) and slowly separate them. If this process is slow enough, we find that due to the *adiabatic theorem* [127, 128] the transformation

preserves all the eigenstates of the Hamiltonian of system and thermal reservoir, and therefore the evolution can be described by an energy-preserving unitary operation.

So far, we have seen that Thermal Operations can be used to describe situations in which the interaction Hamiltonian commutes with the total Hamiltonian, or where the interaction coupling is either weak or changes very slowly in time. We still need to consider the case in which the interaction between system and environment is strong and undergoes a sudden quench. This situation cannot be described with Thermal Operations unless we add a bit more structure to our model. If the operation changes the energy of the system, but does not introduce any coherence in the energy eigenbasis, then the transformation can be implemented with Thermal Operations by adding a battery to the framework, see Sec. 2.4. If, instead, the operation also introduces coherence in the energy eigenbasis, then we need to add to the picture a "control system", i.e., an additional system able to coherently compensate for the energy change in system and environment due to their interaction. Within the framework of Thermal Operations, this system is known as a *coherence reservoir*, that we describe in more details in Sec. 2.2.5.

2.2.4 Controlling the system's Hamiltonian

When the agent is equipped with Thermal Operations, they can act over the system and transform the state. However, without adding additional structure, the agent cannot change the global Hamiltonian of system and environment during the process. This seems to be a reasonable assumption for the Hamiltonian of the environment H_B, since an experimentalist does not, in general, have access to it. However, during a realistic thermodynamic process, the Hamiltonian of the main system can change. Indeed, this is often the case in an experiment, where the system is driven out of equilibrium by changing its Hamiltonian. We now show that if a *clock* [129, 130] is added to the framework, it becomes possible to describe changes in the system's Hamiltonian using Thermal Operations.

A clock acts like a register for the agent transformation; each eigenstate of the clock is associated with a different Hamiltonian of the system. For example, if we need to change the Hamiltonian of the system only once, we can use a two-dimensional clock, with a total Hamiltonian of system and clock given by $H_{SC} = H_S^{\text{in}} \otimes |0\rangle \langle 0|_C + H_S^{\text{fin}} \otimes |1\rangle \langle 1|_C$. Then, in

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order to map the state of the system $\rho \in \mathcal{S}\left(\mathcal{H}_S\right)$ into the state $\sigma \in \mathcal{S}\left(\mathcal{H}_S\right)$, while also changing the Hamiltonian of the system, the agent can perform the following global state transformation

$$\rho_S \otimes |0\rangle \langle 0|_C \rightarrow \sigma_S \otimes |1\rangle \langle 1|_C \, , \tag{2.10}$$

so that the clock is rotated from the state $|0\rangle$ into the state $|1\rangle$, while the state of the system is transformed as intended. Notice that the unitary operations used in the above transformation need to commute with the Hamiltonian H_{SC} of system and clock. The clock described in the above example is quite rudimentary, and a current line of research consists in improving the description of clocks for the quantum regime [101], and studying the ultimate limitations imposed by quantum mechanics on these devices [131, 103, 105].

2.2.5 Creating coherence

An interesting feature of Thermal Operations, which follows from the fundamental operations 1 and 2, is the fact that coherence over the energy eigenbasis cannot be created. Indeed, as we show in the following, these operations are symmetric with respect to the time translations generated by the system Hamiltonian. The fact that Thermal Operations cannot create coherence implies that coherence itself represents an additional resource in thermodynamics. In order to manipulate coherence in this resource theory, the agent needs to have access to an additional system, known as a *coherence reservoir* [132]. This coherence reservoir is a large system with non-degenerate Hamiltonian, and is described by a state in a coherent superposition. The agent can then exchange coherence between this reservoir and the main system, while not degradating the reservoir, which can therefore be re-used an arbitrary number of times.

We now show that Thermal Operations cannot create coherence. This was first shown in Refs. [133, 134]. Let us introduce the notion of a time-translation covariant map [129, 135].

Definition 24 (Time-translation covariant map)**.** *Consider an Hilbert space $\mathcal{H}$ with Hamiltonian H, and a quantum operation $\varepsilon : \mathcal{S}\left(\mathcal{H}\right) \rightarrow \mathcal{S}\left(\mathcal{H}\right)$. We say that the map ε is* time-translation *covariant iff*

$$e^{-iHt}\, \varepsilon(\rho)\, e^{+iHt} = \varepsilon(e^{-iHt}\, \rho\, e^{+iHt}), \quad \forall \rho \in \mathcal{S}\left(\mathcal{H}\right),\ \forall\, t \in \mathbb{R}. \tag{2.11}$$

where e^{-iHt} is the unitary evolution generated by the Hamiltonian H at time t.

If a map is time-translation covariant, we can apply it before the evolution, and then evolve the state, or vice versa we can evolve the state and then apply the map. In any case, the final state we obtain is the same. An example of a map which is clearly time-translation covariant is the unitary evolution of the state with respect to the system Hamiltonian, $\varepsilon(\cdot) = e^{-iHs} \cdot e^{+iHs}$, for any $s \in \mathbb{R}$. This is not the sole map to be time-translation covariant, and we now show that Thermal Operations satisfy Eq. (2.11), see Ref. [133].

Lemma 5. *Consider the resource theory of thermodynamics acting on a finite-dimensional system S with Hamiltonian H_S. Then, the maps in the set of allowed operations of the theory, that is, Thermal Operations, are time-translation covariant.*

Proof. Let us use the definition of time-translation covariant map, given in Eq. (2.11), and the fact that the most general Thermal Operation is of the form given in Eq. (2.6) – here we map state in $\mathcal{S}(\mathcal{H}_S)$ into states in $\mathcal{S}(\mathcal{H}_S)$, for simplicity. For all $\rho_S \in \mathcal{S}(\mathcal{H}_S)$, and for all $t \in \mathbb{R}$, we have,

$$
\begin{aligned}
\varepsilon_{TO}(e^{-iH_S t}\, \rho_S\, e^{+iH_S t}) &= \mathrm{Tr}_B\left[U\left(e^{-iH_S t}\, \rho_S\, e^{+iH_S t}\right) \otimes \tau_\beta\, U^\dagger \right] \\
&= \mathrm{Tr}_B\left[U\left(e^{-iH_S t}\, \rho_S\, e^{+iH_S t}\right) \otimes \left(e^{-iH_B t}\, \tau_\beta\, e^{+iH_B t}\right) U^\dagger \right] \\
&= \mathrm{Tr}_B\left[U\left(e^{-i(H_S+H_B)t}\, \rho_S \otimes \tau_\beta\, e^{+i(H_S+H_B)t}\right) U^\dagger \right] \\
&= \mathrm{Tr}_B\left[e^{-i(H_S+H_B)t}\, U \rho_S \otimes \tau_\beta\, U^\dagger\, e^{+i(H_S+H_B)t} \right] \\
&= e^{-iH_S t}\, \mathrm{Tr}_B\left[U\rho_S \otimes \tau_\beta\, U^\dagger \right] e^{+iH_S t} = e^{-iH_S t}\, \varepsilon_{TO}(\rho_S)\, e^{+iH_S t}, \quad (2.12)
\end{aligned}
$$

where the second equality follows from the fact that $\tau_\beta = e^{-\beta H_B}/Z$, and therefore it commutes with H_B, while the fourth equality follows from the fundamental operation 2 composing Thermal Operations, which requires the unitary $U \in \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_B)$ to commute with the total Hamiltonian $H_S + H_B$. $\qquad \square$

Using the result of the above lemma, we can now show that Thermal Operations are unable to create coherence in the energy eigenbasis, unless the Hamiltonian is degenerate.

Proposition 4. *Consider the resource theory of thermodynamics acting on a finite-dimensional system S with a non-degenerate Hamiltonian H_S. The allowed operations of the theory, Thermal Operations, are unable to create coherence in the eigenbasis of H_S.*

Proof. Consider a state $\rho_S \in \mathcal{S}(\mathcal{H}_S)$ that commutes with the Hamiltonian H_S, and therefore is diagonal in the energy eigenbasis (since H_S is non-degenerate). If we use the time-translation invariance of Thermal Operations, Lem. 5, we find that for all ε_{TO}, and for all $t \in \mathbb{R}$,

$$e^{-iH_S t} \varepsilon_{TO}(\rho_S) e^{+iH_S t} = \varepsilon_{TO}(e^{-iH_S t} \rho_S e^{+iH_S t}) = \varepsilon_{TO}(\rho_S), \tag{2.13}$$

where the second equality follows from the fact that $[\rho_S, H_S] = 0$. However, Eq. (2.13) needs to be valid for all $t \in \mathbb{R}$, which implies that $\varepsilon_{TO}(\rho_S)$ commutes with H_S, and therefore is diagonal in the energy eigenbasis. $\qquad\square$

Since coherence is a resource in thermodynamics, efforts have been spent to study how this quantity evolves under Thermal Operations [136], and whether it can be traded for another resource, for example, for work [137].

An additional question naturally arises, namely, how the agent can create a state with non-zero coherence over the energy eigenbasis within the formalism of Thermal Operations. This problem is equivalent to that considered in the last paragraph of Sec. 2.2.3, on the implementation of unitary operations that do not commute with the system's Hamiltonian. Coherence manipulation with Thermal Operations was first considered in Ref. [132], where it is shown that an additional system, referred to as a *coherence reservoir*, is needed in order to modify the coherence of the main system. In its simplest form, this ancillary system is infinite-dimensional, with a Hamiltonian which is unbounded both from below and above, and the state describing this system is in a uniform superposition over a large subset of energy eigenstates. In the following, we consider the easiest case in which the main system S is a qubit with Hamiltonian $H_S = E_0 |0\rangle \langle 0| + E_1 |1\rangle \langle 1|$, with energy gap $\Delta E = E_1 - E_0$, the coherence reservoir C has Hamiltonian $H_C = \sum_{\ell \in \mathbb{Z}} \ell \, \Delta E \, |\ell\rangle \langle \ell|$, and the state describing this system is $|\Psi\rangle = \sum_\ell^L \frac{1}{\sqrt{L}} |\ell\rangle$, where $L >> 1$, see Fig. 2.2.

With the help of this coherence reservoir, the agent can implement a unitary operation over the main system S which creates coherence. Suppose, for instance, that the agent wants to implement an Hadamard $U_H \in \mathcal{B}(\mathcal{H}_S)$ over the main system S, mapping $|0\rangle \rightarrow |+\rangle$ and $|1\rangle \rightarrow |-\rangle$. This transformation can be realised, using Thermal Operations, by applying the

$$|\Psi\rangle_C = \tfrac{1}{\sqrt{L}}\sum_\ell^L |\ell\rangle_C \qquad\qquad |\tilde{\Psi}\rangle_C$$

$$|0\rangle_S \qquad\qquad |+\rangle_S$$
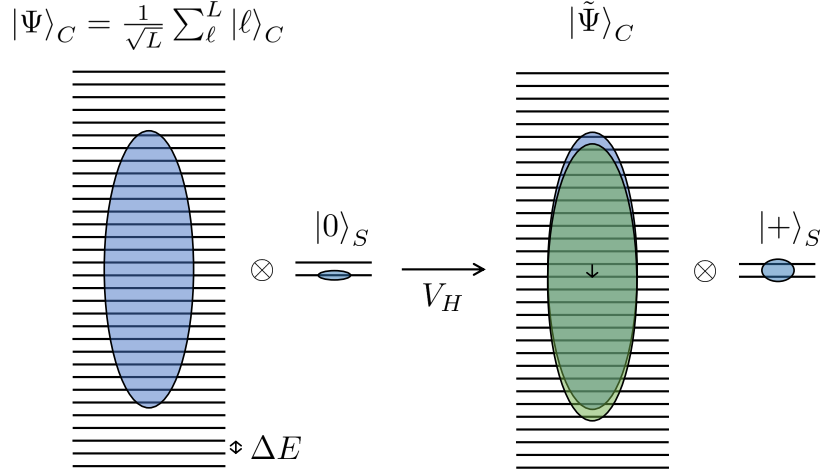
$$\otimes \qquad V_H \qquad \otimes$$

$$\Delta E$$

Figure 2.2: In order to create coherence over the energy eigenbasis of the main system, we need to use a coherence reservoir. This is an ancillary system whose Hamiltonian is a double-infinite ladder – modification to this Hamiltonian can be made so as to obtain a more physical system – described by the state $|\Psi\rangle_C = \sum_\ell^L \frac{1}{\sqrt{L}}|\ell\rangle$, which is in a large ($L \gg 1$) superposition of its energy eigenstates (represented by the blue ellipse on the left-side ladder). In order to create coherence on the main system, and to map its state from $|0\rangle_S$ into $|+\rangle_S$, we can use the energy-preserving unitary operation V_H described in Eq. (2.14). The effect of this unitary over the reference frame is to create a superposition between the original state $|\Psi\rangle_C$ and the same state slightly displaced (the green ellipse on the right-side ladder). Since these two states significantly overlap, the final state is approximately equal to $|\Psi\rangle_C \otimes |+\rangle_S$.

following global operation over the system S and the coherence reservoir C,

$$V_H = \sum_{\ell \in \mathbb{Z}} \sum_{n,m=0}^{1} |n\rangle \langle n| U_H |m\rangle \langle m|_S \otimes |\ell - (n-m)\rangle \langle \ell|_C \,. \tag{2.14}$$

It is easy to show that this operation is energy preserving, since it commutes with the global Hamiltonian of system and coherence reservoir $H_S + H_C$. Furthermore, when V_H is applied to the initial global state $|0\rangle_S \otimes |\Psi\rangle_C$, we get

$$V_H |0\rangle_S \otimes |\Psi\rangle_C = \frac{1}{\sqrt{2}}|0\rangle_S \otimes \left(\sum_\ell^L \frac{1}{\sqrt{L}}|\ell\rangle_C\right) + \frac{1}{\sqrt{2}}|1\rangle_S \otimes \left(\sum_\ell^L \frac{1}{\sqrt{L}}|\ell-1\rangle_C\right)$$

$$\approx |+\rangle_S \otimes |\Psi\rangle_C \,, \tag{2.15}$$

where the last approximate equality follows from the fact that, for $L \to \infty$, the displaced state of the coherence reservoir, $\sum_{\ell}^{L} \frac{1}{\sqrt{L}} |\ell - 1\rangle_C$, almost completely overlap with the state $|\Psi\rangle_C$, see Fig. 2.2. Thus, coherence can be created with Thermal Operations if the agent has access to a coherence reservoir, and if we consider approximate transformations along with exact ones.

Notice that, at a first glance, the above coherence reservoir might seem unphysical, and a too-strong resource that cannot be accessed in a laboratory. However, coherence manipulation is possible even in the case in which the reservoir has a Hamiltonian that is not unbounded from below, making it a physically meaningful system [137]. This system allows for the same power an unbounded reservoir provides, although it gets degraded with time and needs energy to be kept functional. Furthermore, a coherence reservoir of this kind can be realised in the laboratory, since the state of the radiation produced by a laser is a good approximation of the state $|\Psi\rangle_C$ used in the above protocol. Finally, it is interesting to notice that the coherence reservoir we have introduced here plays a very similar role to the one *reference frames* play in asymmetry theory [138, 139]. Indeed, reference frames are systems that can be used to lift the super-selection rules imposed by some conservation laws on the main system, which is the same function the coherence reservoir fulfils in the context of thermodynamics.

2.2.6 Thermalisation and the free states

An important aspect of Thermal Operations is the fact that the agent can use the thermal reservoir to thermalise any ancillary system, and later couple the ancilla to the main system. Previously, we have seen that this paradigm is unable to describe every possible physical situations, since correlations between system and ancilla might be present from the beginning. Furthermore, this class of operations is the result of an additional idealisation, namely the fact that any ancillary system can be thermalised for free, irrespectively of the time-scale of this process. It is known that, for some physical systems, this process can be very long (with respect to the time-scale set by the system's Hamiltonian), and that there exist systems, such as *integrable systems*, which never reach thermal equilibrium [100]. Therefore, by allowing any system to thermalise at no costs, the paradigm of Thermal Operations represents a more powerful set of operations than the one an agent can access in the laboratory. As such, this paradigm is useful for providing lower bounds to the amount of resource used during a thermodynamic

transformation, as we have already stressed in the previous section.

The fact that Thermal Operations allow any system to thermalise at the background temperature, and that these operations are insensitive to the time this process takes, make them unsuitable tools for the study of thermalisation and equilibration of physical systems. These phenomena have been, and still are, extensively researched by the quantum thermodynamics and statistical mechanics community, since they concern almost any system with an open dynamics. Indeed, different quantum mechanical models exist to describe the process of thermalisation, for both open systems [140, 141, 142], and isolated ones [143, 144]. For a review on the topic of thermalisation and equilibration, see Ref. [100].

Finally, it is worth noting that the framework of Thermal Operations assumes that the equilibrium state of any ancillary system is thermal. However, thermalisation is a special case of equilibration, and the equilibrium state of a system does not, in general, need to be thermal [98, 99]. In fact, depending on the constraints a system is subjected to, its equilibrium state can be different. For example, when multiple quantities are conserved, the equilibrium state reached by an open system is not thermal, but rather is described by the grand-canonical ensemble. In recent years, there have been efforts to build resource theories able to describe scenarios in which multiple conserved quantities, even non-commuting ones, are present [93, 94, 96, 95].

2.3   Thermodynamic monotones

Having defined the set of allowed operations for thermodynamics, we can now move to the study of state transformations. Within the resource theoretic framework, one is interested in finding necessary and sufficient conditions for state transformations to be realisable using the class of allowed operations. This is the case of thermodynamics as well, and in this section we present the current results on the conditions for state transformations. In Sec. 2.2.5 we have seen that Thermal Operations cannot create coherence in the energy eigenbasis, unless a coherence reservoir is added to the picture. Thus, the first set of results we present concerns states than are diagonal in the eigenbasis of the Hamiltonian of the system, known as *semi-classical* states. These results are based on a generalisation of majorization, the pre-order relation introduced

in Def. 19, which is known as *d-majorisation* [145].

Before introducing the notion of *d*-majorization, and understanding the link between it and Thermal Operations, we first need to introduce the notion of *d*-stochastic matrix,

Definition 25 (*d*-stochastic matrix). *Consider the set of $n \times n$ real square matrices $M_{n \times n}(\mathbb{R})$, and a n-dimensional vector d with positive elements. A matrix $A \in M_{n \times n}(\mathbb{R})$ with non-negative elements is called d-stochastic iff*

1. *$\sum_{i=0}^{n-1} A_{i,j} = 1, \quad \forall j \in \{0, \ldots, d-1\}$ (stochastic matrix)*

2. *$\sum_{j=0}^{n-1} A_{i,j} \, d_j = d_i, \quad \forall i \in \{0, \ldots, d-1\}$ (d-preserving matrix)*

Notice that, when the vector d is uniform (all entries are equal), the above definition coincides with the one for doubly-stochastic matrices, see Def. 18. We can now introduce a generalisation of the majorization conditions, which is based on the above class of matrices.

Definition 26 (*d*-majorization). *Consider the set V of n-dimensional vectors with positive elements and with ℓ_1-norm equal to 1. Given two vectors $x, y \in V$, we say that x d-majorizes y, in symbols $x \succ_d y$, if and only if there exists a d-stochastic matrix A such that $y = A x$.*

It is easy to show that this notion of majorization coincides with the one of Def. 19 when the matrix A is doubly-stochastic.

We can now provide the necessary and sufficient conditions for a state transformation between semi-classical states to be possible under Thermal Operations. These conditions, known as *thermo-majorisation*, involve the notion of *d*-majorization, and where first introduced in Ref. [8, Thm. 2], where we refer the reader for further details.

Theorem 5. *Consider the resource theory of thermodynamics acting on a d_S-dimensional system S, with Hamiltonian $H_S = \sum_{i=0}^{d_S-1} E_i \, |i\rangle \langle i|_S$. The allowed operations are Thermal Operations, see Def. 22. Given two semi-classical states $\rho, \sigma \in \mathcal{S}(\mathcal{H}_S)$, a state transformation mapping ρ into σ is possible if and only if the population vector of ρ d-majorizes the population vector of σ, where d is the population vector of the thermal state $\tau_\beta = e^{-\beta H_S}/Z$. This condition is known as thermo-majorization.*

This result can be derived using Thm. 3, which concerns standard majorization. The idea is that, when considering system and thermal reservoir together, we can identify subspaces with a

fixed total energy. In these subspaces, we can apply any unitary operation (since the total energy is conserved by definition), and we can trace out part of the reservoir. These operations act over the projection (onto each fixed-energy subspace) of the system's state as doubly-stochastic matrices. This implies that the necessary and sufficient conditions for transforming the state of the system inside these fixed-energy subspaces are given by majorization. By considering all subspaces together, one obtains the thermo-majorization conditions.

These conditions can be checked with the help of a two-dimensional diagram. To each state considered in the transformation, one assigns a curve in the diagram. If the curve of the initial state coincides or lays above the curve of the final state, then we say that the initial state thermo-majorizes the final one, and we can perform the transformation using Thermal Operations. For standard majorization, an equivalent way of representing the conditions exists, in terms of Lorenz curves [145, Ch. I]. We now show how to represent a semi-classical state in the two-dimensional diagram for studying thermo-majorization. We consider a d_S-dimensional system with Hamiltonian $H_S = \sum_{i=0}^{d_S-1} E_i \, |i\rangle \, \langle i|_S$, and a semi-classical state $\rho = \sum_{i=0}^{d_S-1} p_i \, |i\rangle \, \langle i|_S$. For each state, we can introduce a curve in the two-dimensional diagram of Fig. 2.3 as follow. First, we construct the vector of elements $\left\{ p_i \, e^{-\beta E_i} \right\}_{i=0}^{d_S-1}$, and we order it in decreasing order. This is known as β-ordering. Then, using this order, we construct the following list of pairs

$$\left\{ \left(\sum_{i=0}^{k} e^{-\beta E_i} \;,\; \sum_{i=0}^{k} p_i \right) \right\}_{k=0}^{d_S-1} , \tag{2.16}$$

which defines a (concave) curve in the two-dimensional diagram. If we follow the same procedure for the state σ, we can study thermo-majorization in a visual way, since ρ thermo-majorizes σ if and only if the curve associated with the former never lies below the one associated to the latter.

One can also study thermodynamic state transformations when catalysts, see Def. 20, are allowed. In Ref. [146], it is shown that the necessary and sufficient conditions expressed by thermo-majorization are replaced by conditions involving a family of monotones related to the Rényi entropies – as expected from the result we showed in Thm. 4. These new conditions apply to the case in which we consider catalytic Thermal Operations acting over semi-classical states, and we study approximate state transformations, rather than exact ones. Let us first introduce the *Rényi divergences* [53], which provide a generalisation of the Kullback-Leibler
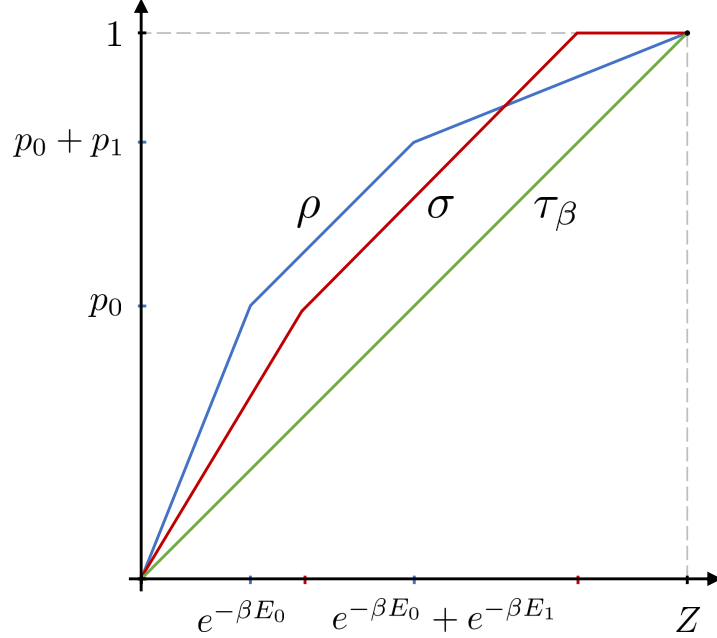
Figure 2.3: The thermo-majorization curves of the semi-classical states ρ, σ, and τ_β, describing the qutrit system with Hamiltonian $H = \sum_{i=0}^{2} E_i \left| i \right\rangle \left\langle i \right|$. The blue line is the curve associated with $\rho = \sum_{i+0}^{2} p_i \left| i \right\rangle \left\langle i \right|$, while the red one is associated with σ. The two lines have a different β-order, as it can be seen by the fact that the elbows are found at different values of the x-axis. In this specific case, neither ρ thermo-majorizes σ, nor the opposite (since the two line intersect). Therefore, no Thermal Operation can map ρ into σ or vice versa. The straight green line is associated with the thermal state of the system, $\tau_\beta = e^{-\beta H}/Z$, and it is easy to see that the β-order is trivial for this state. Since the curves of both ρ and σ lie completely above the line of τ_β (they thermo-majorize this state) we can always find a Thermal Operation mapping these states into the thermal state. For example, this transformation can be performed using the thermalising map shown in Eq. (2.7).

divergence – the classical equivalent of the quantum relative entropy shown in Eq. (1.9).

Definition 27 (Rényi divergences). *Consider the set V of d-dimensional vectors with positive elements, and with ℓ_1-norm equal to 1. We define the family of α-Rényi divergences, for $\alpha \geq 0$, as*

$$D_\alpha(x\|y) = \frac{1}{\alpha - 1} \log \sum_{i=0}^{d-1} x_i^\alpha\, y_i^{1-\alpha}, \quad x, y \in V. \tag{2.17}$$

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For $\alpha = 1$, the Rényi divergence coincides with the Kullback-Leibler divergence.

We can now introduce the family of monotones which plays a pivotal role in catalytic Thermal Operations. These monotones are known as *free energies*, since in the macroscopic limit (when many i.i.d. copies of the state are considered), they all become proportional to the Helmholtz free energy, defined as $F := E - T\,S$, where E is the internal energy of the system, T is the temperature of the surrounding thermal reservoir, and S is the Von Neumann entropy of the system.

Definition 28 (Free energies). *Consider the finite-dimensional system described by the Hilbert space $\mathcal{H}$, with Hamiltonian H. For a semi-classical state $\rho \in \mathcal{S}(\mathcal{H})$, we define the family of free energies as*

$$F_\alpha(\rho) = \frac{1}{\beta}\left(D_\alpha(p\|q) - \log Z\right), \tag{2.18}$$

where p and q are the population vectors of ρ and τ_β, respectively, τ_β is the thermal state of the system, and Z is the partition function.

The free energies are monotones for the resource theory of Thermal Operations. This follows from the fact that the Rényi divergences are contractive under CPTP maps [147, 148], and that the free energies have been constructed following the recipe given in Eq. (1.7) (modulo a constant factor). As a result, Prop. 2 of the previous chapter applies, so that these quantities are indeed monotonic under Thermal Operations.

When the agent is equipped with catalytic Thermal Operations, the necessary and sufficient conditions are expressed in terms of the free energies of Eq. (2.18). These conditions are known as the *second laws* of thermodynamics, see Ref. [146].

Theorem 6. *Consider the resource theory of thermodynamics acting on a finite-dimensional system described by the Hilbert space $\mathcal{H}$, with Hamiltonian H. The allowed operations are catalytic Thermal Operations, i.e., the set of operations given in Def. 22 together with the possibility of using catalysts. Given two semi-classical states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$, a state transformation mapping ρ into σ is possible if and only if*

$$F_\alpha(\rho) \geq F_\alpha(\sigma), \quad \forall\,\alpha \geq 0. \tag{2.19}$$

When states with coherence are considered, the above conditions are still necessary, but are not sufficient any more [134]. It is worth noting that to deal with states with coherence one needs to generalise the Rényi divergences of Eq. (2.17) to the quantum case, see Ref. [149] for further details. In recent work [150], necessary and sufficient conditions for any thermodynamic state transformation have been obtained, for a different class of allowed operations than Thermal Operations. These conditions involve a family of entropies parametrised by two quantum states, associated with a reference frame. The set of operations used, called Generalised Thermal Processes, extends the one presented in Def. 22, and is composed by those maps that *(i)* preserve the energy of the system, *(ii)* have an equilibrium state, *(iii)* do not exploit any source of coherence. The conditions for this class of operations have been obtained using results from thermodynamics and asymmetry theory, together with a notion of majorization which applies to quantum states [151, 152, 153].

When the many-copy limit is considered, instead, one can show that the family of second laws of Thm. 6 collapses into a single relation, which is the well-known Second Law of Thermodynamics, stating that the Helmholtz free energy of a system never increases during a thermodynamic transformation. This is due to the fact that thermodynamics in the many-copy limit is a reversible theory [7], see Def. 16, and as such the state transformations depends on a single monotone, as shown in Thm. 1.

2.4   Batteries and the notion of work

Classical thermodynamics studies the energy transfers occurring during a state transformation between the system and its environment [15]. In particular, in the classical theory we have two well-known kinds of energy transfer, *work* and *heat*. Work is an energy flow which does not carry with it any entropy and can be used, for example, to lift a weight in the gravitational field. Heat, on the other hand, is an entropic energy flow exchanged between system and thermal reservoir. These notions are useful to quantify the amount of resource needed to realise a thermodynamic transformation, and therefore it is interesting to extend them to the quantum realm.

In this section, we introduce different notions of work for the quantum case, and we leave

the discussion about heat to Ch. 4. In order to provide a meaningful definition of work in the quantum regime, we first need to understand how work can be quantified in the classical case. The basic idea is that work in thermodynamics is energy that can be stored somewhere, and subsequently used to perform some useful task. For example, when we lift a weight, we store work (in the form of potential energy) in that system, and we can later use the stored energy to perform some kind of thermodynamic transformation, for example to compress the gas inside a canister with a piston, by connecting the piston to the weight and letting the weight free to move. The work spent during the compression is then quantified by the energy change in the weight.

We here focus on the notion of *deterministic work* [8, 9, 154], i.e., the amount of work that allows the agent to perform a given state transformation with (almost) certainty, as opposed to the notion of *fluctuating work*, which is related to the probability distribution of the work exchanged during the process. In particular, we consider the deterministic work that we can extract from a state, and the deterministic work that we need in order to create the same state. Since we are working in the microscopic regime, where thermal fluctuations are comparable to the energy exchanged, we find that these two notions of deterministic work do not always coincide, implying a fundamental irreversibility of the theory when single quantum systems are considered.

In order to quantify the deterministic work exchanged during a process, we introduce an additional system, that plays the role of the weight of the previous paragraph. In general, we call this system a *battery*, since we can store/extract energy from it. In the following, we specialise the battery to a two-level system with Hamiltonian $H_W = \Delta W \left| 1 \right\rangle \left\langle 1 \right|$, often referred to as a *wit* (a work-bit). If the state of the wit is mapped from $\left| 0 \right\rangle$ to $\left| 1 \right\rangle$, then an amount of work ΔW is stored in this system. Vice versa, if the state of the wit undergoes the transformation $\left| 1 \right\rangle \rightarrow \left| 0 \right\rangle$, an amount ΔW has been extracted from the wit. We now introduce two notions of deterministic work, the *work of formation* and the *extractable work*.

Definition 29 (Deterministic work). *Consider a system S with Hamiltonian H_S, in contact with a thermal reservoir at temperature β^{-1}, and a two-level battery system W, a wit, with Hamiltonian $H_W = \Delta W \left| 1 \right\rangle \left\langle 1 \right|$. Given the state of the system $\rho \in \mathcal{S}\left(\mathcal{H}_S\right)$, we define*

- *The* extractable work $W_{\mathrm{ext}}(\rho)$ *is the largest ΔW for which the following transformation*

is possible,

$$\rho \otimes |0\rangle \langle 0|_W \rightarrow \tau_\beta \otimes |1\rangle \langle 1|_W. \tag{2.20}$$

- *The* work of formation $W_{\text{form}}(\rho)$ *is the smallest ΔW for which the following transformation is possible,*

$$\tau_\beta \otimes |1\rangle \langle 1|_W \rightarrow \rho \otimes |0\rangle \langle 0|_W. \tag{2.21}$$

It is worth noticing that in Eqs. (2.20) and (2.21) the system is mapped to/from the thermal state τ_β. Indeed, this is the result of optimising the extractable work and the work of formation with respect to the final and initial state of the system, respectively. Thus, the extractable work is the maximum amount of energy we can store in the battery when we transform a state ρ into the thermal state τ_β, which is the free state of our resource theory. Likewise, the work of formation is the minimum amount of work we need to extract from the battery in order to create the state ρ starting from the free state τ_β. Notice that, due to quantum fluctuations, these two quantities do not need to be equal, and indeed they are in general different.

When the allowed operations of the theory are Thermal Operations, we have that the extractable work and the work of formation are linked to different Rényi divergences, see Ref. [8] for the proof of the following theorem, and also Refs. [9, 154, 155].

Theorem 7. *Consider the resource theory of thermodynamics equipped with Thermal Operations. We consider a finite-dimensional system S with Hamiltonian H_S, and a wit W with Hamiltonian $H_W = \Delta W |1\rangle \langle 1|$. Given a generic state $\rho \in \mathcal{S}(\mathcal{H}_S)$, we have that the extractable work is given by*

$$W_{\text{ext}}(\rho) = F_0(\rho') - F_0(\tau_\beta), \tag{2.22}$$

where ρ' is a diagonal state, obtained from ρ by de-cohering it in the energy eigenbasis, and F_0 is the $\alpha = 0$ free energy, see Def. 28. The work of formation for a semi-classical state $\rho \in \mathcal{S}(\mathcal{H}_S)$ is given, instead, by

$$W_{\text{form}}(\rho) = F_\infty(\rho) - F_\infty(\tau_\beta), \tag{2.23}$$

where F_∞ is the $\alpha = \infty$ free energy.

From the results of the above theorem, we immediately see that in the single-copy case, thermodynamics is not a reversible theory, since we exchange two different amounts of work

to first map a semi-classical state into the thermal state τ_β, and subsequently map it back. In particular, the work of extraction is always smaller or equal to the work of formation, so that we always lose work if we do a cyclic state transformation. It is worth noting that the above results are valid for exact transformations, and the smoothed version of the free energies should be considered in the case of approximate state transformations [8].

We can express the two kinds of work in terms of two Rényi divergences. Specifically, the extractable work $W_{\text{ext}}(\rho)$ is equal (modulo a multiplicative factor, the temperature of the reservoir) to the *min-relative entropy* [156, 157],

$$D_0(p\|q) = -\log \sum_{i \ : \ p_i > 0} q_i, \qquad (2.24)$$

and the work of formation $W_{\text{form}}(\rho)$ is equal (modulo the same multiplicative factor) to the *max-relative entropy*,

$$D_\infty(p\|q) = -\log \sup_i \frac{p_i}{q_i}. \qquad (2.25)$$

In the above equations, p is the population vector of the semi-classical state ρ, and q is the population vector of the thermal state τ_β.

When energy fluctuations are negligible, thermodynamics becomes a reversible resource theory, and the amount of work used during a state transformation is uniquely quantified by the Helmholtz free energy, given by the $\alpha = 1$ free energy [158]. This is the case, for example, of Ref. [7], where thermodynamics is studied in the many-copy limit. Notice that, in this limit, the agent is also allowed to create states with coherence in the energy eigenbasis, provided they have access to a "small" source of coherence (see Ch. 4 for more details on this source). In Ref. [111], instead, fluctuations are neglected by equipping the agent with a different set of allowed operations than Thermal Operations. The agent is there allowed to use unitary operations which preserve the energy of the global system on average. The amount of work exchanged during a state transformation is then given, predictably, by the Helmholtz free energy. Other notions of work can be defined, for example when the system is interacting with a finite-sized reservoir, see Refs. [159, 160] and Ch. 4 of this thesis, or to quantify the energy used during any processing of quantum information [161, 162].

Finally, as briefly mentioned at the beginning of the section, another way of characterising the work exchanged during a thermodynamic process is to consider its probability distribution,

rather than its average, or the value at the tails of such distribution. This is a common approach in thermodynamics, since the outcomes of an experiment involving a thermodynamic transformation often consists of a work distribution. Results exist that link these work distributions to equilibrium properties of the system, known as fluctuation theorems [88, 89]. In these theorems, one considers an initial and final equilibrium state for the thermodynamic system under examination, together with the forward and backward processes which map between these two states. The processes considered are very general, and they can drive the system out of equilibrium. The theorems then link the work distribution of the forward/backward process to the equilibrium properties of the initial and final state, specifically, to their Helmholtz free energy. Since measuring work during an experiment is easy, while obtaining information on the equilibrium properties of a system is not, these theorems play a fundamental role in thermodynamic experiments. Fluctuation theorems can be extended to the quantum realm when the forward and backward processes used are described by CPTP maps. Quantum fluctuation theorems have been extensively studied both in statistical mechanics (see Ref. [90] for a review on the topic), and resource theories [91, 92].

Part II

Resource theories and thermodynamics

Chapter 3

A framework for multi-resource theories and the first law

In the first part of this thesis, we have introduced a general formalism for resource theories, together with different examples of such theories. For instance, we have considered the resource theory of entanglement with LOCC, the theory of asymmetry, and the theory of thermodynamics with Thermal Operations, that we have explored in full detail in Ch. 2. These resource theories all have in common the fact that they quantify a single kind of resource. For example, the theory of entanglement with LOCC only quantifies the amount of entanglement needed in order to perform a protocol or a state transformation. Likewise, thermodynamics as a resource theory solely quantifies the amount of athermality contained in a system.

It is often the case, in physics, that a specific task or phenomenon depends on multiple quantities or resources, rather than on a single one. For instance, quantum computers, in order to achieve a computational advantage over their classical counterparts, need to initialise their qubits in a pure state and to apply, over these qubits, gates which create coherence in the computational basis, see for instance Ref. [163]. Thus, one might be interested in a resource theoretic framework able to quantify both purity and coherence, so as to study the demands of different quantum algorithms in terms of these quantities. Likewise, in thermodynamics one can be interested in both the amount of work and heat, or, similarly, of energy and entropy, exchanged during a transformation. In this chapter, we introduce a framework to build and

describe multi-resource theories [164], i.e., theories in which more than one resource is accounted for and quantified.

Our framework applies to any physical task or scenario where several constraints and conservation laws can be identified. To build a multi-resource theory describing a given physical task, we first consider multiple single-resource theories, each one for a different constraint or conservation law. Then, we realise the multi-resource theory by identifying its class of allowed operations with the intersection of the sets of allowed operations of the different single-resource theories. For example, in thermodynamics we might consider an isolated system, so that energy needs to be preserved, and the dynamic needs to be reversible. Given these two constraints, we can build two single-resource theories, one with energy-preserving operations, and the other with unitary operations. The intersection of these two classes of operations gives us the set of allowed operations of a multi-resource theory for thermodynamics, whose properties are analysed in the next chapter.

After introducing, in Sec. 3.1, the formalism of multi-resource theories, we study how the resources can be quantified, and what it means for these theories to be reversible, see Sec. 3.2. To approach these questions in a meaningful way, we introduce a property, which we refer to as *asymptotic equivalence*, see Def. 30. When this property is satisfied, the agent can quantify the amounts of resources exchanged during an asymptotic state transformation by considering the difference between the initial and final values of a given set of monotones. We show that, to quantify the resources, the agent needs to introduce batteries, one for each resource. We then move to the study of reversibility. In our formalism, a theory is reversible if, during any cyclic state transformation, no resource is lost, so that the amount spent to asymptotically map a state into another is gained when performing the reverse transformation. We show that multi-resource theories which satisfy the asymptotic equivalence property are reversible. Furthermore, when the monotones appearing in this property obey some natural assumptions, we can prove that they are the unique quantifiers of the resources in the theory.

We then proceed to study the problem of interconversion of resources; suppose the agent is given two batteries, each of them storing a different kind of resource. For example, in thermodynamics we could have a battery storing energy, similar to the one described in Sec. 2.4, and one that stores entropy. We investigate which kind of additional system the agent needs

to be able to exchange one resource for the other. In thermodynamics, this additional system is a thermal reservoir, and the process of exchanging energy for entropy (rather, neg-entropy, or information) is known as Landauer's erasure, while the opposite process is described by a Maxwell's demon type of scenario [165]. Within our framework, we generalise the role played by the thermal reservoir in thermodynamics, and we introduce the notion of *bank* systems, which allows us to exchange one resource for another. We show that interconversion relations exist, which define the exchange rate at which one resource is converted into another.

Finally, we consider asymptotic state transformations when batteries and banks are available. We show that, in this situation, whether or not a state transformation is realisable depends on a single relation, connecting the change of a specific monotone over the main system to the weighted sum of the resources required for the transformation. We call this relation the *first law for multi-resource theories*. Indeed, when the multi-resource theory of thermodynamics is considered, one finds that the relation we obtain coincides with the First Law of Thermodynamics,

$$\Delta F = \Delta E - T \Delta S, \tag{3.1}$$

where the change in the Helmholtz free energy F in the main system is equal to the sum of the amounts of energy E and entropy S provided during the transformation, weighted by the temperature T of an external thermal reservoir which plays the role of the bank. The results here obtained are applied, in the next chapter, to the study of thermodynamics as a multi-resource theory.

3.1 Framework for multi-resource theories

Let us now introduce the formalism we use to create resource theories with multiple resources. These theories are built to describe physical tasks where several constraints and conservation laws are present. Suppose the task at hand involves a quantum system described by the Hilbert space $\mathcal{H}$, and a number $m > 1$ of constraints are present. Then, given the i-th constraint, we can introduce the corresponding single-resource theory R_i, defined by a set of allowed operations $\mathcal{A}_i$ acting on the state space $\mathcal{S}(\mathcal{H})$. Each of these single-resource theories comes with its own set of free states $\mathcal{F}_i$, see Def. 10 in the background chapter, which is invariant under the corresponding

set of allowed operations $\mathcal{A}_i$, see Prop. 1. Once all single-resource theories R_i's are defined, we can build the multi-resource theory R_multi describing the task under investigation. The multi-resource theory is defined by the set of allowed operations $\mathcal{A}_\text{multi}$, which is obtained by taking the intersection between the classes of allowed operations of the m single-resource theories,

$$\mathcal{A}_\text{multi} = \overset{m}{\underset{i=1}{\cap}} \mathcal{A}_i. \tag{3.2}$$

We can extend the multi-resource theory to the many-copy case. To do so, we first have to extend the single-resource theories R_i's, following the same procedure used in Sec. 1.3 of the background material. For each of these theories, we define the class of allowed operations $\mathcal{A}_i^{(n)}$, acting on n copies of the system, where $n \in \mathbb{N}$. The set of free states for the n-copy case is referred to as $\mathcal{F}_i^{(n)}$, and it is invariant under the operations in $\mathcal{A}_i^{(n)}$. Then, the class of allowed operations for the multi-resource theory R_multi, when acting on n copies of the system, is given by the intersection between the sets of allowed operations $\mathcal{A}_i^{(n)}$ of the different single-resource theories, that is, $\mathcal{A}_\text{multi}^{(n)} = \cap_{i=1}^m \mathcal{A}_i^{(n)}$.

We now focus on the sets of free states $\mathcal{F}_i$'s, and their role in the multi-resource theory R_multi. It follows from our definition of the class of allowed operations $\mathcal{A}_\text{multi}$, Eq. (3.2), that each set of free states $\mathcal{F}_i$ is invariant under these operations. However, it is worth noting that the states contained in the $\mathcal{F}_i$'s might not be free for the multi-resource theory. Indeed, some of the states contained in a given set $\mathcal{F}_i$ might not be contained by the other sets $\mathcal{F}_j$'s, $j \neq i$, and therefore it would be impossible to obtain them with the allowed operations $\mathcal{A}_\text{multi}$. In Fig. 3.1 we show the different configurations for the invariant sets of a multi-resource theory with two resources. While the multi-resource theories associated with the left and central panels have free states, the one associated with the right panel does not. The fact that the set of free states might be empty represents one of the main differences between single- and multi-resource theories.

The multi-resource theory R_multi inherits the monotones of the single-resource theories that compose it. This follows trivially from the choice we made in defining the class of allowed operations $\mathcal{A}_\text{multi}$, see Eq. (3.2). Furthermore, other monotones, that are only valid for the multi-resource theory, can be obtained from the ones inherited from the single-resource theories R_i's. For example, if f_i is a monotone for the single-resource theory R_i, and f_j is a monotone for the theory R_j, their linear combination, where the linear coefficients are positive, is a monotone
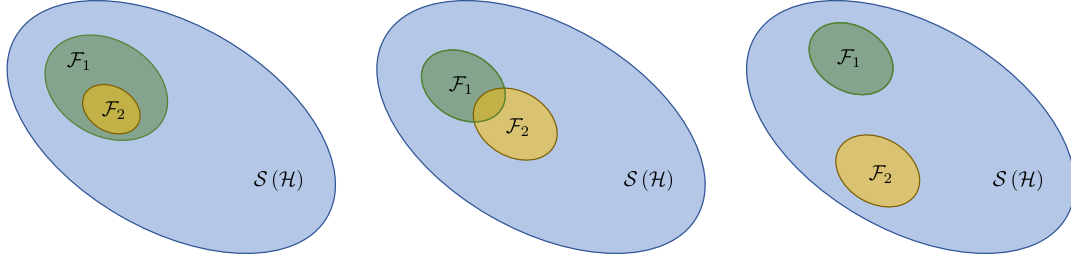
Figure 3.1: The structure of the invariant sets for a multi-resource theory with two resources. For theories with $m > 2$ resources, the structure of the invariant sets can be obtained by composing the three fundamental scenarios presented here. **Left.** The invariant set $\mathcal{F}_2$ is a subset of $\mathcal{F}_1$. This multi-resource theory has a set of free states, which coincides with $\mathcal{F}_2$. An example of such a theory is that of coherence [33] and purity [37], where the invariant sets are incoherent states with respect to a given basis and the maximally-mixed state, respectively. **Centre.** The two invariant sets intersect each other. This theory has a set of free states which coincides with the intersection, $\mathcal{F}_1 \cap \mathcal{F}_2$. An example of multi-resource theory with this structure concerns tripartite entanglement for systems A, B, and C. The allowed operations of this theory are defined by the intersection of the operations associated with the theories of bipartite entanglement for systems AB and C, systems AC and B, and systems A and BC. Notice that this theory does not coincide with the theory of tripartite LOCC, since some of the free states are entangled [166]. **Right.** The two invariant sets are separated. Consequently, the theory does not have any free states. In this situation, one can find an interconversion relation between the resources, as shown in Sec. 3.3. An example of a multi-resource theory with this structure is thermodynamics of closed systems. If the agent does not have perfect control on the reversible operations they implement, and the closed system is coupled to a sink of energy (an ancillary system which can only absorb energy), then the allowed operations are given by the intersection between the set of mixtures of unitary operations, and the set of average-energy-non-increasing maps. In this case, the maximally-mixed state and the ground state of the Hamiltonian are the two invariant sets of the theory. Notice that the set of energy-preserving unitary operations, considered in the next chapter, is a subset of this bigger set.

for the multi-resource theory R_{multi}. Interestingly, in Sec. 3.3 we show that a specific linear combination of monotones of different single-resource theories plays an important role in the

interconversion of resources.

