A cut finite element method with boundary value correction for the incompressible Stokes equations

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Abstract. We design a cut finite element method for the incompressible Stokes equations on curved domains. The cut finite element method allows for the domain boundary to cut through the elements of the computational mesh in a very general fashion. To further facilitate the implementation we propose to use a piecewise affine discrete domain even if the physical domain has curved boundary. Dirichlet boundary conditions are imposed using Nitsche's method on the discrete boundary and the effect of the curved physical boundary is accounted for using the boundary value correction technique introduced for cut finite element methods in Burman, Hansbo, Larson, *A cut finite element method with boundary value correction*, Math. Comp. 87(310):633–657, 2018.

1 Introduction

Let Ω be a domain in $\mathbb{R}^d$ with smooth boundary $\partial\Omega$ and exterior unit normal n. We will consider a cut finite element method (CutFEM) for Stokes problem on Ω with Dirichlet boundary conditions. See [3] and the references therein for an introduction to CutFEM. The Stokes problem takes the form: find $u : \Omega \to \mathbb{R}^d$ and $p : \Omega \to \mathbb{R}$ satisfying the weak form of the system of equations

$$-\Delta u + \nabla p = f \qquad \text{in } \Omega, \tag{1}$$

$$\nabla \cdot u = 0 \qquad \text{in } \Omega, \tag{2}$$

$$u = g \qquad \text{on } \partial\Omega, \tag{3}$$

where $f \in [H^{-1}(\Omega)]^d$ and $g \in [H^{1/2}(\partial\Omega)]^d$, $\int_{\partial\Omega} g \cdot n \, \mathrm{d}s = 0$, are given data. It follows from the Lax-Milgram Lemma that there exists a unique solution $u \in [H^1(\Omega)]^d$ and from Brezzi's Theorem that there exists a unique solution $p \in L^2_0(\Omega)$. We also have the elliptic regularity estimate

$$\|u\|_{H^{s+2}(\Omega)} + \|p\|_{H^{s+1}(\Omega)} \lesssim \|f\|_{H^s(\Omega)} + \|g\|_{H^{s+3/2}(\partial\Omega)}, \qquad s \geq -1. \tag{4}$$

See [4] for further details. Here and below we use the notation $\lesssim$ to denote less or equal up to a multiplicative constant and we let $(\cdot,\cdot)_X$ denote the L^2-scalar product over X with associated norm $\|\cdot\|_X$.

The objective of the present paper is to propose a cut finite element method for the problem (1)-(3). Unfitted finite element methods for incompressible elasticity was first discussed in [1], for the coupling over an internal (unfitted) interface. The fictitious domain problem for the Stokes equations was then considered in [6,11,5] and more recently the inf-sup stability for several different well-known elements on unfitted meshes was proved [8] and further work on the Stokes interface problem was presented in [9]. The upshot in the present contribution is that we, following [7], use a piecewise affine representation of the physical boundary and introduce a correction in the Nitsche formulation to correct for the low order geometry error. This allows us to use for instance a piecewise affine levelset for the geometry representation used in the integration over the cut elements, while retaining the accuracy of a (known) higher order representation of the boundary. This provides an alternative to representing curved boundaries using isoparametric mappings, see [10] for an application of this technique to the Stokes equations. We will focus herein on the derivation of the CutFEM boundary value correction method for the Stokes system, this is the topic of Section 2. We then state some fundamental results (without proof) in Section 3 and finally we report some numerical examples in Section 4.

2 The CutFEM for Stokes equations - derivation

Here we will give the elements of the numerical modelling that leads to the cut boundary value correction method for Stokes equations.

2.1 The domain

We let ϱ be the signed distance function to $\partial\Omega$, negative on the inside of the domain and positive on the outside, and we let $U_\delta(\partial\Omega)$ be the tubular neighborhood $\{x \in \mathbb{R}^d : |\varrho(x)| < \delta\}$ of $\partial\Omega$. Then there is a constant $\delta_0 > 0$ such that the closest point mapping $\mathfrak{p}(x) : U_{\delta_0}(\partial\Omega) \to \partial\Omega$ is well defined and we have the identity $\mathfrak{p}(x) = x - \varrho(x) n(\mathfrak{p}(x))$. Here we recall that $n(\mathfrak{p}(x))$ denotes the outward pointing normal of $\partial\Omega$ at the point $\mathfrak{p}(x)$. We assume that δ_0 is chosen small enough so that $\mathfrak{p}(x)$ is well defined.

