Seeing the infinite

by

Emmanuel Ordóñez Angulo

Thesis submitted for the degree of Master of Philosophy

University College London

I, Emmanuel Ordóñez Angulo, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

Abstract

Against the consensus in the epistemology of mathematics, this thesis argues that knowledge of infinity by acquaintance is possible. Even if knowledge of small cardinal numbers by acquaintance or via experience is possible, the consensus goes, knowledge of large numbers, and importantly of infinite numbers, is only possible by description or via theory. The thesis starts by taking up Stewart Shapiro's view that cardinal numbers can be understood in terms of structures, and hence that knowledge of small cardinals by experience may be understood as experience of small structures, to suggest that a purported experience of the first infinite cardinal, $\aleph_0$, may be understood as an experience of the corresponding infinite structure. This suggestion is reached obliquely by focusing on a particular puzzling case concerning knowledge of infinity: a subject's perceptual report of 'encountering' it. The main hurdle in explaining this puzzle will be that infinities are widely understood in both philosophy and mathematics to exist actually rather than potentially, such that in a framework of perceptual experience requiring both relata of the perceptual relation to be co-present, actual infinities can of course *not* be. Unlike mathematical abstracta, however, perceptual experience is not placeless and timeless but perspectival and extended in time. Hence, an account of the subject's mental state in terms of potential infinity needn't entail anything about the (actually infinite) structure purportedly perceived. I develop such a modal-logic-based account drawing on Shapiro and Øystein Linnebo's explication of potential infinity and on a version of Jaakko Hintikka's explication of knowledge modified for knowledge by acquaintance. The broader consequences of making sense of the possibility of acquaintance with infinity are sketched in terms of Fraser MacBride's 'access' challenge against Shapiro's epistemology.

Impact statement

This thesis develops a challenge to extant views within the philosophy of mathematics and attempts to point us towards unexplored roads for the study of humans' access to mathematical knowledge.

Table of contents

List of figures

I shut my eyes — I opened them. Then I saw the Aleph.
I arrive now at the ineffable core of my story. […]
How, then, can I translate into words the limitless Aleph,
which my floundering mind can scarcely encompass?

—Jorge Luis Borges

1. Introduction

1.1

In 'Epistemology of Mathematics: What are the Questions? What count as Answers?' (2011), Stewart Shapiro offers a helpful opportunity to take a pause, step back, and think about what philosophers of mathematics are doing, and what they should be doing, when they offer or reject theories about the ways in which we acquire mathematical knowledge. Shapiro does this as part of his response to the challenge Fraser MacBride (2008) throws at him demanding to please say how his (Shapiro's) philosophy of mathematics addresses what MacBride calls the 'access problem', the problem 'of explaining how mathematicians can reliably access truths about an abstract realm to which they cannot travel and from which they receive no signals' (MacBride 2008, p. 156).[1] In short, Shapiro tells us that this question is misleading. It is certainly a serious task for a realist about mathematical objects such as Shapiro to give an account of how concrete, finite beings such as mathematicians have any confidence that their beliefs about abstract things such as numbers are true. In the canonical work where he states his view, Shapiro himself asks: '[m]ost of us believe that every natural number has a successor, and I would hope that at least some

[1] MacBride's problem is arguably in essence what elsewhere in the literature gets called 'Benacerraf's problem' after Paul Benacerraf (1973), even though Benacerraf focused on knowledge of mathematical *truths* rather than of their truth-makers. At any rate, we avoid this label for simplicity because we focus on the MacBride - Shapiro exchange here.

of us are fully justified in this belief. But *how*?' (Shapiro 1997, pp. 109-10, my italics). It is misleading to say, however, that the task of offering such an account, i.e. the task of offering a satisfactory epistemology of mathematics, *is* the task of explaining how flesh-and-bone subjects' minds latch on to 'an abstract realm to which [they] cannot travel and from which they receive no signals' (Shapiro 2011, p. 132). As Shapiro points out, philosophers tend to get lost in the 'realm' metaphor. From the fact that abstract mathematical objects aren't located in time and space it doesn't follow that they are located *elsewhere*, like, say, in 'a Platonic heaven' (ibid).

Although Shapiro's paper does address MacBride's worry after clearing out the above confusion, in order to do so he reflects on a broader, seemingly meta-philosophical pair of issues: the issue of what the goals of an epistemology of mathematics might be and the issue of by what standards those goals might be considered met. Disagreements over *these* questions, he says, may explain disagreements over first-order ones such as the question whether a certain approach to the access problem successfully deals with it or not.

This thesis is perhaps best presented as concerned with the access problem. I use this phrasing because, as the reader will see, the way the thesis is concerned with the access problem is slightly oblique. But if what I say is of interest to the philosophy of mathematics at all, it will be because of how it speaks to that problem. Hence, before introducing, as one must, the 'thesis of my thesis', I suggest to start by briefly laying out the context within which the thesis might make sense, which context consists, I suggest, of what I've said Shapiro's paper provides a pause to think about: the issue of what questions an epistemologist of mathematics aims to answer and the issue of by what standards she might be considered as doing her job well.

1.2

The first thing to note is Shapiro's point that one's view on the epistemology of mathematics should dovetail with one's view on what the subject matter of mathematics is. Our epistemology, in other words, should agree with our metaphysics. 'If you say that mathematics is about *ps*, then your epistemology should show how it is that mathematicians manage to know things about these *ps*' (ibid., p. 131). From the various views on the metaphysics of mathematics, let us only mention Shapiro's because only that one will be of interest to our discussion of the access problem here; call it 'mathematical realism'.[2] Roughly, realism is the view according to which mathematical statements should be taken at face value, which means that, according to that view, the singular terms of mathematics refer to mind-independently existing things: abstract mathematical objects. There are various brands of realism too but the relevant point here is just that Shapiro *is* a realist and that this explains the demand MacBride puts on him: if Shapiro weren't a realist about mathematical objects, i.e. if he didn't think the term '4' in a statement like '2+2=4' really stands for some*thing*, he wouldn't be bound to tell us how it is that mathematicians come to know stuff about that thing, which, real as it is, does not depend on anybody's mind in order to exist and have the properties it does.

Shapiro's discussion, then, will aim at clarifying what a *realist* epistemology of mathematics' goals might be —what it is meant to establish— and, as he puts it, what its 'burdens' might be —by what standards it is meant to establish it. With regards to its goals, Shapiro wants to propose a middle ground between two problematic positions. The first,

[2] I avoid 'platonism' for simplicity and 'structuralism' because that term will be introduced later on for different purposes.

seemingly espoused by MacBride, demands an account of mathematical knowledge that doesn't pre-suppose any mathematical knowledge, i.e. an account that *reduces* or *grounds* it[3] on something else: '[o]ne must describe knowers, and the processes used to obtain mathematical knowledge, in thoroughly non-mathematical terms, and then show that knowers do indeed end up with mathematical knowledge' (ibid., p. 132). Perhaps one reason MacBride demands a reductive account of this sort is Shapiro's explicit commitment to a 'naturalised epistemology', which he states elsewhere thus: 'any faculty that the knower has and can invoke in pursuit of knowledge must involve only natural processes amenable to ordinary scientific scrutiny' (Shapiro 1997, p. 110). This requirement, however, Shapiro tells us, needn't exclude the use of mathematics. He writes: 'ordinary scientific scrutiny of just about anything is going to involve mathematics' (2011 p. 133); hence, when explaining mathematical knowledge in natural terms, the philosopher can use mathematics 'just as anyone else' (ibid.). The second problematic position is an epistemology that simply takes it that we philosophers are not in a position to justify mathematics on more 'secure' grounds any more than we would be in a position to criticise it from outside the field. Hence, this view goes, all epistemology can do is describe the way actual knowers —in particular, professional mathematicians— acquire mathematical knowledge, for instance, by describing their techniques, and stay content with that.[4] This view leaves no room for an access problem, but that is because the whole enterprise seemingly shrugs off the normative aspect of its duty since normativity is already contained in the description of *correct* mathematical practice.

[3] The italicised are Shapiro's terms for characterising this position.
[4] For examples of this view, Shapiro refers us to Burgess and Rosen (1997) and Maddy (1997, 2007).

In contrast with those two views, Shapiro's proposal for a —*his*— realist philosophy is a position that neither merely describes how mathematicians end up with mathematical knowledge nor aspires to ground it elsewhere but simply aims to 'show how it is plausible that both ordinary folk and mathematicians end up with knowledge of [mathematical objects]' (ibid., p. 134). This takes us to the second issue: by what standards has the epistemologist succeeded in showing such plausibility?

A proponent of the reductivist view will insist here that the standard is to say 'what exactly constitutes (corrigible) warrant for mathematical beliefs' (MacBride op. cit., p. 164). Note that this standard is independent of epistemological foundationalism, since showing 'how it is ever more than a coincidence that our beliefs about mathematical objects are true' (ibid.) needn't involve pointing further down the tree of knowledge for the source of mathematical knowledge's non-fortuitousness. Shapiro's answer is nevertheless anti-foundationalist: he proposes a kind of holism, where our ability to coherently talk about mathematical objects *is itself* evidence (to know) that they exist (Shapiro op. cit., p. 135). So that is supposed to deal with the issue of justification. As to the further demand for warrant, Shapiro borrows Crispin Wright's words to claim that 'the right response [...] is not to conclude that the acquisition of genuine warrant is impossible, but rather to insist that it does not require this elusive kind of security' (Wright quoted in Shapiro op. cit., p. 144).

The details of Shapiro's holism won't concern us here. What's interesting is that the differences just described between Shapiro's and MacBride's meta-philosophical positions explain their first-order disagreement over whether the mental processes Shapiro lays out in his epistemology really yield knowledge. Obviously, MacBride thinks they don't. Now *this* debate is where things really get thorny.

1.3

In short, Shapiro proposes a 'stratified epistemology'. Because we will follow him in understanding an epistemology as dovetailing with a corresponding metaphysics, we will now introduce his brand of realism, though with the warning that it will just be in the service of making sense of the mental processes he describes as conducive to knowledge of what the (real) subject matter of mathematics is. In Shapiro's view, they are structures. Structures 'exist objectively, independently of the community of mathematicians and scientists, their minds, languages, forms of life, etc.' (ibid., p. 130). Because structures are abstract objects consisting in places that stand in certain relations to each other, the referents of singular terms in mathematical sentences are places in those structures. The subject-matter of arithmetic, for example, is the structure of the natural numbers, and natural number terms refer to places in that structure, which places could nevertheless form not only the set of natural numbers but 'any countably infinite system of objects that has a certain successor relation obeying certain principles' (ibid).

So, for Shapiro, mathematical knowledge is knowledge about the abstract entities structures are. The access problem for him becomes then how we acquire *that*. Shapiro's answer comes in steps.[5] The first step is a mental process he calls 'pattern recognition': '[a] subject observes one or more systems of objects arranged in various ways, and abstracts a pattern, or structure, from the systems' (ibid., p. 136). You observe a pair of apples

[5] Here we are just concerned with (MacBride's criticism of) the first two. See MacBride (2008), pp. 161-4 for the third.

and a pair of pears, or trios of them, and recognise a pattern common to the pairs or trios; thereby, you acquire knowledge of the structures consisting in the cardinal numbers 2 and 4.

The next step is the process Shapiro calls 'projection':

> The subject mentally arranges the first few cardinality structures, say, and realizes that they themselves exhibit a pattern. Each such pattern seems to be extendable to a larger one, by adding a place. The subject then projects this pattern of patterns far beyond those hitherto encountered via simple pattern recognition. This yields knowledge of large finite structures, such as the cardinal-9,422 structure, and eventually knowledge of the natural number structure itself (ibid.).

Projection, then, is meant to allow the formation of, first, a singular thought about some cardinal number that pattern recognition has previously given us knowledge of: the belief that *it* is extendable 'by the addition of a next longest pattern' (MacBride op. cit., p. 159), that is, the belief that *it* has a successor; and, second, the formation of a general thought about cardinal numbers: that *they all* have a distinct successor. MacBride's main challenge to Shapiro here is that it is not clear how a mathematical novice is warranted in forming the second thought from having the first. His scepticism is actually more pervasive: already at the stage of pattern recognition he thinks that 'an illuminating philosophical description of the process remains to be given' (ibid.); but even granting that, he says, the problem with Shapiro's account of projection is that someone who formed the general thought 'all cardinal numbers have a successor' from having the singular thoughts '2 has a successor', '3 has a successor', …, would be *already* exhibiting 'knowledge of the principles which are employed by mathematicians to infer the general from the particular' (ibid.). Indeed, '[i]n the absence of any grasp of these principles, there can be no assurance that the features displayed by a given finite structure are representative of the features characteristic of the infinite structure of which it is an initial fragment' (ibid.). Some such principles are the Dedekind-Peano axioms of arithmetic: statements constituting knowledge of the infinite structure of the natural numbers. At the heart of the matter, then, seems to be the problem of knowledge of infinity. So the access problem MacBride raises for Shapiro is twofold: although it starts by asking how you know the structures denoted by the numerals '2' and '3' from seeing pairs and trios of things ('please offer an illuminating philosophical account of that'), its *punch* is to ask, even if that query is satisfied, how you know the *infinite* structure denoted by the numeral '$\aleph_0$' from knowing the finite structures denoted by '1', '2', '3', etc. As MacBride's and Shapiro's above meta-philosophical commitments suggest, Shapiro will be satisfied with his own answer to this whereas MacBride won't be. Shapiro's answer is basically to concede that the subject is not *warranted* in forming the general thought that all cardinals have a distinct successor, that is, that she doesn't have knowledge of infinity in the sense in which MacBride seems to understand knowledge (as grounded by something which doesn't presuppose it) and to say that, still, this needn't be a problem. Instead, Shapiro thinks, the subject might form such a general thought by 'just having a hunch' (Shapiro op. cit., p. 140), or perhaps on the basis of some 'innate knowledge' (ibid.). For him, of course, it doesn't matter. Shapiro is at any rate prepared to call the general thought whatever MacBride likes if he (MacBride) doesn't accept it as knowledge. 'The critic can call [it] a hypothesis (for what that is worth)' (ibid., p. 142). For Shapiro, the justification of this general thought does not come from grounding it on something else but from 'recogniz[ing] its role in our intellectual enterprise' (ibid.).

So here we reach an impasse. Indeed, even an account that might satisfy what I called the first, less pressing aspect of MacBride's access problem, the demand for an illuminating account of how we acquire knowledge of finite structures in the first place, would fail to satisfy his second worry. But let us pause and oblige him anyway. Such an account is provided by Marcus Giaquinto (2001, 2007, 2012). Giaquinto basically develops what Shapiro calls pattern recognition as a general method to acquire knowledge of universals, but he calls it, instead, 'abstraction'. Giaquinto takes up Russell's distinction between knowledge by description and knowledge by acquaintance, which terms he uses interchangeably with knowledge via theory and knowledge via experience, but amends Russell's characterisation of the two to claim that one can count as knowing some*thing* via experience not only when being presented with *it* directly, as when *it* is a concrete object, but also when being presented with an *instance of it*, as when *it* is a universal (2007, p. 215; 2012, p. 503). So one counts as knowing whiteness via experience from being presented with a white horse and with white sugar and from having then performed the process of abstraction, the 'mental elimination of irrelevant properties when thinking of a body [or plurality of bodies]' (2012, p. 501)[6] which yields direct awareness of just the salient property they share: in this case, the colour white. When it comes to universal structures, one counts as knowing one via experience if one has directly perceived two systems instantiating the structure (say, again, a pair of apples and a pair of pears) and if then one has performed the process of abstraction that yields direct awareness of the property common to the two systems—in this case, the structure they share.

[6] Let the term 'perform' here not mislead the reader. For Giaquinto (as for Russell), abstraction is not a mental *act* but a mental process as involuntary as sense perception itself (Giaquinto 2012, p. 501).

If we may linger on this a little: drawing on Giaquinto's detailed account of the notions of structure and of isomorphism—the phenomenon of two systems sharing a structure (2007, pp. 214-229)—, we might say that the process of abstraction achieves, in formal terms, the detection of an order-preserving correlation between two sets. A set here is just the collection of a system's elements, e.g. the two apples or the two pears. A *structured* set, more specifically, is a set considered under specific relations, functions or constants. The set of natural numbers, for example, is structured under the successor function and contains the constant zero. Two structured sets share their structure, or are isomorphic, whenever there is an order-preserving mapping from one structured set onto the other.[7] This yields an identity: if the condition is met, the structure of the first set *is* the structure of the other. The structure of the set of natural numbers considered under the successor function, for example, is identical with the structure of the set of von Neumann ordinals considered under the membership relation. Now those two systems are abstract, but of course concrete collections can form structured sets and be isomorphic too: the structure of the set consisting of a cell and two generations of cells formed from the initial one via mitosis considered under the 'x is a parent of y' relation can be identical with the structure of the set consisting in the nodes of a hand-drawn diagram considered under the 'there is a hand-drawn arrow from x to y' relation.[8] It is systems like these, concrete and observable, that allow for Giaquinto's proposed process of abstraction. When a subject detects an order-preserving correlation between the sets consisting in the elements of two systems she perceives considered under their respective relations, the subject is effectively *abstracting* ('extracting,

[7] For a formal definition of 'order-preserving correlation' please see Giaquinto (2007), pp. 215-16. The details are ignored for present purposes.
[8] Both examples are Giaquinto's (ibid).

extrapolating, teasing out'[9]) from those two concrete entities—those two collections—, a third: the abstract entity—the structure—instantiated by them.

Something like this, incidentally, is what authors in the literature on abstractionism in the philosophy of mathematics have suggested 'principles of abstraction' state: the identity between two *abstracta* given a certain condition. Borrowing Ebert and Rossberg's (2016, pp. 3-4) way of putting it: '[t]he general form of an abstraction principle can be symbolized like this:

$$\S\alpha = \S\beta \leftrightarrow \alpha \sim \beta$$

Where "§" is a [singular-]term-forming operator applicable to expressions of the type of α and β, and ~ is an equivalence relation on entities denoted by expressions of that type. [...] The *abstracta* denoted by the terms featuring in the identity statement on the left are taken to be introduced, in some sense, by the abstraction principle, giving the equivalence on the right-hand side conceptual priority over them' (emphasis original). Following this schema let me suggest, then, that when a subject performs Giaquinto's abstraction to gain knowledge of a structure, she conforms to something like the following principle:

Structure of set A is Structure of set B	if and only if	there is an order-preserving correlation between set A (under binary relation R) and set B (under binary relation S)

[9] These three phrasings are Salmon's (2018) way to introduce abstraction principles.

where the *abstracta* on the left-hand side, the structures, are 'introduced' in the content of the subject's mental state, as it were, whenever the subject recognises an order-preserving correlation between two systems that she's observed and which must hence be in the content of her mental state prior.[10] Putting abstraction in these terms shows why in Giaquinto's account only *finite* structures can be known via experience: because knowledge of a structure arises only from prior knowledge of the systems whose elements form the sets on the right-hand side of the principle, and perceptual knowledge of a system is only possible if it is finite—and finitely *small,* at that. As Giaquinto writes: 'most structures,

[10] Precisely because, as I've phrased it, this principle mirrors Gottlob Frege's (1884/1953, §64) original example (viz. the direction of line A is the direction of line B iff A and B are parallel), it is not meant as a definition of 'structure' but simply as the specification of the semantic contents of whole sentences where that term occurs. Now *that* is also called 'contextual' or 'implicit' definition in the literature, and is thought to be faulty *qua (reference-fixing) definition* due to its liability to the Julius Caesar problem. An analogue of it for us would be that even if our principle successfully specifies that the content of a sentence about structures, e.g. 'the structure of the set of natural numbers is Julius Caesar', comes down to 'there is an order-preserving correlation between the set of natural numbers and Julius Caesar', it (our principle) fails to determine whether *that* is true—'and, in failing to do so, it fails to determine the references of the terms for [structure]' (Dummett 1991, pp. 156-7). Here we won't be concerned with that problem because we're not interested in fixing the reference of abstract object terms or picking out their identity conditions but simply in making clear how an abstract entity's (a structure's, in this case) featuring in the content of a subject's state can arise from other two entities' (two sets', in this case) featuring in it, which is exactly what our principle, if I got it right, helps to specify. As Frege himself remarks of his own example: even though we don't get a 'demarcated concept of direction', 'we have in our definition the means to *recognize* this object [the direction of A] when it should occur in another guise as the direction of B' (ibid. §66, my emphasis). On a related note, though, perhaps it helps to say that just as one can define 'parallel lines' independently so as to avoid that Frege's principle be taken, contrary to his intention, as a definition—even a contextual definition—of 'parallel' in terms of 'direction' (cf. Salmon op. cit., pp. 1637-8), so too we find in Giaquinto an independent definition of 'order-preserving correlation' that allows us to translate his view on structure into a principle of this form. Although, again, I do not intend it as a definition, I *do* intend, like Frege, to put the priority (*which* contents determine or introduce *which*) on the right-hand side, which is why I make this second clarification.

even most finite structures, are too big and too complicated for visual cognition' (2007, p. 236). It is the same point Shapiro makes when he says of his label for the same process that '[a]t most, pattern recognition accounts for knowledge of small finite structures' (2011, p. 136). Now, Giaquinto and Shapiro offer different accounts of the way we *can* know a bigger structure from knowing two isomorphic sets. Shapiro's way, as mentioned, is projection, where we might not know but suspect or have a hunch or otherwise form beliefs about bigger structures from knowing smaller ones, which eventually leads to forming a belief about the infinite structure of the natural numbers. Shapiro mentions in this same vein Charles Parsons' (2007) account,[11] which we won't discuss (yet) but is akin to Shapiro's in that the step from knowledge of a finite cardinality to knowledge of the infinite is justified by a purported faculty of *intuition*—and I suspect that, in MacBride's book, Parsons' intuition wouldn't be any more secure than Shapiro's hunch. By contrast, Giaquinto's way would stick to the abstraction principle stated above and simply say that if the knowledge of the sets on the right-hand is not perceptual, then it will be theoretical, in which case we'll know the infinite structures introduced by those sets via theory or by description as well. Thus, we might know the infinite structure corresponding to the cardinal $\aleph_0$ by the definite description 'the structure common to all models of Σ', if we let Σ be a set of sentences constituting a version of the Dedekind-Peano axioms for arithmetic (Giaquinto 2007, pp. 215-6). Following Giaquinto's Russellian steps, then, we could say that if knowledge by description allows only *descriptive thoughts* about the object of knowledge, i.e. thoughts one can hold about an object in virtue of its satisfying a descriptive condition, whereas knowledge by acquaintance allows *singular thoughts*, i.e. thoughts

[11] See especially ch. 5-6, and for a more recent defence, Jeshion (2014).

one can hold about an object in virtue of being presented with it,[12,13] then we have that subjects may hold singular thoughts about small, finite structures only, because they may only know small finite structures by acquaintance, whereas knowledge of infinite structures, by contrast, is possible only via theory, and thoughts about them can only be descriptive. To use Russell's colourful way of putting the difference between what the two types of knowledge involve, this means that we cannot, unfortunately, 'bring [infinity] itself before the mind' (1911, p. 127).