Examples of multi-resource theories that are described within our formalism can be found in the literature. In Ref. [167], for instance, the authors study the problem of state-merging when the parties can only use LOCC, and they restrict the local operations to be incoherent operations, i.e., operations that cannot create coherence (in a given basis). This theory coincides with the multi-resource theory obtained from combining two single-resource theories, the one of entanglement, whose set of allowed operations only contains quantum channels built out of LOCC, and the one of coherence, whose set of allowed operations only contains maps which do not create coherence. Depending on the class of Incoherent Operations we chose, the structure of the invariant sets is given by either the left or central panel of Fig. 3.1. Another example is the one studied in Ch. 4, where thermodynamics is described by a multi-resource theory whose allowed operations are given by the intersection between energy-preserving maps and unitary maps. One can also extend this multi-resource theory to the case in which additional quantities (even not commuting ones) are conserved, such as the number of particles, or the angular momentum [168].

3.2   Reversibility of multi-resource theories

We now study reversibility in the context of multi-resource theories. As we noticed in the background chapter, Sec. 1.4, reversibility is generally studied in the many-copy regime, where fluctuations are negligible. In order to study reversibility for multi-resource theories, we first introduce a property, which we refer to as the *asymptotic equivalence property*. When the theory satisfies this property, we show that the resources exchanged during an asymptotic state transformation can be quantified in terms of a specific set of monotones, and that this quantification is unique. As a result, additional devices, called batteries, can be added to the theory so as to individually store each resource. We show that a theory satisfying asymptotic equivalence is also reversible, i.e, the resources exchanged with the batteries during an asymptotic state transformation are equal, with negative sign, to the resources exchanged when performing the inverse transformation.

3.2.1 The asymptotic equivalence property

Let us consider the multi-resource theory R_{multi} introduced in Sec. 3.1, describing a physical task where m constraints and conservation laws are present. We are interested in studying whether the theory is reversible, i.e., whether no resources are lost during an arbitrary cyclic transformation. However, to study this notion of reversibility, one first needs to be able to quantify the amount of resources exchanged in a state transformation. In general, as we have seen in Sec. 1.2 of the background chapter, each resource is quantified by several monotones, and there is not a unique way to assign a value to each state. This reflects the fact that, in resource theories, we can define a partial order between states, rather than a total one.

For single-resource theories, we have shown that if a theory is reversible (in terms of rate of conversion), then there exists a unique quantifier for the resource exchanged during each state transformation, see Thm. 1. However, this result does not apply to multi-resource theories, mainly because defining a rate of conversion for these theories does not seem to be always possible. Indeed, a rate of conversion can be defined only if the theory has a non-empty set of free states, see Def. 15, since the number of copies of the system before and after the transformation are allowed to change. For example, being able to map n copies of ρ into k copies of σ, with $n < k$, implies that we have the possibility to add $k - n$ copies in a free state to the initial n copies of ρ, and to act globally to produce k copies of σ. In multi-resource theories, the set of free states can be empty, see for example the invariant set structure of the right panel of Fig. 3.1, and therefore we cannot define a rate of conversion, nor we can use the results of Thm. 1 about the uniqueness of the resource quantifier.

For this reason, we start our investigation of reversible multi-resource theories by demanding the following property, which is related to the notion of "seed regularisation" of Ref. [16, Sec. 6],

Definition 30 (Asymptotic equivalence). *The multi-resource theory R_{multi} satisfies the asymptotic equivalence property if there exists a set of monotones $\{f_i\}_{i=1}^m$, where each f_i is a monotone for the corresponding single-resource theory R_i, such that, for all $\rho, \sigma \in S(\mathcal{H})$, we have that the following two statements are equivalent,*

- *$f_i^\infty(\rho) = f_i^\infty(\sigma)$ for all $i = 1, \ldots, m$.*

- *There exist a sequence of maps $\{\tilde{\varepsilon}_n : \mathcal{S}\left(\mathcal{H}^{\otimes n}\right) \to \mathcal{S}\left(\mathcal{H}^{\otimes n}\right)\}_n$ such that*

$$\lim_{n \to \infty} \left\| \tilde{\varepsilon}_n(\rho^{\otimes n}) - \sigma^{\otimes n} \right\|_1 = 0, \tag{3.3}$$

  as well as a sequence of maps performing the reverse process. The maps $\{\tilde{\varepsilon}_n\}$ are defined as

$$\tilde{\varepsilon}_n(\cdot) = \mathrm{Tr}_A \left[\varepsilon_n(\cdot \otimes \eta_n^{(A)}) \right], \tag{3.4}$$

  where A is an ancilla composed by a sub-linear number $o(n)$ of copies of the system, and it is described by an arbitrary state $\eta_n^{(A)} \in \mathcal{S}\left(\mathcal{H}^{\otimes o(n)}\right)$, such that $f_i(\eta_n^{(A)}) = o(n)$ for all $i = 1, \ldots, m$. The map $\varepsilon_n \in \mathcal{A}_{\mathrm{multi}}^{(n+o(n))}$ is an allowed operation of the multi-resource theory.

Here, f_i^∞ is the regularisation of the monotone f_i, and $\| \cdot \|_1$ is the trace norm, see Def. 7.

When a multi-resource theory satisfies the above property, we have that all asymptotic state transformations are regulated by the values of specific monotones (one for each resource), which can be used to quantify the resources. Then, given a theory that satisfies this property, we can study reversibility, since we have a well-defined notion of resources. An example of a multi-resource theory that satisfies asymptotic equivalence is thermodynamics (even in the case in which multiple conserved quantities are present), as shown in Refs. [169, 168]. We consider this multi-resource theory in the next chapter.

It is worth noting that, in the above property, we are allowing the agent to act over many copies of the system with more than just the set of allowed operations; we assume the agent to be able to use a small ancillary system, sub-linear in the number of copies of the main system. Roughly speaking, the role of this ancilla is to absorb the fluctuations in the monotones f_i^∞'s during the asymptotic state transformation. It is important to notice that this ancillary system only contributes to the transformation by exchanging a sub-linear amount of resources. Thus, its contribution per single copy of the system is negligible when $n \gg 1$, which justifies the use of this additional tool.

Few comments are in order about the meaning of this property. First, the asymptotic equivalence property implies that the state space can be divided into different equivalence classes of states. Each class is labelled by the value of the regularised monotones f_i^∞'s, and within these classes we can freely move between states in a reversible manner, since we are

not consuming any resource. Secondly, the property only refers to the transformations between states with the same values of these monotones. To study the asymptotic transformations between states with different values of the monotones f_i^∞'s, we need to introduce the notion of a battery, see the next section. Finally, while the above property allows us to focus on the sole monotones f_i^∞'s when studying asymptotic state transformations, it alone does not seem to imply the existence of unique resource quantifiers. For example, a priori one might think that other monotones g_i's exist which have constant values over the same equivalence classes singled out by the monotones f_i's, but order these classes in a different way. However, we show in Sec. 3.2.3 that, when the monotones satisfy some natural assumptions, they are the unique quantifiers for a theory satisfying asymptotic equivalence.

3.2.2 Quantifying resources with batteries

Let us show how, for a multi-resource theory satisfying asymptotic equivalence, the monotones f_i^∞ can be used to quantify the resources contained in the system. We first need to introduce some additional systems, which only store a single kind of resource each, and can be independently addressed by the agent. These additional systems are referred to as batteries [19], and in the background chapter on thermodynamics we presented an example of such systems, see Sec. 2.4. To incorporate the batteries into our theory, one possibility is to divide the system under examination into $m + 1$ partitions. The first partition is the main system S, and the remaining ones are the batteries B_i's. Thus, the Hilbert space is partitioned as $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_{B_1} \otimes \ldots \otimes \mathcal{H}_{B_m}$[1].

First of all, we introduce some natural properties for batteries and monotones that, if satisfied, allow us to uniquely quantify the resources contained in the system S. Since each resource is associated with a different monotone, we can forbid a battery to store more than one resource by constraining the set of states describing this system to only those with a fixed value of all but one monotones.

M1 Consider two states $\omega_i, \omega_i' \in \mathcal{S}(\mathcal{H}_{B_i})$ describing the battery B_i. Then, the value of the

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[1]Alternatively, we can take many copies of a fundamental system described by the unpartitioned Hilbert space $\mathcal{H}$, and divide these copies into the main system S and the batteries B_i's.

regularisation of any monotone f_j (where $j \neq i$) over these two states is fixed,

$$f_j^\infty(\omega_i') = f_j^\infty(\omega_i), \quad \forall j \neq i. \tag{3.5}$$

In this way, the battery B_i is only able to store and exchange the resource associated with the monotone f_i. It would be natural to extend the condition of Eq. (3.5) to the monotones themselves, rather than to use their regularisations. However, this stronger condition is not required in our proofs. Furthermore, in order to address each battery as an individual system, we ask the value of the monotones over the global system to be given by the sum of their values over the individual components,

M2 The regularisations of the monotones f_i's can be separated between main system and batteries,

$$f_i^\infty(\rho \otimes \omega_1 \otimes \ldots \otimes \omega_m) = f_i^\infty(\rho) + f_i^\infty(\omega_1) + \ldots + f_i^\infty(\omega_m), \tag{3.6}$$

where $\rho \in \mathcal{S}(\mathcal{H}_S)$ is the state of the main system, and $\omega_i \in \mathcal{S}(\mathcal{H}_{B_i})$ is the state of the battery B_i.

The above property allows us to separate the contribution given by each subsystem to the amount of i-th resource present in the global system. Notice that this property does not a priori imply that the monotones are additive[2] over the states of the system S, or over the states of the individual batteries. We then ask the monotones to satisfy an additional property, so as to simplify the notation. Namely, we ask the zero of each monotone f_i to coincide with its value over the states in the invariant set $\mathcal{F}_i$,

M3 For each $n \in \mathbb{N}$ and $i \in \{1, \ldots, m\}$, the monotone f_i is equal to 0 when computed over the states of $\mathcal{F}_i^{(n)}$,

$$f_i(\gamma_{i,n}) = 0, \quad \forall \gamma_{i,n} \in \mathcal{F}_i^{(n)}. \tag{3.7}$$

This property sets the zero of the monotones, so that a resourceful state always has a non-negative amount of resource. Notice that property M3 is trivially satisfied by any monotone after a translation. The next property requires that tracing out part of the system does not increase the value of the monotones f_i's,

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[2]A real-valued function f is *additive* if, for any two states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$, we have that $f(\rho \otimes \sigma) = f(\rho) + f(\sigma)$.

M4 For all $n, k \in \mathbb{N}$ where $k < n$, the monotones f_i's are such that

$$f_i(\mathrm{Tr}_k\,[\rho_n]) \leq f_i(\rho_n), \quad \forall\, i \in \{1, \ldots, m\}\,. \tag{3.8}$$

where $\rho_n \in \mathcal{S}\,(\mathcal{H}^{\otimes n})$ and $\mathrm{Tr}_k\,[\rho_n] \in \mathcal{S}\,(\mathcal{H}^{\otimes n - k})$.

This property implies that the resources contained in a system cannot increase if we discard/forget part of it. Additionally, we want the monotones to satisfy sub-additivity, namely

M5 For all $n, k \in \mathbb{N}$, the monotones f_i's are such that

$$f_i(\rho_n \otimes \rho_k) \leq f_i(\rho_n) + f_i(\rho_k), \quad \forall\, i \in \{1, \ldots, m\}\,. \tag{3.9}$$

where $\rho_n \in \mathcal{S}\,(\mathcal{H}^{\otimes n})$ and $\rho_k \in \mathcal{S}\,(\mathcal{H}^{\otimes k})$.

That is, the amount of resource contained in two uncorrelated systems, when measured on the two systems independently, is bigger than or equal to the value measured on the two systems together. This is the case, for example, of the relative entropy of entanglement [170]. Another property we require is that the monotones f_i's scale linearly in the number of systems considered,

M6 Given any sequence of states $\{\rho_n \in \mathcal{S}\,(\mathcal{H}^{\otimes n})\}$, the monotones f_i's are such that

$$f_i(\rho_n) = O(n), \quad \forall\, i \in \{1, \ldots, m\}\,. \tag{3.10}$$

When this property is satisfied, we have that the resources scale extensively. Furthermore, the monotones that satisfy this property can also be regularised, although their regularisation might be identically zero on the whole state space. The last property we demand concerns a particular kind of continuity, introduced in Def. 14, that the monotones need to satisfy,

M7 The monotones f_i's are *asymptotic continuous*, that is, for all sequences of states $\rho_n, \sigma_n \in \mathcal{S}\,(\mathcal{H}^{\otimes n})$ such that $\|\rho_n - \sigma_n\|_1 \to 0$ for $n \to \infty$, where $\|\cdot\|_1$ is the trace norm, we have

$$\frac{|f_i\,(\rho_n) - f_i\,(\sigma_n)|}{n} \to 0 \text{ for } n \to \infty, \quad \forall\, i \in \{1, \ldots, m\}\,. \tag{3.11}$$

This property implies that the monotones are physically meaningful, since their values per single copy over sequences of states converge if the sequences of states converge asymptotically.

We can use this formalism to discuss how resources can be quantified in a multi-resource theory, and consequently how the asymptotic equivalence property implies that the theory is reversible. Let us consider any two states $\rho, \sigma \in \mathcal{S}(\mathcal{H}_S)$, which do not need to have the same values of the monotones f_i's. Then, we choose the initial and final states of each battery B_i such that

$$f_i^\infty \left(\rho \otimes \omega_1 \otimes \ldots \otimes \omega_m\right) = f_i^\infty \left(\sigma \otimes \omega_1' \otimes \ldots \otimes \omega_m'\right), \quad \forall\, i = 1, \ldots, m, \tag{3.12}$$

where $\omega_i,\ \omega_i' \in \mathcal{S}(\mathcal{H}_{B_i})$, for $i = 1, \ldots, m$. Under these conditions, due to the asymptotic equivalence property of $\mathrm{R_{multi}}$, we have that the two global states can be asymptotically mapped one into the other in a reversible way,

$$\rho \otimes \omega_1 \otimes \ldots \otimes \omega_m \xleftrightarrow{\text{asympt}} \sigma \otimes \omega_1' \otimes \ldots \otimes \omega_m', \tag{3.13}$$

where the symbol $\xleftrightarrow{\text{asympt}}$ means that we can find a small ancillary system, and an allowed operation, able to asymptotically map the state on the lhs into the state of the rhs, and vice versa, see the second statement of Def. 30.

During the above transformation, the resource associated with the monotone f_i can only be exchanged between system S and battery B_i. We define the amount of i-th resource exchanged during the transformation as

$$\Delta W_i := f_i^\infty(\omega_i') - f_i^\infty(\omega_i), \tag{3.14}$$

where $\omega_i, \omega_i' \in \mathcal{S}(\mathcal{H}_{B_i})$ are the initial and final state of the battery B_i, respectively. Then, the amount of the i-th resource needed to map the state of the main system from ρ into σ can be computed.

Proposition 5. *Consider a theory* $\mathrm{R_{multi}}$ *with* m *resources and allowed operations* $\mathcal{A}_{\text{multi}}$, *equipped with batteries* $B_1, \ldots, B_m$. *If the theory satisfies the asymptotic equivalence property with respect to the set of monotones* $\{f_i\}_{i=1}^m$, *and these monotones satisfy the properties* M1 *and* M2, *then the amount of* i-*th resource needed to perform the asymptotic state transformation* $\rho \to \sigma$ *is equal to*

$$\Delta W_i = f_i^\infty(\rho) - f_i^\infty(\sigma). \tag{3.15}$$

Proof. Due to asymptotic equivalence, a transformation mapping the global state $\rho \otimes \omega_1 \otimes \ldots \otimes \omega_m$ into $\sigma \otimes \omega_1' \otimes \ldots \otimes \omega_m'$ exists iff the conditions in Eq. (3.12) are satisfied. For a given i,

using the property M2 of the monotone f_i, we can re-write the condition as

$$f_i^\infty(\rho) + f_i^\infty(\omega_1) + \ldots + f_i^\infty(\omega_m) = f_i^\infty(\sigma) + f_i^\infty(\omega_1') + \ldots + f_i^\infty(\omega_m'). \tag{3.16}$$

Then, we can use the property M1, which guarantees that the only systems for which f_i changes are the main system S and the battery B_i. Thus, we find that

$$f_i^\infty(\rho) + f_i^\infty(\omega_i) = f_i^\infty(\sigma) + f_i^\infty(\omega_i'). \tag{3.17}$$

By rearranging the factors in the above equation, and using the definition of ΔW_i given in Eq. (3.14), we prove the proposition. $\qquad\square$

It is now easy to show that, if $\mathrm{R}_{\mathrm{multi}}$ satisfies the asymptotic equivalence property, any state transformation on the main system S is reversible. Indeed, from Eq. (3.15) it follows that the amounts of resources used to map the state of this system from ρ to σ are equal, but with negative sign, to the amounts of resources used to perform the reverse transformation, from σ to ρ. Therefore, any cyclic state transformation over the main system leaves the resources contained in the batteries unchanged.

This formalism also allows us to quantify the resources contained in the main system S. Indeed, if the system is described by the state $\rho \in \mathcal{S}(\mathcal{H}_S)$, the amount of i-th resource contained in the system is given by the amount of i-th resource exchanged, ΔW_i, while mapping ρ into a state in $\mathcal{F}_i$. Using property M3 and Prop. 5, it follows that

Corollary 1. *Consider a theory* $\mathrm{R}_{\mathrm{multi}}$ *with m resources and allowed operations* $\mathcal{A}_{\mathrm{multi}}$*, equipped with batteries* $B_1, \ldots, B_m$*. If the theory satisfies the asymptotic equivalence property with respect to the set of monotones* $\{f_i\}_{i=1}^m$*, and these monotones satisfy the properties M1, M2, and M3, then the amount of the i-th resource contained in the main system, when described by the state* ρ*, is given by* $f_i^\infty(\rho)$*.*

It is worth noting that, in general, one cannot extract all the resources contained in the main system at once. Indeed, this is only possible when the multi-resource theory contains free states, like for example in the cases depicted in the left and centre panels of Fig. 3.1.

Being able to quantify the resources contained in a given quantum state allows us to represent the whole state space of the theory in a *resource diagram* [16, 169]. Indeed, from the
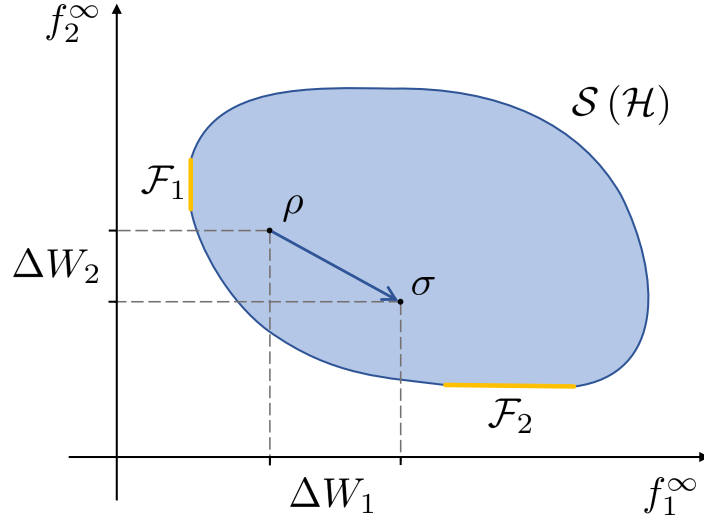
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Figure 3.2: In the figure we represent the state space $\mathcal{S}(\mathcal{H})$ of a multi-resource theory $\mathrm{R_{multi}}$ with two resources. In order for the diagram to be a meaningful representation of this state space, we need the theory to satisfy the asymptotic equivalence property of Def. 30 with respect to the monotones f_1 and f_2. When the theory satisfies this property, we can divide $\mathcal{S}(\mathcal{H})$ into equivalence classes of states with the same value of the regularised monotones f_1^∞ and f_2^∞, which become the x- and y-axis of the diagram, respectively. The state space of the theory is represented by the blue region, and the yellow segments are the invariant sets $\mathcal{F}_1$ and $\mathcal{F}_2$. These sets are disjoint, since the two segments do not intersect each other, and the resource theory $\mathrm{R_{multi}}$ thus corresponds to the one depicted in the right panel of Fig. 3.1. Two equivalence classes, respectively associated to the states ρ and σ, are represented in the diagram. The amounts of resources that are exchanged when transforming from one state to the other, Eq. (3.15), are given in the diagram by the difference between the coordinates of these two points.

definition of asymptotic equivalence it follows that, if two states contain the same resources, i.e., if they have the same values of the monotones f_i^∞'s, then we can map between them using the allowed operations $\mathcal{A}_{\mathrm{multi}}$. This property implies that we can divide the entire state space into equivalence classes, i.e., sets of states with the same value of the m monotones. Then, we can represent each equivalence class as a point in a m-dimensional diagram, with coordinates given by the value of the monotones. By considering all the different equivalence classes, we

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can represent the state space of the main system in the diagram, see for example Fig. 3.2.

3.2.3   Uniqueness results

When a theory satisfies asymptotic equivalence, and the monotones f_i's satisfy the proper-ties M1 – M7, these monotones become the unique quantifiers of the resources. This means that one cannot find other monotones g_i's that give the same equivalence classes of the f_i's, but order them in a different way. Examples of monotones that satisfy all these properties are given in the next sections.

The following theorem, proved in the appendix B.1, states that the monotones f_i's are the unique resource quantifiers when the properties presented in the previous section are satisfied.

Theorem 8. *Consider the resource theory* $\mathrm{R}_{\mathrm{multi}}$ *with m resources, equipped with the batteries* B_i's, where $i = 1, \ldots, m$. *Suppose the theory satisfies the asymptotic equivalence property with respect to the set of monotones* $\{f_i\}_{i=1}^m$. *If these monotones satisfy the properties M1 – M7, and their regularisations are not identically zero over the whole state space, then the amount of i-th resource contained in the main system S is uniquely quantified by the regularisation of the monotone* f_i, *i.e., every other regularised monotone is equal to* f_i^∞ *up to a multiplicative constant.*

It remains to show that the properties we are demanding (M1 – M7) are satisfied by some class of monotones. Here, we present two such classes, both relevant for the results of the next Ch. 4, where asymptotic equivalence is shown for states with same average energy and von Neumann entropy.

Relative entropy distance from the invariant set

Let us first consider the relative entropy distance from the set $\mathcal{F}_i$, which we refer to as $E_{\mathcal{F}_i}$, whose definition is given in Eq. (1.10), and we report it here for clarity,

$$E_{\mathcal{F}_i}(\rho) = \inf_{\gamma_i \in \mathcal{F}_i} D(\rho \,\|\, \gamma_i), \quad \text{where } \rho \in \mathcal{S}\left(\mathcal{H}\right).$$

We now show that, if the invariant set $\mathcal{F}_i$ satisfies the assumptions F1 – F4 introduced in the background chapter, Sec. 1.3, then $E_{\mathcal{F}_i}$ satisfies the properties M1 – M7. Notice, however, that

in order for this monotone to independently measure the amount of resource contained in system and batteries, property M2, we need to impose the following structure. We demand that both system S and batteries B_i's are composed by a certain number of copies of a fundamental system, described by $\mathcal{H}$. Moreover, we ask the regularisation of the monotone $E_{\mathcal{F}_i}$ to be additive[3], so that the contributions given by system and batteries can be separated. This property is satisfied by any theory whose invariant set is composed by a single state (for instance, thermodynamics, or purity theory), and also by the resource theory of bipartite entanglement, if we restrict the state space to a subset of Bell-diagonal states [173, 174, 175], or to the set of pure states.

First of all, we should show that $E_{\mathcal{F}_i}$ is a monotone for the multi-resource theory R_{multi}. This follows from Prop. 2 in the background chapter, and from the fact that the set of allowed operations $\mathcal{A}_{\mathrm{multi}}$ is obtained from the intersection of all the other classes of allowed operations, and particularly from $\mathcal{A}_i$, for which $\mathcal{F}_i$ is an invariant set. Let us now focus on the other properties.

- Property M1 concerns the batteries rather than the monotones. We simply need to choose the states of the battery B_i to have a fixed value of the monotones $f_{j\neq i}^{\infty}$.

- Property M2 is a consequence of the structure of system and batteries, and of the fact that we demand the regularised relative entropy distance to be additive.

- Property M3 follows straightforwardly from the definition of relative entropy distance.

- Property M4 follows from the fact that $\mathcal{F}_i$ is closed under partial tracing, assumption F4. This is proven in Lem. 1.

- Property M5 similarly follows from the fact that $\mathcal{F}_i$ is closed under tensor product. For any two states $\rho_n \in \mathcal{S}\left(\mathcal{H}^{\otimes n}\right)$ and $\rho_k \in \mathcal{S}\left(\mathcal{H}^{\otimes k}\right)$ we have that

$$E_{\mathcal{F}_i}(\rho_n \otimes \rho_k) = \inf_{\gamma_{n+k} \in \mathcal{F}_i^{(n+k)}} D(\rho_n \otimes \rho_k \,\|\, \gamma_{n+k}) \leq \inf_{\gamma_n \in \mathcal{F}_i^{(n)}, \gamma_k \in \mathcal{F}_i^{(k)}} D(\rho_n \otimes \rho_k \,\|\, \gamma_n \otimes \gamma_k)$$

$$= \inf_{\gamma_n \in \mathcal{F}_i^{(n)}} D(\rho_n \,\|\, \gamma_n) + \inf_{\gamma_k \in \mathcal{F}_i^{(k)}} D(\rho_k \,\|\, \gamma_k) = E_{\mathcal{F}_i}(\rho_n) + E_{\mathcal{F}_i}(\rho_k), \qquad (3.18)$$

where the inequality follows from property F3 of the set $\mathcal{F}_i$.

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[3]We would like to weaken this assumption, possibly by requiring property F5 to hold, and by using tools such as the approximate de Finetti's theorems [171, 172]. This will be addressed in future work.

- Property M6 follows from the fact that the set $\mathcal{F}_i$ contains a full-rank state. We have that, for all $\rho_n \in \mathcal{S}(\mathcal{H}^{\otimes n})$,

$$E_{\mathcal{F}_i}(\rho_n) = \inf_{\gamma_n \in \mathcal{F}_i^{(n)}} D(\rho_n \,\|\, \gamma_n) \leq D(\rho_n \,\|\, \gamma_{\text{full}-\text{rank}}^{\otimes n}) = -S(\rho_n) - \text{Tr}\left[\rho_n \log \gamma_{\text{full}-\text{rank}}^{\otimes n}\right]$$

$$\leq -\text{Tr}\left[\rho_n \log \gamma_{\text{full}-\text{rank}}^{\otimes n}\right] \leq n \log \lambda_{\min}^{-1}, \tag{3.19}$$

where the first inequality follows from the fact that $\mathcal{F}_i$ contains a full-rank state, property F2, the second one from the fact that the von Neumann entropy is non-negative, and the last one from the fact that the optimal case is obtained when ρ_n is given by n copies of the eigenstate of $\gamma_{\text{full}-\text{rank}}$ associated with its minimum eigenvalue $\lambda_{\min}$.

- Property M7 for $E_{\mathcal{F}_i}$ has been discussed in the background chapter, particularly in Lem. 4.

Thus, when a multi-resource theory satisfies asymptotic equivalence with respect to the relative entropy distances from the invariant sets $\mathcal{F}_i$'s, and the regularisations of these monotones are additive, we have that Thm. 8 guarantees that these monotones uniquely quantify the different amounts of resources in the theory, provided they are not identically zero over the whole state space[4]. In this case, we have that the amount of i-th resource used to map the main system from the state ρ into the state σ is given by

$$\Delta W_i = E_{\mathcal{F}_i}^{\infty}(\rho) - E_{\mathcal{F}_i}^{\infty}(\sigma), \tag{3.20}$$

which follows from Prop. 5.

Average monotones

There are situations, when we consider specific resource theories, in which some of the properties of the invariant sets $\mathcal{F}_i$'s used in the previous section are not satisfied. In particular, the set of free states might not contain a full-rank state, i.e., property F2 might not be satisfied. An example is the resource theory of energy-non-increasing maps, for a system with Hamiltonian H,

$$\mathcal{A}_H = \{\varepsilon_H : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \,|\, \text{Tr}\left[\varepsilon_H(\rho)H\right] \leq \text{Tr}\left[\rho H\right] \,\, \forall \rho \in \mathcal{S}(\mathcal{H})\}. \tag{3.21}$$

---

[4]An example where the regularised relative entropy from an invariant set is identically zero for all states in $\mathcal{S}(\mathcal{H})$ is the resource theory of asymmetry, see Ref. [6].

A subset of the class $\mathcal{A}_H$ are unitary operations which commute with the Hamiltonian H, that are the allowed operations of the resource theory we consider in the next chapter. If the Hamiltonian H has a non-degenerate ground state $|g\rangle$, then it is easy to show that this state is fixed, that is,

$$\varepsilon_H \left(|g\rangle \langle g| \right) = |g\rangle \langle g| . \tag{3.22}$$

Indeed, the operation $\varepsilon_g(\cdot) = \mathrm{Tr}_A \left[S(\cdot \otimes |g\rangle \langle g|_A) S^\dagger \right]$, where S is the unitary operation implementing the swap between the two states, belongs to $\mathcal{A}_H$ and maps all states into the ground state. Thus, the set of free states does not contain a full-rank state, which implies that the relative entropy distance from this set is ill-defined, and is not asymptotic continuous. Notice that the above argument holds even in the case of a degenerate ground state, with the difference that the invariant set would be composed by any state with support on this degenerate subspace.

We can introduce a different monotone for this kind of resource theory, that is, the average of the observable which is not increased by the allowed operations (modulo a constant factor). For the example we are considering, this monotone would be

$$M_H(\rho) = \mathrm{Tr} \left[H\rho \right] - E_g, \tag{3.23}$$

where H is the Hamiltonian of the system, and $E_g = \langle g| H |g\rangle$ is the energy of the ground state. When n copies of the system are considered, we define the total Hamiltonian as

$$H_n = \sum_{j=1}^n H^{(j)} \text{ where } H^{(j)} = \mathbb{I}_1 \otimes \ldots \otimes \mathbb{I}_{j-1} \otimes H \otimes \mathbb{I}_{j+1} \otimes \ldots \otimes \mathbb{I}_n \tag{3.24}$$

We can now show that this monotone satisfies the properties M1 – M7.

- As stated before, property M1 concerns the batteries rather than the monotones. We need to choose the states of the battery B_i to have a fixed value of the other monotones.

- Property M2 is satisfied by taking the Hamiltonian to be composed by a sum of operators, one acting on the system S, the others acting on the batteries B_i's, with no interaction terms,

$$H = H_S + H_{B_1} + \ldots + H_{B_m} \tag{3.25}$$

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- Property M3 is satisfied by construction, since in Eq. (3.23) we subtract the amount of energy associated with the fixed state $|g\rangle$.

- Properties M4 – M6 all follow from our choice of H_n, see Eq. (3.24), in particular from the absence of interaction terms between different copies of the system.

- Property M7 is shown in Prop. 6, see the appendix B.2.

Then, if we consider a multi-resource theory satisfying asymptotic equivalence with respect to monotones of the form of Eq. (3.23), we have that, due to Thm. 8, these monotones uniquely quantify the corresponding resources.

3.3   Resource interconversion and the first law

When a physical task requires multiple resources, a natural question to ask is whether we can invest more of one of these resources in order to save the others. In the previous section, we have seen that, when the agent has access to the sole system S and to the batteries B_i's, they need to provide a fixed amount of each resource in order to perform a given state transformation, see Prop. 5. In this section, we investigate whether an additional tool can be given to the agent, so as to allow them to inter-convert between resources. If this tool exists, then the agent is able to perform state transformations using flexible amounts of resources. We show that resource interconversion is not possible for every multi-resource theories, but rather only for those theories where the invariant sets do not intersect, see right panel of Fig. 3.1.

In this kind of resource theories, the agent needs to have access to an additional system, which we call a *bank*, in order to trade one resource for another. The agent can pay a given amount of one resource to the bank, and gain back a different amount of another resource, at an exchange rate which depends on the state of the bank. During the exchange, we want the state of the bank to change infinitesimally with respect to a specific measure, which is linked to the exchange rate offered by the bank. We refer to this measure as the *bank monotone*, and we discuss its connection to the bank system in the next section. If the state of the bank does not change with respect to this monotone, then the agent can keep using the bank to exchange resources, and the exchange rate is only infinitesimally modified by each interconversion.

An example of resource interconversion, relevant in thermodynamics, is Landauer's erasure. During this process, the unknown state of a bit, which is in contact with a thermal reservoir at temperature T, is mapped into a known state with certainty, and work (pure energy) is consumed to achieve the erasure. We can interpret this process as an interconversion of resources, where we trade energy ΔE, or work, for information ΔI, or neg-entropy, and we use this second resource to re-set the state of the bit. The interconversion of resources is achieved using the thermal reservoir, whose temperature sets the exchange rate, so that $\Delta E = -T \Delta I$. This example shows that, in thermodynamics, the role of the bank is played by the thermal reservoir.

When the agent has access to the bank, they can exchange resources and therefore they do not need to provide a fixed amount of each resource for a given state transformation. In this situation, the agent can realise a state transformation provided that a single relation is satisfied; we refer to this relation as the *first law of (multi-) resource theories*. Specifically, we find that a state transformation over the system S is possible if the (weighted) sum of the resources exchanged during the transformation is equal to the change, in terms of the bank monotone, between the final and initial state of the system. In the case of thermodynamics, this relation coincides with the First Law of Thermodynamics. Indeed, the bank monotone of this theory is the Helmholtz free energy F, and the relation that needs to be satisfied for a transformation to be possible is given by

$$\Delta F = \Delta E + T \Delta I, \tag{3.26}$$

where ΔF is the change in free energy over the system S, while ΔE and ΔI are the amounts of energy and information exchanged, respectively. The weight of this sum is given by the temperature T characterising the thermal reservoir, i.e., the bank system.

3.3.1 Bank systems and interconversion relations

We now introduce the bank system for the multi-resource theory framework we are considering, and show how this additional tool allows the agent to perform interconversion between resources. To simplify the problem, we only focus on a theory with two resources. However, the results we obtain can be applied to theories with more than two resources, since in that case we can select two resources and perform interconversion while keeping the others fixed. We consider a

resource theory R_{multi} with two invariant sets $\mathcal{F}_1$ and $\mathcal{F}_2$, and allowed operations $\mathcal{A}_{\mathrm{multi}}$. We assume the theory to satisfy the asymptotic equivalence property of Def. 30 with respect to $E_{\mathcal{F}_1}$ and $E_{\mathcal{F}_2}$. Additionally, we demand these invariant sets to be convex, property F1, to contain a full-rank state, property F2, and to satisfy the following property,

F3b The invariant sets $\mathcal{F}_i$'s are such that $\mathcal{F}_i^{(n)} = \mathcal{F}_i^{\otimes n}$, for all $n \in \mathbb{N}$.

Notice that this last property is more demanding than properties F3 and F4, and it implies that the monotones $E_{\mathcal{F}_i}$'s are additive, that is, for all $\rho, \sigma \in \mathcal{S}(\mathcal{H})$,

$$E_{\mathcal{F}_i}(\rho \otimes \sigma) = E_{\mathcal{F}_i}(\rho) + E_{\mathcal{F}_i}(\sigma). \tag{3.27}$$

Consequently, the regularisation of these monotones coincides with the monotones themselves, i.e., for all $\rho \in \mathcal{S}(\mathcal{H})$ we have that $E_{\mathcal{F}_i}^{\infty}(\rho) = E_{\mathcal{F}_i}(\rho)$. Furthermore, the properties F1 and F3b together imply that the invariant set is composed by a single state, i.e., $\mathcal{F}_i = \{\rho_i\}$, where $\rho_i \in \mathcal{S}(\mathcal{H})$, for $i = 1, 2$.

It is important to stress that property F3b is not satisfied by every multi-resource theory. For example, the property is satisfied by the multi-resource theory of thermodynamics, see the next chapter, but it is violated by other theories, like entanglement theory, where the set of free states is composed of separable states. We would like to replace property F3b with the less demanding property F5, which requires the invariant sets to be closed under permutations of copies, together with the implications of the approximate de Finetti's theorems [171].

We make use of property F3b in Lem. 9, shown in appendix B.4, which itself is used to prove some essential properties of the set of bank states, see Def. 31. Furthermore, this property is ultimately used to show that the exchange rate between resources is given by the relative entropy distance from the set of states describing the bank, see Cor. 2. For example, in thermodynamics we find that the exchange rate of the bank, which is described by a thermal state with temperature T, is linked to the relative entropy distance from such a state. Finally, it is worth noting that all the results we obtain in this section also apply if one of the monotones is of the form shown in Eq. (3.23). Indeed, these monotones satisfy the same properties of the relative entropy distances, with the difference that the corresponding invariant set can be composed by multiple states, and these states do not need to have full rank.

Let us now consider an example of resource interconversion which should highlight the properties we are searching for in a bank system. Suppose we have a certain amount of euros and pounds in our wallet, and we want to convert one into the other, for example, from pounds to euros. In order to convert these two currencies we need to go to the bank, which should satisfy the following basic properties. First of all, if we do not give any pounds, we cannot receive any euros (and vice versa). Secondly, the bank converts the two currencies at a certain exchange rate, and this exchange rate can be different depending on the bank we go to. Finally, the exchange rate of the bank should not change as a consequence of our transaction[5].

With the help of the three properties listed above, we can now define a bank system within our framework. We consider a tripartite system consisting of a bank P and two batteries B_1 and B_2. These three systems are all composed by many copies of the same fundamental system, described by the Hilbert space $\mathcal{H}$. This condition allows us to unambiguously define the bank monotone over the global system, see Eq. (3.31). First of all, we demand that the states describing the bank be *passive*, meaning that we should not be able to extract from this system both resources at the same time, since we always need to pay one resource to gain another one. Thus, the set of *bank states* is defined as

Definition 31 (Bank states)**.** *Consider a multi-resource theory* $\mathrm{R}_{\mathrm{multi}}$ *satisfying the asymptotic equivalence property with respect to the monotones* $E_{\mathcal{F}_1}$ *and* $E_{\mathcal{F}_2}$*. The set of* bank states *of the theory is a subset of the state space* $\mathcal{S}(\mathcal{H})$ *defined as,*

$$
\mathcal{F}_{\mathrm{bank}} = \big\{ \rho \in \mathcal{S}(\mathcal{H}) \mid \forall \sigma \in \mathcal{S}(\mathcal{H}),\ E_{\mathcal{F}_1}(\sigma) > E_{\mathcal{F}_1}(\rho) \ \text{or}
$$
$$
E_{\mathcal{F}_2}(\sigma) > E_{\mathcal{F}_2}(\rho) \ \text{or}
$$
$$
E_{\mathcal{F}_1}(\sigma) = E_{\mathcal{F}_1}(\rho) \ \text{and}\ E_{\mathcal{F}_2}(\sigma) = E_{\mathcal{F}_2}(\rho) \big\}. \qquad (3.28)
$$

Within the set $\mathcal{F}_{\mathrm{bank}}$ *we can find different subsets of bank states with a fixed value of* $E_{\mathcal{F}_1}$ *and* $E_{\mathcal{F}_2}$*. We define each of these subsets as*

$$
\mathcal{F}_{\mathrm{bank}}\big(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\big) = \big\{ \rho \in \mathcal{F}_{\mathrm{bank}} \mid E_{\mathcal{F}_1}(\rho) = \bar{E}_{\mathcal{F}_1} \ \text{and}\ E_{\mathcal{F}_2}(\rho) = \bar{E}_{\mathcal{F}_2} \big\}. \qquad (3.29)
$$

Notice that Eq. (3.28) implies that no state can be found with smaller values of both monotones $E_{\mathcal{F}_i}$'s. In this way, the agent is not able to transform the state of the bank in a way

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[5] This last property is approximately satisfied by real banks, at least for the amount exchanged by average costumers.

in which both resources are extracted from it and stored in the batteries. Instead, they always need to trade resources. The set of bank states $\mathcal{F}_{\mathrm{bank}}$ can be visualised in the resource diagram of the theory, see Fig. 3.3. This set is represented by a curve on the boundary of the state space, connecting the points associated with $\mathcal{F}_1$ to those associated with $\mathcal{F}_2$. In appendix B.3 we show that, under the current assumptions, this curve is always convex. For the resource theory under examination to admit an interconversion relation, however, we need to demand a more stringent constraint, namely, that the curve of bank states (or at least part of it) is strictly convex[6], like the one shown in the figure. When this is the case, we can define a bank monotone and extend it to the many-copy case.

The subsets $\mathcal{F}_{\mathrm{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)$'s represent individual points in the resource diagram describing the multi-resource theory, and they obey many of the properties satisfied by the invariant sets $\mathcal{F}_i$'s. Indeed, one can show that

- For all $n \in \mathbb{N}$, we have that each subset of bank states is such that

$$\mathcal{F}_{\mathrm{bank}}^{(n)}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right) = \mathcal{F}_{\mathrm{bank}}^{\otimes n}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right), \tag{3.30}$$

  that is, these subsets satisfy property F3b. This equality is proved in Prop. 9 of appendix B.4, and it requires the curve of bank states to be strictly convex (otherwise, we could take the tensor product of different points on this curve and still obtain a point on such curve).

- Every subset $\mathcal{F}_{\mathrm{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)$ is convex, property F1, as shown in Prop. 10 in appendix B.4.

- Every subset $\mathcal{F}_{\mathrm{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)$, and its extensions to the many-copy case, is invariant under the class of allowed operations $\mathcal{A}_{\mathrm{multi}}$ of the multi-resource theory, as shown in Prop. 11 in appendix B.4.

The second property we demand for a bank is that the exchange rate only depends on which state of the bank we choose to use. Indeed, we now show that each bank state can be associated with a specific function, and that the linear coefficients of this function uniquely

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[6]Given a set V and a function $f : V \to \mathbb{R}$, the function is *strictly convex* if for all $x, y \in V$, $x \neq y$, we have that $f(\lambda x + (1 - \lambda) y) < \lambda f(x) + (1 - \lambda) f(y)$, for any $\lambda \in (0, 1)$.
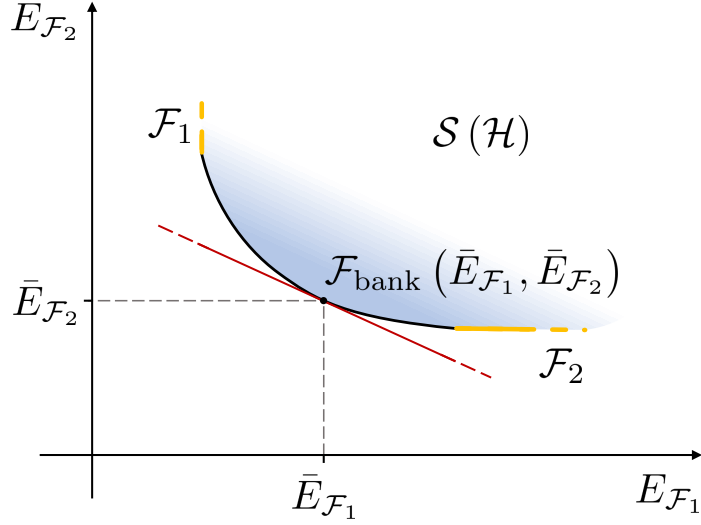
Figure 3.3: The set of bank states introduced in Eq. (3.28) is represented in the $E_{\mathcal{F}_1}$–$E_{\mathcal{F}_2}$ diagram. Only part of the state space $\mathcal{S}(\mathcal{H})$ is shown, in blue, together with the invariant sets of the theory $\mathcal{F}_1$ and $\mathcal{F}_2$, the two yellow segments. Notice that we represent these sets as segments since, in the case in which one of the relative entropy distances is replaced by an average monotone, see Eq. (3.23), the invariant set does not need to be composed by a single state. The black (strictly convex) curve connecting these segments is the set of all the bank states of the theory $\mathcal{F}_{\text{bank}}$. A specific subset of bank states, labelled by $\mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)$, is shown on the curve, see Eq. (3.29). Notice that, graphically, a bank state is one for which there exists no other state in its south-west quadrant. The red line, which is tangent to the set of bank states and passes through the point $\mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)$, is parametrised by $f_{\text{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}} = 0$, see Eq. (3.31).

define the exchange rate at which resources are inter-converted, as we show in Thm. 9. Given a bank described by the subset $\mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)$, let us introduce the following function acting on $\mathcal{S}(\mathcal{H})$,

$$f_{\text{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}(\rho) := \alpha\, E_{\mathcal{F}_1}(\rho) + \beta\, E_{\mathcal{F}_2}(\rho) - \gamma, \tag{3.31}$$

where α, β, and γ are non-negative constant factors that depend on the subset of bank states we have chosen. These coefficients are completely determined (up to a global multiplicative factor) by the following two properties,

B1 The function $f_{\text{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}$ is equal to zero over the subset $\mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)$.

B2 The value of this function on the states contained in the subset $\mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)$ is minimum.

Notice that property B1 is there to set the zero of the function, and implies that

$$\gamma = \alpha\,\bar{E}_{\mathcal{F}_1} + \beta\,\bar{E}_{\mathcal{F}_2}. \tag{3.32}$$

Property B2, instead, fixes the ratio between the constants α and β. This condition can be visualised in the resource diagram, and is equivalent to the request that, in such diagram, the bank monotone is tangent to the state space, so that

$$f_{\text{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}(\rho) \geq f_{\text{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}(\sigma), \quad \forall\,\rho \in \mathcal{S}\left(\mathcal{H}\right), \forall\,\sigma \in \mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right). \tag{3.33}$$

The above property is always satisfied under our working assumptions, since the curve of bank states is convex, see Fig. 3.3. We refer to this function as the *bank monotone*.

