EXTREME EIGENVALUES OF AN INTEGRAL OPERATOR

ALEXANDER V. SOBOLEV

Abstract. We study the family of compact operators $B_\alpha = VA_\alpha V$, $\alpha > 0$ in $\mathsf{L}^2(\mathbb{R}^d)$, $d \ge 1$, where A_α is the pseudo-differential operator with symbol $a^{(\alpha)}(\boldsymbol{\xi}) = a(\alpha\boldsymbol{\xi})$, and both functions a and V are real-valued and decay at infinity. We assume that a and V attain their maximal values $A_0 > 0$, $V_0 > 0$ only at $\boldsymbol{\xi} = 0$ and $\mathbf{x} = 0$. We also assume that

$$a(\boldsymbol{\xi}) = A_0 - \Psi_\gamma(\boldsymbol{\xi}) + o(|\boldsymbol{\xi}|^\gamma),\ |\boldsymbol{\xi}| \to 0,$$
$$V(\mathbf{x}) = V_0 - \Phi_\beta(\mathbf{x}) + o(|\mathbf{x}|^\beta),\ |\mathbf{x}| \to 0,$$

with some functions $\Psi_\gamma(\boldsymbol{\xi}) > 0$, $\boldsymbol{\xi} \ne 0$ and $\Phi_\beta(\mathbf{x}) > 0$, $\mathbf{x} \ne 0$ that are homogeneous of degree $\gamma > 0$ and $\beta > 0$ respectively. The main result is the following asymptotic formula for the eigenvalues $\lambda_\alpha^{(n)}$ of the operator B_α (arranged in descending order counting multiplicity) for fixed n and $\alpha \to 0$:

$$\lambda_\alpha^{(n)} = A_0V_0^2 - \mu^{(n)}\alpha^\sigma + o(\alpha^\sigma), \alpha \to 0,$$

where $\sigma^{-1} = \gamma^{-1} + \beta^{-1}$, and $\mu^{(n)}$ are the eigenvalues (arranged in ascending order counting multiplicity) of the model operator T with symbol $V_0^2\Psi_\gamma(\boldsymbol{\xi}) + 2A_0V_0\Phi_\beta(\mathbf{x})$.

1. Introduction and main result

Let $a = a(\boldsymbol{\xi}), \boldsymbol{\xi} \in \mathbb{R}^d$, $V = V(\mathbf{x}), \mathbf{x} \in \mathbb{R}^d$, $d \ge 1$, be bounded real-valued functions such that $a(\boldsymbol{\xi}) \to 0$, $V(\mathbf{x}) \to 0$ as $|\boldsymbol{\xi}| \to \infty, |\mathbf{x}| \to \infty$. Consider the self-adjoint operator on $\mathsf{L}^2(\mathbb{R}^d)$ defined by

$$B_\alpha = V\mathcal{F}^*a^{(\alpha)}\mathcal{F}V,\ a^{(\alpha)}(\boldsymbol{\xi}) = a(\alpha\boldsymbol{\xi}), \alpha > 0, \tag{1.1}$$

where $\mathcal{F}$ is the unitary Fourier transform

$$(\mathcal{F}u)(\boldsymbol{\xi}) = \hat{u}(\boldsymbol{\xi}) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int e^{-i\boldsymbol{\xi}\cdot\mathbf{x}}u(\mathbf{x})d\mathbf{x}.$$

2010 *Mathematics Subject Classification.* Primary 45C05, 47B06; Secondary 47A75, 45M05.

Key words and phrases. Eigenvalues, asymptotics.

