INDEX ONE MINIMAL SURFACES AND THE ISOPERIMETRIC PROBLEM IN SPHERICAL SPACE FORMS

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Declaration

I, Celso Dos Santos Viana, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

Celso dos Santos Viana

I have had my results for a long time: but I do not yet know how I am to arrive at them.

Carl Friedrich Gauss

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Abstract

The research carried out in this thesis concerns two important class of stationary surfaces in Differential Geometry, namely isoperimetric surfaces and index one minimal surfaces. The former are solutions of the so called isoperimetric problem, which is to determine the regions of least perimeter among regions of same volume in a given manifold. The latter are critical points of the area functional with Morse index one, i.e., minimal surfaces which admits only one direction where the surface can be deformed so to decrease its area. These are usually constructed via mountain pass arguments. This work focus on the study of these objects when the ambient space is a 3-dimensional spherical space forms, i.e., space form with positive curvature. Our main results classify, at the level of topology, such stationary surfaces in the spherical space forms with large fundamental group.

Our first result proves that the solutions of the isoperimetric problem in spherical space forms with large fundamental group are either spheres or tori. It was previously known that solutions with genus zero and one are respectively totally umbilical and flat. Combining our result and this geometric description, we derive that the solutions of the isoperimetric problem are either geodesic spheres or quotients of Clifford tori. Our second result proves that orientable minimal surfaces with index one in the aforementioned spherical space forms have genus at most two. This is a sharp estimate as one can use the continuous one-parameter min-max theory to construct in every 3-dimensional spherical space form an index one minimal surface with genus equal the Heegaard genus of such space which is known to be at most two. Our result confirms a conjecture of R. Schoen for an infinite class of 3-manifolds.

Contents

Introduction

0.1 The isoperimetric problem

The isoperimetric problem is a classical subject in Differential Geometry with its origin in ancient Greece. It consists in finding on a Riemannian manifold M the regions that minimize the perimeter among sets enclosing the same volume. The solutions are called isoperimetric regions and their boundaries isoperimetric hypersurfaces. The Euclidean plane is historically the first space where the problem started to be investigated rigorously. It is now a well known fact that the round circles are the optimal curves for the problem. This geometric fact is often seen through the following classical inequality:

$$L^2 \geq 4\pi A,$$

where L and A stand for the length and the enclosed area of a simple closed curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ respectively.

The framework of geometric measure theory and its tools work successfully well in tackling the aspects of existence and regularity of this variational problem. When M^{n+1} is closed or homogeneous, then isoperimetric hypersurfaces do exist and are smooth up to a closed set of Hausdorff dimension $n-7$. The regular part is a stable hypersurface of constant mean curvature. This major contribution to the isoperimetric problem was achieved thanks to the efforts of many people, including F. Almgren, R. Schoen, L. Simon, F. Morgan, and others (see [43] for a comprehensive list). Despite the long history of the problem, it remains largely open with few 3-manifolds where the problem is completely understood.

The simply connected space forms, $\mathbb{S}^{n+1}$, $\mathbb{R}^{n+1}$ and $\mathbb{H}^{n+1}$, are the most appealing spaces to begin the study of the isoperimetric problem. It turns out that their symmetries are enough to characterize the geodesic spheres as the isoperimetric hypersurfaces.

A complete solution on $\mathbb{S}^2 \times \mathbb{S}^1$ with the standard product metric can be found in [49]. For other homogeneous manifolds with certain product structure, such as $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{S}^1$, and $\mathbb{S}^n \times \mathbb{R}$, see [30], [49], and [48] respectively. The case $\mathbb{S}^1 \times \mathbb{R}^n$ is also treated in [49] and they show that when $n \geq 9$ unduloids are minimizers rather than cylinders for certain volumes. One key idea exploited in the results listed above is the use of symmetry to reduce the problem to an ODE analysis. The case $T^2 \times \mathbb{R}$ where T^2 is a flat torus is not solved in full; great progress can be found in [56, 53, 28]. More generally, it is known that boundaries of small isoperimetric regions in closed manifolds are nearly round spheres, see [45, 47]. To finish this brief and not exhaustive account of results on the isoperimetric problem we mention that Bray and Morgan ([7] and [8]) classified the horizon homologous isoperimetric surfaces in the Schwarzschild manifold. The works in [7, 21] highlighted the interesting relationship between isoperimetric surfaces and the concept of mass in general relativity.

We will be interested in spherical space forms in this paper. A significant result in this direction was given in [56] where A. Ros and M. Ritoré solved the isoperimetric problem in the projective space $\mathbb{RP}^3$. They show that the solutions are geodesic spheres or flat tori. This geometric description for the spheres and tori which are solutions of the isoperimetric problem holds true for every 3-dimensional space form. Later, A. Ros [58] used the above result to give a proof of the Willmore conjecture in $\mathbb{S}^3$ for the special case of surfaces that are invariant by the antipodal map.

The real projective space is a special case of an important family of Riemannian manifolds, namely the Lens spaces $L(p, q)$. These are spherical space forms obtained as a quotient of $\mathbb{S}^3$ by a finite group of isometries that are isomorphic to $\mathbb{Z}_p$ but which also depend on q. They are, along with $\mathbb{S}^2 \times \mathbb{S}^1$, characterized by having Heegaard genus one.

We give a complete solution for the isoperimetric problem in the Lens spaces with large fundamental group:

Theorem 1. *There exists a positive integer p_0 such that for every $p \geq p_0$ and every $q \geq 1$ the isoperimetric surfaces in $L(p,q)$ are either geodesic spheres or quotients of Clifford tori.*

The isoperimetric problem in dimension three was previously solved for only a finite number of non-diffeomorphic 3-manifolds.

We also add to the literature the observation that the proof of the Willmore conjecture by Marques and Neves [38] can be applied to extend the work of Ros and Ritoré [56], on the classification of stable cmc surfaces in $\mathbb{RP}^3$, to $L(3,1)$ and $L(3,2)$:

Theorem 2. *The only immersed stable cmc surfaces in $L(3,1)$ and $L(3,2)$ are either geodesic spheres or projections of Clifford tori. Moreover, the projection of the minimal Clifford torus is, up to ambient isometries, the only index one minimal surface in $L(3,1)$ and $L(3,2)$.*

A version of Theorem 2 for the special case of isoperimetric surfaces in the lens space $L(3,1)$ appeared first in [59, Theorem 15].

The idea of the proof of Theorem 1 is as follows. Stability implies that every isoperimetric surface is connected and its genus is $0, 1, 2$ or 3. If follows from a classical result of Hopf that if the genus is 0, then it is a geodesic sphere. From [56] we know that if the genus is 1, then it is flat, and this forces the surface to be a quotient of a Clifford tori. We are left to rule out other topological types. To do so we argue by contradiction. Because of the algebraic complexity of the lens spaces we divide the lens space in two classes for which we give separately proofs. We assume that there exists a sequence of Lens spaces with increasing fundamental group containing isoperimetric surfaces of genus 2 or 3. After a suitable rescaling on the metrics we use compactness results to obtain a limit for the sequence of Lens spaces which will be a flat three manifold of rank one. In the same way, the sequence of isoperimetric surfaces will converge to a flat surface in the respective ambient manifold. The topology of the surfaces will force the limit to be an union of planes. On the other hand, the minimization property rules this configuration out. This argument works for most sequence of lens spaces. For the remaining cases we use standard compactness applied to the pullback surfaces in the round three sphere.

A natural question after Theorem 1 is whether its proof can be extended to the case of spherical space forms with large fundamental group. From the classification of spherical space forms we observe that these spaces share with the lens spaces the same crucial properties needed in proof of Theorem 1. Theorem 1 is now a corollary of the following result:

Theorem 3. *There exists p_0 such that if M^3 is an spherical space form with $|\pi_1(M)| \geq p_0$, then the isoperimetric surfaces in M^3 are either geodesic spheres or quotient of Clifford tori.*

To the author knowledge, it is not known whether an isoperimetric surface of genus two exists in any spherical space form. It would be interesting to investigate this question in the Poincaré Homology Sphere.

The arguments in the proof of Theorem 1 generalize naturally also for the Berger Spheres $\mathbb{S}^3_\varepsilon$. This is a well known one parameter family of homogeneous metrics on the 3-sphere; the case $\varepsilon = 1$ corresponds to the round metric. When ε is not too small, spheres are the only solutions of the isoperimetric problem. When ε is very small, some tori are better candidates to solve the isoperimetric problem rather than spheres for certain volumes.

Theorem 4. *There exists $\varepsilon_0 > 0$ such that for every $\varepsilon < \varepsilon_0$ the isoperimetric surfaces in the Berger spheres $\mathbb{S}^3_\varepsilon$ are either rotationally invariant spheres or tori.*

Remark 1. It follows from the work in [44] and [49] that the minimal Clifford torus in $L(p, 1)$ (resp. $\mathbb{S}^3_\varepsilon$) is isoperimetric for every $p \geq 3$ (resp. $\varepsilon \leq \frac{1}{3}$).

0.2 Index one minimal surfaces

The Morse index is an important analytic quantity in the study of minimal surfaces. Roughly speaking, it counts the maximal number of directions a minimal surface can be deformed in order to decrease its area. Under this analytical point of view, the simplest minimal surfaces are those with small index, namely zero or one. Index zero minimal surfaces, also known as stable, are an well studied topic in Differential Geometry. Among classical results we mention the Bernstein problem on the classification of complete minimal graphs in $\mathbb{R}^n$ and those connecting stable minimal surfaces and the topology of manifolds admitting positive scalar curvature metrics due to Schoen-Yau. The existence of stable minimal surfaces depends on the geometry and topology of the ambient space and is in general obtained via a minimization procedure. Such surfaces do not exist in manifolds with positive Ricci curvature. Index one minimal surfaces, on the other hand, do exist is this setting and are produced by the one parameter min-max construction of Almgren-Pitts and Simon-Smith [14, 32, 39, 51]. An important feature about these surfaces is that they provide optimal geometric Heegaard splitting of closed 3-manifolds.

A guiding principle in the theory asserts that in positively curved manifolds, the index of a minimal hypersurface controls its topology and geometry. For instance, it is proved in [12] that the set of minimal surfaces with bounded index in a closed 3-manifold with positive scalar curvature cannot contain sequences of surfaces with unbounded genus or area. More generally, it is conjectured in [37, 47] that if Σ is a minimal hypersurface in a closed manifold with positive Ricci curvature M, then $\mathrm{index}(\Sigma) \geq C\, b_1(\Sigma)$, where $b_1(\Sigma)$ is the first Betti number of Σ and C is a constant which depends only on M. Estimates of this type have been studied by many authors, see [3, 11, 33, 57] and references therein for further discussion. These estimates are, however, far from being optimal when the index is small in general. A related problem is to describe the geometry and topology of the minimal surfaces with the smallest index. In this direction, we mention the classical result that flat planes and the catenoid are, respectively, the only embedded minimal surfaces with index zero and one in $\mathbb{R}^3$, see [20, 23, 52, 57, 36]. Similar classification has also been proved in other non-compact flat space forms, see [54]. In higher dimensions, we mention the works [19, 66] on the classification of compact minimal hypersurfaces with index one in $\mathbb{RP}^n$ and $\mathbb{S}^n$, respectively.

Using test functions coming from meromorphic maps and harmonic forms, Ros [57] proved that two sided index one minimal surfaces in 3-manifolds with non-negative Ricci curvature have genus ≤ 3. This result is sharp as the P Schwarz's minimal surface in $\mathbb{R}^3$ projects to a closed minimal surface with genus three and index one in the cubic 3-torus [61]. On the other hand, when the ambient space has positive Ricci curvature, the right estimate is given by the following conjecture:

Conjecture 1 (Schoen [47])**.** *Let M^3 be a closed three manifold with positive Ricci curvature. If Σ is an orientable minimal surface with index one in M^3, then genus$(\Sigma) \leq 2$.*

The interest in this conjecture is in part motivated by its implications for the classification of 3-manifolds. Namely, it is proved in [32] that every closed 3-manifold with positive Ricci curvature contains an index one minimal surface realizing its Heegaard genus. If Conjecture 1 is true, then this Heegaard genus is at most two. Combining this result with the classification of genus two 3-manifolds, one recovers the following classical result of Hamilton:

Theorem 5 (Hamilton [27])**.** *If (M^3, g) is a closed 3-manifold with positive Ricci curvature, then M is diffeomorphic to $\mathbb{S}^3/G$, where G is a finite group of isometries acting freely on $(\mathbb{S}^3, g_0)$.*

Remark 2. With the exception of lens spaces, which has Heegaard genus one, any other spherical space form has Heegaard genus two [46].

The list of 3-manifolds where the Conjecture 1 is verified is small. In the case of spherical space forms, the only examples are the sphere $\mathbb{S}^3$, the projective space $\mathbb{RP}^3$, and the lens spaces $L(3,1)$ and $L(3,2)$ [56, 70]. The conjecture has also been proved on sufficiently pinched convex hypersurfaces in $\mathbb{R}^4$, see [3, Section 5].

Our main result confirms Schoen's Conjecture in the class of spherical space forms with large fundamental group.

Theorem 6. *There exists an integer p_0 so that if Σ is an orientable index one minimal surface embedded in a spherical space form M^3 with $|\pi_1(M^3)| \geq p_0$, then genus$(\Sigma) \leq 2$.*

Remark 3. Theorem 6 also holds for metrics which are perturbations of the round metric on the aforementioned spherical space forms by the compactness theorem of Choi-Schoen [13].

Remark 4. The orientability assumption seems to be necessary in Theorem 6. It is pointed out in [57, 62] that for every integer n, there are lens spaces containing nonorientable area minimizing surfaces with genus greater than n.

The proof of Theorem 6 is inspired by Ritoré and Ros' work on the compactness of the space of index one minimal surfaces in flat three torus, see [55]. Among other results, they proved that the flat three torus with small injectivity radius and unit volume do not contain orientable index one minimal surfaces. Following similar ideas, we show that any rescaled sequence of orientable index one minimal surfaces with genus three in spherical space forms with large fundamental group converges to a totally geodesic surface in a non-compact flat 3-manifold. We contradict this statement by showing that the curvature of such surfaces is large somewhere by an application of the Rolling Theorem for minimal surfaces in $\mathbb{S}^3$, see Chapter 3 Proposition 6.

Chapter 1

Preliminaries

In this chapter, we present the definitions of the objects of interest and state the basic facts which will be used throughout this work. In Section 1, we list the basics related to a Riemannian manifold and its submanifolds. In Section 2, we recall the well known first and second variation formulas for the area functional. In Secton 3, we prove some background result regarding surfaces with constant mean curvature is space forms.

1.1 Geometry of submanifolds

Let (M^n, g) be a Riemannian manifold of dimension n. The Levi-Civita connection associated to the metric g is denoted by $\overline{\nabla}$. The Riemannian curvature tensor, denoted by R, is the tensor defined as:

$$R(X, Y, Z) = \overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]} Z,$$

for every $X, Y, Z \in \mathcal{X}(M)$. Here $\mathcal{X}(M)$ denotes the space of smooth vector fields on M. The sectional curvature of M at a point $x \in M$ in the direction of a 2-dimensional plane $\sigma \subset T_x M$ is given by:

$$K_M(\sigma, x) = g(R(e_1, e_2, e_1), e_2),$$

where $\{e_1, e_2\}$ is an orthonormal basis for σ. The Ricci tensor, denoted by Ric, is the symmetric two tensor defined by:

$$\mathrm{Ric}(X,Y)(x) = \sum_{i=1}^{n} g(R(X,e_i,Y),e_i)(x),$$

where $X, Y \in \mathcal{X}(M)$ and $x \in M$. Here, $\{e_1, \ldots, e_n\}$ is an orthonormal basis for T_xM. Similarly, the scalar curvature of M, denoted by R_g, is the scalar function defined by:

$$R_g(x) = \sum_{i=1}^{n} \mathrm{Ric}(e_i,e_i)(x),$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis for T_xM.

The Levi-Civita connections of a submanifold $\Sigma \subset M$ with the induced Riemannian metric of M is given by:

$$\nabla_X Y = (\overline{\nabla}_X Y)^{\top},$$

where $X, Y \in \mathcal{X}(\Sigma)$. The second fundamental form of Σ, denoted by B, is then defined by:

$$B(X,Y) = (\overline{\nabla}_X Y)^{\perp},$$

where $X, Y \in \mathcal{X}(\Sigma)$. The second fundamental form plays an important role in comparing the intrinsic curvatures of M^n and Σ^k as indicated in the Gauss equation:

Proposition 1 (Gauss Equation)**.** *Given $x \in \Sigma^k$ and σ a 2-dimensional plane in $T_x\Sigma$, then*

$$K_M(\sigma,x) - K_\Sigma(\sigma,x) = \langle B(e_1,e_1), B(e_2,e_2)\rangle - |B(e_1,e_2)|^2,$$

where $\{e_1, e_2\}$ is an orthonormal basis for σ.

Let N be a local unit normal vector field along Σ near a point $x \in \Sigma$. The mapping B is induces on $T_x\Sigma$ the symmetric bilinear form $h : T_x\Sigma \times T_x\Sigma \to \mathbb{R}$ given by $h(X,Y) = \langle B(X,Y), N\rangle$. The bilinear map h associates a self-adjoint linear operator $A_N : \mathcal{X}(\Sigma) \to \mathcal{X}(\Sigma)$ given by $h(X,Y) = \langle A_N(X), Y\rangle$. A formula for this linear map is $A_N(X) = -\overline{\nabla}_X N$. The map A_N is also called the second fundamental form of Σ. When the codimension of Σ in M is one, we omit the subindex and denote the second fundamental form simply by A. The eigenvalues of A are called the principal curvatures of Σ.

The mean curvature vector of $\Sigma \subset M$ at a point x is the normal vector field $\overrightarrow{H}(x)$ defined to be

$$\overrightarrow{H}(x) = \frac{1}{k}\sum_{i=1}^{n} B(e_i, e_i)(x),$$

where k is the dimension of Σ and $e_1, \dots, e_k$ is an orthonormal basis for $T_x\Sigma$. The *mean curvature* H of Σ in the direction of N is given by $H := \langle \overrightarrow{H}_\Sigma, N\rangle$. The geometric significance of the mean curvature vector is justified by the first variation formula for the area functional as we will se below.

1.2 Variational formulae for submanifolds

A smooth variation of a hypersurface Σ^n in M^{n+1} is a smooth map $\varphi : \Sigma \times [0, \varepsilon) \to M^{n+1}$ such that $\varphi(x, 0) = x$. When Σ is non-compact, then we say that φ has compact support if $\varphi(x, t) = x$ outside a compact set. Using the map φ, we construct for every $t \in [0, \varepsilon)$ the hypersurface $\Sigma_t = \varphi(\Sigma, t)$. The vector field $X = \frac{\partial\varphi}{\partial t}(x, t)$ is called the variational vector field. Associated to the variation φ we define the area functional $A(t) = \mathrm{Area}(\Sigma_t)$. The change in the area, up to first order, is given by the following proposition:

First variation of area. If Σ_t is a compactly supported variation of Σ in the direction of X, then

$$\left.\frac{d}{dt}\right|_{t=0} A(t) = -\,n\int_\Sigma \left\langle \overrightarrow{H}, X\right\rangle d_\Sigma. \tag{1.1}$$

It follows from (1.1) that $\Sigma \subset M$ is a critical point of the area functional if, and only if, $\overrightarrow{H} = 0$. A hypersurface Σ is called *minimal* if $\overrightarrow{H} = 0$.

A hypersurface $\Sigma \subset M$ is called two-sided if it has a globally defined unit normal vector field N along Σ. A variation of Σ is called *normal* if $X \perp T\Sigma$. When Σ is two-sided, then the function $f_t = \langle N_t, X_t\rangle$ is well defined; it is called the lapse function.

First variation of the mean curvature. If Σ_t is a normal variation of Σ with lapse function $f \in C^\infty(\Sigma)$, then

$$\left.\frac{d}{dt}\right|_{t=0} H_t = \Delta_\Sigma f + (\mathrm{Ric}_M(N,N) + |A_\Sigma|^2) f. \tag{1.2}$$

The operator $L_\Sigma = \Delta_\Sigma + \mathrm{Ric}_M(N,N) + |A|^2$ is called the *Jacobi operator* of Σ.

Second variation of area. If Σ_t is a normal variation of a minimal surface Σ with lapse function $f \in C^\infty(\Sigma)$, then

$$\left.\frac{d^2}{dt^2}\right|_{t=0} A(t) = -\int_\Sigma f\, L_\Sigma f = \int_\Sigma |\nabla f|^2 - (\mathrm{Ric}_M(N,N) + |A|^2) f^2\, d_\Sigma. \tag{1.3}$$

The second variation of area induces on the minimal surface Σ the quadratic form $I(f,f)$ given by

$$I(f,f) := \left.\frac{d}{dt}\right|_{t=0} A(t).$$

Definition 1. *The Morse index of a closed orientable minimal hypersurface $\Sigma \subset M$ is defined as the maximal dimension of a linear subspace W where $I(\cdot,\cdot)$ is negative definite.*

We say $\lambda \in \mathbb{R}$ is an eigenvalue of the Jacobi operator L if there exists a smooth function $\phi \in C^\infty(\Sigma)$ such that $L\phi + \lambda\phi = 0$. It is an well known fact that the spectrum of L satisfies

$$\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \ldots \to \infty.$$

Moreover, the first eigenspace of L is simple and generated by a positive function.

Lemma 7. *If Σ is a compact orientable minimal hypersurface in M, then the index of Σ is equal the number of negative eigenvalues of L counted with multiplicities.*

Minimal hypersurfaces are a special examples of hypersurfaces with constant mean curvature (known as *cmc hypersurfaces*). Like minimal hypersurfaces, cmc hypersurfaces also enjoy a variational characterization in terms of the area functional. To precise this, we introduce the volume functional associated to a variation $\varphi : \Sigma \times [0, \varepsilon) \to M$. Namely,

$$V(t) = \int_{\Sigma \times [0,t]} \varphi^*(d_M).$$

Geometrically, $V(t)$ measures the enclosed volume between Σ and $\Sigma_t = \varphi(\Sigma, t)$.

First variation of the volume. If Σ_t is a compactly supported variation of Σ, then

$$\left.\frac{d}{dt}\right|_{t=0} V(t) = \int_\Sigma \langle X, N \rangle d_\Sigma.$$

We say φ is a volume preserving variation if $V(t) = 0$ for every t. The existence of volume preserving variation is given below:

Lemma 8. *If $f \in C^\infty(\Sigma)$ satisfies $\int_\Sigma f d_\Sigma = 0$, then there exists a volume preserving variation φ such that $X = fN$.*

It follows from Lemma 8 and from the first variation of area (1.1) that Σ is a cmc hypersurface if, and only if, Σ is a critical point of area functional among volume preserving variations. This is equivalent to require that $A'(0) = 0$ for every variation of Σ such that $\int_\Sigma f d_\Sigma = 0$. Similarly, if we define the functional

$$J(t) = A(t) + nH_0 V(t), \quad \textit{with} \quad H_0 = \frac{1}{|\Sigma|} \int_\Sigma H d_\Sigma,$$

then Σ has constant mean curvature H_0 if, and only if, it is a critical point of J for every smooth variation.

1.3 Alexandrov and Hopf Theorems

Theorem 9 (Alexandrov)**.** *A closed hypersurface with constant mean curvature embedded in $\mathbb{R}^{n+1}$ is a round sphere.*

The proof of Theorem 9 below is from Montiel and Ros [42].

Proof. For simplicity let us restrict to the case $n + 1 = 3$, the proof in the general case is essentially the same. The proof relies on the following lemma:

Lemma 10. *If $\Sigma = \partial\Omega$ is a closed surface embedded in $\mathbb{R}^3$ such that $H > 0$ (not necessarily constant), then*

$$3\,|\Omega| \leq \int_\Sigma \frac{1}{H}\, d_\Sigma.$$

Moreover, the equality holds if and only if Ω is a geodesic ball.