The bank monotone can be easily extended to the state space of n copies of the system. The main difference is that, when we consider states in $\mathcal{S}\left(\mathcal{H}^{\otimes n}\right)$, the coefficient γ is proportional to the number of copies n, and we write $\gamma = n\left(\alpha\,\bar{E}_{\mathcal{F}_1} + \beta\,\bar{E}_{\mathcal{F}_2}\right)$. This follows from property B1, together with the fact that the subset $\mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)$ satisfies property F3b, see Eq. (3.30).

We can now list the additional properties of the bank monotone, that follows from the fact that this function is linear in $E_{\mathcal{F}_1}$ and $E_{\mathcal{F}_2}$, and from the assumption we made on the invariant sets $\mathcal{F}_i$'s. These properties are proved in Prop. 12, in appendix B.4.

B3 The function $f_{\text{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}$ is additive.

B4 The function $f_{\text{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}$ is monotonic under partial tracing.

B5 The function $f_{\text{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}$ scales extensively, i.e., for a sequence of states $\{\rho_n \in \mathcal{S}\left(\mathcal{H}^{\otimes n}\right)\}$, we have

$$f_{\text{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}(\rho_n) = O(n). \tag{3.34}$$

B6 The function $f_{\text{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}$ is asymptotic continuous.

B7 The function $f_{\text{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}$ is monotonic under the set of allowed operations $\mathcal{A}_{\text{multi}}$, since α and β are non-negative.

The third and last property we demand from a bank concerns the back-reaction it experiences during interconversion of resources. We want that, after the transformation, the state of the bank only changes infinitesimally with respect to the bank monotone associated with it. If this is the case, we can show that the exchange rate only changes infinitesimally, and therefore we can keep using the bank to inter-convert between resources at the same exchange rate. More concretely, the global system we consider is composed by many-copies of the same fundamental system described by $\mathcal{H}$, for which we have defined the notion of bank states. We group these copies in three partitions, namely, the bank P and the two batteries B_1 and B_2. The bank is described by $\mathcal{H}_P = \mathcal{H}^{\otimes n}$, $n \in \mathbb{N}$, and its initial state is given by n copies of the bank state $\rho_P \in \mathcal{F}_{\text{bank}} \left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2} \right)$. The batteries are described by $\mathcal{H}_{B_i} = \mathcal{H}^{\otimes m_i}$, $m_i \in \mathbb{N}$, where $i = 1, 2$. The states describing the batteries are $\omega_1 \in \mathcal{S} \left(\mathcal{H}_{B_1} \right)$, and $\omega_2 \in \mathcal{S} \left(\mathcal{H}_{B_2} \right)$, respectively.

A *resource interconversion* is an asymptotically reversible transformation

$$\rho_P^{\otimes n} \otimes \omega_1 \otimes \omega_2 \xleftrightarrow{\text{asympt}} \tilde{\rho}_P^{\otimes n} \otimes \omega_1' \otimes \omega_2', \tag{3.35}$$

where $\tilde{\rho}_P \in \mathcal{S} \left(\mathcal{H} \right)$, $\omega_1' \in \mathcal{S} \left(\mathcal{H}_{B_1} \right)$, and $\omega_2' \in \mathcal{S} \left(\mathcal{H}_{B_2} \right)$, which satisfies the following property,

X1 The state of the bank changes infinitesimally during the resource interconversion.

If $\rho_P \in \mathcal{F}_{\text{bank}} \left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2} \right)$, then the state $\tilde{\rho}_P \in \mathcal{S} \left(\mathcal{H} \right)$ is such that

$$f_{\text{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}} (\tilde{\rho}_P^{\otimes n}) = f_{\text{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}} (\rho_P^{\otimes n}) + \delta_n, \tag{3.36}$$

where $\delta_n > 0$ is such that $\delta_n \to 0$ as $n \to \infty$.

We are now ready to derive the interconversion relation, which regulates the trading of resources, and defines the exchange rate.

Theorem 9. *Consider a resource theory* R_{multi} *with two resources, equipped with the batteries* B_1 *and* B_2. *Suppose the theory satisfies asymptotic equivalence with respect to the monotones* $E_{\mathcal{F}_1}$ *and* $E_{\mathcal{F}_2}$, *i.e. the relative entropy distances from the invariant sets of the theory, and that these sets satisfy the properties F1, F2, and F3b. Then, the resource interconversion of Eq.* (3.35), *where the bank has to transform in accord to condition X1, is solely regulated by the following relation,*

$$\alpha \, \Delta W_1 = -\beta \, \Delta W_2 + \delta_n. \tag{3.37}$$

Furthermore, when the number of copies of the bank system n is sent to infinity, we have that the above equation reduces to the following one, which we refer to as the interconversion relation,

$$\Delta W_1 = -\frac{\beta}{\alpha} \Delta W_2, \tag{3.38}$$

where the amounts of resources exchanged ΔW_i are non-zero.

Proof. Let us consider the resource interconversion of Eq. (3.35), where a global operation is performed over bank and batteries, and the sole constraint over the bank system is given by condition X1. As we discussed in Sec. 3.2.2, in order for the transformation to happen, the conditions of Eq. (3.12) need to be satisfied for both monotones $E_{\mathcal{F}_1}$ and $E_{\mathcal{F}_2}$, which in particular implies that the resources exchanged with the batteries are

$$\Delta W_i = n\left(E_{\mathcal{F}_i}(\rho_P) - E_{\mathcal{F}_i}(\tilde{\rho}_P)\right), \qquad \text{for } i = 1, 2, \tag{3.39}$$

where we have used property F3b. Furthermore, since $f_{\text{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}$ is monotonic under the set of allowed operations, property B7, we find that

$$f_{\text{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}(\rho_P^{\otimes n} \otimes \omega_1 \otimes \omega_2) = f_{\text{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}(\tilde{\rho}_P^{\otimes n} \otimes \omega_1' \otimes \omega_2'). \tag{3.40}$$

Then, since the global system is given by many copies of $\mathcal{H}$, and since the bank monotone is additive, property B3, we can separate the contribution given by bank and batteries. Furthermore, from the definition of bank monotone, Eq. (3.31), and the property of the batteries, condition M1, it follows that

$$f_{\text{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}(\rho_P^{\otimes n}) + \alpha E_{\mathcal{F}_1}(\omega_1) + \beta E_{\mathcal{F}_2}(\omega_2) = f_{\text{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}(\tilde{\rho}_P^{\otimes n}) + \alpha E_{\mathcal{F}_1}(\omega_1') + \beta E_{\mathcal{F}_2}(\omega_2'). \tag{3.41}$$

Now, if we re-order the terms in the above equation, and we use Eq. (3.31) again, we obtain

$$f_{\text{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}(\rho_P^{\otimes n}) - f_{\text{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}(\tilde{\rho}_P^{\otimes n}) = \alpha\left(E_{\mathcal{F}_1}(\omega_1') - E_{\mathcal{F}_1}(\omega_1)\right) + \beta\left(E_{\mathcal{F}_2}(\omega_2') - E_{\mathcal{F}_2}(\omega_2)\right). \tag{3.42}$$

If we use property X1 together with the definitions of ΔW_1 and ΔW_2 given in Eq. (3.14), we get that

$$\alpha\,\Delta W_1 = -\beta\,\Delta W_2 + \delta_n, \tag{3.43}$$

where $\delta_n \to 0$ as n tends to infinity. It remains to show that, when $n \to \infty$, trading non-zero amounts of resources is still possible.

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Let us first recall that property B2 of the bank monotone implies that $f_{\text{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}} = 0$ is a line, tangent to the state space and passing through the states describing the bank, see Fig. 3.4. As a result, the curve of all bank states given in Eq. (B.56) of appendix B.3 can be approximated, in the neighbourhood of $\mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)$, by this line. This implies that, if we take the state $\tilde{\rho}_P$ to be in the set of bank states $\mathcal{F}_{\text{bank}}$, with a value of $E_{\mathcal{F}_1}$ given by

$$E_{\mathcal{F}_1}(\tilde{\rho}_P) = E_{\mathcal{F}_1}(\rho_P) - \epsilon, \tag{3.44}$$

where $\epsilon \ll 1$, we find that the value of the monotone $E_{\mathcal{F}_2}$ for this state is

$$E_{\mathcal{F}_2}(\tilde{\rho}_P) = E_{\mathcal{F}_2}(\rho_P) + \frac{\alpha}{\beta}\epsilon + O(\epsilon^2). \tag{3.45}$$

Then, it is easy to see that, if we map ρ_P into $\tilde{\rho}_P$ during the resource interconversion, we obtain the following

$$\Delta W_1 = n\,\epsilon \quad , \quad \Delta W_2 = -n\frac{\alpha}{\beta}\epsilon + O(n\,\epsilon^2) \quad , \quad \delta_n = O(n\,\epsilon^2), \tag{3.46}$$

where the first two equations follow from Eq. (3.39), while the last one is given by Eq. (3.36). Thus, if we take $\epsilon \propto \frac{1}{n}$, and we send n to infinity, we find that the resources exchanged during the transformations, the ΔW_i's, are finite and their value is arbitrary, while the change in the bank monotone over the bank system, equal to δ_n, is infinitesimally small. $\qquad\square$

Let us analyse the interconversion relation of Eq. (3.38). Since both parameters α and β are non-negative we find that, during a resource interconversion, we always increase the amount contained in one of the batteries (for example, $\Delta W_1 > 0$) while decreasing the amount contained in the other ($\Delta W_2 < 0$). The interconversion also depends on the state of the bank, and particularly on the amounts of resources contained in this system. During the transformation of Eq. (3.35), the global value of the monotones $E_{\mathcal{F}_i}$'s does not change, and therefore the increase in the amount of one resource contained in the batteries corresponds to the decrease of the same resource in the bank. Thus, to achieve resource interconversion in both directions, we need the bank state ρ_P to be such that $E_{\mathcal{F}_1}(\rho_P) > 0$ and $E_{\mathcal{F}_2}(\rho_P) > 0$. For this reason, the only kind of multi-resource theories for which interconversion is possible is the one where the invariant sets are disjoint, see the right panel of Fig. 3.1. Finally, notice that, as far as the resources contained in the bank state are non-zero, the agent is able to exchange any amounts,
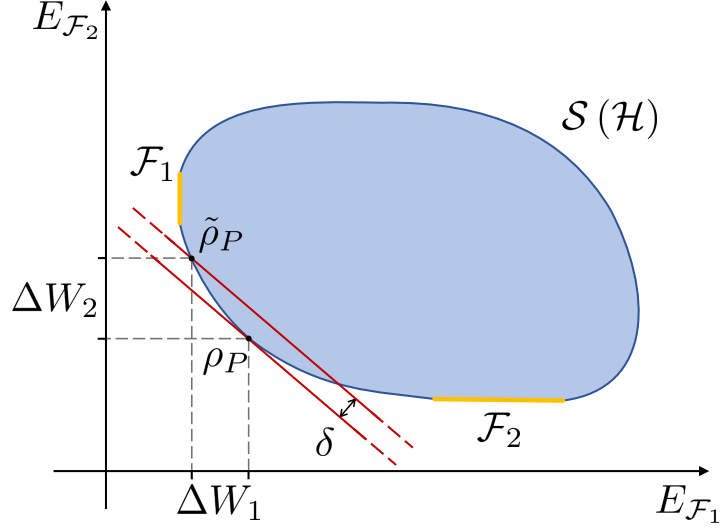
Figure 3.4: The state space of the theory R_{multi} is represented in the $E_{\mathcal{F}_1}$–$E_{\mathcal{F}_2}$ diagram. The invariant sets of the theory, $\mathcal{F}_1$ and $\mathcal{F}_2$, are represented by the two yellow segments. The set of bank states $\mathcal{F}_{\mathrm{bank}}$ lies on the boundary of the state space, and is represented by the curve connecting the two invariant sets. The subset of bank states $\mathcal{F}_{\mathrm{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)$, where ρ_P is contained, is represented by a point in the diagram. The red line, tangent to the state space and passing through the point associated with ρ, represents the set of states with $f_{\mathrm{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}} = 0$. The other line is given by all those states with a value $\delta > 0$ of this monotone. We see that, by mapping ρ_P into $\tilde{\rho}_P$, we can extract an amount ΔW_1 of the first resource, while paying an amount ΔW_2 of the second resource. Furthermore, one can show that when $\delta \to 0$, these two quantities tend to 0 as $\delta^{\frac{1}{2}}$, i.e., with a slower rate. It is then possible to keep the ΔW_i's finite if we take $n \propto \delta^{-\frac{1}{2}}$ copies of the bank states, see the proof of Thm. 9. Thus, in the limit $n \to \infty$, the overall back-action on the bank can be made arbitrarily small.

since we take the number of copies n of the bank to be infinite. This is the case, for example, in thermodynamics, where the bank is composed by an infinite number of thermal states, each containing a positive amount of both energy and information, the two resources of the theory.

3.3.2 Bank monotones and the relative entropy distance

A key requirement for achieving resource interconversion is that the state of the bank only changes infinitesimally with respect to the bank monotone of Eq. (3.31). We now show that this function, obtained from the linear combination of the monotones $E_{\mathcal{F}_i}$'s, is intimately related to the relative entropy distance from the set of states describing the bank. To show this correspondence, we introduce a general procedure for constructing single-resource theories out of multi-resource theories. For example, the procedure we introduce allows us to move from the multi-resource theory of thermodynamics, where $\mathcal{A}_{\mathrm{multi}}$ is composed by energy-preserving unitary operations, to the single-resource theory equipped with Thermal Operations, that we introduced in Ch. 2.

In order to move from a multi-resource theory to a single-resource theory, we proceed by modifying the class of allowed operations. In particular, aside from the possibility of using the operations in $\mathcal{A}_{\mathrm{multi}}$, we allow the agent to prepare an arbitrary number of ancillary systems described by the states in a given subset $\mathcal{F}_{\mathrm{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)$. Notice that providing this additional freedom does not trivialise the theory, since we have seen before that the agent cannot extract resources from these states for free. The single-resource theory $\mathrm{R}_{\mathrm{single}}$ that we obtain has a set of allowed operations defined as follow.

Definition 32 (Collapsed single-resource theory). *Consider the two-resource theory $\mathrm{R}_{\mathrm{multi}}$ with allowed operations $\mathcal{A}_{\mathrm{multi}}$ and invariant sets $\mathcal{F}_1$ and $\mathcal{F}_2$ which satisfy the properties F1, F2, and F3b. Consider the bank set $\mathcal{F}_{\mathrm{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)$ introduced in Eq. (3.29). We define the single-resource theory $\mathrm{R}_{\mathrm{single}}$ as that theory whose class of allowed operations $\mathcal{A}_{\mathrm{single}}$ is composed by the following three fundamental operations,*

1. Add an ancillary system described by $n \in \mathbb{N}$ copies of a bank state $\rho_P \in \mathcal{F}_{\mathrm{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)$.

2. Apply any operation $\varepsilon \in \mathcal{A}_{\mathrm{multi}}$ to system and ancilla.

3. Trace out the ancillary systems.

The most general operation in $\mathcal{A}_{\mathrm{single}}$ which does not change the number of systems between its input and output is

$$\varepsilon^{(\mathrm{s})}(\rho) = \mathrm{Tr}_{P^{(n)}}\left[\varepsilon\left(\rho \otimes \rho_P^{\otimes n}\right)\right], \tag{3.47}$$

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where we are partial tracing over the degrees of freedom $P^{(n)}$, that is, over the ancillary system initially in $\rho_P^{\otimes n}$.

The bank monotone associated with the bank set $\mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)$, see Eq. (3.31), is the unique quantifier for the single-resource theory R_{single}. In order to show the uniqueness of this monotone, we first have to show that the single-resource theory satisfies asymptotic equivalence.

Theorem 10. *Consider the two-resource theory R_{multi} with allowed operations $\mathcal{A}_{\text{multi}}$, and invariant sets $\mathcal{F}_1$ and $\mathcal{F}_2$ which satisfy the properties F1, F2, and F3b. Suppose the theory satisfies the asymptotic equivalence property with respect to the monotones $E_{\mathcal{F}_1}$ and $E_{\mathcal{F}_2}$. Then, given the subset of bank states $\mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)$, the single-resource theory R_{single} with allowed operations $\mathcal{A}_{\text{single}}$ satisfies the asymptotic equivalence property with respect to $f_{\text{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}$.*

The proof can be found in appendix B.1. We can now use the fact that the single-resource theory R_{single} satisfies asymptotic equivalence, and that the bank monotone satisfies the properties M2 – M7. Notice that property M1 is not required in this setting, since we are working with a single resource. Then, from Thm. 8 it follows that the bank monotone is the unique quantifier of the resource associated with R_{single}. As a result, we can understand property X1, which regulates the process of resource interconversion, as the demand that the resource associated with the bank states only changes infinitesimally during the process. In the case of thermodynamics, for example, this property states that the Helmholtz free energy F of the thermal bath only changes infinitesimally during the resource interconversion.

As a side remark, notice that both $E_{\mathcal{F}_1}$ and $E_{\mathcal{F}_2}$ are not monotonic under the set of allowed operations $\mathcal{A}_{\text{single}}$. This follows from the fact that we can now replace any state of the system with a state in $\mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)$, since we are free to add an ancillary system in such state, and to perform a swap between main system and ancilla (this operation belongs to $\mathcal{A}_{\text{multi}}$). Since the bank state always contains non-zero amounts of both resources, we have that there always exist states in $\mathcal{S}\left(\mathcal{H}\right)$ with lower value of either $E_{\mathcal{F}_1}$ or $E_{\mathcal{F}_2}$. Then, this operation would increase the value of one of the two monotones if applied to these states.

We can use the uniqueness of the resource quantifier for the single-resource theory R_{single} to show that the bank monotone coincides (modulo a multiplicative factor) with the relative

entropy distance from the set of bank states $\mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)$. In Prop. 13, in appendix B.4, we show that this relative entropy distance is monotonic under the set of allowed operations $\mathcal{A}_{\text{multi}}$. Furthermore, in order for the correspondence between these two monotones to hold, we need the set of bank states to contain a full-rank state, otherwise the relative entropy distance would be ill-defined on some states in $\mathcal{S}\left(\mathcal{H}\right)$. When the set of bank states has a full-rank state, we can use the same argument used in Sec. 3.2.3, to show that the relative entropy distance satisfies the same properties of the bank monotone, M2 – M7. Then, it follows from Thm. 8 that these two quantities coincide.

Corollary 2. *Consider the two-resource theory* R_{multi} *with allowed operations* $\mathcal{A}_{\text{multi}}$, *and invariant sets* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *which satisfy the properties F1, F2, and F3b. Suppose the theory satisfies the asymptotic equivalence property with respect to the monotones* $E_{\mathcal{F}_1}$ *and* $E_{\mathcal{F}_2}$. *If the subset of bank states* $\mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)$ *contains a full-rank state, then the bank monotone* $f_{\text{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}$ *coincides with the relative entropy distance from this subset of states, modulo a multiplicative constant.*

3.3.3 The first law of general resource theories

In this last section, we introduce the general first law for multi-resource theories, which applies to those scenarios in which the agent has access to the batteries, the bank, and the main system. In such a situation, the agent is able to perform a state transformation over the main system using variable amounts of resources, since they can trade one resource for another using the bank. This freedom is reflected in our formalism by the fact that, for the state transformation to be possible, a single equality, involving the different amounts of resources exchanged ΔW_i's, needs to be satisfied. Clearly, this is a less demanding constraint than the ones regulating state transformations when a bank is not available, see Eq. (3.15).

Using the tools developed in the previous sections we obtain the following corollary, which is a result of Thm. 9, regulating the process of resource interconversion, and of Cor. 2, linking the bank monotone to the relative entropy distance from the set of bank states.

Corollary 3. *Consider the two-resource theory* R_{multi} *with allowed operations* $\mathcal{A}_{\text{multi}}$, *and invariant sets* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *which satisfy the properties F1, F2, and F3b. Suppose the theory*

satisfies the asymptotic equivalence property with respect to the monotones $E_{\mathcal{F}_1}$ and $E_{\mathcal{F}_2}$, and that the global system is divided into a main system S, a bank described by the set of states $\mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)$ (which contains at least one full-rank state), and two batteries B_1 and B_2. Given any two states $\rho, \sigma \in \mathcal{S}\left(\mathcal{H}_S\right)$, we consider the asymptotic transformation

$$\rho \otimes \rho_P^{\otimes n} \otimes \omega_1 \otimes \omega_2 \xleftrightarrow{\text{asympt}} \sigma \otimes \tilde{\rho}_P^{\otimes n} \otimes \omega_1' \otimes \omega_2' \tag{3.48}$$

where $\omega_1, \omega_1' \in \mathcal{S}\left(\mathcal{H}_{B_1}\right)$ and $\omega_2, \omega_2' \in \mathcal{S}\left(\mathcal{H}_{B_2}\right)$, and we use $n \gg 1$ copies of the bank state $\rho_P \in \mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)$, which is modified by the operation according to condition X1. This transformation is possible if and only if the following condition is satisfied,

$$\alpha\,\Delta W_1 + \beta\,\Delta W_2 = E_{\mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)}(\rho) - E_{\mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)}(\sigma), \tag{3.49}$$

where each ΔW_i is defined as the difference in the monotone $E_{\mathcal{F}_i}$ over the final and initial state of the battery B_i, see Eq. (3.14), and $E_{\mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)}$ is the relative entropy distance from the set of states describing the bank.

We refer to Eq. (3.49) as the first law of multi-resource theories. For the resource theory of thermodynamics, where energy and information (or neg-entropy) are the two resources, and the bank is given by an infinite thermal reservoir with a given temperature T, this equation corresponds to the First Law of Thermodynamics. Indeed, in such scenario we have that $\Delta W_1 = \Delta E$ is the energy exchanged with the system, while $\Delta W_2 = \Delta I$ is the information exchanged with the system. Furthermore, since information and entropy are related one with the other, we have that $\Delta W_2 = -\Delta S$. Finally, the change in relative entropy distance on the main system is proportional to the change in Helmholtz free energy ΔF. The linear coefficients in the equation can be computed from Eq. (3.31), using our knowledge that the bank monotone is equal to the relative entropy distance from the thermal state with temperature T. It follows that $\alpha = T^{-1}$ and $\beta = 1$, and the relation in the above corollary become $\Delta F = \Delta E - T\Delta S$, that is, the First Law of Thermodynamics. This example is worked out in more details in the next chapter.

3.4 Chapter summary

In this chapter, we introduced a framework for general multi-resource theories, where the class of allowed operations is defined by considering the intersection of the sets of operations of different single-resource theories, see Eq. (3.2). An interesting feature we find is that, in theories with multiple constraints, there is a difference between the sets of free states and the invariant sets (in contrast with the case of single-resource theories), and a multi-resource theory can have multiple invariant sets and no free states, Fig. 3.1. With the framework derived in this chapter, we hope to find new resource theories for describing scenarios with multiple constraints and conservation laws, for example in many body-physics or in quantum computation.

We then focus on those theories which satisfy the asymptotic equivalence property of Def. 30, and we show that for these theories it is possible to introduce batteries which quantify the different resources exchanged during a state transformation over the system. Furthermore, we show that these theories are reversible, i.e., the resources spent in a forward state transformation are always re-gained during the backward transformation, and that there is a unique quantifier for each resource of the theory, Thm. 8. We know of multi-resource theories that satisfy asymptotic equivalence, for example the one we introduce in the next chapter. However, it would be interesting to study which of the other, already existing, multi-resource theories satisfy this property. Ultimately, one would hope to find some general conditions according to which a multi-resource theory is reversible, similarly to those shown in Thm. 2 of the background chapter.

We study a class of resource theories where it is possible to exchange between two resources, and we show that, in order to inter-convert one resource for another, the agent needs a bank system, Def. 31. When this additional system is available, exchanging resources at a specific rate (which depends on the state of the bank) is possible, see Thm. 9. Furthermore, we show that for a bank to act as a catalyst, i.e., to allow for multiple uses, its state needs not to change with respect to a specific distance, the relative entropy from the bank states, see Cor. 2. Our results apply to a restricted class of multi-resource theories, due to the assumption F3b we had to make. We are interested in finding ways to weaken this assumption, possibly by exploiting the closeness, under permutations, of the invariant sets of the theories – a property satisfied by several resource theories – and using de Finetti's theorems.

We find that resource interconversion is only possible when the invariant set of the theory

are disjoint, see the right panel of Fig. 3.1. It would be interesting to know whether it is possible to define interconversion for theories with a different structure of invariant sets, for instance by relaxing the assumptions made on the bank. One could consider bank states from which both resources could in principle be extracted, and forbid such extraction by further constraining the class of allowed operations.

Finally, using the notion of bank and batteries, we show that only a single relation needs to be satisfied in order to perform a state transformation, Cor. 3. This relation, which we call the first law for resource theories, connects the change in the property of the system – namely, the change in relative entropy distance from the bank states – to the sum of resources exchanged during the transformation. Notice that the results presented in this chapter on reversibility and interconversion of resources are only valid in the asymptotic limit, where many independent and identically distributed copies of a system are considered. However, the general framework we introduced to describe resource theories with multiple resources and batteries can also be applied to scenarios with a single system. Understanding how resources can be exchanged in the single-copy regime, and studying the corrections to the first law in such a regime are worthwhile questions to pursue.

Chapter 4

Thermodynamics as a multi-resource theory

In this chapter, we present a multi-resource theory for thermodynamics [169]. The theory describes physical scenarios in which the agent is acting over an isolated system, and therefore the operations they apply need to be both reversible (unitary) and energy-preserving. Since the isolated system under investigation is arbitrary, we can use this theory to study very general situations. For example, we can partition the isolated system into a main system and a generic environment. Then, if we consider the case in which the environment is thermal and its size is much larger than the size of the system, we recover the single-resource theory of thermodynamics with Thermal Operations. Otherwise, we can take the size of the environment to be comparable to the one of the main system, and in this way we can study the back-reaction experienced by the environment when it interacts with the system. Different kinds of environment can be studied with this multi-resource theory, such as those correlated with the main system, or those that are not thermal – see the next chapter for an example of equilibrium states which are athermal. Thus, our theory provides a general framework to study thermodynamics which encompasses, but is not limited to, the one of Thermal Operations.

In Sec. 4.1 we present the set of allowed operations, namely, unitary operations which preserve the energy of the system. We equip the agent with these operations since they are acting on an isolated system. Indeed, the principle of energy conservation implies that the

operations are energy-preserving, while the fact that the system is closed implies that these operations are unitary. In Sec. 4.2 we show that this theory satisfies the property of asymptotic equivalence, presented in the previous chapter, see Def. 30, with respect to the average energy and the von Neumann entropy (or equivalently, the information). Equipped with this result, we make use of the tools developed in the previous chapter to represent the state space with an *energy-information diagram*, see Sec. 4.3. Using the diagram, we can find which states can describe a bank, i.e., a system which allows us to exchange between energy and entropy, and we derive the interconversion relations, which are linked to Landauer's erasure and Maxwell's demon type of scenarios.

We then move, in Sec. 4.4, to consider two different resources, which are linked to energy and entropy, namely, *work* – a pure energy transfer, with no entropy transfer associated – and *heat* – an entropic energy transfer. We show how these resources are linked to energy and entropy, and we study their dependence on the size of the thermal reservoir the system is coupled to. We find that, for a thermal reservoir of infinite size, the amounts of work and heat exchanged during a state transformation coincide with the classical ones, while we obtain corrections to these quantities in the case in which the size of the reservoir is finite. Finally, we apply our results to the study of heat engines exchanging heat between two finite-sized thermal reservoirs.

4.1 Framework and allowed operations

Let us now introduce the allowed operations of this resource theory, which describes the scenario in which an agent is acting over an isolated system. Since the agent needs to preserve the energy of the system and to act reversibly over it, the class of operations of the theory can be obtained (as shown in Ch. 3) from the intersection of two sets of allowed operations, energy-preserving maps and unitary operations. The resulting set of allowed operations for our theory is therefore composed by every energy-preserving unitary operation acting over the system.

In more detail, we consider a finite-dimensional quantum system described by the Hilbert space $\mathcal{H}$, with $d = \dim \mathcal{H}$, whose Hamiltonian H is completely general. The set of allowed operations is obtained from the intersection between the set of unitary operations

$$\mathcal{A}_{\text{unit}} = \left\{ \varepsilon : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H}) \mid \varepsilon(\cdot) = U \cdot U^\dagger, \text{ where } UU^\dagger = U^\dagger U = \mathbb{I} \right\}, \qquad (4.1)$$

and the set of energy-preserving maps, given by those maps with Krauss operators which individually commute with the Hamiltonian of the system,

$$\mathcal{A}_{\text{ener}} = \left\{ \varepsilon : \mathcal{S}\left(\mathcal{H}\right) \to \mathcal{S}\left(\mathcal{H}\right) \mid \varepsilon(\cdot) = \sum_i K_i \cdot K_i^\dagger, \text{ where } \sum_i K_i^\dagger K_i = \mathbb{I} \text{ and } [K_i, H] = 0 \ \forall i \right\}.$$
(4.2)

It is worth noting that there exist other sets of operations which are equivalent to $\mathcal{A}_{\text{ener}}$, and are possibly more operational. For example, in Ref. [176] it was shown that the above set of operations coincides with the one composed by maps which preserve the probability distribution of the energy of a system.

It is then easy to show that the resulting set of allowed operations $\mathcal{A}_{\text{thermo}} = \mathcal{A}_{\text{unit}} \cap \mathcal{A}_{\text{ener}}$ is

$$\mathcal{A}_{\text{thermo}} = \left\{ \varepsilon : \mathcal{S}\left(\mathcal{H}\right) \to \mathcal{S}\left(\mathcal{H}\right) \mid \varepsilon(\cdot) = U \cdot U^\dagger, \text{ where } UU^\dagger = U^\dagger U = \mathbb{I} \text{ and } [U, H] = \mathbb{I} \right\}. \quad (4.3)$$

This class of operations preserves the value of both the von Neumann entropy of the system S and of the average energy E, that we define as

$$E(\rho) = \text{Tr}\left[H\rho\right]. \tag{4.4}$$

Clearly, these quantities are not the only ones to be conserved, since for example all Rényi entropies are conserved, as well as all energy momenta. Nevertheless, in the following section we show that, in the asymptotic limit, only these two quantities need to be preserved in order for a state transformation to be possible.

It is worth noting that both the class of unitary operations and the class of energy-preserving maps have an empty set of free states. However, we can look at these classes as being, respectively, a subset of the unital maps[1] and of the average-energy-non-increasing maps introduced in the previous chapter, Sec. 3.2.3. These bigger classes of operations define two single-resource theories with non-empty sets of free states, namely, the maximally-mixed state $\frac{\mathbb{I}}{d}$ for the former class, and the ground state of the Hamiltonian for the latter, which can be a pure state if the Hamiltonian is non-degenerate, or a set of mixed states otherwise.

In the following, we study the multi-resource theory equipped with $\mathcal{A}_{\text{thermo}}$, and we investigate asymptotic state transformations. We consider n i.i.d. copies of a state $\rho \in \mathcal{S}\left(\mathcal{H}\right)$, where

---

[1] A unital map is a CPTP map which preserve the identity, i.e., a map $\varepsilon : \mathcal{S}\left(\mathcal{H}\right) \to \mathcal{S}\left(\mathcal{H}\right)$ such that $\varepsilon(\mathbb{I}) = \mathbb{I}$.

$n \gg 1$, and allow the agent to act over these copies with global (allowed) operations. When we consider multiple copies of the system, we define the global Hamiltonian as

$$H_n = \sum_{j=1}^{n} H^{(j)} \tag{4.5}$$

where $H^{(j)} = \mathbb{I}_1 \otimes \ldots \mathbb{I}_{j-1} \otimes H \otimes \mathbb{I}_{j+1} \otimes \ldots \otimes \mathbb{I}_n$ is the single-copy Hamiltonian H applied to the j-th copy of the system. Notice that this Hamiltonian does not include any interaction term between different copies. The allowed operations of this multi-resource theory, when we consider n copies of the system, are defined as

$$\mathcal{A}_{\text{thermo}}^{(n)} = \left\{ \varepsilon_n : \mathcal{S}\left(\mathcal{H}^{\otimes n}\right) \to \mathcal{S}\left(\mathcal{H}^{\otimes n}\right) \mid \varepsilon_n(\cdot) = U \cdot U^\dagger, \text{ where } UU^\dagger = U^\dagger U = \mathbb{I} \text{ and } [U, H_n] = \mathbb{I} \right\}. \tag{4.6}$$

Again, these operations preserve both the von Neumann entropy and the average energy, which is extended to the many-copy case as $E(\cdot) = \text{Tr}\left[H_n \cdot\right]$.

The multi-resource theory we study in this chapter applies to scenarios in which the system is isolated, and thus it differs from the single-resource theory of Thermal Operations, due to the lack of an infinite thermal reservoir. However, if we divide the system under investigation into two partitions, one being the main system and the other the environment, we can study situations where the system interacts with either a finite- or infinite-sized thermal reservoir, or even with an athermal environment. Thus, the multi-resource theory here introduced represents a versatile tool for studying thermodynamics of systems which are coupled with different kinds of environment, see for example Ref. [168]. In this chapter, we focus on the case in which the environment is thermal and has a size comparable to the one of the main system, so that any interaction between the two causes a back-reaction on the environment. Finally, it is worth noting that our study concerns the asymptotic regime, and therefore the results we obtain hold for ideal systems composed of many identical and non-interacting particles, as it is evident from our choice of Hamiltonian H_n.

4.2   Microscopic and macroscopic states in thermodynamics

In this section we present the main result of the chapter, where we show that the multi-resource theory of thermodynamics satisfies the property of asymptotic equivalence, see Def. (30), with

respect to the von Neumann entropy S and the average energy E. This is shown in the following theorem, which is proved in the next section.

Theorem 11. *Consider a d-dimensional quantum system described by the Hilbert space $\mathcal{H}$, with Hamiltonian H. For states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$, the following are equivalent:*

- *The states have equal von Neumann entropy and average energy,*

$$S(\rho) = S(\sigma) \quad , \quad E(\rho) = E(\sigma). \tag{4.7}$$

- *There exist an energy-preserving unitary operation U, as well as a sub-linear ancillary system A described by a state η, with Hamiltonian H_A whose operator norm $\|H_A\| = o(n)$, such that*

$$\lim_{n \to \infty} \left\| \mathrm{Tr}_A \left[U(\rho^{\otimes n} \otimes \eta) U^\dagger \right] - \sigma^{\otimes n} \right\|_1 = 0. \tag{4.8}$$

Since the theory we are considering satisfies asymptotic equivalence, we can apply to it the results developed in the previous chapter. In particular, we can define batteries for both energy and entropy, which satisfy the property M1, and we can also represent the state space in a resource diagram, which we call the *energy-information diagram* – since entropy and information are two equivalent quantities, as we see in the next section.

Before analysing the consequences of asymptotic equivalence for our theory, let us briefly comment on the ancillary system. As we show in the proof of the theorem, the ancilla is essential for the theorem, as it absorbs sub-linear fluctuations in energy and entropy while the process maps ρ into σ. However, this system does not trivialise the theory, since its contribution per single copy of the main system is negligible as we take the asymptotic limit. The role of the ancilla is threefold. Part of this system is used to provide randomness, which allows us to modify the probability distribution of ρ into the one of σ. Another part is used as a register, to make the transformation reversible, and therefore implementable with unitary operations. Finally, the last part of the ancilla is used to absorb the energy fluctuations associated with the state transformation. When mapping from or to a state with coherence in the energy eigenbasis, this part of the ancilla also allows us to modify the coherence.

As a consequence of the above result, we find that in our theory we can (asymptotically) classify any quantum state solely in terms of energy and entropy. Such passage from the

quantum state ρ to the pair $(E(\rho), S(\rho))$ can be interpreted as the thermodynamic passage from microscopic to macroscopic states. Interestingly, our result seems to capture this passage, despite being obtained in the idealised scenario of non-interacting copies. Thus, we identify the many-copy limit that we take when considering asymptotic equivalence with the standard macroscopic limit of thermodynamics.

4.2.1 Proof of asymptotic equivalence in thermodynamics

We now prove Thm. 11 for the case in which the two states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ are diagonal in the energy eigenbasis. At the end of the section we additionally comment on the case in which the states have coherence. Notice that, unlike the other results shown in this part of the thesis, the following protocol has been derived in collaboration with Tobias Fritz, who contributed to this effort substantially.

Before proving the theorem, we introduce a lemma on coarse-graining maps acting on probability spaces, which is used in the main proof when we study the problem of asymptotically modifying the probability distribution of ρ into the one of σ. In this lemma we make use of the Rényi entropies defined in the background chapter, see Def. 21. Specifically, given a unit vector x with d positive elements, we have

$$H_\infty(x) = -\log \max_i x_i, \tag{4.9a}$$

$$H_0(x) = \log |\{x_i : x_i > 0\}|, \tag{4.9b}$$

$$H_{-\infty}(x) = -\log \min_i x_i, \tag{4.9c}$$

where $|\cdot|$ is the cardinality of the set. An important property of the above Rényi entropies, that we use in the proof of Thm. 11, is

$$H_\infty(x) \leq H_0(x) \leq H_{-\infty}(x). \tag{4.10}$$

Then, the lemma on coarse-graining maps states,

Lemma 6. *Consider two finite probability spaces (X, p) and (Y, q). Then there exists a coarse-graining map $f : X \to Y$ such that*

$$\|f_\star(p) - q\|_1 \leq 2^{H_0(q) - H_\infty(p)}, \tag{4.11}$$

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where $f_\star(p)$ is the probability distribution on Y obtained by applying the map f over the elements of X on which p is defined, and $\| \cdot \|_1$ is the total variation distance defined in Def. 5. Furthermore, we have that

$$\left| f^{-1}(y) \right| \leq 2^{H-\infty(p)} \left(2^{-H_\infty(q)} + 2^{-H_\infty(p)} \right), \quad \forall\, y \in Y, \tag{4.12}$$

where $\left| f^{-1}(y) \right|$ corresponds to the number of element $x \in X$ mapped into $y \in Y$ by the map f.

Proof. The coarse-graining map $f : X \to Y$ is defined as follow. Suppose we order the elements of the set X as $x_1, \ldots, x_d$, where d is the cardinality of X. We then define the map f by specifying its action over the ordered elements of X, from $j = 1$ to $j = d$. In particular, the function f acts on the j-th element by mapping it into an element $y \in Y$ such that

$$q_y > \sum_{x \in f_j^{-1}(y)} p_x, \tag{4.13}$$

where the set $f_j^{-1}(y) = f^{-1}(y) \cap \{x_1, \ldots, x_{j-1}\}$, i.e., it is composed of all those elements prior to x_j that have been mapped by f into y, see Fig. 4.1 for an example of such coarse-graining action.

From the defining property of the coarse-graining map f, Eq. (4.13), it follows that

$$\sum_{x \in f^{-1}(y)} p_x \leq q_y + \max_x p_x. \tag{4.14}$$

From this equation, it is easy to show that

$$\left| f^{-1}(y) \right| \min_x p_x \leq \max_y q_y + \max_x p_x, \tag{4.15}$$

which proves Eq. (4.12) in the lemma's thesis. Furthermore, if we consider the total variation distance between $f_\star(p)$ and q, we have that

$$\| f_\star(p) - q \|_1 = \sum_y \max \left\{ 0, \sum_{x \in f^{-1}(y)} p_x - q_y \right\} \leq \sum_y \max_x p_x = |Y| \max_x p_x, \tag{4.16}$$

where the inequality follows from Eq. (4.14). If we take Y to be equal to the support of q, so that $|Y| = |\{q_y : q_y > 0\}|$, we obtain the bound shown in Eq. (4.11) in the lemma's thesis. $\square$

We are now ready to prove Thm. 11 for states diagonal in the energy eigenbasis.
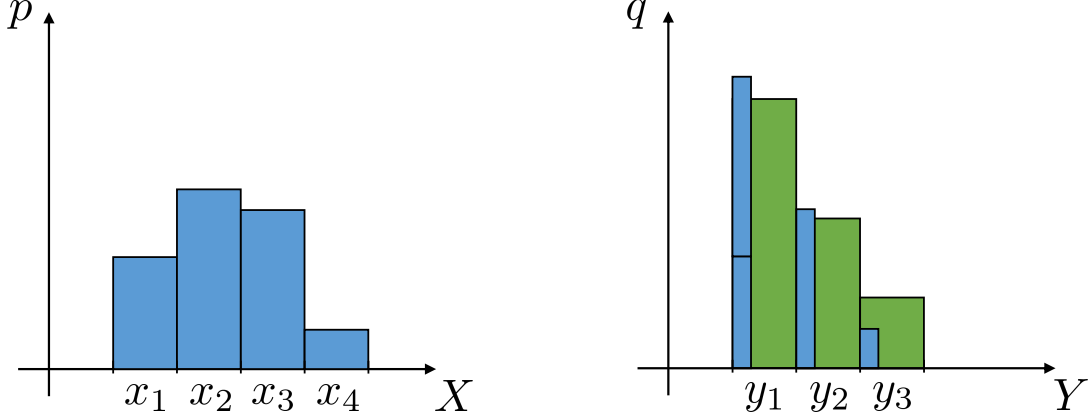
Figure 4.1: The action of a coarse-graining map f over the elements of X, and the resulting distribution $f_\star(p)$ on Y. **Left.** The initial probability distribution p over the element of X is represented in blue. **Right.** The target probability distribution q over the element of Y is represented in green, while the final probability distribution $f_\star(p)$ is represented in blue. The function f we use is such that it maps the elements x_1 and x_2 into y_1, since $p_{x_1} < q_{y_1}$ but $p_{x_1} + p_{x_1} > q_{y_1}$, the element x_3 into y_2, and the element x_4 into y_3. Notice that other coarse-graining choices could have been made.

Proof of Thm. 11. **a.** We start by proving the easy part of the theorem, i.e., that if a map of the form of Eq. (4.8) exists such that we can asymptotically map ρ into σ, then the two states have the same value of average energy and von Neumann entropy. Let us consider a map of the form $\varepsilon(\cdot) = \mathrm{Tr}_A \left[U \left(\cdot \otimes \eta \right) U^\dagger \right]$, where η is the arbitrary state of a sub-linear ancilla, with Hamiltonian H_A such that the operator norm is $\|H_A\| = o(n)$. Furthermore, the operator U is an energy preserving unitary operator, i.e., $[U, H_n + H_A] = 0$, where H_n is defined in Eq. (4.5).

We now introduce a generic function f satisfying the following properties,

1. The function f is sub-additive, i.e., given a state $\rho_{AB} \in \mathcal{S} \left(\mathcal{H}_A \otimes \mathcal{H}_B \right)$,

$$f(\rho_{AB}) \leq f(\mathrm{Tr}_B \left[\rho_{AB} \right]) + f(\mathrm{Tr}_A \left[\rho_{AB} \right]). \tag{4.17}$$

Compare this property with the less demanding M5 in previous chapter.

2. The function f is such that, given a state $\rho_{AB} \in \mathcal{S} \left(\mathcal{H}_A \otimes \mathcal{H}_B \right)$,

$$f(\mathrm{Tr}_B \left[\rho_{AB} \right]) - f(\mathrm{Tr}_A \left[\rho_{AB} \right]) \leq f(\rho_{AB}). \tag{4.18}$$

3. The function f is invariant under any energy-preserving unitary operation U,

$$f(U \rho U^\dagger) = f(\rho), \quad \forall \rho \in \mathcal{S}(\mathcal{H}). \tag{4.19}$$

4. The function f scales extensively with the number of copies of the system, i.e., given $\rho_n \in \mathcal{S}(\mathcal{H}^{\otimes n})$, we have that $f(\rho_n) = O(n)$. This is equivalent to property M6 of the previous chapter.