1.4

I have suggested that at the heart of the access problem for (Shapiro's) realism is the problem of knowledge of infinity. The access problem demanded first an illuminating philosophical account of how we know small, finite structures; we appealed to Giaquinto for that. But the more pressing demand was to explain how we know infinite structures, in particular the structure denoted by the cardinal $\aleph_0$, which Shapiro says we count as knowing when we hold the general thought that all cardinals have a distinct successor. As we have seen, Shapiro doesn't satisfy this stronger demand with the secure warrant MacBride wants. Giaquinto's answer, which is that we know the infinite structure $\aleph_0$ via theory, also fails to satisfy MacBride, who by 'warrant' means a justification that does not presuppose mathematical knowledge—so, of course, no *theory*. Of the two epistemic methods proposed by Giaquinto for knowledge of structures, then, only acquaintance would have satisfied MacBride, but again, we saw

[12] For more on the distinction between singular and descriptive thought, see e.g. Davies (2017), §2. For more on the view that only acquaintance allows the former (at least as the source of a chain of testimony or memory), see e.g. (ibid), §3.1 and Jeshion (2010).

[13] Russell's (1911) original view that the objects of acquaintance are sense-data and universals (and possibly the self) is not discussed here.

that knowledge of structures by acquaintance only applies to finite, small structures. Hence, if there is one thing all three authors seem to agree on, it is that, as Shapiro says—quoting MacBride's channelling of Hume—, 'experience provides no corresponding impression from which the idea of the infinite may be derived' (Shapiro 2011, p. 142).

This brings us finally to the thesis of my thesis. I will argue that, contrary to what the philosophers above suppose, knowledge of infinity by acquaintance is indeed possible in Giaquinto's sense. Whether Shapiro would be very interested in this result, which his meta-philosophical commitments suggest he might not, does not detract from the benefit of satisfying, or getting closer to satisfying, MacBride's demand. But I will end up arguing for this conclusion by focusing on a particular puzzle—specifically, a particular puzzling case—concerning knowledge of infinity. Thus, in the rest of this thesis I will not deal with issues quite in the philosophy of mathematics but mostly to do with the acquisition of conceptual capacities and the role of perceptual experience in it.

I will start by presenting the puzzle. It consists in what appears to be a perceptual encounter with infinity. Then, we will consider some prima facie compelling ways to go about explaining the puzzle, which will all end up proving unsatisfactory but should help us illuminate it further. Then I will argue for a distinct account. Hopefully, towards the end, the connection of that account to the issues presented in this introductory chapter will become clearer. On the whole, though, the aim of the thesis is not to provide MacBride with the non-mathematical warrant his access problem demands but just to understand a little more about what are in themselves two fascinating facts: first, that we, finite creatures, do seem to have true, justified beliefs about the infinite, and second, that sometimes our senses tell us more —perhaps even about *that*— than has traditionally been assumed.

2. What Sarah didn't know

2.1

> When I first encountered infinity, I was a four-year-old in the Latter-day Saints temple in Atlanta, Georgia. My parents, my sister and I were [there for] a ceremony meant to unite us as a family forever. I didn't really understand what 'forever' meant. […] My sister and I knelt on the floor with our elbows on a pedestal, and my parents arranged themselves the same way across from us. Behind them was a mirror, and behind us was a mirror. The two surfaces reflected each other's images back and forth, creating infinite reproductions of our family together. I looked into the images and was amazed to see that they never ended; they just got smaller and smaller. At some point I could no longer discern the individual reflections, but I intuitively grasped that I was merely running into the limits of my vision. The reflections kept going and going. 'Oh,' I thought, with a chill of understanding. 'Forever.'

There are various things to learn from this story. The author, science writer Sarah Scoles (2016), cites it to illustrate her claim that '[s]tudents need to experience math —not just hear about it, as typically happens in the classroom— to understand it' (ibid.). The experience of the ceremony is supposed to have endowed young Sarah with some kind of mathematical understanding she couldn't have gained in the classroom or in her parent's car as they explained what 'forever' meant on the way to the temple: an understanding, in particular, of the infinite. '[N]o mathematical concept is more intense than infinity', she writes, '[w]hich makes infinity uniquely relevant […] for sharpening mathematical literacy […] more generally'. But the gains don't end there:

> Although it seems to be one of the most confounding things in mathematics, infinity can be a gateway drug to deeply personal mathematical experiences. It connects instantly to big, personal questions about life and death, power and control, the beginning of time and the end of the Universe (ibid.).

These claims on the epistemic significance of young Sarah's experience might strike one as going from the modest to the grandiose, but consider what seem to be some prima facie echoes of them in philosophy. On the centrality of the infinite for mathematics, for example, we find definitions of set theory, considered often to be the *foundation* of mathematics, as 'the science of the infinite'.[14] On the centrality of *knowledge* of the infinite for mathematical knowledge, on the other hand, we find views like Charles Parsons' according to which intuitive knowledge of the infinite can ground intuitive knowledge of truths analogous to the Dedekind-Peano axioms of arithmetic,[15, 16] the statements that Frege himself set out to derive from logic in his project of grounding all of mathematics on it. And on the connection of the infinite to *personal* questions about life and death, and time and the universe, we find in A. W. Moore's (2001) monograph on the subject the claim that our idea of the infinite arises contrastively from our awareness of our own finitude, which involves crucially awareness of both *death* and the passage of *time*. Because in thinking about death and time, moreover, we're thinking about *our* mortality and *our* temporality, 'in thinking about the infinite', Moore writes, 'we are thinking, at a very deep level, about ourselves' (p. xviii).

Take these broad brushstrokes on the importance of understanding infinity simply as a hint of the potential interest of Sarah's intuitions on the epistemic significance of her experience. Before any attempt is made to make sense of those intuitions, however, we'll have to focus on one basic feature of the story that *is* more straightforwardly striking and which, presumably, gives rise to the general interest of the phenomenon behind the experience altogether. I'm referring to Sarah's claim, at the very beginning of the passage,

---

[14] See Bagaria (2014).
[15] The phrasing of 'intuiting the infinite' here is found in Jeshion's (2017) recent defence of Parsons' view.
[16] At least the first four axioms are thought by Parsons to be grounded in intuition.

of having ‘encountered infinity’, where by ‘infinity’ she seems to refer to what she *saw* at the temple, and so, by ‘encountering’, to a perceptual experience. Let me elaborate on this a little bit. What we have in the passage is not, of course, a theoretical description of a phenomenon, let alone an explanation of it. What we have is a subject’s first-person report of an experience she underwent as a younger self: a description, that is, of what a phenomenon *seemed like* to the subject who experienced it. If it is right to understand Sarah’s claim of ‘encountering infinity’ as the claim that she had a perceptual experience of infinity, then, by saying that *that* is striking we don’t mean to say that it’s striking that she in fact had a perceptual experience of infinity—which would no doubt be very striking but which is a description of the phenomenon we shall for the moment neither endorse nor disallow. What we mean instead is just that it’s striking that it *seemed* to Sarah to have perceptually encountered infinity. That is striking in and of itself because, however ‘the infinite’ turns out to be construed later in this thesis (a mathematical object, concept, property, or whatever), if it is a mathematical entity at all, then it is prima facie, like all mathematical entities, abstract, so not something standardly thought of as perceptible or ‘encounterable’. What calls out for an answer, then, might be put as the question: what could the nature of the phenomenon that occurred that day in that temple be such that it afforded this striking phenomenology to its experiencer—such that it *seemed* to its experiencer to be an encounter with an abstract entity?

Consider, relatedly, that one of Moore’s aims in his monograph is to make sense of the notion that finite creatures like us can be ‘shown the infinite’. His aim in doing that, he says, is to ease a tension caused by, on the one hand, our ‘urge’ to acknowledge that there exists such a thing as the infinite and our impossibility, on the other, to actually encounter such a thing. But, crucially, Moore’s way of making sense of that notion is to simply explain that our seeming awareness of the infinite is a kind of insight belonging to the class of 'inexpressible states of knowledge', which are inexpressible precisely because what they are knowledge of does not actually correspond to a fact of the world.[17] Hence, even in what seems to be the closest effort in the literature to make sense of an epistemic state like the one Sarah is reporting, the existence of such an *encounterable* thing as the infinite is explicitly rejected.

This also explains why for Moore the relation a subject has with what she is 'shown' is not like the subject-object relation a perceiver has with something she is presented with. It is also clear, then, why Moore's 'showing' account could be of no use to our case anyway. As suggested above, Sarah's epistemic state involves what seems —*to her* at least— a straightforward case of a *perceptual encounter*. Since she's telling us, moreover, that this purported encounter gave her understanding of something she hadn't understood theoretically before, the report involves also what seems —*to her* at least— something like *perceptual knowledge*, or at least the epistemic upshot of that perceptual experience.

To rephrase, then, our puzzle is that what Sarah reports to have perceptually encountered, and thereby understood, infinity, is something standardly thought of as imperceptible because it is abstract—and, for that same reason, knowable only theoretically. Accordingly, the two tasks here will be to explore, first, the nature of Sarah's experience, and second, the nature of the understanding that experience gave her. Whether *this* connects to the larger issues she thinks it connects with might depend on our answers to those questions.

---

[17] See Moore (ibid.), xv-xvii and ch. 13.

2.2

The first thing to say about the concept of the infinite is that, in the variety of the concept we will be considering here, it has to do with numbers. As Moore tells us in his historical survey, although in ancient Greece there are instances of philosophers who thought of the infinite as an entity in its own right, a sort of substance,[18] ever since Aristotle's time the expression 'the infinite' has tended to be understood as something more like a predicate that we apply to other things. This predicate could be applied to single substances, if philosophers wanted to speak about things of infinite magnitude, or it could be applied to pluralities, to speak about collections of infinitely many things. This third variety of the concept seems to be the most pervasive today: it is the idea not of a substance or an infinite magnitude but of an infinite *number*. By 'number' here we refer to cardinal numbers, because those are the numbers we use to speak about the sizes of collections. So, henceforth, when we speak of the concept of 'the infinite' or 'infinity' we will mean the concept of an infinite cardinal number.

Leaving complications about the concept of number aside, if we understand numbers intuitively as properties displayed jointly by things in a collection, and so, number words as names for the cardinality of sets, then the concept of the infinite involves the notion of a set having no limit in size. Hence, writing at the dawn of the concept of infinite number, Aristotle says that 'generally the infinite is as follows: there is *always another and another* to be taken. And the thing taken will always be finite, but always different' (*Physics* 3.6, 206a27- 29, emphasis added). Note the resemblance with Sarah's phrasing that 'the reflections kept going and going'.

What the four-year-old hadn't understood before her experience at the temple but she did after it might have been what it is for a collection to be

[18] Paradigmatically, Anaximander.

unlimited in size, or in other words, for a cardinality to be infinite. Picture it: there she is, all dressed up in the back seat of her parents' car, listening to the words. 'Bla bla bla ceremony bla bla together as a family *forever'*. Seemingly a derivation of the concept of infinity, 'forever' is just the notion of time having no limit. There is no reason to suppose young Sarah wasn't linguistically competent enough to parse the meaning of her parents' words correctly: by functional application, her mind goes, apply negation to the thought of something reaching a limit. But the result of that matches no concept in Sarah's four-year-old repertoire. Consider Chomsky's concepts 'colourless green idea' or 'furious sleep'. Picture your own parents explaining to you on the way somewhere: 'bla bla ceremony bla bla *colourless green idea*'. What on Earth could that be? Or as we say in Spanish: how do you eat it? 'Oh', your parents might have been wise to say, smiling, 'you'll understand when you see it'.

And understand by seeing Sarah did. Or so she claims. If the concept of infinity she grasped is, as suggested above, the idea of an infinite cardinal number, and cardinal number terms name the cardinality of sets, then Sarah's understanding put her in the position or *disposed* her, I suggest, to think of the concept of the infinite as a possible constituent of an answer to the question 'how many?'. Let me put this in a different way. If cardinal number terms name cardinalities of sets, then the possession of a given concept number, e.g. the concept of the number 4, contributes to having the disposition to answer 'four' when confronted with a question such as 'how many Beatles are there?'. Similarly, the possession of the concept of an infinite number should put you in the position to answer 'infinitely many' when confronted with a question such as 'how many natural numbers are there?'. It is of course required for answering these two questions not only the possession of the relevant number concepts but also the possession of the relevant popular and mathematical knowledge, respectively—knowing, that is, whether a set exemplifies that cardinality, and so whether it falls under that concept. If you have the number concept '4' but are not acquainted with any set of things in the world that displays that cardinality, then you won't be able to answer the question about the Beatles. You may have heard of them but ignore that they're a four-member band. So my suggestion that Sarah's epistemic gain consisted, in part at least, in the disposition to use the concept of infinity in response to a question about cardinality requires that her epistemic gain involve *both* (i) the acquisition of the concept of an infinite number *and* (ii) acquaintance with something that she could apply that concept to.[19] Whatever else Sarah's experience may have taught her, I'd like to focus in the rest of this chapter on this suggestion: the suggestion that Sarah's experience disposed her to give, if confronted with the question 'how many reflections are there?', the answer 'infinitely many'.

2.3

Part of I've suggested just now is a claim about concept possession. I suggested that Sarah did not possess the concept of infinity before and that the temple experience fixed it for her, and I took it for granted that Sarah's 'chill of understanding' proved this claim. However, one might have some reservations. Recall Frank Jackson's (1986) neuroscientist Mary, who knew everything there is to know about the theory of colour vision but did not know what seeing red

---

[19] This might seem to imply that Sarah gained acquaintance with something that *does* in fact exemplify the concept of infinity, but earlier we said this is something we won't either assert or reject yet. Still, because the fixing of the concept did not occur when she was explained it theoretically in the car but when she underwent the relevant perceptual experience, it seems that during that experience she gained acquaintance with something that at least *seemed* to her to exemplify that concept, and to which, in fact, she *did* apply it. From the latter follows, I suggest, that what she saw *could* be applied the concept. Why or how is part of what we'll explore here.

was like. One way of interpreting Jackson's thought experiment has been to say that whereas Mary didn't strictly speaking learn any new facts upon leaving her black-and-white room, what she learned was a new way of understanding the facts she already knew. Thus, Brian Loar (1990) makes a distinction between possessing a scientific or theoretical concept for a given subject matter, in this case an experience, and possessing a phenomenal concept for it. By understanding the experience of seeing red in the descriptive mode —by understanding, say, a complete scientific description of the brain state that realises that experience—, Mary counted as having the theoretical concept of 'seeing red' but as lacking the phenomenal concept *for the same thing* before she left the room. Afterwards, understanding the experience of seeing red in the subjective mode by actually undergoing it fixed in Mary the phenomenal concept as well. Now, although phenomenal concepts refer to mental entities —to experiences or to their qualitative properties, or 'qualia'—, one might argue that in undergoing the experience of seeing red Mary must have also gained a new concept for the property she saw itself. Hence, following David Papineau's (2002, 2009) distinction, we might say that in addition to having a theoretical concept for redness —in addition to, say, knowing which reflectance properties give rise to which brain state when light bounces from a tomato's surface to a subject's eyes—, Mary gained the perceptual concept for that colour property when she left her room.

So, while Mary may have similarly reported a 'chill of understanding' upon seeing red for the first time, we know by stipulation that she *already* understood what both seeing red and redness itself are even though she didn't have the corresponding phenomenal and perceptual concepts. Her understanding, then, did not consist in acquiring an entirely new concept but just the phenomenal and perceptual counterparts to (theoretical) concepts she already had. Why, then, this line of argument would go, can't pre-temple-experience Sarah also be said to already understand what 'forever' or 'infinity' is even though she hasn't had an experience that corresponds to *that*? In other words: couldn't her post-experience 'oh!' express the thrill of gaining a new *way* of understanding something rather than the thrill of accessing something entirely unknown to her before?

It is a good question. One cannot perhaps, after all, entirely trust the report of a four-year-old. And perhaps it is in fact counterintuitive that a linguistically competent subject would have the concepts of 'end' or 'limit' and the concept of negation but fail to grasp the concept of something not reaching an end or a limit. To tackle this worry, a word on conditions for concept possession generally and one on the concept of infinity in particular are in order.

2.3.1

Concept possession is a complex topic. For present purposes, let us just assume a couple of conditions often viewed as linked to it. First: if we take concepts to be sub-propositional mental representations, and if we take propositions to be the objects of attitudes we hold when we represent the world, then possessing concepts is a matter of certain representational capacities. Then, in virtue of allowing us to represent things in certain ways, concepts allow us to undertake certain epistemic tasks. One such task is sorting things in the world into categories. When a subject is capable of sorting things into triangular and non-triangular ones, for example, she meets one condition for possessing the concept 'triangle'.[20] Although this means crucially that the subject is capable of perceiving triangles as triangles, it needn't be the case that the latter epistemic-perceptual capacity explains her sorting behaviour; rather, the subject sees

[20] This first condition, call it sorting, is discussed by e.g. Fodor (1990, 1994), Peacocke (1992), Weiskopf and Bechtel (2004), and Ginsborg (2006).

things as triangular partly *in virtue of* having the capacity to sort them in a way sensitive to the property that concept picks out. But, of course, the subject might successfully sort together triangular things not while seeing them as triangular but as falling under the distinct, co-extensional concept 'three-sided'. So additionally to the capacity to get the extension right, in order to count as possessing 'triangle' the subject would have to be disposed to infer that those things have angles rather than, say, being disposed to infer that they have closed sides, which would be a mark, instead, of possession of the concept 'three-sided'. Possessing distinct concepts, then, comes with the ability to make inferences that are allowed by the concepts' distinct 'constitutive structures': the concept 'angle' is structurally constitutive of 'triangle', and the concepts 'three' and 'side' of 'three-sided'.[21,22]

This second feature of concepts, call it inference licensing, is of particular interest with regards to concepts which might not be analysable in terms of sorting because they're non-empirical but which we possess just in virtue of being disposed to use them appropriately in our rational lives. Take the non-empirical concept 'and'. Jerry Fodor writes: 'a sufficient condition for a speaker's meaning *and* by 'and' [is] that, ceteris paribus, he takes 'P and Q' to be true iff he takes 'P' to be true and 'Q' to be true' (1990, p. 111). So one understands the concept 'and' simply if one makes inferences that conform, as Fodor puts it, to its rules of introduction and elimination, regardless of whether one can explicitly state these rules.

Mathematical concepts are arguably like logical concepts in this respect. Regardless of whether mathematics itself can indeed be reduced to logic (and/or set theory), both mathematical and logical concepts certainly seem to

[21] The example is Weiskopf and Bechtel's (ibid.), p. 52.