Proof. Let N be the unit normal vector of Σ pointing inwards, define the set $A_c = \{x + tN(x) : x \in \Sigma, \quad 0 \leq t \leq c\}$ where c is a smooth positive function on Σ. Given a point $y \in \Omega$, pick $x \in \Sigma$ such that $d(y, \Sigma) = |y - x|$. Let $\alpha(s)$ be a curve on Σ such that $\alpha(0) = x$ and consider the function $g(s) = |\alpha(s) - y|^2$. One can easily check that $g'(s) = 2\langle \alpha'(s), \alpha(s) - y\rangle$ and that $y - x = tN(x)$ for some constant t. Since $g'(0) = 0$. In the same way, we check that $g''(s) = 2\langle \alpha''(s), \alpha(s) - y\rangle + 2|\alpha'(s)|^2$ and since x is the closest point to y, we have that

$$0 \leq g''(0) = -2\langle \alpha'(0), A_N(\alpha'(0))\rangle\langle N(x), y - x\rangle + 2|\alpha'(0)|^2.$$

Let λ_1 and λ_2 be the principal curvatures of Σ. Choosing $\alpha'(0)$ to be the principal direction associated to the principal curvature λ_2, we observe that $0 \leq -2\lambda_2 t + 2$ and this implies that $t \leq \frac{1}{\lambda_2(x)} \leq \frac{1}{H(x)}$. Let $F : \Sigma \times [0, a] \to \mathbb{R}^3$ be the map defined by $F(x, t) = x + tN(x)$. The previous computations show that $\Omega \subset A_{\frac{1}{\lambda_2}}$. The Jacobian of F satisfies $|Jac(F)(x)| = |(1 - \lambda_1(x)t)(1 - \lambda_2(x)t)| \leq (1 + Ht)^2$. The equality occur if, and only if, x is an umbilical point of Σ. By standard integration properties, we have

$$vol(\Omega) \leq vol(A_{\frac{1}{H}}) \leq \int_\Sigma \int_0^{\frac{1}{H}} |Jac(F)| dt d_\Sigma \leq \int_\Sigma \int_0^{\frac{1}{H}} (1 + tH)^2 dt d_\Sigma.$$

It follows that $3|\Omega| \leq \int_\Sigma \frac{1}{H}\, d_\Sigma$. Moreover, the equality occurs if, and only if, Σ is totally umbilical. $\square$

Using Lemma 10, we complete the proof of the theorem. Let us consider in $\mathbb{R}^3$ the vector field $X = (x, y, z)$ (the position vector). A simple computation gives that $\mathrm{div}_{\mathbb{R}^3} X = 3$ and $div_\Sigma X = 2$. If we compute the variation of Σ in the direction of X, then $J'(0) = A'(0) + 2HV'(0) = 0$, where $A'(0) = \int_\Sigma div_\Sigma = 2|\Sigma|$ and $V'(0) = \int_\Sigma \langle X, N\rangle = -\int_\Omega div_{\mathbb{R}^3} = -3vol(\Omega)$. Hence, $3|\Omega| = \frac{1}{H}|\Sigma|$ and Σ is totally umbilical by Lemma 10. It is a classical result from geometry that this implies that Σ is a round sphere. $\square$

We point out that Lemma 10 holds true in much more general situations:

Theorem 11 (A. Ros [60]). *Let $\Omega \subset M^{n+1}$ be a compact domain with smooth boundary Σ^n. If $Ric_M \geq 0$ and the mean curvature H of Σ is positive,then*

$$(n+1)vol_g(\Omega) \leq \int_\Sigma \frac{1}{H}\, d_M.$$

The equality holds if, and only if, Ω is isometric to a round ball in $\mathbb{R}^{n+1}$.

The remaining of this section is devoted to immersed cmc surfaces inside a 3-dimensional space form $M^3(c)$. Let $\phi : \Sigma \to M^3(c)$ be an isometric immersion and $z = x + yi$ be local conformal coordinates on Σ. In this coordinate system, we define the quadratic differential $A^{2,0}$ as follows

$$A^{2,0} = 4\, A(\partial_z, \partial_z)\, dz^2 \;=\; \Big(A(\partial_x, \partial_x) - A(\partial_y, \partial_y) - 2A(\partial_x, \partial_y) i \Big)\, dz^2.$$

One can check that $A^{2,0}$ is globally well defined over Σ. The importance of this quadratic differential relies on the following fact:

Proposition 2. *$A^{2,0}$ is holomorphic if, and only if, H is constant.*

Proof. By definition a function $\Phi(z)$ is holomorphic if, and only if, $\partial_{\bar z}\Phi(z) = 0$. Recall that $2\,\partial_{\bar z} = \partial_x + \partial_y\, i$. Here we consider Φ to be

$$\Phi(z) = A(\partial_x, \partial_x) - A(\partial_y, \partial_y) - 2A(\partial_x, \partial_y)\, i.$$

A direct computation gives

$$\begin{aligned}
\partial_{\bar z}\Phi(z) &= \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\, i\right)\Big(A(\partial_x, \partial_x) - A(\partial_y, \partial_y) - 2A(\partial_x, \partial_y)\, i \Big) \\
&= \frac{1}{2}\frac{\partial}{\partial x} A(\partial_x, \partial_x) - \frac{1}{2}\frac{\partial}{\partial x} A(\partial_y, \partial_y) + \frac{\partial}{\partial y} A(\partial_x, \partial_y) \\
&+ \left(\frac{1}{2}\frac{\partial}{\partial y} A(\partial_x, \partial_x) - \frac{1}{2}\frac{\partial}{\partial y} A(\partial_y, \partial_y) - \frac{\partial}{\partial x} A(\partial_x, \partial_y)\right) i.
\end{aligned}$$

We will use the following notation $E = \langle \partial_x, \partial_x \rangle$, $F = \langle \partial_x, \partial_y \rangle$ and $G = \langle \partial_y, \partial_y \rangle$. Because we are using conformal coordinates, we have that $E = G$ and $F = 0$. In the same way, we denote $e = A(\partial_x, \partial_x)$, $f = A(\partial_x, \partial_y)$ and $g = A(\partial_y, \partial_y)$. Using these notations the above expression takes the form

$$\partial_{\bar{z}} \Phi(z) = \Big(\frac{1}{2} e_x - \frac{1}{2} g_x + f_y \Big) + \Big(\frac{1}{2} e_y - \frac{1}{2} g_y - f_x \Big)\, i. \tag{1.4}$$

We use the *Codazzi equation* to expand (1.4). Recall first that *Codazzi equation* in this setting takes the following form:

$$\nabla_X A(Y) - \nabla_Y A(X) = A([X,Y]) \Longrightarrow \nabla_{\partial_x} A(\partial_y) = \nabla_{\partial_y} A(\partial_x).$$

Using that $A(\partial_x, \partial_y) = \langle A(\partial_x), \partial_y \rangle$, we obtain that

$$\partial_y e = \partial_y \langle A(\partial_x), \partial_x \rangle = \langle \nabla_{\partial_y} A(\partial_x), \partial_x \rangle + \langle A(\partial_x), \nabla_{\partial_y} \partial_x \rangle.$$

$$\partial_x f = \partial_x \langle A(\partial_x), \partial_y \rangle = \langle \nabla_{\partial_x} A(\partial_y), \partial_x \rangle + \langle A(\partial_y), \nabla_{\partial_x} \partial_x \rangle.$$

Subtracting these two equations and applying the *Codazzi equation* we arrive at

$$e_y - f_x = \langle A(\partial_x), \nabla_{\partial_y} \partial_x \rangle - \langle A(\partial_y), \nabla_{\partial_x} \partial_x \rangle.$$

Now expressing $\nabla_{\partial_y} \partial_x$ in the base $\{\partial_x, \partial_y\}$ we get

$$\begin{aligned} e_y - f_x = {} & \frac{1}{E} \langle \nabla_{\partial_y} \partial_x, \partial_x \rangle \, e + \frac{1}{E} \langle \nabla_{\partial_y} \partial_x, \partial_y \rangle \, f \\ & - \frac{1}{E} \langle \nabla_{\partial_x} \partial_x, \partial_x \rangle \, f - \frac{1}{E} \langle \nabla_{\partial_x} \partial_x, \partial_y \rangle \, g. \end{aligned}$$

Recalling that $H = \frac{1}{2E}\Big(e + g\Big)$, we obtain

$$e_y - f_x = \frac{1}{2E} E_y \Big(e + g \Big) = E_y \, H. \tag{1.5}$$

The same argumentation gives

$$f_y - g_x = -E_x \, H. \tag{1.6}$$

On the other hand, a derivation of the equation $E\, H = \frac{e+g}{2}$ gives

$$\frac{e_x + g_x}{2} = E_x H + E H_x \quad \textit{and} \quad \frac{e_y + g_y}{2} = E_y H + E H_y. \tag{1.7}$$

Therefore, (1.5), (1.6), and (1.7) combined implies that

$$EH_x = \frac{e_x - g_x}{2} + f_y \quad and \quad EH_y = \frac{g_y - e_y}{2} + f_x.$$

Finally, substituting this information in (1.4) we obtain that

$$\partial_{\bar{z}}\Phi(z) = EH_x - EH_y\, i = 2\, E\, \partial_z\, H.$$

This completes the prof of the proposition. □

Corollary 1 (Hopf). *If $\phi : \mathbb{S}^2 \to M^3(c)$ is a immersion with constant mean curvature H, then $\phi(\Sigma)$ is a totally umbilical and, hence, a geodesic sphere.*

Lemma 12. *Let Σ be a closed orientable surface and $\phi : \Sigma \to M^3(c)$ an immersion with constant mean curvature in a space form $M^3(c)$. If $g(\Sigma) > 0$ then the number of umbilical points, counted with multiplicities is $4g - 4$.*

1.4 Stable cmc tori in space forms

As we saw in previous sections, a cmc hypersurface is a critical point of the area functional among volume preserving variations. In this section, we study stable critical points, namely minimize area up to second order.

Let Σ^n be a two sided cmc hypersurface in a given manifold M^{n+1}. If Σ_t is a smooth variation of Σ, then

$$A'(t) = -\int_{\Sigma_t} nH_t f_t d_{\Sigma_t} \quad and \quad V'(t) = \int_{\Sigma} f_t d_{\Sigma_t} + \int_0^t \int_{\Sigma} \partial_t f_t d_{\Sigma_t} + \int_0^t \int_{\Sigma} f_t \partial_t d_{\Sigma_t},$$

where $f_t = \langle X_t, N_t \rangle$ and X_t is the variational vector field. The precise computations are given in the Appendix. Computing the second derivative, we obtain

$$A''(0) = -\int_{\Sigma} n \frac{d}{dt}|_{t=0} H_t f d_{\Sigma} - nH \int_{\Sigma} \partial_t f_t d_{\Sigma} - nH \int_{\Sigma} f \partial_t d_{\Sigma_t}.$$

Similarly, we check that $V''(0) = 2\int_{\Sigma} \partial_t f_t d_{\Sigma} + 2\int_{\Sigma} f_t \partial_t d_{\Sigma_t}$. If Σ_t is volume preserving, then $V(t) \equiv 0$. Therefore,

$$\left.\frac{d^2}{dt^2}\right|_{t=0} |\Sigma_t| = \int_{\Sigma} |\nabla f|^2 - \Big(Ric_M(N,N) + |A|^2\Big) f^2 d_{\Sigma} = -\int_{\Sigma} f L_{\Sigma} f d_{\Sigma}.$$

A hypersurface Σ with constant mean curvature is called *stable cmc* if

$$\frac{d^2}{dt^2}|\Sigma_t|\bigg|_{t=0} \geq 0$$

for any volume preserving variation Σ_t. By Lemma 8, this is equivalent to have

$$I(f,f) := -\int_\Sigma f L_\Sigma f d_\Sigma = \int_\Sigma |\nabla f|^2 - \Big(Ric_M(N,N) + |A|^2\Big) f^2 d_\Sigma \geq 0. \quad (1.8)$$

for any $f \in C^\infty(\Sigma)$ with $\int_\Sigma f d_\Sigma = 0$.

The study of stable cmc hypersurfaces was initiated in [4, 5] with a new characterization of the round spheres as the only immersed stable cmc hypersurfaces in the space forms $\mathbb{H}^n$, $\mathbb{R}^n$, and $\mathbb{S}^n$.

Theorem 13 (Ros-Ritore [56])**.** *Let Σ^2 be either a closed stable cmc surface or a closed orientable index one minimal surface immersed in (M^3, g). If $Ric_M > 0$, then $g(\Sigma) \leq 3$.*

Proof. By Riemann Surface Theory, there exists a meromorphic map $\phi : \Sigma \to \mathbb{S}^2$ with degree

$$deg(\phi) \leq 1 + \left[\frac{g+1}{2}\right].$$

Lemma 14. *Let w be a smooth positive function on Σ. There exists a conformal diffeomorphism $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$ such that*

$$\int_\Sigma w\, \varphi \circ \phi \, d_\Sigma = 0.$$

Proof. Let $\Pi_a : \mathbb{S}^2 \to \mathbb{R}^2$ be the stereographic projection with respect the pole $a \in \mathbb{S}^2$. For every $t \in \mathbb{R}$, we consider the map $\gamma_t^a : \mathbb{S}^2 \to \mathbb{S}^2$ defined by $\gamma_t^a(x) = \Pi_a^{-1}\left(e^t \Pi_a(x)\right)$. For a fixed $x \in \mathbb{S}^2$ we see $\gamma_t^a(x)$ as a curve on $\mathbb{S}^2$ which at $t = 0$ pass through x with velocity $a - \langle x, a\rangle x$. Hence, $\gamma_t^a(x)$ is the tangent flow of the vector field $\overline{a}(x) = a - \langle x, a\rangle x$ on $\mathbb{S}^2$. Thus, for every t γ_t^a represents a conformal diffeormorphism of $\mathbb{S}^2$. We claim that at least one map γ_t^a satisfies the property of the lemma. Arguing by contradiction, we assume that $\int_\Sigma w(x)\gamma_t^a \circ \phi(x) d_\Sigma \neq 0$ for every t and every $a \in \mathbb{S}^2$. Let us consider the following map

$$F : [0,\infty) \times \mathbb{S}^2 \to \mathbb{S}^2 \quad \textit{defined by} \quad F(t,a) = \frac{\int_\Sigma w(x)\gamma_t^a \circ \phi(x) d_\Sigma}{\| \int_\Sigma w(x)\gamma_t^a \circ \phi(x) d_\Sigma \|}.$$

The map F satisfies the following properties:

$$F(0,a) = \frac{\int_\Sigma w(x)\phi(x)\,d_\Sigma}{\|\int_\Sigma w(x)\phi(x)\,d_\Sigma\|} = \text{const} \quad \textit{and} \quad F(+\infty,a) = a.$$

In other words, F induces a homotopy between the identity map and the constant map. This is a contradiction since $\mathbb{S}^2$ is non-contractible. $\square$

From now on we work with the map $\gamma_t^a \circ \phi$ given in Lemma 14, note that $\gamma_t^a \circ \phi$ and ϕ have the same degree. We abuse notation and denote it still by ϕ. If Σ is a stable cmc, we choose $w = 1$ in Lemma 14. If Σ is an index one minimal surface, we choose $w = u_1$, the first eigenfuction of the Jacobi operator L. Thus, by Lemma 14, the coordinate functions of ϕ are valid test functions for the stability inequality (2.1). Hence,

$$\int_\Sigma \left(Ric(N,N) + |A|^2\right)\sum_{i=1}^{3}\phi_i^2 d_\Sigma \le \int_\Sigma \sum_{i=1}^{3}|\nabla\phi_i|^2 d_\Sigma.$$

Now we observe that $\sum_{i=1}^{3}\phi_i^2 = 1$ since $\phi(\Sigma) = \mathbb{S}^2$. Since ϕ is conformal, $\langle d\phi(v), d\phi(u)\rangle = \lambda\langle v,u\rangle$ for some non-negative function λ.. One can check that $\sum_{i=1}^{3}|\nabla\phi_i|^2 = 2\lambda$ and $Jac(\phi) = \lambda$. Hence,

$$\int_\Sigma \sum_{i=1}^{3}|\nabla\phi_i|^2 d_\Sigma = 2\int_\Sigma Jac(\phi)d_\Sigma = 2\int_\Sigma \phi^*\left(d_{\mathbb{S}^2}\right) = 2\,deg(\phi)\int_{\mathbb{S}^2} d_{\mathbb{S}^2} = 8\pi\,deg(\phi).$$

Let $\{e_1, e_2\}$ be a orthonormal base for $T\Sigma$. By the *Gauss equation* 1, we can write

$$Ric(N,N) + |A|^2 = Ric(e_1) + Ric(e_2) - 2K_\Sigma + 4H^2,$$

where K_Σ denotes the *Gaussian curvature* of Σ. Hence,

$$\int_\Sigma \Big(\mathrm{Ric}(e_1) + \mathrm{Ric}(e_2) - 2K_\Sigma + 4H^2\Big)\,d_\Sigma \le 8\pi\,deg(\phi).$$

The Gauss-Bonnet theorem then implies that

$$\left(\frac{1}{2}\inf_\Sigma Ric_M + H^2\right)|\Sigma| \le 2\pi\left(2 - g + \left[\frac{g+1}{2}\right]\right). \tag{1.9}$$

Since the left hand side is strictly positive, we conclude that $g(\Sigma) \le 3$. $\square$

Corollary 2. *Under the same assumptions of Theorem 13, if $g = 2$ or $g = 3$, then*
$$\Big(\frac{1}{2}\inf_{\Sigma} Ric_M + H^2\Big)|\Sigma| \leq 2\pi.$$

By the Hopf Theorem, stable cmc surfaces with genus zero in space forms are totally umbilical. For the remaining of the section, we focus on the geometry of stable cmc tori in space forms. The main result, due to Ritore and Ros [56], asserts that such surfaces are flat.

Let us start with stable cmc immersions $\phi : \Sigma \to M^3(c)$, where $M^3(c)$ is a space form with curvature c and satisfying $c + H^2 > 0$. In this case, we set $b^2 = 4(c + H^2)$. Let ds_0^2 be a metric on Σ defined by

$$ds_0^2 = \, b\,|\Phi(z)|\,|dz^2|,$$

where $\Phi(z)$ is the map discussed in Proposition 2. Let P be the set of umbilical points of the immersion ϕ. The metric ds_0^2 is well defined in $\Sigma - P$, Note that ds^2 is conformal to ds_0^2. Hence, there exists a smooth function $u \in C^\infty(\Sigma - P)$ such that
$$ds^2 = \frac{e^{2u}}{4(c+H^2)} ds_0^2.$$
It is a standart computation to check that

$$\widetilde{g} = e^{2u}\, g \Longrightarrow\; e^{2u}\widetilde{K} = K - \Delta_g u,$$

where $\widetilde{K}$ and K are the Gaussian curvatures of the metrics $\widetilde{g}$ and g respectively. In our setting we have
$$\frac{e^{2u}}{4(c+H^2)} K = K_0 - \frac{1}{2}\Delta_0 \ln\Big(\frac{e^{2u}}{4(c+H^2)}\Big).$$

Lemma 15. *ds_0^2 is a flat metric.*

Proof. Recall that, in our local conformal coordinates, $ds_0^2 = b\,|\Phi(z)||dz^2|$, where $|dz^2| = dx^2 + dy^2$ is the canonical flat metric on $\mathbb{R}^2$. Therefore,

$$b\,|\Phi| K_0 = -\frac{1}{2}\Delta \ln\, b\,|\Phi| \quad \textit{where} \quad \Delta = \partial_x^2 + \partial_y^2.$$

We proved in Propostion 2 that $\Phi(z) = f + g\,i$ is holomorphic. Hence, f and g are harmonic functions. The lemma will follows from the claim below:
Claim:

$$\Delta \ln\Big(f^2 + g^2\Big) = 0$$

To see this we use the following information $\Delta f = \Delta g = 0$ and the Cauchy-Riemann equations: $|\nabla f|^2 = |\nabla g|^2$ and $\langle \nabla f, \nabla g\rangle = 0$. Since it is standard computation we will omit it. The claim implies $K_0 = 0$ and ds_0^2 is flat. $\square$

The function u from $ds^2 = \frac{e^{2u}}{b^2} ds_0^2$ satisfies an elliptic differential equation:

Proposition 3. *(Sinh-Gordon equation)*

$$\Delta_0 u + \sinh(u)\cosh(u) = 0. \tag{1.10}$$

Proof. The first information we have is that

$$\frac{e^{2u}}{4(c+H^2)} K = -\Delta_0 u. \tag{1.11}$$

The next step is to express K in terms of the function u. To do so we use the following two facts

$$\frac{|\Phi(z)|^2}{E^2} = 4\Big(H^2 - \det(A)\Big) \quad \textit{and} \quad c = K - \det(A).$$

The first identity follows from the definition of Φ and the second identity is just the Gauss equation. Hence,

$$K = c + H^2 - \frac{|\Phi(z)|^2}{4E^2}$$

On the other hand, we have that

$$E|dz^2| = ds^2 = \frac{e^{2u}}{4(c+H^2)} ds_0^2 = \frac{e^{2u}}{4(c+H^2)} 2\sqrt{c+H^2}|\Phi(z)||dz^2|$$

Therefore,

$$\frac{|\Phi(z)|}{E} = \frac{2\sqrt{c+H^2}}{e^{2u}}.$$

The Gaussian curvature can now be expressed as

$$K = (c+H^2)\Big(1 - \frac{1}{e^{4u}}\Big). \tag{1.12}$$

The equations (1.11) and (1.12) imply

$$\Delta_0 u + \frac{1}{2}\frac{e^{2u}-e^{-2u}}{2} = 0,$$

which is precisely equation (1.10). □

It follows from the proof that the Gaussian curvature K of Σ is given by

$$K = \frac{b^2}{4}\Big(1 - e^{-4\,u}\Big).$$

In particular, K and u have the same sign. Moreover, at a umbilical point we have that $u = +\infty$ and $K = c + H^2$.

Let us now use the conformal change of the metric to write the stability inequality (2.1) in terms of the metric ds_0^2. Firstly, the volume element d_Σ and the gradients in terms of the new metric is given by

$$d_\Sigma = \frac{e^{2u}}{b^2} dA_0 \quad \textit{and} \quad \nabla f = \frac{b^2}{e^{2u}}\nabla^0 f.$$

In particular,

$$|\nabla f|^2 = \frac{b^2}{e^{2u}}|\nabla^0 f|^2.$$

The Gauss equation implies $Ric(N,N) + |A|^2 = 2c + |A|^2 = b^2 - 2K$. Hence,

$$\begin{aligned} I(f,f) &= \int_\Sigma |\nabla f|^2 - \Big(b^2 - 2K\Big) f^2\, d_\Sigma \\ &= \int_\Sigma \left(\frac{b^2}{e^{2u}}|\nabla^0 f|^2 - \Big(b^2-2K\Big)f^2\right)\frac{e^{2u}}{b^2} dA_0. \end{aligned}$$

Using the expression for K we obtain

$$\frac{e^{2u}}{b^2}\Big(b^2 - 2K\Big) = \frac{e^{2u}+e^{-2u}}{2} = \cosh(2u) = \cosh^2(u) + \sinh^2(u)$$

Therefore, the index form $I(f,f)$ takes the form

$$I(f,f) = \int_\Sigma |\nabla_0 f|^2 - \Big(\cosh^2(u) + \sinh^2(u)\Big) f^2\, dA_0$$

for every $f \in C_0^\infty(\Sigma/P)$ satisfying $\int_\Sigma f\, \frac{e^{2u}}{b^2}\, dA_0 = 0$.