5. The function f is asymptotic continuous, see Def. (14). This property coincides with M7 in the previous chapter.

It is easy to show that both the von Neumann entropy S and the average energy E satisfy the properties listed above. In particular, the von Neumann entropy satisfies property 2, as it follows from the Araki-Lieb triangle inequality [177]. Furthermore, the entropy is asymptotic continuous, property 5, which follows from Fannes inequality [178]. The average energy trivially satisfies both properties 1 and 2, since this quantity is additive, i.e., Eq. (4.17) is saturated for all states $\rho_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ when $f = E$. Furthermore, the asymptotic continuity of the energy follows from Prop. 6, in Sec. B.2 of the appendix. We can group the properties 1 and 2 in a single equation,

$$f(\mathrm{Tr}_B [\rho_{AB}]) - f(\mathrm{Tr}_A [\rho_{AB}]) \leq f(\rho_{AB}) \leq f(\mathrm{Tr}_B [\rho_{AB}]) + f(\mathrm{Tr}_A [\rho_{AB}]). \tag{4.20}$$

For a function f satisfying the above properties, we can prove the following chain of inequalities,

$$
\begin{aligned}
\frac{1}{n} \left| f\left(\rho^{\otimes n}\right) - f\left(\sigma^{\otimes n}\right) \right| &\leq \frac{1}{n} \left| f\left(\rho^{\otimes n} \otimes \eta\right) - f\left(\sigma^{\otimes n}\right) \right| + \frac{f(\eta)}{n} \\
&= \frac{1}{n} \left| f\left(U\left(\rho^{\otimes n} \otimes \eta\right) U^\dagger\right) - f\left(\sigma^{\otimes n}\right) \right| + \frac{f(\eta)}{n} \\
&\leq \frac{1}{n} \left| f\left(\mathrm{Tr}_A\left[U\left(\rho^{\otimes n} \otimes \eta\right) U^\dagger\right]\right) - f\left(\sigma^{\otimes n}\right) \right| + \frac{f(\eta)}{n} + \frac{f(\tilde{\eta})}{n} \\
&= \frac{1}{n} \left| f\left(\varepsilon\left(\rho^{\otimes n}\right)\right) - f\left(\sigma^{\otimes n}\right) \right| + o(1). \tag{4.21}
\end{aligned}
$$

where $\tilde{\eta} = \mathrm{Tr}_S \left[U\left(\rho^{\otimes n} \otimes \eta\right) U^\dagger\right]$ is the state of the ancilla after we apply the energy-preserving unitary operation. The first and second inequalities follow from Eq. (4.20), the first equality follows from property 3, the second equality follows from property 4 and from the fact that

the ancillary system is sub-linear in the number of copies n. If we now use the asymptotic continuity of f, property 5, together with the fact that $\|\varepsilon(\rho^{\otimes n}) - \sigma^{\otimes n}\|_1 \to 0$ as $n \to \infty$ (which is our starting hypothesis), we find that

$$f^\infty(\rho) = f^\infty(\sigma), \tag{4.22}$$

where f^∞ is the regularised version of the function f, see Def. 13. Since both the von Neumann entropy and the average energy are extensive, i.e., $f(\rho^{\otimes n}) = nf(\rho)$, they coincide with their regularisation. As a result, we find that $E(\rho) = E(\sigma)$, and $S(\rho) = S(\sigma)$, which proves the first part of the theorem, and applies to both states with and without coherence in the energy eigenbasis.

b. We now prove that, given two diagonal states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ with same von Neumann entropy and average energy, there exists an allowed operation, i.e., an energy-preserving unitary operation, together with a sub-linear ancillary system, which asymptotically maps ρ into σ. Since we are working with n i.i.d copies of the system, we make use of the typical states ρ_{typ} and σ_{typ}, associated with $\rho^{\otimes n}$ and $\sigma^{\otimes n}$, respectively. Suppose that $\rho = \sum_{i=1}^{d} p_i |i\rangle \langle i|$, and $\sigma = \sum_{i=1}^{d} q_i |i\rangle \langle i|$, then their typical states are defined as

$$\rho_{\text{typ}} = \sum_{x \in T_\rho} p_x |x\rangle \langle x|, \tag{4.23a}$$

$$\sigma_{\text{typ}} = \sum_{y \in T_\sigma} q_y |y\rangle \langle y|, \tag{4.23b}$$

where the set T_ρ contains all the typical sequences $x = x_1, \ldots, x_n$, in which the value i occurs n_i times, where $n_i \in [(n - \delta)p_i, (n + \delta)p_i]$, and $\lim_{n \to \infty} \delta = \infty$, $\delta = o(n)$. The same applies for the sequences $y \in T_\sigma$, where the n_i's now depend on the distribution q. In appendix C, we show that the trace distance between the n-copy state $\rho^{\otimes n}$ and the typical state ρ_{typ} is arbitrarily close to 1 as $n \gg 1$, see Eq. (C.2). Thus, in the rest of the proof we focus on finding an energy-preserving unitary operation mapping the typical state ρ_{typ} into the typical state σ_{typ}, rather then considering the exact problem.

Given the state ρ_{typ} of Eq. (4.23a), we can focus on the probability distribution associated to it, $p_{\text{typ}} = \{p_x : x \in T_\rho\}$, and similarly for σ_{typ} we have the distribution $q_{\text{typ}} = \{q_x : x \in T_\sigma\}$. Using the properties of the Rényi entropies of Eq. (4.10) and the fact that the probabilities in

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the distribution p_{typ} are bounded in a specific range, see Eq. (C.6) in appendix, we can show that

$$(n - \delta)\, S(\rho) \leq H_\infty(p_{\text{typ}}) \leq H_0(p_{\text{typ}}) \leq H_{-\infty}(p_{\text{typ}}) \leq (n + \delta)\, S(\rho), \qquad (4.24)$$

where $\delta = o(n)$ is the same parameter introduced when we defined the range of the n_i's. Notice that the same chain of inequalities applies to q_{typ}, since our hypothesis is that $S(\rho) = S(\sigma)$.

As a first step in our protocol, we want to map p_{typ} into q_{typ}. To do so we use an additional ancillary system of $3\,\delta S(\rho)$ qubits, described by the state η_1 associated with the uniform distribution r_1, and a coarse-graining function f of the kind introduced in Lem. 6. The dimension of the ancilla we use is such that

$$\|f_\star\, (p_{\text{typ}} \otimes r_1) - q_{\text{typ}}\|_1 \leq 2^{H_0(q_{\text{typ}})-H_\infty(p_{\text{typ}} \otimes r_1)} = 2^{H_0(q_{\text{typ}})-H_\infty(p_{\text{typ}})-H_\infty(r_1)}$$

$$\leq 2^{-\delta S(\rho)} \qquad (4.25)$$

where the first inequality follows from Lem. 6, Eq. (4.11), the equality follows from the additivity of H_∞, and the last inequality follows from Eq. (4.24) and from the fact that r_1 is the uniform distribution. Notice that the above equation implies that the total variation distance between the two probability distributions tends to 1 as n tends to infinity, as desired.

However, the function f cannot be implemented with a unitary operation as it is, since, as shown in the proof of Lem. 6, it maps multiple x's into the same y. To make this function a bijection, we need to add an additional sub-linear ancilla in an pure state η_2, with associated (deterministic) distribution r_2, which serves as a register. Whenever we map two sequences x and x' into the same y, we map the register in a different state, so as to make f reversible. The number of qubits required by this second ancilla are upper-bounded by Eq. (4.12), since for all $y \in T_\sigma$ we have that

$$\left|f^{-1}(y)\right| \leq 2^{H_{-\infty}(p_{\text{typ}} \otimes r_1)} \left(2^{-H_\infty(q_{\text{typ}})} + 2^{-H_\infty(p_{\text{typ}} \otimes r_1)}\right) \leq 2^{5\delta\, S(\rho)} + 2^{2\delta\, S(\rho)}, \qquad (4.26)$$

where the second inequality follows from Eq. (4.24), from the additivity of H_∞ and $H_{-\infty}$, and from the fact that r_1 is the uniform distribution. Thus, it is easy to see that also the second ancillary system is composed by $o(n)$ qubits. When we act over the n copies of the system and the two ancillae, we can therefore implement a unitary operation U_f which acts over the probability distribution of ρ_{typ} as the function f, and therefore prepares the state σ_{typ} as desired.

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So far our considerations did not concern the energy associated with the x and y sequences. We now show that, in order to make U_f an energy-preserving unitary operation, we only need to add a third sub-linear ancillary system, whose Hamiltonian has a spectrum bounded from above and below by a sub-linear amount. Notice that, for simplicity, we ask the other two ancillary systems to have a fully-degenerate Hamiltonian, so that any energy change during the transformation implemented by U_f is due to a change on the main system. Recall that, for n copies, the global Hamiltonian of the system H_n is of the form given in Eq. (4.5), sum of the single-copy Hamiltonian $H = \sum_{i=1}^{d} E_i \ket{i}\bra{i}$ over each copy of the system. It is then easy to show that the average energy of a typical sequence is such that

$$(n - \delta)\, E(\rho) \le E(x) \le (n + \delta)\, E(\rho), \quad \forall\, x \in T_\rho, \tag{4.27}$$

where $E(x) = \sum_{i=1}^{d} n_i E_i$, and the bounds follow from the range of the n_i's. Notice that, since $E(\rho) = E(\sigma)$ in our hypothesis, we have that the same bounds apply to the average energy $E(y)$ of the typical sequences $y \in T_\sigma$. As a result, mapping a generic x into a generic y is followed by a change in average energy which is never higher that $2\delta\, E(\rho)$, an amount that is sub-linear in n. Thus, we can make the unitary operation U_f energy-preserving by dilating it so as to act over the previous systems and an additional one described by a pure state η_3, whose Hamiltonian H_3 is such that its operator norm $\|H_3\| = o(n)$. When the operation over the main system map a sequence x into a sequence y with different average energy, the difference is absorbed into the third ancilla.

To conclude, the protocol can be represented as a CPTP map $\varepsilon : \mathcal{S}(\mathcal{H}^{\otimes n}) \to \mathcal{S}(\mathcal{H}^{\otimes n})$ acting on the main system, defined as

$$\varepsilon(\cdot) = \mathrm{Tr}_A\left[\tilde{U}_f \left(\cdot \otimes \eta \right) \tilde{U}_f^\dagger \right], \tag{4.28}$$

where $\eta = \eta_1 \otimes \eta_2 \otimes \eta_3$ is the state of the sub-linear ancilla A, and $\tilde{U}_f$ is the energy-preserving unitary operation acting on system and ancilla, so that $\left[H_n + H_3, \tilde{U}_f \right] = 0$. This proves the second part of the theorem for states diagonal in the energy eigenbasis. $\square$

4.2.2   Discussion on the proof for states with coherence

We now prove, for the case of qubits, that Thm. 11 holds even when we consider states which have coherence on the energy eigenbasis. This proof has been first introduced, in the context of

thermodynamics, in Ref. [7], and is inspired by the work in entanglement theory on *entanglement spread* [179, 180, 181, 182]. The proof can be extended to qudits [169], where the sole additional step consists in showing that the ancillary system used to create/destroy coherence is still sub-linear in the number of copies of the main system. This involves some tools from combinatorics, knowns as Minkowski sums, see for example [183].

To prove the theorem, we simply show that it is possible to map a state with coherence into a state without coherence, while keeping fixed the entropy and average energy of the system. This map is achieved using a sub-linear ancillary system described by a large superposition of energy eigenstates – a reference frame, see Sec. 2.2.5 of the background material – and an energy-preserving unitary operation acting on the two systems. Since the transformation does not correlate ancilla and main system, we can use the same procedure (and the inverse unitary operation) to map a state without coherence into a state with coherence. If we combine this procedure with the one presented in the previous section, which allows us to change the spectrum of the state but not its entropy, we prove the theorem for general qubits states.

We consider a qubit system with Hamiltonian $H = E_0 \lvert 0 \rangle \langle 0 \rvert + E_1 \lvert 1 \rangle \langle 1 \rvert$, with $E_0 < E_1$, described by the state

$$\rho = p \lvert \phi_0 \rangle \langle \phi_0 \rvert + (1 - p) \lvert \phi_1 \rangle \langle \phi_1 \rvert, \tag{4.29}$$

where the two states $\lvert \phi_0 \rangle$ and $\lvert \phi_1 \rangle$ are obtained from a superposition of the eigenstates of the Hamiltonian, $\lvert 0 \rangle$ and $\lvert 1 \rangle$. In the limit of many copies $n \gg 1$, the state $\rho^{\otimes n}$ is indistinguishable from its typical state,

$$\rho_{\mathrm{typ}} = \sum_{t,g} p_t \lvert \psi_{t,g} \rangle \langle \psi_{t,g} \rvert, \tag{4.30}$$

where, for convenience, we express the typical state in a slightly different way than that used in the previous section, Eq. (4.23a). To compare the two, notice that $x = (t, g)$, where t labels the type of the state, and g is the degeneracy label. A state labelled by t contains n_t copies of $\lvert \phi_0 \rangle$, and $n - n_t$ copies of $\lvert \phi_1 \rangle$, where $n_t \in [n\,p - \sqrt{n}, n\,p + \sqrt{n}]$. This typical range coincides with the one of the previous section, when we chose $\delta \propto \sqrt{n}$. The degeneracy label g is associated with the permutations that we can perform on the copies of the states $\lvert \phi_0 \rangle$ and $\lvert \phi_1 \rangle$, so that we can re-express the state as

$$\lvert \psi_{t,g} \rangle = P_{\pi_g} \lvert \phi_0 \rangle^{\otimes n_t} \otimes \lvert \phi_1 \rangle^{\otimes n - n_t} \tag{4.31}$$

where π_g is an element of the symmetric group S_n, and P_{π_g} acts over the system as shown in Eq. (1.19) of the background chapter.

Let us now focus on the state $|\psi_{t,g}\rangle$, for a fixed value of t and g. It is easy to show, using the central limit theorem, that the squared amplitudes of this state cluster around the eigenstates of H_n whose energy is close to

$$\bar{E}_t = n_t \langle \phi_0 | H | \phi_0 \rangle + (n - n_t) \langle \phi_1 | H | \phi_1 \rangle , \tag{4.32}$$

which is the average energy of the state $|\psi_{t,g}\rangle$. It is worth noting that $\bar{E}_t$ might not be an eigenvalue of the n-copy Hamiltonian H_n, but we can always find an energy eigenvalue E_t such that $|\bar{E}_t - E_t| < \Delta E$, where $\Delta E = E_1 - E_0$ is the energy gap of the single-system Hamiltonian H. More formally we have that, in the limit of $n \gg 1$, the state $|\psi_{t,g}\rangle$ is indistinguishable from the typical state

$$|\psi_{t,g}^{\text{typ}}\rangle = \sum_{\ell,s} c_{\ell,s}^{t,g} |E_\ell, s\rangle , \tag{4.33}$$

where the state $|E_\ell, s\rangle$ is an eigenstate of H_n associated with the eigenvalue E_ℓ, and s is a degeneracy index. In particular, for a fixed value of t we have that the energy E_ℓ takes values in the range $[\bar{E}_t - O(\sqrt{n}), \bar{E}_t - O(\sqrt{n})]$. Furthermore, it is worth noting that the energy difference between any two values E_ℓ and $E_{\ell'}$ is a multiple of ΔE. The size of the energy range together with the discrete nature of the values of E_ℓ play a fundamental role in the sub-linearity of the ancillary system we are going to introduce.

The unitary operation U_S we want to implement over the n copies of the system is such that, on the typical states $|\psi_{t,g}^{\text{typ}}\rangle$'s, it acts as

$$U_S |\psi_{t,g}^{\text{typ}}\rangle = |E_t, g\rangle , \quad \forall t, g, \tag{4.34}$$

and we extend it to act on the remaining Hilbert space in an arbitrary way. Notice that $|E_t, g\rangle$ is one of the eigenstates of H_n whose associated eigenvalue E_t is such that $|\bar{E}_t - E_t| < \Delta E$. We will later show that, even if the unitary operation U_S is changing the average energy of each state $|\psi_{t,g}^{\text{typ}}\rangle$, the average energy per single copy of the system is only infinitesimally modified. The unitary operation U_S does not commute with H_n, since it maps states with coherence in the energy eigenbasis into eigenstates of the Hamiltonian, see Eq. (4.34). To make this unitary operation energy-preserving, we follow the steps presented in Sec. 2.2.5 of the background

chapter. First, we introduce an ancillary system with Hamiltonian

$$H_A = \sum_{h=-\frac{3}{2}H}^{\frac{3}{2}H} h\,\Delta E\,|h\rangle\,\langle h|\,, \tag{4.35}$$

described by a large superposition of energy eigenstates,

$$|H\rangle = \frac{1}{\sqrt{H}} \sum_{h=-\frac{1}{2}H}^{\frac{1}{2}H} |h\rangle\,. \tag{4.36}$$

This state is a reference frame, and allows us to address and modify the coherence on the main system. Let us now introduce an isometry, acting on system and reference frame,

$$U_{SA} = \sum_{t',g',\ell',s',h} |E_{t'},g'\rangle\,\langle E_{t'},g'|\,U_S\,|E_{\ell'},s'\rangle\,\langle E_{\ell'},s'|\otimes|h+E_{\ell'}-E_{t'}\rangle\,\langle h|\,, \tag{4.37}$$

where it is easy to show that this operation commutes with the global Hamiltonian $H_n + H_A$. We can extend the isometry to act on the remaining part of the Hilbert space of system and ancilla in an arbitrary way, so as to obtain an energy-preserving unitary operation $\tilde{U}_{SA}$. It is worth noting that this operation modifies the energy of the ancilla by at most $\pm O(\sqrt{n})$, as it follows from the range of values of E_ℓ introduced above. Furthermore, the energy of the ancilla is always modified by multiple of ΔE. As we are going to show, this facts allow us to compute the dimension H of the reference frame.

We now want to prove that, with the help of the ancillary system and of the unitary operation $\tilde{U}_{SA}$, it is possible to map the initial state ρ_{typ} of Eq. (4.30) into a state which is diagonal in the energy eigenbasis, and has the same spectrum,

$$\rho_{\mathrm{typ}}^{\mathrm{diag}} = \sum_{t,g} p_t\,|E_t,g\rangle\,\langle E_t,g|\,. \tag{4.38}$$

Furthermore, the operation we apply on system and ancilla leaves the state of the latter almost unchanged. We prove the above by showing that the following distance, between the state obtained by the protocol and the target state,

$$\delta_{\mathrm{target}} = \left\| \tilde{U}_{SA}\left(\rho_{\mathrm{typ}}\otimes|H\rangle\,\langle H|\right)\tilde{U}_{SA}^\dagger - \rho_{\mathrm{typ}}^{\mathrm{diag}}\otimes|H\rangle\,\langle H| \right\|_1\,, \tag{4.39}$$

tends to zero as n tends to infinity, for an appropriate choice of the size H of the ancilla. Notice that we can re-arrange the above equation using the fact that the initial and final states have

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the same spectrum, and that the trace norm satisfies the triangle inequality, obtaining

$$
\begin{aligned}
\delta_{\text{target}} &= \left\| \sum_{t,g} p_t \left(\tilde{U}_{SA} \, |\psi_{t,g}^{\text{typ}}\rangle \otimes |H\rangle \, \langle\psi_{t,g}^{\text{typ}}| \otimes \langle H| \, \tilde{U}_{SA}^\dagger - |E_t,g\rangle \otimes |H\rangle \, \langle E_t,g| \otimes \langle H| \right) \right\|_1 \\
&\leq \sum_{t,g} p_t \left\| \tilde{U}_{SA} \, |\psi_{t,g}^{\text{typ}}\rangle \otimes |H\rangle \, \langle\psi_{t,g}^{\text{typ}}| \otimes \langle H| \, \tilde{U}_{SA}^\dagger - |E_t,g\rangle \otimes |H\rangle \, \langle E_t,g| \otimes \langle H| \right\|_1 \\
&= \sum_{t,g} p_t \sqrt{1 - \left| \langle E_t,g| \otimes \langle H| \, \tilde{U}_{SA} \, |\psi_{t,g}^{\text{typ}}\rangle \otimes |H\rangle \right|^2},
\end{aligned}
\tag{4.40}
$$

where the last equality follows from the fact that the trace norm can be expressed in terms of the fidelity, so that $\| |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi| \|_1 = \sqrt{1 - |\langle\psi|\phi\rangle|^2}$.

Let us now focus on the overlap between the state we obtain with our protocol and the target state,

$$
\langle E_t,g| \otimes \langle H| \, \tilde{U}_{SA} \, |\psi_{t,g}^{\text{typ}}\rangle \otimes |H\rangle
$$

$$
= \langle E_t,g| \otimes \langle H| \left(\sum_{t',g',\ell',s',h} |E_{t'},g'\rangle \, \langle E_{t'},g'| \, U_S \, |E_{\ell'},s'\rangle \, \langle E_{\ell'},s'| \otimes |h + E_{\ell'} - E_{t'}\rangle \, \langle h| \right) |\psi_{t,g}^{\text{typ}}\rangle \otimes |H\rangle
$$

$$
= \sum_{\ell,s} c_{\ell,s}^{t,g} \, \langle E_t,g| \, U_S \, |E_\ell,s\rangle \times \left(\frac{1}{\sqrt{H}} \sum_{h'} \langle h'| \right) \left(\frac{1}{\sqrt{H}} \sum_{h} |h + E_\ell - E_t\rangle \right).
\tag{4.41}
$$

Notice that, from the definition of U_S, Eq. (4.34), and the definition of $|\psi_{t,g}^{\text{typ}}\rangle$, Eq. (4.33), it follows that

$$
(c_{\ell,s}^{t,g})^\star = \langle E_t,g| \, U_S \, |E_\ell,s\rangle.
\tag{4.42}
$$

Therefore, we find that

$$
\begin{aligned}
\left| \langle E_t,g| \otimes \langle H| \, \tilde{U}_{SA} \, |\psi_{t,g}^{\text{typ}}\rangle \otimes |H\rangle \right| &= \left| \sum_{\ell,s} |c_{\ell,s}^{t,g}|^2 \left(\frac{1}{\sqrt{H}} \sum_{h'} \langle h'| \right) \left(\frac{1}{\sqrt{H}} \sum_{h} |h + E_\ell - E_t\rangle \right) \right| \\
&= \sum_{\ell,s} |c_{\ell,s}^{t,g}|^2 \left| \left(\frac{1}{\sqrt{H}} \sum_{h'} \langle h'| \right) \left(\frac{1}{\sqrt{H}} \sum_{h} |h + E_\ell - E_t\rangle \right) \right|,
\end{aligned}
\tag{4.43}
$$

where the last equality follows from the fact that all terms in the sum are positive. We can now focus on the overlap between the initial and final state of the reference frame. Since the unitary operation maps between energy levels with an energy difference of $O(\sqrt{n})$, it is easy to

see that this overlap is

$$\left| \left(\frac{1}{\sqrt{H}} \sum_{h'} \langle h'| \right) \left(\frac{1}{\sqrt{H}} \sum_{h} |h + E_\ell - E_t\rangle \right) \right| \geq \frac{H - O(\sqrt{n})}{H}, \tag{4.44}$$

see Fig. 2.2 in the background chapter for a visualisation of the overlap. As a result, we find that

$$\left| \langle E_t, g| \otimes \langle H| \tilde{U}_{SA} |\psi_{t,g}^{\text{typ}}\rangle \otimes |H\rangle \right| \geq \frac{H - O(\sqrt{n})}{H} \sum_{\ell,s} |c_{\ell,s}^{t,g}|^2 = \frac{H - O(\sqrt{n})}{H}. \tag{4.45}$$

In order for the overlap to tend to one as n tends to infinity, we can take the number of states H in the superposition to scale with a higher power than $1/2$. For example, if we take $H = n^{\frac{2}{3}}$, we find that the overlap is $1 - n^{-\frac{1}{6}}$. Thus, we can now turn to the trace distance between the state obtained through our protocol and the target state, where it is easy to show that $\delta_{\text{target}} \leq O(n^{-\frac{1}{12}})$, that is, the protocol maps the state of the system into a state which is indistinguishable, in the limit of $n \to \infty$, from the diagonal one which we wanted to achieve, $\rho_{\text{typ}}^{\text{diag}}$.

Now we can show that the unitary operation U_S, defined in Eq. (4.34), does not change the average energy per single copy of the system, when n tends to infinity. To do so, we compute the energy difference per single copy between the initial (typical) state ρ_{typ}, see Eq. (4.30), and the final (typical) state $\rho_{\text{typ}}^{\text{diag}}$, see Eq. (4.38). This difference is given by

$$\begin{aligned}
\frac{1}{n} \left| E\left(\rho_{\text{typ}}\right) - E\left(\rho_{\text{typ}}^{\text{diag}}\right) \right| &= \frac{1}{n} \left| \sum_{t,g} p_t \big(E\left(|\psi_{t,g}\rangle \langle \psi_{t,g}|\right) - E\left(|E_t, g\rangle \langle E_t, g|\right) \big) \right| \\
&\leq \frac{1}{n} \sum_{t,g} p_t \left| E\left(|\psi_{t,g}\rangle \langle \psi_{t,g}|\right) - E\left(|E_t, g\rangle \langle E_t, g|\right) \right| \\
&= \frac{1}{n} \sum_{t,g} p_t \left| \bar{E}_t - E_t \right| \leq \frac{1}{n} \sum_{t,g} p_t \Delta E = \frac{\Delta E}{n}, \tag{4.46}
\end{aligned}$$

where the last inequality follows from the fact that we have chosen each E_t to be the closest eigenvalue to the average energy $\bar{E}_t$, defined in Eq. (4.32). Thus, when n tends to infinity we find that, per single copy, the difference between the average energy of ρ_{typ} and $\rho_{\text{typ}}^{\text{diag}}$ tends to zero.

The above protocol allows us to map a state with coherence into another one with the same spectrum and average energy, using an ancilla described by a large (but still sub-linear)

superposition over the energy eigenstates of H_A, where the operator norm of the Hamiltonian is $\|H_A\| = O(n^{\frac{2}{3}})$, compare with Eq. (4.35) when $H = n^{\frac{2}{3}}$. Since the final state of this ancilla is infinitesimally changed, we can use the inverse of the unitary operation presented in Eq. (4.37) over system and ancilla in order to perform the reverse transformation. Thus, by combining this procedure with the one presented in the previous section, we are able to asymptotically map between any two states with same energy and entropy, using energy-preserving unitary operations and sub-linear ancillae.

4.3   The energy-information diagram

In the previous chapter we showed that, when the theory satisfies asymptotic equivalence, and the monotones are well-behaved in the sense of properties M1 – M7, then we can uniquely represent the whole state space in a resource diagram, see Sec. (3.2.2). In this section, we show the form of the diagram for the multi-resource theory of thermodynamics, we introduce batteries for both energy and entropy, and we consider the problem of resource interconversion.

Before describing the state space, let us introduce the following quantity, which we refer to as *information*, or neg-entropy,

$$I(\rho) = \log d - S(\rho), \tag{4.47}$$

where the state $\rho \in \mathcal{S}(\mathcal{H})$, and $d = \dim \mathcal{H}$. This quantity is equal to zero for the maximally-mixed state $\frac{\mathbb{I}}{d}$, and takes the maximum value $\log d$ over any pure state $|\psi\rangle \in \mathcal{H}$. Clearly, one can replace, in Thm. 11, the von Neumann entropy S with the information I, since these two quantities are interchangeable. Thus, for the resource diagram of this theory, we use information in place of entropy, so as to simplify the comparison with the diagrams presented in the previous chapter. Finally, notice that information I is also given by the relative entropy distance from the maximally-mixed state, $I(\rho) = E_{\frac{\mathbb{I}}{d}}(\rho) = D(\rho \| \frac{\mathbb{I}}{d})$. The state $\frac{\mathbb{I}}{d}$ is the fixed state for the class of unital maps, of which unitary operations are a subset.

Let us first identify the boundary of the state space with respect to energy and information. We know that $I \in [0, \log d]$, and its maximum value is reached on the pure states. Furthermore, the energy E varies over the set of pure states, and we can achieve any value of energy between the lowest value $E_{\min}$ and the highest one, $E_{\max}$. To do so, we just need to construct a suitable
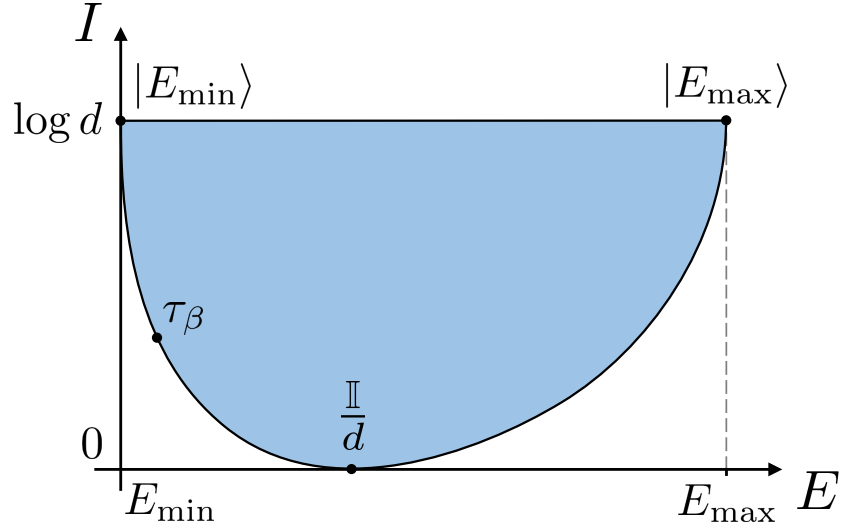
Figure 4.2: We represent the state space of a quantum system with a non-degenerate Hamiltonian H. The state-state space $\mathcal{S}(\mathcal{H})$ is represented by the blue region, and each point is an equivalence class of states labelled by average energy E and information I, i.e., a macroscopic state. The straight segment of points with maximum information is the set of pure states $|\psi\rangle \in \mathcal{S}(\mathcal{H})$, while the curve of points with minimum information (for fixed energy) is given by the set of thermal states τ_β with positive and negative inverse temperature β. Thermal states with a positive temperature are such that the ground state has the highest population, while the excited states are less and less populated as their energy increases. On the contrary, for thermal states with a negative temperature the populations are inverted, so that the maximally-excited state is highly populated, and the population of the other levels decreases with energy. A single state has zero information associated with it, the maximally-mixed state $\frac{\mathbb{I}}{d}$. Likewise, since the Hamiltonian H is here non-degenerate, we have a single state with minimum energy, the ground state, and a single one with maximum energy, the maximally excited state.

superposition between the ground state of the Hamiltonian and the most-excited state. Then, the set of pure states is represented by a straight line in the diagram, which is one of the boundaries of the state space. The other boundary is obtained by minimising the information (or maximising the entropy) for fixed energy. This is a well-known problem in thermodynamics, whose solution is given by the set of Gibbs states of the Hamiltonian H at positive and negative

temperatures. Thus, the set of thermal states $\tau_\beta = e^{-\beta H}/Z$ with $\beta \in \mathbb{R}$ is represented as a curve in the energy-information diagram which provides the other boundary for the state space. In between these boundaries, any point represents one or more states, see Fig. 4.2.

We now turn to the problem of building batteries for energy and information, where the main requirement for these systems is to satisfy property M1, i.e., each battery needs to store a single kind of resource. With the aid of the diagram, we can easily pinpoint the useful states for building batteries. Let us consider the energy battery B_E first. We build it using $m \gg 1$ copies of the system, where k copies ($k < m$) are initialised in the ground state $|E_{\min}\rangle$, and $m - k$ copies are in the most-excited state $|E_{\max}\rangle$. We define this battery state as

$$\omega_E(k) = |E_{\min}\rangle \langle E_{\min}|^{\otimes k} \otimes |E_{\max}\rangle \langle E_{\max}|^{\otimes m-k} . \tag{4.48}$$

Storing energy in the battery corresponds to decreasing the number k of ground states, and consequently increasing the number of most-excited states. On the other hand, extracting energy from the battery corresponds to increasing the number of ground states, and decreasing the most-excited states. Clearly, this battery is able to store energy, but its entropy (or information) is always fixed, so that this quantity is not stored by the battery.

Similarly, we can construct an information battery B_I. This is built out of $\ell \gg 1$ copies of the system. The state of the battery is composed of $h < \ell$ maximally-mixed states $\frac{\mathbb{I}}{d}$, and $\ell - h$ pure states $|\psi\rangle$ with the same average energy of $\frac{\mathbb{I}}{d}$; this is always possible, since it is sufficient to take a suitable superposition of ground state and most-excited state. The state of the information battery is therefore

$$\omega_I(h) = \frac{\mathbb{I}}{d}^{\otimes h} \otimes |\psi\rangle \langle \psi|^{\otimes \ell-h} . \tag{4.49}$$

Thus, storing information in the battery corresponds to decreasing the number of maximally-mixed states h, and extracting information corresponds to increasing this number.

We can now consider a state transformation between two states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ associated with different macroscopic states. To do so, we follow the procedure shown in Sec. 3.2.2 of the previous chapter, where it is shown how resources can be quantified when using batteries. We consider the main system S, and the two batteries B_E and B_I. The asymptotic state transformation we are interested in is

$$\rho^{\otimes n} \otimes \omega_E(k) \otimes \omega_I(h) \xleftrightarrow{\text{asympt}} \sigma^{\otimes n} \otimes \omega_E(k') \otimes \omega_I(h'), \tag{4.50}$$

which is possible, according to Thm. 11, if and only if the energy and the information in the global system (main system and batteries) are preserved. Recall that the amount of energy exchanged by the battery during the transformation is given by, see Eq. (3.14) in the previous chapter,

$$\Delta W_E := E\left(\omega_E(k')\right) - E\left(\omega_E(k)\right) = \left(k - k'\right)\left(E_{\max} - E_{\min}\right), \tag{4.51}$$

while the amount of information exchanged is defined as

$$\Delta W_I := I\left(\omega_I(h')\right) - I\left(\omega_I(h)\right) = \left(h - h'\right)\log d. \tag{4.52}$$

Per single-copy of the main system, we find that the amount of energy and information exchanged to map ρ into σ is given by, respectively,

$$\Delta W_E(\rho \to \sigma) = \frac{1}{n}\Delta W_E = \frac{k - k'}{n}\left(E_{\max} - E_{\min}\right) = E(\rho) - E(\sigma), \tag{4.53a}$$

$$\Delta W_I(\rho \to \sigma) = \frac{1}{n}\Delta W_I = \frac{h - h'}{n}\log d = I(\rho) - I(\sigma), \tag{4.53b}$$

where, clearly, we can satisfy the second equality in each line up to an infinitesimal error if we use $n, m, l \in \mathbb{N}$ big enough.

Before we move to study the interconversion relations for this theory, we briefly comment on the physical interpretation of the batteries presented in this section. The energy battery, which resemble the one introduced in the background chapter on Thermal Operations, see Sec. 2.4, can be thought as the (microscopic) equivalent of a weight, or a collection of weights, in a gravitational field. When we store energy in this battery, we lift some of these weights, increasing the potential energy of the battery. To extract this energy, instead, we lower some of the weights. Likewise, the information battery can be understood as a memory system, where the pure states provide information which can be accessed and used, while the maximally-mixed states describe that part of the memory which has been used, and is now useless. In practical thermodynamic applications, it is usually the case that we can realise the first kind of battery, the one storing energy, while the entropy or information is exchanged using a thermal reservoir. The information exchanged with the thermal reservoir comes at an energy cost, which is usually referred to as *heat*.

In the next section, we consider the case in which the agent has access to both an energy and an information battery, and we show that the thermal reservoir plays the role of the bank,

allowing us to exchange between resources. In Sec. 4.4, instead, we assume that the agent has not access to an entropy battery, and that they can only use an energy battery and a thermal reservoir. In this second case, we see that other kinds of resources, known as *work* and *heat*, arise.

4.3.1 Trading energy and entropy

Suppose that, in our theory, the agent has access to the energy and information batteries introduced in the previous section. We are now interested in studying the interconversion of these two resources, which can be realised only if an additional system, the bank, is introduced, see Ch. 3 for further details. In particular, we need to identify the set of states which can be used to describe the bank. However, it is worth noting that in the current formulation of the multi-resource theory, there seems to be some arbitrariness in the definition of this set, which is composed of those states with the minimum possible values of both resources, see Eq. (3.28) in the previous chapter. Indeed, the operations the agent can use always preserve energy and information, so that storing and extracting any one of them seem to be equally important. For example, we have already noticed that, in this theory, entropy and information are equivalent, although they actually quantify the same resource in opposite way.

In order to unambiguously define the bank system, we now consider a multi-resource theory whose allowed operations are a superset of $\mathcal{A}_{\text{thermo}}$. This theory describes the situation in which the agent is acting on a closed system, and has a coarse-grained control over the unitary operations they can perform. Thus, the operations they implement are described by mixtures of unitary operations, a strict subset of the unital maps. Furthermore, we assume that the system can exchange energy (but no other quantities) with an ancillary system which acts as an energy sink, so that the operations the agent performs cannot increase the energy of the system. As a result, the allowed operations we consider in this section are given by the intersection between the set of mixtures of unitary operations, and the set of average-energy-not-increasing maps, introduced in Sec. 3.2.3 of the previous chapter. Within this new resource theory, energy and information are the resources required to perform state transformations, since they never increase under the allowed operations. Clearly, this resource theory satisfies the asymptotic equivalence property, since the class of allowed operations is larger than the

one of energy-preserving unitary operations. Then the bank states are the ones with minimum values of information and energy, and it is easy to show, for example by using the method of Lagrange multipliers, that this set is given by the Gibbs state of the system Hamiltonian at a positive temperature,

$$\mathcal{F}_{\text{bank}} = \left\{ \tau_\beta \in \mathcal{S}\left(\mathcal{H}\right) \ | \ \tau_\beta = \frac{e^{-\beta H}}{Z}, \quad \forall \beta \in \mathbb{R}^+ \right\}. \tag{4.54}$$

The inverse temperature β is a continuous label for this set, and for each fix value of this parameter we obtain a different bank, with a different exchange rate.

Following the procedure in Sec. 3.3.1 of the previous chapter, we can now define the bank monotone associated with the state τ_β. We can use the fact that the bank monotone is represented, in the energy-information diagram, as a tangent to the state space in the point associated with τ_β, see Fig. 4.3. We recall that the von Neumann entropy of a thermal state can be expressed as $S(\tau_\beta) = \beta\, E(\tau_\beta) + \log Z$. Using this information, is is easy to show that

$$\frac{\mathrm{d}I}{\mathrm{d}\beta} = \beta \langle \Delta^2 H \rangle_\beta, \tag{4.55a}$$

$$\frac{\mathrm{d}E}{\mathrm{d}\beta} = -\langle \Delta^2 H \rangle_\beta, \tag{4.55b}$$

where $\langle \Delta^2 H \rangle_\beta = \mathrm{Tr}\left[H^2\, \tau_\beta \right] - \left(\mathrm{Tr}\left[H\, \tau_\beta \right] \right)^2$ is the variance of the energy over the system. The linear coefficient of the tangent line we are interested in is then given by

$$\frac{\mathrm{d}I}{\mathrm{d}E} = \frac{\mathrm{d}I}{\mathrm{d}\beta} \left(\frac{\mathrm{d}E}{\mathrm{d}\beta} \right)^{-1} = -\beta. \tag{4.56}$$

The absolute value of this coefficient gives the exchange rate at which the agent can inter-convert the resources. Furthermore, with this linear coefficient we can define the bank monotone,

$$f^\beta_{\text{bank}}(\rho) = (E(\rho) - E(\tau_\beta)) + \beta^{-1}\left(I(\rho) - I(\tau_\beta) \right) = F(\rho) - F(\tau_\beta), \tag{4.57}$$

where the last equality follows from the definition of Helmholtz free energy.

Resource interconversion is obtained, in this theory, through the following transformation,

$$\tau_\beta^{\otimes n} \otimes \omega_E(k) \otimes \omega_I(h) \xleftrightarrow{\text{asympt}} \tilde{\tau}_\beta^{\otimes n} \otimes \omega_E(k') \otimes \omega_I(h'), \tag{4.58}$$

where $\Delta k = k - k'$ and $\Delta h = h - h'$ are finite, while we send the number of copies of the bank state, n, to infinity, so that $\left| f^\beta_{\text{bank}}(\tau_\beta) - f^\beta_{\text{bank}}(\tilde{\tau}_\beta) \right| \to 0$, which is required by condition X1.

Figure 4.3: The state space $\mathcal{S}(\mathcal{H})$ is represented in the energy-information diagram (blue region), together with the line (in red) tangent to this convex region in the point associated with the bank state τ_β. The tangent line represent the set of states with bank monotone equal to zero. The bank monotone for the thermodynamic theory is proportional to the Helmholtz free energy, see Eq. (4.57), which is indeed a linear combination of the two monotones we are considering, energy and information. The absolute value of the linear coefficient of the line is the exchange rate at which the agent can inter-convert the resources, and from the figure it is clear that this rate changes when we change the state of the bank to be at a different temperature – since the tangent line in the new point as a different slope.

From the transformation of Eq. (4.58), we find that the interconversion relation of this theory is given by

$$\Delta W_E = -\beta^{-1}\,\Delta W_I. \tag{4.59}$$

This equation regulates the amounts of resources that the agent can exchange using a bank. As expected, the agent needs to provide one resource in order to extract the other, and the exchange rate is given by the inverse temperature, i.e., by the linear coefficient of the tangent line in Fig. 4.3. As already mentioned, examples of resource interconversion are Landauer's erasure, where energy is traded with a thermal reservoir in order to gain neg-entropy, which in turn is used to erase the state of an unknown bit, and the Maxwell demon, where information

is traded with the reservoir in order to extract energy.

Let us now consider the relative entropy distance from the bank state. It is easy to show that Cor. 2 is satisfied in the multi-resource theory we are studying, and therefore the bank monotone coincides (modulo a multiplicative constant) with the relative entropy distance from τ_β, which is

$$E_{\tau_\beta}(\rho) = D(\rho \,\|\, \tau_\beta) = \beta \left(E(\rho) - \beta^{-1} S(\rho) + \beta^{-1} \log Z \right) = \beta \, f_{\text{bank}}^\beta(\rho). \tag{4.60}$$

This monotone is a measure of athermality for states, meaning that condition X1 introduced in the previous chapter corresponds, in this multi-resource theory, to the demand that the bank system only changes its athermality by an infinitesimal amount.

We conclude the section with the derivation of the First Law of Thermodynamics within the multi-resource theory we are considering. Recall that, as shown in Cor. 3, an agent who has access to bank and batteries can modify the state of the main system if the amounts of resources exchanged satisfy a single relation, which is the first law for general multi-resource theories, see Eq. (3.49). This relation, for our theory, is given by

$$\Delta W_E + \beta^{-1} \, \Delta W_I = F(\rho) - F(\sigma), \tag{4.61}$$

which tell us that an asymptotic transformation mapping ρ into σ can be achieved if the weighted sum of the resources ΔW_E and ΔW_I exchanged with the batteries is equal to the athermality change in the system. Notice that the weight in the lhs is given by the inverse of the exchange rate, see Eq. (4.59), and it is proportional to the temperature of the thermal reservoir (the bank) which the agent has access to.

4.4   Work and heat as resources

In the previous section, we considered the case in which the agent has access to both an energy and an information battery. However, when considering thermodynamic tasks performed in a laboratory, it is often the case that the agent has access to the energy battery, but not to the entropy one. In this scenario, the entropy required to perform a state transformation is exchanged with the thermal reservoir, which again, plays the role of the bank. In this setting, we have that energy is the sole resource the agent is considering, but this quantity can be

divided into two contributions; part of the energy is directly used to change the state of the main system, while the other part is used to transfer entropy in/out the main system using the thermal reservoir as a source/sink. The first kind of energy contribution is referred to as *work*, and does not carry any entropy with it, and the second contribution is referred to as *heat*, which consists of an entropic transfer of energy with the system. In this section, we consider the case where the agent has access to an energy battery and a thermal reservoir only, and we define the amount of work and heat necessary to perform a generic state transformation. Additionally, since our formalism allows us to study the case in which the thermal reservoir has a finite size, we analyse the back-reaction experienced by the reservoir during a state transformation over the main system.

We consider a tripartite global system, composed by a main system S, a thermal reservoir B at temperature T, and an energy battery B_E. Each partition is itself composed of many copies of the same subsystem equipped with Hamiltonian H. The initial state of the global system is

$$\Omega_{\text{in}} = \rho^{\otimes n} \otimes \tau_{\beta_1}^{\otimes \ell} \otimes \omega_E(k), \tag{4.62}$$

where ρ is the initial state of the main system, the thermal reservoir has an inverse temperature β_1, and the energy battery is described by the state $\omega_E(k)$ introduced in Eq. (4.48). The final state of this system is

$$\Omega_{\text{fin}} = \sigma^{\otimes n} \otimes \tau_{\beta_2}^{\otimes \ell} \otimes \omega_E(k'), \tag{4.63}$$

where the state of the main system is σ, the temperature of the thermal reservoir has changed due to the interaction with the system, and the battery has exchanged energy, so that the number of ground states in it is changed. In order to use asymptotic equivalence, we ask the number of copies of system n, reservoir ℓ, and battery m to tend to infinity. Notice that, depending on the number of copies of the reservoirs, relative to the number of copies of the main system, one can study situations where the thermal reservoir has an infinite or finite size. For example, demanding $\frac{\ell}{n} = \text{const}$ for $n, \ell \to \infty$ corresponds to the case in which the thermal reservoir and the main system have a comparable size. When, instead, the ratio $\frac{\ell}{n} \to \infty$ for $n, \ell \to \infty$, we have that the system is coupled to an infinite-sized reservoir.

It is worth noting that, in general, the final state of the reservoir does not have to be thermal, since the interaction with the system might have driven the environment out of equilibrium.

However, if the final state of the reservoir is athermal, one could extract additional work from it, while keeping its entropy unchanged. Here, we are interested in the optimal amount of work we need to exchange when mapping ρ into σ, and therefore we demand the transformation to map the state of the reservoir into the state with minimum energy (for a given entropy).

We can now use the asymptotic equivalence property of Thm. 11, which tells us that the transformation $\Omega_{\text{in}} \to \Omega_{\text{fin}}$ is possible if the energy and entropy of the initial and final states are equal. Let us first consider the entropy condition enforced by asymptotic equivalence. This condition allows us to link the relative size of the reservoir $R_{\text{size}} = \frac{\ell}{n}$ with the change in entropy of system and reservoir,

$$R_{\text{size}} = \frac{S(\sigma) - S(\rho)}{S(\tau_{\beta_1}) - S(\tau_{\beta_2})}. \tag{4.64}$$

First of all, it is worth noting that the relative size R_{size} is a positive quantity, and therefore the inverse temperature of the reservoir is such that $\beta_1 < \beta_2$ when $S(\rho) > S(\sigma)$, and vice versa – assuming that $\beta_1, \beta_2 > 0$. Physically, this implies that when we dump entropy from the system into the thermal reservoir, we increase its temperature, and vice versa, as we would expect in the case of a finite-sized thermal reservoir. When the relative size tends to infinity, i.e., the reservoir is super-linear in the number of copies of the system, it is easy to show that $\beta_2 \to \beta_1$. Therefore, in this scenario, the thermal reservoir is able to absorb entropy from the system while its temperature is left unchanged, as we would expect from a reservoir with infinite size [7].

The energy condition, instead, allows us to compute the work extracted during the asymptotic process mapping ρ into σ. Per single copy of the main system, we find that the energy exchanged with the battery, i.e., the work extracted from the system, is given by

$$W_{\beta_1, \beta_2}(\rho \to \sigma) := \frac{k - k'}{n} \Delta E = (E(\rho) - E(\sigma)) - \frac{E(\tau_{\beta_1}) - E(\tau_{\beta_2})}{S(\tau_{\beta_1}) - S(\tau_{\beta_2})} (S(\rho) - S(\sigma)), \tag{4.65}$$

where $\Delta E = E_{\text{max}} - E_{\text{min}}$ is the energy unit of the battery. Likewise, the heat absorbed from the environment is given by the change in average energy between the initial and final state of the reservoir, and it is given by

$$Q_{\beta_1, \beta_2}(\rho \to \sigma) = R_{\text{size}} \left(E(\tau_{\beta_1}) - E(\tau_{\beta_2}) \right) = \frac{E(\tau_{\beta_1}) - E(\tau_{\beta_2})}{S(\tau_{\beta_1}) - S(\tau_{\beta_2})} (S(\sigma) - S(\rho)). \tag{4.66}$$

Both work and heat depend on the initial and final state of the system, as well as on the initial and final temperature of the reservoir. Notice that, even in this setting, our definition of work

and heat is consistent with the First Law of Thermodynamics, since we have that

$$E(\rho) - E(\sigma) = Q_{\beta_1,\beta_2}(\rho \to \sigma) - W_{\beta_1,\beta_2}(\rho \to \sigma), \tag{4.67}$$

where the lhs is independent of the inverse temperatures β_1 and β_2.