Here and further on the integral without indication of the domain means integration over the entire space $\mathbb{R}^d$. The operator $\mathcal{F}^* a^{(\alpha)} \mathcal{F}$ is also described as a pseudo-differential operator with symbol $a^{(\alpha)}$. This description however is not helpful for us as we do not use calculus of pseudo-differential operators. It is useful to point out the following scaling invariance of the operator B_α: a straightforward change of variables $\mathbf{x} \to r\mathbf{x}$, $\boldsymbol{\xi} \to r^{-1}\boldsymbol{\xi}$ with an arbitrary $r > 0$, reduces the operator B_α to the unitarily equivalent operator

$$V(r\mathbf{x})\mathcal{F}^* a(\alpha r^{-1}\boldsymbol{\xi})\mathcal{F} V(r\mathbf{x}). \tag{1.2}$$

In particular, the choice $r = \alpha$ transfers the scaling onto the function V. We use operator (1.2) further on with a specific choice of the parameter r.

It is clear that B_α is compact for all $\alpha > 0$. We are interested in the asymptotics of the extreme top eigenvalues of the operator B_α as $\alpha \to 0$. More precisely, denote by $\lambda_\alpha^{(1)}, \lambda_\alpha^{(2)}, \dots$ the eigenvalues of B_α arranged in descending order counting multiplicity. The associated normalized pair-wise orthogonal eigenfunctions are denoted by $\psi_\alpha^{(1)}, \psi_\alpha^{(2)}, \dots$. We study the asymptotics of $\lambda_\alpha^{(n)}$ as $\alpha \to 0$ for a fixed n. This problem has been addressed in the literature in different contexts under different conditions on the functions a and V. For example, if a and V are indicator functions of bounded intervals in $\mathbb{R}$, the behaviour of the eigenvalues was studied by D. Slepian and H.O. Pollak in [11]. For $d \geq 2$ this problem was analyzed by D. Slepian in [12] with a, V being indicator functions of balls. In both cases (one- and multi-dimensional) the eigenvalues $\lambda_\alpha^{(n)}$ are exponentially close to 1 as $\alpha \to 0$.

In [13] H. Widom considered the function V which was the indicator of an interval I, and symbol $a = a(\xi)$, $\xi \in \mathbb{R}$, having one global maximum at $\xi = 0$, and satisfying the condition

$$a(\xi) = A_0 - \Psi|\xi|^\gamma + o(|\xi|^\gamma), \ |\xi| \to 0, \tag{1.3}$$

with $A_0 = a(0) = \max a(\xi) > 0$, and some $\Psi > 0$, $\gamma > 0$. It was proved that

$$\lambda_\alpha^{(n)} = A_0 - \alpha^\gamma \Psi \mu^{(n)} + o(\alpha^\gamma), \ \alpha \to 0, \tag{1.4}$$

where $\mu^{(n)}, n = 1, 2, \dots$ are eigenvalues of the fractional Dirichlet Laplacian $(-\Delta)^{\frac{\gamma}{2}}$ on I, arranged in ascending order counting multiplicity. A multi-dimensional analogue of this result was obtained by H. Widom in [14]. We omit its formulation for the sake of brevity. A result of the type (1.4) also holds if V is not assumed to be a simple indicator function, but attains its (positive) maximum on a set of positive measure, see [7].

For applications to transport problems (see [2] and [5]) it is also useful to investigate the case where both functions a and V have unique power-like maxima. This is exactly the case that we study in the present paper. Note that this problem is very close to the quasi-classical study of low-lying eigenvalues for the Schrödinger operator, see e.g. [3], [9]. The latter are usually analysed for smooth symbols, which in some cases allows one to obtain a complete expansion in powers of the parameter (as in [3]).

Our assumptions on a and V are described below. By C, c (with or without indices) we denote various positive constants whose precise value is of no importance.

Condition 1.1. (1) *a and V are real-valued continuous functions such that $a(\boldsymbol{\xi}) \to 0$ as $|\boldsymbol{\xi}| \to \infty$, and $V(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$.*

(2) *The functions a and V attain their global maxima only at $\boldsymbol{\xi} = \mathbf{0}$ and $\mathbf{x} = \mathbf{0}$ respectively:*

$$A_0 := \max a(\boldsymbol{\xi}) > 0, \ V_0 := \max V(\mathbf{x}) > 0.$$

The function V satisfies the condition $-V_0 + c \le V(\mathbf{x}) \le V_0$, $\mathbf{x}$ a.e., with a positive constant c.