[22] This second condition is discussed by the same authors cited two footnotes above except for Ginsborg, who focuses just on sorting because she deals just with empirical concepts.

be, at least in part, non-empirical. Possession of them is displayed when one's reasoning conforms to the rules of inference that govern them. You don't understand the concept '4', for example, if you cannot infer from 'there are three Beatles dead and only one still alive' that there are in total four Beatles. But you count as possessing the concept if you can make that inference even if you can't state the axioms of arithmetic that allow it. When it comes to *geometrical* mathematical concepts such as 'triangle', then, perhaps one counts as understanding the concept partially if one can sort objects exemplifying them but cannot make the inferences they allow.

At any rate, the inference licensing condition seems to be closely related to a topic we touched on briefly above: linguistic competence. It seems natural to think that if one is disposed to use a concept appropriately in reasoning and one knows a term that refers to it, one should be able to use this term appropriately in expressing one's reasoning via speech. This is because, assuming a standard compositional view of semantics, concepts are constituents of propositions or thoughts just as terms for them are constituents of written or spoken sentences. Thus, it seems natural to think that if a subject displays competence in the use of a term, she masters the concept it expresses.[23]

But consider this toy scenario. Suppose a child is presented with sets of triangular and square objects and she is asked which of those have three sides. Suppose she correctly points at triangular objects only. Then you, the adult, pick up one of the triangular objects and ask her to count the sides. The child correctly obliges. But then you ask her to say which objects have three angles, and suppose that now she hesitates. Maybe she'll take a guess and point at the triangular objects again because she'll suppose the 'three' in both questions offers a clue. 'So these have three angles?', you ask to confirm. 'Yes, those have

[23] I equate mastering or understanding a concept with (fully) possessing it here, and distinguish that from the ability to define it or state its associated rules of inference.

three angles', the child replies. And now you ask her to count the angles. It is not clear, I submit, that she'd do this right. After all, 'angle' is a less basic concept than 'side': one way of defining the concept of an angle is by appealing to the initial and the terminal sides that constitute the rays whose intersection forms it. Perhaps the point is more compelling if you imagine asking the child to count the sides of a table and then asking her to count its angles. In short: the child's correct use of the term 'angle' in answering 'yes, those have three angles' above needn't show her mastery of the concept 'angle'.

So correctly using a term might not in fact show mastery of the concept it expresses. To make this point more precisely: even *semantic* competence, which for simplicity I'll assume constitutes linguistic competence together with knowledge of rules of syntax, doesn't entail *conceptual* competence. And this might be related to the fact that the meaning of a term and the concept it expresses are distinct things.[24] Delving into this would take us too far afield, but to take a recent example from the literature: people can surely be said to have possessed the *concept* 'whale' before scientists discovered whales are not fish even if only now do dictionaries define the *meaning* of the term expressing that concept by reference to the property of being a mammal rather than a type of fish.[25] Conversely, an alien who reads a pictureless 21st-century dictionary may be said to learn this meaning but not to possess the concept, because unless she's acquainted with those creatures it's not clear that she'd meet the sorting and inference conditions described above. The point then is that one can correctly use a term, and so thereby deploy its meaning, without possessing the concept it expresses. James Higginbotham (1998) puts this by saying that whereas semantic competence is 'the state of mind that is attained when one

[24] This distinction has roots of course in Frege's distinction between sense and reference, but for a recent articulation see e.g. Sawyer (forthcoming). For the related distinction between conceptual and semantic competence, see e.g. Higginbotham (1998).

[25] The example is Sawyer's (forthcoming).

knows the meaning of one's own words', conceptual competence is 'the state of mind of one who knows the *nature* of his own concepts' (p. 150, italics mine). This is consistent with the compositional view of semantics assumed above: consider a subject that understands the concept 'end' and the logical concept of negation and knows the meaning of terms expressing them, and who is *now* introduced to the concept 'forever' in those terms. 'For our family to be together forever', this four-year-old subject's parents explain, 'is exactly for the duration of our family's union not to have an end'. Presumably, the subject understands the *meaning* of this. And she might be able to use the term in other sentences: 'if this lane doesn't have an end', she asks shyly of the motorway leading to the temple, 'we would drive on forever?' 'Yes!', the parents reply excitedly, believing the four-year-old has grasped the concept.

Well, I want to insist she hasn't. Even if pre-temple-experience Sarah may have begun to understand the *meaning* of the term in the car, she still hasn't grasped, to use Higginbotham's phrase, the *nature* of infinity. It is not clear she would grasp, for instance, that for a collection to be infinite in size is for every member to have a successor, or that there are as many even and odd numbers as there are whole numbers even if there are infinitely many of all three types. And yet, she seems to meet something like the inference enabling condition for concept possession discussed above—she has just explicitly drawn an inference involving what it would *mean* for the lane not to have an end, i.e. that it would be infinite. So what more do we need?

2.3.2

Consider now a different toy scenario. You're a first-year philosophy student and want to impress your friends, so you enrol in an advanced Kant seminar where the lecturer speaks right from the start of one of Kant's maximally obscurely named notions. She calls it 'schematism'.[26] She says disconcerting things like: 'one way to see categorial schemata is as pure intuitions', and 'it isn't clear Kant establishes the transcendental schematism of the judgment', and 'the schematism is meant to bring together categories and appearances'. Naturally, after the first lecture you'll barely know how to even spell the word. After a few, you may start to tell your friends a thing or two about what you're studying: something to do with something like mental processes. Over time, your competence in the use of the term will improve: you'll ask a cogent question in class; you'll correct a classmate who says the schematism is concerned with empirical concepts; you'll answer 'schematism!' to the pub quiz question: what did Jacobi praise as 'the most wonderful and most mysterious of all unfathomable mysteries and wonders'? At some point, finally, you might start to feel confident not just in your use of the term but also in your understanding of the *nature* of the concept, even if that will probably take years.

What this change in confidence reflects is a change in semantic and conceptual competence. Those changes, in turn, reflect a change in your meeting what appears to be a third condition for concept possession, or perhaps a condition underlying the other two: a subject's being aware that her way of sorting and reasoning —her way of conceiving— is *appropriate* to what she is sorting and reasoning about. Hannah Ginsborg (2006) puts this roughly by saying that acquiring a concept comes with a sense of normativity. This might seem to presuppose possession of the acquired concept if *what it is* for a subject to take herself to conceive appropriately were for her to take her way of conceiving to latch on to the truth, which would be to say that, for example, a child tasked with sorting triangular from square objects as part of the very process of learning the concept 'triangle' *already* thinks that those things are

[26] I owe this example to my supervisor, Mark Kalderon.

triangles and ought to be perceived as such. But the child needn't think this, and this condition does not presuppose that. It suffices that the subject 'take it that she ought to perceive [the object] *this* way, […] where her taking it that she ought to perceive the object in this way does not depend on any prior appreciation—implicit or explicit—of how it ought to be perceived' (ibid.). This sense of normativity, then, doesn't involve a judgement about what the subject is representing —not a judgement that the Fs are in fact Fs and ought to be perceived as Fs— but is a species of self-awareness, something more like the demonstrative judgement '*this* way of perceiving is appropriate'. Ginsborg calls this prior normativity underlying the one at play when the subject does *now* take it that the thing falls under a certain concept and ought to be perceived thus 'primitive normativity'. Call the latter 'conceptual normativity'. Note that although awareness of primitive normativity is theoretically —and perhaps chronologically— prior to conceptual normativity, both are present once the subject fully possesses the concept. At that point, she can make *both* normative judgements 'this way of conceiving is appropriate' and 'those Fs are Fs and ought to be conceived as such'.

Return to our cases now. As Sarah begins to understand the meaning of 'infinity' in her parents' car, I suggest, she's like the philosophy student after the first few Kant lectures. Although she might indeed be able to deploy the meaning of the term in a sentence that expresses an inference ('if this lane doesn't have an end, we'll drive on forever?'), this would be due only to her progress in semantic competence. Her lack of confidence, however, would show she still lacks the primitive normativity that even the child who doesn't have the concept 'triangle' *does* display awareness of when she's confident that these things should be sorted apart from those when she's in the process of acquiring the concept. Because Sarah still doesn't understand the nature of infinity, further, she is also unaware of the conceptual normativity that would allow her to know for sure, rather than guess, what it would be for a cardinality to be infinite in size. Because she hasn't reached conceptual competence, in other words, she still doesn't have the *conceivability* capacity that understanding the concept of infinity would give her: the capacity, that is, 'that enables us to represent scenarios to ourselves using words or concepts or sensory images, scenarios that purport to involve actual or non-actual things in actual or non-actual configurations' (Gendler & Hawthorne 2002, p. 1). The representational aspect of the conceivability capacity is key: 'what would it be —what would it *look like*—', Sarah might wonder, 'for a size —for the length of our time together— not to have a limit?'

I'm arguing then that infinity was in fact an entirely unknown concept for the four-year-old. Against the line of argument sketched earlier on, I submit that she did not have any variety of the concept prior to the temple experience. If there is an analogy between our case and Jackson's Mary case, then, it may not be that experience fixed the phenomenal counterpart of a theoretical concept the subject already had, as suggested by Loar, but that experience fixed a new capacity or ability in the subject, as suggested by David Lewis (1990): the capacity to 'visualize red'[27] in Mary's case, and the capacity to conceive of cardinalities of infinite size, in Sarah's.[28] And as suggested by our discussion of concept possession above, this capacity may be tested for in view of the discussed conditions. Because the concept of infinite cardinality is a mathematical concept, we said, understanding it involves importantly the

[27] The phrasing is in Lewis' quoting of Laurence Nemirow.

[28] These two alternatives map Lewis' distinction between what he calls the 'phenomenal information hypothesis' and his own 'ability hypothesis' as alternative ways to make sense of Jackson's thought experiment. Strictly speaking, our discussion remains neutral on that debate, since Sarah's acquiring abilities needn't exclude her acquiring also phenomenal and/or perceptual concepts. As will be clear soon, however, we will in fact endorse a sort of ability hypothesis in making sense of our case, since the conceivability capacity posited will involve both the inferential capacity discussed here and a sort of *recognitional* capacity. Wait for the next section.

ability to make certain inferences that the concept allows. If a child can't make those inferences, she doesn't have the concept. And here I think empirical research on children's understanding of infinity agrees.

A Finnish study by Pehkonen et al (2006) shows that one can distinguish between three levels of children's such understanding:

> The lowest level is when they do not understand infinity, but use only finite numbers. In the intermediate level, the students understand potential infinity, and use processes that have no end. Those students who have reached the third level are able to conceptualise actual infinity and the final resultant state of the infinite process (Pehkonen et al 2006, p. 347).

Disregard for now the potential/actual distinction. By actual infinity, Pehkonen et al seem to mean what we've introduced before as the concept of infinite magnitude: 'a realised "thing"' (p. 345) or 'the final state of the infinite process' (p. 347), whereas by potential infinity they seem to mean what we've introduced as the concept of an infinite cardinal number: 'the [...] infinite process of counting *more and more* numbers' (p. 345, my italics) or an 'unending number' (p. 348). In their study, they measured these levels of understanding among 11-12 and 13-14 year-olds, or 5th and 7th graders. The results were that only 20% of 5th graders had an understanding of the infinite cardinality of the set of natural numbers. And '[t]he situation is not much better in the seventh grade' (p. 350). Note additionally that these subjects are significantly older than the subject in our case; they are pre- and adolescents—in a country famous for its educational system, at that.

A more in-depth study by British professor of mathematical thinking David Tall (2001) offers an interesting reflection on his own son's understanding of infinity, who first showed an inkling of the concept at age seven (p. 7). After some light instruction by his father, the boy 'returned with a new view of infinity as a single large entity that is bigger than anything else and has no bigger number' (p. 8). Tall's earlier research (1980) agrees with Pehkonen et al that adolescents' first understanding of infinity involves 'infinite processes' (p. 12), which intuitions, however, he found to 'clash with the introduction of infinite cardinals' (ibid.). The concept of an infinite iteration, then, seemed less hard to grasp than the concept of an infinite fixed size. After more drilling, however, Tall's son managed to overcome this clash and understand precisely that: the concept of an infinite cardinality. Tall elicits this knowledge through the following dialectic, having introduced his son to the name for the first infinite cardinal (p. 17):

> "How many whole numbers are there?"
> "Aleph."
> "Aleph. That's right! Well, how many even numbers are there then?"
> "Aleph?"
> "… and how many odd numbers are there?"
> "Aleph."

As the intermediate hesitance here (and elsewhere in the study) shows, the boy's conceptual competence is clearly in the process of growing. In the early stages, his understanding of infinity grounded a view of arithmetic 'whereby "infinity plus infinity is two infinity"' (p. 18), which view he updated once his grasp of infinity was firmer to infer *now* 'the conflicting idea that "aleph plus aleph is aleph"' (ibid.):

> "I don't believe infinity plus one is bigger than infinity any more."
> "What is it then?" I asked.
> "Infinity," he replied. "I've been talking with my pals and we all think that you can't have bigger than infinity."

To be sure, the last remark suggests that although the seven-year-old had come to grasp the concept of infinity sufficiently enough to judge that $\aleph_0 + \aleph_0 = \aleph_0$, he hadn't yet grasped the notion that there are indeed some infinities bigger than others—he might have not been able to judge, for instance, that $\aleph_1$

$> \aleph_0$. But the latter judgement would be licensed plausibly by possession of a finer concept than just the concept of infinite cardinality or infinite number, which *did* license the former judgement once it (the concept of infinite number) was fixed in the boy's repertoire by the dialectic reported in the study. To sum up: as both empirical research and Sarah's own account suggest, the four-year-old was too young to have the concept of infinity before her experience of the ceremony at the temple. What she may have indeed understood prior to the experience, if at all, was just the meaning of the term, which her four-year-old semantic resources —knowledge of terms expressing the notion of 'limit' or 'end' and the logical concept of negation— did allow her to grasp. This, nevertheless, was insufficient for Sarah to have the ability to represent to herself —to *conceive* of— scenarios involving sets of infinite size: the set, for example, of all points in time where her family would be together. Thus, she could not have met the required sorting or inference-licensing conditions discussed above, the latter of which Tall's seven-year-old son did come to meet. At risk of being repetitive: towards the end of the study, Tall's son wasn't just able to deploy the meaning of the term —he wasn't just able to use his knowledge of 'limit' and negation— but made correct inferences involving the *nature* of the concept of infinite number, such as the judgement: 'aleph plus aleph is aleph'. Still, to be sure, saying that Sarah —or Tall's son, for that matter— didn't meet the sorting condition is tricky. How can *anyone* meet the sorting condition anyway for what seems a non-empirical concept?

This is of course part of our puzzle. To recap: my suggestion in the previous section was that the epistemic gain of Sarah's experience consisted, in part at least, in the disposition to use the concept of infinity in response to a question about cardinality. This, in turn, involved *both* (i) the acquisition of the concept of an infinite number *and* (ii) acquaintance with something that she could apply that concept to. Point (i) seems to be what Tall's son displays when answering correctly the question 'how many even/odd numbers are there?', hence, what proves his meeting the inference-licensing condition. And point (ii) is what *might* prove the sorting condition. Although this, again, is strange, it does seem to be what Sarah is reporting. She seems to be reporting having acquired the ability to perceive a set of infinite members as distinct in kind from a set of finite members. With this came, also mysteriously, the confidence that Ginsborg's proposed double awareness of normativity affords a new concept possessor: '"*Oh*," I thought, with a chill of understanding. "Forever."' But just like a child can acquire the ability to perceive triangles as triangles precisely *by* being presented with triangles and then being tasked to sort them apart from squares, the question of how a perceptual experience seems to have endowed Sarah with the concept of infinite number might lie in the answer to the question of what exactly she was presented with in that foundational encounter such that she *thereby* learned, at that very moment, to perceive what she saw as infinite in size.

Thus we return to the puzzle as we'd phrased it last time we touched base with it. How exactly did Sarah's experience dispose her to give, if confronted with the question 'how many reflections are there?', the answer 'infinitely many'? We'd better start thinking about the concept-fixing properties of perceptual experience, then.

2.4

To say that, after the experience but not before, Sarah was in the position to answer a question about cardinality —because she'd acquired both a cardinality-relevant concept and knowledge of something that she could apply the concept to—, to say that is to say that during the experience Sarah acquired a new epistemic capacity —specifically, a new conceptual capacity, i.e. a new disposition. Following her description of the events, we conjectured earlier that the relation between the fixing of this capacity and her perceptual experience was not just a correlation or a co-occurrence but a causal relation: somehow, it was the experience what *endowed* her with this capacity.

The thought that experience endows us with epistemic capacities theory can't endow us with is found, for example, in David Lewis (ibid.). Although Lewis speaks of various types of capacities, viz. imagining, remembering and recognising, the one relevant here seems to be a kind of recognitional capacity, which I'll paraphrase for present purposes as the capacity to recognise instances of concepts when one is presented with them in experience. In Lewis' example, the concept in question is the one corresponding to the Australian version of Marmite: '[the] abilities to remember and imagine and recognize [Vegemite] are abilities you cannot gain (unless by super-neurosurgery, or by magic) except by tasting Vegemite and learning what it's like. You can't get them by taking lessons on the physics or the parapsychology of the experience, or even by taking comprehensive lessons that cover the whole of physics and parapsychology' (p. 18). To be sure, Lewis is focusing here on the capacity to recognise a new instance of the *experience* of Vegemite-tasting rather than on the capacity to recognise new instances of the stuff itself; however, Lewis' story seems to imply that the latter, first-order capacity is also fixed when the former is: '[s]ome know how to recognize a C-38 locomotive by sight, others don't. If you don't, it won't much help if you memorize a detailed geometrical description of its shape' (p. 19). The experiences of tasting Vegemite and of seeing C-38 locomotives, then, don't only teach us what those two experiences are like but also contribute the capacity to recognise instances of those two concepts.[29] Only experience can do this because recognitional capacities are not a matter of possessing

[29] Because both are types of knowledge-how, they might require degrees of practice, such that one such experience might not suffice to fix the capacity. This seems irrelevant for the main discussion, though.

'information', by which Lewis means 'scientific' or theoretical information, which in turn means it is not knowledge-that but knowledge-how. As Lewis says: '[l]essons impart information; ability is something else' (ibid., p. 18).

Notice, however, that in Lewis' story, subjects needn't have grasped by experience the concepts they do learn to recognise instances of experientially. Indeed, Lewis himself starts the article by competently using the concept 'Vegemite' and telling us that, because he doesn't know what it tastes like, he wouldn't be able to recognise a sample. This only shows that, of course, the capacity to recognise Fs is not equivalent with grasping the concept F. We understand the concept 'unmarried' even if —jokes aside— we can't recognise unmarried people by sight. For some concepts, however, our understanding of them *does* involve the ability to recognise their instances, or as Steve Yablo puts it, the 'ability to work out [their] extension in *perceptually* (as opposed to intellectually) presented scenarios' (2002, p. 461). These are what Yablo calls, following other philosophers, 'observational concepts'.

Yablo's example is the concept 'oval'. [30] Just like Lewis thinks recognising C-38-locomotive instances is not something you could achieve by reading their geometrical description but only by looking at them, Yablo thinks that 'what marks a figure as oval is not its satisfaction of some objective geometric condition, but the fact that when you look at it, it *looks* egg-shaped' (ibid., p. 465, emphasis added). However, whereas you can grasp what C-38-locomotiveness is without being able to recognise one by sight, if you don't know how to recognise oval things, then you don't understand ovalness.

Our grasp of observational concepts, then, is at least partly constituted by how their instances appear to us in experience (ibid.). If Lewis is right that

[30] Chalmers (2002, p. 190) voices a reasonable doubt about Yablo's choice of 'oval' as an example, but perhaps it helps to point out that Yablo thinks 'oval' is not a pure geometric concept, like 'triangle' is: 'Why are the oval things picked out experientially? There is no in-principle reason, but only a practical one: we have no other way [so no intellectual way] of roping in the intended shapes' (p. 466).

recognising Fs is not just a matter of having information or knowledge-that about Fs but requires instead some other kind of epistemic relation with Fs, a relation established in experience, and if recognising observational-Fs is necessary for grasping an observational-concept F, then there are cases of concept learning, for example the case of observational concepts, in which experience provides *both* the concept-fixing and the fixing of the instance-recognition capacity.

Here's how this relates to our discussion. I have suggested that Sarah's experience endowed her with an understanding her parents' explanation didn't. This understanding seemed to involve the capacity to answer a cardinality question using the concept of infinity. Following Lewis and Yablo, it would appear then that Sarah's new capacity would amount to her grasping a concept and her acquiring the ability to recognise its instances, but because both things were fixed by experience, that would appear to end up categorising 'infinity' as something like an observational concept. Now that is unlikely. Presumably, mathematicians count as grasping the concept even though they learn it from set *theory* textbooks. And not only that: prima facie, it doesn't even make sense to speak of infinity's instances being 'experienced' because, as mentioned earlier, mathematical entities are largely viewed as non-experientiable. By those standards, moreover, infinity is not alone: no number, infinite or finite, can be experienced, and so no recognitional capacity can constitute, even partly, our grasp of their concepts.