Theorem 16 (Ros-Ritore [56]). *Let $\phi : \Sigma \to M^3(c)$ be a stable cmc immersion or an orientable index one minimal surface such that $c + H^2 > 0$. Then $\{p \in \Sigma : K(p) < 0\}$ is connected and each component of $\{p \in \Sigma : K(p) > 0\}$ contains an umbilical point.*

Proof. We may assume that $g(\Sigma) \geq 1$ and that Σ is not flat. Let $\Omega = \{p \in \Sigma : K(p) \neq 0\}$ and let Ω_1 be the component of Ω without umbilic points. It follows from the assumptions and from the Gauss-Bonnet Theorem that $\Omega_1 \neq \emptyset$. This allows us to define

$$f = \sinh(u) \quad on \quad \Omega^1 \quad and \quad f \equiv 0 \quad on \quad \Sigma - \Omega^1$$

The boundary of Ω_1 is contained in the set where the Gaussian curvature vanishes, hence where u vanishes. This implies that f is in the Sobolev space $\mathcal{H}^1(\Sigma)$. Hence,

$$\begin{aligned}
I(f,f) &= \int_{\Omega^1} |\nabla_0 \sinh(u)|^2 - \Big(\cosh^2(u) + \sinh^2(u)\Big) \sinh^2(u)\, dA_0 \\
&= \int_{\Omega^1} \Big\langle \nabla_0 f, \nabla_0 \sinh(u) \Big\rangle_0 - \Big(\cosh^2(u) + \sinh^2(u)\Big) \sinh^2(u)\, dA_0 \\
&= \int_{\Omega^1} -\sinh(u)\Delta_0 \sinh(u) - \Big(\cosh^2(u) + \sinh^2(u)\Big) \sinh^2(u)\, dA_0 \\
&= -\int_{\Omega^1} \sinh(u)\Big(\sinh(u)|\nabla_0 u|^2 + \cosh(u)\Delta_0 u\Big)\, dA_0 \\
&\qquad\qquad - \int_{\Omega^1} \Big(\cosh^2(u) + \sinh^2(u)\Big) \sinh^2(u)\, dA_0 \\
&= -\int_{\Omega^1} \sinh^2(u)|\nabla_0 u|^2\, dA_0 - \int_{\Omega^1} \sinh^4(u)\, dA_0 \; < \; 0.
\end{aligned}$$

The last equality follows from the Sinh-Gordon equation (1.10). Therefore, we cannot find two disconnected regions without umbilical points otherwise we could change f, by multiplying it by constants in each connected component, so that f has mean zero on Σ. □

Corollary 3. *Let $\phi : \Sigma \to M^3(c)$ be a immersion with constant mean curvature. If $g(\Sigma) = 1$ then Σ is flat.*

Proof. Since $g(\Sigma) = 1$, mo umbilical points exist on Σ. If $b^2 = 4(c + H^2) > 0$ and K is not identically zero, then we can construct f such that $I(f,f) < 0$ as shown in the proof above, contradiction. If $c + H^2 \leq 0$, then the Gauss equation implies that $K = 2(H^2 + c) - (c + \frac{1}{2}|A|^2) \leq 0$. Therefore, by the Gauss-Bonnet Theorem, $K \equiv 0$. □

Example 1 (Clifford Torus)**.** For each $r \in (0, \frac{\pi}{2})$ we define the Clifford Torus T_r as:

$$T_r = \mathbb{S}^1(\cos(r)) \times \mathbb{S}^1(\sin(r)) \subset \mathbb{S}^3. \tag{1.13}$$

Let us study the intrinsic and extrinsic geometry of T_r as a submanifold of $\mathbb{S}^3$. A parametrization for T_r is given by

$$X(\theta, \varphi) = \Big(\cos(r)\cos(\theta), \cos(r)\sin(\theta), \sin(r)\cos(\varphi), \sin(r)\sin(\varphi)\Big).$$

The correspondent coordinate base $\{X_\theta, X_\varphi\}$ is computed below:

$$X_\theta = \cos(r)\Big(-\sin(\theta), \cos(\theta), 0, 0\Big) \text{ and } X_\varphi = \sin(r)\Big(0, 0, -\sin(\varphi), \cos(\varphi)\Big).$$

One can now check that $E = \cos^2(r)$, $F = 0$ and $G = \sin^2(r)$. Next, we compute the unit normal vector

$$N = \Big(\sin(r)\cos(\theta), \sin(r)\sin(\theta), -\cos(r)\cos(\varphi), -\cos(r)\sin(\varphi)\Big).$$

A simple computation gives:

$$\begin{aligned} A_N(X_\theta) &= -N_\theta = \sin(r)\Big(\sin(\theta), -\cos(\theta), 0, 0\Big) = -\frac{\sin(r)}{\cos(r)} X_\theta \\ A_N(X_\varphi) &= -N_\varphi = \cos(r)\Big(0, 0, -\sin(\varphi), \cos(\varphi)\Big) = \frac{\cos(r)}{\sin(r)} X_\varphi. \end{aligned}$$

It follows immediately from above formulas that the principal curvatures of T_r are $\lambda_2 = \frac{\cos(r)}{\sin(r)}$ and $\lambda_1 = -\frac{\sin(r)}{\cos(r)}$. The mean curvature and norm do the second fundamental form satisfy:

$$H = \frac{\cos(r)}{\sin(r)} - \frac{\sin(r)}{\cos(r)} \textit{ and } |A|^2 = \frac{4 - 2\sin^2(2r)}{\sin^2(2r)}.$$

The Gauss equation implies that the Gauss curvature of T_r is identically zero.

Proposition 4. *Every flat tori with constant mean curvature in $\mathbb{S}^3$ is congruent to a Clifford torus T_r.*

Proof. This follows from the Rigidity theorem, pg. 49 in [10]. Indeed, if $\Sigma \subset \mathbb{S}^3$ is flat and with constant mean curvature H, then its principal curvatures are constant. Hence, the Clifford torus T_r with mean curvature H has the same second fundamental form as Σ. $\square$

Chapter 2

The isoperimetric problem for lens spaces

In this chapter we present the first contribution of this thesis. We solve the isoperimetric problem in the Lens spaces with large fundamental group. Namely, we prove that the isoperimetric surfaces are geodesic spheres or tubes about geodesics. The isoperimetric problem in other spherical space forms will be discussed in Chapter 3.

2.1 The isoperimetric problem

Let (M^{n+1}, g) be an orientable Riemannian manifold of dimension $n+1$. The $n+1$-dimensional Hausdorff measure of a region $\Omega \subset M$ is denoted by $|\Omega|$. Similarly, we denote the n-dimensional Hausdorff measure of the hypersurface $\partial\Omega \subset M$ by $|\partial\Omega|$. The class of regions considered here are those of finite perimeter, see [66].

A region $\Omega \subset M$ is called an *Isoperimetric region* if

$$|\partial\Omega| = \inf\{|\partial\Omega'| : \Omega' \subset M \quad and \quad |\Omega'| = |\Omega|\}.$$

In this case, the hypersurface $\Sigma = \partial\Omega$ is called an *Isoperimetric hypersurface.*

The existence of isoperimetric hypersurfaces is, in general, handled by a compactness theorem from geometric measure theory. For non-compact manifolds one needs to be careful since a minimizing sequence of regions of fixed volume may drift off to infinity. We recommend [43] for a recent reference on the regularity of isoperimetric hypersurfaces:

Theorem 17. *Let (M^{n+1}, g) be a closed Riemannian manifold. For every $0 < t < vol(M)$ there exists an isoperimetric region Ω satisfying $|\Omega| = t$. Moreover, $\Sigma = \partial\Omega$ is smooth up to a closed set of Hausdorff dimension $n-7$.*

Stability

Isoperimetric hypersurfaces are stable critical points of the area functional for variations that preserve the enclosed volume; thus, the regular part of isoperimetric hypersurfaces has constant mean curvature. More generally, we say a two-sided isometric immersion $\phi : \Sigma^n \to M^{n+1}$, i.e., it has a globally defined unit normal vector field, has *constant mean curvature* (cmc) if it is a critical point of the area functional for volume preserving variations. Critical points are called *stable cmc* if the second derivative of the area is non-negative for such variations.

Equivalently, ϕ is stable if for every $f \in C^\infty(\Sigma)$ with compact support such that $\int_\Sigma f\, dvol_\Sigma = 0$, we have

$$I(f,f) = -\int_\Sigma f\, L\, f\, d_\Sigma := \int_\Sigma |\nabla f|^2 - (Ric(N,N) + |A|^2)\, f^2\, d_\Sigma \geq 0. \quad (2.1)$$

N is the unit normal vector of Σ and A is the *second fundamental form* of the immersion ϕ. The *mean curvature* of Σ, denoted by H, is defined by $2\,H = trace(A)$.

The study of immersed stable cmc hypersurfaces started in [4] and [5] with a new characterization of the geodesic spheres in the simply connected space forms $\mathbb{R}^{n+1}$, $\mathbb{S}^{n+1}$ and $\mathbb{H}^{n+1}$. The classification of stable cmc surfaces is often a way to approach the isoperimetric problem in reasonable spaces. With this purpose in mind, A. Ros and M. Ritoré in [56] used the Hersch-Yau trick to study orientable stable cmc surfaces on 3-manifolds with positive Ricci curvature.

Theorem 18 (Ros-Ritoré [56]). *Let (M, g) be a closed three manifold with positive Ricci curvature. If $\phi : \Sigma \to M^3$ is a stable cmc immersion, then $g(\Sigma) \leq 3$. Moreover, if $g(\Sigma) = 2$ or 3, then*

$$\Big(\frac{1}{2}\inf_{\Sigma} Ric_M + H^2\Big)|\Sigma| \leq 2\pi.$$

We use the recent resolution of the Willmore conjecture by Marques and Neves to improve the main result in [56]:

Proposition 5. *Let $\phi : \Sigma \to M^3$ be a stable immersion with constant mean curvature H into an elliptic space form $M = \mathbb{S}^3/G$. Then*

1. *If $g(\Sigma) = 2$ or 3, then $\Big(1 + H^2\Big)|\Sigma| \leq 2\pi$.*
2. *If $g(\Sigma) = 2$ or 3 and $|G| \leq 4$, then ϕ is an embedding. Moreover, if $|G| \leq 6$, then the pullback of Σ, through the covering map $\Pi : \mathbb{S}^3 \to M^3$, is connected.*
3. *If $|G| = 2$ or 3, then $g(\Sigma) = 0$ or 1.*

Proof. The first statement follows from the previous theorem since $Ric_M = 2$. Let $\phi_* : \pi_1(\Sigma) \to \pi_1(M)$ be the induced map in fundamental groups. As $K = Ker(\phi_*)$ has finite index there exists a finite covering $\psi : \widetilde{\Sigma} \to \Sigma$ such that $Im\Big(\psi_*\Big) = K$ and $\Big(\phi \circ \psi\Big)_* = 0$. This means there exists a lifting of this map into $\mathbb{S}^3$ and we denote it by $\widetilde{\phi} : \widetilde{\Sigma} \to \mathbb{S}^3$. It follows that $\Big(1 + H^2\Big)|\widetilde{\Sigma}| \leq |G|\, 2\pi$. If ϕ is not an embedding, then $\widetilde{\phi}$ is not embedding either. By the work of Li and Yau [35] the Willmore energy of the immersed surface $\widetilde{\Sigma}$, i.e. $\mathcal{W}(\widetilde{\Sigma}) = \int_{\widetilde{\Sigma}} \Big(1 + H^2\Big) dvol_{\Sigma}$, is strictly greater[1] than 8π. Therefore, if $|G| \leq 4$, we obtain a contradiction and $\widetilde{\phi}$ is an embedding. Moreover, for closed surfaces with genus greater than or equal to 1 in $\mathbb{S}^3$ the Willmore conjecture, recently proved in [38], states that $\mathcal{W}(\Sigma) \geq 2\pi^2$. Let $\cup_{i=1}^{l} \widetilde{\Sigma}_i$ be the pre-image of Σ by the universal covering map, then

$$2\, l\, \pi^2 \leq \sum_{i=1}^{l} \mathcal{W}\Big(\widetilde{\Sigma}_i\Big) = |G|\mathcal{W}(\Sigma) \leq |G| 2\pi \Rightarrow \frac{|G|}{l} \geq \pi$$

[1] The case $\mathcal{W}(\Sigma) = 8\pi$ is discussed in [55].

Therefore, if $|G| \leq 6$, then $l = 1$ and $|G| \geq \pi$. In particular, if $|G| = 2$ or 3, then there exist no stable cmc surface Σ with $g(\Sigma) \geq 2$. □

Corollary 4. *The stable cmc surfaces in $L(3,1)$ and $L(3,2)$ are totally umbilical spheres or flat tori. In addition, the index one minimal surfaces in $L(3,1)$ and $L(3,2)$ are congruent to the projection of minimal Clifford torus.*

Proof. Let $\Sigma \subset L(3,q)$, $q = 1, 2$, be in the conditions of the corollary. By Proposition 5, $g(\Sigma) = 0$ or 1. If $g(\Sigma) = 0$, then it follows from the Hopf holomorphic quadratic differential that Σ is totally umbilical. If $g(\Sigma) = 1$, then by Corollary 3 Σ is flat, the result now follows from Proposition 4. □

2.1.1 The isoperimetric profile

The isoperimetric properties of M can be encapsulated in a single function called the *isoperimetric profile*. This is the function $I_M : [0, \mathrm{vol}(M)] \to [0, +\infty)$ defined by

$$I_M(v) = \inf\{|\partial\Omega| \,:\, \Omega \subset M \quad \text{and} \quad |\Omega| = v\}. \tag{2.2}$$

We finish the section with some well known facts on the analytic nature of I_M. These will be used later in Section 3.

Let Ω be an isoperimetric region in M such that $|\Omega| = v$ for some $v \in (0, \mathrm{Vol}(M))$. The function I_M has left and right derivatives $(I_M)'_-(v)$ and $(I_M)'_+(v)$. In addition, if H is the mean curvature of $\Sigma = \partial\Omega$ in the direction of the inward unit vector, then

$$(I_M)'_+(v) \leq 2H \leq (I_M)'_-(v). \tag{2.3}$$

The second derivative also exists but weakly in the sense of comparison functions. More precisely, we say $f'' \leq h$ weakly at x_0 if there exists a smooth function g such that $f \leq g$, $f(x_0) = g(x_0)$, and $g'' \leq h$. In this sense we have

$$I_M(v)^2\, I''_M(v) + \int_\Sigma \left(Ric_g(N,N) + |A|^2\right) d_\Sigma \leq 0. \tag{2.4}$$

The equations (2.3) and (2.4) are first presented on [6] (see also Section 5 in [27]). We sketch the proof of (2.3) and (2.4).

Let Σ_V be the variation $\Sigma_t = exp_\Sigma(tN)$ of Σ reparametrized in terms of the enclosed volume $v(t)$. In addition, let $\phi(t)$ (resp. $\phi(v)$) be the area of Σ_t (resp. Σ_v). By the first variation formula for the area and volume we have $\phi'(0) = 2\,H\,|\Sigma|$ and $v'(0) = |\Sigma|$ respectively. Since $\phi'(t) = \phi'(v)v'(t)$, we conclude that $\phi'(v(0)) = 2H$ and also that $v'(0)^2\phi''(v(0)) = \phi''(0) - \phi'(v(0))v''(0)$. On the other hand, the second derivative of area for general variations implies the following:

$$\begin{aligned}\phi''(0) &= -\int_\Sigma 1\,L\,1\,d_\Sigma + 2H\,v''(0)\\ &= -\int_\Sigma \Big(Ric_g(N,N) + |A|^2\Big)d_\Sigma + 2H\,v''(0).\end{aligned}$$

Hence, in the sense of comparison functions, (2.4) follows from:

$$\phi(v(0))^2\,\phi''(v(0)) + \int_\Sigma \Big(Ric_g(N,N) + |A|^2\Big)d_\Sigma = 0. \tag{2.5}$$

2.2 Some aspects of the lens spaces

In order to define the Lens spaces, we first recall the round three sphere as:

$$\mathbb{S}^3 = \{(z,w) \in \mathbb{C}^2 \,:\, |z|^2 + |w|^2 = 1\}.$$

Fix p, q integers with the following property $1 \leq q < p$ and $gcd(p,q) = 1$. Let $\mathbb{Z}_p$ be the group $\mathbb{Z}/p\mathbb{Z}$ acting on $\mathbb{S}^3$ as follows:

$$m \in \mathbb{Z}_p \mapsto m \cdot (z,w) = (e^{\frac{2\pi iqm}{p}} z, e^{\frac{2\pi im}{p}} w). \tag{2.6}$$

The group $\mathbb{Z}_p$ acts freely and properly discontinuously on $\mathbb{S}^3$. The orbit space $\mathbb{S}^3/\mathbb{Z}_p$ is a closed three manifold called the *lens space*, it is denoted by $L(p,q)$.

The Hopf fibration, which is the Riemannian submersion $h : \mathbb{S}^3 \to \mathbb{S}^2(\frac{1}{2}) = \mathbb{C}\cup\{\infty\}$ defined by $h(z,w) = \frac{z}{w}$, can be extended naturally to $L(p,q)$. Indeed, the group $\mathbb{Z}_p$ acts on the set of *Hopf fibers* through the cyclic action of $\Gamma_p = \langle e^{\frac{2\pi i(q-1)}{p}}\rangle$ on $\mathbb{S}^2(\frac{1}{2})$ given by

$$e^{\frac{2\pi i(q-1)}{p}} : \mathbb{C} \longrightarrow \mathbb{C}, \quad \lambda \longmapsto e^{\frac{2\pi i(q-1)}{p}} \cdot \lambda.$$

The Hopf fibration for $L(p,q)$ is then defined as $h : L(p,q) \to \mathbb{S}^2(1/2)/\Gamma_p$. The set $\mathbb{S}^2(1/2)/\Gamma_p$ is a two dimensional orbifold with conical singularities at the north and south pole when $q \neq 1$. The pre-image of each of these singularities is called a critical fiber of h.

The preimage of horizontal slices of $\mathbb{S}^2(1/2)/\Gamma_p$ via h corresponds to the Clifford torus described in (1.13). They are natural candidates to solve the Isoperimetric problem in $L(p,q)$.

2.2.1 Comments on Steiner Symmetrization

Steiner and Schwarz symmetrization theorems were proved in [44] for certain fiber bundles such as the Lens spaces. To explain this symmetrization procedure we restrict to the case $L(p,1)$ where the Hopf fibration $h : L(p,1) \to \mathbb{S}^2(\frac{1}{2})$ is a smooth Riemannian submersion.

The symmetrization consists of associating to each set of finite perimeter $R \subset L(p,1)$ the set $Sym(R)$ in the product manifold $\mathbb{S}^2(\frac{1}{2}) \times \mathbb{S}^1(\frac{1}{p})$ defined by replacing the slice of R in each fiber with a ball of the same volume about the respective fiber in the product. The coarea formula for Riemannian submersions implies that $Sym(R)$ and R have the same volume. It is proved in [44] that $Sym(R)$ has no greater perimeter than R.

One immediate consequence is that $I_{L(p,1)} \geq I_{\mathbb{S}^2(\frac{1}{2})\times\mathbb{S}^1(\frac{1}{p})}$. Applying the classification of the isoperimetric problem on $\mathbb{S}^2(\frac{1}{2}) \times \mathbb{S}^1(\frac{1}{p})$, [49], we conclude that $I_{L(p,1)} = I_{\mathbb{S}^2(\frac{1}{2})\times\mathbb{S}^1(\frac{1}{p})}$ in a interval around $V = \frac{\mathrm{Vol}(L(p,1))}{2}$. In particular, the minimal Clifford torus is isoperimetric in $L(p,1)$ for every $p \geq 3$. The isoperimetric profiles, however, do not coincide as the profile of geodesic spheres on the respective spaces are different. Therefore, this technique is not enough to completely solve the isoperimetric problem.

It is important to point out that, for general Lens spaces $L(p,q)$, there is no analogue of [49] for $\mathbb{S}^2(\frac{1}{2})/\Gamma_p \times \mathbb{S}^1(\frac{1}{n_p})$ which is a manifold having codimension two singularities.

2.2.2 Description and geometry

For every $x \in L(p,q)$ the injectivity radius of $L(p,q)$ at x satisfies $\mathrm{inj}_x L(p,q) \geq \frac{\pi}{p}$, with equality only at points in the critical fibers. Indeed, for $\theta = e^{\frac{2\pi i}{p}}$ we have:

$$\begin{aligned} a^2 := d^2_{\mathbb{R}^4}[(\theta^{kq}z, \theta^k w), (z,w)] &= |\theta^{kq}-1|^2|z|^2 + |\theta^k - 1|^2|w|^2 \geq |\theta - 1|^2 \\ d_{\mathbb{S}^3}[(\theta^{kq}z, \theta^k w), (z,w)] &= 2\arcsin(\frac{a}{2}) \geq 2\arcsin\Big(\frac{2}{2}\sin\frac{2\pi}{2p}\Big) = \frac{2\pi}{p}. \end{aligned}$$

However, it is not true in general that $inj(L(p,q),x) = O(\frac{1}{p})$ as $p \to \infty$.

Example 2. Let's consider $L(k^2, k+1)$, $k \in \mathbb{Z}_+$. We show that the injectivity radius at points far away from the critical fibres are $O(\frac{1}{k})$. If the round metric is scaled by the factor k^2, then we have the Riemannian submersion:

$$h : \Big(L(k^2, k+1), k^2 g_0, x_k\Big) \to \Big(\mathbb{S}^2/\mathbb{Z}_k, k^2 g_{\mathbb{S}^2}, h(x_k)\Big).$$

The fibers have constant length 2π except the critical fibres which have length $\frac{2\pi}{p}$. The right hand side will converge, as $k \to \infty$, to $\mathbb{S}^1 \times \mathbb{R}$. It follows from the coarea formula for Riemannian submersions that the volume of the geodesic ball $B_{4\pi}(x_k)$ in $\Big(L(k^2, k+1), k^2 g_0, x_k\Big)$ is bounded from below. Therefore, by Cheeger's inequality, Lemma 51 in [50], the injectivity radius of the sequence $\Big(L(k^2, k+1), k^2 g_0, x_k\Big)$ is bounded from below. This sequence converges to a flat $T^2 \times \mathbb{R}$.

If $x, y \in T_{\frac{\pi}{4}}/\mathbb{Z}_p \subset L(p,q)$, then $d_{L(p,q)}(x,y) \geq C d_{T_{\frac{\pi}{4}}/\mathbb{Z}_p}(x,y)$ for some constant $C > 0$ independent of p, q. Thus intrinsic and extrinsic distances on $T_{\frac{\pi}{4}}/\mathbb{Z}_p$ are equivalent.