The obtained equations for work and heat are similar to the standard ones. Indeed, work is given by the free energy difference between ρ and σ, for an external effective temperature $\beta_{\mathrm{eff}}^{-1}$ depending on the initial and final temperatures of the thermal reservoir,

$$\beta_{\mathrm{eff}} = \frac{S(\tau_{\beta_1}) - S(\tau_{\beta_2})}{E(\tau_{\beta_1}) - E(\tau_{\beta_2})}. \tag{4.68}$$

The effective inverse temperature $\beta_{\mathrm{eff}}^{-1}$ can be visualised as a slope in the energy-information diagram, see Fig. 4.4. With the help of this effective temperature, we can re-express work and heat in a more familiar way, as

$$W_{\beta_1,\beta_2}(\rho \to \sigma) = F_{\beta_{\mathrm{eff}}}(\rho) - F_{\beta_{\mathrm{eff}}}(\sigma), \tag{4.69a}$$

$$Q_{\beta_1,\beta_2}(\rho \to \sigma) = \beta_{\mathrm{eff}}^{-1}\left(S(\sigma) - S(\rho)\right), \tag{4.69b}$$

where $F_{\beta_{\mathrm{eff}}} = E - \beta_{\mathrm{eff}}^{-1} S$ is the Helmholtz free energy of a system coupled to a reservoir with the effective temperature.

We now study the limiting case of an infinite reservoir, whose temperature only changes infinitesimally. In this case, we can express $\beta_2 = \beta_1 + \varepsilon$, where $|\varepsilon| \ll 1$. It is then straightforward to show that $\beta_{\mathrm{eff}} = \beta_1 + O(\varepsilon)$, and that work and heat are equal to the standard ones (up to first order in ε), that is,

$$W_{\mathrm{std}}(\rho \to \sigma) = F_{\beta_1}(\rho) - F_{\beta_1}(\sigma) + O(\varepsilon), \tag{4.70a}$$

$$Q_{\mathrm{std}}(\rho \to \sigma) = \beta_1^{-1}\left(S(\sigma) - S(\rho)\right) + O(\varepsilon). \tag{4.70b}$$

Furthermore, from Eq. (4.64) it follows that, in order for the temperature to change by an infinitesimal amount $|\varepsilon| \ll 1$, the relative size of the thermal reservoir needs to tend to infinity as

$$R_{\mathrm{size}} = \frac{S(\sigma) - S(\rho)}{\beta_1 \langle \Delta^2 H \rangle_{\beta_1}} \frac{1}{\varepsilon} + O(1), \tag{4.71}$$

where the expectation value in the denominator is the variance of energy in the state τ_{β_1}, linked to the heat capacity for a single copy of the system by the relation $C := \frac{\mathrm{d}E(\tau_{\beta_1})}{\mathrm{d}T} = \beta_1^2 \langle \Delta^2 H \rangle_{\beta_1}$, see Eq. (4.55b).

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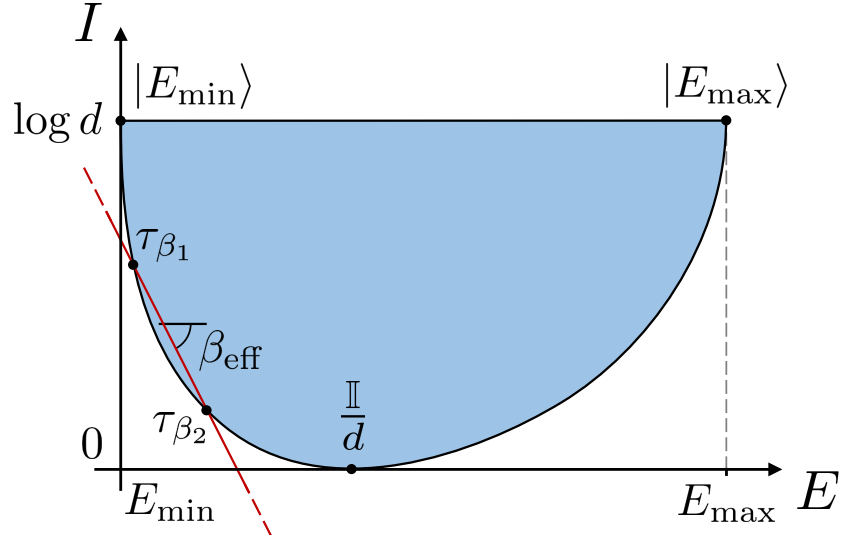
Figure 4.4: We represent the effective temperature β_{eff} in the energy-information diagram. When the size of the reservoir is comparable with the one of the system, i.e., $R_{\text{size}} < \infty$, we have that the temperature of the thermal reservoir changes during the transformation mapping ρ into σ. If the initial state of the thermal reservoir is τ_{β_1}, and the final state is τ_{β_2}, where β_1 and β_2 are two different positive temperatures, the effective temperature is given by the linear coefficient of the line connecting the two points associated with the corresponding thermal states. When $\beta_2 = \beta_1 + \varepsilon$, for $|\varepsilon| \to 0$, the two points get closer and closer, and the line approaches the tangent to the curve of thermal states. In this case, β_{eff} tends to β_1.

4.4.1 Heat engines between finite-sized reservoirs

We now show how the results of the previous section can be used to analyse the efficiency of heat engines and refrigerators exchanging heat between two finite-sized thermal reservoirs. In the following, we do not assume that the engine performs any specific kind of cycle, nor we consider in detail the structure of the thermal machine. Instead, we consider the case in which the agent has access to two finite-sized thermal reservoirs, whose relative size is comparable, and to a battery where work can be stored or extracted – depending on whether the machine is a heat engine or a refrigerator. The working body, in this picture, is represented by the sub-linear ancillary system which allows us to perform the asymptotic transformation. Notice that this ancilla is not returned in its original state, so that the efficiency we find for our devices

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is an upper bound to the efficiency of any actual cyclic engine, where work needs to be used in order to restore the state of the device. Furthermore, the efficiency we find is optimal, since our transformation is reversible, and if we were able to find a better efficiency we could extract an infinite amount of work for free (modulo the consumption of sub-linear ancillary systems). However, it is worth noting that, due to the finite size of the reservoirs, the optimal efficiency we find is always lower than Carnot efficiency.

Our model consists of the same tripartite system presented in the previous section, in the case where both the initial state ρ and the final state σ are thermal. Thus, the initial state is given by

$$\Omega_{\text{in}}^{\text{engine}} = \tau_{\beta_{\text{cold}}}^{\otimes n} \otimes \tau_{\beta_{\text{hot}}}^{\otimes \ell} \otimes \omega_E(k), \tag{4.72}$$

where we ask $\beta_{\text{cold}} > \beta_{\text{hot}}$. The final state, instead, is given by

$$\Omega_{\text{fin}}^{\text{engine}} = \tau_{\beta_{\text{less-cold}}}^{\otimes n} \otimes \tau_{\beta_{\text{less-hot}}}^{\otimes \ell} \otimes \omega_E(k'), \tag{4.73}$$

where we demand the following chain of inequalities to be satisfied,

$$\beta_{\text{cold}} > \beta_{\text{less-cold}} > \beta_{\text{less-hot}} > \beta_{\text{hot}}, \tag{4.74}$$

so as to have a heat engine which extracts work. Physically, the above inequalities mean that the engine uses the hot and cold reservoirs to extract work, but in the meanwhile it degrades these reservoirs, evening out their temperatures. Due to asymptotic equivalence, Thm. 11, we can consider both the forward and backward transformations from $\Omega_{\text{in}}^{\text{engine}}$ to $\Omega_{\text{fin}}^{\text{engine}}$, which describe the mechanism of a heat engine and of a refrigerator, respectively.

In order to evaluate the efficiency of these two devices, we need to evaluate the heat exchanged with the hot reservoir Q_{hot}, the work produced or consumed W, and the heat exchanged with the cold reservoir Q_{cold}. Due to reversibility, these three quantities are the same (in absolute value) for both devices. Using Eq. (4.65) we find that the work stored into the battery is given by

$$W = (E(\tau_{\beta_{\text{cold}}}) - E(\tau_{\beta_{\text{less-cold}}})) - \beta_{\text{eff-hot}}^{-1} \left(S(\tau_{\beta_{\text{cold}}}) - S(\tau_{\beta_{\text{less-cold}}}) \right), \tag{4.75}$$

where the effective inverse temperature $\beta_{\text{eff-hot}}$ is obtained from Eq. (4.68) for $\beta_1 = \beta_{\text{hot}}$ and $\beta_2 = \beta_{\text{less-hot}}$. Likewise, the heat exchanged with the hot reservoir is given by Eq. (4.66),

$$Q_{\text{hot}} = \beta_{\text{eff-hot}}^{-1} \left(S(\tau_{\beta_{\text{less-cold}}}) - S(\tau_{\beta_{\text{cold}}}) \right). \tag{4.76}$$

Notice that both W and Q_{hot} are defined per single copy of the cold reservoir. Since the system S is now the cold reservoir, the single-copy heat Q_{cold} exchanged with the reservoir is equal to

$$Q_{\text{cold}} = E(\tau_{\beta_{\text{less-cold}}}) - E(\tau_{\beta_{\text{cold}}}). \tag{4.77}$$

We can now turn to the analysis of both the efficiency of the heat engine, which is defined as $\eta_{\text{engine}} = \frac{W}{Q_{\text{hot}}}$, and the one of the refrigerator, $\eta_{\text{refrigerator}} = \frac{Q_{\text{cold}}}{W}$. It is easy to show that these efficiencies are equal to

$$\eta_{\text{engine}} = 1 - \frac{\beta_{\text{eff-hot}}}{\beta_{\text{eff-cold}}}, \tag{4.78a}$$

$$\eta_{\text{refrigerator}} = \left(\frac{\beta_{\text{eff-cold}}}{\beta_{\text{eff-hot}}} - 1 \right)^{-1}, \tag{4.78b}$$

where $\beta_{\text{eff-cold}}$ is defined as in Eq. (4.68), for the choice of $\beta_1 = \beta_{\text{cold}}$ and $\beta_2 = \beta_{\text{less-cold}}$. If we use the relation between the different temperatures shown in Eq. (4.74), it is easy to show that the efficiencies we have found are sub-Carnot. This is due to the fact that the temperature of the two finite-sized reservoirs changes during the process. However, in the limit where these temperatures change by only an infinitesimal amount, both efficiencies approach the Carnot value, as we expect since our process is reversible.

4.5 Chapter summary

In this chapter we show how thermodynamics can be recast as a multi-resource theory, whose two resources are energy and information (or entropy). The class of allowed operations, Eq. (4.3), is composed by energy-preserving unitary operations. Interestingly, this theory allows us to describe scenarios where the system is isolated, as well as where it interacts with an environment, which does not need to be thermal nor have an infinite size – a crucial difference from the settings where Thermal Operations can be applied, see Ch. 2. Here, we focus on the case in which the system is coupled to an environment, which is thermal but not infinite-sized. It would be interesting to use this same formalism to study scenarios where the environment is not thermal, for example, we might consider it to be passive, see next chapter for more details on this scenario.

One of the main results of the chapter is Thm. 11, where we prove that thermodynamics as a multi-resource theory satisfies asymptotic equivalence. To prove the theorem we construct a

protocol that allows us to asymptotically transform between any two states with same energy and information. In this protocol we make explicit which ancillary systems are used, how they are used, and what their dimension is. In particular, we show that, in order to asymptotically map between two states, we only need ancillae whose dimension (and energy) is sub-linear in the number of copies of the main system. Our theorem considers a single conserved quantity, energy, and in a following paper it has been shown that this theorem can be extended to many conserved quantities [168].

Since the theory satisfies asymptotic equivalence, we are able to apply the general results shown in the previous chapter. In particular, we explicitly build batteries for energy and information, and we study the interconversion of these resources, see Eq. (4.59). In order to exchange resources, the agent needs a bank, which in this scenario consists of a thermal state at a given temperature. We also study the first law of this multi-resource theory, and we show that it corresponds (as expected) to the First Law of Thermodynamics.

We then consider the case in which the agent has only access to an energy battery and a thermal reservoir whose size is finite. We show that, in this setting, the two main resources described by the theory can be transformed to be work (energy exchanged with the battery) and heat (energy exchanged with the thermal reservoir). For a given state transformation, we find the corrections given by the finiteness of the reservoir to the amount of work and heat exchanged, see Eqs. (4.69). Finally, we use these results to derive the efficiencies of heat engines and refrigerators exchanging heat between two finite-sized reservoirs, Eqs. (4.78), and we find that these efficiencies are always sub-Carnot, due to the fact that the cycle modifies the properties (the temperature) of the reservoirs.

It is worth noting that the results we obtain are valid in a specific regime delineated by several idealising assumptions, such as the assumption that all energy-preserving unitary operations are available, and the presence of many non-interacting and identical copies of the system. One can think of dropping some of these assumptions, and for example investigate the theory when arbitrary states and interactions are allowed, or when one has a much more realistic class of operations not requiring such fine-grained control of system and reservoir [124].

Chapter 5

Energetic instability of passive states in thermodynamics

We now use the resource theoretic framework to study a notion of equilibrium for closed systems, known as *passivity*. When the state of a closed system is passive, it is not possible to lower its average energy by means of unitary operations. Thus, if the dynamics of a thermodynamic system is reversible and energy-decreasing (perhaps because the system is in contact with an energy sink), we find that the system is in equilibrium when it is described by a passive state. Nevertheless, in this chapter we provide a protocol that enables the agent to extract energy from any closed system described by a passive state [184]. To achieve energy extraction, we allow the agent to add a catalyst (see Sec. 1.5.2 in the background) to the main system, and to use a unitary operation acting on both system and catalyst, in such a way that the local state of the catalyst is preserved at the end of the transformation. With this set of operations, we show that energy can always be extracted from any passive state, with the exception of completely passive states, i.e., thermal states with non-negative temperature $T \geq 0$. In this way, we show that passive states are energetically unstable, and that thermal states are the only equilibrium states when catalytic unitary operations are considered.

The energy-extraction scheme we consider is reminiscent of the cycles occurring inside heat engines. An engine is a device which exchanges heat between two thermal reservoirs at different temperatures, and during this exchange it turns part of the heat into work. This process is

cyclic, since the machine returns in its initial state after exchanging heat between the two reservoirs, and it can be repeated forever. The protocol for extracting energy from passive states is analogous to these cycles. Indeed, one can associate a positive temperature (or virtual temperature) to any pair of levels in a passive state. We can then identify two pairs of levels, one associated with the highest temperature, and one associated with the lowest temperature. We can then think of these two pairs of levels as if they were two thermal reservoirs, one with a cold temperature, the other with a hot temperature. In our protocol, we use a catalyst which acts as the machine of a heat engine. This catalyst interacts with the two pairs of levels in the passive state, and at the end of the interaction the state of the catalyst returns in its original state, while the average energy of the system described by the passive state is reduced.

During the interaction, correlations are created between system and catalyst. However, we show that it is possible to make these correlations negligible if we are allowed to use a catalyst whose dimension tends to infinity. When we have access to such catalysts, we can map any passive state into a thermal state, while creating almost zero correlations between the two systems. This result is particularly interesting, since it shows that the sole presence of a catalyst makes the process of thermalisation for closed systems possible. Therefore, the protocol we present in this chapter can be understood as a possible explanation for how closed systems reach thermal equilibrium when driven by reversible dynamics. Furthermore, our thermalisation process does not require the presence of multiple copies of the system, since we focus on a single passive state, and thus it provides a way to recover the notion of temperature outside the thermodynamic limit.

Other protocols for energy-extraction from passive states can be conceived. In particular, energy-extraction schemes are known for situations in which it is possible to act oven many copies of a passive state [119, 185, 186], using only global unitary operations. The protocol we present here is different, since it focuses on energy extraction for a single copy of passive state. However, since the local state of the catalyst we use is preserved during the operation, we can also apply our scheme on many copies of a passive state, by individually performing a cyclic transformation on each copy. Thus, our protocol could be applied to situations where the agent has access to a reservoir composed by many copies of a passive state, and for some reasons they are allowed to act on these copies individually. The agent can then couple the catalyst with

one copy of the passive state, and perform the energy-extraction cycle. Correlations between catalyst and individual system are created, but the agent can simply discard the system and take a fresh copy, over which performing the same cycle. In this way, the agent consumes passive states as fuel, while extracting energy from them and storing it in a battery. The above scenario might also be relevant to the field of quantum technologies, since microscopic machines, such as heat engines, can nowadays be realised in the laboratory [187, 188]. Furthermore, the scheme we present could be used to extract energy from the left-over states of a quantum computation, or as a process to obtain thermal states with a preferred temperature.

5.1   Passivity and complete passivity

Let us first introduce *passivity* [189, 190], a notion of equilibrium for closed systems which is weaker than thermal equilibrium.

Definition 33 (Passive state). *Consider a finite-dimensional system associated with the Hilbert space $\mathcal{H}$, with Hamiltonian $H = \sum_{i=0}^{d-1} E_i \ket{i}\bra{i}$, where $d = \dim \mathcal{H}$. We say that a state $\rho \in \mathcal{S}(\mathcal{H})$ is* passive *iff its average energy cannot be lowered by acting on it with unitary operations, that is,*

$$\mathrm{Tr}\,[H\,\rho] \leq \mathrm{Tr}\left[H\,U\rho\,U^\dagger\right], \quad \forall U \in \mathcal{B}(\mathcal{H})\ ,\ UU^\dagger = U^\dagger U = \mathbb{I}. \tag{5.1}$$

On the contrary, we say that a state is *active* if the average energy can be lowered with a unitary operation. We can link the notion of passivity to the possibility of extracting work from a closed system. In the background material, we have introduced the notion of batteries, and we have considered the case in which the battery is explicit, meaning that it enters the transformation together with the system, see Def. 29 for example. Here, we consider an implicit battery. In this case, we do not represent the battery with an additional system, and we allow any unitary operation to act over the state of the system. When the average energy of the system is reduced/increased by the unitary operation, we assume that the difference in energy is stored/extracted from the implicit battery. Thus, work is accounted by the change in average energy of the main system. As a result, Def. 33 implies that work cannot be extracted, by means of unitary operations, from a system that is described by a passive state.

We can also introduce a more restrictive notion of passivity, known as *k-passivity*, involving many i.i.d copies of the same state.

Definition 34 (*k*-passive state)**.** *Consider a finite-dimensional system associated with the Hilbert space $\mathcal{H}$, with Hamiltonian $H = \sum_{i=0}^{d-1} E_i |i\rangle \langle i|$, where $d = \dim \mathcal{H}$. A passive state $\rho \in \mathcal{S}(\mathcal{H})$ is k-passive iff the state $\rho^{\otimes k} \in \mathcal{S}(\mathcal{H}^{\otimes k})$ is passive with respect to the global Hamiltonian $H_k = \sum_{i=1}^{k} H^{(i)}$, where $H^{(i)}$ is a single-system Hamiltonian acting on the i-th copy of the system.*

When a state is *k*-passive, the agent is not able to extract energy even if they are allowed to act over this state with a global unitary operation. In the limit of k tending to infinity, we recover the strongest notion of passivity, known as *complete passivity.*

Definition 35 (Completely passive states)**.** *Consider a finite-dimensional system associated with the Hilbert space $\mathcal{H}$, with Hamiltonian $H = \sum_{i=0}^{d-1} E_i |i\rangle \langle i|$, where $d = \dim \mathcal{H}$. A state $\rho \in \mathcal{S}(\mathcal{H})$ is* completely passive *iff it is k-passive for all $k \in \mathbb{N}$.*

It has been shown, see Ref. [190], that the completely passive states of a system with Hamiltonian H are the ones satisfying the KMS condition [191, 192, 193]. These states are the ground state and the thermal states with inverse temperature $\beta \geq 0$. Thus, the notion of thermal equilibrium for open systems and of complete passivity for closed systems coincide.

Passive states are characterised by some interesting properties that allow us to easily represent them. It can be shown [189, 190, 119], that a passive state is diagonal in the energy eigenbasis, its spectrum is such that the ground state has the highest probability of being occupied, and the probability of occupation decreases as the energy associated with the eigenstates of H increases, see Fig. 5.1. Therefore, we have that a state ρ is passive iff $\rho = f(H)$, where f is a monotonic non-increasing function. Simply put, this means that the state can be expressed as

$$\rho = \sum_{i=0}^{d-1} p_i |i\rangle \langle i|, \quad \text{such that } p_i \geq p_{i+1} \ \forall \, i = 0, \ldots, d-2, \tag{5.2}$$

where $\{|i\rangle\}_{i=0}^{d-1}$ are the eigenvectors of H, ordered so that $E_i \leq E_{i+1}$ for all i[1].

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[1]When an energy level is degenerate, so that $E_i = E_{i+1}$, we must make an additional stability assumption to ensure that $p_i = p_{i+1}$.
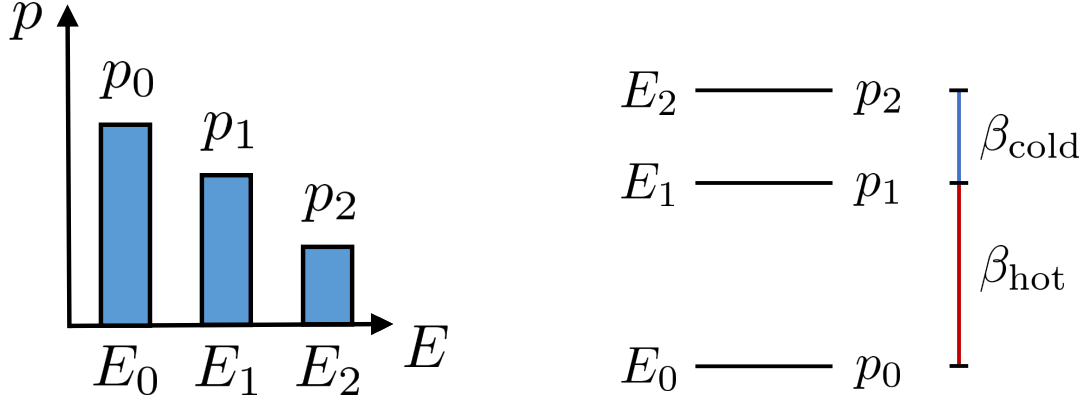
Figure 5.1: **Left.** The spectrum of a qutrit passive state $\rho = \sum_{i=0}^{2} p_i \, |i\rangle \langle i|$ over the eigenbasis of its Hamiltonian $H = \sum_{i=0}^{2} E_i \, |i\rangle \langle i|$. The occupation probabilities are ordered in a decreasing order, from the one associated with the ground state of H, to the one associated with the maximally-excited one, as per definition in Eq. (5.2). **Right.** A passive state can equally be described by *virtual temperatures*. Indeed, for each pair of eigenvalues of ρ, say p_i and p_j, we can define a virtual temperature β_{ij} through the relation $p_i/p_j = e^{-\beta_{ij}(E_i - E_j)}$, where E_i (E_j) is the energy level associated with the eigenstate $|i\rangle$ ($|j\rangle$). In the figure, the pair of eigenstates $(|0\rangle, |1\rangle)$ is associated with the hot temperature β_{hot}^{-1}, while the pair $(|1\rangle, |2\rangle)$ is associated with the cold temperature β_{cold}^{-1}. The temperature associated with $(|0\rangle, |2\rangle)$ is an average of the other two temperatures.

An additional way to describe the probability distribution of a passive state ρ is to use virtual temperatures [194, 195]. Indeed, for any given passive state, we can associate a (non-negative) virtual temperature with each pair of eigenstates. For example, if we consider the pair $(|i\rangle, |j\rangle)$, we define the virtual temperature associated with it as the parameter $\beta_{ij}^{-1} \geq 0$ such that

$$\frac{p_i}{p_j} =: e^{-\beta_{ij}(E_i - E_j)}, \tag{5.3}$$

where p_i is the probability of occupation of the state $|i\rangle$, and E_i is the energy associated with the state (similarly for j). When all pairs of states have the same virtual temperature, the state is completely passive, i.e., is the thermal state of H at that temperature.

5.2 Energy extraction from passive states

We now present the protocol we use to extract energy from a passive state. In the following, we focus on the simplest system that can be described using passive states, namely, a qutrit system described by the Hilbert space $\mathcal{H} \equiv \mathbb{C}^3$. Notice, however, that the results we obtain extend to any d-dimensional system, with $d \geq 3$, as we show in Sec. 5.2.3. The Hamiltonian of the qutrit system is

$$H_P = \sum_{i=0}^{2} E_i \left| i \right\rangle \left\langle i \right|_P, \tag{5.4}$$

where we order the energy eigenvalues in increasing order, $E_i \leq E_{i+1}$. We also define the energy gap between ground state and first excited state as $\Delta E_{10} = E_1 - E_0 \geq 0$, and the one between first and second excited state as $\Delta E_{21} = E_2 - E_1 \geq 0$. Since the state describing the system is passive, we have that, according to Def. 33, it is diagonal in the energy eigenbasis,

$$\rho_P = \sum_{i=0}^{2} p_i \left| i \right\rangle \left\langle i \right|_P, \tag{5.5}$$

and its spectrum is decreasingly ordered, i.e., $p_i \geq p_{i+1}$, see the left plot of Fig. 5.1.

For our goals, a more convenient description of the passive state ρ_P is given in terms of the virtual temperatures associated to the pair of states $(\left| 0 \right\rangle_P, \left| 1 \right\rangle_P)$ and $(\left| 1 \right\rangle_P, \left| 2 \right\rangle_P)$. If the state is passive but not completely passive, which is the scenario we are interested in, these two temperatures are both positive, but their value is different. In this section we assume that the virtual temperature of the pair $(\left| 0 \right\rangle_P, \left| 1 \right\rangle_P)$ is higher than the virtual temperature of the pair $(\left| 1 \right\rangle_P, \left| 2 \right\rangle_P)$, but the protocol can be easily adjusted to consider the opposite scenario. We refer to these virtual temperatures as T_{hot} and T_{cold}, respectively, and we define them as

$$\frac{p_1}{p_0} =: e^{-\beta_{\text{hot}} \Delta E_{10}}, \tag{5.6a}$$

$$\frac{p_2}{p_1} =: e^{-\beta_{\text{cold}} \Delta E_{21}}, \tag{5.6b}$$

where $\beta_{\text{hot}} = T_{\text{hot}}^{-1}$, and $\beta_{\text{cold}} = T_{\text{cold}}^{-1}$. As we highlighted in the previous section, the protocol we present is reminiscent of the cyclic processes occurring inside a heat engine. From this point of view, we have that the pair of states $(\left| 0 \right\rangle_P, \left| 1 \right\rangle_P)$ is the "hot thermal reservoir" from which the machine extracts energy, and the pair of states $(\left| 1 \right\rangle_P, \left| 2 \right\rangle_P)$ is the "cold thermal reservoir" where the machine dumps energy.

The machine we use in our protocol is a catalyst. This is an additional d-dimensional system, initially in a state $\rho_M = \sum_{j=0}^{d-1} q_j |j\rangle \langle j|_M$, whose spectrum is defined later. During the interaction with the system, the state of the catalyst changes, but we demand that its final state is exactly equal to the initial one. Notice that, however, correlations between the system and the catalyst can be created by the protocol, and in fact they generally are. Since the initial and final states of the catalyst are the same, the average energy of this system after a cycle does not change, independently of which Hamiltonian H_M we chose. Thus, we can simply forget about the Hamiltonian of the catalyst, or equally we can assume that it is fully-degenerate. Furthermore, notice that no interaction Hamiltonian between the system and the catalyst is present in this scheme, so that the creation of correlations between these two systems does not influence the global energy.

Instead of first defining the state of the catalyst, we introduce the global unitary operation that we apply over system and catalyst. This operation is composed by a sequence of "hot" swaps and "cold" swaps. The former swap between pairs of states in the catalyst and the pair of states representing the hot reservoir $(|0\rangle_P, |1\rangle_P)$. The latter, instead, swap between other pairs of levels in the catalyst and the pair of states representing the cold reservoir $(|1\rangle_P, |2\rangle_P)$, see Fig. 5.2. The idea is that, during each hot swap, energy is extracted from the system (since its average energy is decreased), and the state of the catalyst is modified. Then, a number of cold swaps are performed, and during this process energy is pumped back into the system, while the state of the catalyst is restored. The key fact here is that the amount of energy we extract during the hot swaps is bigger than the amount of energy we pay during the cold ones, in analogous fashion with the processes taking place inside heat engines. It is worth noting that, however, the cycle we perform needs to be tailored to the specific passive state we are considering in order to extract energy. In particular, the number of hot and cold swaps we perform depend on the virtual temperatures of the passive state, as well as on the energy gaps of the system's Hamiltonian. The global unitary operation we apply on system and catalyst is the following,

$$S_{m,n} = S_{(1,2)}^{(0,m)} \circ S_{(1,2)}^{(m,m+1)} \circ \ldots \circ S_{(1,2)}^{(m+n-2,m+n-1)} \circ S_{(0,1)}^{(m-1,m+n-1)} \circ S_{(0,1)}^{(m-2,m-1)} \circ \ldots \circ S_{(0,1)}^{(0,1)}, \quad (5.7)$$

where $S_{(a,b)}^{(c,d)}$ is a partial swap operation between system and catalyst, realised through the permutation $|a\rangle_P |d\rangle_M \leftrightarrow |b\rangle_P |c\rangle_M$. The parameter $m \in \mathbb{N}$ is the number of "hot" swaps $S_{(0,1)}^{(c,d)}$

$\uparrow$: Swap with $(|0\rangle_P, |1\rangle_P)$

$\downarrow$: Swap with $(|1\rangle_P, |2\rangle_P)$

$|m+n-1\rangle_M$
$|m+n-2\rangle_M$
$\cdots$
$|m+1\rangle_M$
$|m\rangle_M$
$|m-1\rangle_M$
$|m-2\rangle_M$
$\cdots$
$|1\rangle_M$
$|0\rangle_M$
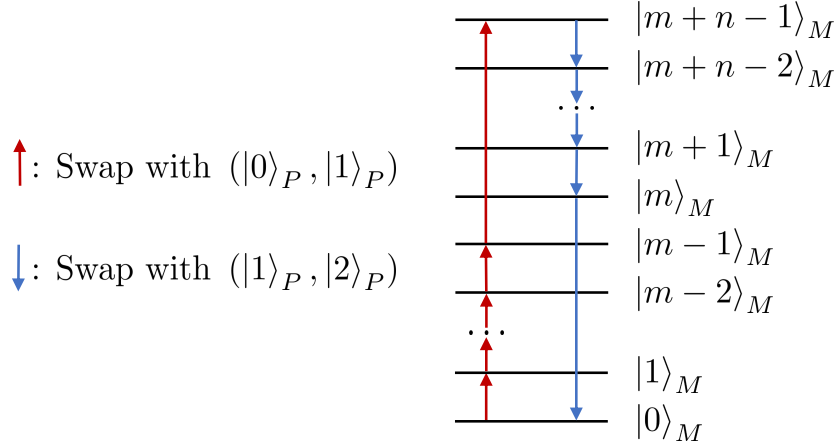
Figure 5.2: The cycle $S_{m,n}$ is represented in a pictorial way over the eigenstates of the d-dimensional catalyst (where $d = m+n$). Notice that the Hamiltonian of the catalyst is arbitrary, and we here order its eigenstates to simplify the visualisation of the cycle. The upward arrow connecting two eigenstates of the catalyst represents a swap between these two states and the pair $(|0\rangle_P, |1\rangle_P)$ of the passive state – a hot swap. The downward arrow connecting two eigenstates of the catalyst represents a swap between these two states and the pair $(|1\rangle_P, |2\rangle_P)$ of the passive state – a cold swap. We initially perform $m-1$ swaps between $(|0\rangle_P, |1\rangle_P)$ and $\{(|j\rangle_M, |j+1\rangle_M)\}_{j=0}^{m-2}$, and one swap between $(|0\rangle_P, |1\rangle_P)$ and $(|m-1\rangle_M, |m+n-1\rangle_M)$. Then, we perform $n-1$ swaps between $(|1\rangle_P, |2\rangle_P)$ and $\{(|j\rangle_M, |j+1\rangle_M)\}_{j=m}^{m+n-2}$, and one swap between $(|1\rangle_P, |2\rangle_P)$ and $(|0\rangle_M, |m\rangle_M)$. If we consider the arrow representation of swaps, we can see that the cycle is close, and this feature allows us to recover the local state of the catalyst M while also extracting energy from the system.

we perform, while the parameter $n \in \mathbb{N}$ is the number of "cold" swaps $S_{(1,2)}^{(c,d)}$. The dimension of the catalyst is $d = m + n$.

By fixing the global operation $S_{m,n}$ and the state of the system ρ_P, one finds that the spectrum of the state of the catalyst ρ_M is completely defined. In particular, the spectrum is obtained from the constraint that, at the end of the transformation, the local state of the catalyst is left unchanged,

$$\rho_M \overset{!}{=} \mathrm{Tr}_P \left[S_{m,n} \left(\rho_P \otimes \rho_M \right) S_{m,n}^\dagger \right], \tag{5.8}$$

where the symbol $\overset{!}{=}$ means that we demand the state of the catalyst to satisfy the equality. We can also compute the amount of energy that can be extracted from the state ρ_P during this process. As we noticed before, we are working in a framework in which the battery is implicit. Thus, in our scheme we only have system and catalyst, and there is no additional system where energy can be stored. To quantify the amount of energy extracted, we simply look at the change in average energy within the global system. In particular, since the catalyst is constrained to have equal initial and final state, we have that the change in energy is solely due to the change in the state of the system. Thus, we define the energy extracted by the cycle as

$$\Delta W = \mathrm{Tr}\left[H_P\left(\rho_P - \tilde{\rho}_P\right)\right], \tag{5.9}$$

where the state of the system at the end of the cycle is $\tilde{\rho}_P$, defined as

$$\tilde{\rho}_P = \mathrm{Tr}_M\left[S_{m,n}\left(\rho_P \otimes \rho_M\right)S_{m,n}^\dagger\right]. \tag{5.10}$$

5.2.1 Extracted energy and efficiency

We now show that, for all passive but not completely passive states, energy can always be extracted from the system by choosing appropriate parameters m and n. In particular, in order to extract energy, the ratio $\frac{m}{n}$ needs to lie within an interval that solely depends on the energy gaps of H_P, and on the virtual temperatures of the passive state. The amount of energy extracted from the cycle, see appendix D.1 for details, is given by

$$\Delta W = \alpha\left(m\,\Delta E_{10} - n\,\Delta E_{21}\right)\left(e^{\beta_{\mathrm{cold}}n\Delta E_{21}} - e^{\beta_{\mathrm{hot}}m\Delta E_{10}}\right), \tag{5.11}$$

where α is a positive coefficient depending non-trivially on the probability distribution of ρ_{P}. From the above equation, it is easy to show that energy can always be extracted from a system with Hamiltonian H_P described by a passive state ρ_P, by choosing an appropriate number of hot and cold swaps. In particular, we have that $\Delta W > 0$ if and only if

$$\frac{\Delta E_{21}}{\Delta E_{10}} < \frac{m}{n} < \frac{\beta_{\mathrm{cold}}}{\beta_{\mathrm{hot}}}\frac{\Delta E_{21}}{\Delta E_{10}}, \tag{5.12}$$

where it is worth noting that we are considering the situation in which the passive state has virtual temperatures $\beta_{\mathrm{cold}} > \beta_{\mathrm{hot}}$. However, one can use the same cycle introduced in the

previous section to deal with passive states whose virtual temperatures are inverted. In that case, the ratio between the number of hot and cold swaps needs to lie within $\left(\frac{\beta_{\text{cold}}}{\beta_{\text{hot}}} \frac{\Delta E_{21}}{\Delta E_{10}}, \frac{\Delta E_{21}}{\Delta E_{10}} \right)$.

We can additionally study the amount of energy that is extracted and provided during each hot and cold swap, respectively. Interestingly, by computing the total amount of energy extracted with the hot swaps, and the total energy pumped back with the cold swaps, we can work out how efficient was the cycle in extracting energy from the system, in analogy with the efficiency of heat machines. During a single hot swap, the energy extracted is given by

$$q_{\text{hot}} = \alpha \, \Delta E_{10} \left(e^{\beta_{\text{cold}} n \Delta E_{21}} - e^{\beta_{\text{hot}} m \Delta E_{10}} \right), \tag{5.13}$$

where α is the same positive coefficient appearing in Eq. (5.11), and this quantity is positive when the ratio $\frac{m}{n}$ satisfies the second inequality in Eq. (5.12). The energy spent during a cold swap, instead, is given by

$$q_{\text{cold}} = \alpha \, \Delta E_{21} \left(e^{\beta_{\text{cold}} n \Delta E_{21}} - e^{\beta_{\text{hot}} m \Delta E_{10}} \right). \tag{5.14}$$

It is easy to see that, when energy is extracted during a hot swap, energy is necessarily spent during a cold swap. For the overall cycle to extract energy, we need the total energy extracted $Q_{\text{hot}} = m \, q_{\text{hot}}$ to be larger than the total energy spent $Q_{\text{cold}} = n \, q_{\text{cold}}$, since ΔW is given by the difference between Q_{hot} and Q_{cold}. This is the case when the first inequality of Eq. (5.12) is satisfied.

We can now compute the energy-extraction efficiency for our protocol. This quantity is given by the ratio between the extracted energy ΔW and the total energy extracted during the hot swaps Q_{hot}, and is clearly inspired by the efficiency of heat engines. It is easy to show that the efficiency of our protocol is

$$\eta = \frac{\Delta W}{Q_{\text{hot}}} = 1 - \frac{n \, \Delta E_{21}}{m \, \Delta E_{10}}. \tag{5.15}$$

Using the above equation, together with the range of $\frac{m}{n}$ for which the protocol extract energy, we find that the efficiency of our protocol always lies below $1 - \frac{T_{\text{cold}}}{T_{\text{hot}}}$, which is the well-known Carnot efficiency. In Sec. 5.3, we show that if the catalyst is taken to be infinite-dimensional, and the number of hot and cold swaps is carefully chosen, we can realise a protocol that extract a non-trivial amount of energy from the system with Carnot efficiency.

Apart from the amount of energy extracted during the process, and the efficiency of such extraction, we can also study how the passive state is modified during the cycle. This study is fundamental for the discussion of the next section, where we show that an infinite-dimensional catalyst allows us to map any passive state into a completely passive state. The final state of the system after the cycle is given by Eq. (5.10), and we can express its spectrum as

$$p_0' = p_0 + m\,\Delta P, \tag{5.16a}$$

$$p_1' = p_1 - (m+n)\,\Delta P, \tag{5.16b}$$

$$p_2' = p_2 + n\,\Delta P, \tag{5.16c}$$

see appendix D.2 for details. Notice that the unit of probability ΔP depends on the virtual temperatures of the initial state ρ_P, on the energy gaps of the Hamiltonian H_P, and on the number of hot and cold swaps performed during the cycle,

$$\Delta P = \alpha \left(e^{\beta_{\mathrm{cold}} n \Delta E_{21}} - e^{\beta_{\mathrm{hot}} m \Delta E_{10}} \right). \tag{5.17}$$

When the protocol extracts energy from the system, the unit $\Delta P > 0$. This easily follows from the second inequality in Eq. (5.12). Thus, our cycle acts over the state of the system by depleting the population of the first excited state, while increasing the populations of the ground state (during the hot swaps), and of the second excited state (during the cold swaps).

5.2.2 Energy extraction from completely passive states

In this section, we study energy extraction from states that are completely passive, see Def. 35, and nearly completely passive. We recall that a completely passive state is a thermal state with Hamiltonian H_P and non-negative temperature. Thus, when a state is completely passive, its virtual temperatures are all equal to a single one. In this situation, our protocol fails to extract energy from the system, as it can be easily seen from Eq. (5.11), where we replace $\beta_{\mathrm{cold}} = \beta_{\mathrm{hot}} = \beta$. The failure of our protocol when completely passive states are considered is not surprising, since according to the Second Law of Thermodynamics we cannot realise a cyclic process whose sole outcome is the extraction of work from a single thermal reservoir (here, a single thermal state).

When the state is close, in trace distance, to a completely passive state, we are still able to extract energy. However, to do so we need to perform an infinite number of hot and cold

swaps, and therefore we require an infinite-dimensional catalyst. Let us consider the case in which the virtual temperatures of the passive state ρ_P are such that $\beta_{\mathrm{cold}} = \beta_{\mathrm{hot}} + \varepsilon$, where $\varepsilon > 0$ is infinitesimal. One can show that, for $\varepsilon \to 0$, the trace distance $\|\rho_P - \tau_{\beta_{\mathrm{hot}}}\|_1 \to 0$. Furthermore, we ask that the energy gaps of the Hamiltonian H_P are such that

$$\exists\, M, N \in \mathbb{N} \;:\; \frac{\Delta E_{21}}{\Delta E_{10}} = \frac{M}{N}, \tag{5.18}$$

i.e., we assume that this ratio is always equal to a rational number. The number of hot and cold swaps necessary to extract energy from the system is given in Eq. (5.12). In particular, the first inequality in the equation implies that $m > n\frac{M}{N}$, and therefore energy is extracted when the number of hot swaps is $m = \left\lceil n\frac{M}{N} \right\rceil + 1$. For simplicity, we take the number of cold swaps to be $n = n' N$, so that $m = n' M + 1$, where $n' \in \mathbb{N}$. We can now consider the second inequality of Eq. (5.12), and we obtain that $n' > \frac{\beta_{\mathrm{hot}}}{\varepsilon\, M}$. When $\varepsilon \to 0$, i.e., when the distance between the passive state ρ_P and the set of thermal states tends to 0, we have that the number of hot and cold swaps we need to perform tends to infinity.

5.2.3 Energy extraction from qudit passive states

The cycle introduced in the previous section allows for energy extraction from any passive, but not completely passive, qutrit states. Here, we show that the same cycle can be used to extract energy from a *qudit* system with Hamiltonian $H_P^{(d)} = \sum_{i=0}^{d-1} E_i \left|i\right\rangle \left\langle i\right|_P$, described by a passive state $\rho_P^{(d)} = \sum_{i=0}^{d-1} p_i \left|i\right\rangle \left\langle i\right|_P$. Indeed, if the state is passive, we can always find a three-dimensional subspace $A_k = \mathrm{span}\left\{ \left|k\right\rangle_P, \left|k+1\right\rangle_P, \left|k+2\right\rangle_P \right\}$ where the virtual temperatures associated with the pairs $(\left|k\right\rangle_P, \left|k+1\right\rangle_P)$ and $(\left|k+1\right\rangle_P, \left|k+2\right\rangle_P)$ are different. If these temperatures are such that β_{hot} is associated with the first pair of states, and β_{cold} with the second pair, then we can use the same construction shown in the previous section, with the same number of hot and cold swaps. If the temperatures are inverted, then the protocol still allows us to extract energy, but we need to adjust the number of hot and cold swaps so that their ratio lies within the correct interval.

As a first step, let us divide the passive state into a mixture of states, one with support on the subspace A_k, the other with support on the complement,

$$\rho_P^{(d)} = \lambda\, \rho_P^{(A_k)} + (1 - \lambda)\, \rho_P^{(A_k^c)}, \tag{5.19}$$

where the coefficient $\lambda = \sum_{i \in A_k} p_i$ takes values between 0 and 1, and the two states are defined as

$$\rho_P^{(A_k)} = \sum_{i \in A_k} \frac{p_i}{\sum_{j \in A_k} p_j} |i\rangle \langle i|_P \,, \tag{5.20a}$$

$$\rho_P^{(A_k^c)} = \sum_{i \notin A_k} \frac{p_i}{1 - \sum_{j \in A_k} p_j} |i\rangle \langle i|_P \,. \tag{5.20b}$$

Likewise, the Hamiltonian of the system can be expressed as a sum of two operators, one with support on the subspace A_k, the other with support on its complement, A_k^c. Thus, we write $H_P^{(d)} = H_P^{(A_k)} + H_P^{(A_k^c)}$, where

$$H_P^{(A_k)} = \sum_{i \in A_k} E_i |i\rangle \langle i|_P \,, \tag{5.21a}$$

$$H_P^{(A_k^c)} = \sum_{i \notin A_k} E_i |i\rangle \langle i|_P \,. \tag{5.21b}$$

It is easy to see that, for the Hamiltonian $H_P^{(A_k)}$, the state $\rho_P^{(A_k)}$ is passive. Then, we can add a catalyst described by the state ρ_M introduced in the previous section, and we can perform the following global transformation over system and catalyst,

$$U = \Pi_{A_k^c} \otimes \mathbb{I}_M + \Pi_{A_k} \otimes \mathbb{I}_M \circ S_{m,n} \circ \Pi_{A_k} \otimes \mathbb{I}_M, \tag{5.22}$$

where Π_{A_k} is the projector onto the subspace A_k, while $\Pi_{A_k^c}$ is the projector onto its complement. The unitary operation $S_{m,n}$ is defined in Eq. (5.7), and the number of hot swaps m and cold swaps n is chosen in order to extract energy from $\rho_P^{(A_k)}$. Furthermore, we choose the state of the catalyst to be such that $\rho_M = \mathrm{Tr}_P \left[S_{m,n} \, \rho_P^{(A_k)} \otimes \rho_M \, S_{m,n}^\dagger \right]$.