Yet, if we stick to Sarah's report, it does seem that she acquired something like the capacity to recognise instances of infinity in the temple, and *that* seems to be part of her having grasped the concept altogether. Indeed, as hinted at earlier on, Sarah seems to have somehow met the sorting condition. So here are two puzzles this raises. First: is there any way to make sense of the idea of our being able to work out the extension of a mathematical concept in perceptually presented scenarios? Secondly: what exactly could an object of perception be like such that it allows the application of a mathematical concept by its perceiver?

2.5

Consider again the concept '4'. We suggested earlier that part of what it is to have that concept is to be able to use it in answering questions about cardinality. Because cardinality questions involve mathematical concepts, the knowledge those questions involve is, presumably, theoretical, or in Lewis' terms, knowledge-that. So, for example, grasping the fact that the set $\{x \in \mathbb{N} : x \leq 3\}$ has a cardinality 4 is to pick out an instance of the concept '4', and that is, to use Yablo's phrase, 'an intellectually presented scenario'. Concepts of which the extension is determinable perceptually, however, must be, if not observational, at least observable: even if a concept's grasp does not require knowledge of how its instances appear in experience, as is the case of the concept 'oval', for its extension to be perceptually determinable the concept must at least be accessible by experience, as is the case of the concept 'Vegemite'. The concepts Lewis argued experience could teach us to recognise instances of, then, must be observable concepts. His other examples (locomotives, skunks, the colour green) also qualify. And, surprisingly, there is a sense in which certain number-concepts seem to be observable in this way.

We're talking about *subitizable* numbers. As James Davies (2017) explains, 'subitizing' is the 'immediate perceptual recognition of the cardinality of a collection of visual or auditory objects' (ibid). The thought is that, when answering questions about cardinality, one can either count the elements of a set, which is a cognitive process, or one can simply *see* how many elements the set has. Although there is a debate on whether there really is a difference in kind between these two processes (one being cognitive and one perceptual), or whether the difference is in degree (both being cognitive)[31], even theories suggesting the latter view leave room for counting and subitizing being *distinct* processes, since the support for joining them together seems to rely on their both being attention-demanding processes, and counting requires *additional* attention-dependent processes that subitizing doesn't.[32] At any rate, we seem allowed to say that subitizing is not merely fast counting. And the phenomenology of each process seems to agree. Consider rolling a dice that lands on its '3' side, and then a pair of dice that land on their '4' and '6' sides. In the first case, but not the second, knowing which number-concept is exemplified by the set of dots you see is as immediate as knowing which shape-concept is exemplified by the dice. Both seeing *that* the dots jointly exemplify the concept '3' and seeing *that* the dice exemplifies the concept 'cube' are forms of *conceptual seeing*, because they're both cases of applying a concept to what is perceived, but that presupposes that they're both also cases of non-conceptual seeing, if we follow Fred Dretske's (1969) —now standard— view that conceptually seeing something implies non-conceptually seeing it. A line of thought like this, then, leads Tyler Burge to claim that '[subitizing] is a form of non-conceptual relation to the cardinal numbers'.[33,34]

In Yablo's terms, this means subitizing is the process of perceptually recognising an instance of a number concept. Although there is room to think additionally that it plays a role in children's process of grasping number-concepts in the first place,[35] it is certainly not the case that experiencing instances of the concept '4' is necessary for its possession. So what we have here

[31] See Vetter (2009) for an example of the latter, and Railo et al (2008) and Piazza et al (2011) for examples of the former.
[32] Railo and Hannula-Sormunen (2012), p. 3234.
[33] The reference here is again to Davies (2017).
[34] A similar view is found in Giaquinto (2001), who defends that cardinalities are sensible properties 'of sets, concept extensions, collections, pluralities', etc., and that the smallest ones are graspable non-inferentially.
[35] See e.g. Railo and Hannula-Sormunen (ibid).

is a case of some mathematical concepts being, if not observational, at least observable.

Could something like subitizing help to explain the epistemic upshot of Sarah's temple experience? Could she, in other words, have learned to recognise infinity by seeing an instance of it in the way Lewis would have learned to recognise Vegemite if he had decided to try some?[36]

Sadly, the answer is no. First of all, it is fairly well established that the subitizing range only reaches up to 4—and, given certain attentional constraints, 2.[37] Although the characterisation I quoted above conveniently omits that range to focus on the recognitional aspect of subitizing, others take it to be part of the very definition.[38] Still, because another part of the definition seems to be the contrast with counting, and the two are jointly exhaustive alternatives for grasping a presented cardinality, one might nevertheless think that perhaps infinity is an exception to the subitizing range as standardly conceived, since our conjecture was that Sarah had acquired a capacity to answer a cardinality question upon undergoing the perceptual experience that she did, which means she acquired it on the basis of a perceptual rather than a cognitive process. She did not have to count the reflections serially, and fail to reach an end, to form the belief that what she saw was a set of infinite size. So, following this line of thought, couldn't Sarah's experience be a case of subitizing an infinite cardinal number?

Here is the second, decisive reason why not. When a subject subitizes a number from the 1-4 range, *if* they are cases of non-conceptual relations to cardinal numbers at all, what the subject is non-conceptually related to is a property displayed jointly by the members of the set she sees. A perceptual

[36] He didn't, in order not to 'spoil a good example'.
[37] Railo et al (2008).
[38] '[Subitizing] refers to a fast and highly accurate, effortless process by which a small number of items can be enumerated without counting', Railo and Hannula-Sormunen (ibid., p. 3233).

relation requires the actual co-presence of its relata. In subitizing, according to Burge, the relata are the subject and the set's cardinality. Because the subject doesn't only see this property instantiated by the set but applies a concept-number to it, it will be helpful to describe her perceptual state as a state of 'seeing that x is F'—seeing, in this case, that set x has cardinality F. And veridical perception is factive, such that from 'S sees that x is F' one can infer 'x is *in fact* F'. If it is true, then, that you see that the set of dots has cardinality 4, it is true that the set of dots *in fact* has cardinality 4. Sadly for Sarah, however, the set of reflections that she saw did *not* in fact have an infinite cardinality.

The reasons are purely physical. To summarise an account of the phenomenon by optical physicist Gregory Gbur (2011), one way to settle whether there are infinitely many reflections in a mirror parallel to another mirror is to calculate the total area of the images reflected to the subject who's facing one of them. That is: if the sum of the areas of $image_1$, $image_2$, $image_3$, …, returns an infinite value, then there are infinitely many images. So consider Figure 1: $image_1$ is the image reflected first in the mirror facing the subject; it appears at distance d from her, and it has width W and height H. $Image_2$ is the one reflected secondly in *that* mirror; it appears at distance $3d$ from the subject and has width 1/3W and a height 1/3H. $Image_3$ is at $5d$ and has 1/5W and 1/5H, $image_4$ $7d$ and 1/7W and 1/7H, and so on, 'in principle to infinity' (ibid.).

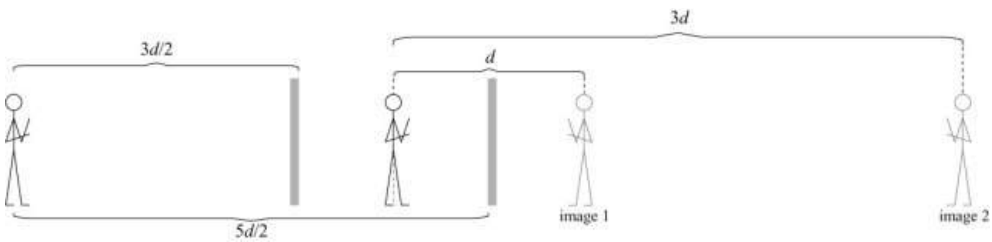


Figure 1, taken from Gbur (2011).

Since multiplying each image's width by its height gives us its area, the total area of images is calculated as:

$$\text{area} = WH(1+1/3^2+1/5^2+1/7^2+\ldots)$$

'This is known in mathematics as an *infinite series*' (ibid.). However, says Gbur, for mathematical reasons not to be developed here, the sum of *this* particular infinite series is determinately finite. So the total area of the collection of images is finite. So there are only finitely many images reflected.

Here's the upshot. For a subject to count as seeing that x is F, x must in fact be F. The set of reflections Sarah was facing was not in fact infinite. So she was not in a perceptual relation with an infinite cardinality. This, of course, makes it irrelevant whether her purported grasp of a cardinality was perceptual or cognitive: since there was no infinite cardinal number being exemplified *there*, she could have neither subitized it nor attempted to count it.

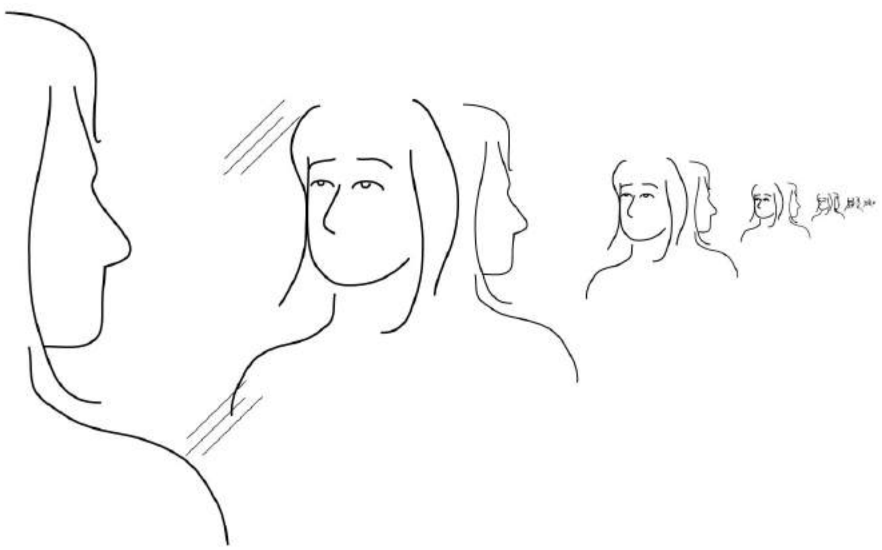
But then we're left with even more puzzling a case. To recap: given Sarah's claim of having understood infinity upon her experience at the temple, we sought ways to understand what epistemic upshot exactly experience alone can have. Lewis and Yablo gave us two: experience can yield, on the one hand, the capacity to recognise instances of observable concepts, but it can also, in the case of observational concepts, help to constitute our grasp of the concepts in the first place. Because Sarah's experience disposed her to answer 'infinitely many' if confronted with the question 'how many reflections are there?', it seemed that the concept she'd grasped and learned to recognise instances of was a cardinal number concept, in particular the concept of infinite cardinal number. We have shown, however, that experience did not in fact present her with an infinite cardinality. So her case can't be like Lewis' case of learning how to recognise instances of 'Vegemite' or Yablo's case of grasping the concept 'oval', because in both cases experience in fact presented Lewis and

Yablo with Vegemite samples and oval things. It continues to be the case, however, that after Sarah left the temple she had grasped the concept of infinity, and that she did apply it to what she'd seen in there. In other words, it's still the case that she acquired the disposition to answer 'infinitely many' to a question about the number of reflections. So it continues to need explaining how the experience in the temple yielded Sarah (i) the acquisition of the concept of an infinite number *and* (ii) acquaintance with something that she could apply that concept to. But now we know that whatever satisfied (ii), that is, whatever she could —because she did— apply the concept of infinity to, was not indeed a set of infinite cardinality. Normally, when experience compels us to apply some concept F to a non-F thing, it is because that thing has an F-like appearance. F-like Gs, along with Fs but unlike non-F-like Gs, invite perceivers to apply the concept F to them. That is one way for something to satisfy the description 'something that Sarah could apply the concept of infinity to'. To illustrate with an example: perhaps just as virtual reality or props help medical students to acquire the concept 'palpitation' without the need to experience the real thing in suspecting patients' chests, so too Sarah acquired the concept of infinity by experiencing an infinite-*like* but finite entity. And perhaps just as part of medical students' grasp of the concept 'palpitation' consists partly in learning to recognise instances of the concept by experiencing non-actual instances, so too Sarah's grasp of infinity consisted partly in her learning to recognise instances of infinity by experiencing something which was, unbeknownst to her, finite. Perhaps, in other words, the way to understand Sarah's experience is as a sort of illusion.

2.6

It has some bearing on the matter at this point that mirrors are involved. As Maarten Steenhagen (2017) explains, there is a significant strand of theorists who ‘take mirrors to introduce some kind of optical illusion to visual experience. Call this view *specular illusionism’* (ibid., p. 1228). Steenhagen identifies two versions of it: one that focuses on the appearance of a left/right reversal in the reflected image, and one that focuses on the appearance of ‘a space opening up before us’ (p. 1229) where there is actually just a surface. The latter version has the further consequence of an illusion of things being located where they’re not: behind the mirror rather than in front of it.

Let us illustrate with our own case. Consider Figure 2, simplified without the rest of Sarah’s family:[39]



l_1 l_2 l_3 l_4 …

Figure 2.

Sarah is at location l_1. According to the illusionist, Sarah’s face (in the first image reflected) appears to her to be at location l_2. But the region at l_2 isn’t

[39] Or actually without Sarah. Clearly, I don’t know how to draw children.

occupied by a face; it is occupied by the mirror. So that face appears to be at l_2 whereas it is in fact at l_1: illusion!

As pointed out, this illusion is a consequence of the broader one that makes it appear as though *there are* such regions as the ones at l_2, l_3, l_4, …, in the first place: the illusion of 'a space opening up before us'. Clare Mac Cumhaill (2011) explains it roughly as follows. Consider Figure 3. When one is in fact facing empty space, as when one looks at a series of trees on the horizon from a vantage point, then we say that the empty space separating us from the trees is the space *through which* we see the trees. That's space *a* in the figure. If the landscape in question is rather desert and there's nothing between those trees, then we see the space separating them as empty as well. That's space *b*. So, even though we see through *a* but not through *b*, both *a* and *b* have a *see-through* appearance or look, and *b* inherits it from *a* in virtue of its looking to be continuous with it (ibid., pp. 488-9).

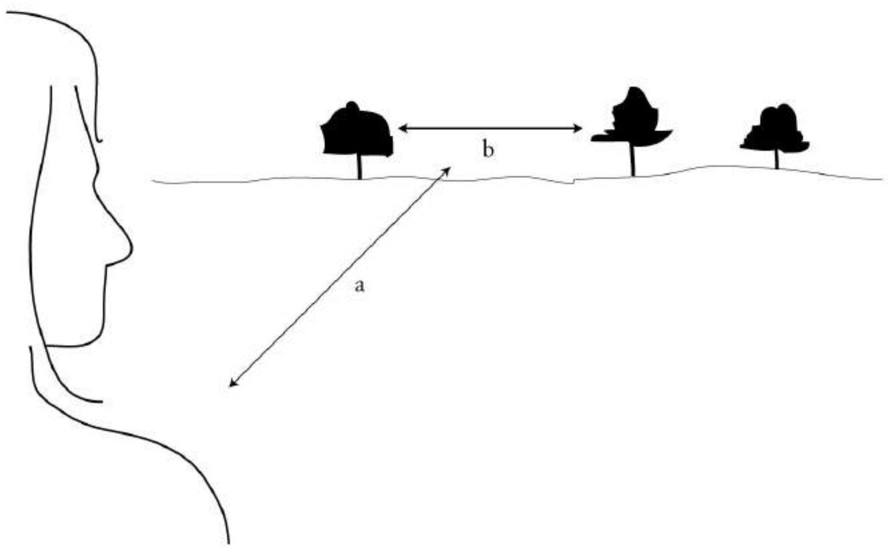


Figure 3.

Similarly, when we're facing a mirror, says Mac Cumhaill, the empty space 'in' the mirror (spaces *b* and *b'* in Figure 4) has a see-through look because it looks continuous with the space one sees the mirror through (space *a*).

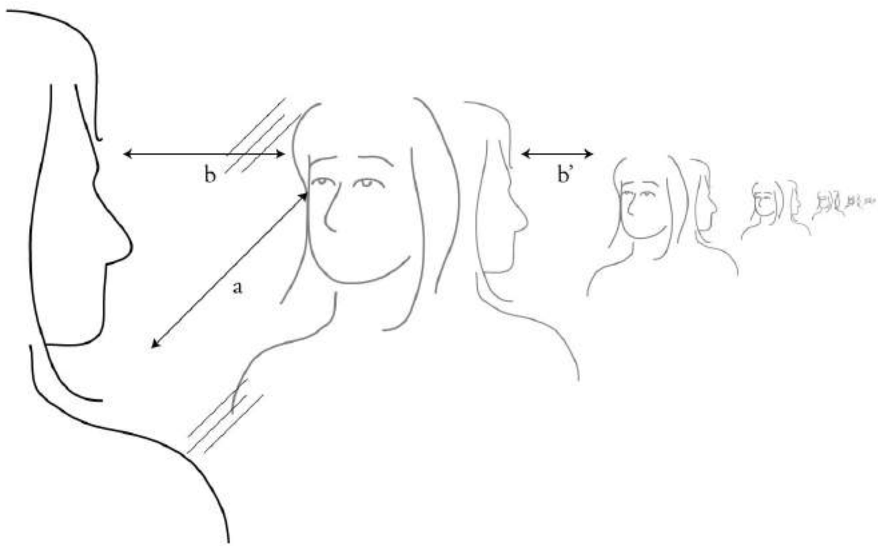


Figure 4.

This phenomenon, viz. the space 'in' the mirror having a see-through look, is in turn related[40] to another feature of veridical empty-space-experience that specular experience replicates: what Mac Cumhaill calls 'elasticity'. Return to the case of seeing the trees. As you move left to right or back and forth but continue to gaze at the horizon, your experience will '[display] characteristic patterns of expansion and contraction that cue awareness of the presence of objects and the empty space that is [there,] "outside"' (ibid., p. 492). The

[40] In fact, not just related but *explained* by it, as I read Mac Cumhaill (p. 492). In that case, though, she offers two explanations: (i) that specular space looks continuous with the space one sees the mirror through, and (ii) the elasticity phenomenon. In my use of her material, though, I remain silent on whether (ii) actually explains the see-through look of specular space.

objects you see, in other words, along with the space that contains and separates them, will appear to change dimensions: grow or shrink horizontally or vertically as you move to the left or right, up and down, and towards and away from them. But this only happens 'when one shares the same space as the objects and regions that one sees — that is, when the empty regions one sees are part of the space *in which one is'* (ibid.). In terms of Figure 3, that means that *a* and *b* are sub-regions of the same space. When we see pictures, by contrast, elasticity doesn't occur—the objects and space you see 'there' do not appear to change as your position relative to them does. This is because the empty space seen in a picture is not and does not look[41] continuous with the space the picture is seen through. In terms of Figure 5, that means *a* and *b* are *not* sub-regions of the same space.

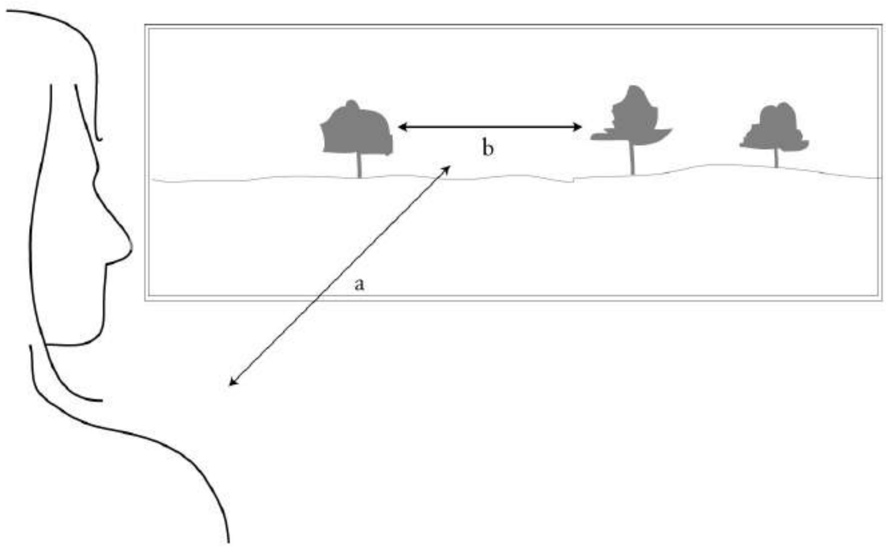


Figure 5.

[41] Mac Cumhaill points us cleverly to an exception that nevertheless does not allow for elasticity: René Magritte's *La Condition Humaine.*

However, because of the phenomenon illustrated in Figure 4, i.e. because space in a mirror *does* look continuous with the space that mirror is seen through, elasticity in mirror experience *does* occur. As you move left to right or back and forth, what you see in the mirror will expand or contract just as if you were seeing through a window. So, even though mirrors are surfaces displaying images, like pictures are, they *behave* rather like windows.[42] This further explains, finally, that just like things you see through a window appear behind it, 'the specular image appears "behind" the surface of the mirror' (ibid, p. 493). This is the case, Mac Cumhaill hastens to say, even if the experiencer is not 'epistemically innocent', i.e. even if she doesn't mistake the reflection and the mirror by material objects and a window.