2.2 The mesh, discrete domains, and finite element spaces

- Let $\Omega_0 \subset \mathbb{R}^d$ be a convex polygonal domain such that $U_{\delta_0}(\Omega) \subset \Omega_0$, where $U_\delta(\Omega) := U_\delta(\partial\Omega) \cup \Omega$. Let $\mathcal{K}_{0,h}, h \in (0, h_0]$, be a family of quasiuniform partitions, with mesh parameter h, of Ω_0 into shape regular triangles or tetrahedra K. We refer to $\mathcal{K}_{0,h}$ as the background mesh.

– Given a subset ω of Ω_0, let $\mathcal{K}_h(\omega)$ be the submesh defined by

$$\mathcal{K}_h(\omega) = \{K \in \mathcal{K}_{0,h} : \overline{K} \cap \overline{\omega} \neq \emptyset\},$$

i.e., the submesh consisting of elements that intersect $\overline{\omega}$, and let

$$\mathcal{N}_h(\omega) = \cup_{K \in \mathcal{K}_h(\omega)} K,$$

be the union of all elements in $\mathcal{K}_h(\omega)$. Below the L^2-norm of discrete functions frequently should be interpreted as the broken norm. For example for norms over $\mathcal{N}_h$ we have

$$\|v\|^2_{\mathcal{N}_h(\omega)} := \sum_{K \in \mathcal{K}_h(\omega)} \|v\|^2_K.$$

– Let Ω_h, $h \in (0, h_0]$, be a family of polygonal domains approximating Ω, possibly independent of the computational mesh. We assume neither $\Omega_h \subset \Omega$ nor $\Omega \subset \Omega_h$, instead the accuracy with which Ω_h approximates Ω will be important.
– Let the active mesh $\mathcal{K}_h$ be defined by

$$\mathcal{K}_h := \mathcal{K}_h(\Omega \cup \Omega_h),$$

i.e., the submesh consisting of elements that intersect $\Omega_h \cup \Omega$, and let

$$\mathcal{N}_h := \mathcal{N}_h(\Omega \cup \Omega_h),$$

be the union of all elements in $\mathcal{K}_h$.
– Let $V^k_{0,h}$ be the space of piecewise continuous polynomials of degree k defined on $\mathcal{K}_{0,h}$ and let the finite element space V^k_h be defined by

$$V^k_h := \{v_h : v_h := \tilde{v}_h|_{\mathcal{N}_h} \text{ for } \tilde{v}_h \in V^k_{0,h}\}.$$

– To each Ω_h we associate the outward pointing normal $\nu_h : \partial\Omega_h \to \mathbb{R}^d$, $|\nu_h| = 1$, and the distance from $\partial\Omega_h$ to $\partial\Omega$ in the direction ν_h, $\varrho_h : \partial\Omega_h \to \mathbb{R}$, such that if $\mathfrak{p}_h(x, \varsigma) := x + \varsigma\nu_h(x)$ then $\mathfrak{p}_h(x, \varrho_h(x)) \in \partial\Omega$ for all $x \in \partial\Omega_h$. We will also assume that $\mathfrak{p}_h(x, \varsigma) \in U_{\delta_0}(\Omega)$ for all $x \in \partial\Omega_h$ and all ς between 0 and $\varrho_h(x)$. For conciseness we will drop the second argument of $\mathfrak{p}_h$ below whenever it takes the value $\varrho_h(x)$. We assume that the following assumptions are satisfied

$$\delta_h := \|\varrho_h\|_{L^\infty(\partial\Omega_h)} = o(h^\zeta), \qquad h \in (0, h_0], \tag{5}$$

and

$$\|\nu_h - n \circ \mathfrak{p}\|_{L^\infty(\partial\Omega_h)} = o(h^{\zeta-1}), \qquad h \in (0, h_0], \tag{6}$$

where $\zeta \in \{1, 2\}$. We also assume that h_0 is small enough for some additional geometric conditions to be satisfied, for details see [7, Section 2.3].

2.3 Numerical modelling

We now proceed to show how to obtain a boundary value correction formulation for the Stokes system.