Lemma 19. *If $x \in T_{\frac{\pi}{4}}/\mathbb{Z}_p \subset L(p,q)$ and the extrinsic diameter of $T_{\frac{\pi}{4}}/\mathbb{Z}_p$ in $L(p,q)$ is bounded from below, then $inj_x L(p,q) = O(\frac{1}{p})$.*

Proof. Let $\lambda_p = \frac{1}{inj_x L(p,q)}$ and recall that $\frac{1}{\lambda_p} \geq \frac{\pi}{p}$. Without loss of generality, let's assume that $\mathrm{diameter}_{L(p,q)}(T_{\frac{\pi}{4}}/\mathbb{Z}_p) \geq 1$. Hence, under the rescaled metric $\lambda_p^2 g_{\mathbb{S}^3}$, the extrinsic diameter of $T_{\frac{\pi}{4}}/\mathbb{Z}_p$ is greater than or equal to λ_p. Let $\gamma_p(t)$ be a geodesic segment realizing the intrinsic diameter of $T_{\frac{\pi}{4}}/\mathbb{Z}_p$. Thus, we can find disjoint balls $B_R(x_i) \subset L(p,q)$, with $R < \frac{C}{2}$, $x_i \in \gamma_p(t)$, and $i = 1, \dots, [\lambda_p] + 1$. Hence,

$$\sum_{i=1}^{[\lambda_p]+1} \mathcal{H}^2(B_R(x_{i_0}) \cap T_{\frac{\pi}{4}}/\mathbb{Z}_p) \leq |T_{\frac{\pi}{4}}/\mathbb{Z}_p| = \lambda_p^2 \frac{2\pi^2}{p}.$$

Therefore, there exists $i_0 \in \{1, \dots, [\lambda_p] + 1\}$ such that:

$$C_1 \leq \mathcal{H}^2(B_R(x_{i_0}) \cap T_{\frac{\pi}{4}}/\mathbb{Z}_p) \leq \frac{2\pi^2 \cdot \lambda_p}{p}.$$

The first inequality follows from the Monotonicity Formula, Proposition 10 in the Appendix, applied to $T_{\frac{\pi}{4}} \subset (L(p,q), \lambda_p^2 g_{\mathbb{S}^3}, x)$. This finishes the proof of the lemma. $\square$

Let's use the notation $\mathbb{Z}_p^q$ to represent the group $\mathbb{Z}_p$ acting on $\mathbb{S}^3$ and its dependence on the parameter q. By using the toroidal coordinate system for $\mathbb{S}^3$,

$$\varphi_r : \mathbb{R}^2 \to T_r : \varphi_r(u,v) = (\cos(r)e^{2\pi i u}, \sin(r)e^{2\pi i v}) \in \mathbb{S}^3,$$

the action of $\mathbb{Z}_p^q$ on T_r corresponds to the following action on $\mathbb{R}^2$:

$$(u,v) \longmapsto (u + \frac{kq}{p}, v + \frac{k}{p}).$$

In these coordinates , the $\mathbb{Z}_p^q$ orbit at the point $(z_0, w_0) = \varphi(\mathbb{Z} \times \mathbb{Z}) \in T_r$ is given by:

$$\text{Orbit}_{p,q}(z_0, w_0) = \{(m,n) + k(\frac{1}{p}, \frac{q}{p}) : m, n, k \in \mathbb{Z}\}. \tag{2.7}$$

Lemma 20. *Given a sequence $\{L(p,q)\}_{p\in\mathbb{N}}$, there exist $b, m_0, n_0 \in \mathbb{Z}$ and a subsequence $\{L(p_l, q_l)\}_{l\in\mathbb{N}}$ such that one of the following holds:*

1. *For every $(z_0, w_0) \in T_r$, $\varphi_r(Orbit_{p_l,q_l}(z_0, w_0))$ is becoming dense on T_r as $l \to \infty$.*

2. *$\varphi_r(Orbit_{p_l,q_l}(z_0, w_0))$ is contained in b integral curves of the vector field $X(z,w) = (m_0\sqrt{-1}\, z, n_0\sqrt{-1}\, w) \in \mathcal{X}(\mathbb{S}^3)$.*

Proof. To prove the lemma it is enough to consider $(z_0, w_0) \in T_{\frac{\pi}{4}}$. If there is a subsequence for which the diameter of $T_{\frac{\pi}{4}}/\mathbb{Z}_p^q$ is going to zero as $p \to \infty$, then $\varphi_{\frac{\pi}{4}}(\mathrm{Orbit}_{p,q}(z_0, w_0))$ is clearly becoming dense on $T_{\frac{\pi}{4}}$ and item 1 is proved.

Let's consider now the case where the diameter of $T_{\frac{\pi}{4}}$ in $L(p,q)$ is bounded away from zero. From the equivalence between extrinsic and intrinsic distance and by Lemma 2.9 we conclude that the Euclidean injectivity radius satisfies $inj_{(z_0,w_0)} T_{\frac{\pi}{4}}/\mathbb{Z}_p = O(\frac{1}{p})$. In particular, there exist $k_p, m_p, n_p \in \mathbb{Z}$ such that

$$0 < ||k_p(\frac{1}{p}, \frac{q}{p}) - (m_p, n_p)||_{\mathbb{R}^2} \leq \frac{C}{p}.$$

Therefore, there exists $(m_0, n_0) \in B_C(0) \cap \mathbb{Z} \times \mathbb{Z} \subset \mathbb{R}^2$ such that $(k_p - pm_p, k_p q - n_p p) = (m_0, n_0)$ infinitely often and $\frac{\sqrt{m_0^2+n_0^2}}{2p}$ is the Euclidean injectivity radius of $T_{\frac{\pi}{4}}/\mathbb{Z}_p$ at (z_0, w_0) for this subsequence. Hence, the sub-orbit generated by the translation $(u,v) \to (u,v) + k_p(\frac{1}{p}, \frac{q}{p})$ is contained in the line $\mathbb{Z} \times \mathbb{Z} + \{t(m_0, n_0) : t \in \mathbb{R}\}$. It follows that the $\mathrm{Orbit}_{p,q}(z_0, w_0)$ is contained in a union of equidistant lines parallel to the one described above by homogeneity. Modulo $\mathbb{Z} \times \mathbb{Z}$ the number of such lines is finite, let's denote it by b_p. Modulo $\mathbb{Z} \times \mathbb{Z}$ there are $\frac{p}{b_p}$ points of $\mathrm{Orbit}_{p,q}(z_0, w_0)$ in each of these lines. Hence,

$$\frac{p}{b_p}(\frac{k_p}{p}, \frac{k_p q}{p}) - \frac{p}{b_p}(m_p, n_p) = \frac{p}{b_p}(\frac{m_0}{p}, \frac{n_0}{p}) \in \mathbb{Z} \times \mathbb{Z}.$$

Therefore, b_p divides m_0 and, hence, it is independent of p. In other words, $\varphi_{\frac{\pi}{4}}(\mathrm{Orbit}_{p,q}(z_0, w_0))$ is contained in b integral curves of the vector field $X(z,w) = (m_0\sqrt{-1}z, n_0\sqrt{-1}w) \in \mathcal{X}(\mathbb{S}^3)$. $\square$

2.2.3 Cheeger-Gromov convergence

A sequence of pointed Riemannian manifolds (M_i, g_i, x_i) (Riemannian manifolds (M_i, g_i) with points $x_i \in M_i$) is said to converge, in the sense of Cheeger-Gromov, to a pointed Riemannian manifold (M, g, x) as $i \to \infty$ if the following two conditions hold true:

1. There exists an exhaustion of M by compact sets $\Omega_i \subset M$: $\overline{\Omega}_i \subset \Omega_{n+1}$ and $\bigcup_{i=1}^{\infty} \Omega_i = M$.

2. There exists a family of smooth maps $\phi_i : \Omega_i \to M_i$ which are diffeomorphic onto their image such that $\phi_i(x) = x_i$ for all i and $\phi_i^*(g_i) \to g$ in the C^∞ topology.

In the next lemma we study the Cheeger-Gromov convergence for a sequence of Lens spaces.

Recall that the rank of an orientable flat 3-manifold $\mathbb{R}^3/G$ is the rank of the subgroup of translations in G.

Lemma 21. *Let $(L(p,q), p^2\, g_{\mathbb{S}^3}, x_p)$ be a sequence of Lens spaces. There exists a flat 3-manifold (M, δ, x_∞) of rank at most one such that after passing to some subsequence $(L(p,q), p^2 g_{\mathbb{S}^3}, x_p) \xrightarrow{C-G} (M, \delta, x_\infty)$.*

Proof. Since the sequence $(L(p,q), p^2\, g_{\mathbb{S}^{\not{3}}})$ has constant curvature which is converging to zero as $p \to \infty$ and $\mathrm{inj}(L(p,q), p^2\, g_{\mathbb{S}^3}) \geq \pi$, the Cheeger-Gromov Compactness Theorem implies that $(L(p,q), p^2 g_{\mathbb{S}^3}, x_p) \xrightarrow{C-G} (M, \delta, x_\infty)$, where (M, δ) is an orientable flat 3-manifold. Every non-compact flat 3-manifold is finitely covered by either $\mathbb{S}^1 \times \mathbb{R}^2$ or $T^2 \times \mathbb{R}$; hence, to show that rank of M is one, it is enough to prove that that the volume growth of balls of large radius are at least quadratic.

Below we denote $T_r/\mathbb{Z}_p$ by T_r. Let T_{r_p} be the Clifford torus through x_p enclosing a region Ω_{r_p}. Under the scaling by $\lambda_p = p^2$ we have that $|\Omega_{r_p}| = 2\pi^2 p^2 \sin^2(r_p)$. If $\lim_{p\to\infty} |\Omega_{r_p}| < \infty$, then $\lim_{p\to\infty} |T_{r_p}| = 2\pi^2 p \sin(2r_p) < \infty$. Moreover, the second fundamental form A_{r_p} of T_{r_p} satisfies $\lim_{p\to\infty} |A_{r_p}|^2 = \lim_{p\to\infty} \frac{1}{p^2}(\frac{\cos^2(r_p)}{\sin^2(r_p)} + \frac{\sin^2(r_p)}{\cos^2(r_p)}) < \infty$. The critical fiber $T_0 \subset \Omega_{r_p}$ is distant from x_p by $O(\frac{1}{p})$ since Ω_{r_p} is converging to a compact region in M. Instead of using base points x_p we choose new base points $y_p \in T_0$; it follows that $(L(p,q), p^2 g_{\mathbb{S}^3}, y_p) \to (N, \delta, y_\infty)$ and $\mathrm{rank}(N) = \mathrm{rank}(M)$. We claim that rank of N is at most one:

$$|B_{2R}(y_\infty)| = \lim_{p\to\infty} |B^p_{2R}(y_p)| \geq \lim_{p\to\infty} |\Omega_{\frac{R}{p}}| = \lim_{p\to\infty} 2\pi^2 p^2 \sin^2(\frac{R}{p}) = cR^2.$$

Let's assume now that $\lim_{p\to\infty} |\Omega_{r_p}| = \infty$, consequently $\lim_{p\to\infty} |T_{r_p}| = \infty$ and $\lim_{p\to\infty} |A_{r_p}|^2 = 0$. Recall the function $r = r(x)$, the distance from the Clifford torus through x to the critical fiber T_0 with respect to the round metric. The unit vector field ∂r is orthogonal to T_r for every r and it is well defined on $L(p,q)-\{T_0\cup T_{\frac{\pi}{2}}\}$. Let $\gamma(r)$ be the geodesic whose velocity is ∂r and such that $\gamma(r_p)=x_p$. Consider $K_{r_p,R}=\{x\in T_r : d_{L(p,q)}(x,\gamma(r))\leq \frac{R}{p} \text{ and } |r-r_p|\leq \frac{R}{p}\}$. By the triangle inequality $K_{r_p,R}\subset B_{2R}(x_p)$ under the metric $p^2 g_{\mathbb{S}^3}$. Applying the coarea formula for $f(r)=p\,r$, $|\nabla f|_{p^2 g_{\mathbb{S}^3}}=1$, we obtain:

$$|K_{r_p,R}|=\int_{r_p-\frac{R}{p}}^{r_p+\frac{R}{p}}|B_R(\gamma(u))\cap T_u|p\,du=|B_R(\gamma(u_0))\cap T_{u_0}|_{p^2 g_{\mathbb{S}^3}}R\geq cR^2,$$

where $u_0\in[r_p-\frac{R}{p},r_p+\frac{R}{p}]$ is from the mean value theorem for integrals. The last inequality is justified as follows. Either the extrinsic diameter of T_{u_0} is going to infinity and T_{u_0} is converging with multiplicity to a flat surface or the extrinsic diameter of T_{u_0} is bounded. The former implies that $|B_R(\gamma(u_0))\cap T_{u_0}|_{p^2 g_{\mathbb{S}^3}}\geq cR$. The latter implies that $B_R(\gamma(u_0))\cap T_{u_0}=T_{u_0}$, which is a contradiction since $|T_{u_0}|_{p^2 g_{\mathbb{S}^3}}\to\infty$. We conclude that $Vol(B_{2R})\geq cR^2$ which implies that the rank of M is at most one. $\square$

2.3 Proof of Theorem 1

Lemma 22. *Let $(L(p,q),p^2 g_{\mathbb{S}^3},x_p)\to(M,\delta,x_\infty)$ be as in Lemma 21 and let Σ_p be an isoperimetric surface in $L(p,q)$ such that $x_p\in\Sigma_p$ and $g(\Sigma_p)\geq 1$. There exists a constant $C>0$ such that $|A_{\Sigma_p}|_{p^2 g_{\mathbb{S}^3}}\leq C$.*

Proof. Let $y_p\in\Sigma_p\subset L(p,q)$ be such that $|A_p|(y_p)=\max_{\Sigma_p}|A_p|^2$ and define $\lambda_p=\max_{\Sigma_p}|A_p|(y_p)$. Arguing by contradiction, let's assume that $\frac{\lambda_p}{p}\to\infty$. In local coordinates around y_p we consider the surface $\Sigma_p'=\lambda_p\Sigma_p$ on the Euclidean ball $B_{\lambda_p\frac{\pi}{10p}}(0)$ endowed with the rescaled metric $\lambda_p^2 g_{\mathbb{S}^3}$. Therefore, $(B_{\lambda_p\frac{\pi}{10p}}(0),\lambda_p^2 g_{\mathbb{S}^3},y_p)$ converges to $(\mathbb{R}^3,\delta,0)$ as $p\to\infty$. The surface Σ_p' now has the property that $\max_{\Sigma_p'}|A_p'(0)|^2=1$.

By the strong compactness for a sequence of isoperimetric surfaces with bounded second fundamental form, see Corollary 6 in the Appendix, there exists a subsequence converging to a properly embedded surface $\Sigma'\subset\mathbb{R}^3$, the convergence is in the sense of graphs and with multiplicity one. Moreover, Σ' is also stable, i.e.:

$$I_{\Sigma'}(f,f) \geq 0, \ \forall f \in C_0^\infty(\Sigma') \ \textit{satisfying} \int_{\Sigma'} f \, d_{\Sigma'} = 0.$$

If Σ' is compact, then it has to be a round sphere by Alexandrov's Theorem, which is a contradiction since strong convergence preserves topology. If Σ' is non-compact, then it has infinite area by the monotonicity formula: indeed, by Proposition 10 in the Appendix there exists a positive constant C such that

$$\frac{d}{dr}\left(\frac{e^{C\,r}\,|\Sigma' \cap B_r(x)|}{r^2}\right) \geq 0.$$

In particular, $|\Sigma' \cap B_r(x)| \geq \pi r^2$. As Σ' is properly embedded, it has infinite extrinsic diameter and the claim follows. Therefore, Σ' is totally geodesic by Da Silveira's Theorem 42 in the Appendix, which is a contradiction since $\max_{\Sigma'} |A| = 1$. □

The following lemma gives a description of $I_{L(p,q)}$ for small volumes:

Lemma 23. *For p large enough there exist v_p and $\varepsilon_p > 0$ such that $I_{L(p,q)}$ is given by the profile of spheres on $(0, v_p]$ and by the profile of flat tori on $[v_p, v_p + \varepsilon_p)$. Moreover, if Σ_p is an isoperimetric surface such that $I_{L(p,q)}(v_p) = |\Sigma_p|$, then $g(\Sigma_p) = 0$ or 1.*

Proof. For each p we consider the first volume, v_p, for which there is transition on topology of isoperimetric surfaces from spheres to something else. If v_* is the volume for which the profile of geodesic spheres intersect the profile of flat tori, then $v_p \leq v_*$. The value of v_* is computed by solving the following system of equations:

$$\frac{2\pi^2}{p}\sin^2(r) = 2\pi s - \pi \sin(2s) \quad \text{and} \quad \frac{2\pi^2}{p}\sin(2r) = 4\pi \sin^2(s). \tag{2.8}$$

The left hand sides (right hand sides) of the identities in (2.8) correspond to the enclosed volume and area of the Clifford torus T_r (geodesic spheres S_s of radius s), respectively. It follows that $s \leq \frac{\pi}{p}$; another way to see this is by recalling that the injectivity radius of $L(p,1)$ is $\frac{\pi}{p}$ at every point. Therefore, $I_{L(p,q)}(v_p) \le \frac{C}{p^2}$.

Let Σ_p be an isoperimetric surface with genus $g(\Sigma_p) \ge 1$ and satisfying $I_{L(p,q)}(v_p) = |\Sigma_p|$. By Lemma 22 the sequence $\{\Sigma_p\}_{p\in\mathbb{N}}$ has bounded second fundamental form in $(L(p,q), p^2 g_{\mathbb{S}^3}, x_p)$; thus, it strongly converges to a properly embedded surface Σ of finite area in some orientable flat three manifold (M,δ) of rank at most one by Lemma 21. By the monotonicity formula, Proposition 10 in the Appendix, Σ has bounded extrinsic diameter. Since Σ is properly embedded, we conclude Σ is closed. It follows that the pre-image $\widehat{\Sigma}$ of Σ in $\mathbb{R}^3$ is contained in a solid cylinder. Hence, $\widehat{\Sigma}$ is an union of round spheres by Alexandrov's Theorem or is a surface of revolution about the axis of the cylinder by Theorem 41 in the Appendix. Therefore, $g(\Sigma) = 0$ or 1. From the strong compactness for isoperimetric surfaces, see Corollary 6, we have that $g(\Sigma_p) = 0 \text{ or } 1$, $v_p = v_*$ and the existence of the desired $\varepsilon_p > 0$. $\square$

Claim 1. *Theorem 1 follows if we can show that the isoperimetric surfaces separating $L(p,q)$ in two regions of the same volume are tori.*

Proof. By the strong compactness for isoperimetric surfaces, Corollary 6, there exists $\widehat{v}$ such that if Σ is an isoperimetric surface enclosing volume $v \in [\widehat{v}, \frac{\pi^2}{p}]$, then Σ is a flat torus. It follows from Lemma 23 and for large p that the isoperimetric profile $I_{L(p,q)}$ is given by the area of geodesic spheres for volumes in $(0, v_p]$ and by the area of flat tori for volumes in $[v_p, \varepsilon_p] \cup [\widehat{v}, \frac{\pi^2}{p}]$. In other words, if $f(v)$ is the function defined by $f(v) = |T_{r(v)}|$, where $T_{r(v)}$ is the Clifford torus enclosing a volume equal to v, then $I_{L(p,q)}(v) = f(v)$ on $[v_p, \varepsilon_p] \cup [\widehat{v}, \frac{\pi^2}{p}]$. It follows that $\phi(v) = f(v) - I_{L(p,q)}(v)$ has a local maximum point at $t_* \in (\varepsilon_p, \widehat{v})$.

The claim will follow by exploring the weak differential equation for $I_{L(p,q)}$. From (2.5) we have

$$f^2(v) f''(v) + \int_{T_{r(v)}} (2 + |A_{r(v)}|^2)\, d_{T_{r(v)}} = 0.$$

Let Σ be an isoperimetric surface such that $I_{L(p,q)}(t_*) = |\Sigma|$. Note that $g(\Sigma) \ge 2$ since the profile of the spheres lies above the profile of Clifford tori. If Σ_v is the unit normal variation of Σ parametrized by the enclosed volume, then we define $h(v) = \mathit{Area}(\Sigma_v)$. Since $h \geq I_{L(p,q)}$, we have that $\phi_1 = f - h$ has also a local maximum point at t_*. Hence, $\phi_1'(t_*) = 0$, i.e., $H_r = H$, and $\phi_1''(t_*) \leq 0$. Applying equation (2.5) for h together with the Gauss equation and the Gauss-Bonnet theorem we obtain:

$$\begin{aligned} \phi_1''(t_*) \leq 0 \quad &\Rightarrow \quad \frac{1}{h^2}\int_\Sigma (2+|A|^2)\, d_\Sigma \leq \frac{1}{f^2}\int_{T_{r(t_*)}} (2+|A_{r(v)}|^2)\, d_{T_{r(t_*)}} \\ &\Rightarrow \quad 4(1+H^2)\, h + 8\pi\,(g-1) \leq \Big(4(1+H_r^2)\, f\Big)\frac{h^2}{f^2} \\ &\Rightarrow \quad 1+H^2+\frac{2\pi(g-1)}{h} \leq (1+H_r^2)\,\frac{h}{f} \leq 1+H^2. \end{aligned}$$

Therefore, $g(\Sigma) = 1$ and $I_{L(p,q)} = f$ in $[v_p, \frac{\pi^2}{p}]$. $\square$

Proof of Theorem 1.1. Arguing by contradiction, let us assume that there exists an infinite sequence of Lens spaces $L(p,q)$ containing an isoperimetric surface Σ_p of genus $g \geq 2$ for each p. By the claim above, we may assume that Σ_p divides $L(p,q)$ in two regions of equal volume.

We consider the pointed manifolds $(L(p,q), p^2 g_{\mathbb{S}^3}, x_p)$, where the base point x_p are chosen to belong to Σ_p. By the Cheeger-Gromov Compactness Theorem, $(L(p,q), p^2 g_{\mathbb{S}^3}, x_p) \xrightarrow{C-G} (M, \delta, x_\infty)$, where (M,δ) is a flat three manifold. The inclusion of Σ_p into M through the diffeomorphism ϕ_p is still denoted by Σ_p.

By Lemma 20, the proof of Theorem 1 reduces to investigating the Cases I and II below.

Case I: There is a subsequence whose $\mathbb{Z}_p^q$ orbit of a point is contained in a finite number (independent of p) integral curves of a vector field $X(z,w) = (m_0\sqrt{-1}\,z, n_0\sqrt{-1}\,w) \in \mathcal{X}(\mathbb{S}^3)$.

We claim that the injectivity radius of $L(p,q)$ at every point is $O(\frac{1}{p})$. Indeed, let $\mathrm{Orbit}_{p,q}(z_0,w_0)$ be the orbit of $(z_0,w_0) \in \mathbb{S}^3$ with respect to $\mathbb{Z}_p^q$. As before, $\frac{p}{b}$ points of $\mathrm{Orbit}_{p,q}(z_0,w_0)$ lie on the curve $\beta(t) = \psi(z_0,w_0,t)$. Here, ψ is the one parameter family of diffeomorphisms associated to X:

$$\psi : \mathbb{S}^3 \times \mathbb{R} \to \mathbb{S}^3 : \psi(z,w,t) = (e^{m_0 i t} z, e^{n_0 i t} w).$$

When ordered according the orientation of $\beta(t)$, those points determine a piecewise closed geodesic $\gamma_p(t)$ with $\gamma_p(0) = (z_0, w_0)$. As $p \to \infty$, $\gamma_p(t)$ converges to $\beta(t)$. The claim now follows from :

$$\lim_{p\to\infty} \frac{p}{b} inj_{(z_0,w_0)} L(p,q) \leq \lim_{p\to\infty} \mathcal{H}^1(\gamma_p) = 2\pi\sqrt{m_0^2|z_0|^2 + n_0^2|w_0|^2}. \tag{2.9}$$

By the Cheeger-Gromov Compactness Theorem we obtain, up to subsequence, the following convergence:

$$(L(p,q),\, p^2 \cdot g_{\mathbb{S}^3},\, x_p) \xrightarrow{C-G} (M,\, \delta,\, x_\infty), \tag{2.10}$$

where $(M,\, \delta,\, x_\infty)$ is a flat three manifold of rank at most one by Lemma 21. By (2.9) $M = \mathbb{R}^3/S_\theta$, where $\theta \in \mathbb{Q}$.

The curves $t \to \beta(t) = \psi(x,t)$ represent integral curves of X through $x \in L(p,q)$; they have bounded geodesic curvature and $\mathcal{H}^1(\beta) = O(\frac{1}{p})$. Hence, the integral curves of X converge to closed geodesics in M under (33). As sets, they coincide with the standard vertical fibers of M.

By the Poincare-Hopf index theorem there exists a zero for the vector field $\frac{X^\top}{|X|} \in \mathcal{X}(\Sigma_p)$ since $g(\Sigma_p) \geq 2$. Hence, we can choose the base points x_p to satisfy $g_{\mathbb{S}^3}(\frac{X}{|X|}(x_p), N(x_p)) = \pm 1$, here N is the unit normal vector of Σ_p.