Steenhagen takes issue with this. Whereas Mac Cumhaill's view makes *all* specular experience illusory, Steenhagen wants to say that if an epistemically innocent perceiver mistakes the reflection and the mirror by material objects and a window, *that* is not explained by a fact about mirrors (that they are illusory) but just by a fact about the perceiver's (lack of) knowledge. According to Steenhagen, mirrors don't introduce illusions into visual experience but just a distinctive phenomenology: the possibility of seeing something by looking in a direction different from where that thing is. In terms of Figure 2 again, this means that it is not the case that Sarah's face appears to her to be located at l_2, where instead there's just a mirror, but rather that by looking in the direction of l_2 she correctly sees something located at l_1. Thanks to their physical properties, mirrors make it possible for the location of things and the direction in which they are visible to come apart.

How does this help explain Sarah's case? Well, it actually makes it harder. In these last few pages, we explored the possibility of explaining the

[42] This phrase is from Mac Cumhaill (ibid., p.492), but she attributes it to cognitive scientist Roberto Casati.

case in terms of the perception of something that looked infinite but wasn't. If Steenhagen is right, however, mirrors do not present us with anything that looks F without being so. So they behave, in this sense, even more like windows than Mac Cumhaill thought: they show nothing that is not the case. Steenhagen's account is reductionist even in a stricter way: whereas the illusionist might think that what you see is either a face located where it's not (it appears behind the mirror but is before it) or a specular image located where it's not (it appears behind the mirror but is where the surface of the mirror is), Steenhagen thinks that what you see in specular experience is numerically identical with what is before the mirror, and that the posited, numerically distinct specular image in fact does not exist. In other words, there is no 'reproduction' in the mirror. There is simply nothing there. *Real* images, says Steenhagen, can be projected onto a surface and affect photosensitive material, which then we can physically store. Virtual, effectively *non-existent* images, by contrast, are a sort of abstraction performed by optical scientists when they construct a geometrical model to study the behaviour of light when it passes through lenses or reflects in mirrors. Here, the scientist posits a 'virtual intersection' of rays —producing a 'virtual image'— that don't *actually* intersect —and so, that don't *actually* produce an image (Steenhagen op. cit., pp. 1234-5).

On this account, then, the object of perception for Sarah wasn't only finite but *very*: at most, she saw a set of cardinality 4. Her family.

2.7

We've been pushed against a wall. Following Sarah's account of the events, we set ourselves the questions of what kind of epistemic upshot her 'chill of understanding' consisted in, and what exactly the nature of the experience that yielded it was. We conjectured that she had acquired the disposition to answer ‘infinitely many’ if confronted with a question about the cardinality of the set of reflections. Given Lewis’ and Yablo’s views on the kind of epistemic upshot experience has, we cashed this out in terms of her acquiring the capacity to recognise instances of the concept of infinity and her correlated grasp of that concept altogether. This required that she be presented with such an actual instance in her foundational experience. But we saw that no such instance was presented. Then we tried an alternative: that she had been presented with something infinite-appearing even if not actually infinite. This proved even less hopeful: we looked at a convincing argument establishing that, when one looks into mirrors, one doesn’t really see anything ‘in’ them that’s not present on the other side anyway.

But we may have been barking at the wrong tree all along. We were assuming that whatever Sarah saw such that it triggered her grasp of the concept of infinity was some *thing* exhibiting that property: a portion of reality, just like the portion of reality her family is displays the cardinality 4. However, and going back now to the issues discussed in the introduction, another way of thinking about infinity is as a *structural feature* of reality rather than as a portion of it. According to the structuralist brand of realism in the philosophy of mathematics, most notably Shapiro’s (1997),[43] the rough idea is that when a realist says that the subject matter of mathematics exists mind-independently she doesn’t mean so much that mathematics is concerned with the ‘internal nature’ of certain objects of study —not so much concerned with the nature of any real *things*— but instead that it’s concerned with the way these objects relate to each other —concerned with the *structural properties* of these real things.[44] Now, to be sure, in Shapiro’s view these relations form a further

[43] Other discussions of structuralism include Resnik (1997) and Parsons (1990). We focus on Shapiro’s *ante rem* (i.e. non-reductive) structuralism, though the *ante rem* label is omitted for simplicity since we introduced it as a type of realism anyway.
[44] I borrow part of this characterisation from Korbmacher and Schiemer (2017).

object of study in its own right: a structure. But this does not come down to the same view. That mathematical objects can be defined in terms of structural properties makes a difference to us because in our search for an explanation to Sarah's puzzling experience we ran out of first-order individual *things* — including collections of them— that could fit into the object side of the subject-object perceptual relation which keeps stubbornly eluding us.

Moreover, as hinted at in the introduction, structuralism is of particular relevance for the philosophical study of that obscure object of infinity. According to Shapiro, a set of countably infinite cardinality such as the set of natural numbers shares its structure with any countably infinite set. This is the cardinality we've been helplessly looking for a perceptible instance of, and it corresponds to the first infinite cardinal number, $\aleph_0$. But whereas subitizing requires that the cardinal number 4, for example, be a sensible property displayed jointly by actually co-present members of a set like Sarah's family, a structure is just defined in terms of how places in that structure stand in relation to each other, such that grasping *that* doesn't require acquaintance with every place in the structure because, precisely, places are not objects in their own right but roles to be played by the objects in whatever system instantiates the structure in question.

Now, yes, taking this avenue might seem too quick. By this I don't only mean that the proposal to understand what Sarah saw as a structure needs more careful elucidation but also that over us looms the very problem we introduced much earlier on. Even from a position of sympathy towards structuralism, Giaquinto told us that 'knowing the natural number structure is different, less direct, from the kind of knowledge of finite structures [because] in this case we cannot experience an entire instance of the structure' (2007, p. 228), which is why grasping $\aleph_0$ could only count as 'knowledge by description, rather than knowledge by acquaintance' (ibid.). And, as you will recall, pretty much everyone else agreed.

But here is a catch. Ever since Cantor's pioneering work in the formal study of infinity in the late 19th century, the concept of infinite size has come with the embedded notion of *completed* size. Mathematically, after all, a set has its cardinality as a fixed property. But this is a relatively recent development. Apart from a few exceptions,[45] before Cantor and ever since Aristotle the prevalent notion of infinity came with the embedded notion of *potentiality* rather than of *actuality*. Because, as you will recall, the concept could refer to either a magnitude or a number, we might put the above in slightly anachronistic jargon by saying that for a magnitude to be infinite was in pre-Cantor times for it to be the object of an operation with infinitely many potential iterations —for example, a line segment's disposition to be again and again divided or extended— whereas for a number to be infinite was for it to correspond to the size of a set with potentially infinite members rather than with a fixed size.[46] Thus, again, Aristotle wrote: 'generally the infinite is as follows: there is always another and another to be taken'.[47] This characterisation contrasts with today's Cantorian concept of the infinite, which does not rely on temporal vocabulary —like Aristotle's 'always'— or on dispositional vocabulary —like Aristotle's 'to be taken'. For Cantor, an infinite set is infinite at any time and possible situation/world. In philosophically realist

[45] Duns Scotus and Gregory of Rimini are just two examples of a few actualists on the margins. See Moore (ibid.), pp. 45-55.
[46] This is anachronistic because, as Moore (op. cit., pp. 36-7) points out, Aristotle endorsed only the idea of infinite magnitude, but I'm applying his process-based reasoning to our discussion of infinite number for purposes of the dialectic. We're not interested in historical accuracy here.
[47] We remain neutral here on the question whether Aristotle located potential infinity in the process, as Jaakko Hintikka (1966) suggests, or in the structure of the magnitude to which the process merely bears witness, as Jonathan Lear (1979) suggests. I hope everything I say is compatible with either exegesis.

terms, this means that our concept of infinity is today such that the structure denoted by the cardinal number $\aleph_0$ is infinite simpliciter or actually infinite rather than potentially infinite. Now *that* is why, as Giaquinto says, 'we cannot experience an entire instance of the [$\aleph_0$] structure'. Not only is actual infinity as a matter of fact uninstantiated by concrete systems such as the reflections in the mirrors, as we saw, but in any case our perceptual capacities are finite.

However —and this is where we perk up—, if we learned anything positive from our unfavourable discussion of specular experience it should be that sometimes we can *experience* things in different ways than they *are*. Just as the idea that one can see one's own face by looking in the direction of a train's window doesn't entail that one's face —or even its image— is *in fact* on the other side, perhaps the idea that Sarah may have *somehow* experienced the structure $\aleph_0$ needn't entail the undesirable result that $\aleph_0$ is in fact not actually infinite.

Mirrors showed us that matters of fact and of experience come apart. If the contemporary consensus in both mathematics and (realist) philosophy is that infinity *is* in fact actual, we're not going to quarrel with that. But nobody said we cannot help ourselves to the notion of potentiality when working elsewhere than on the object side of the perceptual relation. After all, unlike mathematical abstracta, perceptual experience is not placeless and timeless but perspectival and extended in time. In what follows, then, I will offer an account of Sarah's experience that relies on the actual/potential infinity distinction and on the insight that mirrors behave like windows—but, most importantly, on Sarah's own first-person account, which now we know we shouldn't treat uncharitably. Once we understand, as we have, that a perceptual report states how things are *experienced* by the subject, we're in a position to see that the key to our puzzle lies in the Aristotelian tone in which Sarah put *that* herself: 'the reflections kept going and going. […] Forever'.

3. What Sarah learned

3.1

Let us start by fleshing out the actual/potential infinity distinction a bit more. As recent work by Øystein Linnebo and Shapiro (2017) shows, the distinction has a very long history, as long as the history of the concept of infinity itself. If I may generalise from their survey slightly, perhaps it's fair to say that although historically the distinction seems to have revolved around a broad metaphysical debate, the debate about whether infinite magnitudes or collections should be understood as actually or just potentially existing, now that *that* debate has been pretty much settled (the actualist view won), the distinction seems to be relevant to more specific —but quite lively— debates within the philosophy of mathematics: for example, the debate over what the totality of all sets is if it is not a set (here a potentialist view would suggest that it is a 'potential hierarchy'),[48] the debate over absolute generality (here a potentialist view would suggest that quantification in ordinary mathematical discourse is implicitly modal and generalises 'over absolutely all mathematical objects' rather than just over the ones generated up to a given stage),[49] and other issues.[50] I mention this just to acknowledge the complexity of the subject. All we'll help ourselves to here, however, is just Linnebo and Shapiro's *explication* of potential infinity. Perhaps just a short word on what I called the 'broad metaphysical debate' over the distinction would then be helpful both to complement the brief remarks

[48] See e.g. Linnebo (2013) and Soysal (2017).

[49] See e.g. Linnebo and Shapiro (op. cit.) and Shapiro and Wright (2006).

[50] A third example of an issue Linnebo and Shapiro themselves mention is Michael Dummett's (1978) contention that the phenomenon of indefinite extensibility provides a *new* argument for intuituionism, i.e. that a concept whose totality of instances is extendable by one more instance via a process that makes reference to the initial totality is a concept the domain over which quantification must be intuitionistic. However, this is part of the very material Linnebo and Shapiro's discussion deals with rather than merely an issue on which it has consequences.

made towards the end of the previous chapter and to introduce us to Linnebo and Shapiro's work.

First off: it is a bit misleading for me to say that the 'broad metaphysical debate' was won by the view that infinity actually exists. That is partly because the dialectic in question was neither properly a 'debate' nor properly 'metaphysical'. It wasn't properly a debate because, in fact, it was simply standard throughout most of the history of the concept of infinity to think of it as applying to the *would-be* results of procedures that *could* be iterated indefinitely. The potential endlessness of the procedure of adding members to a collection, say, justified the concept of infinite number, and the procedure of extending or bisecting a line, say, justified the concept of infinite magnitude. But at any rate no such *completed* infinite procedure was thought to exist; hence, nor completed infinite magnitudes or numbers. But this wasn't properly a metaphysical contention because ever since Aristotle's rejection of mathematical objects as existing eternally in 'a Platonic heaven', it was generally not up for grabs whether such infinite magnitudes or numbers were to be found in reality. If authors bothered to express a rejection of actual infinity, we might say, these views were meant to be just conceptual. As authors as late as Gauss (1831) suggest, the point wasn't so much that there exist no actual infinities but that it doesn't make much mathematical sense to take the concept of infinity as actual: 'I protest against the use of infinite magnitude as something completed, which is never permissible in mathematics' (quoted in Linnebo and Shapiro op. cit., p. 1). When Cantor challenged this view and argued exactly the opposite, that *only* a notion of actual infinity is sensible, he was likewise making a conceptual (mathematical) claim rather than a philosophical (metaphysical) one:[51]

[51] It is said that Cantor associated what he called 'absolute infinity' (as opposed to transfinite cardinalities) with God, but we'll ignore that for present purposes.

> I cannot ascribe any being to the indefinite, the variable, the improper infinite […] because they are […] never adequate *ideas* (Cantor 1883, p. 205, note 3, my italics).
>
> [E]very potential infinite, if it is to be applicable in a rigorous mathematical way, presupposes an actual infinite (Cantor 1887, pp. 410–411).[52]

Because, as we've said, Cantor gave a precise, formal characterisation of infinity that allowed studying it in a way that no characterisations of it had allowed before, the Cantorian orientation became dominant in the field. But the phenomenon of mathematics going back to a Platonic stance on the way they speak of their subject matter is arguably more generalised. Originally, Plato had criticised the geometers of his day for speaking as though they could *do* things with mathematical objects, 'squaring and applying and adding and the like' (Linnebo and Shapiro op. cit., p. 5). Euclid's first postulate in the *Elements*' list of what geometers *can do*, for example, was '[t]o draw a straight line from any point to any point' (ibid.), which, for Plato, made no sense as mathematical objects were eternal existents. Similarly, David Hilbert follows Cantor's drive in the late 19th century in saying that, contrary to Euclid, '[f]or every two points A, B there [simply] exists a line a that contains each of the points A, B.' (ibid., p. 28). Linnebo and Shapiro call the Platonic/contemporary way of speaking 'static', by which we can take them to mean atemporal or non-modal, and the Aristotelian way 'dynamic', by which we can take them to mean temporal or modal.[53]

An any rate, even more radically than Hilbert's view on the non-modal nature of points and the lines that connect them, when it comes to infinity,

---

[52] Both quotes are taken from Linnebo and Shapiro (op. cit..), p. 6.

[53] Linnebo and Shapiro's 'dynamic/static' labels reflect a conflation we follow them in doing here between dispositional/modal and temporal notions, which we hope is allowed by the structural analogies there are between modal and tense logic as noted e.g. by Arthur Prior (1957), but also, more importantly, by Aristotle's own perceived connection between modality and time. More on the latter issue shortly.

most contemporary thinkers take it that the notion of potential infinity is difficult to even be made sense of without appeal to actuality, and hence that if the Aristotelian dynamic notion was supposed to, precisely, bypass *that*, then it was never a very clear concept in the first place. Prima facie, this squares with what we've said so far. Cantor's formal characterisation of infinity is no less than what we might say constitutes the foundations of set theory, arguably inaugurated by his (1874) proof of the uncountability of the real numbers. One consequence of this proof is that the infinite set of real numbers is larger than the infinite set of natural numbers. Because, as Moore (op. cit., pp. 50-54) points out, the pre-Cantor view took it implicitly that all infinities were equinumerous, accepting Cantor's proof came down to changing one's very concept of infinity—by, arguably, *refining* it. If, again, we accept the realist view here that mathematical facts are objective and timeless, and if the concept of infinity being refined in 1874 means the judgements this concept allows weren't available for people to make prior to that date, then *that* means Aristotle himself didn't grasp the concept of infinity to its full extent. So that's what it means for the notion of potential infinity to be mathematically unclear or confused, whether in Aristotle's writings or any other potentialist's. In this vein, and speaking for most of his colleagues, mathematician Karl-George Niebergall says: '"a clear meaning has never been given to" the phrases "x is potentially infinite" and "T makes an assumption of the potentially infinite" (quoted in Linnebo and Shapiro op. cit., p. 7).

In short, Linnebo and Shapiro summarise the problem thus:

> Niebergall [...] argues that, on some straightforward attempts at definition, the potentially infinite just collapses into the actually infinite (or the finite). [...] Everything is either finite or infinite—nothing can fit between those (ibid.).

This seems straightforward enough. Intuitively, collections are either finite or not. The main obstacle to the intelligibility of the notion of potential infinity, however, may be put simply as the fact that potentiality is a modal notion, and, as Linnebo and Shapiro acknowledge in deference to Niebergall again, this 'clashes with the dominant contemporary view that, in mathematics, "talk of possibility and necessity becomes dispensable" since "a mathematical sentence is regarded as necessary if true"' (p. 8). Hence, in tasking themselves to offer an intelligible notion of potential infinity, Linnebo and Shapiro take up the prior task of defending the mathematical use of modal vocabulary.

To delve into (my presentation of) Linnebo and Shapiro's explication of potential infinity now, their way to do this first thing (defending the mathematical use of modal vocabulary) is rather modest. Linnebo and Shapiro's 'single controversial claim' (ibid, p .9) on behalf of the potentialist is that the non-modal language of contemporary mathematics is not fully explicit. Hence, in order to make sense of the notion she advocates, the potentialist will have to provide a modal language that can nevertheless be translated into the non-modal language of ordinary mathematics. To put it differently, the potentialist's claim about the need for a translation is the claim that two mathematical statements, one involving modal vocabulary and the other involving none, can express the same proposition but one —the modal statement— will do it more explicitly. When we say the natural numbers are infinitely many, for example, this view goes, we *really* mean they are potentially infinitely many. To this, again, mainstream mathematics replies by asking what exactly that means. More specifically: what mathematical claims does a potentialist view allow? Do we get to quantify over all natural numbers, for example, if they purportedly constitute a domain of not fixed but modally variant size? Because, again, the potentialist's position is not that infinity is potential rather than actual, and so not that their view about the infinity of the natural numbers is correct whereas a Cantorian view isn't, but instead that a modal expression of infinity is merely more explicit, the inferences the potentialist's proposed view will allow will have to match the ordinary mathematician's. The standard by which her explication of potential infinity can be deemed logically sound, then, will be whether the entailment relations it determines as obtaining in the modal language correspond to the entailment relations that obtain in ordinary mathematics when translated into its non-modal language.

This squares with Linnebo and Shapiro's modest description of potentialism's task as providing just an explication —making logical sense or defending the coherence— of potential infinity. The competing view, 'actualism', is in these terms just the view that 'the non-modal language of ordinary mathematics is already fully explicit' (ibid.). But the disagreement over potentialists' 'single controversial claim' reflects a more substantive disagreement over the nature of infinity. According to actualism, infinities exist actually, which is, as we noted before, for them to be infinite *simpliciter*. Hence actualism's rejection of a need for fuller explicitness in mathematics. The set of natural numbers, this view goes, is actually infinite if it is infinite at all and that's all there is to be said about it. According to potentialism, by contrast, 'the objects with which mathematics is concerned are generated successively' (ibid.), and in the Aristotelian version of potentialism considered here,[54] that process —the process of generating the next object— is always available, which is to say that, *necessarily*, at any given point it is *possible* to add one more element to the collection. Hence their need for modal vocabulary.

---

[54] For simplicity, I gloss over the authors' distinction among several brands of potentialism, which disagree over whether all processes can be completed and whether mathematical statements are made true only when the objects they're concerned with are generated, among other presently irrelevant issues.

To illustrate: for an actualist, to say that the set of natural numbers is infinite is to say that every natural number has a successor, which we might express by:

$$\forall m \exists n \, \text{Succ}(m,n)$$

For the potentialist, to say that the set of natural numbers is infinite is to say that, at any point in time at which we have finitely many natural numbers, we can always generate or construct —there can exist— one more. And we might express this by:

$$\Box \forall m \Diamond \exists n \, \text{Succ}(m,n)$$

The potentialist's view, then, is that when the actualist —or the ordinary mathematician, really— says that '$\forall m \exists n$ Succ(m,n)', she *really* means that '$\Box \forall m \Diamond \exists n$Succ(m,n)', or that *that* is what she would say were she to use a fully explicit form of expression. Hence, again, the need for a translation that guarantees that whenever an inference is valid in the fully explicit modal language, it will also be valid in the less explicit, ordinary non-modal one.

In short, the task of explicating the notion of potential infinity comes down for present purposes to the task of explicating the logic of the 'fully explicit' claim about the infinity of the natural numbers: 'necessarily, for any natural number a successor can be generated'; or in other words, it comes down to the task of providing an 'analysis of quantification over a potentially infinite domain' (ibid., p. 16). To do this, Linnebo and Shapiro adopt the possible worlds heuristic.