Derivation. Let $f = Ef$, $u = Eu$, and $p = Ep$, be the extensions of f and u from Ω to $U_{\delta_0}(\Omega)$. For $v \in V_h$ we have using Green's formula

$$\begin{aligned}(f, v)_{\Omega_h} &= (f + \Delta u - \nabla p, v)_{\Omega_h} - (\Delta u - \nabla p, v)_{\Omega_h} \\ &= (f + \Delta u - \nabla p, v)_{\Omega_h \setminus \Omega} + (\nabla u, \nabla v)_{\Omega_h} - (p, \nabla \cdot v)_{\Omega_h} \\ &\quad - (\nu_h \cdot \nabla u - \nu_h p, v)_{\partial\Omega_h},\end{aligned}$$

where we used the fact $f + \Delta u - \nabla p = Ef - \Delta Eu - \nabla Ep$, which is not in general equal to zero outside Ω. Now the boundary condition $u = g$ on $\partial\Omega$ may be enforced weakly as follows

$$\begin{aligned}(f, v)_{\Omega_h} = (f + \Delta u - \nabla p, v)_{\Omega_h \setminus \Omega} + (\nabla u, \nabla v)_{\Omega_h} - (p, \nabla \cdot v)_{\Omega_h} \\ - (\nu_h \cdot \nabla u - \nu_h p, v)_{\partial\Omega_h} - (u \circ \mathfrak{p}_h - g \circ \mathfrak{p}_h, \nu_h \cdot \nabla v)_{\partial\Omega_h} \\ + \beta h^{-1}(u \circ \mathfrak{p}_h - g \circ \mathfrak{p}_h, v)_{\partial\Omega_h}.\end{aligned}$$

Since we do not have access to $u \circ \mathfrak{p}_h$ we use a Taylor approximation in the direction ν_h

$$u \circ \mathfrak{p}_h(x) \approx T_k(u)(x) := \sum_{j=0}^{k} \frac{D^j_{\nu_h} u(x)}{j!} \varrho^j_h(x),$$

where $D^j_{\nu_h}$ is the jth partial derivative in the direction ν_h. Thus it follows that the solution to (1)-(3) satisfies

$$\begin{aligned}(f, v)_{\Omega_h} &= (f + \Delta u - \nabla p, v)_{\Omega_h \setminus \Omega} + (\nabla u, \nabla v)_{\Omega_h} - (p, \nabla \cdot v)_{\Omega_h} \\ &- (\nu_h \cdot \nabla u - \nu_h p, v)_{\partial\Omega_h} \\ &- (T_k(u) - g \circ \mathfrak{p}_h, \nu_h \cdot \nabla v)_{\partial\Omega_h} + \beta h^{-1}(T_k(u) - g \circ \mathfrak{p}_h, v)_{\partial\Omega_h} \\ &- (u \circ \mathfrak{p}_h - T_k(u), \nu_h \cdot \nabla v)_{\partial\Omega_h} + \beta h^{-1}(u \circ \mathfrak{p}_h - T_k(u), v)_{\partial\Omega_h},\end{aligned}$$

for all $v \in V_h$. Rearranging the terms we arrive at

$$\begin{aligned}(\nabla u, \nabla v)_{\Omega_h} - (p, \nabla \cdot v)_{\Omega_h} - (\nu_h \cdot \nabla u - p\nu_h, v)_{\partial\Omega_h} \\ - (T_k(u), \nu_h \cdot \nabla v)_{\partial\Omega_h} + \beta h^{-1}(T_k(u), v)_{\partial\Omega_h} \\ + (f + \Delta u - \nabla p, v)_{\Omega_h \setminus \Omega} \\ - (u \circ \mathfrak{p}_h - T_k(u), \nu_h \cdot \nabla v)_{\partial\Omega_h} + \beta h^{-1}(u \circ \mathfrak{p}_h - T_k(u), v)_{\partial\Omega_h} \\ = (f, v)_{\Omega_h} - (g \circ \mathfrak{p}_h, \nu_h \cdot \nabla v)_{\partial\Omega_h} + \beta h^{-1}(g \circ \mathfrak{p}_h, v)_{\partial\Omega_h},\end{aligned} \tag{7}$$

for all $v \in V^k_h$. The discrete method is obtained from this formulation by dropping the consistency terms of highest order, i.e. those on lines three and four of equation (7).

Bilinear Forms. We define the forms

$$
\begin{aligned}
a_0(v,w) &:= (\nabla v, \nabla w)_{\Omega_h} \\
&\quad - (\nu_h \cdot \nabla v, w)_{\partial\Omega_h} - (T_k(v), \nu_h \cdot \nabla w)_{\partial\Omega_h} \\
&\quad + \beta h^{-1}(T_k(v), w)_{\partial\Omega_h},
\end{aligned} \tag{8}
$$

$$
j_{\pm}^k(v,w) := \gamma_j \sum_{F\in\mathcal{F}_h} \sum_{l=1}^{k} h^{2l\pm 1}([D_{n_F}^l v],[D_{n_F}^l w])_F, \tag{9}
$$