Lemma 24. *There exists a properly embedded surface $\Sigma_\infty \subset M$ such that $(\Sigma_p, x_p) \to (\Sigma_\infty, x_\infty)$ with multiplicity one. Moreover, Σ_∞ is totally geodesic and perpendicular to the standard fibers of M.*

Proof. By Lemma 22, the isoperimetric surfaces $(\Sigma_p, x_p) \subset (L(p,q), p^2 g_{\mathbb{S}^3}, x_p)$ have uniformly bounded second fundamental form. Applying the strong compactness theorem for isoperimetric surfaces, Corollary 6, we conclude that Σ_p converges smoothly and with multiplicity one to a properly embedded stable CMC surface $\Sigma_\infty \subset \mathbb{S}^1 \times \mathbb{R}^2$. If $\liminf_{p\to\infty} |\Sigma_p|_{p^2 g_{\mathbb{S}^3}} < \infty$, then the monotonicity formula, Proposition 10, implies that the extrinsic diameter of Σ_p and Σ_∞ are bounded. This is impossible since the sequence Σ_p separates $L(p,q)$ in two regions of the same volume that goes to infinity as $p \to \infty$. Therefore, Σ_∞ is a complete properly embedded stable CMC surface in M with infinite area. Applying Da Silveira's Theorem 42 once more, we obtain that Σ_∞ is totally geodesic. As $g_{\mathbb{S}^3}(\frac{X}{|X|}(x_p), N(x_p)) = \pm 1$, we conclude that Σ_∞ is orthogonal to the standard fibers of M. □

We claim that Σ_∞ separates M. If it does not separate, then there exists a loop γ intersecting Σ_∞ at a single point. As $\Sigma_p \to \Sigma_\infty$ with multiplicity one, the same conclusion holds for Σ_p, which contradicts the fact that Σ_p separates $L(p,q)$. Therefore, there is $k \geq 1$ such that $\Sigma_\infty = \partial\Omega_\infty = \bigcup_{i=1}^{2k} \sigma_i$, where σ_i is a flat plane for each i.

Claim 2. *This configuration cannot be a limit of isoperimetric surfaces.*

We regard M as a slab in $\mathbb{R}^3$ with height 2π. Now, we construct a deformation of Σ_∞ which decreases its area as follows. First, we cut off the k solid cylinders obtained from the intersection of Ω_∞ with a vertical solid torus of radius R. To balance the enclosed volume we add a vertical solid torus of radius r, see Figure 1 for the case $k = 1$. If a_i is the distance between σ_{2i} and σ_{2i-1}, then the radius r is given by:

$$\sum_{i=1}^{k} \pi\, R^2 a_i = \pi\, r^2 2\pi \Rightarrow r = R\sqrt{\frac{\sum_i a_i}{2\pi}}.$$

The boundary of this new region is denoted by $\widetilde{\Sigma}_\infty$ and

$$Area(\widetilde{\Sigma}_\infty \cap K) = Area(\Sigma_\infty \cap K) - 2k\pi\, R^2 + 2\pi\, R \cdot \sum_i a_i + 2\pi\, r \cdot 2\pi.$$

If R is large enough, then $\widetilde{\Sigma}_\infty$ has less area than Σ_∞. This is impossible since the strong multiplicity one convergence allow us to carry out this deformation of Σ_∞ to Σ_p which contradicts the fact that Σ_p is an isoperimetric surface.

Case II: There is a subsequence $p \to \infty$ where the $\mathbb{Z}_p^q$ orbit of a point is becoming dense on the Clifford torus containing such point.

We use geometric measure theory methods to analyse the pre-image sequence $\{\Sigma_p'\}_{p\in\mathbb{N}} \subset \mathbb{S}^3$.

It is proved in [60] that if $\Sigma = \partial\Omega$ has positive mean curvature, then

$$3\,|\Omega| \leq \int_\Sigma \frac{1}{H}\, d_\Sigma.$$

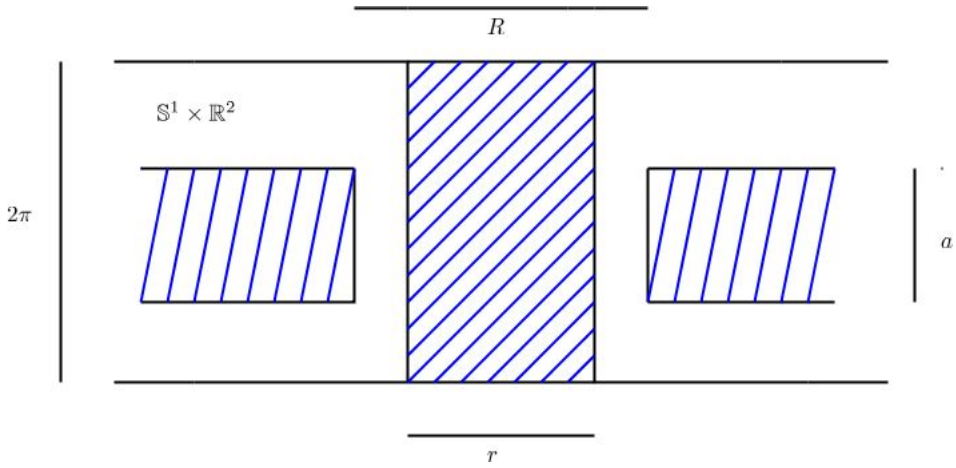

Figure 2.1: Compact support deformation of Σ_∞.

Applying this formula to the sequence of isoperimetric surfaces, we conclude that the mean curvature of Σ_p satisfies $H_p \leq \frac{|\Sigma_p|}{3|\Omega_p|} \leq \frac{2}{3}$ since $|\Sigma_p| \leq \frac{2\pi^2}{p}$ and $|\Omega_p| = \frac{\pi^2}{p}$. We have used that Σ_p separates $L(p,q)$ into regions of the same volume.

As $\{\Sigma_p\}_{p\in\mathbb{S}^3}$ has area and mean curvature bounded, we apply *Allard's compactness theorem*, Theorem 42.7 and Remark 42.8 in [67], to obtain an integral varifold $0 \neq \mathcal{V}^2 \subset \mathbb{S}^3$ that, up to subsequence, is $\mathcal{V}^2 = \lim_{p\to\infty} \Sigma'_p$. Recall that $\mathbb{S}^1 \times \mathbb{S}^1$ acts on $\mathbb{S}^3$ via $(z,w) \to (\alpha_1, \alpha_2)(z,w) := (\alpha_1\, z, \alpha_2\, w)$. We claim that $\mathcal{V}^2$ is $\mathbb{S}^1 \times \mathbb{S}^1$ invariant: indeed, if $(z,w) \in \operatorname{supp}(\mathcal{V}^2)$ and $(\alpha_1,\alpha_2) \in \mathbb{S}^1 \times \mathbb{S}^1$, then for each p there is $l_p \in \mathbb{Z}$ such that $\lim_{p\to\infty}(e^{\frac{2\pi i l_p q}{p}} z, e^{\frac{2\pi i l_p}{p}} w) = (\alpha_1 z, \alpha_2 w)$. On the other hand, as $(z,w) \in \mathcal{V}^2$, there is $(z_p,w_p) \in \Sigma'_p$ such that $(e^{\frac{2\pi i l_p q}{p}} z_p, e^{\frac{2\pi i l_p}{p}} w_p) \in \Sigma'_p$ and $\lim_{p\to\infty}(z_p,w_p) = (z,w)$. It follows that $\lim_{p\to\infty}(e^{\frac{2\pi i l_p q}{p}} z_p, e^{\frac{2\pi i l_p}{p}} w_p) = (\alpha_1 z, \alpha_2 w)$ and $(\alpha_1,\alpha_2)(z,w) \in \operatorname{supp}(\mathcal{V}^2)$ by the monotonicity formula Proposition 10. In particular, *supp* $\mathcal{V}^2 = \bigcup_{j=1}^k T_{r_j}$ where T_{r_j} is a Clifford torus. The monotonicity formula implies that the convergence $\Sigma'_p \to \mathcal{V}^2$ is also in Hausdorff distance; hence, we have that $\Sigma'_p = \cup_{j=1}^k \Sigma'^j_p$ and $\operatorname{supp}(\lim_{p\to\infty} \Sigma'^j_p) = T_{r_j}$. Since Σ'_p is $\mathbb{Z}_p^q$ invariant there exists $\theta_p^{j_1 j_2} \in \mathbb{Z}_p^q \subset \mathbb{S}^1 \times \mathbb{S}^1$ for which $\theta_p^{j_1 j_2}(\Sigma'^{j_2}_i) = \Sigma'^{j_1}_p$. By taking the limit we obtain $(\alpha_1,\alpha_2)(T_{r_{j_2}}) = T_{r_{j_1}}$ for some $(\alpha_1,\alpha_2) \in \mathbb{S}^1 \times \mathbb{S}^1$. As this is impossible, we conclude that $k = 1$ and all Σ'_p are connected for p large.

Now we consider $\{\Sigma'_p\}_{p=1} \subset \mathbf{I}_2(\mathbb{S}^3, \mathbb{Z})$, the space of 2-dimensional integral *currents* on $\mathbb{S}^3$. Each $\Sigma'_p = \partial\Omega'_p$ and $\Omega'_p \in \mathbf{I}_3(\mathbb{S}^3, \mathbb{Z})$. As Ω'_p is a region of finite perimeter ($\mathcal{X}_{\Omega'_p}$ is BV function with uniform bounded variation), then $\Omega'_i \to \Omega'$ and $\Sigma'_p \to \partial\Omega'$ as currents, Ω' is an open set of finite perimeter, see Theorem 6.3 and proof of Theorem 37.2 in [67]. Since $|\Omega_p| = \pi^2$ we conclude that $|\Omega'| = \pi^2$. Applying the *Constancy theorem*, Theorem 26.27 in [67], we conclude that Ω' is the handlebody bounded by the Clifford torus T_{r_1}, and consequently $r_1 = \frac{\pi}{4}$.

We proved that $\mathcal{V}^2 = m\, T_{\frac{\pi}{4}}$ for some positive integer $m \in \mathbb{N}$. Since Σ_p is isoperimetric, it follows that $m = 1$. Indeed,

$$m\,|T_{\frac{\pi}{4}}| = |\mathcal{V}^2| = \lim_{p\to\infty} |\Sigma'_p| \leq |T_{\frac{\pi}{4}}|.$$

As $T_{\frac{\pi}{4}}$ is smooth, we have for $r > 0$ sufficiently small that the density $\theta(T_{\frac{\pi}{4}}, r, x) \leq 1 + \frac{\epsilon}{2}$, where $\epsilon > 0$ is from Theorem 40 in the Appendix. On the other hand, as Σ'_p is converging to $T_{\frac{\pi}{4}}$ with multiplicity one, then $\theta(\Sigma'_p, x, r) \leq 1 + \epsilon$ for p large enough. Now we invoke the smooth version of *Allard's Regularity Theorem*, Theorem 40, to concluded that the convergence $\Sigma'_p \to T_{\frac{\pi}{4}}$ is strong, i.e., graphical with multiplicity one. As strong convergence preserves topology, we conclude that $g(\Sigma_p) = 1$. This completes the proof of Theorem 1. $\square$

2.4 Berger spheres

Let g_0 be the round metric on $\mathbb{S}^3$ and J the vector field on $\mathbb{S}^3$ defined as $J(z, w) = (\sqrt{-1}\,z, \sqrt{-1}\,w)$. Recall that J is tangent to the fibers of the Hopf fibration $h : \mathbb{S}^3 \to \mathbb{S}^2(\frac{1}{2})$.

The *Berger metrics* are Riemannian metrics g_ε on $\mathbb{S}^3$ defined as:

$$g_\varepsilon(X, Y) = g_0(X, Y) + (\varepsilon^2 - 1)g_0(X, J)g_0(Y, J), \;\; \varepsilon > 0.$$

The Riemannian manifolds $(\mathbb{S}^3, g_\varepsilon)$ are called the *Berger spheres*, they are denoted by $\mathbb{S}^3_\varepsilon$. Geometrically, the metric g_ε shrinks the Hopf fibers to have length $2\pi\,\varepsilon$.

The Berger metrics are also homogeneous and their group of isometries has dimension four. It follows from the work of Abresch and Rosenberg [1] that every constant mean curvature surface in $\mathbb{S}^3_\varepsilon$ admits a holomorphic quadratic differential. In particular, every CMC sphere in $\mathbb{S}^3_\varepsilon$ is rotationally invariant.

A precise study of closed orientable surfaces with constant mean curvature on the Berger spheres is given in [69]. It is proved there the existence of $\varepsilon_1 > 0$ with the following property: if $\varepsilon \in [\varepsilon_1, 1]$, then every stable constant mean curvature surface in $\mathbb{S}^3_\varepsilon$ has genus zero or one. Moreover, if $\varepsilon^2 \in [\frac{1}{3}, 1]$, then these stable CMC surfaces are totally umbilical spheres or the minimal Clifford torus, the latter only occurring when $\varepsilon^2 = \frac{1}{3}$. In particular, rotationally invariant spheres are the only solutions of the isoperimetric problem in $\mathbb{S}^3_\varepsilon$ for $\varepsilon^2 \in [\frac{1}{3}, 1]$.

Theorem 25. *There exists $\varepsilon_0 > 0$ such that for every $\varepsilon < \varepsilon_0$ the isoperimetric surfaces in the Berger spheres $\mathbb{S}^3_\varepsilon$ are either rotationally invariant spheres or tori.*

Proof of Theorem 2. Arguing by contradiction, let us assume the existence of a sequence $\varepsilon \to 0$ such that for every ε there exists an isoperimetric surface Σ_ε in $\mathbb{S}^3_\varepsilon$ with $g(\Sigma_\varepsilon) \geq 2$.

We rescale the metric g_ε of $\mathbb{S}^3_\varepsilon$ by the factor $\lambda_\varepsilon = \frac{1}{\varepsilon^2}$. The Hopf fibers have constant length equal to 2π under the new metric $\lambda_\varepsilon g_\varepsilon$. It follows that the injectivity radius of $\mathbb{S}^3_\varepsilon$ at a point p is equal to $inj_p \mathbb{S}^3_\varepsilon = \pi$ for every $p \in \mathbb{S}^3_\varepsilon$.

Since h is a local trivial fibration, we have that for each $p_\varepsilon \in \mathbb{S}^3_\varepsilon$ there exist a neighbourhood V of $h(p_\varepsilon)$ and a diffeomorphism $\phi_\varepsilon : V \times \mathbb{S}^1 \to h^{-1}(V)$ such that $h \circ \phi_\varepsilon = \pi_1$, where $\pi_1 : V \times \mathbb{S}^1 \to V$ given by $\pi(x, y) = x$. Moreover, $\phi^*_\varepsilon(\frac{1}{\varepsilon^2} g_\varepsilon) \to \delta$ in the C^∞ topology. Therefore, in the sense of Cheeger-Gromov we have:

$$\Big(\mathbb{S}^3_\varepsilon, \frac{1}{\varepsilon^2} g_\varepsilon, p_\varepsilon\Big) \to \Big(\mathbb{S}^1 \times \mathbb{R}^2, \delta, 0\Big).$$

We pick the points $p_\varepsilon \in \Sigma_\varepsilon$ with the property that $g_\varepsilon(J, N_\varepsilon)(p_\varepsilon) = \pm\,\varepsilon$, this means J and N_ε are parallel at p_ε. These points exist by the Poincaré-Hopf index theorem. By Lemma 22 the inclusion of Σ_ε in $\Big(\mathbb{S}^1 \times \mathbb{R}^2, \phi^*_\varepsilon(\varepsilon^{-2} g_\varepsilon)\Big)$ has the following property:

$$\textit{There exists } C > 0 \textit{ such that } \sup_{\Sigma_\varepsilon} |A_\varepsilon| \le C \text{ for every } \varepsilon.$$

By the strong compactness theorem for isoperimetric surfaces, Corollary 6 in the Appendix, we can extract a subsequence, $\{\Sigma_{\varepsilon_n}\}$, which converges with multiplicity one to a properly embedded surface $\Sigma_\infty \subset (\mathbb{S}^1 \times \mathbb{R}^2, \delta)$.

If $Area(\Sigma_\infty) < \infty$, then the monotonicity formula, Proposition 10, implies that Σ_∞ is compact. We apply Theorem 41 to conclude that Σ_∞ is either a round sphere or torus. This is impossible since we have strong convergence and $g(\Sigma_\varepsilon) = 2\, or\, 3$. Therefore, Σ_∞ is a complete non-compact surface with infinite area. Moreover, Σ_∞ is also a stable CMC surface in $\mathbb{S}^1 \times \mathbb{R}^2$:

$$I_{\Sigma_\infty}(f, f) \ge 0, \;\; \forall f \in C_0^\infty(\Sigma_\infty) \textit{ such that } \int_{\Sigma_\infty} f \, d_{\Sigma_\infty} = 0.$$

It follows from Theorem 42 that Σ_∞ is totally geodesic. By the choice of p_ε we conclude that Σ_∞ is orthogonal to the $\mathbb{S}^1$ fibers of $\mathbb{S}^1 \times \mathbb{R}^2$. Since Σ_∞ separates $\mathbb{S}^1 \times \mathbb{R}^2$, we also conclude that Σ_∞ is an union of at least two totally geodesic planes. As shown in the proof of Theorem 1.1, this configuration cannot be a limit of isoperimetric surfaces. $\square$

Chapter 3

Index one minimal surfaces in spherical space forms

In this chapter we study classification results for orientable minimal surfaces with Morse index one embedded in spherical space forms with large fundamental group. In the last section we extend the classification of isoperimetric surfaces from previous chapter to spherical space forms with large fundamental group.

3.1 Preliminaries

3.1.1 Morse index

A surface $\Sigma \subset (M^3, g)$ is called minimal when the trace of its second fundamental form is identically zero. Equivalently, the first variation of its area is zero for all variations generated by flows of compact supported vector fields $X \in \mathcal{X}_0(M)$. If Σ is two sided, then its second variation formula is given by:

$$I(f) := \frac{d^2}{dt^2}\bigg|_{t=0} \text{area}(\Sigma_t) = \int_\Sigma |\nabla f|^2 - (\text{Ric}(N, N) + |A|^2) f^2 \, d_\Sigma, \tag{3.1}$$

where $f = \langle X, N \rangle$ is the normal component of X, $\text{Ric}(\cdot, \cdot)$ is the Ricci curvature of M , and A is the second fundamental form of Σ. The quantity $I(f)$ is called the Morse index form of Σ and is the quadratic form associated to the Jacobi operator

$$L = \Delta + (\mathrm{Ric}(N, N) + |A|^2).$$

The Morse index of Σ is defined as the number of negative eigenvalues of L.

3.1.2 Spherical space forms

We regard $\mathbb{S}^3$ as the unit quaternions, i.e., $(z, w) = z_1 + z_2\, i + (w_1 + w_2\, i)\, j$ and $|z|^2 + |w|^2 = 1$. Let $\phi : \mathbb{S}^3 \times \mathbb{S}^3 \to SO(4)$ be the homomorphism of groups which associate for each pair $(u_1, u_2) \in \mathbb{S}^3 \times \mathbb{S}^3$ the isometry $\phi(u_1, u_2) \in SO(4)$ given by $x \mapsto \phi(u_1, u_2)(x) = u_1 x u_2^{-1}$. The map ϕ is surjective and $\mathrm{Ker}\phi = C = \{(\pm 1, \pm 1)\}$. Similarly, one can construct the homomorphism $\psi : \mathbb{S}^3 \to SO(3) \subset SO(4)$ defined by $x \in \mathbb{S}^3 \mapsto \psi(u)(x) = uxu^{-1}$. This map is also surjective and its kernel is $\{\pm 1\}$. It follows that there exists an unique homomorphism $\varphi : SO(4) \to SO(3) \times SO(3)$ such that $\varphi \circ \phi = \psi \times \psi$.

For each finite subgroup $G \subset SO(4)$ we associate $H = \varphi(G) \subset SO(3) \times SO(3)$. The projection of H on each factor of $SO(3) \times SO(3)$ is denoted by H_1 and H_2 respectively. If G acts freely on $\mathbb{S}^3$, then H_1 or H_2 must be cyclic [65]. The pre-images in $\mathbb{S}^3$ of H_1 and H_2 via the homomorphism ψ are denoted by $\widehat{H}_1$ and $\widehat{H}_2$ respecively. Since H_1 and H_2 are finite subgroups of $SO(3)$, they must be isomorphic to either the cyclic group, the dihedral group D_n, the tetrahedral group T, the octahedral group O or the icosahedral group I.

It is showed in [65] that any finite subgroup $G \subset SO(4)$ is conjugated in $SO(4)$ to a finite subgroup of either $\phi(\mathbb{S}^1 \times \mathbb{S}^3)$ or $\phi(\mathbb{S}^3 \times \mathbb{S}^1)$. Two important remarks that we will use are the following: $\phi(\mathbb{S}^1 \times \mathbb{S}^3)$ preserves the Hopf fibers in $\mathbb{S}^3$ and left multiplication by unit quaternions leaves the Hopf fibers invariant. Recall that the Hopf map $h : \mathbb{S}^3 \to \mathbb{S}^2(\frac{1}{2})$ sends (z, w) to z/w where we think of $\mathbb{S}^2$ as $\mathbb{C} \cup \{\infty\}$. In particular, up to conjugation in $O(4)$, we may assume that G is a subgroup of $\phi(\mathbb{S}^1 \times \mathbb{S}^3)$. The following describes the classical classification of 3-dimensional spherical space forms:

Theorem 26. *Let G be a finite subgroup of $\phi(\mathbb{S}^1 \times \mathbb{S}^3)$ acting freely on $\mathbb{S}^3$. Then one of the following holds:*

1. *G is cyclic;*

2. *H_2 is T, O, I, or D_n and H_1 is cyclic of order coprime to the order of H_2. Moreover, $G = \phi(\widehat{H}_1 \times \widehat{H}_2)$;*

3. *$H_2 = T$ and H_1 is cyclic. Moreover, G is a subgroup of index three in $\phi(\widehat{H}_1 \times \widehat{H}_2)$;*

4. *$H_2 = D_n$ and H_1 is cyclic. Moreover, G is a subgroup of index two in $\phi(\widehat{H}_1 \times \widehat{H}_2)$.*

Proof. See page 455 in [65]. □

The spherical space forms obtained when G is cyclic are called lens spaces. If p and q are relative primes, then we denote by $L(p,q)$ the lens space defined by the action of $\mathbb{Z}_p$ on $\mathbb{S}^3$ as follows: given $m \in \mathbb{Z}_p$, we define $m \cdot (z,w) = (e^{2\pi\frac{mq\,i}{p}} z, e^{2\pi\frac{m\,i}{p}} w)$. The Clifford torus $T_r \subset \mathbb{S}^3$, defined as

$$T_r := \mathbb{S}^1(\cos(r)) \times \mathbb{S}^1(\sin(r)) \subset \mathbb{S}^3 \tag{3.2}$$

where $r \in [0, \frac{\pi}{2}]$, is invariant by the group $\mathbb{Z}_p$ and the projection of this family foliates $L(p,q)$ by tori of constant mean curvature. One can check that $T_{\frac{\pi}{4}}$ projects to an index one minimal tori in $L(p,q)$ for every $p \geq 2$ and $q \geq 1$.

Example 3 (Immersed minimal tori with index one)**.** Let T be a Clifford torus in $\mathbb{S}^3$ containing the geodesics T_0 and $T_{\frac{\pi}{2}}$. For each p, let $\mathcal{V}_p$ be the varifold defined by $\mathcal{V}_p = \cup_{g\in\mathbb{Z}_p} g \cdot T$, where $\mathbb{Z}_p$ is the group defined above. The projection of $\mathcal{V}_p$ in $L(p,q)$ is a minimal immersed torus which fails to be embedded at the critical fibers T_0 and $T_{\frac{\pi}{2}}$ when $q \neq 1$. If p is even, then $\mathrm{Index}(\mathcal{V}_p/\mathbb{Z}_p) = 1$. Moreover, if p, q are chosen so that $\lim_{p\to\infty} \mathrm{diam}(T_{\frac{\pi}{4}}/\mathbb{Z}_p) = 0$, then the varifold $\mathcal{V} = \lim_{p\to\infty} \mathcal{V}_p$ is the foliation of $\mathbb{S}^3$ where the leaves are Clifford torus containing T_0 and $T_{\frac{\pi}{2}}$.