> [T]he idea is that a "possible world" has access to other possible worlds that contain objects that have been constructed or generated from those in the first world. From the perspective of the earlier world, the "new" objects in the second exist only potentially (ibid., p. 11).

Because Linnebo and Shapiro are drawing an analogy between Aristotle's temporal vocabulary and their own mathematical modal vocabulary, they call possible worlds accessible from the current one *later* worlds, such that, in the case of constructing the successors of natural numbers, 'the later world [contains] the successor of the largest natural number in the first world' (ibid.). Following also Aristotle's rejection of actual infinity, potentialism would hold in this heuristic that 'every possible world is finite, in the sense that it contains finitely many objects' (ibid.). And following, finally, 'ordinary mathematical talk about construction', it is assumed that no objects are destroyed when new ones are constructed, such that the world containing the successor of the current world's largest natural number contains also the original collection. Hence, 'the domains of the possible worlds grow along the accessibility relation. So we assume:

$$w_1 \leq w_2 \rightarrow D(w_1) \subseteq D(w_2),$$

where '$w_1 \leq w_2$' says that w_2 is accessible from w_1, and for each world w, D(w) is the domain of w' (ibid.).

Now, one prima facie issue with this heuristic is that it might seem to imply that at least some mathematical objects are contingent (because for them to be constructed is for them to be contained in some worlds but not in others), whereas, as we know, the dominant view of the metaphysics of mathematical objects is that they exist at all possible worlds if they exist at all—which is also, again, why Linnebo and Shapiro admit they have to justify their use of modal vocabulary and say something about its interpretation. But precisely because they want to keep their account as a mere explication of potential infinity rather than as a substantive view with consequences about its *nature*, let alone about the nature of 'constructed' mathematical objects, their modal vocabulary, they warn, can hardly be interpreted as ordinary metaphysical modality. Thus, their 'a successor can be generated' phrase, for example, should not be interpreted as a substantive metaphysical claim about the coming into existence of objects.[55] Instead, they suggest,

> we might regard [the modality invoked in explicating potential infinity] as an altogether distinct kind of modality, say the logico-mathematical modality of Putnam (1967) or Hellman (1989), or the interpretational modality of Fine (2005) or Linnebo (2013) (ibid., p. 12).

Ultimately, though, they refrain from offering a definitive answer to what the best interpretation of the modality in question might be and stick to 'identifying some structural features of any plausible interpretation' (ibid.), which serves us here of course just as well. The most important such feature is, arguably, the accessibility relation, as that is what will determine which system of modal logic our notion of potential infinity operates with. As we've said, the worlds in our heuristic are related by the ≤ relation, which is a partial order, i.e. an accessibility relation that is reflexive, transitive and anti-symmetric. These properties correspond to the system of modal logic S4. Additionally, though, Linnebo and Shapiro want the accessibility relation to have the property of directed convergence, that is, they want us to be able to get to w_1 if we choose to construct object o_1 from w_0, or to get to w_2 if we choose to construct object

[55] The reason Linnebo and Shapiro say 'it is doubtful the modality in question can be "ordinary" metaphysical modality' (ibid., p. 12) is actually not the one I've just presented but a technical one, namely, that the introduced conditional [viz. $w_1 \leq w_2 \rightarrow D(w_1) \subseteq D(w_2)$] implies the validity of the converse Barcan formula (CBF): $\exists x \Diamond \varphi(x) \rightarrow \Diamond \exists x \varphi(x)$, paraphrased as 'if there is some object (at a given world) that possibly has a certain property, then it's possible that there is an object with that property'. This is problematic for a metaphysical interpretation of the present modality because in our heuristic we have that there are objects that may have not come into existence, i.e. we have that $\exists x \Diamond \neg\varphi(x)$, and the CBF would make it follow from *that* that $\Diamond \exists x \neg\varphi(x)$, i.e. that it is possible that there exists something that doesn't exist, which is absurd. Both my informal and their technical reasons, however, come down to disallowing the modality's being interpreted as making substantial claims about the contingent (or not) existence of the objects this heuristic treats as potential first and actual once constructed.

o_2 instead, without losing the option in either w_1 or w_2 to construct the neglected object later, which means that both w_1 and w_2 can access —converge in— a world w_3 that contains both o_1 and o_2. Adding this fourth constraint to the accessibility relation results in our operating with the system of modal logic known as S4.2. The reason Linnebo and Shapiro originally present to motivate convergence is that, in the construction of geometrical objects, we might have various options at every step that nevertheless converge in the same result: '[f]or example, given two intervals that don't yet have bisections, we can choose to bisect one or the other of them, or perhaps to bisect both simultaneously' (p. 13). Here we're only concerned with the successive construction of natural number successors, so, to be sure, our case doesn't motivate convergence in the same way geometrical construction does; however, as Linnebo and Shapiro point out, having the accessibility relation being convergent ensures also the following principle of S4.2:

$$\Diamond \Box p \rightarrow \Box \Diamond p$$

which says that if a proposition is possibly necessary, then it is necessarily possible. This principle will be key to help the potentialist ensure that entailments in the non-modal language of ordinary mathematics obtain whenever they obtain in her modal language, which claim she will state in what Linnebo and Shapiro call the 'potentialist mirroring theorem'. To get this theorem we just need to add to S4.2 a twofold principle ensuring the stability of our language's formulas: '[s]ay that a formula φ is *stable* if the necessitations of the universal closures of the following two conditionals hold:

$$\varphi \rightarrow \Box\varphi$$
$$\neg\varphi \rightarrow \Box\neg\varphi$$

Intuitively, a formula is stable just in case it never "changes its mind", in the sense that, if the formula is true (or false) of certain objects at some world, it remains true (or false) of these objects at all "later" worlds as well' (ibid, p. 14).

Assuming S4.2 and the stability principles, then, allows us to state the mirroring theorem:

> Let $\vdash$ be the relation of classical deducibility in a non-modal first-order language *L*. Let $L^{\Diamond}$ be the corresponding modal language, and let $\vdash^{\Diamond}$ be deducibility in this language by $\vdash$, S4.2, and axioms asserting the stability of all atomic predicates of *L*. [Then,] for or any formulas φ_1, …, φ_n, ψ of *L*, we have:
>
> $$\varphi_1, \ldots, \varphi_n \vdash \psi \text{ iff } \varphi_1^{\Diamond}, \ldots, \varphi_n^{\Diamond} \vdash^{\Diamond} \psi^{\Diamond}$$ [56]

which we might paraphrase simply as the claim that an entailment obtains in the non-modal language of ordinary mathematics whenever its modal counterpart obtains in the potentialist's language.

Consider now the question we raised before. The ordinary mathematician's qualms were put as the question of what 'the clear meaning' of the notion of potential infinity might be. If knowing the clear meaning of the notion of actual infinity —or, for this mathematician, infinity simpliciter— is related to knowing what inferences that concept allows, then knowing what inferences the notion of potential infinity allows, or knowing its logical behaviour, might help us to get closer to knowing what the 'clear meaning' of the notion of potential infinity is. Thus, we asked: within a potentialist view, are we entitled to quantify over all the natural numbers, for example? Linnebo and Shapiro's answer is yes. On behalf on the potentialist, they've clarified that her view is just that the ordinary quantifiers of mathematics are implicitly modal but that they are quantifiers all the same. So, in the potentialist's 'fully explicit' language, 'the modalized quantifiers $\Box\forall$ and $\Diamond\exists$ behave logically just as ordinary quantifiers, except that they generalize across all (accessible)

[56] Ibid., p. 15. We are referred to Linnebo (2013) for a proof.

possible worlds rather than a single world' (ibid., p. 15). So *that's* what it is to quantify over a potentially infinite domain.

And that is pretty much the gist of it.

Now this might seem a little disappointing. As Linnebo and Shapiro point out, 'it might be objected that on this [...] explication, potential infinity is scarcely different from actual infinity [...] and that the resulting theory [is] just actualism in potentialist garb' (ibid., p. 16). To tackle this worry, the authors distinguish between two brands of potentialism and develop their consequences, but that is beyond the scope of our interests here. One important result we've reaped already from Linnebo and Shapiro's effort to elucidate the potentialist's 'single controversial claim' is that one can make sense of the notion of potential infinity in terms of ordinary classical first-order logic rather than it having to be made sense of in intuitionistic terms, as has often been assumed.[57] This is beneficial for an Aristotelian view of infinity: '[w]e take it [that Aristotle's] notion is based on some form of metaphysical modality, which behaves classically. Given this and the fact that Aristotle does not seem to allow any exceptions to the Law of Excluded Middle, he —and all the thinkers he inspired— are entitled to take the logic of potential infinity to be classical' (ibid.).

How does this all relate to our discussion? Earlier we hinted at an Aristotelian interpretation of our puzzle but have since come a long way off, so let us backtrack a little. The previous chapter ended with the suggestion that Sarah may have somehow experienced the structure of the natural numbers. The main problem with this suggestion was that the structure of the natural numbers is infinite, whereas, as Giaquinto and Shapiro —along with everyone else— have pointed out, even if knowledge of finite structures via experience is

[57] The assumption dates back to Dummett's (op. cit.) contention that quantification over a potentially infinite domain *must* be intuitionistic, which claim Linnebo and Shapiro say was never 'properly substantiated' (op. cit., p. 23).

possible, knowledge of infinite structures isn't. The source of this impossibility, I suggested, was that infinity is implicitly understood as actual. And, sure enough, actually infinite structures simply can't be instantiated by finite concrete systems, let alone be objects of perception. But this thought was shown to be slightly Platonic in spirit. In his rejection of Plato's view that mathematical objects are independent from empirical reality, Aristotle argued that infinity existed only potentially, never actually, and, we might say, that our belief in it is indeed empirically justified by the sorts of things we can, in fact, do: bisecting and extending lines, he thought, are in principle never-ending processes. Now, we noticed in passing a connection between Aristotle's idea and the puzzle we're trying to solve. Both an Aristotelian mathematician in the process of extending a line and Sarah are, unlike mathematical abstracta, concrete beings embedded in time. Hence, what they are experiencing and whatever thoughts arise from that experience must be understood in dynamic rather than in static terms. Aristotle could not have come up with his view had he existed only at some point in time rather than experiencing, in a temporally extended fashion, the processes he described. Picture him by a river in Athens, brooding over Platonic forms. 'Sure, because the world is finite, the amount of water in this river is finite too', we find him thinking. 'But there is nothing in principle to stop this finite amount of water from running the course of the river over and over again. And if I were immortal, I would be free to sit here and watch it do that *forever*'.

As Moore points out, the connection of Aristotle's actual/potential infinity distinction with time, and hence with lived experience, was not fortuitous:

> For Aristotle, the infinite was the *untraversable*. But traversal takes time. So there is no making sense of the claim that something is untraversable save with respect to the whole of time (op. cit., p. 40, my italics).

This contrasts with processes the length of which you can in fact 'traverse', such as, contrary to Zeno, the process of Achilles reaching the finish line of the race against the tortoise.[58] For the river's running of its course to be infinite, then, is for one not to be able to traverse the length of its duration. But note, again, that this ability is to be understood as a matter of principle. There is nothing in the river's running of its course itself, or, as Jonathan Lear (1979) puts it, nothing in 'the structure of [its] magnitude' (p. 193) that impedes its being untraversable. Only Aristotle's mortality —and perhaps his capacity for boredom— impede him from witnessing this going on forever. Hence, his potential/actual infinity distinction is to be understood *essentially* in temporal terms:

> The actual infinite is that whose infinitude exists, or is given, at some point in time. The potential infinite is that whose infinitude exists, or is given, *over* time; it is never wholly present (ibid.).

As we know, belief in the latter is what Aristotle argued was empirically justified and the former what wasn't.[59]

Return now to Sarah's case. Suppose we begin the inspection of her temporal experience at the point after she is carried by her dad to the centre of the mirror arrangement. Suppose her eyes are closed in that process. 'Open your eyes', then they tell her. Boom: she opens them and is suddenly confronted with a system of what we've learned is a finite number n of

[58] This is a reference to Zeno's paradox, arguably the oldest of the paradoxes of infinity (of which we omit the details here because it is so well known). It is Zeno of Elea's conclusion that Achilles could never overtake the tortoise what Aristotle's distinction serves to reject.

[59] Indeed, Moore argues that, for Aristotle, time is not just connected with the potential/actual infinity distinction but modality generally: 'Aristotle believed that questions of possibility and impossibility were themselves intimately connected with time, so that asking whether or not something was possible was akin to asking whether or not it would be so—at some time' (ibid.).

reflections, as illustrated roughly by Figure 2 earlier, repeated below as Figure 6. Call this point in time t_0.

Figure 6.

Astounded, she takes a second to absorb what she is seeing. But then astonishment leads the way to curiosity and she moves a little to the right. Call that point t_1. Because of what we learned about mirrors' window-like behaviour in visual experience, what she sees between t_0 and t_1 displays 'elasticity'. Just to refresh our memory, this means that, exactly as it would were she seeing through a window, Sarah's experience of staring at the mirror as she moves further to the right displays the 'characteristic patterns of expansion and contraction that cue awareness [*as*] of the presence of objects and the empty space that is "outside"' (Mac Cumhaill op. cit., p. 492). So the reflections she sees stretch and contract a little in her visual field just like present objects would on the other side of a window rather than being experienced as constant in size, as they would were she seeing, instead, a picture. But an additional crucial feature of the experience of seeing through a window is that when we move, for example, to its right, the objects on the other side previously occluded by the window's left edge come into sight. Likewise, the system of n reflections becomes for Sarah at t_1 a system of n+1. Presumably, *that* is when she gasps. Then, of course, she moves further still, and the system becomes one of n+2 reflections at t_2, n+3 at t_3, n+4 at t_4. Perhaps she reaches finally an awkward position at which the angles of the mirrors don't let her see one more reflection because then she'd start to see behind the mirror, or because she'd bump into someone outside the centre of the arrangement, but she understands this makes her seeing one more reflection impossible as a matter of fact rather than as a matter of principle. If the platform where she stands were rotating, or if the mirrors rotated with her as she moves left or right, she would continue to see one more reflection, over and over again, forming a circle with the rotating arrangement of mirrors—'forever'. In other words, there is nothing in the structure of the system of reflections itself that puts any limits on the process of getting the next one into sight. It is, as we might now say, *untraversable*.

So Sarah is having an experience of Aristotelian infinity. Interpreting our case this way would be of little use if we didn't have a cogent account of what exactly that means, but thanks to Linnebo and Shapiro —and contrary to most mathematicians—, now we do. I suggest, in short, that for the system of reflections that Sarah sees to be potentially infinite is for it to be always possible to see one reflection more. In terms of the heuristic we presented above, and if we take the reflections to stand for the mathematical objects of the model, *that* means that for the system of reflections to be potentially infinite is for any world w_0 containing n reflections to access another world w_1 containing n+1. At w_0, the n reflections exist actually whereas the extra reflection exists potentially. Then, at w_1, the process of mathematical construction —which here consists of the process of the subject moving a bit further to get one more reflection into sight— has resulted in that extra reflection being actualised and yet another one coming into potential existence because now w_1 accesses w_2. And so on. Because we can generalise with our quantifiers across all accessible possible worlds rather than just over a single world, however, even at w_0 already we can make the claim that every reflection can always have a successor, which is to say that the set is, in effect, potentially infinite.

I should ask the reader for some patience at this point. What I've presented just now is simply what it would mean theoretically to say that the set of reflections is potentially infinite. Hence, if what I've said is right, then all I've done is shown that such a claim makes logical sense, and, to be fair, that is the extent of my ambitions in this section (which ambitions I inherit from Linnebo and Shapiro's work). But as we've emphatically repeated, an Aristotelian take on this thesis' puzzle would require not only making logical or mathematical sense of the phenomenon Sarah seems to have experienced but also making sense of the nature of the experience itself. For Aristotle, as we've seen, infinity is intimately linked to our lived temporal embeddedness: the infinite is the *untraversable*, traversing being an action verb; and it is that whose infinitude is given *over time*.[60] Now, unlike Aristotle, we're not interested in making any claims about the nature of infinity, but like him, we're interested in the experiential aspect of it because we're dealing with a puzzle concerning a subject's acquisition of *knowledge* of infinity. This, in turn, involves two things. To recap: our task was to explain the facts that (i) Sarah seems to have gained acquaintance with something that she could apply the concept of infinity to, and that (ii) *that* fixed the concept in her repertoire. In the last chapter, we saw how (i) could not be explained by examining the material, perceptible things in Sarah's surroundings. The suggestion came up to explain (ii) by appeal,

[60] Moore's experience-alluding 'given' phrase here is not coincidental: 'I use the phrase 'is given' advisedly. The metaphor of reception has often been felt to go naturally with this account, since reception takes place in and over time' (ibid., p. 40).

instead, to Giaquinto and Shapiro's notion of experience of structure. The main obstacle to this suggestion was infinite structures' *actuality*, which we've taken a first step towards circumventing by defending the coherence of the idea of infinity being potential. Whatever success we may have had in making sense of the potential infinity of the set of reflections in this section, then, it will only prove fruitful if it serves us to come up with the account we're seeking of (i) and, consequently, of (ii).

So let us turn from the logic of potentially infinite systems to that of Sarah's mental state now.

3.2

Modal logic proved a natural way to make sense of potential infinity. It is, after all, the method of choice for the formal study of certain features of reality that go beyond what's directly there for us to see: those which don't correspond to what is the case but what must or could be the case. Similarly, it is often the method of choice for studying our *knowledge* of those features of reality —i.e. knowledge of what can and must be the case—, as well as analogous aspects of knowledge itself —i.e. what can and must be ruled out as being or not the case given one's evidence. Modal logic deals paradigmatically with the necessity or possibility of propositions. Hence, since Jaakko Hintikka (1962) first took up the project of using modal logic to study knowledge taken as a representational mental state, most of the ensuing research programme has dealt with the analysis of what we saw Lewis call knowledge-*that*, or theoretical knowledge.[61] Attention has also been paid more recently to the study of what we saw him call knowledge-*how*, or practical knowledge, but even this programme has been similarly focused on the question whether this latter type of knowledge is

[61] See van Benthem (2006) for a survey.

reducible or not to the theoretical type.[62] But the tendency might be partly due to Hintikka's non-neutral stance in his seminal work. For Hintikka, theoretical knowledge is the *only* type of knowledge, or at least the conceptually fundamental one. And that is related, of course, to his view (1969) of the very semantics of 'knows' as expressing a *propositional* attitude: a binary relation between a subject, the attitude holder, and a proposition.[63] A consequence of that view is that the kind of knowledge we're interested in here, knowledge by acquaintance, might have to tag along practical knowledge in being somehow analysed in terms of theoretical knowledge. As Hintikka himself notes:

> [My views] are incompatible with many well-known philosophical doctrines. To mention only one, if my analysis […] of the direct-object construction with 'knows' is essentially correct, I have in a sense disproved Russell's claim [that knowledge by acquaintance] is 'logically independent of knowledge of truths'. For in [my analysis] the *only* construction in which 'knows' occurs is (a shorthand for) 'knows that' (Hintikka 1970, p. 883, my italics).

In following Giaquinto's Russellian view, however, we've been assumed here precisely the doctrine Hintikka admits his view doesn't allow. Indeed, in our account, knowledge by acquaintance *must* be independent of theoretical knowledge because part of what we're doing may be understood as trying to ground the mathematical knowledge our subject seems to have acquired on something other than theoretical knowledge, which, as MacBride complains, cannot be appealed to in justifying the subject's formation of beliefs about the infinite lest the account be accused of circularity ('ahem, the word is *holism*', I can hear Shapito interject).

[62] See Bengson and Moffet (2012) for a survey.
[63] In short, the view is that different attitudes towards propositions (knowledge, belief, memory, hope, wish, etc.) are correlated with the different types of accessibility relations the world of evaluation has towards the worlds at which the proposition is true in the model.

Now, as the previous section showed, the propositional use of modal logic and the possible worlds heuristic needn't be a rule. But just to recall how this use works: Hintikka's analysis of knowledge attributions essentially consists in dividing the set of all possible worlds into those compatible and those incompatible with the subject's body of theoretical knowledge. The idea is to map a subject x and a world w into the set of possible worlds w' compatible with what x knows at w, which in turn means to take the set of possible worlds w' to be compatible with what x knows at w if and only if all of the propositions x knows at w are true in w'. So, in Hintikka's epistemic modal logic, the worlds accessed by the world of the subject's epistemic state, the 'world of evaluation', are the ones at which the known propositions are true. Schematically, this is not too different from the way we analysed potential infinity before. In Linnebo and Shapiro's modal logic, the world accessed by the actual world is the one at which the mathematical object to be constructed exists already, which means that it exists potentially at the actual world. To bring the parallel between both modal logics to the fore, this means that the *existence* of the next natural number is to the actual world in Linnebo and Shapiro's analysis of potential infinity as the *truth* of known propositions is to the world of evaluation in Hintikka's analysis of knowledge.