If we apply the global unitary operation U on qudit and catalyst, we obtain the following final state

$$\tilde{\rho}_P^{(d)} = \mathrm{Tr}_M \left[U \, \rho_P^{(d)} \otimes \rho_M \, U^\dagger \right] = \lambda \, \mathrm{Tr}_M \left[S_{m,n} \, \rho_P^{(A_k)} \otimes \rho_M \, S_{m,n}^\dagger \right] + (1 - \lambda) \, \rho_P^{(A_k^c)}$$

$$= \lambda \, \tilde{\rho}_P^{(A_k)} + (1 - \lambda) \, \rho_P^{(A_k^c)}, \tag{5.23}$$

where the state of the catalyst is left unchanged, and the cycle has been designed in such a way that the qutrit final state $\tilde{\rho}_P^{(A_k)}$ has lower average energy than the initial state $\rho_P^{(A_k)}$, which is always possible as shown in the previous section. The amount of energy extracted by this

protocol is then given by

$$\Delta W = \text{Tr}_P \left[H_P^{(d)} \left(\rho_P^{(d)} - \tilde{\rho}_P^{(d)} \right) \right] = \lambda \, \text{Tr}_P \left[H_P^{(A_k)} \left(\rho_P^{(A_k)} - \tilde{\rho}_P^{(A_k)} \right) \right], \tag{5.24}$$

where the weight $\lambda > 0$ is the probability that the qudit passive state $\rho_P^{(d)}$ have support on the subspace A_k.

5.3   Instability of passive states

In the previous section, we have shown that energy can be extracted from a system with an arbitrary Hamiltonian H_P, described by any passive (but not completely passive) state ρ_P. In this section, we focus on the problem of optimal energy extraction, and we build on the previous cycle to obtain a protocol able to extract the maximum amount of energy possible from the system. This protocol requires an infinite-dimensional catalyst, and it allows us to transform the passive state along continuous trajectories in state space, whose end lies inside the set of thermal states. When the optimal amount of energy is extracted, we find that the protocol maps the passive state into a thermal state with the same von Neumann entropy and a lower average energy. However, we can tune the protocol to obtain different thermal states, for example we can also use it to map the passive state into a thermal state with same average energy and higher von Neumann entropy.

Let us consider the cycle introduced in the previous section, and send the number of hot swaps m and cold swaps n to infinity, while keeping their ratio finite. From Eq. (5.12) we know that the cycle allows us to extract energy if the ratio between hot and cold swaps is

$$\gamma = \frac{m}{n} \in \left(\frac{\Delta E_{21}}{\Delta E_{10}}, \frac{\beta_{\text{cold}}}{\beta_{\text{hot}}} \frac{\Delta E_{21}}{\Delta E_{10}} \right), \tag{5.25}$$

where we are considering the case in which the qutrit passive state ρ_P is parametrised by β_{hot} and β_{cold} as in Eq. (5.6), and $\beta_{\text{hot}} < \beta_{\text{cold}}$. When the number of swaps performed tends to infinity, we have that the dimension of the catalyst tends to infinity too, while the change ΔP in the populations of the passive state can be shown to tend to 0, with an exponential scaling in the number of swaps performed in the cycle.

Our protocol for optimal energy extraction uses the cycle described above, involving an infinite number of hot and cold swaps. Specifically, we apply this cycle on the system N times,

for $N \to \infty$, and each time we adjust the parameter γ. At each iteration, the population of the passive state changes infinitesimally, and a different (infinite-dimensional) catalyst is used. We can think of using a number N of different catalysts, or to use a single catalyst with dimension $d = N(m + n)$, where N, m, and n tend to infinity. The evolution of the passive state is given by Eqs. (5.16), which can be expressed as a set of differential equations, since ΔP is infinitesimally small. In appendix D.3 and D.4, we show that the evolution of the passive state during this protocol is described by the following differential equation

$$\frac{\mathrm{d}p_1}{\mathrm{d}t} = -\left(1 + \gamma(t)^{-1}\right) \frac{\mathrm{d}p_0}{\mathrm{d}t}, \tag{5.26}$$

where t is a continuous parameter labelling the sequence of cycles that infinitesimally modify the passive state.

We provide now the solution of the above equation for the case in which the parameter γ is equal to the extremal values of the energy-extraction interval of Eq. (5.25). In particular,

1. Consider $\gamma(t) = \frac{\Delta E_{21}}{\Delta E_{10}}$, i.e, the ratio between the hot and cold swaps used in each cycle is constant during the whole evolution of the system. In this case, Eq. (5.26) constrains the state of the system to evolve along a trajectory of constant energy, and it can be expressed as

$$\mathrm{Tr}\left[H_P \, \rho_P\right] = \mathrm{Tr}\left[H_P \, \tilde{\rho}_P(t)\right], \quad \forall \, t \geq 0. \tag{5.27}$$

   Thus, for this choice of $\gamma(t)$ we have that the protocol map the passive state ρ_P toward the set of thermal states while keeping the energy fixed.

2. Consider $\gamma(t) = \frac{\beta_{\mathrm{cold}}(t)}{\beta_{\mathrm{hot}}(t)} \frac{\Delta E_{21}}{\Delta E_{10}}$, where the value of the virtual temperatures changes after each cycle. For this choice of $\gamma(t)$, one can show that the differential equation can be re-expressed as a constraint over the entropy of the system, namely,

$$S\left(\rho_P\right) = S\left(\tilde{\rho}_P(t)\right), \quad \forall \, t \geq 0. \tag{5.28}$$

   Therefore, in this case we have that the state of the system evolves toward the set of thermal states while keeping fixed the von Neumann entropy.

These two trajectories can be visualised in a two-dimensional diagram representing the set of passive states, see the left panel of Fig. 5.3. Furthermore, any intermediate trajectory between

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the ones defined by Eqs. (5.27) and (5.28) can be achieved with our protocol, if we select an appropriate parameter $\gamma(t)$. In this way, starting from a passive state ρ_P we can reach a subset of thermal states. These are the stationary states of our evolution, as we have shown in the last part of Sec. 5.2.1. By interrupting the cycle before reaching a thermal state, we can produce any passive state with both lower or equal average energy, and greater or equal entropy than the initial state. This set of states is shown in the resource diagram for energy and entropy introduced in the previous chapters, see the right panel of Fig. 5.3.

Let us now consider the final states obtained when the parameter $\gamma(t)$ takes one of its extremal values. In situation 1, when the energy is preserved, the evolution maps the state into a thermal state of the Hamiltonian H_P with inverse temperature $\beta_{\min}$, defined as

$$\beta_{\min} \; : \; \text{Tr} \left[H_P \, \tau_{\beta_{\min}} \right] = \text{Tr} \left[H_P \, \rho_P \right]. \tag{5.29}$$

Then, this protocol does not extract energy from the system, but instead it increases the entropy of the state, by creating correlations with the catalyst. In terms of efficiency, we have that this protocol has $\eta = 0$, see Eq. (5.15). Although this evolution is not relevant for the ultimate goal of energy extraction, it represents a possible mechanism for a closed system to reach thermal equilibrium. In particular, the thermal state is obtained here as a result of the onset of classical correlations between system and catalyst. Notice, however, that this mechanism is extremely fine-tuned, since we are using a specific catalyst and a specific interaction. To show that this process can explain the phenomenon of thermalisation in nature, one should prove that the same trajectory can be obtained for (almost) all passive states when a random catalyst is used, and a Haar-random unitary is applied on system and catalyst.

In situation 2, instead, we find that energy is extracted from the system. In particular, the initial state ρ_P is mapped, following a trajectory of constant entropy, into the thermal state with inverse temperature $\beta_{\max}$, defined as

$$\beta_{\max} \; : \; S(\tau_{\beta_{\max}}) = S(\rho_P). \tag{5.30}$$

This protocol allows us to extract energy from the system. In particular, one can show that the amount of energy extracted is optimal, i.e., no other protocol using a catalyst and a sequence of unitary operations can extract more energy. Indeed, the final state of any such protocol is of
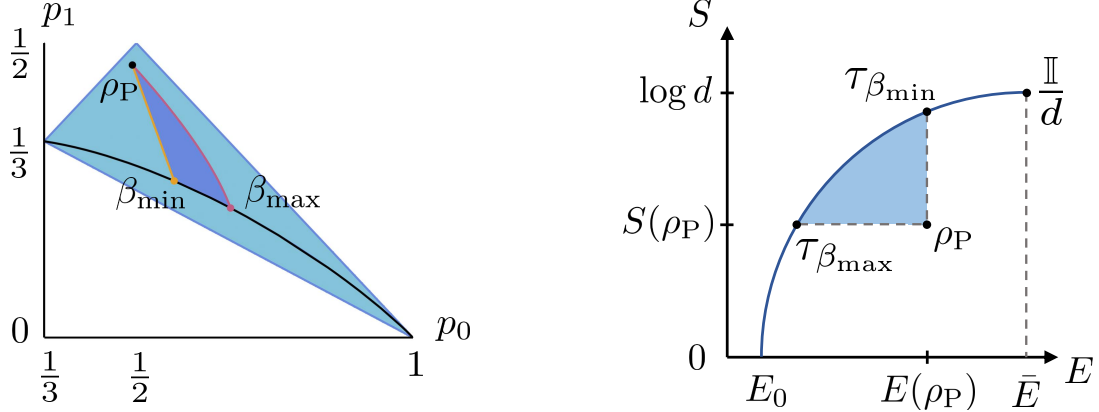
Figure 5.3: **Left.** The state space of a qutrit system, where the region of passive states is highlighted in light blue. The black line contained in the passive region is the set of thermal states. We fix an initial state ρ_P, represented by the black point in the diagram. Then, we evolve this state by applying the protocol described in this section. The evolution is modulated by the parameter $\gamma(t)$ defined in Eq. (5.25). For $\gamma(t) = \frac{\Delta E_{21}}{\Delta E_{10}}$, the system evolves along the yellow trajectory, and the final state is the thermal state at temperature $\beta_{\min}$ (with same average energy of ρ_P). For $\gamma(t) = \frac{\beta_{\mathrm{cold}}(t)}{\beta_{\mathrm{hot}}(t)} \frac{\Delta E_{21}}{\Delta E_{10}}$, the system evolves along the purple trajectory, and the final state is the thermal state at temperature $\beta_{\max}$ (with same von Neumann entropy of ρ_P). The dark blue region represents the subset of achievable states for the initial state is ρ_P. **Right.** A partial representation of the state space of a d-level quantum system in the energy-entropy diagram. On the x-axis we have the average energy $E(\rho) = \mathrm{Tr}\,[H_P\,\rho]$, and on the y-axis we have the von Neumann entropy $S(\rho)$. Each point in the diagram is an equivalence class of states with fixed average energy and entropy. Here, we solely represent the states with average energy lower than $\bar{E} = \mathrm{Tr}\,\left[H_P \frac{\mathbb{I}}{\mathrm{d}}\right]$, where $\frac{\mathbb{I}}{\mathrm{d}}$ is the maximally-mixed state, since all passive states are contained within this set. For a given initial state ρ_P, the light blue region contains all the passive states which can be achieved with our protocol.

the form $U\,\rho_P \otimes \rho_M\,U^\dagger$, where U is any unitary operation acting on system and catalyst, and ρ_M is a generic state of the catalyst such that $\rho_M = \mathrm{Tr}\,\left[U\,\rho_P \otimes \rho_M\,U^\dagger\right]$. Then, we have that

$$S(\rho_P) + S(\rho_M) = S(\rho_P \otimes \rho_M) = S(U\,\rho_P \otimes \rho_M\,U^\dagger) \le S(\tilde{\rho}_P) + S(\rho_M), \qquad (5.31)$$

where the inequality follows from subadditivity of the von Neumann entropy, and $\tilde{\rho}_P$ is the final

state of the system alone. The above equation implies that these protocols never decrease the entropy of the system, so that $S(\rho_P) \leq S(\tilde{\rho}_P)$. It is easy to show that, for a fixed entropy, the states with minimum energy are thermal states. Furthermore, if we restrict ourselves to the set of thermal states, we have that the average energy increases as the entropy increases, see the right panel of Fig. 5.3. Thus, the optimal amount of energy extracted by our protocol is obtained when the entropy associated with the final thermal state is minimum, i.e., when it is equal to the entropy of the initial state ρ_P, which is the case we are considering in Eq. (5.30).

As a result, the maximum energy we can extract from a passive state using catalysts and unitary operations is

$$\Delta W_{\max} = \text{Tr} \left[H_P \left(\rho_P - \tau_{\beta_{\max}} \right) \right]. \tag{5.32}$$

We refer to the above quantity as the *catalytic ergotropy* – see Ref. [196] – associated with the passive state ρ_P, since this is the maximum energy extracted from a single copy of the system when catalytic reversible operations are allowed. The energy-extraction efficiency of this protocol is given by the Carnot efficiency $\eta_\text{Carnot} = 1 - \frac{\beta_\text{hot}}{\beta_\text{cold}}$, as one might expect as this process, which resembles the ones taking place in heat engines, is optimal. Furthermore, since the entropy of both system and catalyst is unchanged, we have that no correlations have been created during the process. Finally, it is worth noting that, in the case of open quantum systems, the problem of catalytic work extraction has been the object of extensive studies, see for instance Refs. [146, 197, 56, 55]. When the agent does not have access to a catalyst, they can nevertheless achieve optimal work extraction from passive states by acting with a global operation over n copies of the system, in the limit of $n \to \infty$, as shown in Ref. [119]. This result also follows from Thm. 11, shown in the previous chapter.

5.4   Chapter summary

In this chapter, we consider energy extraction from a class of equilibrium states known as passive states. These states describe a closed system, and were thought to be energetically stable, since no unitary evolution can lower their energy. We design an explicit protocol, which makes use of a catalyst, for extracting energy from any single copy of an athermal passive state, see Sec. 5.2. This ancillary system participates in the energy extraction process, and

we require the local state of this catalyst to be recovered at the end of the cycle (although correlations can arise during the process). In this way, the cycle can be run multiple times, and each time it acts over a new copy of the passive state. The cycle could find applications in energy extracting devices, for example in situations where some almost-thermal garbage states are produced by the processes occurring in the device. Realising our protocol in the laboratory might even be possible nowadays, at least for those passive states which require catalysts with a small dimension.

We then show that, when the agent has access to an infinite-dimensional catalyst, they are able to smoothly evolve the passive state toward the set of thermal states. Interestingly, the evolution is tuned by a free parameter γ, see Sec. 5.3, and different trajectories can be achieved. For example, the agent can evolve the system toward the thermal states while keeping the energy fixed (and therefore increasing its entropy), or vice versa, by keeping the entropy fixed. In this latter case, the amount of energy extracted is optimal, and the final state is a thermal state with the same entropy of the initial passive state. This result provides a way to single out thermal states, and consequently to recover the notion of temperature, without having to consider the thermodynamic limit.

Our results provide some evidence that a resource theory for thermodynamics with an imperfect thermal reservoir, such as the one suggested at the end of the previous chapter, presents non-trivial challenges. Such a resource theory could be realised by providing passive states for free. However, an obvious restriction we should make in this resource theory consists in the fact that we could not provide more than k copies of a k-passive state, see Def. 34, otherwise energy might be extracted with unitary operations from this free state. Moreover, our results show that, even in the case in which a single passive state is provided, an ancillary system exists such that energy can be extracted from it. Thus, in order to build a sensible resource theory, passive states should be always provided at an energy cost, equal to the optimal amount of energy extractable from them when a machine is present.

Part III

Appendices

Appendix A

Hypothesis testing and quantum Stein's lemma

In this appendix, we present some results (well-known in the literature) on *hypothesis testing*, which find use in the context of reversible resource theories. Hypothesis testing, a branch of information theory, provides tools to quantitatively study those scenarios where an observer needs to distinguish between two sets of probability distributions describing a random variable [198, Chp. 12]. These tools also extend to the quantum theory, where the observer needs to distinguish between two sets of quantum states describing the system under investigation. In particular, here we focus on quantum Stein's lemma, which links the asymptotic scaling of the error in distinguishing between two states with their relative entropy. This connection is then used, in the background chapter on resource theories, to build a class of allowed operations which makes a theory reversible, see Sec. 1.4.

A.1   Quantum Stein's lemma

Consider the situation in which an observer is given a source of quantum systems described by the Hilbert space $\mathcal{H}$. Suppose that each use of the source produces independent and identically distributed copies of either the quantum state $\rho \in \mathcal{S}(\mathcal{H})$, referred to as the *null hypotheses*, or $\sigma \in \mathcal{S}(\mathcal{H})$, referred to as the *alternative hypotheses*. Finding the optimal way to learn whether the source is producing ρ or σ is the main goal of that branch of quantum information theory

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known as *quantum hypothesis testing*. In order to learn which of the two states the source is producing, the observer is allowed to perform a measurement over many copies of the system. Suppose the observer uses the source n times, and subsequently performs a POVM over the global system, see Def. 4. This POVM is composed by two effects, $E_n \in \mathcal{B}(\mathcal{H}^{\otimes n})$, associated with the null outcome ("the state is ρ"), and $\mathbb{I}_n - E_n$, associated with the alternative outcome ("the state is σ"). The measurement can then be optimised over the following two errors,

Type I The probability of obtaining the alternative outcome when the measured state is ρ.

$$\mathrm{p}_n^{\mathrm{Type\ I}}(E_n) = \mathrm{Tr}\left[\rho^{\otimes n}\left(\mathbb{I}_n - E_n\right)\right]. \tag{A.1}$$

Type II The probability of obtaining the null outcome when the measured state is σ.

$$\mathrm{p}_n^{\mathrm{Type\ II}}(E_n) = \mathrm{Tr}\left[\sigma^{\otimes n} E_n\right]. \tag{A.2}$$

Depending on the physical situation, the observer might need to optimise the measurement in different ways. We now specialise to the case in which the observer needs to minimise the probability of the type II error, while keeping the probability of the type I error below a fixed threshold. This specific case is known in the literature as *asymmetric hypothesis testing*. The relevant quantity in this scenario is therefore

$$\beta_n(\delta) := \min_{0 \le E_n \le \mathbb{I}_n}\left\{\mathrm{p}_n^{\mathrm{Type\ II}}(E_n) \mid \mathrm{p}_n^{\mathrm{Type\ I}}(E_n) \le \delta\right\}, \tag{A.3}$$

where $\delta > 0$. How this quantity scales, as the number of copies n measured by the observer tends to infinity, is described by quantum Stein's lemma [199, 200],

Theorem 12. *Consider a finite dimensional Hilbert space $\mathcal{H}$, and two states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$. Then, the quantity $\beta_n(\delta)$ defined in Eq. (A.3) has the following scaling, for all $\delta > 0$,*

$$\lim_{n \to \infty} -\frac{1}{n}\log\beta_n(\delta) = D(\rho \,\|\, \sigma), \tag{A.4}$$

where $D(\rho \,\|\, \sigma)$ is the quantum relative entropy defined in Eq. (1.9).

The above theorem provides an operational meaning to the quantum relative entropy. Indeed, in the current setting this quantity represents the asymptotic decay rate of the probability of type II error, in the case in which the probability of type I error is bounded by a constant factor, arbitrarily close to zero.

A.2   A generalisation of quantum Stein's Lemma

We now consider a generalisation of the previous scenario, which has been studied in Ref. [38]. Suppose that after n uses of the source, the observer is left with a global quantum system which is described by either n i.i.d. copies of the state $\rho \in \mathcal{S}(\mathcal{H})$ (the null hypotheses), or by a random state belonging to the set $\mathcal{F}^{(n)} \subset \mathcal{S}(\mathcal{H}^{\otimes n})$ (the alternative hypotheses). In particular, the family of sets $\mathcal{F}^{(n)}$, parametrised by the number of uses n of the source, is supposed to satisfy the assumptions F1 – F5 introduced in Sec. 1.3 of the background chapter on resource theories. Among these assumptions, the most important one for the present discussion is the last one. Indeed, the closeness of the sets $\mathcal{F}^{(n)}$ under permutations implies that we can make use of a number of results known as de Finetti's theorems [201, 171, 45], connecting the states contained in these sets to i.i.d. copies of single-system states. Using this connection, one can derive for the current scenario a very similar statement to the one of Thm. 12.

The measurement performed by the observer in this scenario is described by a POVM with two effects, $\{E_n, \mathbb{I}_n - E_n\}$, associated with the null and alternative outcomes, respectively. While the probability of type I error is still given by Eq. (A.1), we have that the probability of type II error defined in Eq. (A.2) is now replaced by the following one,

$$\tilde{p}_n^{\text{Type II}}(E_n) = \max_{\gamma_n \in \mathcal{F}^{(n)}} \text{Tr}\left[\gamma_n E_n\right]. \tag{A.5}$$

In the asymmetric hypothesis testing scenario, the observer want to minimise the type II error while bounding the probability of type I error by an arbitrarily small constant factor $\delta > 0$. The relevant quantity the observer needs to minimise in this case is

$$\tilde{\beta}_n(\delta) := \min_{0 \leq E_n \leq \mathbb{I}_n} \left\{ \tilde{p}_n^{\text{Type II}}(E_n) \mid p_n^{\text{Type I}}(E_n) \leq \delta \right\}. \tag{A.6}$$

The following theorem, proved in Ref. [38, Thm. 1], links the asymptotic decay rate of $\tilde{\beta}_n(\delta)$ to the regularised relative entropy distance from the set $\mathcal{F} = \mathcal{F}^{(1)}$, defined in Eq. (1.14). This theorem is a generalisation of quantum Stein's Lemma to the current scenario.

Theorem 13. *Consider a finite dimensional Hilbert space $\mathcal{H}$, a state $\rho \in \mathcal{S}(\mathcal{H})$, and a family of sets $\mathcal{F}^{(n)} \subset \mathcal{S}(\mathcal{H}^{\otimes n})$ satisfying the assumptions F1, F2, F3, F4, and F5. Then, for all $\delta > 0$,*

$$\lim_{n \to \infty} -\frac{1}{n} \log \tilde{\beta}_n(\delta) = E_{\mathcal{F}}^{\infty}(\rho), \tag{A.7}$$

where $E_{\mathcal{F}}^{\infty}$ is the regularised relative entropy distance from the set $\mathcal{F} = \mathcal{F}^{(1)}$.

The implications of this theorem do not solely concern the field of hypothesis testing. Indeed, as we briefly sketch in Sec. 1.4, the theorem can be used in the context of resource theories [44, 45, 32], in order to build a set of allowed operations which make the theory reversible in the sense of Def. 16.

Appendix B

Structure of multi-resource theories

In this appendix we present some additional information on the tools presented in Ch. 3, together with the proofs of some of the main results of the chapter.

B.1 Uniqueness proofs

In this part of the appendix we prove the main results on the uniqueness of resource quantifiers, which have been used in Ch. 3. Let us start by proving a trivial lemma, which can be found in Ref. [202, Prop. 13] as well, useful for the proofs of the main theorems we consider in this section.

Lemma 7. *Given a regularisable function* $f : \mathcal{S}\left(\mathcal{H}^{\otimes n}\right) \to \mathbb{R}$*, the regularised version is extensive,*

$$f^\infty(\rho^{\otimes k}) = k\, f^\infty(\rho)\ ,\ \forall\, \rho \in \mathcal{S}\left(\mathcal{H}\right)\ ,\ \forall\, k \in \mathbb{N}. \tag{B.1}$$

Proof. Consider a function $h : \mathbb{R} \to \mathbb{R}$, such that $\lim_{n\to\infty} h(n) = L < \infty$. This is equivalent to saying that

$$\forall\, \epsilon > 0, \exists\, c \in \mathbb{R}\ :\ |h(n) - L| < \epsilon,\ \forall\, n > c. \tag{B.2}$$

Let us now consider an invertible function $g : \mathbb{R} \to \mathbb{R}$, and consider $m \in \mathbb{R}$ such that $n = g(m)$. Then, we can rewrite Eq. (B.2) as

$$\forall\, \epsilon > 0, \exists\, c \in \mathbb{R}\ :\ |h(g(m)) - L| < \epsilon,\ \forall\, g(m) > c, \tag{B.3}$$

and by defining $\tilde{c} = g^{-1}(c)$, we get

$$\forall \epsilon > 0, \exists \tilde{c} \in \mathbb{R} \ : \ |h(g(m)) - L| < \epsilon, \ \forall m > \tilde{c}. \tag{B.4}$$

Therefore, we have $\lim_{m \to \infty} h(g(m)) = L$.

If we choose $h(n) = \frac{1}{n} f(\rho^{\otimes n})$, whose limit is $L = f^{\infty}(\rho)$, and we use the invertible function $g(m) = k \cdot m$ where $k \in \mathbb{N}$ is fixed, we get

$$f^{\infty}(\rho) = \lim_{m \to \infty} \frac{1}{k \cdot m} f(\rho^{\otimes k \cdot m}) = \frac{1}{k} \lim_{m \to \infty} \frac{1}{m} f((\rho^{\otimes k})^{\otimes m}) = \frac{1}{k} f^{\infty}(\rho^{\otimes k}), \tag{B.5}$$

which proves the lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We can now prove Thm. 8, stating that a multi-resource theory satisfying asymptotic equivalence has a unique quantifier for each of the resources present in the theory.

Theorem 8. *Consider the resource theory* $\mathrm{R_{multi}}$ *with m resources, equipped with the batteries* B_i*'s, where $i = 1, \ldots, m$. Suppose the theory satisfies the asymptotic equivalence property with respect to the set of monotones $\{f_i\}_{i=1}^m$. If these monotones satisfy the properties M1 – M7, and their regularisations are not identically zero over the whole state space, then the amount of i-th resource contained in the main system S is uniquely quantified by the regularisation of the monotone f_i, i.e., every other regularised monotone is equal to f_i^{∞} up to a multiplicative constant.*

Proof. Let us prove that f_1^{∞} uniquely quantifies the amount of 1-st resource contained in the main system (the proof for the other $f_{i \neq 1}$'s is analogous). We prove the theorem by contradiction. Suppose that there exists two monotones f_1 and g_1 satisfying the properties M1 – M7, such that

1. $\exists \rho \in \mathcal{S}(\mathcal{H}_S)$, where $\rho \notin \mathcal{F}_1$, for which $f_1^{\infty}(\rho) = g_1^{\infty}(\rho)$ (this is always possible by rescaling the monotone g).

2. $\exists \sigma \in \mathcal{S}(\mathcal{H}_S)$, where $\sigma \notin \mathcal{F}_1$, for which $f_1^{\infty}(\sigma) \neq g_1^{\infty}(\sigma)$ (that is, f_1 is not unique).

Consider now the values of $f_1^{\infty}(\rho)$ and $f_1^{\infty}(\sigma)$. If these are equal, it is easy to see, using the asymptotic equivalence property, that f_1 is unique. Suppose instead that they are not equal.

Then, there exists $n, k \in \mathbb{N}$[1] such that

$$n f_1^\infty(\rho) = k f_1^\infty(\sigma). \tag{B.6}$$

Let us consider the system together with the batteries B_i's, initially in the state $\rho^{\otimes n} \otimes \omega_1 \otimes \ldots \otimes \omega_m$. Then, we take the states $\omega_i' \in \mathcal{S}(\mathcal{H}_{B_i})$, where $i = 1, \ldots, m$, such that

$$f_i^\infty(\rho^{\otimes n} \otimes \omega_1 \otimes \ldots \otimes \omega_m) = f_i^\infty(\gamma_n \otimes \omega_1' \otimes \ldots \otimes \omega_m') \,, \; \forall \, i \in \{1, \ldots, m\} \,, \tag{B.7}$$

$$f_j^\infty(\omega_i) = f_j^\infty(\omega_i') \,, \; \forall \, i, j \in \{1, \ldots, m\} \,, \; i \neq j, \tag{B.8}$$

where $\gamma_n \in \mathcal{F}_1^{(n)}$. Due to the asymptotic equivalence property, the conditions in Eq. (B.7) imply that there exists a sequence of maps $\{\tilde{\varepsilon}_N\}_N$ of the form of Eq. (3.4) such that

$$\lim_{N \to \infty} \left\| \tilde{\varepsilon}_N \left(\left(\rho^{\otimes n} \otimes \omega_1 \otimes \ldots \otimes \omega_m \right)^{\otimes N} \right) - \left(\gamma_n \otimes \omega_1' \otimes \ldots \otimes \omega_m' \right)^{\otimes N} \right\|_1 = 0, \tag{B.9}$$

as well as another sequence of maps performing the reverse transformation. From the asymptotic continuity of g_1, property M7, it then follows that

$$g_1 \left(\tilde{\varepsilon}_N \left(\left(\rho^{\otimes n} \otimes \omega_1 \otimes \ldots \otimes \omega_m \right)^{\otimes N} \right) \right) = g_1 \left(\left(\gamma_n \otimes \omega_1' \otimes \ldots \otimes \omega_m' \right)^{\otimes N} \right) + o(N). \tag{B.10}$$

Let us consider the lhs of the above equation, and recall that the map $\tilde{\varepsilon}_N$ is obtained by applying an allowed operation to N copies of the system together with a sub-linear ancilla $\eta_N^{(A)}$. For simplicity, in the following chain of inequalities we refer to $\rho^{\otimes n} \otimes \omega_1 \otimes \ldots \otimes \omega_m$ as Ω,

$$g_1 \left(\tilde{\varepsilon}_N \left(\Omega^{\otimes N} \right) \right) = g_1 \left(\mathrm{Tr}_A \left[\varepsilon_N \left(\Omega^{\otimes N} \otimes \eta_N^{(A)} \right) \right] \right) \leq g_1 \left(\varepsilon_N \left(\Omega^{\otimes N} \otimes \eta_N^{(A)} \right) \right)$$
$$\leq g_1 \left(\Omega^{\otimes N} \otimes \eta_N^{(A)} \right) \leq g_1 \left(\Omega^{\otimes N} \right) + g_1 \left(\eta_N^{(A)} \right) \leq g_1 \left(\Omega^{\otimes N} \right) + o(N) \tag{B.11}$$

where the first inequality follows from property M4, the second one from the monotonicity of g_1 under allowed operations, the third one from the sub-additivity of g_1, property M5, and the last inequality from property M6 and the fact that the ancilla is sub-linear in N. If we now combine this equation with the previous one, we divide both sides by N, and we send it to infinity, we obtain that the regularised version of g_1 is such that,

$$g_1^\infty \left(\rho^{\otimes n} \otimes \omega_1 \otimes \ldots \otimes \omega_m \right) \geq g_1^\infty \left(\gamma_n \otimes \omega_1' \otimes \ldots \otimes \omega_m' \right). \tag{B.12}$$

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[1]Where we assume that all physically meaningful values of the f_i^∞'s are in $\mathbb{Q}$, which we recall is dense in $\mathbb{R}$.

By using the same argument for the sequence of maps performing the reverse transformation, we find that the above equation needs to hold as an equality, that is,

$$g_1^\infty \left(\rho^{\otimes n} \otimes \omega_1 \otimes \ldots \otimes \omega_m \right) = g_1^\infty \left(\gamma_n \otimes \omega_1' \otimes \ldots \otimes \omega_m' \right). \tag{B.13}$$

We can now separate each contribution to g_1 thanks to the property M2, use the fact that the batteries $B_{i \neq 1}$'s are not changing their value of g_1, property M1, and the fact that the final state of the system does not contain any resource associated with g_1, property M3. Then, we find that

$$n\, g_1^\infty (\rho) = g_1^\infty \left(\omega_1' \right) - g_1^\infty \left(\omega_1 \right), \tag{B.14}$$

where we have also used Lem. 7. The same result follows for f_1, so that we find that

$$n\, f_1^\infty (\rho) = f_1^\infty \left(\omega_1' \right) - f_1^\infty \left(\omega_1 \right). \tag{B.15}$$

If we now consider Eqs. (B.6) and (B.15), we find that

$$k\, f_1^\infty (\sigma) = f_1^\infty \left(\omega_1' \right) - f_1^\infty \left(\omega_1 \right). \tag{B.16}$$

We can add to the above equation the term $f_1^\infty (\gamma_k)$, where $\gamma_k \in \mathcal{F}_1^{(k)}$, since this term is equal to zero due to property M3. Then, we find

$$k\, f_1^\infty (\sigma) + f_1^\infty (\omega_1) = f_1^\infty (\gamma_k) + f_1^\infty \left(\omega_1' \right). \tag{B.17}$$

Now, we want to introduce the initial and final states of the batteries $B_{i \neq 1}$'s, so as to be sure that the transformation from $\sigma^{\otimes k}$ into γ_k does not violate the conservation of the other resources. Specifically, we introduce $\omega_i, \omega_i'' \in \mathcal{S}(\mathcal{H}_{B_i})$ for $i \neq 1$, such that

$$f_i^\infty \left(\sigma^{\otimes k} \otimes \omega_1 \otimes \omega_2 \otimes \ldots \otimes \omega_m \right) = f_i^\infty \left(\gamma_k \otimes \omega_1' \otimes \omega_2'' \otimes \ldots \otimes \omega_m'' \right) , \ \forall\, i \in \{2, \ldots, m\}, \tag{B.18}$$

$$f_1^\infty(\omega_i) = f_1^\infty(\omega_i'') , \ \forall\, i \in \{2, \ldots, m\}, \tag{B.19}$$

$$f_j^\infty(\omega_i) = f_j^\infty(\omega_i'') , \ \forall\, i, j \in \{2, \ldots, m\}, \ i \neq j. \tag{B.20}$$

Then, using the constraints of Eq. (B.19) over the states of the $B_{i \neq 1}$'s batteries, we can re-write Eq. (B.17) as

$$k\, f_1^\infty (\sigma) + f_1^\infty (\omega_1) + f_1^\infty (\omega_2) + \ldots + f_1^\infty (\omega_m) = f_1^\infty (\gamma_k) + f_1^\infty \left(\omega_1' \right) + f_1^\infty \left(\omega_2'' \right) + \ldots + f_1^\infty \left(\omega_m'' \right). \tag{B.21}$$

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If we now use Lem. 7 and property M1, we find that

$$f_1^\infty \left(\sigma^{\otimes k} \otimes \omega_1 \otimes \omega_2 \otimes \ldots \otimes \omega_m \right) = f_1^\infty \left(\gamma_k \otimes \omega_1' \otimes \omega_2'' \otimes \ldots \otimes \omega_m'' \right). \tag{B.22}$$

From Eqs. (B.18) and (B.22) it follows, using the asymptotic equivalence property, that there exists a sequence of maps $\{\tilde{\varepsilon}_N'\}_N$ such that

$$\lim_{N\to\infty} \left\| \tilde{\varepsilon}_N' \left(\left(\sigma^{\otimes k} \otimes \omega_1 \otimes \omega_2 \otimes \ldots \otimes \omega_m \right)^{\otimes N} \right) - \left(\gamma_k \otimes \omega_1' \otimes \omega_2'' \otimes \ldots \otimes \omega_m'' \right)^{\otimes N} \right\|_1 = 0, \tag{B.23}$$

as well as a related sequence of maps performing the reverse transformation. Using the properties of g_1, as we did before, we find that

$$k\, g_1^\infty \left(\sigma \right) = g_1^\infty \left(\omega_1' \right) - g_1^\infty \left(\omega_1 \right). \tag{B.24}$$

Then, combining Eqs. (B.14) and (B.24), we obtain that

$$n\, g_1^\infty \left(\rho \right) = k\, g_1^\infty \left(\sigma \right). \tag{B.25}$$

Finally, using Eq. (B.6) and the initial assumption on the state ρ, we find that

$$f_1^\infty \left(\sigma \right) = g_1^\infty \left(\sigma \right), \tag{B.26}$$

which contradicts our initial assumption. Therefore, f_1^∞ uniquely quantifies the amount of 1-st resource contained in the main system. $\qquad\square$

The next result we prove, Thm. 10, concerns the passage from a multi-resource theory to a single-resource one. The passage is obtained by defining a new set of allowed operations, see Def. 32, where the agent is allowed to add an arbitrary number of bank states. We show that such single-resource theory satisfies asymptotic equivalence with respect to the bank monotone defined in Eq. (3.31).

Theorem 10. *Consider the two-resource theory* $\mathrm{R}_{\mathrm{multi}}$ *with allowed operations* $\mathcal{A}_{\mathrm{multi}}$, *and invariant sets* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *which satisfy the properties F1, F2, and F3b. Suppose the theory satisfies the asymptotic equivalence property with respect to the monotones* $E_{\mathcal{F}_1}$ *and* $E_{\mathcal{F}_2}$. *Then, given the subset of bank states* $\mathcal{F}_{\mathrm{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2} \right)$, *the single-resource theory* $\mathrm{R}_{\mathrm{single}}$ *with allowed operations* $\mathcal{A}_{\mathrm{single}}$ *satisfies the asymptotic equivalence property with respect to* $f_{\mathrm{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}$.

Proof. **(a)** We start the proof by showing that, for the single resource theory R_{single}, the second statement in Def. 30 implies the first one. In other words, we want to show that for any two states $\rho, \sigma \in S(\mathcal{H})$ which can be asymptotically mapped into one another with the allowed operations $\mathcal{A}_{single}$, the value of the bank monotone on the two states is the same. Suppose there exists a sequence of operations $\left\{\tilde{\varepsilon}_N^{(s)}\right\}_N$ such that $\lim_{N\to\infty} \left\| \tilde{\varepsilon}_N^{(s)}(\rho^{\otimes N}) - \sigma^{\otimes N} \right\|_1 = 0$, where these maps are of the form

$$\tilde{\varepsilon}_N^{(s)}(\cdot) = \mathrm{Tr}_A\left[\varepsilon_N^{(s)}(\cdot \otimes \eta_N^{(A)})\right],\tag{B.27}$$

with $\eta_N^{(A)} \in S\left(\mathcal{H}^{\otimes o(N)}\right)$ an arbitrary state of a sub-linear ancilla, and $\varepsilon_N^{(s)}$ an allowed operation for R_{single}. Likewise, suppose there is a sequence of maps that perform the reverse transformation. If we use the asymptotic continuity of the bank monotone, property B6, it follows that

$$f_{bank}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}\left(\tilde{\varepsilon}_N^{(s)}(\rho^{\otimes N})\right) = f_{bank}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}\left(\sigma^{\otimes N}\right) + o(N).\tag{B.28}$$

Then, by using the properties B1 − B7 of the bank monotone, we can prove the following chain of inequalities for the lhs of the above equation

$$\begin{aligned}f_{bank}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}\left(\tilde{\varepsilon}_N^{(s)}(\rho^{\otimes N})\right) &= f_{bank}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}\left(\mathrm{Tr}_A\left[\varepsilon_N^{(s)}(\rho^{\otimes N} \otimes \eta_N^{(A)})\right]\right) \le f_{bank}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}\left(\varepsilon_N^{(s)}(\rho^{\otimes N} \otimes \eta_N^{(A)})\right)\\ &\le f_{bank}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}\left(\rho^{\otimes N} \otimes \eta_N^{(A)}\right) = f_{bank}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}\left(\rho^{\otimes N}\right) + f_{bank}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}\left(\eta_N^{(A)}\right)\\ &\le f_{bank}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}\left(\rho^{\otimes N}\right) + o(N)\end{aligned}\tag{B.29}$$

where the first inequality follows from monotonicity under partial trace, property B4, the second one from monotonicity under the allowed operations $\mathcal{A}_{single}$ (that we still need to show), the equality follows from additivity, property B3, and the last inequality from the extensivity of the monotone, property B5. If we use the same argument for the sequence of maps performing the reverse transformation, and we regularise the monotones by dividing the equations by the number of copies N, and sending N to infinity, we find that

$$f_{bank}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}\left(\rho\right) = f_{bank}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}\left(\sigma\right),\tag{B.30}$$

which proves the asymptotic equivalence property in one direction.

We still need to show that the bank monotone is monotonic under the allowed operations $\mathcal{A}_{single}$ of the single-resource theory. Recall that the most general of these operations, Eq. (3.47),

is given by

$$\varepsilon^{(\mathrm{s})}(\rho) = \mathrm{Tr}_{P^{(n)}} \left[\varepsilon(\rho \otimes \rho_P^{\otimes n}] \right], \tag{B.31}$$

where $\varepsilon \in \mathcal{A}_{\mathrm{multi}}$, and we add $n \in \mathbb{N}$ copies of the bank state $\rho_P \in \mathcal{F}_{\mathrm{bank}} \left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2} \right)$. Then, using the properties of the bank monotone, we can show that

$$\begin{aligned}
f_{\mathrm{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}} \left(\varepsilon^{(\mathrm{s})}(\rho) \right) &= f_{\mathrm{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}} \left(\mathrm{Tr}_{P^{(n)}} \left[\varepsilon(\rho \otimes \rho_P^{\otimes n}] \right) \right) \leq f_{\mathrm{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}} \left(\varepsilon(\rho \otimes \rho_P^{\otimes n}) \right) \\
&\leq f_{\mathrm{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}} \left(\rho \otimes \rho_P^{\otimes n} \right) = f_{\mathrm{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}} (\rho) + f_{\mathrm{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}} \left(\rho_P^{\otimes n} \right) \\
&= f_{\mathrm{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}} (\rho),
\end{aligned} \tag{B.32}$$

where the first inequality follows from property B4, the second one from the monotonicity under the allowed operations $\mathcal{A}_{\mathrm{multi}}$, property B7, and the last two equalities from additivity, property B3, and the fact that the bank monotone is equal to zero over the bank states, property B1, respectively.

(b) We now want to prove the other direction of the asymptotic equivalence property for the resource theory $\mathrm{R}_{\mathrm{single}}$, i.e., that the first statement in Def. 30 implies the second one. In other words, we want to show that for all states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ such that $f_{\mathrm{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}} (\rho) = f_{\mathrm{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}} (\sigma)$, there exists a sequence of operations $\left\{ \tilde{\varepsilon}_N^{(s)} \right\}_N$ of the form given in Eq. (B.27), mapping N copies of ρ into N copies of σ, where $N \to \infty$. Before proving this part of the theorem, we recall that, given the bank state $\rho_P \in \mathcal{F}_{\mathrm{bank}} \left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2} \right)$, all other bank states $\tilde{\rho}_P \in \mathcal{F}_{\mathrm{bank}}$ are such that, if $E_{\mathcal{F}_1}(\tilde{\rho}_P) = E_{\mathcal{F}_1}(\rho_P) + \delta$ with $\delta \ll 1$, then

$$E_{\mathcal{F}_2}(\tilde{\rho}_P) = E_{\mathcal{F}_2}(\rho_P) - \frac{\alpha}{\beta} \delta + O(\delta^2), \tag{B.33}$$

which follows from the fact that $f_{\mathrm{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}} = 0$ parametrises the line which is tangent to the state space and passes through the point $\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2} \right)$, see appendix B.3.

Given the two states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ with same value of the monotone $f_{\mathrm{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}$, let us introduce the sequences of states $\{ \sigma_n \in \mathcal{S}(\mathcal{H}) \}_n$ and $\{ \tilde{\rho}_{P,n} \in \mathcal{F}_{\mathrm{bank}} \}_n$ such that, for $n \in \mathbb{N}$ big enough, we have

$$E_{\mathcal{F}_1}(\sigma_n) = E_{\mathcal{F}_1}(\sigma) \tag{B.34}$$

$$E_{\mathcal{F}_1}(\rho \otimes \rho_P^{\otimes n}) = E_{\mathcal{F}_1}(\sigma_n \otimes (\tilde{\rho}_{P,n})^{\otimes n}), \tag{B.35}$$

$$E_{\mathcal{F}_2}(\rho \otimes \rho_P^{\otimes n}) = E_{\mathcal{F}_2}(\sigma_n \otimes (\tilde{\rho}_{P,n})^{\otimes n}), \tag{B.36}$$

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where $\rho_P \in \mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)$. From the above equations, and from the additivity of $E_{\mathcal{F}_1}$, which follows from property F3b, we obtain that

$$E_{\mathcal{F}_1}(\tilde{\rho}_{P,n}) = E_{\mathcal{F}_1}(\rho_P) + \frac{1}{n}\left(E_{\mathcal{F}_1}(\rho) - E_{\mathcal{F}_1}(\sigma)\right). \tag{B.37}$$

Notice that, for $n \to \infty$, we have that $\frac{1}{n}\left(E_{\mathcal{F}_1}(\rho) - E_{\mathcal{F}_1}(\sigma)\right) \to 0$, and therefore, for n sufficiently big, it follows from Eq. (B.33) that

$$E_{\mathcal{F}_2}(\tilde{\rho}_{P,n}) = E_{\mathcal{F}_2}(\rho_P) - \frac{\alpha}{\beta}\frac{1}{n}\left(E_{\mathcal{F}_1}(\rho) - E_{\mathcal{F}_1}(\sigma)\right) + O(n^{-2}). \tag{B.38}$$

If we now combine Eq. (B.36) and (B.38) together, we use the additivity of $E_{\mathcal{F}_2}$, and we use the fact that ρ and σ have the same value of the bank monotone, we obtain the following

$$E_{\mathcal{F}_2}(\sigma_n) = E_{\mathcal{F}_2}(\sigma) + O(n^{-1}). \tag{B.39}$$

Let us now focus on the operations mapping ρ into σ. We do this in two steps. First, we use the fact that the theory R_{multi} satisfies asymptotic equivalence, and we consider the Eqs. (B.35) and (B.36). These equations imply that, for all $n \in \mathbb{N}$, there exists of a sequence of maps $\{\tilde{\varepsilon}_{N,n}\}_N$ such that

$$\lim_{N \to \infty}\left\|\tilde{\varepsilon}_{N,n}\left(\left(\rho \otimes \rho_P^{\otimes n}\right)^{\otimes N}\right) - \left(\sigma_n \otimes (\tilde{\rho}_{P,n})^{\otimes n}\right)^{\otimes N}\right\|_1 = 0. \tag{B.40}$$

As per definition of asymptotic equivalence, the maps $\tilde{\varepsilon}_{N,n} : \mathcal{S}\left(\mathcal{H}^{\otimes N(n+1)}\right) \to \mathcal{S}\left(\mathcal{H}^{\otimes N(n+1)}\right)$ are of the form

$$\tilde{\varepsilon}_{N,n}(\cdot) = \text{Tr}_A\left[\varepsilon_{N,n}\left(\cdot \otimes \eta_N^{(A)}\right)\right] \tag{B.41}$$

where the map $\varepsilon_{N,n}$ is an allowed operation of R_{multi} acting on system and ancilla, and the state of the ancilla is $\eta_N^{(A)} \in \mathcal{S}\left(\left(\mathcal{H}^{\otimes n+1}\right)^{\otimes f(N)}\right)$, where $f(N) = o(N)$. Notice that, in particular, we can take n to be a monotonic function of N, $n = g(N)$, such that $\lim_{N \to \infty} g(N) = \infty$ and $f(N)g(N) = o(N)$. For example, if $f(N) \propto N^{1/2}$, we can chose $g(N) \propto N^{1/4}$, so that their product is $N^{3/4} = o(N)$.