$$
a_h(v,w) := a_0(v,w) + j_-^k(v,w), \tag{10}
$$

$$
b_\kappa(q,w) := (q, \nabla\cdot v)_{\Omega_h} - \kappa(q, v\cdot\nu_h)_{\partial\Omega_h}, \tag{11}
$$

$$
s(y,q) := j_+^m(y,q) + \gamma_p \sum_{F\in\mathcal{F}_h} h^3([n_F\cdot\nabla y],[n_F\cdot\nabla q])_F, \tag{12}
$$

$$
l_h(w) := (f,w)_{\Omega_h} - (g\circ\mathfrak{p}_h, \nu_h\cdot\nabla w)_{\partial\Omega_h} + \beta h^{-1}(g\circ\mathfrak{p}_h, w)_{\partial\Omega_h}, \tag{13}
$$

where γ_j, γ_p, and β, are positive constants. Here we used the notation:

- $\mathcal{F}_h$ is the set of all internal faces to elements $K\in\mathcal{K}_h$, i.e. faces that are not included in the boundary of the active mesh $\mathcal{K}_h$, that intersect the set $\Omega\setminus\Omega_h\cup\partial\Omega_h$, and n_F is a fixed unit normal to $F\in\mathcal{F}_h$.
- $D_{n_F}^l$ is the partial derivative of order l in the direction of the normal n_F to the face $F\in\mathcal{F}_h$.
- $[v]|_F = v_F^+ - v_F^-$, with $v_F^\pm = \lim_{s\to 0^+} v(x\mp s n_F)$, is the jump of a discontinuous function v across a face $F\in\mathcal{F}_h$.
- The stabilizing terms $j_\pm^k(v,w)$ are introduced to extend the coercivity of $a_0(\cdot,\cdot)$, to all of $\mathcal{N}_h$ and similarly for the pressure. Thanks to this property one may prove that the condition number is uniformly bounded independently of how Ω_h intersects the mesh following the ideas of [2,11].
- Observe the presence of the penalty coefficient β in (8) and (13). In order to guarantee coercivity β has to be chosen large enough and due to the Taylor expansions we also have to require that $h\in(0,h_0]$ with h_0 sufficiently small.

The Method. Find: $(u_h,p_h)\in W_h := [V_h^k]^d\times V_h^m$ such that

$$
a_h(u_h,v) - b_1(p_h,v) + b_0(q,u_h) + s(p_h,q) = l_h(v), \qquad \forall (v,q)\in W_h, \tag{14}
$$

where a_h is defined in (10), b_0 and b_1 in (11), s in (12) and finally l_h in (13).

For the analysis below it will be convenient to use the compact formulation: find: $(u_h,p_h)\in W_h$ such that

$$
A_h[(u_h,p_h),(v,q)] + s(p_h,q) = l_h(v), \qquad \forall (v,q)\in W_h,
$$

where

$$
A_h[(u_h,p_h),(v,q)] := a_h(u_h,v) - b_1(p_h,v) + b_0(q,u_h).
$$

Remark 1. Note that different forms $b.(\cdot,\cdot)$ are used in the moment and mass equations and that

$$-b_1(p_h, u_h) + b_0(p_h, u_h) = (p_h, u_h \cdot \nu_h)_{\partial\Omega_h}.$$

It follows that the velocity pressure coupling term is skew-symmetric only up to a boundary term that is essential for consistency. The reason this term is omitted in the mass equation is that it is not consistent and must either be improved using a special boundary value correction, or omitted. For simplicity we here chose the latter option.

3 Theoretical results

In this section we will report on some fundamental theoretical results that hold for the formulation (14). Due to space limitations the proof will be given elsewhere.

For stability we will use the following norm defined for functions restricted to the discrete space W_h or (v,q) in $[H^{k+\frac{1}{2}+\epsilon}(\Omega_0)]^d \times H^{m+\frac{1}{2}+\epsilon}(\Omega_0)$, $\epsilon > 0$,

$$\begin{aligned}|||(v,q)||| := \|v\|_{H^1(\Omega_h)} + \|h^{-\frac{1}{2}} v\|_{\partial\Omega_h} \\ + \|h\nabla q\|_{\Omega_h} + \|q\|_{\Omega_h} + j_-^k(v,v)^{\frac{1}{2}} + s(q,q)^{\frac{1}{2}}.\end{aligned}$$

3.1 Inf-sup stability

Key to discrete well-posedness and to the error analysis is the following inf-sup stability result that is robust with respect to how the discrete domain intersects the mesh. The main difficulty in the proof of this result is to handle the lack of skew symmetry between the terms b_1 and b_0. It follows however that the perturbation can be absorbed by the L^2-norm of the pressure and the boundary penalty term when β is sufficiently large and h sufficiently small.