This is all just to say that the possible worlds framework serves just as well to accommodate propositional content as it does to accommodate objectual content. And if this is so, then there is no obvious reason why we couldn't take up Hintikka's own epistemic logic and tweak it to analyse knowledge by acquaintance. Given the materials we've got already, that shouldn't be too hard. Putting together again, on the one hand, Linnebo and Shapiro's idea of analysing the potentiality of object o at w as requiring its existence at (accessed world) w', and on the other, Hintikka's idea of analysing knowledge of p at w as requiring its truth at (accessed world) w', we might suggest as a first pass the idea of analysing *acquaintance* with object o at w as requiring its *existence* at (accessed world) w'.

Now, to be sure, all we're borrowing here from the last section's modal logic is the —perhaps trivial— point that one world can access another world's objects rather than just its truths. But the accessibility relation between worlds in the logic of knowledge by acquaintance would have to be of the same type as that of propositional epistemic logic. Just as in the latter the accessibility relation is reflexive to reflect the required truth of known propositions, so it would have to be in the former to reflect the required existence of known objects. This makes the modal system for both epistemic logics M/T.

But this yields a plausible epistemic logic for acquaintance as understood classically: what we can know by acquaintance is only the *actual*— what exists at *our* world. Indeed, in their formalisation of essentially the same logic for acquaintance I've just proposed, Iaquinto and Spolaore (forthcoming) point out that both requirements that a known proposition be true and that a known object exist come down to 'a more general principle that, echoing Parmenides [...], we could voice as "thou couldst not know that which is not"' (p. 3). This is the same constraint we'd encountered in our discussion of subitizing. Furthermore, because acquaintance is, as Russell put it, 'the converse of the relation of object and subject which constitutes *presentation*' (1911, p. 108, my italics), what we can know by acquaintance must not only exist in our world but also be *present* before us. This second point is important. Iaquinto and Spolaore require only that the known object exist because what they're analysing is a subject's entire body of knowledge by acquaintance, that is, the domain of objects she's acquainted with, whereas our case is one of a subject's *acquiring* such a piece of knowledge, that is, that domain growing in virtue of the subject's being presented with something new, which is why we have to add that the object acquired knowledge of must not only exist but be co-present with the subject to respect Russell's remark and, relatedly, the common view that the presence of objects before knowers is key to perceptual knowledge of them. As Crane and French (2017, §1.1.2 and §3.4) summarise the point, 'it seems that you can only see or hear or touch what is *there'* (my italics).[64]

The challenge for us is, of course, that the phenomenon we're suggesting our subject is acquiring acquaintance of, potential infinity, is modal in nature. It does not meet Crane and French's description as being directly *there*. This is an impasse, however, only if we think of our subject's mental state in static terms, or to think, again, of the domain of objects she knows as it is at *one* point in time. Just as we understood the phenomenon itself dynamically in the last section, then, we'd do better to understand our subject's knowledge of it dynamically as well, or to think of the domain of objects she knows as growing over time.

So let's see how the logic of that would work. To be sure, at any one point Sarah is only confronted with a finite number n of reflections. That is the set we might say forms the object of acquaintance actually present to her *there*. Return to the point at which Sarah opens her eyes. Call that point t_0. Let us attribute to Sarah knowledge by acquaintance of the nth reflection she sees at t_0 and analyse this attribution according to the epistemic logic suggested before as being true iff any world w' compatible with Sarah's epistemic state at w contains a set of n reflections.[65] Then she moves a little to the right. At t_1, we can say Sarah has acquired knowledge by acquaintance of the n+1th reflection iff any world w' compatible with Sarah's epistemic state at w contains a set of n+1 reflections. And so on. So at no individual point is Sarah acquainted with

[64] The addition to the existence requirement makes no difference to the logic, however, as the possible worlds framework is just a heuristic that we may interpret as modelling *situations* where the subject and object of knowledge are co-present. The logic will indeed change in virtue of the domain growing, but more on that shortly.
[65] I use the phrase 'w contains o' as the converse of 'o exists at w' for convenience.

more than finitely many reflections. But just as the system of reflections can be understood as potentially infinite if the worlds of our framework connect *over time* rather than it (the system) being only understandable as finite at any individual point, we can understand Sarah's state as acquiring perceptual knowledge of this potentially infinite system if we tweak our epistemic logic to analyse Sarah's state over time rather than analysing it statically. To do this, we have to take into account the process of Sarah's moving a bit from instant t_n to instant t_{n+1} while getting one more reflection into sight. As mentioned before, that process is the analogue of mathematical construction in Linnebo and Shapiro's logic. Similarly, the domain of objects Sarah is presented with grows from t_n to t_{n+1} just like the domain of natural numbers in Linnebo and Shapiro's logic grows as the actual world's largest number's (potential) successor is constructed and, hence, actualised. Finally, the analogue to Linnebo and Shapiro's assumption that no objects get destroyed as the worlds' domains grow is that Sarah doesn't forget there were prior reflections and, ultimately, an initial one. Putting both processes together, then, it turns out Sarah's epistemic state considered dynamically or when she is in motion may be represented as accessing, as it were, more than one world: the actual world, where the domain of presented objects contains only a finite number n of reflections, and the world that is accessed by the actual world, which contains its (the actual world's) last reflection's successor. This might seem pretty contentious, so let me pause and clarify it a bit. At any one point at which we pause Sarah's experience to examine it, she will certainly turn out to be presented with only that instant's actually finite number n of reflections. However, because she's aware of how this number grew to n after being n-1 only an instant before, she's aware already at the actual world w_n that the number will grow to n+1 just an instant from now; hence, she's aware already at the actual world w_n *of* this extra reflection contained in the later world w_{n+1}.[66] But this awareness is not a form of hunch or intuition. It is an instance, I submit, of bona fide perceptual knowledge. Only because we've artificially paused Sarah's experience *now* to examine it is the extra reflection not contained in the world accessed at this point, i.e. in the actual world; but perceptual experience is *constitutively temporal*. Even as I stand here looking at my screen, light takes time to travel from and around it to my eyes so that I can enter the state of seeing it; likewise, when I see a clock ticking, I can see the second-hand moving over time rather than just seeing it at its initial and final positions. [67] There's no obvious reason to include *those* times in our examinations of whether I am perceptually justified in believing there's a computer before me and believing the second-hand has moved but exclude the time Sarah is taking to move an inch, during which time she becomes aware of one more reflection, in our examination of whether she's perceptually justified in believing there's a successor to the last reflection she'd seen and a successor to that one and so on, and indeed, in her believing there *can always* be a successor.

So the intrinsic temporally extended nature of Sarah's experience is really the core of the argument here. Just as we can make sense of the system of reflections being potentially infinite, I am arguing we can make sense of Sarah's

---

[66] Starting now, I use 'S is aware of o' synonymously with 'S is acquainted with o' or 'S has knowledge by acquaintance of o', all of which I take to be true iff o exists at any world compatible with the subject's epistemic state at the world of evaluation.

[67] These two examples are meant to support, respectively, that perceptual experience itself is temporally extended —assuming, as Prosser (2016) puts it, that 'causal influence takes time to travel'—, and that so are its contents. This (twofold) claim is contentious and I am assuming it without much by way of argument here, but it goes back at least to William James' (1890) suggestion that conscious experience is stream-like to account for what he called, uncoincidentally, our experience of *succession* or of change. For discussion see e.g. Broad (1923), whose minute-hand example I'm borrowing, as well as Soteriou (2013), esp. pp. 28-30 and 135-154, and Prosser (op cit.), among others.

acquiring knowledge by acquaintance of *that*. If this seems convincing, we need only to upgrade the system of our epistemic logic from M/T to S4 to match it as closely as possible to the logic of the potential infinity of the reflections.[68] On the one hand, S4 already includes the reflexivity of the accessibility relation we need in order to ensure that all objects known in the world of evaluation exist in that world; and on the other, it includes the transitive property of the relation that we need in order to ensure that if the world of evaluation accesses the domain of the world of the potential successor of the last actual reflection, then it accesses the world next and the world next such that our subject counts as being aware not only of the next reflection but, ultimately, of the potential infinity of them.

Figure 7 illustrates the contrast between the dynamic nature of the logic of potential infinity and the logic of Sarah's awareness of it and the static nature of Iaquinto and Spolaore's epistemic logic of knowledge by acquaintance (which we admit, again, is rightly justified by their aim of analysing the subject's body of known objects at some point in time).

(All of the accessibility relations that obtain from any world w_n to any world $w_{m>n}$ due to transitivity are omitted for simplicity in the figure.)

[68] We cannot upgrade all the way to S4.2 because while that system is fit for the logic of mathematical construction, the temporal vocabulary with which we describe *that* does not quite match the vocabulary we can use to describe temporal progress in empirical reality, which is constrained in concrete ways mathematical construction isn't. Whereas in mathematical construction a neglected step remains always available for later use, such that both worlds w_1 and w_2 accessed from w_0 may converge in w_3, our behaviour in time is irreversible and tends to take us down non-convergent paths. Because we can think of the instants over which Sarah's experience is extended as a partial order, however, the properties of the accessibility relation in S4 should suffice to capture her epistemic state.

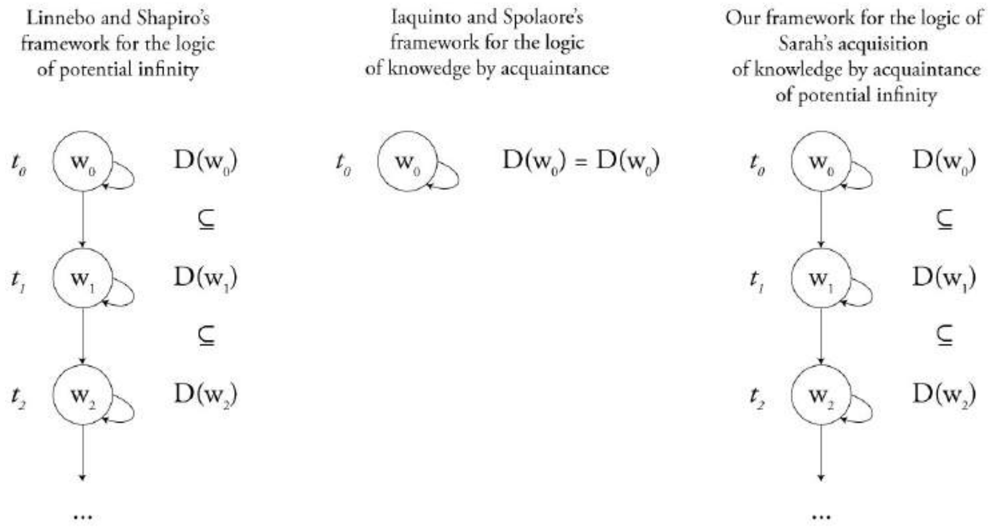


Figure 7.

The ascriptions of acquaintance work the same way as before. We can ascribe Sarah awareness of an *n*th reflection at w if and only if, for any world w' compatible with her epistemic state at w, it (the *n*th reflection) is in the domain of w'. The difference is that this description now turns out to be satisfied not just by the actual world but also the world containing the actual world's last reflection's successor, and the next, and the next. However, just as Linnebo and Shapiro's framework still marks a difference between the actual and potential existence of mathematical objects according to whether they are contained in the actual/current world or in potential/later worlds, that is, whether they have been constructed or not, so we can mark a difference between Sarah's actual awareness of reflections and her potential awareness of reflections according similarly to whether they are contained in the world of evaluation or in later worlds, that is, whether they've been brought into sight by her motion or not.

This difference can be made clear in the logic we've assumed as follows (borrowing partially from Linnebo and Shapiro op. cit., p. 27). Suppose that w

$\leq$ w' and let *a* be an object that exists at w' (so that $a \in D(w'))$. Then $\varphi(a)$ holds at w'. So $\Diamond\varphi(a)$ holds at w. Now suppose φ stands for the natural language predicate 'has been brought into existence' or 'has been constructed' in Linnebo and Shapiro's logic of potential infinity, and for the predicate 'is seen' in our logic of Sarah's acquaintance with the potential infinity of the reflections. Just as what it is for *a* to exist potentially at w (at t_0) is for it to have been constructed at w' (at t_1), what it is for Sarah to be potentially aware of *a* at w (at t_0) is for *a* to be actually seen by her at w' (at t_1).

Nevertheless, the predicates these modal operators qualify are not trivial. The successor of w's largest natural number *does* exist, even if potentially, at w as long as it is contained in w'. And Sarah *is* aware of the successor of w's last reflection, even if potentially, at w as long as it is actually brought into sight at w'. Whether this obtains, whether this 'as long as' condition is satisfied, is understood here not as depending on whether the arrangement of mirrors as a matter of *fact* allows it via rotation or otherwise or not but, again, as depending on the structure of the system itself—as a matter of *principle*. Sarah does not think she can, unlike Aristotle looking at the river, stay there and *actually* watch the reflections going on forever. Likewise, she did not need to be presented with an *actually* infinite number of reflections for her to grasp the structure under which they were ordered. What she became acquainted with *can* be understood, however, as a potentially infinite system, and it is *that*, I argue, what she was able to apply the concept of infinity to.

Whether this counts as an experience describable quite as strongly as being presented with an abstract entity is something we will refrain from claiming, but if asked whether it supports the (realist) belief that infinite structures *exist*, because the suggestion here is that we can in some way be acquainted with them, then we won't shy away from winking. The matter is for philosophers of mathematics to settle, though. Here, I hope only to have offered a plausible account of (that part of) our puzzle.

3.3

The foregoing was an effort to make sense of the claim that not only is potential infinity conceptually coherent, as Linnebo and Shapiro have argued, but that in fact a system such as the set of reflections formed by the mirror arrangement at the temple can be meaningfully understood as potentially infinite. Furthermore, we have attempted to make sense of the claim that it is possible for a subject to become perceptually aware of that property of such a system.

So what is left is to explain how that experience may fix the concept of infinity in a subject who has undergone it. We'd suggested towards the end of the last chapter that this concept may have been fixed in virtue of our subject's awareness *not* of a purported infinite portion of reality but of some of reality's structural features. If we can take ourselves to have established that she became acquainted with a potentially infinite system over the last two sections, then what is relevant about *that* is not the collection of reflections itself but the structural property they collectively displayed: the property of being ordered in the way the natural numbers are ordered, that is, their property of displaying the structure $\aleph_0$. And the system of reflections displayed this structural property just as legitimately as the set of natural numbers itself has it even though we've understood the former as potentially infinite and the latter as actually so. The modal qualification, in other words, does not affect our attribution of this structural property to these two systems because modality affects, as we'd put it, the 'internal nature' of the members forming those systems rather than the way they (the members) are related, and a system is attributed a structural property only as a matter of how the objects in it, whatever their internal nature, stand in relation to each other.

If this all seems right, then we can explain Sarah's acquisition of the concept of infinity straightforwardly by appeal to how Giaquinto and Shapiro explain that the process of abstraction from small, finite collections yields knowledge of small, finite structures. The one worry with this might be that, as we had put the process of abstraction following Giaquinto before, a structure is abstracted only upon acquaintance with (at least) *two* systems instantiating it, because to conform to the relevant abstraction principle is to exercise the ability to *recognise*, to borrow Frege's phrasing, 'this object [the structure of Set A] when it should occur in another guise [as the structure of Set B]'. Similarly, Shapiro's very name for the same process —'pattern *recognition*'— seems to follow Frege and imply that acquaintance with a structure requires acquaintance with more than one instance in more than one occasion. Indeed, as Giaquinto tells us, in Russell's original view, 'abstraction is an involuntary process resulting from exposure over time to *many* instances of the property that one comes to grasp' (Giaquinto 2012, p. 501, my italics). Our case fails to meet that description both in that the system of reflections was the first and only system displaying the structure $\aleph_0$ known to our subject (either by acquaintance *or* by description, as our discussion of children's knowledge of infinity showed), and in that her abstracting that structure from that system seems to have occurred instantly rather than over time.

To tackle this worry I don't have much more to say than to point out Giaquinto's own acknowledgement of Jackson's Mary case as an instance of abstraction of a property despite the fact that it is a case, like ours, of acquisition of the corresponding concept (a phenomenal and/or perceptual concept, for Mary) upon observation of an instance of it for the first time: '[t]he notion of broad acquaintance captures precisely what Frank Jackson's

Mary lacks with regard to scarlet [before she's been allowed out her black and white room]' (ibid., p. 503). Consider, additionally, that both Giaquinto and Shapiro are in the business of simply showing how it is *plausible* that knowledge via experience of small cardinals might work, which explains why they do not claim it a *necessary* condition that this knowledge arise from experience of more than one instance. Their project is not strongly normative in that way.

But here's another route. Consider what acquiring the concept of infinite number would carry. Acquiring the cardinal number concept '4' by abstracting the finite structure corresponding to that cardinal from exposure to quadruples of things, we'd said, carries the ability to recognise instances of that concept either in perceptually presented scenarios, such as the sight of quadruples of things, or in intellectually presented scenarios, such as the written string $\{x \in \mathbb{N} : x \leq 3\}$ (provided one knows additionally set theory notation). It also carries the ability to infer from e.g. 'there are three Beatles dead and only one still alive' that there are in total four Beatles (provided one knows dead and alive are mutually exclusive and jointly exhaustive properties Beatles can have). Both types of ability amount to conforming to the rules governing that concept.

Now, what about the concept of infinite number? As pointed out before, the rules of inference associated with that concept can be identified with none other than the Dedekind-Peano axioms of arithmetic. One way to see whether Sarah really did abstract knowledge of this structure from acquaintance with that one instance at the temple, then, is to see whether she thereby became able to make inferences that conform to the Dedekind-Peano axioms, just like Tall's son became able to via exposure to theory.

This would of course require actually testing Sarah's four-year-old self. Consider, however, that Sarah's perceptual awareness of the potential infinity of the reflections is equivalent to what Charles Parsons argues constitutes the non-theoretical awareness of a similarly infinite set, a sequence of stroke-strings '|, ||, |||, …' forming an ω-sequence we can 'intuit' just like we intuit other 'quasi-concrete' abstract objects such as 'letters, words, sounds and shapes',[69] which non-theoretical awareness he thinks grounds, '*in a certain sense*, our knowledge of the first four of the Dedekind-Peano axioms', that is, 'our knowledge of the infinitude of the natural numbers'.[70]

Parsons' view is complex and we will not attempt to defend it. Most of the objections against it, however, turn on the obscurity of the faculty of intuition he appeals to. As Robin Jeshion tells us, whereas we know what it is to come to know things via theory or via experience, '[t]he notion of intuition', by contrast, 'seems murky and obscure, and is widely deemed a creature of darkness' (2014, p. 330). Given we're not relying on Parsons' intuition (or Shapiro's hunches and/or innate knowledge) here, perhaps we can borrow Parsons' explication of how the purported intuition of the mathematical objects his stroke-strings are[71] grounds *propositional* 'knowledge of statements analogous to the Dedekind-Peano axioms' (ibid., p. 336), and adapt this explication to ground Sarah's knowledge of the concept of infinity on her one-off experience of the potentially infinite reflections.

After introducing a language *L* of which the stroke '|' is the only symbol and the well-formed expressions are strings such as '|, ||, |||, …', Parsons basically interprets the string formed by a single stroke as zero and the operation of adding one stroke to the right as the successor operation. This renders the sequence '|, ||, |||, …' isomorphic to the set of the natural numbers. Then, the statement '||| is the successor of ||' is justifiedly believed to be true in

[69] This reference is to Jeshion's (2014, p. 330) defence of Parsons' view.
[70] Ibid, pp. 327 and 336.
[71] They count as arithmetical objects '[i]nsofar as they together constitute an ω-sequence' (ibid., p. 335).

virtue of one's intuition that each string can be extended by one stroke more, which constitutes 'the weakest expression of the idea that our "language" is potentially infinite' (Parsons 1980, p. 105). Knowing this about *L* amounts to knowing the following:

> (PA1') | is a stroke string.
> (PA2') | is not the successor of any stroke string.
> (PA3') Every stroke string has a successor that is also a stroke string.
> (PA4') Different stroke strings have different successors.