We can now define the sequence of maps $\left\{\tilde{\varepsilon}_N^{(\text{s})}\right\}_N$ acting on $\mathcal{S}\left(\mathcal{H}^{\otimes N}\right)$. These maps are defined as

$$\tilde{\varepsilon}_N^{(\text{s})}(\rho^{\otimes N}) = \text{Tr}_P\left[\tilde{\varepsilon}_{N,g(N)}\left(\rho^{\otimes N} \otimes \rho_P^{\otimes Ng(N)}\right)\right], \tag{B.42}$$

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where we are tracing out the part of the system which was initially in the state $\rho_P^{\otimes Ng(N)}$. It is interesting to notice that this system is super-linear in the number of copies N of ρ, a condition that seems to be necessary to achieve the conversion, see Ref. [7] for an example in thermodynamics. We can re-write these maps as

$$\tilde{\varepsilon}_N^{(\text{s})}(\rho^{\otimes N}) = \text{Tr}_A \left[\varepsilon_N^{(\text{s})} \left(\rho^{\otimes N} \otimes \eta_N^{(A)} \right) \right], \tag{B.43}$$

where we recall that the ancillary system still lives on a sub-linear number of copies of $\mathcal{H}$, due to our choice of the function $g(N)$, and the operation $\varepsilon_N^{(\text{s})}$ is an allowed operations for the theory R_{single} – compare it with Eq. (3.47) – defined as

$$\varepsilon_N^{(\text{s})}(\cdot) = \text{Tr}_P \left[\varepsilon_{N,g(N)} \left(\cdot \otimes \rho_P^{\otimes Ng(N)} \right) \right]. \tag{B.44}$$

If we now use Eq. (B.40) together with the monotonicity of the trace distance under partial tracing, we find that

$$\lim_{N \to \infty} \left\| \tilde{\varepsilon}_N^{(\text{s})}(\rho^{\otimes N}) - \left(\sigma_{g(N)} \right)^{\otimes N} \right\|_1 = 0. \tag{B.45}$$

To conclude the proof, we notice that the sequence of states $\left\{ \sigma_{g(N)} \right\}_N$ does not need to converge to σ with respect to the trace distance. However, if we consider the regularisation of the $E_{\mathcal{F}_i}$'s on these states, we find that

$$\lim_{N \to \infty} \frac{1}{N} E_{\mathcal{F}_i}(\sigma_{g(N)}^{\otimes N}) = E_{\mathcal{F}_i}(\sigma), \quad i = 1, 2, \tag{B.46}$$

which follows from Eqs. (B.34) and (B.39). Then, we can use the asymptotic equivalence of R_{multi}, which tells us that there exists a second sequence of allowed operations, and a sub-linear ancilla, such that we can asymptotically transform the state of the system into σ. This concludes the proof. $\square$

B.2   Asymptotic continuity of average monotone

In this section of the appendix we prove that the monotone defined in Sec. 3.2.3, Eq. (3.23), is asymptotic continuous, see Def. 14 in the background chapter.

Proposition 6. *Consider an Hilbert space $\mathcal{H}$ with dimension d, an Hermitian operator $A \in \mathcal{B}(\mathcal{H})$, and the function $M_A : \mathcal{S}(\mathcal{H}) \to \mathbb{R}$ defined as*

$$M_A(\rho) = \text{Tr}\,[A\rho] - a_0, \tag{B.47}$$

where $\rho \in \mathcal{S}(\mathcal{H})$ is an element of the state space, and a_0 is the minimum eigenvalue of A. When n copies of the Hilbert space are considered, $\mathcal{H}_n = \otimes_{i=1}^n \mathcal{H}^{(i)}$, the above operator is extended as $A_n = \sum_{i=1}^n A^{(i)}$, where $A^{(i)} \in \mathcal{B}(\mathcal{H})$ acts on the i-th copy of the Hilbert space. Then, the function M_A is asymptotic continuous.

Proof. Consider two states $\rho_n, \sigma_n \in \mathcal{S}(\mathcal{H}^{\otimes n})$, such that $\|\rho_n - \sigma_n\|_1 \to 0$ for $n \to \infty$. We are interested in the difference between the value of the function M_A evaluated on ρ_n and σ_n. By definition,

$$|M_A(\rho_n) - M_A(\sigma_n)| = |\text{Tr}\left[(\rho_n - \sigma_n) A_n\right]|. \tag{B.48}$$

Now, we can diagonalise the operator $\rho_n - \sigma_n = \sum_\lambda \lambda |\psi_\lambda\rangle \langle\psi_\lambda|$. Then, we find

$$|\text{Tr}\left[(\rho_n - \sigma_n) A_n\right]| = \left|\sum_\lambda \lambda \langle\lambda| A_n |\lambda\rangle\right| \le \sum_\lambda |\lambda| |\langle\lambda| A_n |\lambda\rangle| \le \sum_\lambda |\lambda| \|A_n\|_\infty, \tag{B.49}$$

where we are using the operator norm $\|O\|_\infty = \sup_{|\psi\rangle \in \mathcal{H}} \frac{\|O|\psi\rangle\|}{\||\psi\rangle\|}$, and the last inequality straightforwardly follows from the definition of this norm. Then, due to the definition of A_n, it is easy to show that $\|A_n\|_\infty = n \|A\|_\infty$, and therefore

$$\sum_\lambda |\lambda| \|A_n\|_\infty = n \|A\|_\infty \sum_\lambda |\lambda| = n \|A\|_\infty \|\rho_n - \sigma_n\|_1. \tag{B.50}$$

Finally, notice that $\dim \mathcal{H}_n = d^n$, where d is fixed by the initial choice of $\mathcal{H}$. Then, we have,

$$|M_A(\rho_n) - M_A(\sigma_n)| \le n \log d \, \|A\|_\infty \|\rho_n - \sigma_n\|_1. \tag{B.51}$$

If we now divide by n both side of the inequality, we get that

$$\frac{|M_A(\rho_n) - M_A(\sigma_n)|}{n} \le \log d \, \|A\|_\infty \|\rho_n - \sigma_n\|_1, \tag{B.52}$$

and if we send $n \to \infty$, we obtain that $\frac{1}{n} |M_A(\rho_n) - M_A(\sigma_n)| \to 0$, which proves the theorem. $\square$

B.3   Convex boundary and bank states

In the following, we consider the case of a two-resource theory R_{multi} defined on the Hilbert space $\mathcal{H}$. The class of allowed operations is $\mathcal{A}_{\text{multi}} = \mathcal{A}_1 \cap \mathcal{A}_2$, where each $\mathcal{A}_i$ is a subset of the

class of all CPTP maps that leave the set of states $\mathcal{F}_i$ invariant. We ask the resource theory R_{multi} to satisfy the asymptotic equivalence property with respect to the monotones $E_{\mathcal{F}_1}$ and $E_{\mathcal{F}_2}$. Furthermore, we assume that the two invariant sets satisfy the properties F1 – F4, and we additionally demand that the two relative entropy distances be extensive,

$$E_{\mathcal{F}_i}(\rho^{\otimes n}) = n \, E_{\mathcal{F}_i}(\rho), \quad \forall \rho \in \mathcal{S}(\mathcal{H}), \ \forall n \in \mathbb{N}, \ \text{for } i = 1, 2. \tag{B.53}$$

Notice that the above property implies that $E_{\mathcal{F}_i}^{\infty} = E_{\mathcal{F}_i}$ for $i \in \{1, 2\}$. This property is weaker than F3b, since the latter implies Eq. (B.53), but not vice versa. It follows from Thm. 8 that the two monotones $E_{\mathcal{F}_1}$ and $E_{\mathcal{F}_2}$ uniquely quantify the resources in our theory. As a result, we can represent the state space of R_{multi} in a two-dimensional diagram, as shown in Fig. 3.2.

We choose the two invariant sets of the theory to be disjoints, i.e., $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$. In this situation, we can find some bank states $\rho \in \mathcal{F}_{\text{bank}}$, see Eq. (3.28), such that both $E_{\mathcal{F}_1}(\rho) > 0$ and $E_{\mathcal{F}_2}(\rho) > 0$, and we can use these states to freely inter-convert (at a given rate) between the two resources. It is easy to show that, in this case, $E_{\mathcal{F}_2}(\rho) > E_{\mathcal{F}_2}(\mathcal{F}_2) = 0 \ \forall \rho \in \mathcal{F}_1$, and similarly $E_{\mathcal{F}_1}(\rho) > E_{\mathcal{F}_1}(\mathcal{F}_1) = 0 \ \forall \rho \in \mathcal{F}_2$. Moreover, we can find in both invariant sets $\mathcal{F}_1$ and $\mathcal{F}_2$ a subset of states with minimum value of, respectively, the monotones $E_{\mathcal{F}_2}$ and $E_{\mathcal{F}_1}$, that is

$$\mathcal{F}_{1,\text{min}} = \left\{ \sigma \in \mathcal{F}_1 \, | \, E_{\mathcal{F}_2}(\sigma) = \min_{\rho \in \mathcal{F}_1} E_{\mathcal{F}_2}(\rho) \right\} \subseteq \mathcal{F}_1, \tag{B.54a}$$

$$\mathcal{F}_{2,\text{min}} = \left\{ \sigma \in \mathcal{F}_2 \, | \, E_{\mathcal{F}_1}(\sigma) = \min_{\rho \in \mathcal{F}_2} E_{\mathcal{F}_1}(\rho) \right\} \subseteq \mathcal{F}_2. \tag{B.54b}$$

Given these two subsets, we can then define the following real intervals,

$$I_1 = [E_{\mathcal{F}_1}(\mathcal{F}_1) = 0 \, ; \, E_{\mathcal{F}_1}(\mathcal{F}_{2,\text{min}})], \tag{B.55a}$$

$$I_2 = [E_{\mathcal{F}_2}(\mathcal{F}_2) = 0 \, ; \, E_{\mathcal{F}_2}(\mathcal{F}_{1,\text{min}})]. \tag{B.55b}$$

Lemma 8. *Consider the multi-resource theory R_{multi} with allowed operations $\mathcal{A}_{\text{multi}}$, and invariant sets $\mathcal{F}_1$ and $\mathcal{F}_2$ which satisfy properties F1 – F4, and $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$. If the theory satisfies the asymptotic equivalence property with respect to the monotones $E_{\mathcal{F}_1}$ and $E_{\mathcal{F}_2}$, and these monotones are extensive, see Eq. (B.53), then for all bank states $\rho \in \mathcal{F}_{\text{bank}}$ we have that $E_{\mathcal{F}_1}(\rho) \in I_1$ and $E_{\mathcal{F}_2}(\rho) \in I_2$.*

Proof. Suppose, for example, that there exists a bank state $\rho \in \mathcal{F}_{\text{bank}}$ such that $E_{\mathcal{F}_1}(\rho) \notin I_1$, that is, $\exists \sigma \in \mathcal{F}_{2,\text{min}}$ such that $E_{\mathcal{F}_1}(\sigma) < E_{\mathcal{F}_1}(\rho)$. By definition of $\mathcal{F}_2$ we also have that $E_{\mathcal{F}_2}(\sigma) \leq E_{\mathcal{F}_2}(\rho)$. These two inequalities, however, contradict the fact that ρ is passive, see Eq. (3.28), and conclude the proof. $\qquad\square$

It is easy to show that for all $\bar{E}_{\mathcal{F}_1} \in I_1$ there exists (at least) one state $\rho \in \mathcal{S}(\mathcal{H})$ such that $E_{\mathcal{F}_1}(\rho) = \bar{E}_{\mathcal{F}_1}$, and the same applies for I_2. However, one ought to be careful, as it is not the case that for any two values $\tilde{E}_{\mathcal{F}_1} \in I_1$ and $\tilde{E}_{\mathcal{F}_2} \in I_2$, there exists a $\sigma \in \mathcal{S}(\mathcal{H})$ such that $E_{\mathcal{F}_1}(\sigma) = \tilde{E}_{\mathcal{F}_1}$ and $E_{\mathcal{F}_2}(\sigma) = \tilde{E}_{\mathcal{F}_2}$. The proof that $\forall \bar{E}_{\mathcal{F}_1} \in I_1, \ \exists \rho \in \mathcal{S}(\mathcal{H}) : E_{\mathcal{F}_1}(\rho) = \bar{E}_{\mathcal{F}_1}$ follows from two facts: (*i*) $\mathcal{S}(\mathcal{H})$ is a compact and path-connected set, and therefore its image under the (asymptotic) continuous function $E_{\mathcal{F}_1}$ is a compact and path-connected set in $\mathbb{R}$, that is, a closed and bounded interval $I_{1,\mathcal{S}(\mathcal{H})}$, and (*ii*) $I_1 \subseteq I_{1,\mathcal{S}(\mathcal{H})}$. As a side remark, we notice that the above results would hold even if the monotones were not extensive, since the only property we need here is continuity, and it has been proved that if $E_{\mathcal{F}_i}$ is asymptotic continuous, so is $E_{\mathcal{F}_i}^\infty$, see Ref. [203, Cor. 8].

Let us now define, in the $E_{\mathcal{F}_1}$–$E_{\mathcal{F}_2}$ diagram, the curve of bank states, which lies on part of the boundary of the state space, as per definition in Eq. (3.28). The curve is defined as

$$\gamma_{\text{bank}} = \{(E_{\mathcal{F}_1}(\rho), E_{\mathcal{F}_2}(\rho)) \mid \rho \in \mathcal{F}_{\text{bank}}\}, \tag{B.56}$$

where $\mathcal{F}_{\text{bank}}$ is the set of bank states of the theory. It is easy to see that this curve is completely contained within the subset of $\mathbb{R}^2$ given by $I_1 \times I_2$. Together with this curve, we can introduce the real-valued function $c_{\text{bank}} : I_1 \to I_2$, defined as

$$c_{\text{bank}}(E_{\mathcal{F}_1}) = \text{if } (\exists P \in \gamma_{\text{bank}} \text{ such that } P[0] = E_{\mathcal{F}_1}) \text{ return } P[1]. \tag{B.57}$$

Essentially, this function checks the first element of the tuples in γ_{bank}, and returns the second element of the tuple whose first element is equal to $E_{\mathcal{F}_1}$. Since I_1 is a closed interval in $\mathbb{R}$, we have that for all $E_{\mathcal{F}_1} \in I_1$, the function c_{bank} is well-defined. See Fig. B.1 for the representation of the above curve of bank states in the resource diagram of the theory.

We now prove the following two propositions, which assure that the monotone $f_{\text{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}$ of Eq. (3.31) satisfies the property B2. This first proposition essentially tells us that the function c_{bank} is monotonic decreasing.
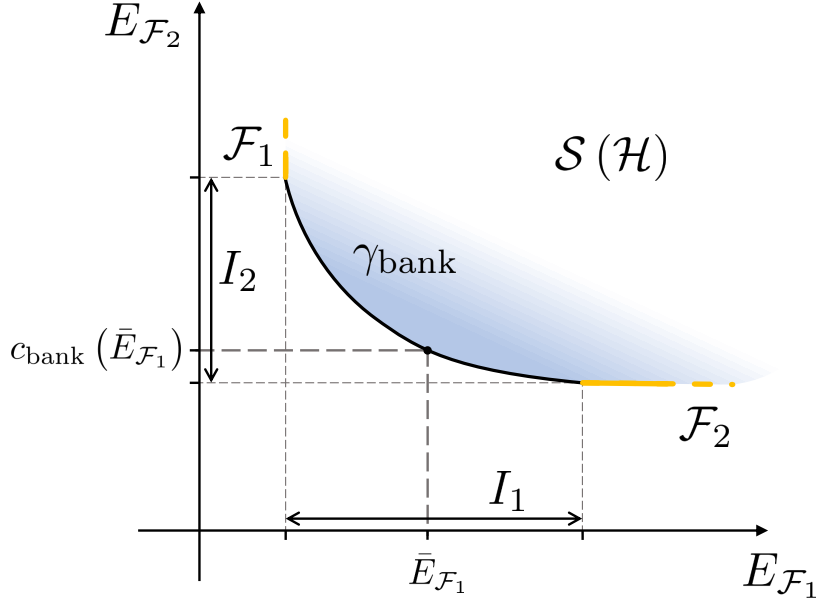
Figure B.1: We represent part of the state space $\mathcal{S}(\mathcal{H})$ in the $E_{\mathcal{F}_1}$–$E_{\mathcal{F}_2}$ diagram. In the figure, the green segment is the invariant set $\mathcal{F}_1$, the yellow one is $\mathcal{F}_2$, and the black curve connecting these two segments is γ_{bank}, the curve of bank states of the theory, see Eq. (B.56). On the $E_{\mathcal{F}_1}$-axis we highlight the interval I_1 defined in Eq. (B.55a), and similarly for the interval I_2 on the $E_{\mathcal{F}_2}$-axis. Furthermore, the action of the function $c_{\text{bank}} : I_1 \to I_2$, defined in Eq. (B.57), is shown for the input value $\bar{E}_{\mathcal{F}_1}$.

Proposition 7. *For all* $P_A, P_B \in \gamma_{\text{bank}}$, *where* $P_A = \left(E_{\mathcal{F}_1}^{(A)}, E_{\mathcal{F}_2}^{(A)}\right)$ *and* $P_B = \left(E_{\mathcal{F}_1}^{(B)}, E_{\mathcal{F}_2}^{(B)}\right)$, *we have that*

$$E_{\mathcal{F}_1}^{(A)} < E_{\mathcal{F}_1}^{(B)} \Leftrightarrow E_{\mathcal{F}_2}^{(A)} > E_{\mathcal{F}_2}^{(B)}. \tag{B.58}$$

Proof. We prove the propositions in a single direction, as the other follows in analogue manner. Suppose that $E_{\mathcal{F}_1}^{(A)} < E_{\mathcal{F}_1}^{(B)}$, and consider the states ρ_A, $\rho_B \in \mathcal{F}_{\text{bank}}$ such that $E_{\mathcal{F}_1}(\rho_A) = E_{\mathcal{F}_1}^{(A)}$, and $E_{\mathcal{F}_1}(\rho_B) = E_{\mathcal{F}_1}^{(B)}$. Since ρ_B belongs to the set of bank states, we have that one of the following conditions, see Eq. (3.28), has to be satisfied for all states $\sigma \in \mathcal{S}(\mathcal{H})$,

1. $E_{\mathcal{F}_1}(\sigma) > E_{\mathcal{F}_1}(\rho_B)$.

2. $E_{\mathcal{F}_2}(\sigma) > E_{\mathcal{F}_2}(\rho_B)$.

3. $E_{\mathcal{F}_1}(\sigma) = E_{\mathcal{F}_1}(\rho_B)$ and $E_{\mathcal{F}_2}(\sigma) = E_{\mathcal{F}_2}(\rho_B)$.

Let us then take $\sigma = \rho_A$. In this case, options 1 and 3 are not possible, since they contradict the hypothesis. Therefore, option 2 has to be valid, which implies that $E_{\mathcal{F}_2}(\rho_A) > E_{\mathcal{F}_2}(\rho_B)$. In a similar manner, if $E_{\mathcal{F}_1}^{(A)} = E_{\mathcal{F}_1}^{(B)}$, the only possible option for ρ_B would have been $E_{\mathcal{F}_2}(\rho_A) = E_{\mathcal{F}_2}(\rho_B)$, which concludes the proof. $\qquad\square$

The second propositions tells us, instead, that the function c_{bank} is convex.

Proposition 8. *For all $P_A, P_B \in \gamma_{\text{bank}}$, where $P_A = \left(E_{\mathcal{F}_1}^{(A)}, E_{\mathcal{F}_2}^{(A)} \right)$ and $P_B = \left(E_{\mathcal{F}_1}^{(B)}, E_{\mathcal{F}_2}^{(B)} \right)$, and for all $\lambda \in [0,1]$, there exists a $P_C \in \gamma_{\text{bank}}$, where $P_C = \left(E_{\mathcal{F}_1}^{(C)}, E_{\mathcal{F}_2}^{(C)} \right)$, such that*

$$E_{\mathcal{F}_1}^{(C)} = \lambda \, E_{\mathcal{F}_1}^{(A)} + (1 - \lambda) \, E_{\mathcal{F}_1}^{(B)}, \tag{B.59}$$

$$E_{\mathcal{F}_2}^{(C)} \leq \lambda \, E_{\mathcal{F}_2}^{(A)} + (1 - \lambda) \, E_{\mathcal{F}_2}^{(B)} \tag{B.60}$$

Proof. Let us consider, without losing in generality, that $E_{\mathcal{F}_1}^{(A)} < E_{\mathcal{F}_1}^{(B)}$, and take $\rho_C \in \mathcal{F}_{\text{bank}}$ such that $E_{\mathcal{F}_1}(\rho_C) = \lambda \, E_{\mathcal{F}_1}^{(A)} + (1 - \lambda) \, E_{\mathcal{F}_1}^{(B)}$. This state always exists since I_1 is a closed interval (and therefore is path-connected). Let us now define ρ_A, $\rho_B \in \mathcal{F}_{\text{bank}}$ such that $E_{\mathcal{F}_1}(\rho_A) = E_{\mathcal{F}_1}^{(A)}$, and $E_{\mathcal{F}_1}(\rho_B) = E_{\mathcal{F}_1}^{(B)}$. By convexity of the relative entropy distance $E_{\mathcal{F}_1}$, it follows that

$$E_{\mathcal{F}_1}(\rho_C) = \lambda \, E_{\mathcal{F}_1}^{(A)} + (1 - \lambda) \, E_{\mathcal{F}_1}^{(B)} \geq E_{\mathcal{F}_1}\left(\lambda \, \rho_A + (1 - \lambda) \, \rho_B \right). \tag{B.61}$$

Then, it is easy to show that

$$E_{\mathcal{F}_2}(\rho_C) \leq E_{\mathcal{F}_2}\left(\lambda \, \rho_A + (1 - \lambda) \, \rho_B \right) \leq \lambda \, E_{\mathcal{F}_2}^{(A)} + (1 - \lambda) \, E_{\mathcal{F}_2}^{(B)}, \tag{B.62}$$

where the first inequality follows from Prop. 7, and the second one from the convexity of $E_{\mathcal{F}_2}$. Since $\rho_C \in \mathcal{F}_{\text{bank}}$, the point $P_C = (E_{\mathcal{F}_1}(\rho_C), E_{\mathcal{F}_2}(\rho_C))$ is a point on the curve γ_{bank}. $\qquad\square$

It is easy to see that the above propositions imply that c_{bank} is (strictly) monotonic decreasing, and convex. Since this function is defined on the closed interval $I_1 \in \mathbb{R}$, we have that c_{bank} is continuous (except, maybe, at its endpoints). Therefore, we can always define the monotone $f_{\text{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}$ of Eq. (3.31), and it always satisfies condition B2. It is worth noticing that the results obtained in this section are based on the convexity of the monotones $E_{\mathcal{F}_1}$ and $E_{\mathcal{F}_2}$. If these monotones are also sub-additive, then the results can be extended to their regularisation $E_{\mathcal{F}_1}^{\infty}$ and $E_{\mathcal{F}_2}^{\infty}$, without the need of asking the extensivity property of Eq. (B.53), as it was shown in Ref. [202, Prop. 13]. Furthermore, all the results apply if one (or both) the monotones are

of the form of Eq. (3.23), since they satisfy all the necessary properties, in particular they are linear in both the tensor product and the admixture of states.

B.4   Properties of the bank states

In this section we prove some of the properties characterising the sets of bank states introduced in Sec. 3.3.1, see Def. 31, when the curve representing this set in the resource diagram is strictly convex. In particular, we show that each set $\mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)$ is additive, property F3b, convex, property F1, and invariant under the allowed operations of the multi-resource theory. Before proving the additivity of the sets of bank states, we prove the following lemma concerning the super-additivity of the monotones $E_{\mathcal{F}_i}$'s, when the corresponding sets $\mathcal{F}_i$'s satisfy additivity, property F3b.

Lemma 9. *Consider a state* $\rho_{S_1,S_2} \in \mathcal{S}\left(\mathcal{H}^{\otimes 2}\right)$, *and suppose that the sets* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *satisfy the property F3b, i.e.,* $\mathcal{F}_i^{(n)} = \mathcal{F}_i^{\otimes n}$ *for all* $n \in \mathbb{N}$, $i = 1, 2$. *Then, the relative entropy distances from these sets,* $E_{\mathcal{F}_1}$ *and* $E_{\mathcal{F}_2}$, *are such that*

$$E_{\mathcal{F}_i}(\rho_{S_1,S_2}) \geq E_{\mathcal{F}_i}(\rho_{S_1}) + E_{\mathcal{F}_i}(\rho_{S_2}), \quad i = 1, 2, \tag{B.63}$$

where $\rho_{S_1} = \text{Tr}_{S_2}[\rho_{S_1,S_2}]$, *and similarly* $\rho_{S_2} = \text{Tr}_{S_1}[\rho_{S_1,S_2}]$. *Furthermore, the above inequality is saturated if and only if* $\rho_{S_1,S_2} = \rho_{S_1} \otimes \rho_{S_2}$. *The result extends trivially to the case in which* $n > 2$ *copies of the system are considered.*

Proof. Let us consider the monotone $E_{\mathcal{F}_1}$, as the following argument can be equally applied to $E_{\mathcal{F}_2}$. By definition of relative entropy distance, we have that

$$E_{\mathcal{F}_1}(\rho_{S_1,S_2}) = \inf_{\sigma_{S_1,S_2} \in \mathcal{F}_1^{(2)}} D(\rho_{S_1,S_2} \| \sigma_{S_1,S_2}) = -S(\rho_{S_1,S_2}) + \inf_{\sigma_{S_1,S_2} \in \mathcal{F}_1^{(2)}} \left(-\text{Tr}\left[\rho_{S_1,S_2} \log \sigma_{S_1,S_2}\right]\right),$$
$$\tag{B.64}$$

where $S(\rho_{S_1,S_2}) = -\text{Tr}\left[\rho_{S_1,S_2} \log \rho_{S_1,S_2}\right]$ is the Von Neumann entropy of the state ρ_{S_1,S_2}. From the sub-additivity of the Von Neumann entropy, we have that

$$-S(\rho_{S_1,S_2}) \geq -S(\rho_{S_1}) - S(\rho_{S_2}), \tag{B.65}$$

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while from the property F3b it follows that

$$\inf_{\sigma_{S_1,S_2} \in \mathcal{F}_1^{(2)}} \left(-\mathrm{Tr}\left[\rho_{S_1,S_2} \log \sigma_{S_1,S_2}\right] \right) = \inf_{\sigma_{S_1}, \sigma_{S_2} \in \mathcal{F}_1} \left(-\mathrm{Tr}\left[\rho_{S_1,S_2} \log \sigma_{S_1} \otimes \sigma_{S_2}\right] \right)$$

$$= \inf_{\sigma_{S_1}, \sigma_{S_2} \in \mathcal{F}_1} \left(-\mathrm{Tr}\left[\rho_{S_1} \log \sigma_{S_1}\right] - \mathrm{Tr}\left[\rho_{S_2} \log \sigma_{S_2}\right] \right)$$

$$= \inf_{\sigma_{S_1} \in \mathcal{F}_1} \left(-\mathrm{Tr}\left[\rho_{S_1} \log \sigma_{S_1}\right] \right) + \inf_{\sigma_{S_2} \in \mathcal{F}_1} \left(-\mathrm{Tr}\left[\rho_{S_2} \log \sigma_{S_2}\right] \right).$$

$$(B.66)$$

From Eqs. (B.64), (B.65), and (B.66) it follows that

$$E_{\mathcal{F}_1}(\rho_{S_1,S_2}) \geq \inf_{\sigma_{S_1} \in \mathcal{F}_1} \left(-S(\rho_{S_1}) - \mathrm{Tr}\left[\rho_{S_1} \log \sigma_{S_1}\right] \right) + \inf_{\sigma_{S_2} \in \mathcal{F}_1} \left(-S(\rho_{S_2}) - \mathrm{Tr}\left[\rho_{S_2} \log \sigma_{S_2}\right] \right)$$

$$= E_{\mathcal{F}_1}(\rho_{S_1}) + E_{\mathcal{F}_1}(\rho_{S_2}). \tag{B.67}$$

$$\square$$

We can now prove the additivity of the sets of bank states.

Proposition 9. *Suppose the sets $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy the property F3b, i.e., $\mathcal{F}_i^{(n)} = \mathcal{F}_i^{\otimes n}$ for all $n \in \mathbb{N}$, $i = 1, 2$. Consider the set of bank states $\mathcal{F}_{\mathrm{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)$, and the relative entropy distances $E_{\mathcal{F}_1}$ and $E_{\mathcal{F}_2}$ from the sets $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. If the curve of bank states is strictly convex, then this set, in the case in which we consider of $n \in \mathbb{N}$ copies of the system, is given by*

$$\mathcal{F}_{\mathrm{bank}}^{(n)} = \left\{ \rho_1 \otimes \ldots \otimes \rho_n \in \mathcal{S}\left(\mathcal{H}^{\otimes n}\right) \mid \exists \bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2} \text{ such that } \rho_1, \ldots, \rho_n \in \mathcal{F}_{\mathrm{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right) \right\}. \tag{B.68}$$

Furthermore, we have that for all subset of bank states $\mathcal{F}_{\mathrm{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right) \subset \mathcal{S}\left(\mathcal{H}\right)$, its extension to n copies of the system is given by

$$\mathcal{F}_{\mathrm{bank}}^{(n)}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right) = \mathcal{F}_{\mathrm{bank}}^{\otimes n}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right). \tag{B.69}$$

Proof. We prove the theorem for $n = 2$, as the argument extends trivially for $n > 2$. Consider a state $\sigma_{S_1,S_2} \in \mathcal{S}\left(\mathcal{H}^{\otimes 2}\right)$. From Lem. 9, it follows that

$$E_{\mathcal{F}_i}(\sigma_{S_1,S_2}) \geq E_{\mathcal{F}_i}(\sigma_{S_1}) + E_{\mathcal{F}_i}(\sigma_{S_2}), \quad i = 1, 2, \tag{B.70}$$

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where $\sigma_{S_1} = \mathrm{Tr}_{S_2}[\sigma_{S_1,S_2}]$, $\sigma_{S_2} = \mathrm{Tr}_{S_1}[\sigma_{S_1,S_2}]$, and the inequality is saturated iff $\sigma_{S_1,S_2} = \sigma_{S_1} \otimes \sigma_{S_2}$. Now, for both the states $\sigma_{S_1}, \sigma_{S_2} \in \mathcal{S}(\mathcal{H})$, select the bank states $\rho_{P_1}, \rho_{P_2} \in \mathcal{F}_{\mathrm{bank}}$ such that

$$E_{\mathcal{F}_i}(\sigma_{S_j}) \geq E_{\mathcal{F}_i}(\rho_{P_j}), \quad i,j = 1,2. \tag{B.71}$$

Recall now that, in the $E_{\mathcal{F}_1}$–$E_{\mathcal{F}_2}$ diagram, the curve of bank state is convex, see Prop. 8, and therefore given $\rho_{P_1}, \rho_{P_2} \in \mathcal{F}_{\mathrm{bank}}$, we can find another $\rho_{P_3} \in \mathcal{F}_{\mathrm{bank}}$ such that

$$\frac{1}{2}E_{\mathcal{F}_i}(\rho_{P_1}) + \frac{1}{2}E_{\mathcal{F}_i}(\rho_{P_2}) \geq E_{\mathcal{F}_i}(\rho_{P_3}), \quad i = 1,2. \tag{B.72}$$

Furthermore, since we are demanding the curve of bank states to be strictly convex, the above inequalities are saturated iff ρ_{P_1}, ρ_{P_2}, and ρ_{P_3} all belong to the same subset $\mathcal{F}_{\mathrm{bank}}(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2})$. By combining Eqs. (B.70), (B.71), and (B.72), together with property F3b of the sets $\mathcal{F}_1$ and $\mathcal{F}_2$ (that implies the additivity of the corresponding relative entropy distances), we find that for all $\sigma_{S_1,S_2} \in \mathcal{S}(\mathcal{H}^{\otimes 2})$, it exists a $\rho_{P_3} \in \mathcal{F}_{\mathrm{bank}}$ such that

$$E_{\mathcal{F}_i}(\sigma_{S_1,S_2}) \geq E_{\mathcal{F}_i}(\rho_{P_3}^{\otimes 2}), \quad i = 1,2 \tag{B.73}$$

where the inequality is saturated iff $\sigma_{S_1,S_2} = \sigma_{S_1} \otimes \sigma_{S_2}$, and both σ_{S_1} and σ_{S_2} belong to the same subset $\mathcal{F}_{\mathrm{bank}}(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2})$. Due to the definition of bank states given in Eq. (3.28), the thesis of this proposition follows. $\qquad \square$

In the following proposition, instead, we show convexity of the sets of bank states.

Proposition 10. *Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are convex sets, property F1, and consider the relative entropy distances from these two sets, $E_{\mathcal{F}_1}$ and $E_{\mathcal{F}_2}$. Then, the set of bank states $\mathcal{F}_{\mathrm{bank}}(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2})$ is convex, as well as its extension to the n-copy case, $\mathcal{F}_{\mathrm{bank}}^{(n)}(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2})$, defined in Eq. (B.69).*

Proof. Let us consider two states $\rho_1, \rho_2 \in \mathcal{F}_{\mathrm{bank}}(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2})$. For these two states, there exists $\sigma_1, \sigma_2 \in \mathcal{F}_1$ such that

$$E_{\mathcal{F}_1}(\rho_1) = D(\rho_1 \,\|\, \sigma_1) = \bar{E}_{\mathcal{F}_1}, \tag{B.74a}$$

$$E_{\mathcal{F}_1}(\rho_2) = D(\rho_2 \,\|\, \sigma_2) = \bar{E}_{\mathcal{F}_1}. \tag{B.74b}$$

Then, for all $\lambda \in [0, 1]$, we have

$$E_{\mathcal{F}_1}\big(\lambda \rho_1 + (1 - \lambda) \rho_2\big) = \inf_{\sigma \in \mathcal{F}_1} D(\lambda \rho_1 + (1 - \lambda) \rho_2 \,\|\, \sigma)$$

$$\leq D(\lambda \rho_1 + (1 - \lambda) \rho_2 \,\|\, \lambda \sigma_1 + (1 - \lambda) \sigma_2)$$

$$\leq \lambda D(\rho_1 \,\|\, \sigma_1) + (1 - \lambda) D(\rho_2 \,\|\, \sigma_2) = \bar{E}_{\mathcal{F}_1}, \tag{B.75}$$

where the first inequality follows from the fact that $\mathcal{F}_1$ is convex, property F1, and the second inequality from the joint convexity of the relative entropy. In the same way, it follows that

$$E_{\mathcal{F}_2}\big(\lambda \rho_1 + (1 - \lambda) \rho_2\big) \leq \bar{E}_{\mathcal{F}_2}. \tag{B.76}$$

Since ρ_1, $\rho_2 \in \mathcal{F}_{\text{bank}}\big(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\big)$, they satisfy the properties of Eq. (3.28), and therefore it has to be that, for all $\lambda \in [0, 1]$,

$$E_{\mathcal{F}_1}\big(\lambda \rho_1 + (1 - \lambda) \rho_2\big) = \bar{E}_{\mathcal{F}_1} \text{ and } E_{\mathcal{F}_2}\big(\lambda \rho_1 + (1 - \lambda) \rho_2\big) = \bar{E}_{\mathcal{F}_2}. \tag{B.77}$$

Thus, we have that $\lambda \rho_1 + (1 - \lambda) \rho_2 \in \mathcal{F}_{\text{bank}}\big(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\big)$. This result can be extended to the case of $n \in \mathbb{N}$ copies of the system, where the bank set $\mathcal{F}_{\text{bank}}^{(n)}\big(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\big)$ is defined as in Eq. (B.69). In this case, the proof is analogous to the one considered above, with the exception that in the rhs of Eqs. (B.74), and of the following ones, we add the multiplicative factor n. $\qquad\square$

The next proposition concerns the invariance, under the allowed operations in $\mathcal{A}_{\text{multi}}$, of the sets of bank states.

Proposition 11. *Consider a resource theory R_{multi} with allowed operations $\mathcal{A}_{\text{multi}}$, and two invariant sets $\mathcal{F}_1$ and $\mathcal{F}_2$. Consider the subset of bank states $\mathcal{F}_{\text{bank}}\big(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\big)$ as defined in Eq. (3.29). Then, for all $\varepsilon \in \mathcal{A}_{\text{multi}}$, we have that $\mathcal{F}_{\text{bank}}\big(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\big)$ is an invariant set, that is*

$$\varepsilon\,(\rho) \in \mathcal{F}_{\text{bank}}\big(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\big), \quad \forall \rho \in \mathcal{F}_{\text{bank}}\big(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\big) \tag{B.78}$$

Analogously, the set of bank states describing n copies of the bank system is invariant under the class of allowed operations $\mathcal{A}_{\text{multi}}^{(n)}$

Proof. Let us consider $\rho \in \mathcal{F}_{\text{bank}}\big(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\big)$, as well as the state $\varepsilon(\rho)$ obtained by applying the map $\varepsilon \in \mathcal{A}_{\text{multi}}$ to the bank state. Due to the monotonicity of $E_{\mathcal{F}_1}$ and $E_{\mathcal{F}_2}$, we have that $E_{\mathcal{F}_1}(\varepsilon(\rho)) \leq E_{\mathcal{F}_1}(\rho)$, and $E_{\mathcal{F}_2}(\varepsilon(\rho)) \leq E_{\mathcal{F}_2}(\rho)$. Recall now that ρ is a bank state, which implies that $\forall \sigma \in \mathcal{S}(\mathcal{H})$, one (or more) of the following options holds

1. $E_{\mathcal{F}_1}(\sigma) > E_{\mathcal{F}_1}(\rho)$.

2. $E_{\mathcal{F}_2}(\sigma) > E_{\mathcal{F}_2}(\rho)$.

3. $E_{\mathcal{F}_1}(\sigma) = E_{\mathcal{F}_1}(\rho)$ and $E_{\mathcal{F}_2}(\sigma) = E_{\mathcal{F}_2}(\rho)$.

However, the monotonicity conditions given by $E_{\mathcal{F}_1}$ and $E_{\mathcal{F}_2}$ implies that $\varepsilon(\rho)$ violates options 1 and 2, so that option 3 is the only possible one. But this implies that $E_{\mathcal{F}_1}(\varepsilon(\rho)) = E_{\mathcal{F}_1}(\rho)$ and $E_{\mathcal{F}_2}(\varepsilon(\rho)) = E_{\mathcal{F}_2}(\rho)$, meaning that $\varepsilon(\rho) \in \mathcal{F}_{\text{bank}}(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2})$. The same argument applies to the set $\mathcal{F}_{\text{bank}}^{(n)}(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2})$, when n copies of the system are considered. Indeed, this case is analogous to the one considered above, with the sole difference that now the state $\rho \in \mathcal{F}_{\text{bank}}^{(n)}(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2})$, the state $\sigma \in \mathcal{S}(\mathcal{H}^{\otimes n})$, and the operations we use are in the class $\mathcal{A}_{\text{multi}}^{(n)}$. □

We now provide a brief proof that the bank monotone introduced in Eq. (3.31) satisfies the properties B3 – B7.

Proposition 12. *Consider a resource theory* R_{multi} *with allowed operations* $\mathcal{A}_{\text{multi}}$, *and two invariant sets* $\mathcal{F}_1$ *and* $\mathcal{F}_2$. *Consider the subset of bank states* $\mathcal{F}_{\text{bank}}(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2})$, *and its associated bank monotone* $f_{\text{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}$, *see Eq. (3.31). Then, the bank monotone satisfies the properties from B3 to B7.*

Proof. That the monotone is additive, property B3, follows from property F3b of both sets $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_{\text{bank}}(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2})$, see Prop. 9. Given $\rho, \sigma \in \mathcal{S}(\mathcal{H})$, we have that

$$
\begin{aligned}
f_{\text{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}(\rho \otimes \sigma) &= \alpha\left(E_{\mathcal{F}_1}(\rho \otimes \sigma) - 2\bar{E}_{\mathcal{F}_1}\right) + \beta\left(E_{\mathcal{F}_2}(\rho \otimes \sigma) - 2\bar{E}_{\mathcal{F}_2}\right) \\
&= \alpha\left(E_{\mathcal{F}_1}(\rho) - \bar{E}_{\mathcal{F}_1}\right) + \beta\left(E_{\mathcal{F}_2}(\rho) - \bar{E}_{\mathcal{F}_2}\right) \\
&\quad + \alpha\left(E_{\mathcal{F}_1}(\sigma) - \bar{E}_{\mathcal{F}_1}\right) + \beta\left(E_{\mathcal{F}_2}(\sigma) - \bar{E}_{\mathcal{F}_2}\right) \\
&= f_{\text{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}(\rho) + f_{\text{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}(\sigma).
\end{aligned} \tag{B.79}
$$

Property B4, i.e., monotonicity under partial tracing, follows from Lem. 9. Indeed, given

any state $\rho_n \in \mathcal{S}\left(\mathcal{H}^{\otimes n}\right)$, we have that

$$
\begin{aligned}
f_{\text{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}(\rho_n) &= \alpha \left(E_{\mathcal{F}_1}(\rho_n) - n\,\bar{E}_{\mathcal{F}_1}\right) + \beta \left(E_{\mathcal{F}_2}(\rho_n) - n\,\bar{E}_{\mathcal{F}_2}\right) \\
&\geq \alpha \left(E_{\mathcal{F}_1}(\text{Tr}_k\left[\rho_n\right]) - (n-k)\,\bar{E}_{\mathcal{F}_1}\right) + \alpha \left(E_{\mathcal{F}_1}(\text{Tr}_{n-k}\left[\rho_n\right]) - k\,\bar{E}_{\mathcal{F}_1}\right) \\
&\quad + \beta \left(E_{\mathcal{F}_2}(\text{Tr}_k\left[\rho_n\right]) - (n-k)\,\bar{E}_{\mathcal{F}_2}\right) + \beta \left(E_{\mathcal{F}_2}(\text{Tr}_{n-k}\left[\rho_n\right]) - k\,\bar{E}_{\mathcal{F}_2}\right) \\
&= f_{\text{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}(\text{Tr}_k\left[\rho_n\right]) + f_{\text{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}(\text{Tr}_{n-k}\left[\rho_n\right]) \geq f_{\text{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}(\text{Tr}_k\left[\rho_n\right]), \quad (\text{B.80})
\end{aligned}
$$

where $k < n$, and $\text{Tr}_k\left[\cdot\right]$ means that we are tracing out k copies of the system. Notice that the first inequality follows from Eq. (B.63), while the second one from property B2, i.e., from the positivity of the bank monotone.

The extensivity of the bank monotone, property B5, follows from the fact that the monotones $E_{\mathcal{F}_i}$'s are extensive (since the $\mathcal{F}_i$'s contain a full-rank state, property F2), and that γ scales linearly in n as we consider states on $\mathcal{S}\left(\mathcal{H}^{\otimes n}\right)$. Property B6, i.e., asymptotic continuity of the bank monotone, follows from the fact that both $E_{\mathcal{F}_i}$'s are asymptotic continuous. Given two states $\rho_n, \sigma_n \in \mathcal{S}\left(\mathcal{H}^{\otimes n}\right)$, we have that

$$
\frac{1}{n}\left|f_{\text{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}(\rho_n) - f_{\text{bank}}^{\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}}(\sigma_n)\right| \leq \frac{\alpha}{n}\left|E_{\mathcal{F}_1}(\rho_n) - E_{\mathcal{F}_1}(\sigma_n)\right| + \frac{\beta}{n}\left|E_{\mathcal{F}_2}(\rho_n) - E_{\mathcal{F}_2}(\sigma_n)\right|, \quad (\text{B.81})
$$

and if $\lim_{n\to\infty}\left\|\rho_n - \sigma_n\right\|_1 = 0$, then the rhs of the above equation tends to 0 as $n \to \infty$. Finally, property B7, i.e., monotonicity under the class of operations $\mathcal{A}_{\text{multi}}$, follows from the monotonicity of the $E_{\mathcal{F}_i}$'s, and the fact that α and β are positive constants. $\qquad\square$

We close this section with the proof that the relative entropy distance from the set of bank states is monotonic under the allowed operations of a single-resource theory R_{single} build as in Def. 32.

Proposition 13. *Consider a multi-resource theory* R_{multi} *with two resources, whose allowed operations* $\mathcal{A}_{\text{multi}}$ *leave the sets* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *invariant. Suppose these invariant sets satisfy the properties F1, F2, and F3b. Then, the relative entropy distance from the subset of bank states* $\mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)$ *is monotonic under both the class of operations* $\mathcal{A}_{\text{multi}}$ *and the class* $\mathcal{A}_{\text{single}}$ *introduced in Def. 32.*

Proof. **1**. Here we show invariance of the relative entropy distance with respect to the addition of an ancillary system described by $n \in \mathbb{N}$ copies of a bank states. Consider the state $\rho \in \mathcal{S}\left(\mathcal{H}\right)$,

and the bank state $\rho_P \in \mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)$. Then, we have

$$
\begin{aligned}
E_{\mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)}\left(\rho \otimes \rho_P^{\otimes n}\right) &= \inf_{\sigma, \sigma_{P_1}, \ldots, \sigma_{P_n} \in \mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)} D\left(\rho \otimes \rho_P^{\otimes n} \, \| \, \sigma \otimes \sigma_{P_1} \otimes \ldots \otimes \sigma_{P_n}\right) \\
&= \inf_{\sigma \in \mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)} D(\rho \, \| \, \sigma) + \sum_{i=1}^{n} \inf_{\sigma_{P_i} \in \mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)} D(\rho_P \, \| \, \sigma_{P_i}) \\
&= \inf_{\sigma \in \mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)} D(\rho \, \| \, \sigma) = E_{\mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)}(\rho), \qquad \text{(B.82)}
\end{aligned}
$$

where the first equality follows from Prop. 9, and the last one from the fact that $\rho_P \in \mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)$.