Proposition 2. *Let either $m = k$ and $\gamma_p > 0$ or $k = 2$ and $m = 1$ and $\gamma_p = 0$ and assume that* (5)-(6) *hold with $\zeta = 1$. Then there exists $\alpha > 0$, $h_0 > 0$ such that for all $(v,q) \in W_h$, when $h < h_0$, there holds*

$$\alpha|||(v,q)||| \leq \sup_{(w,y)\in W_h} \frac{A_h[(v,q),(w,y)] + s(q,y)}{|||(w,y)|||}.$$

3.2 A priori error estimate

In this section we will present an optimal error estimate in the natural energy norm of the problem on the physical domain. The proof of the estimate follows the ideas of [7], this time combining inf-sup stability on the discrete error, Galerkin orthogonality, continuity of A_h, estimation of geometry errors and finally approximability. The error on the physical domain is then controlled by adding and subtracting an interpolant of the exact solution and using the triangle inequality. If i_h denotes the Lagrange interpolant on $[V_h^k]^d$,

$$\begin{aligned}\|u-u_h\|_{H^1(\Omega)} &\leq \|u-i_hEu\|_{H^1(\Omega)} + \|i_hEu-u_h\|_{H^1(\mathcal{N}_h)} \\ &\lesssim \|u-i_hEu\|_{H^1(\Omega)} + |||(i_hEu-u_h,0)|||.\end{aligned}$$

Theorem 3. *Let $(u,p) \in [H^s(\Omega)]^d \times H^{s-1}(\Omega)$, with $s \geq 2$ be the solution to* (1)-(3)*. Assume that the hypothesis of Proposition 2 are satisfied and that in addition* (4) *holds with $s \geq 1$ and* (5)-(6) *hold with $\zeta = 2$. Let $(u_h, p_h) \in [V_h^k]^d \times V_h^m$ be the solution of the finite dimensional problem* (14)*. Then there holds*

$$\|u-u_h\|_{H^1(\Omega)} + \|p-p_h\|_\Omega \lesssim h^\sigma(|u|_{H^{\sigma+1}(\Omega)} + |p|_{H^\sigma(\Omega)}),$$

where $\sigma = \min\{k, s-1\}$.

4 Numerical example

In our numerical example we consider a two dimensional problem discretized by the lowest order (inf–sup stable) Taylor–Hood element: piecewise quadratic, continuous, approximation of the velocity and piecewise linear, continuous, approximation of the pressure, together with a piecewise linear approximation of the domain. We shall study the convergence with and without boundary modification.

We consider a problem from [6] with exact solution (with $f = 0$),

$$u_1 = 20xy^3, \quad u_2 = 5x^4 - 5y^4, \quad p = 60x^2y - 20y^3.$$

Our computational domain is a disc with center at the origin. The exact velocities are used as Dirichlet data on the boundary of the domain. Note that since the exact solution is given everywhere, setting Dirichlet data on the approximate boundary is not a problem in this (special) case; to simulate the knowledge of data on the boundary only, we take the boundary data from the edge of the exact domain and use as boundary conditions on the approximate boundary, using the closest point projection. We choose the method parameters $\gamma_j = 10^{-3}$, $\gamma_p = 0$ and $\beta = 100$.

In Figure 1 we show the convergence obtained with and without boundary modification. We note that without boundary modification we lose optimal convergence in velocities but retain optimal convergence for pressure, which is expected since the approximation of the boundary is piecewise linear leading to an $O(h^2)$ geometric consistency error. With boundary modification we recover optimal order convergence also for the velocity.

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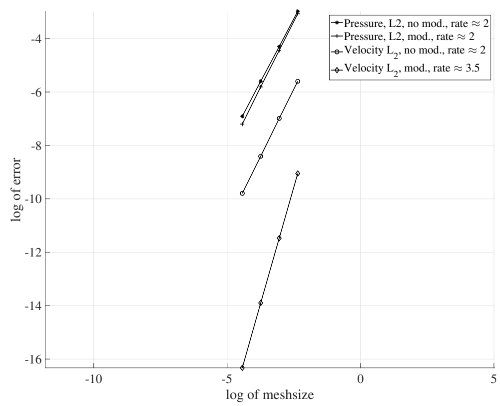

Fig. 1: Convergence results.

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