Remark 5 (Doubling the Clifford torus)**.** If the group G satisfies item (4) in Theorem 26, i.e., $H_2 = D_n$, then we call $\mathbb{S}^3/G$ a Prism manifold. These spherical space forms are double covered by lens spaces. In particular, one can sweep-out each Prism manifold with surfaces whose area does not exceed twice the volume of $\mathbb{S}^3/G$, see [32]. Applying the min-max theory, one obtains an orientable index one minimal surface with area bounded as above. If the order of G is sufficiently large, then Theorem 6 implies that the genus of these min-max surfaces is two. We remark that these manifolds do not contain minimal tori by Frankel's Theorem [24]. One can visualize these surfaces better when the Prism manifold they live have a double cover $L(p,q)$ which satisfies $\lim_{p\to\infty} \operatorname{diam}(T_{\frac{\pi}{4}}/\mathbb{Z}_p) = 0$. In this case, the orbit of a point $x \in T_r$ with respect to G_p is becoming dense in the Clifford torus T_r. For every p, let $\hat{\Sigma}_p$ be the pre-image of these index one minimal surfaces in $\mathbb{S}^3$. By Frankel's Theorem $\hat{\Sigma}_p \cap T_{\frac{\pi}{4}} \neq \emptyset$ for every p; hence, $\hat{\Sigma}_p$ converges as varifolds to the Clifford torus $T_{\frac{\pi}{4}}$ with multiplicity two. The surface $\hat{\Sigma}_p$ pictures like a doubling of the minimal Clifford torus.

Remark 6 (Desingularizing stationary varifolds)**.** Another family of spherical space forms is given by the quotients $\mathbb{S}^3/(I^* \times \mathbb{Z}_m)$, where m satisfies $(m, 30) = 1$. By Frankel's Theorem, there are no minimal spheres or minimal tori in $\mathbb{S}^3/(I^* \times \mathbb{Z}_m)$. With the help of the Hopf fibration $h : \mathbb{S}^3 \to \mathbb{S}^2$, it is possible to construct a sweep-out of $\mathbb{S}^3$ which is invariant by $I^* \times \mathbb{Z}_m$ and that projects to a sweep-out in $\mathbb{S}^3/(I^* \times \mathbb{Z}_m)$ by surfaces with genus two and area bounded from above by $\frac{C}{m}$, see [31, Section 6]. Applying the min-max theory, one obtains an index one minimal surface Σ_m with genus two in $\mathbb{S}^3/(I^* \times \mathbb{Z}_m)$ and area satisfying $|\Sigma_m| \leq \frac{C}{m}$. Its pre-image $\hat{\Sigma}_m \subset \mathbb{S}^3$ has uniform bounded area and converges, as $m \to \infty$, to a stationary varifold $\mathcal{V}$ which is invariant by the Hopf fibration. In particular, $\mathcal{V} = h^{-1}(\mathcal{T})$, where $\mathcal{T}$ is a I^* invariant geodesic net in $\mathbb{S}^2$. By Allard's Regularity Theorem, the genus of $\hat{\Sigma}_m$ is concentrated near $h^{-1}(V)$, where V is the set of vertices of $\mathcal{T}$. The surface $\hat{\Sigma}_m$ pictures like a desingularization of $h^{-1}(\mathcal{T})$ near $h^{-1}(V)$ through Scherk towers.

3.1.3 Non compact flat space forms

Every non-compact orientable flat space form is the quotient of $\mathbb{R}^3$ by a discrete subgroup G of the group $\operatorname{Iso}(\mathbb{R}^3)$ of affine orientation preserving isometries acting properly and discontinuously in $\mathbb{R}^3$. For every subgroup G we denote by $\Gamma(G)$ the subgroup of translations in G. The following describe all the possible types of affine diffeomorphic complete non compact orientable flat three manifolds (see [73] for a comprehensive discussion):

If $\operatorname{rank}(\Gamma(G)) = 0$ or 1, then either $G = \{\mathrm{Id}\}$ or $G = S_\theta$, with $0 \leq \theta \leq \pi$, where S_θ is the subgroup generated by a screw motion given by a rotation of angle θ followed by a non trivial translation in the direction of the rotation axis.

If $\operatorname{rank}(\Gamma(G)) = 2$, then either G is generated by two linearly independent translations and $\mathbb{R}^3/G$ is the Riemannian product $T^2 \times \mathbb{R}$, where T^2 is a flat torus, or G is generated by a screw motion with angle π and a translation orthogonal to the axis of the screw motion.

Theorem 27 (Ritoré [54])**.** *If Σ is a complete orientable index one minimal surface properly embedded in a non-compact orientable flat 3-manifold $\mathbb{R}^3/G$, then*

$$-8\pi < \int_\Sigma K_\Sigma \, d_\Sigma \leq -2\pi.$$

Remark 7. By [40, 41], the total curvature of a properly embedded minimal surface in $\mathbb{R}^3/G$ is a multiple of 2π if finite.

Example 4 (Index one Helicoids with total curvature -2π)**.** Let Σ the helicoid in $\mathbb{R}^3$ parametrized by $X(u,v) = (u\cos(v), u\sin(v), v)$. One can check that

$$\int_{\Sigma\cap\{0\leq v\leq 4\pi\}} K_\Sigma d_\Sigma = -4\pi. \tag{3.3}$$

Now consider $\Sigma/\mathbb{Z}_{4\pi}$ in $\mathbb{R}^3/\mathbb{Z}_{4\pi}$, where $\mathbb{Z}_{4\pi}$ is the group of vertical translations by multiples of 4π. Recall that the Gauss map $N : \Sigma/\mathbb{Z}_{4\pi} \to \mathbb{S}^2$ is conformal and with degree one by (3.3). A standard argument implies that $\operatorname{ind}(L_{\Sigma/\mathbb{Z}_{4\pi}}) = \operatorname{ind}(L_0)$, where $L_\Sigma = \Delta + |\nabla N|^2$ is the Jacobi operator of Σ and L_0 is the operator $L_0 = \Delta + 2$ on $\mathbb{S}^2$. Hence, $\operatorname{ind}(\Sigma/\mathbb{Z}_{4\pi}) = 1$. Let S_π be the subgroup of isometries generated by the screw motion $R(x,y,z) = (-x,-y,z+2\pi)$. Using that S_π is a subgroup of order two in $\Sigma/\mathbb{Z}_{4\pi}$, we conclude that Σ/S_π is a minimal surface with index one and total curvature -2π in $\mathbb{R}^3/S_\pi$.

Remark 8. It is an open question weather there exists an index one minimal surface Σ in $\mathbb{R}^3/G$ such that $\int_\Sigma K_\Sigma \, d_\Sigma = -6\pi$. If Σ is an index one minimal surface in a non-compact flat 3-manifold $\mathbb{R}^3/G$ where G contains only translations, then $\int_\Sigma K_\Sigma \, d_\Sigma = -4\pi$ [55].

Example 5. Let us show that $\mathbb{R}^3/S_{\frac{2\pi}{l}}$ can be obtained as a limit of Lens spaces under the Cheeger-Gromov convergence. To see this, consider the sequence of Lens spaces $(L(p_k, k), p_k^2 \, g_0, x_k)$, where x_k lies on the critical fiber $T_{\frac{\pi}{2}}$ and $p_k = l(k-1)$. This sequence has curvature close to zero and injectivity radius at x_k bounded from below by π. We claim that

$$(L(p_k, k), p_k^2 \, g_0, x_k) \xrightarrow{C-G} (\mathbb{R}^3/S_{\frac{2\pi}{l}}, \delta, x_\infty).$$

The observation is that the critical fiber $T_{\frac{\pi}{2}}$ has length 2π whereas the nearby Hopf fibers are equidistant and have length $2\pi \, l$.

3.2 Proof of Theorem 6

Proposition 6. *Let Σ be a closed minimal surface in $\mathbb{S}^3$ and $N : \Sigma_p \to \mathbb{S}^3$ be the unit normal vector field of Σ. If we denote by*

$$c = c(\Sigma) = \min \left\{ \arctan \left(\frac{1}{\max\{\lambda_2(x) : x \in \Sigma\}} \right), \frac{\pi}{4} \right\},$$

where $\lambda_2(x)$ is the non-negative principal curvature at x, and by $F : \Sigma \times [0, c) \to \mathbb{S}^3$ the exponential map on Σ, which is given by

$$(x, t) \mapsto F(x, t) = \cos(t)x + \sin(t)N(x),$$

then F is a diffeomorphism onto its image.

Proof. We may assume that $g(\Sigma) \geq 1$ since a minimal sphere in $\mathbb{S}^3$ is an equator and the Proposition trivially holds.

Let $\{e_1, e_2\}$ be an orthonormal basis with eigenvectors of the second fundamental form A_Σ and $\{\lambda_1, \lambda_2\}$ the respective eigenvalues. It follows that $dF(e_i) = (\cos(t) - \sin(t)\lambda_i)e_i$ and $dF(\partial t) = -\sin(t)x + \cos(t)N(x)$. Since

$\tan(t) \leq 1/\max_\Sigma\{\lambda_2\}$ for every $t \in (0, c)$, we conclude that F is a local diffeomorphism. The unit normal vector field along $\Sigma_t = F(\Sigma, t)$ is $N_t = -\sin(t)x + \cos(t)N(x)$. Moreover, if we denote the mean curvature of $\Sigma_t =$ by H_t, then

$$H_t = \frac{1}{2}\frac{(1+\lambda_2^2)\sin(2t)}{(\cos^2(t) - \sin^2(t)\lambda_2^2)} > 0,$$

for every $t \in (0, c)$. Let $t_0 = \sup\{t > 0 : F : \Sigma \times [0, t] \to \mathbb{S}^3 \text{ is injective}\}$. If $t_0 < c$, then there exist (x_1, t_1) and (x_2, t_2) in $\Sigma \times [0, t_0]$ with the same image under F. Since Σ separates $\mathbb{S}^3$, these points must lie on $\Sigma \times \{t_0\}$. Hence, we may assume that $F(x_1, t_0) = F(x_2, t_0)$ and that $x_1 \neq x_2$. Since Σ_{t_0} has a tangential self intersection at $F(x_1, t_0)$, we conclude that $N_{t_0}(x_1) = \pm N_{t_0}(x_2)$. If $N_{t_0}(x_1) = N_{t_0}(x_2)$, then $t_0 = \frac{\pi}{4}$, contradiction. Consequently, $N_{t_0}(x_1) = -N_{t_0}(x_2)$ since $x_1 \neq x_2$. Hence, Σ_{t_0} is locally at $F(x_1, t_0)$, an union of two tangential surfaces Γ_1 and Γ_2 with $\Gamma_1 \leq \Gamma_2$. Moreover, the mean curvatures in the $N_{t_0}(x_1)$ direction say satisfies $H_{\Gamma_1} \leq 0 \leq H_{\Gamma_1}$. Applying the Maximum Principle [63, Lemma 1], we conclude the existence of neighborhoods of x_1 and x_2 in Σ with the same image under F_{t_0} and $H_{t_0} = 0$ there. This is a contradiction and the result follows. $\square$

Lemma 28. *Let Σ be a orientable minimal surface embedded in $\mathbb{S}^3$. If $R < c(\Sigma)$, then there exists $C > 0$ independent of Σ such that vol$(B_{2R}(x)) \geq C\,R \cdot$ area$(\Sigma \cap B_R(x))$ for every $x \in \Sigma$.*

Proof. By Proposition 6, the following map is a diffeomorphism onto its image:

$$F : \Sigma \cap B_R(x) \times [0, \frac{R}{2}) \to \mathbb{S}^3.$$

Let us denote the image by Ω. By the change of variables formula,

$$\mathrm{Vol}(\Omega) = \int_0^{\frac{R}{2}} \int_{\Sigma \cap B_R(x)} \left(\cos^2(s) - \lambda_2^2 \sin^2(s)\right) d_\Sigma\, ds.$$

Hence, we can choose $0 < C < \min\{\cos^2(s)(1 - \frac{\tan^2(s/2)}{\tan^2(s)}) : s \in [0, \frac{c}{2}]\}$ such that $\mathrm{Vol}(\Omega) \geq CR\mathrm{Area}(\Sigma \cap B_R(x))$. As $\Omega \subset B_{2R}(x)$, the lemma is proved. $\square$

Let $\{\Sigma_n\}$ be a sequence of minimal hypersurfaces in a Riemannian manifold (M, g). We say that $\{\Sigma_n\}$ converges, in the C^∞ topology, to a surface Σ if for every $x \in \Sigma$ and for n large, the hypersurface Σ_n can be written locally as graphs over an open set of $T_p\Sigma$, and these graphs converge smoothly to the graph of Σ.

We say that $\{\Sigma_n\}$ satisfy *local area bounds* if there exist $r > 0$ and $C > 0$ such that $|\Sigma_n \cap B_r(x)| \leq C$ for every $x \in M$.

Proposition 7. *Let $\{\Sigma_n\} \subset (M, g_n)$ be a sequence of properly embedded minimal surfaces such that $\sup_{\Sigma_n} |A_n| \leq C$ and with local area bounds. Assume that g_n converges to g, in the C^∞ topology.*

If $\{\Sigma_n\}_{n=1}$ has an accumulation point, then we can extract a subsequence which converges to a minimal surface Σ properly embedded in (M, g).

Recall from Theorem 13 that the genus of an orientable index one minimal surface in a 3-manifold with positive Ricci curvature is at most 3.

Lemma 29. *Let M_p be a 3-manifold with positive Ricci curvature and $\Sigma_p \subset M_p$ a closed orientable minimal surface with index one and genus h. Assume that (M_p, g_p, x_p) converges, in the Cheeger-Gromov sense, to a flat manifold (M, δ, x_∞) and that (Σ_p, x_p) converges graphically with multiplicity one to a properly embedded minimal surface $(\Sigma_\infty, x_\infty)$ in (M, δ, x_∞).*

1. *If $h = 2$, then $\int_{\Sigma_\infty} K_\infty \, d_{\Sigma_\infty} = -6\pi$, -4π, or 0.*

2. *If $h = 3$, then $\int_{\Sigma_\infty} K_\infty \, d_{\Sigma_\infty} = 0$.*

Proof. Since the Cheeger-Gromov convergence preserves topology in the compact setting, we conclude that M is non compact. It follows that Σ_∞ is a complete non compact minimal surface in M since $h \geq 2$. The multiplicity one convergence implies that Σ_∞ is two sided. Moreover, the index of Σ_∞ is at most one by the lower semi continuity of the index. If $\mathrm{Ind}(\Sigma_\infty) = 0$, then Σ_∞ is flat and we are done. Hence, we may assume that $\mathrm{Ind}(\Sigma_\infty) = 1$. By classical arguments in [22], Σ_∞ is conformally equivalent to $\Sigma - \{q_1, \ldots, q_l\}$, where Σ is a closed Riemann surface. Let $D_i(q_i)$ be conformal disks on Σ centered at q_i. Given $\varepsilon > 0$ we define U_ε to be $\Sigma - \cup_{i=1}^{l}\{z \in D_i(q_i); |z_i| \le \varepsilon\}$. On the set U_{ε^2} we define the function u_ε by:

$$u_\varepsilon = 0 \quad \text{on} \quad U_\varepsilon \quad \text{and} \quad u_\varepsilon = \frac{\ln(\frac{|z|}{\varepsilon})}{\ln(\varepsilon)} \quad \text{for} \quad z \in U_{\varepsilon^2} - U_\varepsilon.$$

One can check that $\lim_{\varepsilon\to 0}\int_\Sigma |\nabla u_\varepsilon|^2 d_\Sigma = 0$. The set U_ε is seen as a subset of Σ_∞ and, by choosing ε small, we may assume that $\mathrm{Index}(U_\varepsilon) = 1$. It follows that for p large depending on ε, there exist $U_p \subset U'_p \subset \Sigma_p$ for which $\mathrm{Index}(U_p) = 1$ and such that U_p and U'_p converge graphically to U_ε and U_{ε^2}, respectively. Moreover, by means of u_ε we can construct, for each p large enough, an function u_p on Σ_p satisfying $u_p = 0$ on U_p, $u_p = 1$ at $\Sigma_p - U'_p$, and such that $\lim_{p\to\infty}\int_{\Sigma_p} |\nabla u_p|^2\, d_{\Sigma_p} = 0$, i.e., $\int_{\Sigma_p} |\nabla u_p|^2\, d_{\Sigma_p} = O_1(\varepsilon)$. As Σ_p and U_p both have index one, we concluded that $\mathrm{Indice}(\Sigma_p - U_p) = 0$. As $\mathrm{supp}(u_p) \subset \Sigma_p - U_p$, we obtain

$$0 \le \int_{\Sigma_p} \left(|\nabla u_p|^2 - (\mathrm{Ric}_{g_p}(N_p, N_p) + |A_p|^2)\, u_p^2 \right) d_{\Sigma_p}.$$

By the Gauss equation, $2\overline{K}_p = 2K_p + |A_p|^2$, where $\overline{K}_p$ is the sectional curvature of M in the direction of $T\Sigma_p$. Therefore,

$$\begin{aligned} 0 \;&\le\; \int_{\Sigma_p} \left(|\nabla u_p|^2 - (\mathrm{Ric}_{g_p}(N_p) + 2\overline{K}_p)\, u_p^2 + 2K_p u_p^2 \right) d_{\Sigma_p} \\ &= \int_{\Sigma_p} (|\nabla u_p|^2 d_{\Sigma_p} + 2\int_{\{K_p \le 0\}} K_p u_p^2) d_{\Sigma_p} + 2\int_{\{K_p > 0\}} |K_p|\, u_p^2 d_{\Sigma_p} \\ &\quad - \int_{K_p \le 0} (\mathrm{Ric}_{g_p}(N_p) + 2\overline{K}_p) u_p^2 - \int_{K_p > 0} (\mathrm{Ric}_{g_p}(N_p) + 2\overline{K}_p) u_p^2. \end{aligned}$$

If $\{e_1, e_2\}$ is an orthonormal base for $T\Sigma_p$, then $\mathrm{Ric}_{g_p}(e_1) = \overline{K}_p + \overline{K}(e_1, N)$, $\mathrm{Ric}_{g_2}(e_2) = \overline{K}_p + \overline{K}_p(e_2, N)$, and $\mathrm{Ric}_{g_p}(N_p) = \overline{K}(e_1, N) + \overline{K}(e_2, N)$. This immediately implies that $2\overline{K} + \mathrm{Ric}_{g_p}(N_p) = \mathrm{Ric}_{g_p}(e_1) + \mathrm{Ric}_{g_p}(e_2)$. Hence,

$$\begin{aligned} 0 &\le \int_{\Sigma_p} |\nabla u_p|^2 d_{\Sigma_p} + 2\int_{\{K_p\le 0\}} K_p u_p^2 d_{\Sigma_p} \\ &\qquad\qquad + \int_{\{K_p>0\}} \left(2|K_p| - \mathrm{Ric}_{g_p}(N_p) - 2\overline{K}_p\right) u_p^2 \\ &= \int_{\Sigma_p} |\nabla u_p|^2 d_{\Sigma_p} + 2\int_{\{K_p\le 0\}} K_p u_p^2 d_{\Sigma_p} \\ &\qquad\qquad\qquad - \int_{\{K_p>0\}} \left(|A_p|^2 + \mathrm{Ric}_{g_p}(N_p)\right) u_p^2. \\ &\le \int_{\Sigma_p} |\nabla u_p|^2 d_{\Sigma_p} + \int_{\{u_p\equiv 1\}\cap\{K_p\le 0\}} 2K_p\, d_{\Sigma_p} \\ &= O_1(\varepsilon) + \int_{\Sigma_p\cap\{K_p\le 0\}} 2K_p\, d_{\Sigma_p} - \int_{U_p\cap\{K_p\le 0\}} 2K_p\, d_{\Sigma_p}. \end{aligned}$$

On the other hand, we have that $\int_{U_p\cap\{K_p\le 0\}} K_p\, d_{\Sigma_p} = \int_{\Sigma_\infty} K_\infty d_{\Sigma_\infty} + O_2(\varepsilon)$, for the total curvature of Σ_∞ is uniformly close to that of U_ε which is uniformly close to that of $\int_{U_p\cap\{K_p\le 0\}} K_p\, d_{\Sigma_p}$. Hence,

$$\int_{\Sigma_\infty} K_\infty\, d_{\Sigma_\infty} \le O_1(\varepsilon) + O_2(\varepsilon) + \int_{\Sigma_p\cap\{K_p\le 0\}} K_p\, d_{\Sigma_p}.$$

This implies that $\int_{\Sigma_\infty} K_\infty\, d_{\Sigma_\infty} \le \int_{\Sigma_p\cap\{K_p\le 0\}} K_p\, d_{\Sigma_p}$ since $\int_{\Sigma_\infty} K_\infty d_{\Sigma_\infty}$ and $\int_{\Sigma_p\cap\{K_p\le 0\}} K_p d_{\Sigma_p}$ are independent of ε. On the other hand,

$$\int_{\Sigma_p\cap\{K_p\le 0\}} K_p\, d_{\Sigma_p} \le \int_{\Sigma_\infty} K_\infty\, d_{\Sigma_\infty}$$

by the upper semi continuity of the limit of non-positive functions. Therefore,

$$-8\pi < \int_{\Sigma_\infty} K_\infty\, d_{\Sigma_\infty} = \lim_{p\to\infty} \int_{\Sigma_p\cap\{K_p\le 0\}} K_p\, d_{\Sigma_p} \le 4\pi(1-h). \tag{3.4}$$

The first strictly inequality is from Theorem 27 and the second inequality if from the Gauss-Bonnet Theorem. If $h=2$, then $\int_{\Sigma_\infty} K_{\Sigma_\infty}\, d_{\Sigma_\infty} = -4\pi$ or -6π by Remark 7. If $h=3$, then (3.4) becomes a contradiction and the lemma is proved. □

Corollary 5. *If M_p is a spherical space form and genus$(\Sigma_p) = 2$, then*

$$\int_{\Sigma_\infty} K_\infty\, d_{\Sigma_\infty} = -4\pi \quad \textit{or} \quad 0.$$

Proof. Since there is no loss of negative Gaussian curvature, then

$$\lim_{p\to\infty}\int_{\{K_p>0\}}(2K_p - \mathrm{Ric}_{g_p}(N_p) - 2\overline{K}_p\, d_{\Sigma_p} = 0.$$

By the scale invariance of this quantity, we can assume that $\overline{K}_p = 1$. The Gauss equation then implies that $\lim_{p\to\infty}|\Sigma_p \cap \{K_p > 0\}| = 0$. Therefore, $\lim_{p\to\infty}\int_{\{K_p>0\}} K_p\, d_{\Sigma_p} = 0$. The corollary now follows from the Gauss-Bonnet Theorem. □

Theorem 30. *There exists an integer p_0 such that if Σ is an orientable index one minimal surface in an spherical space form M^3 with $|\pi_1(M)| \geq p_0$, then genus$(\Sigma) \leq 2$.*

Proof. By Theorem 13, we only need to rule out orientable index one minimal surfaces of genus 3. The proof is by contradiction. In what follows M_p denotes an spherical space form such that $|\pi_1(M_p)| = p$, i.e., $M_p = \mathbb{S}^3/G_p$ and $|G_p| = p$. Arguing by contradiction, let us assume the existence of a sequence of spherical space forms $\{M_{p_i}\}_{i=1}^{\infty}$ such that each M_{p_i} contains an index one minimal surface Σ_{p_i} of genus three and that $\lim_{i\to\infty} p_i = \infty$.

We consider the rescaled sequence $(M_{p_i}, \lambda_{p_i}^2\, g_{\mathbb{S}^3}, x_{p_i})$, where $x_{p_i} \in \Sigma_{p_i}$ and $\lambda_{p_i} > 0$ is such that $\lim_{i\to\infty}\lambda_{p_i}\mathrm{inj}_{x_{p_i}} M_{p_i} > 0$. Similarly, we consider $(\Sigma_{p_i}, x_{p_i}) \subset (M_{p_i}, \lambda_{p_i}^2\, g_{\mathbb{S}^3}, x_{p_i})$. By Cheeger-Gromov's compactness theorem, there exists a subsequence $\{M_{p_i}\}_{i\in\mathbb{N}}$ which converges in the Cheeger-Gromov sense to a flat manifold (M, δ, x_∞).