2. Now we show monotonicity of the relative entropy distance with respect to the allowed operations $\mathcal{A}_{\text{mulit}}$. Let us consider a state $\rho \in \mathcal{S}(\mathcal{H})$, together with an operation $\varepsilon \in \mathcal{A}_{\text{multi}}$. Then, we have that

$$
\begin{aligned}
E_{\mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)}\left(\varepsilon(\rho)\right) &= \inf_{\sigma \in \mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)} D\left(\varepsilon(\rho) \, \| \, \sigma\right) \leq \inf_{\sigma \in \mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)} D\left(\varepsilon(\rho) \, \| \, \varepsilon(\sigma)\right) \\
&\leq \inf_{\sigma \in \mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)} D(\rho \, \| \, \sigma) = E_{\mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)}(\rho), \qquad \text{(B.83)}
\end{aligned}
$$

where the first inequality follows from Prop. 11, and the second one from the monotonicity of the relative entropy under CPTP maps. This result trivially extends to the case in which we have multiple copies of the system, since in Prop. 11 we have shown that $\mathcal{F}_{\text{bank}}^{(n)}$ is invariant under the allowed operations $\mathcal{A}_{\text{multi}}^{(n)}$ for all $n \in \mathbb{N}$.

3. We show the monotonicity of the relative entropy with respect to partial tracing when the ancillary system is composed by just one copy. However, the result straightforwardly extends to the case in which the ancillary system is composed by $n \in \mathbb{N}$ copies. Let us consider the state $\rho_{S_1, S_2} \in \mathcal{S}\left(\mathcal{H}^{\otimes 2}\right)$. Then, we have that

$$
\begin{aligned}
E_{\mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)}\left(\text{Tr}_{S_2}\left[\rho_{S_1, S_2}\right]\right) &= \inf_{\sigma_{S_1} \in \mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)} D\left(\text{Tr}_{S_2}\left[\rho_{S_1, S_2}\right] \, \| \, \sigma_{S_1}\right) \\
&= \inf_{\sigma_{S_1}, \sigma_{S_2} \in \mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)} D\left(\text{Tr}_{S_2}\left[\rho_{S_1, S_2}\right] \, \| \, \text{Tr}_{S_2}\left[\sigma_{S_1} \otimes \sigma_{S_2}\right]\right) \\
&\leq \inf_{\sigma_{S_1}, \sigma_{S_2} \in \mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)} D\left(\rho_{S_1, S_2} \, \| \, \sigma_{S_1} \otimes \sigma_{S_2}\right) \\
&= E_{\mathcal{F}_{\text{bank}}\left(\bar{E}_{\mathcal{F}_1}, \bar{E}_{\mathcal{F}_2}\right)}(\rho_{S_1, S_2}), \qquad \text{(B.84)}
\end{aligned}
$$

where the second equality follows from Prop. 9, while the inequality follows from the monotonicity of the relative entropy distance under CPTP maps. $\qquad \square$

Appendix C

Typical states

In this section we introduce the notion of *typical states*, which provides an efficient way to represent and deal with many i.i.d. copies of a quantum state. For a more detailed analysis on this topic, we refer the reader to Ref. [48, Ch. 12]. Consider a finite-dimensional quantum system described by the Hilbert space $\mathcal{H}$, such that $d = \dim \mathcal{H}$, and a quantum state $\rho \in \mathcal{S}(\mathcal{H})$ of the form

$$\rho = \sum_{i=1}^{d} p_i \, |i\rangle \langle i| . \tag{C.1}$$

Given $n \gg 1$ copies of this state, we can introduce the typical state $\rho_{\text{typ}} \in \mathcal{S}(\mathcal{H}^{\otimes n})$ such that

$$\|\rho^{\otimes n} - \rho_{\text{typ}}\|_1 \to 1, \quad \text{for } n \to \infty, \tag{C.2}$$

where $\| \cdot \|_1$ is the trace norm. The above equation implies that the typical state is a good approximation of $\rho^{\otimes n}$ in the asymptotic limit, since for $n \gg 1$ the two states are indistinguishable.

The typical state ρ_{typ} is defined as

$$\rho_{\text{typ}} = \sum_{x \in T_\rho} p_x \, |x\rangle \langle x| , \tag{C.3}$$

where $x = x_1, \ldots, x_n$ are sequences of n characters[1] which belongs to the set of typical sequences T_ρ. This set contains any sequence x such that the value i occurs a number of times

$$n_i \in [(n - \delta)p_i, (n + \delta)p_i] , \tag{C.4}$$

---

[1] Each character x_j can assume a value from 1 to d.

where $\delta = O(n^x)$, and $x \in \left[\frac{1}{2}; 1\right)$, see Fig. C.1. For example, if we are considering a qubit system such that $p_1 = \frac{1}{4}$ and $p_2 = \frac{3}{4}$, and we consider $n = 4$ copies of the system, we have that an instance of (strongly) typical sequences is 1211 (as well as any other permutation of the characters). We can also compute the probability associated with a sequence x containing n_i times the value i. This probability is given by

$$p_x = \Pi_{i=1}^d p_i^{n_i}. \tag{C.5}$$

From this equation, it is easy to show that the probability associated with a sequence $x \in T_\rho$ is such that

$$2^{-(n+\delta)\,S(\rho)} \le p_x \le 2^{-(n-\delta)\,S(\rho)}, \tag{C.6}$$

where $S(\rho)$ is the von Neumann entropy of the state ρ.

Given the definition of typical state ρ_{typ}, Eq. (C.3), it is possible to show that this state is indistinguishable from $\rho^{\otimes n}$ as n tends to infinity, i.e., we can prove Eq. (C.2). To do so, we need an additional tool, namely, the *law of large numbers*.

Theorem 14. *Consider the i.i.d. random variables $X_1, \ldots, X_n$ with finite first and second moments, $\mathbb{E}(X) < \infty$ and $\mathbb{E}(X^2) < \infty$, respectively. Then, for all $\varepsilon > 0$, we have that*

$$\text{prob}\left(|S_n - \mathbb{E}(X)| > \varepsilon\right) \to 0, \quad n \to \infty, \tag{C.7}$$

where $S_n = \sum_{j=1}^n \frac{X_j}{n}$ is a random variable as well.

Using the above theorem, we can show that the probability that a generic sequence x extracted from $\rho^{\otimes n}$ is in T_ρ tends to 1 as $n \to \infty$, and consequently that Eq. (C.2) holds. First, recall that each character in the sequence x is a i.i.d. random variable x_j which can assume value i with probability p_i. Then, we can define the random variable $-\log p(x_j)$, which assumes values $-\log p_i$ with probability p_i. The first momenta of this random variable is $\mathbb{E}(-\log p(x_j)) = -\sum_{i=1}^d p_i \log p_i = S(\rho)$. If we now use the law of large numbers, we obtain

$$\text{prob}\left(\left|\sum_{j=1}^n \frac{-\log p(x_j)}{n} - S(\rho)\right| > \varepsilon\right) \to 0, \quad n \to \infty. \tag{C.8}$$

Using the properties of the logarithm, it is easy to show that $\sum_{j=1}^n \log p(x_j) = \log p(x_1, \ldots, x_n) = p(x)$, where x is the sequence of characters x_j. As a result, we have that

$$\text{prob}\left(\left|\frac{-\log p(x)}{n} - S(\rho)\right| > \varepsilon\right) \to 0, \quad n \to \infty, \tag{C.9}$$
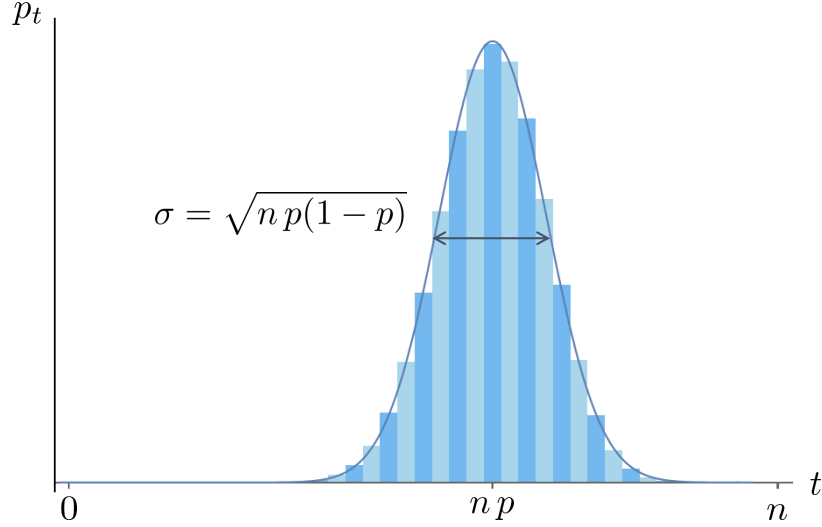
Figure C.1: The spectrum of n copies of the qubit state $\rho = p \left|0\right\rangle \left\langle 0\right| + (1-p) \left|1\right\rangle \left\langle 1\right|$, for $n \to \infty$, over different types. A *type* is given by the set of states containing a fixed number of copies of $\left|0\right\rangle$. The spectrum is given by the binomial distribution, which is well-approximated, in the limit of n tending to infinity, by a Gaussian distribution with mean value $\mu = n\,p$ and variance $\sigma^2 = n\,p(1-p)$. Due to the properties of the Gaussian distribution, cutting the tails of the spectrum at a distance from the mean value proportional to $\sigma = O(\sqrt{n})$ gives us an approximate spectrum which is infinitesimally close to the original one, in total variation distance. For this reason, in Eq. (C.4) we use a $\delta = O(n^x)$, where $x \geq \frac{1}{2}$. This qualitative statement is usually rigorously proved using Hoeffding's inequality [204].

and if we compare the argument of the probability in the lhs of the above equation with the range for the probabilities of the typical sequences, Eq. (C.6), we find that the two coincide for $\varepsilon = \frac{\delta}{n} S(\rho) > 0$. Therefore, we have that the probability that a sequence x is in T_ρ tends to 1 as n tends to infinity, and therefore the typical approximation of $\rho^{\otimes n}$ is a good one, since the two states are indistinguishable in the limit.

Appendix D

Cycle for energy extraction from passive states

In this appendix, we present in full details the cycle used in Ch. 5 to extract energy from passive states. Energy extraction from a qutrit system is achieved by using a d-dimensional catalyst, and by acting on this global system with a unitary operation that preserves the local state of the catalyst. In the following, we do not make use of virtual temperatures to describe the passive state of the system, and instead we focus on its populations. We recall that the system's Hamiltonian is $H_P = \sum_{i=0}^{2} E_i |i\rangle \langle i|_P$, where we define $\Delta E_{10} = E_1 - E_0 > 0$ and $\Delta E_{21} = E_2 - E_1 > 0$. The state of the system is passive, and therefore we can represent it as a diagonal state in the energy eigenbasis $\rho_P = \sum_{i=0}^{2} p_i |i\rangle \langle i|_P$, where $p_i \geq p_{i+1}$ for $i = 0, 1$, a direct consequence of the no-energy-extraction condition of Eq. (5.1).

The catalyst we use is a d-level system described by the state

$$\rho_M = \sum_{j=0}^{d-1} q_j |j\rangle \langle j|_M. \tag{D.1}$$

Since the local state of the catalyst does not change, the Hamiltonian of this system can be taken to be arbitrary. The global unitary operation has been described in Sec. 5.2, Eq. (5.7), and we re-write it here,

$$S_{m,n} = S_{(1,2)}^{(0,m)} \circ S_{(1,2)}^{(m,m+1)} \circ \ldots \circ S_{(1,2)}^{(m+n-2,m+n-1)} \circ S_{(0,1)}^{(m-1,m+n-1)} \circ S_{(0,1)}^{(m-2,m-1)} \circ \ldots \circ S_{(0,1)}^{(0,1)},$$

where the operation $S_{(a,b)}^{(c,d)}$ is a swap between system and catalyst, whose action is $|a\rangle_P |d\rangle_M \leftrightarrow$

$|b\rangle_P |c\rangle_M$. Notice that, in order for the cycle to be possible, the dimension of the catalyst has to be at least equal to $m + n$, and in the following we fix $d = m + n$.

For the given unitary evolution, we can easily evaluate the final state of the global system. This final state presents classical correlations between system and catalyst, but in the following we only consider their marginal states, which are the sole information we need. Indeed, the energy of the global system only depends on the Hamiltonian H_P of the system, since the catalyst state is locally unchanged, and we do not have an interaction term H_{int}. The final state of the system is given by $\tilde{\rho}_P = \text{Tr}_M \left[S_{m,n} (\rho_P \otimes \rho_M) S_{m,n}^{\dagger} \right]$, a diagonal state in the energy eigenbasis with the following populations,

$$p_0' = p_0 + \sum_{j=1}^{m-1} (p_1 q_{j-1} - p_0 q_j) + (p_1 q_{m-1} - p_0 q_{m+n-1}), \tag{D.2a}$$

$$p_1' = p_1 - \sum_{j=1}^{m-1} (p_1 q_{j-1} - p_0 q_j) - (p_1 q_{m-1} - p_0 q_{m+n-1}) - \sum_{j=m+1}^{m+n-1} (p_1 q_j - p_2 q_{j-1}) - (p_1 q_m - p_2 q_0),$$
$$\tag{D.2b}$$

$$p_2' = p_2 + \sum_{j=m+1}^{m+n-1} (p_1 q_j - p_2 q_{j-1}) + (p_1 q_m - p_2 q_0). \tag{D.2c}$$

The final state of the catalyst is given by $\tilde{\rho}_M = \text{Tr}_P \left[S_{m,n} (\rho_P \otimes \rho_M) S_{m,n}^{\dagger} \right]$, and we ask this state to be equal to the initial one, so that $\tilde{\rho}_M = \rho_M$. This constraint provides the following set of equations,

$$q_0 = p_0 q_0 + p_0 q_1 + p_1 q_m \tag{D.3a}$$

$$q_j = p_1 q_{j-1} + p_0 q_{j+1} + p_2 q_j, \quad j = 1, \ldots, m - 2 \tag{D.3b}$$

$$q_{m-1} = p_1 q_{m-2} + p_0 q_{m+n-1} + p_2 q_{m-1} \tag{D.3c}$$

$$q_m = p_0 q_m + p_2 q_0 + p_1 q_{m+1} \tag{D.3d}$$

$$q_j = p_0 q_j + p_2 q_{j-1} + p_1 q_{j+1}, \quad j = m + 1, \ldots, m + n - 2 \tag{D.3e}$$

$$q_{m+n-1} = p_1 q_{m-1} + p_2 q_{m+n-2} + p_2 q_{m+n-1}, \tag{D.3f}$$

which, if solved, allows for the probability distribution of ρ_M to be expressed in terms of the distribution of the passive state ρ_P.

D.1   Energy extraction

The energy extracted from the system is defined by the difference in average energy between its initial and final state, as we show in Sec. 5.2, Eq. (5.9), and we re-write it here for convenience,

$$\Delta W = \mathrm{Tr}\left[H_P\left(\rho_P - \tilde{\rho}_P\right)\right].$$

We can express the amount of extracted energy in terms of the gaps of the Hamiltonian H_P,

$$\Delta W = \Delta E_{10}\left(p_0' - p_0\right) - \Delta E_{21}\left(p_2' - p_2\right), \tag{D.4}$$

where this expression has been obtained by applying the normalisation constraint to the initial and final state of the system. If we now replace the probability distribution of the final state $\tilde{\rho}_P$, given in Eqs. (D.2), into the above equation, we find

$$\Delta W = \Delta E_{10}\left(\sum_{j=1}^{m-1}\left(p_1 q_{j-1} - p_0 q_j\right) + \left(p_1 q_{m-1} - p_0 q_{m+n-1}\right)\right)$$
$$- \Delta E_{21}\left(\sum_{j=m+1}^{m+n-1}\left(p_1 q_j - p_2 q_{j-1}\right) + \left(p_1 q_m - p_2 q_0\right)\right). \tag{D.5}$$

This expression can be highly simplified if we use the properties of the probability distribution of the catalyst, Eqs. (D.3). In particular, from Eq. (D.3b) we find that

$$p_1 q_{j-1} - p_0 q_j = p_1 q_j - p_0 q_{j+1}, \quad \forall\, j = 1, \ldots, m-2, \tag{D.6}$$

while from Eq. (D.3c) we have that

$$p_1 q_{m-2} - p_0 q_{m-1} = p_1 q_{m-1} - p_0 q_{m+n-1}. \tag{D.7}$$

Together, these equations reduce the first bracket of Eq. (D.5) into a single term,

$$\sum_{j=1}^{m-1}\left(p_1 q_{j-1} - p_0 q_j\right) + \left(p_1 q_{m-1} - p_0 q_{m+n-1}\right) = m\left(p_1 q_{m-1} - p_0 q_{m+n-1}\right). \tag{D.8}$$

If we consider Eq. (D.3e), instead, we find that

$$p_1 q_j - p_2 q_{j-1} = p_1 q_{j+1} - p_2 q_j, \quad \forall\, j = m+1, \ldots, m+n-2, \tag{D.9}$$

while Eq. (D.3d) implies that

$$p_1 q_{m+1} - p_2 q_m = p_1 q_m - p_2 q_0. \tag{D.10}$$

These two equations simplify the second bracket of Eq. (D.5),

$$\sum_{j=m+1}^{m+n-1} (p_1 q_j - p_2 q_{j-1}) + (p_1 q_m - p_2 q_0) = n \left(p_1 q_{m+n-1} - p_2 q_{m+n-2} \right). \tag{D.11}$$

We can now use Eq. (D.3f) to show that

$$p_1 q_{m-1} - p_0 q_{m+n-1} = p_1 q_{m+n-1} - p_2 q_{m+n-2}, \tag{D.12}$$

which allows us to express the energy we extract as

$$\Delta W = \left(m \, \Delta E_{10} - n \, \Delta E_{21} \right) \left(p_1 q_{m+n-1} - p_2 q_{m+n-2} \right). \tag{D.13}$$

From the above equation we notice that the energy extracted is factorised into an Hamiltonian contribution and another contribution associated with the probability distribution of the passive state. Then, for a given Hamiltonian H_P such that $m \, \Delta E_{10} > n \, \Delta E_{21}$, we will find that only certain passive states allow for energy extraction, i.e., the ones in which $p_1 q_{m+n-1} > p_2 q_{m+n-2}$. Therefore, for every given Hamiltonian, i.e., for every ΔE_{10} and ΔE_{21}, and for every given cycle, i.e., for every n and m, we find that the set of passive states is divided into two subsets, those that allow for energy extraction, and those which do not.

We now express the probability distribution of the state of the catalyst ρ_M in terms of the probability distribution of the passive state ρ_P. As a first step, we express the first $m - 2$ elements of the sequence $\{q_j\}_{j=0}^{m-1}$ in terms of last two elements, q_{m-2} and q_{m-1}. Moreover, we express the first $n - 2$ elements of $\{q_j\}_{j=m}^{m+n-1}$ in terms of q_{m+n-2} and q_{m+n-1}. This can be done using the equalities of Eqs. (D.3b) and (D.3e), which we recast in the following way.

$$q_j = \left(1 + \frac{p_0}{p_1} \right) q_{j+1} - \frac{p_0}{p_1} q_{j+2}, \quad \forall j = 0, \ldots, m - 3, \tag{D.14}$$

$$q_j = \left(1 + \frac{p_1}{p_2} \right) q_{j+1} - \frac{p_1}{p_2} q_{j+2}, \quad \forall j = m, \ldots, m + n - 3. \tag{D.15}$$

It can be proved, see Sec. D.5, that the elements of these sequences can be expressed as

$$q_j = T_1 \left(m - (j+2) \right) q_{m-2} - \frac{p_0}{p_1} T_1 \left(m - (j+3) \right) q_{m-1}, \quad \forall j = 0, \ldots, m - 3, \tag{D.16}$$

$$q_j = T_2 \left(m + n - (j+2) \right) q_{m+n-2} - \frac{p_1}{p_2} T_2 \left(m + n - (j+3) \right) q_{m+n-1}, \quad \forall j = m, \ldots, m + n - 3,$$

$$\tag{D.17}$$

where we define the two functions $T_1(h) = \sum_{l=0}^{h} \left(\frac{p_0}{p_1}\right)^l$ and $T_2(h) = \sum_{l=0}^{h} \left(\frac{p_1}{p_2}\right)^l$.

We can now express, using Eqs. (D.3c) and (D.3f), the elements q_{m-2} and q_{m-1} in terms of q_{m+n-2} and q_{m+n-2}. From Eq. (D.3c) we obtain that

$$q_{m-2} = T_1(2)\, q_{m+n-1} - \frac{p_2}{p_1}\, T_1(1)\, q_{m+n-2}. \tag{D.18}$$

From Eq. (D.3f), instead, we get that

$$q_{m-1} = T_1(1)\, q_{m+n-1} - \frac{p_2}{p_1}\, T_1(0)\, q_{m+n-2}. \tag{D.19}$$

Then, we can finally express q_{m+n-2} in terms of q_{m+n-1} through Eq. (D.3d), and we obtain

$$q_{m+n-2} = D(m,n)\, q_{m+n-1}, \tag{D.20}$$

where the coefficient $D(m,n)$ is defined as

$$D(m,n) = \frac{p_1}{p_2} \frac{T_1(m) + \frac{p_1}{p_2} T_2(n-2)}{T_1(m-1) + \frac{p_1}{p_2} T_2(n-1)}. \tag{D.21}$$

Thanks to the above result, we can express the overall probability distribution of the catalyst's state ρ_M in terms of the occupation probability of the state $|m+n-1\rangle_M$. Thus, we have that

$$q_j = \left(T_1(m-j) - \frac{p_2}{p_1} D(m,n)\, T_1(m-(j+1)) \right) q_{m+n-1}, \quad j = 0, \ldots, m-1, \tag{D.22}$$

$$q_j = \left(T_2(m+n-(j+2))\, D(m,n) - \frac{p_1}{p_2} T_2(m+n-(j+3)) \right) q_{m+n-1}, \quad j = m, \ldots, m+n-3, \tag{D.23}$$

$$q_{m+n-2} = D(m,n)\, q_{m+n-1}, \tag{D.24}$$

where it is possible to show that each q_j, with $j = 0, \ldots, m+n-2$, is positive if q_{m+n-1} is positive (see the technical result Prop. 15). From the normalisation condition it then follows that the sequence $\{q_j\}_{j=0}^{m+n-1}$ is a proper probability distribution. Moreover, the normalisation condition allows us to evaluate q_{m+n-1} as a function of the probability distribution of the passive state ρ_P,

$$q_{m+n-1} = \frac{T_1(m-1) + \frac{p_1}{p_2} T_2(n-1)}{\left(T_1(m) + \frac{p_1}{p_2} T_2(n-2) \right)^2 + \left(\left(\frac{p_1}{p_2}\right)^n - \left(\frac{p_0}{p_1}\right)^m \right) \left(\sum_{j=0}^{m} T_1(j) - \frac{p_1}{p_2} \sum_{j=0}^{n-3} T_2(j) \right)}. \tag{D.25}$$

From Eq. (D.25) we can express all the other elements of $\{q_j\}_{j=0}^{m+n-1}$ in terms of the probability distribution of ρ_P.

We can now further characterise the amount of energy extracted during our cycle. If we apply Eq. (D.24) into Eq. (D.13), we obtain

$$\Delta W = (m\,\Delta E_{10} - n\,\Delta E_{21})\,\frac{p_1\left(\left(\frac{p_1}{p_2}\right)^n - \left(\frac{p_0}{p_1}\right)^m\right)}{\mathrm{T}_1(m-1) + \frac{p_1}{p_2}\mathrm{T}_2(n-1)}\,q_{m+n-1}, \tag{D.26}$$

where the sign of ΔW solely depends on the terms $(m\,\Delta E_{10} - n\,\Delta E_{21})$ and $\left(\left(\frac{p_1}{p_2}\right)^n - \left(\frac{p_0}{p_1}\right)^m\right)$, since the other factors are always positive. We can group the other terms in a single, positive coefficient, which we refer to as α in Ch. 5,

$$\alpha = \frac{p_1\,q_{m+n-1}}{\mathrm{T}_1(m-1) + \frac{p_1}{p_2}\mathrm{T}_2(n-1)}. \tag{D.27}$$

As a result, we find that energy can be extracted from a system with Hamiltonian H_P, described by the passive state ρ_P, using a cycle with m hot swaps and n cold swaps, when one of the following is satisfied,

1. The probability distribution is such that $\left(\frac{p_1}{p_2}\right)^n > \left(\frac{p_0}{p_1}\right)^m$ and the energy gaps are such that $m\,\Delta E_{10} > n\,\Delta E_{21}$.

2. The probability distribution is such that $\left(\frac{p_1}{p_2}\right)^n < \left(\frac{p_0}{p_1}\right)^m$ and the energy gaps are such that $m\,\Delta E_{10} < n\,\Delta E_{21}$.

For example, for the class of passive states considered in Ch. 5 we have that energy is extracted from the system when the first condition is satisfied, as it can be seen by comparing this condition with Eq. (5.12).

D.2 Final state of the system

Let us consider the final state of the system after we have applied the cycle $S_{m,n}$. In Eq. (D.2) we have shown the probability distribution of $\tilde{\rho}_P$ as a function of $\{q_i\}_{i=0}^{m+n-1}$. Thanks to the

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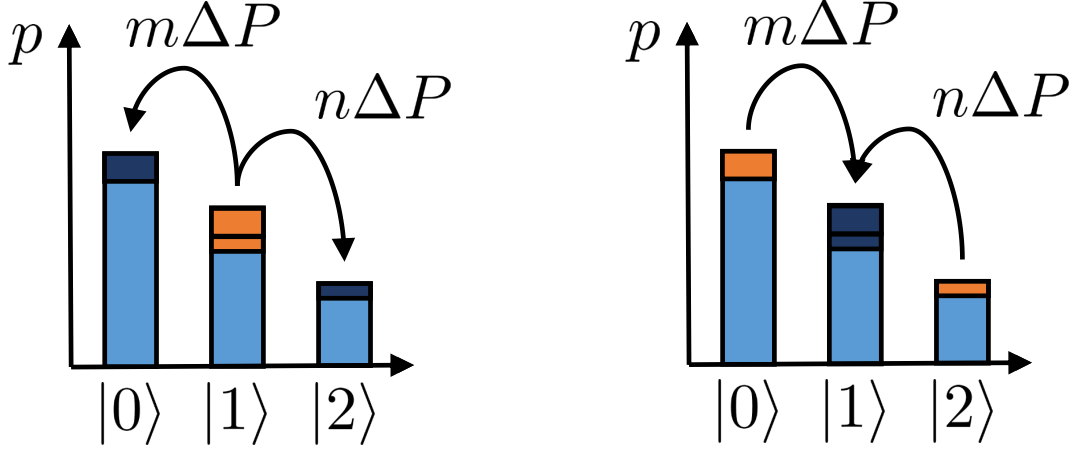
Figure D.1: The action of the cycle $S_{m,n}$ on the qutrit passive state ρ_P. **Left.** When the Hamiltonian H_P is such that $m\,\Delta E_{10} > n\,\Delta E_{21}$, then energy is extracted by reducing the probability of occupation of $|1\rangle$ by an amount $(m+n)\,\Delta P$ (orange), while increasing the probabilities of occupation of $|0\rangle$ and $|2\rangle$ by an amount $m\,\Delta P$ and $n\,\Delta P$ (dark blue), respectively. **Right.** The action of the same cycle on the passive state ρ_P, when the system's Hamiltonian H_P is such that $m\,\Delta E_{10} < n\,\Delta E_{21}$. In this case, energy is extracted when the cycle acts on the populations of the system in the opposite way compared to the previous scenario.

constraints introduced in Eqs. (D.3), we can simplify the form of $\tilde{\rho}_P$, so that we obtain

$$p'_0 = p_0 + m\,\Delta P, \tag{D.28}$$

$$p'_1 = p_1 - (m+n)\,\Delta P, \tag{D.29}$$

$$p'_2 = p_2 + n\,\Delta P. \tag{D.30}$$

Thus, the cycle acts on the passive state by modifying the original probabilities by multiples of

$$\Delta P = \frac{p_1\,q_{m+n-1}}{\mathrm{T}_1(m-1) + \frac{p_1}{p_2}\mathrm{T}_2(n-1)}\left(\left(\frac{p_1}{p_2}\right)^n - \left(\frac{p_0}{p_1}\right)^m\right). \tag{D.31}$$

The expression of the final state $\tilde{\rho}_P$ allows us to understand how the cycle operates over the system when energy is extracted. In particular, we can consider the evolution of the system in two different situations, linked to the two possible scenarios described in the previous section.

Suppose that we apply a cycle composed by m hot swaps and n cold swaps. When H_P is such that $m\,\Delta E_{10} > n\,\Delta E_{21}$, then energy is extracted if the unit $\Delta P > 0$, so that the

protocol is depleting the population p_1, while increasing the populations p_0 and p_2, see the left panel of Fig. D.1. Energy is extracted from the cycle because the energy gained while moving $m \, \Delta P$ from p_1 to p_0 is bigger than the energy spent to move $n \, \Delta P$ from p_1 to p_2. When the Hamiltonian of the system is such that $m \, \Delta E_{10} < n \, \Delta E_{21}$, we have that energy extraction is possible if $\Delta P < 0$. In this case, both populations p_0 and p_2 are depleted, while the population p_1 is increased, see the right panel of Fig. D.1. Energy is extracted from the cycle since the energy gained while moving $n \, |\Delta P|$ from p_2 to p_1 is bigger than the energy spent to move $m \, |\Delta P|$ from p_0 to p_1.

D.3   Asymptotic behaviour of the protocol

We are now interested in the study of the cycle $S_{m,n}$ when the dimension of the catalyst, as well as the number of hot and cold swaps, tends to infinity. In the following, we consider a system whose Hamiltonian H_P and state ρ_P are such that energy can be extracted if the parameters m and n of the cycle satisfy condition 1 of the previous section. These constraints imply that the ratio of hot and cold swaps, $\gamma = \frac{m}{n}$, needs to be finite and to lie within a well-defined range,

$$\frac{\Delta E_{21}}{\Delta E_{10}} < \gamma < \frac{\log \frac{p_1}{p_2}}{\log \frac{p_0}{p_1}}, \tag{D.32}$$

which can be compared with the range in Eq. (5.25) of Ch. 5. In the following we study the asymptotic probability distribution of the catalyst, the energy extracted, and the final state of the passive system. As a first step, let us consider the coefficient $D(m,n)$ introduced in Eq. (D.21). When both m and n tends to infinity, we find that

$$D(m,n) \approx 1 + \left(\frac{p_0}{p_1}\right)^m \left(\frac{p_2}{p_1}\right)^n \frac{(p_0 - p_2)(p_1 - p_2)}{(p_0 - p_1)} + O\left(\left(\frac{p_2}{p_1}\right)^n ; \left(\frac{p_0}{p_1}\right)^{2m} \left(\frac{p_2}{p_1}\right)^{2n}\right), \tag{D.33}$$

where it is easy to verify that the term $(p_0/p_1)^m (p_2/p_1)^n \to 0$ as $m, n \to \infty$, and that both $(p_2/p_1)^n$ and $(p_0/p_1)^{2m} (p_2/p_1)^{2n}$ tends to 0 faster that this first term. However, we cannot say which one is the fastest without further assumptions, and that is the reason we keep both terms in the big-O.

Once the expansion of $D(m,n)$ is known, we can focus on the probability distribution of the catalyst. For simplicity, we consider the distribution in Eqs. (D.22), (D.23), and (D.24),

where q_{m+n-1} is not defined yet; we define it through the normalisation condition, once the asymptotic expansion has been performed. We find that

$$q_j \approx q_{m+n-1} \left(\frac{p_0 - p_2}{p_0 - p_1} + O\left(\left(\frac{p_0}{p_1} \right)^m \left(\frac{p_2}{p_1} \right)^n \right) \right) \left(\frac{p_0}{p_1} \right)^{m-j}, \quad j = 0, \ldots, m-1, \tag{D.34}$$

$$q_j \approx q_{m+n-1} \left(\frac{p_2}{p_1} \frac{p_0 - p_2}{p_0 - p_1} + O\left(\left(\frac{p_0}{p_1} \right)^m \left(\frac{p_2}{p_1} \right)^n \right) \right) \left(\frac{p_0}{p_1} \right)^m \left(\frac{p_1}{p_2} \right)^{m-j}, \quad j = m, \ldots, m+n-3, \tag{D.35}$$

$$q_{m+n-2} \approx q_{m+n-1} \left(1 + O\left(\left(\frac{p_0}{p_1} \right)^m \left(\frac{p_2}{p_1} \right)^n \right) \right). \tag{D.36}$$

We are now able to obtain the value of q_{m+n-1} by imposing the normalisation condition over the asymptotic probability distribution of the catalyst. We find that

$$q_{m+n-1} \approx \left(\frac{(p_1 - p_2)(p_0 - p_1)^2}{p_1(p_0 - p_2)^2} + O\left(\left(\frac{p_0}{p_1} \right)^m \left(\frac{p_2}{p_1} \right)^n \right) \right) \left(\frac{p_1}{p_0} \right)^m, \tag{D.37}$$

which implies that q_{m+n-1} tends to 0 as $(p_1/p_0)^m$ for $m \to \infty$. Notice that the same result can be obtained by directly expanding Eq. (D.25). If we send m and n to infinity, we find that the asymptotic probability distribution of the catalyst is

$$q_j \approx \frac{(p_1 - p_2)(p_0 - p_1)}{p_1(p_0 - p_2)} \left(\frac{p_0}{p_1} \right)^{-j}, \quad j = 0, \ldots, m-1, \tag{D.38}$$

$$q_j \approx \frac{p_2(p_1 - p_2)(p_0 - p_1)}{p_1^2(p_0 - p_2)} \left(\frac{p_1}{p_2} \right)^{m-j}, \quad j = m, \ldots, m+n-3, \tag{D.39}$$

$$q_{m+n-2} \approx q_{m+n-1} \approx \frac{(p_1 - p_2)(p_0 - p_1)^2}{p_1(p_0 - p_2)^2} \left(\frac{p_1}{p_0} \right)^m. \tag{D.40}$$

We can now investigate how the probability distribution of the main system changes, and evaluate the asymptotic energy extracted ΔW during on cycle. Let us consider the probability unit ΔP, introduced in Eq. (D.31). If we set m and n to infinity, we have that

$$\Delta P \approx \frac{(p_1 - p_2)^2(p_0 - p_1)^2}{p_1(p_0 - p_2)^2} \left(\frac{p_1}{p_0} \right)^m, \tag{D.41}$$

that tends to 0 with an exponential scaling. Therefore, when we run the protocol with an infinite-dimensional catalyst, the passive states is modified by an infinitesimal amount. As a consequence, the energy extracted during the cycle is infinitesimal as well. Indeed, from Eq. (D.26) it follows that ΔW tends to 0 as $m, n \to \infty$, since ΔW is proportional to ΔP (modulo a multiplying factor proportional to m, which tends to infinity more slowly than $(p_1/p_0)^m$ tends to 0).

D.4   Energy extraction through multiple cycles

We now study the evolution of a passive state when the asymptotic cycle presented in the previous section is applied to the system an infinite number of times. After each cycle, the state of the system is infinitesimally modified, and we find that its evolution can be described by a set of differential equations. Indeed, when $\Delta P \to 0$, the Eqs. (D.28) and (D.29) can be re-cast as follow

$$\frac{\mathrm{d}p_0}{\mathrm{d}t} = \frac{(p_1 - p_2)^2 (p_0 - p_1)^2}{p_1 (p_0 - p_2)^2}, \tag{D.42}$$

$$\frac{\mathrm{d}p_1}{\mathrm{d}t} = -\left(1 + \gamma(t)^{-1}\right) \frac{(p_1 - p_2)^2 (p_0 - p_1)^2}{p_1 (p_0 - p_2)^2}, \tag{D.43}$$

where $\gamma(t)$ can be changed at each iteration of the cycle, and therefore depends on the parameter t. The $\gamma(t)$ coefficient takes values in the range given by Eq. (D.32). We define

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = \lim_{m \to \infty} \frac{p_i' - p_i}{\Delta \mathrm{p}(m)}, \quad \text{where } \Delta\mathrm{p}(m) = m \left(\frac{p_1}{p_0}\right)^m, \quad \text{for } i = 0, 1. \tag{D.44}$$

The continuous parameter t is here related to the number of cycles we perform on the system. It is worth noting that Eqs. (D.42) and (D.43) share a common (positive) factor. Therefore we have that, as time goes on, the probability of occupation of $|0\rangle_P$ increases, while the one of $|1\rangle_P$ decreases. Moreover, since $\gamma(t) > 0$, the increase in the former population is slower than the decreasing of the latter.

The two differential equations can be arranged in a single one, shown in the main text, Eq. (5.26), and reported here for convenience,

$$\frac{\mathrm{d}p_1}{\mathrm{d}t} = -\left(1 + \gamma(t)^{-1}\right) \frac{\mathrm{d}p_0}{\mathrm{d}t}.$$

We can investigate the solution of this equation for $\gamma(t)$ close to its limiting values. First, consider the case in which $\gamma(t) = \frac{\Delta E_{21}}{\Delta E_{10}} + \frac{1}{m} \approx \frac{\Delta E_{21}}{\Delta E_{10}}$. Then, the solution of the above equation is

$$p_1(t) = -\left(1 + \frac{\Delta E_{10}}{\Delta E_{12}}\right) \left(p_0(t) - p_0(t = 0)\right) + p_1(t = 0), \tag{D.45}$$

where $\{p_i(t)\}$ is the probability distribution of the state after a number of cycles, parametrised by t, have been performed. If we rearrange Eq. (D.45), we see that it is equivalent to the following constraint

$$\mathrm{Tr}\left[H_\mathrm{P}\, \rho_\mathrm{P}(t)\right] = \mathrm{Tr}\left[H_\mathrm{P}\, \rho_\mathrm{P}\right] \quad \forall\, t \geq 0, \tag{D.46}$$

which implies that the evolution conserves the energy of the system, or, equivalently, that no energy is extracted during the process. The other extremal evolution of the state is achieved for $\gamma(t) = \frac{\log p_1 - \log p_2}{\log p_0 - \log p_1} - \frac{1}{m} \approx \frac{\log p_1 - \log p_2}{\log p_0 - \log p_1}$. In this case, $\gamma(t)$ depends on the probability distribution of the passive state, and therefore its value changes at each iteration of the cycle. Replacing $\gamma(t)$ in Eq. (5.26) result in the following

$$\log p_0 \, \frac{\mathrm{d}p_0}{\mathrm{d}t} + \log p_1 \, \frac{\mathrm{d}p_1}{\mathrm{d}t} + \log p_2 \, \frac{\mathrm{d}p_2}{\mathrm{d}t} = 0, \tag{D.47}$$

which, if integrated between 0 and t, gives the following constraint on the entropy of the evolved states

$$S\left(\rho_{\mathrm{P}}(t)\right) = S\left(\rho_{\mathrm{P}}\right) \quad \forall \, t \geq 0, \tag{D.48}$$

where $S(\cdot)$ is the Von Neumann entropy. Therefore, the evolution of the passive state has to preserve the entropy of the system.

Notice that any value of $\gamma(t)$ within the range of Eq. (D.32) can be chosen, so that a passive state can evolve along any intermediate trajectory between that of constant energy and that of constant entropy. The trajectories bring the passive state toward the completely passive states, that are the fixed points of this evolution.

D.5   Technical results

In this section, we prove some technical results used in the analysis of the energy-extracting cycle.

Proposition 14. *Consider the sequence of real numbers $\{x_j\}_{j=a}^{b}$, those elements are linked by the following set of equations,*

$$x_j = (1 + \lambda) \, x_{j+1} - \lambda \, x_{j+2}, \quad j = a, \ldots, b - 2,$$

where $\lambda \in \mathbb{R}$ and $a, b \in \mathbb{N}$, $a \leq b - 2$. Then, the elements of this sequence can be expressed in terms of x_{b-1} and x_b as

$$x_j = \mathrm{T}(b - (j+1), \lambda) \, x_{b-1} - \lambda \, \mathrm{T}(b - (j+2), \lambda) \, x_b, \quad j = a, \ldots, b - 2,$$

where $\mathrm{T}(h, \lambda) = \sum_{l=0}^{h} \lambda^l = \frac{1 - \lambda^{h+1}}{1 - \lambda}$.

Proof. If we insert the solution into the set of equations, we find

$$\mathrm{T}(b-(j+1),\lambda)\,x_{b-1} - \lambda\,\mathrm{T}(b-(j+2),\lambda)\,x_b = (1+\lambda)\mathrm{T}(b-(j+2),\lambda)\,x_{b-1}$$
$$- \lambda(1+\lambda)\,\mathrm{T}(b-(j+3),\lambda)\,x_b$$
$$- \lambda\,\mathrm{T}(b-(j+3),\lambda)\,x_{b-1}$$
$$+ \lambda^2\,\mathrm{T}(b-(j+4),\lambda)\,x_b$$

for j taking values from a to $b-2$. We can re-organise the above equation, and we find that it is satisfied iff

$$\mathrm{T}(b-(j+1),\lambda) = (1+\lambda)\,\mathrm{T}(b-(j+2),\lambda) - \lambda\,\mathrm{T}(b-(j+3),\lambda), \quad j = a,\dots,b-2, \quad \text{(D.49)}$$
$$\mathrm{T}(0,\lambda) = (1+\lambda)\,\mathrm{T}(-1,\lambda) - \lambda\,\mathrm{T}(-2,\lambda). \tag{D.50}$$

These two equalities easily follow from the definition of $\mathrm{T}(h,\lambda)$, as it can be check by replacing this coefficient with its explicit form in both Eqs. (D.49) and (D.50). $\qquad\square$

Proposition 15. *The probability distribution of the state ρ_M is positive and normalised.*

Proof. Let us consider the probabilities q_j for $j = 0,\dots,m-1$, as given in Eq. (D.22). If we replace j with $j' = m-j$, then the main coefficient in the equation becomes

$$\mathrm{T}_1(j') - \frac{p_2}{p_1}\,D(m,n)\,\mathrm{T}_1(j'-1) = \frac{\mathrm{T}_1(j')\,\mathrm{T}_1(m-1) - \mathrm{T}_1(j'-1)\,\mathrm{T}_1(m)}{\mathrm{T}_1(m-1) + \frac{p_1}{p_2}\mathrm{T}_2(n-1)}$$
$$+ \frac{p_1}{p_2}\frac{\mathrm{T}_1(j')\,\mathrm{T}_2(n-1) - \mathrm{T}_1(j'-1)\,\mathrm{T}_2(n-2)}{\mathrm{T}_1(m-1) + \frac{p_1}{p_2}\mathrm{T}_2(n-1)}.$$

It is clear that the denominator is positive, as $\mathrm{T}_1(h)$ and $\mathrm{T}_2(h)$ are positive for all $h \in \mathbb{Z}$. We need to show that the nominator is positive as well. The nominator of the first term can be reduced to

$$\mathrm{T}_1(j')\,\mathrm{T}_1(m-1) - \mathrm{T}_1(j'-1)\,\mathrm{T}_1(m) = \mathrm{T}_1(m-1) - \mathrm{T}_1(j'-1) = \sum_{l=j'}^{m-1}\left(\frac{p_0}{p_1}\right)^l \geq 0,$$

where the last equality follows from the fact that $j' = 1,\dots,m$. The nominator of the second term can be expressed as

$$\mathrm{T}_1(j')\,\mathrm{T}_2(n-1) - \mathrm{T}_1(j'-1)\,\mathrm{T}_2(n-2) = \mathrm{T}_1(j'-1)\left(\frac{p_1}{p_2}\right)^{n-1} + \mathrm{T}_2(n-2)\left(\frac{p_0}{p_1}\right)^{j'}$$
$$+ \left(\frac{p_0}{p_1}\right)^{j'}\left(\frac{p_1}{p_2}\right)^{n-1} > 0.$$

Thus, the probabilities $\{q_j\}_{j=0}^{m-1}$ are positive when q_{m+n-1} is positive.

We can now focus on the probabilities q_j for $j = m, \ldots, m + n - 3$, as given in Eq. (D.23). By replacing j with $j' = m + n - (j + 2)$ we obtain that the main coefficient in the equation becomes

$$\mathrm{T}_2(j')\, D(m, n) - \frac{p_1}{p_2}\, \mathrm{T}_2(j' - 1) = \left(\frac{p_1}{p_2}\right) \frac{\mathrm{T}_2(j')\, \mathrm{T}_1(m) - \mathrm{T}_2(j' - 1)\, \mathrm{T}_1(m - 1)}{\mathrm{T}_1(m - 1) + \frac{p_1}{p_2}\mathrm{T}_2(n - 1)}$$
$$+ \left(\frac{p_1}{p_2}\right)^2 \frac{\mathrm{T}_2(j')\, \mathrm{T}_2(n - 2) - \mathrm{T}_2(j' - 1)\, \mathrm{T}_2(n - 1)}{\mathrm{T}_1(m - 1) + \frac{p_1}{p_2}\mathrm{T}_2(n - 1)}.$$

As before, the denominator is positive, as $\mathrm{T}_1(h)$ and $\mathrm{T}_2(h)$ are both positive $\forall\, h \in \mathbb{Z}$. The nominator of the first term can be reduced to

$$\mathrm{T}_2(j')\, \mathrm{T}_1(m) - \mathrm{T}_2(j' - 1)\, \mathrm{T}_1(m - 1) = \mathrm{T}_2(j' - 1) \left(\frac{p_0}{p_1}\right)^m + \mathrm{T}_1(m - 1) \left(\frac{p_1}{p_2}\right)^{j'}$$
$$+ \left(\frac{p_1}{p_2}\right)^{j'} \left(\frac{p_0}{p_1}\right)^m > 0.$$

The nominator of the second term can be expressed as

$$\mathrm{T}_2(j')\, \mathrm{T}_2(n - 2) - \mathrm{T}_2(j' - 1)\, \mathrm{T}_2(n - 1) = \mathrm{T}_2(n - 2) - \mathrm{T}_2(j' - 1) = \sum_{l = j'}^{n-2} \left(\frac{p_1}{p_2}\right)^l \geq 0,$$

where the last equality follows from the fact that $j' = 1, \ldots, n - 2$. Thus, the probabilities $\{q_j\}_{j=m}^{m+n-3}$ are positive when $q_{m+n-1} > 0$.

In Eq. (D.24), we showed that q_{m+n-2} is related to q_{m+n-1} by the multiplicative coefficient $D(m, n)$, which can be easily shown to be positive for any integer $m, n \geq 1$. Finally, the normalisation condition force $q_{m+n-1} > 0$, and implies the probability distribution of ρ_M to be positive and normalised. $\qquad\square$

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