These statements are analogous to the first four Dedekind-Peano axioms:

> (PA1) Zero is a natural number.
> (PA2) Zero is not the successor of any natural number.
> (PA3) Every natural number has a successor that is also a natural number.
> (PA4) Different natural numbers have different successors.[72]

Seeing how intuitive awareness of the stroke strings yields awareness of the propositions (PA1') and (PA2') is, Jeshion says, 'straightforward'; '[t]he axiom that presents the most complexities is (PA3')' (ibid., pp. 336-7). That is because (PA3') amounts to the general thought we saw McBride demand a non-theoretical warrant for in the introduction, which very worry Bob Hale and Crispin Wright press against Parsons' view as well: '[t]he problem, of course, is to see how, following Parsons' intuitive route, knowledge of *general* truths about intuited objects [...] can be achieved' (2002, p. 106). Jeshion's and Parsons' answers to this challenge rely on imagination's —the basis for intuition— capacity to deliver 'vague', i.e. 'arbitrary', i.e. *non-particular*[73] mental representations (of the stroke strings), which non-particularity Hale and Wright regard —even putting their many qualms about intuition aside— as

[72] Both this formulation of the axioms and their *L* analogues are Jeshion's (op. cit.., p. 336).
[73] This is *my* way of putting it.

insufficient for the requisite *generality* (of knowledge about *all* the stroke strings), or in other words, as insufficient to justify genuine universal quantification (over *all* stroke strings).[74]

We shall interrupt our discussion of Parsons' view here and simply point back at the previous section to show how we've already circumvented the analogous problem. Our Linnebo and Shapiro-inspired epistemic logic was designed precisely to explicate how Sarah was able to gain awareness not just of the current and next reflection but of their potential infinity, that is, awareness —in this modally qualified sense— of *all* of the reflections, because, recall, 'the modalized quantifiers […] generalize across all (accessible) possible worlds rather than a single world'.

Hence, if we can take the first reflection seen by Sarah to stand for zero and her seeing of every successive reflection to stand for the successor operation, then, by Jeshion's (presentation of Parsons') demonstration of how awareness of those two things yields awareness of the PA' truths (Jeshion op. cit., p. 336), we have that Sarah has come to grasp that:

> (PA1") [Demonstrative referring to first reflection] is a reflection.
> (PA2") [Demonstrative referring to first reflection] is not the successor of any reflection.
> (PA3") Every reflection has a successor that is also a reflection.
> (PA4") Different reflections have different successors.

(One caveat: (PA3") is actually the result of translating what I suggested Sarah comes to perceptually grasp, the Aristotelian thought that every reflection *can always* have a successor, into the *non-modal* language of ordinary mathematics, which we know we can do unproblematically thanks to Linnebo and Shapiro's mirroring theorem: an entailment obtains in the non-modal language of ordinary mathematics whenever its modal counterpart obtains in

[74] This is just one of their diagnosed problems with it. See Hale and Wright (op. cit.), pp. 108-11.

the potentialist's language—which, in other words, states that 'classical first-order Dedekind-Peano arithmetic [...] is *equivalent* [to a potentialist translation of that theory]' (Linnebo and Shapiro op. cit., p. 16, my italics).)

If this seems right, then by grasping the truths (PA1") - (PA4"), Sarah also grasped what Jeshion's defence of Parsons' view implies might be the hardest —though in her view, not impossible— truth for Parsons to justify intuitive knowledge of, the stroke-string analogue of the Axiom of Infinity:

> (Axiom of Infinity') There are infinitely many distinct strings of stroke (ibid., p. 337),
>
> or, in Sarah's case:
>
> (Axiom of Infinity") There are infinitely many distinct reflections,

which knowledge of is sufficient for knowledge of the general thought (PA3"), as knowing (PA3") is necessary for knowledge of *it*,[75] and which is exactly what Sarah is telling us she came to understand:

> I looked into the images and was amazed to see that they never ended; [...] the reflections kept going and going. 'Oh,' I thought, with a chill of understanding. 'Forever'.

One last worry here might be that knowledge of (PA1") – (PA4") is knowledge about the system of reflections and that it is not obvious this is equivalent to knowledge of (PA1) – (PA4), i.e. to genuine mathematical knowledge. But recall that, in the structuralist framework we've assumed, mathematical knowledge is knowledge about the structures shared by isomorphic systems, either concrete or abstract. Hence, because the set of reflections is isomorphic to the set of natural numbers, *both* (PA1") – (PA4")

[75] Along with two other truths, in the case of the stroke language: that 'we can indefinitely iterate the operation of adding one new string' and 'the new string of strokes obtained from adding a new stroke is in fact new'. See ibid., p. 336, fn. 14.

and (PA1) – (PA4) constitute knowledge about the structure $\aleph_0$, i.e. knowledge of (the rules of inference governing the concept of) infinity.

If Sarah became aware of those rules, then, we may suppose she would have indeed been able to answer the questions Tall's son did had she additionally acquired relevant —but non-arithmetical— knowledge such as the name for the first infinite cardinal. That is how she can be said to meet the inference-enabling condition we'd discussed earlier on. And by learning what a (potentially) infinite structure *looked like*, and thereby becoming able to recognise instances of the concept in differently presented scenarios —e.g. the perceptually presented scenario of the mirrors and the intellectually presented scenario of being *told* she and her family would be together forever—, she can be said to meet the sorting condition. Finally, her '"Oh," I thought, with a chill of understanding' phrasing suggests she also acquired a sense of normativity. After the experience, it seems she became *confident* that she could make, as we had put it, both normative judgements 'this way of conceiving is appropriate' and 'that F is an F and ought to be conceived as such'.

So this is how, I submit, Sarah's experience fixed the concept of infinite number in her repertoire—or at least, if it is a matter of degree, how she started to understand it.

And that is also how our story comes to its end.

4. Conclusion

In his original defence of the claim that knowledge by acquaintance 'is essentially simpler than [...] and logically independent of knowledge of truths', Russell acknowledges that, still, 'it would be rash to assume that human beings ever, in fact, have acquaintance with things without at the same time knowing some truth about them' (Russell op. cit., p. 25).

Here I have claimed that the puzzle we presented constitutes a case of a subject's acquiring perceptual knowledge of the structural properties of a potentially infinite system, in particular, her acquiring knowledge by acquaintance of the infinite structure corresponding to the cardinal $\aleph_0$, which suffices to generalise to the 'thesis of my thesis': that knowledge of infinity by acquaintance is indeed possible. I have also claimed, as Russell predicted, that this perceptual knowledge came with some knowledge of truths about that structure, which we can spell out as truths analogous to the first four Dedekind-Peano axioms. To count as aware of these truths, our subject needn't be able to state them; it suffices that her rational behaviour conform to them. The fact that she was able to recognise both the system of reflections and the concept of 'forever' as, respectively, perceptually and intellectually presented instances of the concept of infinity, I took it suggested that she had grasped its *nature*, which means that she would be able to make something like the correct —i.e., the rule-following— judgement '$\aleph_0 + \aleph_0 = \aleph_0$', which Tall's son came to be able to make only via theory at age 7. Importantly, because this is a cardinal number concept, I also took it that our subject acquired the disposition to answer, if confronted with the question 'how many reflections are there?', 'infinitely many'.

Now, this might all seem a little pointless if our aim had just been to give an idiosyncratic explication of Sarah's anecdote. I'm not going to argue with that. However, let me point out some consequences of broader interest of that explication. First: as Hale and Wright point out, it is widely assumed that in providing an epistemology of mathematics, '[t]wo broad approaches seem possible: *intuitional* and *intellectual*' (Hale and Wright op. cit., p. 104). Parsons' and Shapiro's views on knowledge of infinity might be roughly put in the first box, Giaquinto's in the second.[76] By pressing his access problem, MacBride seemed to demand a third approach. How can we know that all the natural numbers have a successor other than by a hunch or intuition *or* by already knowing the principles that allow inferring the general from the particular? In grounding our subject's knowledge of the foregoing truth about the structure $\aleph_0$ on neither intuition nor theory but on perceptual experience, we have, in effect, challenged that wide assumption—we have suggested that perceptual access to mathematical knowledge is possible. Strictly speaking, that idea had been defended by Shapiro and Giaquinto themselves before insofar as knowledge of small cardinals counts as mathematical knowledge; however, in suggesting that perceptual access to knowledge of the structure $\aleph_0$ is possible, we have made the stronger suggestion that perceptual access to (some)[77] knowledge of *arithmetic* is possible, and have thereby tackled the heart of McBride's access problem: the problem of knowledge of infinity. If the argument for this is convincing, I take it that it would constitute an exciting result.

Moreover, the way to get there passed by another contentious claim: that perceptual knowledge of modality is possible too. By appealing to the temporally extended nature of perceptual experience, I suggested that our

[76] I'm referring to the hunch and/or innate knowledge we saw Shapiro suggest to account for the jump to the general thought. He also offers another way in earlier work, 'characterisation', which is straightforwardly intellectual —Hale and Wright (op. cit., p. 112-3) call it a 'canonical axiomatic *description*'—, but which is not relevant to our discussion.

[77] We've failed, like Parsons, to justify knowledge of an analogue to the fifth Dedekind-Peano axiom, without which, as Hale and Wright (op. cit., p. 108) point out, *full* knowledge of arithmetic isn't accounted for.

subject's epistemic state represented a potential infinity of objects as well as actual ones. She didn't only learn of infinity by perception but also of a modal fact—the fact we represented as the Aristotelian claim '$\Box\forall m \Diamond \exists n$ SUCC(m,n)'.

True: both the mathematical and the modal knowledge perceptually accessed by our subject involved rather peculiar circumstances. So perhaps the interest of our test case lies in the questions it opens up rather than in our explication of it. Does grounding a subject's knowledge of arithmetic on what is viewed as a more fundamental type of knowledge prove any better epistemologically than Shapiro's holism—is there any value in doing what we've done? If so, under what other, more common circumstances could we say we acquire perceptual knowledge of mathematics? On the other hand, would our subject's purported perceptual access to modal knowledge count against Tim Williamson's (2007) well-known view that there is no evidential role for sense experience in the epistemology of modality? Specifically: should we understand Sarah's experience as providing *perceptual evidence*, i.e. as providing justification or support of some degree for a modal belief, or should we rather, compatibly with Williamson's claim that perception's role is merely enabling here, understand the experience as simply *making* the modal fact *evident*, i.e. as enabling Sarah to know it?[78] Finally, more specifically to our case: does Sarah know anything mathematicians who learn of $\aleph_0$ via theory do not know—is there any added epistemic value to 'mathematical experiences', as her adult self calls them? Did she, furthermore, learn anything deep about 'life and death, power and control, the beginning of time and the end of the Universe'?

[78] Thanks to Mark Kalderon for (Austinian) illumination on this question, which is of course more complex than I've presented it here. For example: perception making something evident might also count as perception providing evidence of it if we accept Schellenberg's (2013) notion of *factive* perceptual evidence.

I shall explore some of these questions in further work. Until then, I hope this thesis has at least pointed us towards what might otherwise seem an improbable research programme: our experience of the structural features of reality.

References

Aristotle. 1942. *The Basic Works of Aristotle*. NY: Random House

Bagaria, J. 2017. Set Theory. *The Stanford Encyclopedia of Philosophy* (Winter 2017 Edition), Edward N. Zalta (ed.), URL = <https://plato.stanford.edu/archives/win2017/entries/set-theory/>.

Benacerraf, P. 1973. Mathematical truth. *Journal of Philosophy*, 70, 661–679

Bengson, J. and Moffett, M. A.. 2012. Two Conceptions of Mind and Action: Knowledge How and the Philosophical Theory of Intelligence. In *Knowing How: Essays on Knowledge, Mind, and Action*. Oxford: Oxford University Press

van Benthem, J. 2006. Epistemic logic and epistemology: the state of their affairs. *Philosophical Studies* 128: 49–76

Broad, C. D. 1923. *Scientific Thought*. London: Kegan Paul

Burgess, J. and Rosen, G. 1997. *A Subject with No Object: Strategies for Nominalistic Interpretation of Mathematics*. Oxford: OUP

Cantor, G. 1874. Uber eine Eigenschaft des Inbegriffes aller reelen algebraischen Zahlen. *Journal für die reine und angewandte Mathematik* 77, 258–62.

Chalmers, D. 2002. Does conceivability entail possibility?. In T. Gendler and J. Hawthorne (eds.). *Conceivability and Possibility*. Oxford: Oxford University Press

Crane, T. and French, C. 2017. The Problem of Perception. In Edward N. Zalta (ed.) *The Stanford Encyclopedia of Philosophy (Winter 2016 Edition)*

Davies, J. E. 2017. Towards a Theory of Singular Thought About Abstract Mathematical Objects. *Synthese,* doi: 10.1007/s11229-017-1644-0

Dretske, F. 1969. *Seeing and Knowing*. London: Routledge and Kegan Paul

Dummett, M. 1978. The philosophical basis of intuitionistic logic. In *Truth and Other Enigmas*. Cambridg, MA: Harvard University Press, pp. 215–247

Dummett, M. 1991. *Frege: Philosophy of Mathematics*. Cambridge, MA: Harvard University Press

Ebert, A. and Rossberg, M., eds. 2016. *Abstractionism: Essays in the Philosophy of Mathematics.* Oxford: OUP

Fodor, J. 1990. A theory of content II. In Jerry Fodor, *A Theory of Content*. Cambridge: MIT Press

Fodor, J. 1994. Concepts: A potboiler. *Cognition*, 50, 95–113

Frege, G. 1884/1953. Translated by J.L. Austin. *The Foundations of Arithmetic*. Oxford: Blackwell

Gbur, G. 2011. Infinity is weird… even in infinity mirrors! Skulls in the Stars: The intersection of physics, optics, history and pulp fiction. Available at https://skullsinthestars.com/2011/07/30/infinity-is-weird-even-in-infinity-mirrors/

Gendler, T. S., and Hawthorne, J. (eds.). 2002. *Conceivability and Possibility*. New York: Oxford University Press

Giaquinto, M. 2001. Knowing Numbers. *Journal of Philosophy*, xcviii

Giaquinto, M. 2007. *Visual thinking in mathematics*. Oxford: OUP

Giaquinto, M. 2012. Russell on Knowledge of Universals by Acquaintance. *Philosophy*, 87, doi:10.1017/S0031819112000381

Ginsborg, H. 2006. Empirical concepts and the content of experience. *European Journal of Philosophy,* 14(3), doi: 10.1111/j.1468-0378.2006.00230.x

Hale, B., and Wright, C. 2002. Benacerraf's dilemma revisited. *European Journal of Philosophy*, 10, 101–129.

Higginbotham, J. 1998. Conceptual Competence. *Philosophical Issues*, 9, pp.149-152

Hintikka, J. 1962. *Knowledge and Belief*. Ithaca, NY: Cornell University Press

Hintikka, J. 1966. Aristotelian Infinity. *Philosophical Review*, 75, pp. 197-218

Hintikka, J. 1969. Semantics for propositional attitudes. In J.W. Davis, D.J. Hockney & W.K. Wilson (eds.), *Philosophical Logic*. Dordrecht: Reidel, pp. 21–45

Hintikka, J. 1970. Objects of knowledge and belief: acquaintances and public figures. *Journal of Philosophy,* 67, pp. 869–83

Iaquinto, S. and Spolaore, G. Forthcoming. Outline for a logic of knowledge of acquaintance. *Analysis*, doi: 10.1093/analys/any005

James, W. 1890/1983. *The Principles of Psychology*. Cambridge, MA: Harvard University Press

Jackson, F. 1986. What Mary Didn't Know. *Journal of Philosophy*, 83(5), pp. 291-295

Jeshion, R. 2010. Acquaintance, semantic instrumentalism, and cognitivism. In R. Jeshion (ed.), *New essays on singular thought*. Oxford: Oxford University Press, pp. 105–140

Jeshion, R. 2017. Intuiting the infinite. *Philosophical Studies,* 171, doi: 10.1007/s11098-013-0274-8

Korbmacher, J. and Schiemer, G. 2017. What are Structural Properties? *Philosophia Mathematica*, nkx011, doi: 10.1093/philmat/nkx011

Lear, J. 1979. Aristotelian Infinity. *Proceedings of the Aristotelian Society*, 80, 187–210

Lewis, D. 1990. What experience teaches. In W.G. Lycan (ed.). *Mind and cognition*. Oxford: Blackwell

Linnebo, Ø. 2013. The potential hierarchy of sets. *The Review of Symbolic Logic*, 6(2), 205-228, doi:10.1017/S1755020313000014

Linnebo, Ø. and Shapiro, S. 2017. Actual and Potential Infinity. *Noûs*, doi: 10.1111/nous.12208

Loar, B. 1990. Phenomenal States. *Philosophical Perspectives*, 4, 81-108, doi: 10.2307/2214188

MacBride, F. 2008. Can Ante Rem Structuralism Solve the Access Problem? *The Philosophical Quarterly*, 58, pp.155-64.

Mac Cumhaill, C. 2011. Specular space. *Proceedings of the Aristotelian Society*, cxi(3), 487–495. doi:10.1111/j.1467-9264.2011.00320.x

Maddy, P. 1997. *Naturalism in Mathematics*. Oxford: OUP

Maddy, P. 2007. *Second Philosophy: a Naturalistic Method*. Oxford: OUP

Moore, A. W. 2001. *The Infinite*. Oxford: OUP

Papineau, D. 2002. Thinking about Consciousness, Oxford: Oxford University Press

Papineau, D. 2009. Phenomenal and perceptual consciousness. In Torin Alter & Sven Walter (eds.), *Phenomenal Concepts and Phenomenal Knowledge: New Essays on Consciousness and Physicalism*. Oxford: Oxford University Press, pp. 111--144

Parsons, C. 1980. Marhematical Intuition. *Proceedings of the Aristotelian Society*, 80, pp. 145-68.

Parsons, C. 1990. The structuralist view of mathematical objects. *Synthese* 84, pp. 303–346

Parsons, C. 2007. *Mathematical Thought and its Objects*. Cambridge: CUP

Peacocke, C. 1992. *A Study of Concepts*. Cambridge: MIT Press

Pehkonen, E., et al. 2006. Infinity of numbers: how students understand it. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, 4, pp. 345-352

Piazza, M. et al. 2011. Subitizing reflects visuo-spatial object individuation capacity. *Cognition* 121, pp.147-153

Prior, A. 1957. *Time and Modality*. Oxford: Clarendon Press

Prosser, S. 2016. Is Experience Temporally Extended? In *Experiencing Time.* Oxford: Oxford University Press

Railo, H. et al. 2008. The role of attention in subitizing. *Cognition* 107, pp.82-104

Railo, H. and Hannula-Sormunen, M. 2012. Subitizing. *Encyclopedia of the Sciences of Learning*, pp. 3233-3235

Resnik, M. 1996. Structural relativity. *Philosophia Mathematica* (3) 3, 83–99

Russell, B. 1911. Knowledge by acquaintance and knowledge by description. *Proceedings of the Aristotelian Society*, 11(5), 108–28.

Salmon, N. 2018. Julius Caesar and the numbers. *Philosophical Studies* 175, doi: 10.1007/s11098-017-0927-0

Sawyer, S. Forthcoming. The Importance of Concepts. *Proceedings of the Aristotelian Society*

Schellenberg, S. 2013. Experience and evidence. *Mind,* 122(487), pp. 699-747

Scoles. S. 2016. How thinking about infinity changes kids' brains on math. *Aeon.* Available at https://aeon.co/ideas/how-thinking-about-infinity-changes-kids-brains-on-math

Shapiro, S. 1997. *Philosophy of Mathematics: Structure and Ontology*. Oxford: Oxford University Press

Shapiro, S. 2011. Epistemology of Mathematics: What are the Questions? What count as Answers? *The Philosophical Quarterly*, 61, doi: 10.1111/j.1467-9213.2010.658.x

Shapiro, S. and Wright, C. 2006. All Things Indefinitely Extensible. In Rayo and Uzquiano (eds.), *Absolute Generality*. Oxford: OUP

Steenhagen, M. 2017. False reflections. *Philosophical Studies,* 174, doi: 10.1007/s11098-016-0752-x

Soteriou, M. 2013. *The Mind's Construction*. Oxford: OUP

Soysal, Zeynep. 2017. Why is the universe of sets not a set? *Synthese*, doi: 10.1007/s11229-017-1513-x

Tall, D. 1980. Mathematical intuition, with special reference to limiting processes. *Proceedings of the Fourth International Congress on Mathematical Education*. Berkeley, CA. 170176. Available at: http://www.warwick.ac. uk/staff/david.tall

Tall, D. 2001. A child thinking about infinity. *Journal of Mathematical Behavior*, 20, pp.7-19

Vetter, P. 2009. How many can you see at a glance? The Role of Attention in Visual Enumeration. Doctoral dissertation, UCL

Weiskopf, D. and Bechtel, W. 2004. Remarks on Fodor on Having Concepts. *Mind and Language* 19, 48-56

Williamson, T. 2007. Philosophical Knowledge and Knowledge of Counterfactuals. In C. Beyer and A. Burri (eds), *Philosophical Knowledge: its Possibility and Scope*. Amsterdam: Rodopi

Yablo, S. 2002. Coulda, Woulda, Shoulda. In T. Gendler and J. Hawthorne (eds.). *Conceivability and Possibility*. Oxford: Oxford University Press

Like the hero of this story, I too encountered infinity at an early age.

This thesis is dedicated to those who've showered me with infinite love since then, especially Bárbara, Darío and Manuel, but also to those who've showered me with so much in more recent times, especially Clémence, José Antonio and, in particular, Josh.

Nothing I can say or do would ever reflect my debt to you.

Endless thanks.