Lemma 31. *Let $(M_{p_i}, \lambda_{p_i}^2\, g_{\mathbb{S}^3}, x_{p_i}) \xrightarrow{C-G} (M, \delta, x_\infty)$ as above and assume that $\liminf_{i\to\infty}\lambda_{p_i} c(\Sigma_{p_i}) > 0$. Then $\{(\Sigma_{p_i}, \lambda_{p_i}^2\, g_{\mathbb{S}^3}, x_{p_i})\}_{i\in\mathbb{N}}$ satisfies local area bounds in $B_R(x_i)$ for some $R > 0$.*

Proof. Since Σ_p' is invariant by the group G_p, then $F : \Sigma_p' \times [0, c(\Sigma_p')) \to \mathbb{S}^3$ is also G_p invariant. Hence, it makes sense to consider $F : \Sigma_p \times [0, c(\Sigma_p)) \to M_p$ which is a diffeomorphism onto its image by Proposition 6. Let $r < \frac{1}{4}\min\{1, \liminf_{i\to\infty}\lambda_{p_i} c(\Sigma_{p_i})\}$, then for every $y_i \in \Sigma_{p_i} \cap B_R(x_i)$ we have that $\mathrm{Vol}(B_{\frac{2r}{\lambda_{p_i}}}(y_p)) \geq C_1\, r\, \mathrm{Area}(\Sigma_p \cap B_{\frac{r}{\lambda_{p_i}}}(y_p))$ by Lemma 28. Since this formula is scale invariant, the lemma is proved. □

Lemma 32. *Let A_p be the second fundamental form of Σ_p in M_p. There exist $C > 0$ such that* $\sup_{\Sigma_p} |A_p|_{\lambda_p^2 g_{\mathbb{S}^3}} = \sup_{\Sigma_p} \frac{1}{\lambda_p^2}|A_p|^2 \leq C$.

Proof. Let $y_p \in \Sigma_p \subset M_p$ be such that $|A_p|(y_p) = \max_{\Sigma_p} |A_p|^2$ and define the quantity $\rho_p = \max_{\Sigma_p} |A_p|(y_p)$. Arguing by contradiction, we assume that $\frac{\rho_p}{\lambda_p} \to \infty$. We consider the surface $\widehat{\Sigma}_p = (\Sigma_p, y_p) \subset (M_p), \rho_p^2 g_{\mathbb{S}^3}, y_p)$. Under this scale, the sequence $(M_p), \rho_p^2 g_{\mathbb{S}^3}, y_p)$ converges to $(\mathbb{R}^3, \delta, 0)$ as $p \to \infty$. Moreover, the surface $\widehat{\Sigma}_p$ satisfies $\max_{\widehat{\Sigma}_p} |A_p'(x)|^2 = |A_p'(0)|^2 = 1$ and enjoys local area bounds by previous lemma. By Proposition 11, $\widehat{\Sigma}_p$ converges to a non-flat properly embedded minimal surface $\Sigma_\infty \subset \mathbb{R}^3$ of index one. The convergence is with multiplicity one. Indeed, applying Proposition 6 to N and $-N$ we obtain that $F : (\Sigma_p, x_p) \times (-\alpha, \alpha) \to (M_{p_i}, \rho_{p_i}^2 g_{\mathbb{S}^3}, x_p)$, with $0 < \alpha < \liminf_{i\to\infty} \rho_{p_i} c(\Sigma_{p_i})$, is a diffeomorphism onto its image. Hence, there exists a tubular neighbourhood of radius α around each Σ_{p_i} in $(M_{p_i}, \rho_{p_i}^2 g_{\mathbb{S}^3})$ and the convergence is with multiplicity one. Since $g(\Sigma_{p_i}) = 3$, Lemma 29 implies that $\int_{\Sigma_\infty} K_\infty \, d_{\Sigma_\infty} = 0$. This contradicts $|A_{\Sigma_\infty}|(0) = 1$. $\square$

Combining Lemma 31, Lemma 32, and Proposition 11 we obtain:

Lemma 33. *There exist a properly embedded orientable minimal surface $\Sigma_\infty \subset (M, \delta, x_\infty)$ such that:*

$$\{\Sigma_{p_l}\}_{l\in\mathbb{N}} \subset (M_{p_l}, \lambda_{p_l}^2 g_{\mathbb{S}^3}, x_{p_l}) \to \Sigma_\infty \text{ in the } C^k \text{ topology.}$$

The convergence is with multiplicity one and the Morse index of Σ_∞ is at most one.

Lemma 34. *Let $x_p \in \Sigma_p$ be such that $\sup_{\Sigma_p} |A_p| = |A_p|(x_p)$. If $\lim_{p\to\infty} \lambda_p c(\Sigma_p) < \infty$, then*

$$\lim_{p\to\infty} \frac{|A_p|^2(x_p)}{\lambda_p^2} > 0.$$

Proof. As $\lim_{p\to\infty} \lambda_p c(\Sigma_p) < \infty$, there exists a positive constant C such that $c(\Sigma_{p_i}) \leq \frac{C\pi}{\lambda_{p_i}}$ for every $i \geq 1$. Hence,

$$c(\Sigma_p) \leq \frac{C\pi}{\lambda_p} \Leftrightarrow \arctan\left(\frac{1}{\lambda_2(x_p)}\right) \leq \frac{C\pi}{\lambda_p} \Leftrightarrow \lambda_2(x_p) \geq \frac{1}{\tan(\frac{C\pi}{\lambda_p})},$$

where $\lambda_2(x)$ is the largest principal curvature of Σ_p at x. Therefore,

$$\lim_{p\to\infty}\frac{|A_p|^2(x'_p)}{\lambda_p^2}=\lim_{p\to\infty}\frac{2\lambda_2^2(x_p)}{\lambda_p^2}\geq\lim_{p\to\infty}\frac{2}{\lambda_p^2\tan^2(\frac{C\pi}{\lambda_p})}=\frac{2C^2}{\pi^2}$$

and the lemma is proved. □

Lemma 35. *If for each p there exists λ_p such that $inj_x M_p \geq \frac{C}{\lambda_p}$ for every $x\in\Sigma_p$, then $\lim_{p\to\infty}\lambda_p\, c(\Sigma_p)=\infty$.*

Proof. Let $x_p\in\Sigma_p$ be such that $\sup_{\Sigma_p}|A_p|=|A_p|(x_p)$. By Lemma 33, the sequence of pointed manifolds $(M_p,\lambda_p^2 g_0,x_p)$ converges to (M,δ,x_∞) in the Cheeger-Gromov convergence and $\Sigma_p\to\Sigma_\infty$ in (M,δ,x_∞). If $\lim_{p\to\infty}\lambda_p\, c(\Sigma_p)<\infty$, then, by Lemma 34, Σ_∞ is not totally geodesic. This contradicts Lemma 29. □

By Theorem 26, we may assume that the subsequence $\{M_{p_i}\}_{i\in\mathbb{N}}$ satisfies either Case I, II, or III below:

Case I: The sequence $\{M_{p_i}\}_{i\in\mathbb{N}}$ is such that $H_2^p=\pi_2(\varphi(G_p))$ is either T, O, or I.

Lemma 36. *If M_p is such that $H_2^p=T$, O, or I, then $inj_x M_p=O(\frac{1}{p})$ for every $x\in M_p$.*

Proof. Since the group G_p preserves the Hopf fibers, the Hopf fibers have size $O(\frac{1}{p})$. Let $h:(M_p,p^2\,g_{\mathbb{S}^3})\to(\mathbb{S}^2(\frac{1}{2})/H_2^p,p^2\,g_{\mathbb{S}^2})$ be the Hopf fibration. Let $B_r(x_p)$ be the ball of radius r in $(M_p,p^2\,g_{\mathbb{S}^3})$. Since $H_2^p=T$, O, or I, there exists $c_0>0$ such that $\mathrm{vol}(h(B_r(x_p))\geq c_0r^2$. By the co-area formula,

$$\mathrm{vol}(B_{2r}(x_p))\geq\int_{h(B_r(x_p))}\mathcal{H}^1(h^{-1}(y))\,d\mathcal{H}^2(y)\geq C\,c_0\,r^2.$$

Cheeger's inequality implies that $\mathrm{inj}_{x_p}(M_p,p^2\,g_{\mathbb{S}^3})\geq i_0$ for $i_0>0$. □

Since $g(\Sigma_p)\geq 3$, there exist a point $y_p\in\Sigma_p$ such that the Hopf fiber through y_p is orthogonal to Σ_p. If we parametrize such fiber by $\gamma:[0,2\pi]\to M_p$, then the map F from Proposition 6 satisfies $F(y_p,t)=\gamma(t)$. It follows from Lemma 36 that $c(\Sigma_p)\leq\frac{C}{p}$. This contradicts Lemma 35.

Case II: The sequence $\{M_{p_i}\}_{i\in\mathbb{N}}$ is such that $H_2^p = \pi_2(\varphi(G_p))$ is $\mathbb{Z}_m$.

This corresponds to a subsequence of Lens spaces $L(p_i, q_i)$. The next lemma is useful for the analysis of this case:

Lemma 37. *If $M_p = L(p,q)$ and diameter$(T_{\frac{\pi}{4}}/\mathbb{Z}_p) > \varepsilon$ for every p, then $inj_{x_p} M_p = O(\frac{1}{p})$ and $(M_p, p^2 g_{\mathbb{S}^3}, x_p) \xrightarrow{C-G} (\mathbb{S}^1 \times \mathbb{R}^2, \delta, x_\infty)$. Moreover, there exist a unit vector field $X \in \mathcal{X}(\mathbb{S}^3)$ which is $\mathbb{Z}_p$ invariant and such that its orbits converge to the standard $\mathbb{S}^1$ fibers of $\mathbb{S}^1 \times \mathbb{R}^2$.*

Proof. See Section 3 in [70]. □

For subsequences as in Lemma 37, we pick y_p such that $g_{\mathbb{S}^3}(N(y_p), X(y_p)) = \pm 1$. The existence of $y_p \in \Sigma_p$ is from the Poincaré-Hopf Index Theorem applied to the vector field $X^T \in \mathcal{X}(\Sigma_p)$. Applying Lemmas 33 and 29, we conclude that Σ_∞ is an union of planes orthogonal to the fibers of $\mathbb{S}^1 \times \mathbb{R}^2$. This implies that $\lim_{p\to\infty} p\, c(\Sigma_p) < \infty$ which contradicts Lemma 35.

It remains to study subsequences of Lens spaces $L(p_i, q_i)$ such that

$$\lim_{i\to\infty} \text{diameter}(T_{\frac{\pi}{4}}/\mathbb{Z}_{p_i}) = 0.$$

Let us prove that the pre-image of Σ_p in $\mathbb{S}^3$, denoted by $\hat{\Sigma}_p$, converges in the Hausdorff sense to $T_{\frac{\pi}{4}}$ as $p \to \infty$.

Lemma 38.

$$\lim_{i\to\infty} d_H(\hat{\Sigma}_{p_i}, T_{\frac{\pi}{4}}) = 0.$$

Proof. Without loss of generality, we assume that $\Sigma_p \cap A_p$, where $A_p = \{x \in L(p,q) : r(x) \geq \frac{\pi}{4}\}$, is stable. Let us define the quantities $a = \liminf_{p\to\infty} \inf\{r(x) : x \in \Sigma_p\}$ and $b = \limsup_{p\to\infty} \sup\{r(x) : x \in \Sigma_p\}$. If $b < \frac{\pi}{2}$, then T_b can be obtained as a limit of $\hat{\Sigma}_p$ as $p \to \infty$ since the curvature of $\Sigma_p \cap \hat{A}_p$ is uniformly bounded and since each orbit of $\mathbb{Z}_{p_i}$ is becoming dense on the Clifford torus that contains it. Thus, $b = \frac{\pi}{4}$ which implies that $a = \frac{\pi}{4}$ and the lemma is proved in this case. Indeed, if $a < \frac{\pi}{4}$, then $T_{\frac{\pi}{4}}$ would be a stable minimal surface, contradiction. Hence, we may assume that $b = \frac{\pi}{2}$.

First we study the case where $\Sigma_p \cap T_{\frac{\pi}{2}} = \emptyset$ for every p. Let $x_p \in \hat{\Sigma}_p$ be the closest point to $T_{\frac{\pi}{2}}$. By the stability assumption, the connected components of $\hat{\Sigma}_p$ in $\hat{A}_p$ converge to leafs of a minimal lamination F in $\hat{A}_p$. Since $T_{\frac{\pi}{2}}$ is tangent to every such leaf that it intersects, we conclude that $T_{\frac{\pi}{2}}$ is contained in a leaf F_α. Let $\Gamma_p \subset \mathbb{S}^3$ be a minimal torus containing the geodesics T_0 and $T_{\frac{\pi}{2}}$ and perpendicular to $\hat{\Sigma}_p$ at x_p. The minimal tori Γ_p is a leaf of the singular lamination $E = \{E_\beta\}$ by the union of minimal tori containing T_0 and $T_{\frac{\pi}{2}}$. By compactness, Γ_p converge to a leaf E_β perpendicular to F_α along $T_{\frac{\pi}{2}}$. Consequently, there exist another leaf E_{β_1} which is tangent to F_α along $T_{\frac{\pi}{2}}$. By the analytical continuation, the lamination F coincide with the singular lamination E, contradiction.

Now we study the case $\Sigma_p \cap T_{\frac{\pi}{2}} \neq \emptyset$. By choosing $x_p \in \Sigma_p \cap T_{\frac{\pi}{2}}$, we have that $(L(p,q), p^2 g_0, x_p) \to (M, \delta, x_\infty)$, where M is a quotient of $\mathbb{R}^3$ by a screw motion with angle θ and $(\Sigma_p, p^2 g_0, x_p) \to (\Sigma_\infty, \delta, x_\infty)$, where $\Sigma_\infty \subset M$ is totally geodesic by Lemma 29. If Σ_∞ is a plane, then $\lim_{p\to\infty} p\, c(\Sigma_p) < \infty$. As this contradicts Lemma 35 (note that $\mathrm{inj}_x L(p,q) \geq \frac{\pi}{p}$ for every x), we conclude that Σ_∞ is flat cylinder. It is enough to proving that $\theta \neq 0$, since there are no totally geodesic cylinders in M in this case. Let T_{r_p} be the Clifford torus such that $\lim_{p\to\infty} p\, d_{L(p,q)}(T_{r_p}, T_{\frac{\pi}{2}}) = c_0$. It follows that $(T_{r_p}, p^2\, g_0)$ converges to a tube of radius c_0 around the central fiber in (M, δ) through x_∞. Recall $h : \mathbb{S}^3 \to \mathbb{S}^2(\frac{1}{2})$ the Hopf fibration. If $\gamma_p : [0,1] \to \mathbb{S}^3$ is the geodesic segment such that $|\gamma_p| = 2\,\mathrm{inj}_{\gamma_p(0)} L(p,q)$ with $\gamma_p(0) \in T_{r_p}$, then $h(\gamma_p)$ is a geodesic whose extremities determine an arc $\beta_p : [0,1] \to h(T_{r_p})$. Note that $\mathrm{inj}_{y_p} L(p,q) \geq |h(\gamma_p)|$. Under the scale $\lambda = p^2$ of the round metric g_0, β_p converge to an arc β in the geodesic circle with radius c_0 centered at the origin in $\mathbb{R}^2$ and $h(\gamma_p)$ converges to a linear segment γ whose extremities are those of β. The angle $\frac{|\beta|}{c_0}$ is independent of the choice of c_0. Hence, $|\gamma|$ increases as c_0 increases. In particular, the injective radius of M is not constant and, hence, $\theta \neq 0$. $\square$

By Lemma 38, for each p there exists $\lambda_p > 0$ such that $\mathrm{inj}_x L(p,q) = O(\frac{1}{\lambda_p})$ for every $x \in \Sigma_p$. Lemmas 29 and 33 imply that $(L(p,q), \lambda_p^2 g_0, x_p) \to$

(M, δ, x_∞) and that $\Sigma_p \to \Sigma_\infty$, where Σ_∞ is a totally geodesic surface in (M, δ). If M is diffeomorphic to $T^2 \times \mathbb{R}$, then $\lim_{p\to\infty} \lambda_p\, c(\Sigma_p) < \infty$ and we reach a contradiction with Lemma 35. The same argument applies if $\mathbb{R}^3/T_v$ and Σ_∞ is a union of planes. Therefore, we assume, regardless the choices of base points, that M is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^2$ and that Σ_∞ is a totally geodesic $\mathbb{S}^1 \times \mathbb{R}$. Let us show that this is incompatible with genus$(\Sigma_p) > 1$.

Lemma 39. *For each j, let Σ_j be a closed minimal surface of genus g in M_j and assume that $(M_j, \lambda_j^2 g_0, x_j) \to (\mathbb{S}^1 \times \mathbb{R}^2, \delta, x_\infty)$ and that $\Sigma_j \to \mathbb{S}^1 \times \mathbb{R}$ for every choice of base points $x_j \in \Sigma_j$. Then $g = 1$.*

Proof. It follows from the assumptions, that there exist positive constants C_1 and C_2 such that $\frac{C_1}{\lambda_j} \le \mathrm{inj}_x \Sigma_j \le \frac{C_2}{\lambda_j}$ and $\frac{C_1}{\lambda_j} \le \mathrm{inj}_x M_j \le \frac{C_2}{\lambda_j}$ for every $x \in \Sigma_j$ and every j. For each j, let $\mathcal{F}_j = \{B_1, \ldots, B_{N_j}\}$ be a maximal disjoint collection of balls $B_i = B_{\frac{R}{\lambda_j}}(x_{ij})$ in M_j where $x_{ij} \in \Sigma_j$ and $R > 4\,C_2$. By the assumption of the lemma, there exists j_0 such that $\Sigma_j \cap B_{\frac{R}{\lambda_j}}(x_{ij})$ is an annular surface for every $j \ge j_0$. For j sufficiently large, let K_j be a connected component of $\Sigma_j - \cup_{i=1}^{N_j} B_i$ and take $y_j \in K_j$. By assumption, $(\Sigma_j, \lambda_j^2 g_0, y_j) \to (\mathbb{S}^1 \times \mathbb{R}, \delta, y_\infty)$. We consider $\mathcal{F}_\infty$ the disjoint collection of regions in Σ_∞ obtained as the limit of $\Sigma_j \cap B_{ij}$. Note that each element of F_∞ is the intersection of geodesic balls in $\mathbb{S}^1 \times \mathbb{R}^2$ centered on Σ_∞ and radius $R \in [C_1, C_2]$, hence, an annulus where each boundary component generates $\pi_1(\Sigma_\infty)$. Moreover, each connected component of $\Sigma_\infty - \mathcal{F}_\infty$ is compact by the maximality of $\mathcal{F}_j$. Since K_j is connected and $y_\infty \notin \mathcal{F}_\infty$, we conclude that K_∞ is also an annulus. Hence, there exists an integer j_2 such that Σ_j is an union of disjoint annulus for every $j \ge j_2$. By the Gauss-Bonnet Theorem, genus$(\Sigma_j) = 1$. □

Case III: The sequence $\{M_{p_i}\}_{i\in\mathbb{N}}$ is such that $H_2^p = \pi_2(\varphi(G_p))$ is $\mathbb{D}_{2n}$.

The spherical space forms in this case are double covered by lens spaces. The arguments in Case II apply *mutatis mutandis.* □

3.3 Proof of Theorem 3

In this section we complete the classification of isoperimetric surfaces in the spherical space forms with large fundamental group.

Proof of Theorem 3. Let $\{M_p\}$ be a sequence of spherical space forms satisfying the following two properties: M_p contains an isoperimetric surface Σ_p of genus 2 or 3 for each p and $\pi_1(M_p) = p$. By Theorem 26 and after passing to a subsequence, we can assume that $\{M_p\}$ is of one of the following types:

- Type A : $H_2(M_p) = \mathbb{Z}_n$.
- Type B : $H_2(M_p) = D_{2n}$.
- Type C : $H_2(M_p) = T$, O, or I.

A sequence of type A corresponds to the case of lens spaces discussed in Chapter 2. Hence, we assume that the sequence $\{M_p\}$ is of Type B or of Type C. Let us first deal with sequences which are of Type B. In this case M_p is double covered by a Lens space $L(\frac{p}{2}, q)$ for each p. One first observation is that

$$\frac{1}{2}\mathrm{inj}_{\widetilde{x}}L(\frac{p}{2}, q) \leq \mathrm{inj}_x M_p \leq \mathrm{inj}_{\widetilde{x}}L(\frac{p}{2}, q).$$

As in the proof of Theorem 1, we can assume that each isoperimetric surface Σ_p divides M_p in two regions of equal volume. In Chapter 2, we show that every sequence of lens space has a subsequence which satisfies item 1 or 2 in Lemma 20. This information was used in the proof of Theorem 1 where we divide the proof in the Cases I and Case II. Let us assume that $\{L(\frac{p}{2}, q)\}$ is as in Case I. In this case, we have that

$$(L(\frac{p}{2}, q), p^2 g_{\mathbb{S}^3}, \widehat{x}_p) \to (M, \delta, x_\infty),$$

where M is orientable flat 3-manifold of rank one regardless the choice of base points $\widehat{x}_p$; topologically $M \cong \mathbb{S}^1 \times \mathbb{R}^2$. Therefore,

$$(M_p, , p^2 g_{\mathbb{S}^3}, x_p) \to (N, \delta, x_\infty),$$

where N is also an orientable flat 3-manifold of rank one. Moreover, the sequence $\{\Sigma_p\}$ has uniform bounded second fundamental form in $(M_p, p^2 g_{\mathbb{S}^3}, x_p)$ and converges with multiplicity one to a totally geodesic surface Σ_∞ in N. As shown in the proof of Theorem 1, it is possible to choose base points $\widetilde{x}_p$ for which $\widetilde{\Sigma}_p$ (the pre-image of Σ_p in $L(\frac{p}{2}, q)$) converges to an union of parallel planes. Hence, for this particular choice of base points we obtain that Σ_p converges to an union of parallel planes in N as well. This configuration cannot be a limit of isoperimetric surfaces.

Next, we assume that $\{L(\frac{p}{2}, q)\}$ is as in Case II. In this case, we take the pre-image of Σ_p in $\mathbb{S}^3$, which we denote by Σ'_p, and take its limit in the sense of varifolds. Following the arguments in the proof of Case II in Chapter 2, we conclude that Σ'_p converge to an union of Clifford torus $T_{r_1} \cup \ldots \cup T_{r_k}$. Since such union is transitive with respect to D_{2n}, we conclude that $k = 2$. The case $k = 1$ cannot happen because otherwise Σ'_p would be Hausdorff close to $T_{\frac{\pi}{4}}$ which contradicts the fact that Σ'_p divides $\mathbb{S}^3$ in two regions of equal volume. Using the theory of currents we conclude that Σ'_p bounds a connected region Ω'_p which converge to the region Ω' bounded by two Clifford tori. Since the projection of Ω' on M_p is a valid competitor for the isoperimetric problem, Σ'_p converges with multiplicity one. The result now follows from Allard's Regularity Theorem, Theorem 40 in the Appendix.

Finally, we assume that $\{M_p\}$ is of Type C. Hence, $\mathrm{inj}_x M_p = O(\frac{1}{p})$ and it is computed by computing the size of the Hopf fibers. Moreover, we claim that

$$(M_p, p^2 g_{\mathbb{S}^3}, x_p) \quad \to \quad (M, \delta, x_\infty),$$

where M is a flat orientable non-compact three manifold with rank one. To see this we consider the Hopf fibration which in this case takes the form $h : M_p \to \mathbb{S}^2/H_2$, where $\mathbb{S}^2/H_2$ is a fixed two dimension orbifold. Applying the Coarea Formula to the fibration h implies that for every $x_p \in M_p$ that the ball $B_r(x_p)$ in $(M_p, p^2 g_{\mathbb{S}^3}, x_p)$ satisfies $\mathrm{vol} B_r(x) \geq C\, r^2$ for every r large enough. This proves the claim. The argument follows the same lines as in the proof of Case I in the proof of Theorem 1. $\square$

Chapter 4

Appendix

In this chapter, now collect some background results for surfaces with constant mean curvature in 3-manifolds.