(3) *Let $\Phi_\beta, \Psi_\gamma \in \mathsf{C}^\infty(\mathbb{R}^d \setminus \{0\})$ be some real-valued functions, homogeneous of degree $\beta > 0$ and $\gamma > 0$ respectively, positive at $\mathbf{x} \ne 0$. The functions V and a satisfy the properties*

$$V(\mathbf{x}) = V_0 - \Phi_\beta(\mathbf{x}) + o(|\mathbf{x}|^\beta), \ |\mathbf{x}| \to 0, \tag{1.5}$$

and

$$a(\boldsymbol{\xi}) = A_0 - \Psi_\gamma(\boldsymbol{\xi}) + o(|\boldsymbol{\xi}|^\gamma), \ |\boldsymbol{\xi}| \to 0. \tag{1.6}$$

With appropriate modifications, the assumption of continuity of a and V can be relaxed, but we have chosen to avoid more complicated formulations.

The results of the paper are described with the help of the following model pseudo-differential operator T defined formally by its symbol

$$t(\mathbf{x}, \boldsymbol{\xi}) = V_0^2 \Psi_\gamma(\boldsymbol{\xi}) + 2A_0 V_0 \Phi_\beta(\mathbf{x}). \tag{1.7}$$

The operator T is essentially self-adjoint on $\mathsf{C}_0^\infty(\mathbb{R}^d)$, and has a purely discrete spectrum (see e.g. [8, Theorems 26.2, 26.3]). The same operator can be also defined (see [1, p. 229, Theorem 1]) as the unique self-adjoint operator associated with the quadratic form

(1.8)
$$T[u,v] = V_0^2 \int \Psi_\gamma(\boldsymbol\xi)\hat u(\boldsymbol\xi)\overline{\hat v(\boldsymbol\xi)}d\boldsymbol\xi + 2A_0V_0\int \Phi_\beta(\mathbf x)u(\mathbf x)\overline{v(\mathbf x)}d\mathbf x,$$

which is closed on $D[T] = \mathsf H^{\frac{\gamma}{2}}(\mathbb R^d)\cap \mathsf L^2(\mathbb R^d, |\mathbf x|^\beta)$, where $\mathsf H^s(\mathbb R^d)$, $s>0$, is the standard Sobolev space notation, whereas $\mathsf L^2(\mathbb R^d, |\mathbf x|^\beta)$ is the weighted $\mathsf L^2$-space with the norm $\sqrt{\int |u|^2|\mathbf x|^\beta d\mathbf x}$. We use the notation $T[u] = T[u,u]$. Recall that in view of the polarization identity, the form $T[w], w\in D[T]$, determines $T[u,v]$ for all $u,v\in D[T]$. Denote by $\mu^{(n)}>0$, $n=1,2,\dots$ the eigenvalues of T arranged in ascending order counting multiplicity, and by $\phi^{(n)}$ – an orthonormal basis of corresponding normalized eigenfunctions.

Let σ be the number found from the equation
$$\frac{1}{\sigma} = \frac{1}{\beta}+\frac{1}{\gamma}.$$
The next theorem constitutes the main result of the paper.

Theorem 1.2. *Suppose that the functions a and V satisfy Condition 1.1. Then for any $n=1,2,\dots$, the asymptotics hold:*

(1.9)
$$\lim_{\alpha\to 0}\alpha^{-\sigma}(A_0V_0^2 - \lambda_\alpha^{(n)}) = \mu^{(n)}.$$

A few remarks are in order.