4.1 Variational formulas

In this section, we consider two-sided hypersurfaces $\Sigma^n \subset M^{n+1}$ and variations of Σ given by smooth maps $\varphi : \Sigma \times (-\epsilon, \epsilon) \to M$ with $\Sigma = \varphi(\Sigma, 0)$ and such that $\varphi_t : x \in \Sigma \longmapsto \varphi(x, t) \in M$ is an immersion of Σ in M for every $t \in (-\epsilon, \epsilon)$.

It will be useful for the computations to introduce local coordinates $x_1, \ldots, x_n$ in Σ. We will also use the simplified notation

$$\partial_t = \frac{\partial \varphi}{\partial_t} \quad \text{and} \quad \partial_i = \frac{\partial \varphi}{\partial x_i},$$

where i runs from 1 to n. The unit normal vector field along $\Sigma_t = \varphi(\Sigma, t)$ is denoted by N_t. Regarding the indexes, we use the usual summation and notational conventions. For example, the mean curvature of Σ is given by $H = g^{ij} A_{ij}$, where $A_{ij} = \langle B(\partial_i, \partial_j), N \rangle$ for every $1 \leq i, j \leq n$.

Let us recall the lapse function $f_t = \langle \partial_t, N_t \rangle$ and the volume functional $V(t)$ associated to the variation φ and defined by

$$V(t) = \int_{\Sigma \times [0,t]} \varphi^*(d_M).$$

Proposition 8.

$$\frac{d}{dt}|\Sigma_t| = -n\int_{\Sigma_t} H_t\, f_t\, d_{\Sigma_t} \quad and \quad V'(0) = \int_{\Sigma} f\, d_{\Sigma}.$$

Proof. The induced metric $g_{ij}(t)$ on Σ is given by $g_{ij} = \langle \partial_{x_i}\varphi, \partial_{x_j}\varphi\rangle$. Consequently, we have

$$\begin{aligned}\partial_t\, g_{ij} &= \partial_t\langle \partial_{x_i}, \partial_{x_j}\rangle = \langle \nabla_{\partial_t}\partial_{x_i}, \partial_{x_j}\rangle + \langle \nabla_{\partial_t}\partial_{x_j}, \partial_{x_i}\rangle \\ &= \langle \nabla_{\partial_{x_i}}\partial_t, \partial_{x_j}\rangle + \langle \nabla_{\partial_{x_j}}\partial_t, \partial_{x_i}\rangle = 2\langle \overline{\nabla}_{\partial_{x_i}}\partial_t, \partial_{x_j}\rangle.\end{aligned}$$

The area of Σ_t is given by

$$|\Sigma_t| = \int_{\Sigma} \sqrt{\det(g_{ij}(t))}\sqrt{\det(g_{ij}(t_0))}^{-1}\, d_{\Sigma_{t_0}}.$$

Recall that $D_{Id}(\det)T = Tr(T)$ for every linear map $T : \mathbb{R}^n \to \mathbb{R}^n$. Without lost of generality, we may assume that $g_{ij}(t_0) = \delta_{ij}$. Therefore,

$$\begin{aligned}\frac{d}{dt}\Big|_{t=t_0}|\Sigma_t| &= \int_{\Sigma}\frac{1}{2}\, Tr\Big(2\,\langle \overline{\nabla}_{\partial_{x_i}}\partial_t, \partial_{x_j}\rangle\Big) d_{\Sigma} = \int_{\Sigma} div_{\Sigma}\, \partial_t\, d_{\Sigma} \\ &= \int_{\Sigma} div_{\Sigma}\, \partial_t^{\top} + div_{\Sigma}\Big(f\, N\Big) d_{\Sigma} = -n\int_{\Sigma} H\, f\, d_{\Sigma}.\end{aligned}$$

To compute the variation of the volume just notice that the volume element of $\Sigma \times [0, t]$ with the induced metric coming from φ is

$$dvol_t = \langle \partial_t, N_t\rangle\, dt \wedge d_{\Sigma_t}.$$

Therefore,

$$V'(0) = \frac{d}{dt}\Big|_{t=0}\int_0^t\int_{\Sigma}\langle \partial_t, N_t\rangle\, d_{\Sigma_t} = \int_{\Sigma} f\, d_{\Sigma}.$$

$\square$

Proposition 9. *Let $\varphi : \Sigma^n \times [-\varepsilon, \varepsilon] \to M^{n+1}$ be a normal variation of Σ. Then*

$$\frac{d}{dt}H_t = \frac{1}{n}\bigg(\Delta_{\Sigma_t} f_t + (Ric_M(N_t, N_t) + |A_{\Sigma_t}|^2) f_t\bigg) = \frac{1}{n}\, L_{\Sigma_t}\, f_t.$$

Proof. In the coordinates given by φ, we have the coordinate base $\{\partial_{x_1}, \ldots, \partial_{x_n}, \partial_t\}$ on M. The mean curvature H_t is given by $n\, H_t = g^{ij} A_{ij}$, so we have

$$n\,\partial_t\, H_t = \partial_t\, g^{ij}\, A_{ij} + g^{ij}\partial_t\, A_{ij}. \tag{4.1}$$

Since $\left(g \cdot g^{-1}\right)_{ij} = \delta_{ij}$, we have that $\partial_t\, g^{-1} \cdot g = -g^{-1} \cdot \partial_t g$. Using this fact and that $\partial_t g_{ij} = -2\, A_{ij}\, f_t$, we obtain

$$\partial_t\, g^{ij} = 2g^{ik} A_{kl}\, g^{lj}\, f_t.$$

On the other hand,

$$\begin{aligned}
\partial_t\, A_{ij} &= \partial_t \langle \overline{\nabla}_{\partial_{x_i}} \partial_{x_j}, N\rangle = \langle \overline{\nabla}_{\partial_t} \nabla_{\partial_{x_i}} \partial_{x_j}, N_t\rangle + \langle \overline{\nabla}_{\partial_{x_i}\partial_{x_j}}, \overline{\nabla}_{\partial_t} N_t\rangle \\
&= R(N_t, \partial_{x_i}, \partial_{x_j}, N_t)\, f_t + \langle \overline{\nabla}_{\partial_{x_i}} \overline{\nabla}_{\partial_{x_j}} \partial_t, N_t\rangle + \langle \overline{\nabla}_{\partial_{x_i}} \partial_{x_j}, \overline{\nabla}_{\partial_t} N_t\rangle.
\end{aligned}$$

Let's study the second term above:

$$\begin{aligned}
\langle \nabla_{\partial_{x_i}} \nabla_{\partial_{x_j}} \partial_t, N_t\rangle &= \partial_{x_i} \langle \nabla_{\partial_{x_j}} f_t\, N_t, N_t\rangle - \langle \nabla_{\partial_{x_j}} f_t\, N_t, \nabla_{\partial_{x_j}} N_t\rangle \\
&= \partial_{x_i} \partial_{x_j}\, f_t - f_t \langle \nabla_{x_i} N_t, \nabla_{x_j} N_t\rangle.
\end{aligned}$$

Let us now study $\overline{\nabla}_{\partial_t} N_t$:

$$\begin{aligned}
\langle \nabla_{\partial_t} N_t, \partial_{x_k}\rangle &= -\langle N_t, \nabla_{\partial_t} \partial_{x_k}\rangle = -\langle N_t, \nabla_{\partial_{x_k}} \partial_t\rangle - \langle N, [\partial_t, \partial_{x_k}]\rangle \\
&= -\langle N_t, \nabla_{\partial_{x_k}} \left(f_t N_t\right)\rangle = -\partial_{x_k} f_t = -\langle \partial_{x_k}, \nabla_\Sigma f_t\rangle.
\end{aligned}$$

Since $\langle \nabla_{\partial_t} N_t, N_t\rangle = 0$, we conclude that $\nabla_{\partial_t} N_t = -\nabla_\Sigma f_t$. Therefore,

$$\begin{aligned}
\partial_t A_{ij} &= R(N_t, \partial_{x_i}, \partial_{x_j}, N_t)\, f_t + \left(\partial_{x_i} \partial_{x_j} - \Gamma^k_{ij} \partial_{x_k}\right) f_t - \langle \nabla_{x_i} N_t, \nabla_{\partial_{x_j}} N_t\rangle\, f_t \\
&= R(N_t, \partial_{x_i}, \partial_{x_j}, N_t)\, f_t + \nabla^{ij}_\Sigma f_t - \langle \nabla_{x_i} N_t, \nabla_{\partial_{x_j}} N_t\rangle\, f_t.
\end{aligned}$$

It follows from above identity that

$$g^{ij} \partial_t\, A_{ij} = Ric(N_t, N_t)\, f_t + \Delta_\Sigma f_t - |A|^2 f_t. \tag{4.2}$$

On the other hand,

$$\partial_t g^{ij} A_{ij} = 2g^{ik}\, A_{kl}\, g^{lj}\, A_{ij} f_t = 2\, |A|^2\, f_t. \tag{4.3}$$

Combining (4.1), (4.2) and (4.3), we obtain

$$\frac{d}{dt} H_t = \frac{1}{n}\Big(\Delta_{\Sigma_t} f_t + (Ric_M(N_t, N_t) + |A_{\Sigma_t}|^2) f_t\Big) = \frac{1}{n} L_{\Sigma_t} f_t.$$

□

4.2 Monotonicity formula

Proposition 10 (Monotonicity Formula)**.** *Let Σ^2 be a smooth surface with bounded mean curvature H inside a 3-manifold M with bounded curvature, i.e. $|K_M| \leq k$, and with positive lower bound on the injective radius $inj(M) \geq i_0$. Then there exists a positive constant $C = C(H, i_0, k)$ such that*

$$\frac{d}{dr}\Big(\frac{e^{C\,r}\,Area(\Sigma \cap B_r(p))}{r^2}\Big) \geq 0, \tag{4.4}$$

for every $p \in \Sigma$ and $r \leq \min\{i_0, \frac{1}{\sqrt{k}}\}$.

Proof. Since $-k \leq K_M \leq k$ then by the Hessian Comparison Theorem, there exists a constant $C = C(k, i_0) > 0$ such that $|\nabla^2 d_M| \leq C$ where this function is smooth. The hessian of d_M with the induced metric satisfies

$$\nabla^2_M d^2_M(X,Y) = \nabla^2_\Sigma d^2_M(X,Y) - A(X,Y)\langle \nabla_M d^2_M, N\rangle.$$

This implies

$$\Delta_\Sigma d^2_M = \nabla^2_M d^2_M(e_1,e_1) + \nabla^2_M d^2_M(e_2,e_2) + 2H\langle \nabla_M d^2_M, N\rangle.$$

Since $|\nabla_M d_M| = 1$, we have that $\nabla^2_M d^2_M = 2d_M \nabla^2_M d_M + 2$ and

$$\begin{aligned} |\Delta_\Sigma d^2_M - 4| &= |\nabla_M d^2_M(e_1,e_1) + \nabla^2_M d^2_M(e_2,e_2) - 4 \, + 2H\langle \nabla_M d^2_M, N\rangle| \\ &= d_M\Big|2\nabla^2_M d_M(e_1,e_1) + 2\nabla^2_M d_M(e_2,e_2) + 4H\langle \nabla_M d_M, N\rangle\Big|. \end{aligned}$$

Hence, there exists $C = C(k, i_0, H) > 0$ such that

$$|\Delta_\Sigma d^2_M - 4| \leq C\, d_M.$$

By the coarea formula we have

$$Area(\Sigma \cap B_r(p)) = \int_{\Sigma \cap B_r(p)} |\nabla_\Sigma d_M|\,|\nabla_\Sigma d_M|^{-1} = \int_0^r \int_{\Sigma \cap \partial B_s} \frac{1}{|\nabla_\Sigma d_M|}.$$

From this we obtain that

$$r\,\frac{d}{dr}\,Area(\Sigma \cap B_r) = \int_{\Sigma \cap \partial B_r} \frac{r}{|\nabla_\Sigma d_M|}.$$

On other hand, by Stoke's theorem

$$\begin{aligned}\int_{\Sigma\cap B_r}\Delta_\Sigma d_m^2 &= \int_{\Sigma\cap\partial B_r} 2r\,\langle\nabla_\Sigma d_M,\nu\rangle = \int_{\Sigma\cap\partial B_r} 2r\,\langle\nabla_\Sigma d_M,\frac{\nabla_\Sigma d_M}{|\nabla_\Sigma d_M|}\rangle\\ &\leq \int_{\Sigma\cap\partial B_r}\frac{2r}{|\nabla_\Sigma d_M|}.\end{aligned}$$

Therefore,

$$2r\frac{d}{dr}Area(\Sigma\cap B_r)\geq\int_{\Sigma\cap B_r}\Delta_\Sigma d_M^2\geq(4-C\,r)Area(\Sigma\cap B_r).$$

A simple computation now gives

$$\frac{d}{dr}\frac{e^{C\,r}Area(\Sigma\cap B_r(p))}{r^2}\geq 0.$$

□

Let Σ be a CMC surface in a closed manifold M^3. The density of Σ at x is given by

$$\theta(\Sigma,x,r)=\frac{\mathrm{Area}(\Sigma\cap B_r(x))}{\pi\,r^2}.$$

Theorem 40 (Allard's Regularity Theorem)**.** *Let M^3 be a closed manifold and $\rho>0$. There exist $\epsilon=\epsilon(M,\rho)>0$ and $C=C(M,\rho)$ with the following property: if $\Sigma\subset M$ is a smooth embedded CMC surface satisfying*

$$\theta(\Sigma,x,r)\leq 1+\epsilon$$

for every $x\in M$ and $r<\rho$, then its second fundamental form is uniformly bounded, i.e., $|A_\Sigma|\leq C$.

Proof. See Theorem 1.1 in [72]. □

4.3 Compactness for cmc surfaces

Let $\{\Sigma_n\}_{n\in\mathbb{N}}$ be a sequence of surfaces in a manifold M. We say that Σ_n converge to Σ in the sense of graphs if near any point $p\in\Sigma$ and for large n the surface Σ_n is locally a graph over an open set of $T_p\Sigma$ and these graphs converge smoothly to the graph of Σ. In addition, we say that $\{\Sigma_n\}$ satisfy *local area bounds* if there exist $r > 0$ and $C > 0$ such that $|\Sigma_n \cap B_r(x)| \leq C$ for every $x \in M$.

A hypersurface Σ is said to be *weakly embedded* if it admits only tangential self intersections.

Proposition 11. *Let $\{\Sigma_n\} \subset (M, g_n)$ be a sequence of embedded surfaces with constant mean curvature satisfying local area bounds and such that $\sup_{\Sigma_n} |A_n| \leq C$. Let's assume that g_n converges to a metric δ in the C^∞ topology. If $\{\Sigma_n\}_{n=1}$ has an accumulation point, then we can extract a subsequence that converges to a properly weakly embedded CMC surface Σ in (M, δ).*

Sketch of the Proof. Let's first recall the constant mean curvature equation for graphs. If Σ' is a surface with constant mean curvature H in (M, g), then Σ' can be written locally as a graph over a neighbourhood $U_p \subset T_p\Sigma'$:

$$\Sigma' = Graph(\, u\,) = \{(x_1, x_2, u(x_1, x_2) : x_1, x_2 \in U_p)\}.$$

In coordinates $g_{ij} := g(e_i, e_j)$ where $\{e_1, e_2, e_3\}$ is the coordinate base associated to (x_1, x_2, x_3). Let $\{E_1, E_2\}$ be the coordinate base for Σ_n, i.e., $E_i = e_i + u_{x_i} e_3 = T_i^l e_l$. The induced metric h is expressed by $h_{ij} = h(E_i, E_j)$. A simple computation gives:

$$g(N, e_i) = \frac{-u_{x_i}}{\sqrt{1 + g^{ij} u_{x_i} u_{x_j}}} \text{ and } g(N, e_3) = \frac{1}{\sqrt{1 + g^{ij} u_{x_i} u_{x_j}}}.$$

We also have

$$\begin{aligned} \nabla_{E_i} E_j &= T_i^l \nabla_{e_l} T_j^m e_m = T_i^l T_j^m \nabla_{e_l} e_m + E_i(T_j^m) e_m \\ &= T_i^l T_j^m \Gamma_{ml}^k e_k + u_{x_i x_j} e_3, \end{aligned}$$

where Γ_{ml}^k are the Christoffel symbols of g. Therefore,

$$g(N, \nabla_{E_i} E_j) = -\frac{\sum_{k=1}^2 T_i^l T_j^m \Gamma_{ml}^k u_{x_k}}{\sqrt{1 + g^{ij} u_{x_i} u_{x_j}}} + \frac{\left(u_{x_i x_j} + T_i^l T_j^m \Gamma_{ml}^3\right)}{\sqrt{1 + g^{ij} u_{x_i} u_{x_j}}}.$$

Finally, the mean curvature equation is written as:

$$H \sqrt{1 + g^{ij} u_{x_i} u_{x_j}} = h^{ij} u_{x_i x_j} + h^{ij} T_i^l T_j^m \Gamma_{ml}^3 - \sum_{k=1}^2 h^{ij} T_i^l T_j^m u_{x_k} \Gamma_{ml}^k. \qquad (4.5)$$

Since $h_{ij} = T_i^l T_j^m g_{ml}$ and $T_i^3 = u_{x_i}$, then $|h^{ij}| \leq C_1(g^{ij}, u_{x_i}, u_{x_j})$. The equation (4.5) is uniformly elliptic as long as $|\nabla u|, |\nabla^2 u| < \widetilde{C}$.

Let $p \in M$ be an accumulation point for the sequence $\{\Sigma_n\}$. By the upper bound on the second fundamental form, $\sup_{\Sigma_n} |A_n| < C$, there exists $r_0 = r_0(C)$ such that for every $q \in B_{r_0}(p) \cap \Sigma_n$ we have that $\Sigma_n \cap B_{r_0}(q)$ is locally a graph u_n over a neighbourhood $U_q \subset T_q\Sigma_n$. Moreover, there exists $C_2 = C_2(C) > 0$ for which $\max\{\nabla u_n, \nabla^2 u_n\} \leq C_2$. As $|g_n - \delta|_{C^{2,\alpha}} \to 0$, the Schauder estimates for solutions of elliptic equations, see [25], imply that u_n have $C^{2,\alpha}$ estimates on $B_{r_0/2}(q)$, i.e., $|u_n|_{C^{2,\alpha}} \leq C_3(|H_n| + |u_n|)$. Therefore, $u_n, \nabla u_n, \nabla^2 u_n$ are uniformly bounded and equicontinuous.

As $\{\Sigma_n\}_{n\in\mathbb{N}}$ satisfy local area bounds, then $|\Sigma_n \cap B(r_0/2)(y)| \leq C_4$. On the other hand, the monotonicity formula, Proposition 10, gives that $|\Sigma_n^j \cap B_{r_0}(y)| \geq C_5 r_0^2$, where Σ_n^j is a connected component of $\Sigma_n \cap B_r$. It follows that the number of components of $\Sigma_n \cap B_{r_0/2}(p)$ is finite and independent of n. By the Arzelà-Ascoli theorem we can extract a subsequence for which $\{u_n^j\}$ converges to u for every j. Moreover, u also satisfies the constant mean curvature equation (4.5). As the set of accumulation points of $\{\Sigma_n\}$ is compact in $B_R(p)$ we can cover this set by finite balls $B_{r_0}(p_k)$ with $k = 1, \ldots, N$. Repeating the arguments in each of these balls and applying a diagonal argument we obtain a properly immersed surface Σ on $B_R(p) \subset M$ with constant mean curvature H. Since the surfaces Σ_n are embedded, it follows that Σ does not cross itself though it may have tangential self-intersections. Therefore, Σ is properly weakly embedded in M. □

Corollary 6. *Let $(\Sigma_n, x_n) \subset (M_n, g_n, x_n)$ be a sequence of isoperimetric surfaces with $|A_n| \leq C$. Assume that (M_n, g_n, x_n) converges, in the sense of Cheeger-Gromov, to a three manifold (M, g, x). There exists a properly embedded surface $\Sigma \subset (M, g, x)$ such that $\Sigma_n \to \Sigma$ in the sense of graphs and the convergence is with multiplicity one.*

Sketch of the Proof. First we remark that $\{\Sigma_n\}$ satisfy local area bounds. Indeed, $\text{Area}(\Sigma_n \cap B_r(p)) \leq 2|\partial B_r(p)|$. By Proposition 11 we only need to rule out possibly multiplicities for the convergence $\Sigma_n \to \Sigma$ and points where

Σ fails to be embedded. If the multiplicity of the limit is bigger than two,

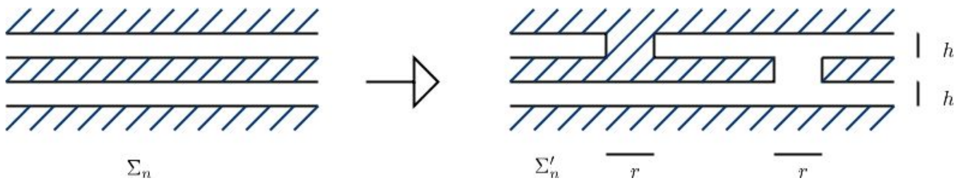

Figure 4.1: Example of higher multiplicity

then $\Sigma_n \cap B_r(p)$ has several components getting arbitrarily close. This allow us to do a local cut and past deformation, as shown in Figure 2, that preserves the enclosed volume. If δ is the Euclidean metric, then $\frac{1}{\widetilde{C}}\delta \leq g_n \leq \widetilde{C}\delta$ and $\frac{1}{C'}\text{Area}_\delta \leq \text{Area}_{g_n} \leq C'\text{Area}_\delta$. Thus, if $h \ll r$, then

$$Area_{g_n}(\Sigma'_n) \leq Area_{g_n}(\Sigma_n) - C'_1 r^2 + C'_2 rh < \text{Area}_{g_n}(\Sigma_n).$$

The deformation needed for the multiplicity two case is shown in Figure 3.

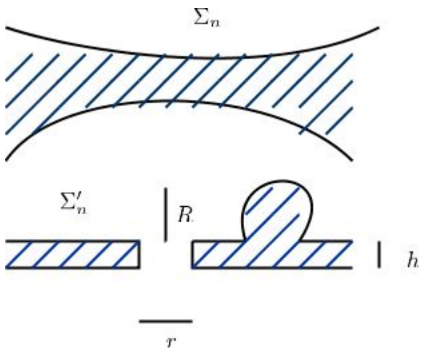

Figure 4.2: Example of multiplicity two

The constraint on the enclosed volume implies that $\frac{4}{3}\pi R^3 \approx \pi r^2 h$. Hence, if $h \ll r$, then

$$Area(\Sigma'_n) \leq\ Area(\Sigma'_n) - C'_1 r^2 + C'_2 r\, h - C'_3 R^2 + C'_4 R^2 < Area(\Sigma'_n).$$

The construction to deal with points where Σ has tangential self intersections is similar to the multiplicity two case. The corollary now follows since these constructions contradict the fact that Σ_n is isoperimetric for every n. $\square$

Theorem 41 (Korevaar-Kusner-Solomon [34])**.** *If Σ is a complete properly embedded constant mean curvature surface contained in a solid cylinder, then Σ is rotationally symmetric with respect to a line parallel to the axis of the cylinder.*

Theorem 42 (Da Silveira [16])**.** *Let (Σ, ds^2) be a complete orientable Riemannnian surface conformally equivalent to a compact Riemann surface punctured at a finite number of points. Let $L = \Delta + q$ be an operator satisfying $q \geq 0$ and $q \neq 0$, let us also assume that Σ has infinite area. Then there exists a piecewise smooth function f with compact support satisfying*

$$-\int_\Sigma f\, L\, f d_\Sigma < 0 \quad \textit{and} \quad \int_\Sigma f\, d_\Sigma = 0.$$

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