Remark 1.3.
(1) Note that formally, the asymptotics (1.9) imply (1.4) if one takes $d=1$ and $\beta=\infty$.
(2) A model operator of the form (1.7) was featured in [6] where the norm of a special self-adjoint integral operator with properties similar to B_α, was studied.
(3) The spectrum of the operator B_α is symmetric in functions V and a in the following sense. Suppose that $a=p^2$ with some real-valued function p. Then $B_\alpha = X_\alpha^*X_\alpha$, where $X_\alpha = p^{(\alpha)}\mathcal F V$. It is well known (see e.g. [1, p. 95, Theorem 5]) that the non-zero spectra of $X_\alpha^*X_\alpha$ and $X_\alpha X_\alpha^*$ coincide. As a consequence, instead of B_α one can study the operator $p^{(\alpha)}\mathcal F Y\mathcal F^* p^{(\alpha)}$, $Y=V^2$, which has the same non-zero eigenvalues. In view of the scaling invariance mentioned at the beginning of the Introduction, this operator is unitarily equivalent to $p\mathcal F Y^{(\alpha)}\mathcal F^* p$. This conclusion reverses the roles of the functions a and V in the initial operator B_α. If one replaces the condition (1.6) with

an appropriate asymptotic condition on the function p, then it is easy to see that formula (1.9) reflects this symmetry as well.

(4) One could also examine the case when one or both of the functions a, V attain their respective maximal values at several points, and have there the asymptotics of the type (1.5) and (1.6). The author believes that this problem can be tackled by standard methods via decoupling distinct maximum points, thereby reducing the issue to the case of a single maximum.

The case of multiple extremal points was extensively studied in the literature for low-lying values of the Schrödinger operators, see e.g. [4], [10] and references therein. In particular, in [4], the potential is assumed to attain its extremum on a collection of surfaces instead of that of points. We do not go into details.

(5) Conceptually, the proof of Theorem 1.2 follows the paper [13], but the technical details are quite different: for instance, the model operator T replaces the fractional Laplacian used in [13].

2. Preliminary estimates. Lower bounds for the top eigenvalues

Throughout the paper we assume that Condition 1.1 is satisfied. Without loss of generality we may assume that $A_0 = V_0 = 1$. Instead of the operator B_α defined in (1.1), we use the operator (1.2) with

$$r = \alpha^{\frac{\gamma}{\gamma+\beta}},$$

so that (1.1) is unitarily equivalent to

$$B_\alpha = W_\alpha \mathcal{F}^* b_\alpha \mathcal{F} W_\alpha,$$

where W_α and b_α are defined in the following way:

$$W_\alpha(\mathbf{x}) = V\big(\alpha^{\frac{\gamma}{\gamma+\beta}}\mathbf{x}\big), \quad b_\alpha(\boldsymbol{\xi}) = a\big(\alpha^{\frac{\beta}{\gamma+\beta}}\boldsymbol{\xi}\big).$$

Note that slightly abusing the notation we use for the unitarily equivalent operator the same notation B_α. This will not cause any confusion. For thus defined functions W_α and b_α the conditions (1.5) and (1.6) imply that

$$\lim_{\alpha\to 0} \alpha^{-\sigma}\big(1 - b_\alpha(\boldsymbol{\xi})\big) = \Psi_\gamma(\boldsymbol{\xi}), \ \forall \boldsymbol{\xi} \in \mathbb{R}^d, \tag{2.1}$$

and

$$\lim_{\alpha\to 0} \alpha^{-\sigma}\big(1 - W_\alpha(\mathbf{x})^2\big) = 2\Phi_\beta(\mathbf{x}), \ \forall \mathbf{x} \in \mathbb{R}^d. \tag{2.2}$$

Both convergences are uniform in $\mathbf{x}$ and $\boldsymbol{\xi}$ varying over compact sets.

Here is another useful property of the family W_α:

Lemma 2.1. *For any $u \in D[T]$, we have*

$$\alpha^{-\sigma} \int |W_\alpha(\mathbf{x}) - 1|^2 |u(\mathbf{x})|^2 d\mathbf{x} \to 0, \ \alpha \to 0. \tag{2.3}$$

Proof. The function $W_\alpha - 1$ is bounded uniformly in $\mathbf{x}$ and α, so that

$$|W_\alpha(\mathbf{x}) - 1|^2 \le C|W_\alpha(\mathbf{x}) - 1|, \ \mathbf{x} \in \mathbb{R}^d.$$

On the other hand,

$$|W_\alpha(\mathbf{x}) - 1| \le C\alpha^\sigma |\mathbf{x}|^\beta, \ \mathbf{x} \in \mathbb{R}^d.$$

Therefore, for any $R > 0$, we can estimate as follows:

$$\begin{aligned}
\alpha^{-\sigma} \int |W_\alpha(\mathbf{x}) &- 1|^2 |u(\mathbf{x})|^2 d\boldsymbol{\xi} \\
&\le \alpha^{-\sigma} \int_{|\mathbf{x}|<R} |W_\alpha(\mathbf{x}) - 1|^2 |u(\mathbf{x})|^2 d\mathbf{x} \\
&\quad + C\alpha^{-\sigma} \int_{|\mathbf{x}|\ge R} |W_\alpha(\mathbf{x}) - 1||u(\mathbf{x})|^2 d\mathbf{x} \\
&\le C\alpha^\sigma \int_{|\mathbf{x}|<R} |\mathbf{x}|^{2\beta} |u(\mathbf{x})|^2 d\mathbf{x} + C \int_{|\boldsymbol{\xi}|>R} |\mathbf{x}|^\beta |u(\mathbf{x})|^2 d\mathbf{x} \\
&\le C\alpha^\sigma R^\beta \int_{|\mathbf{x}|<R} |\mathbf{x}|^\beta |u(\mathbf{x})|^2 d\mathbf{x} + C \int_{|\mathbf{x}|>R} |\mathbf{x}|^\beta |u(\mathbf{x})|^2 d\mathbf{x}.
\end{aligned}$$

Both integrals on the right-hand side are finite, since $u \in D[T]$, and the second one tends to zero as $R \to \infty$. Thus, passing first to the limit $\alpha \to 0$, and then taking $R \to \infty$, we conclude that the right-hand side tends to zero as $\alpha \to 0$, as claimed. □

Now we show that in some suitable sense the operator B_α can be approximated by the operator $I - \alpha^\sigma T$ as $\alpha \to 0$. Define the form

$$R_\alpha[u] = (B_\alpha u, u) - \|u\|^2 + \alpha^\sigma T[u], \tag{2.4}$$

which is closed on the domain $D[T]$, and two more forms

$$K_\alpha[u, v] = \alpha^{-\sigma} \int (1 - b_\alpha(\boldsymbol\xi)) \hat{u}(\boldsymbol\xi) \overline{\hat{v}(\boldsymbol\xi)} d\boldsymbol\xi, \tag{2.5}$$

$$S_\alpha[u, v] = \alpha^{-\sigma} \int \big(1 - W_\alpha(\mathbf{x})^2\big) u(\mathbf{x}) \overline{v(\mathbf{x})} d\mathbf{x}, \tag{2.6}$$

that are defined for all $u, v \in \mathsf{L}^2(\mathbb{R}^d)$. It is easily checked that with $w_\alpha = W_\alpha u, y_\alpha = W_\alpha v$, we have

$$\alpha^{-\sigma}\big((u, v) - (B_\alpha u, v)\big) = K_\alpha[w_\alpha, y_\alpha] + S_\alpha[u, v], \tag{2.7}$$

and

$$R_\alpha[u, v] = \alpha^{\sigma}\big(T[u, v] - K_\alpha[w_\alpha, y_\alpha] - S_\alpha[u, v]\big). \tag{2.8}$$

Note that $K_\alpha[u] \ge 0$ and $S_\alpha[u] \ge 0$ for all $\alpha > 0$. Also, due to (2.1) and (2.2), for any $u \in D[T]$ we have

$$K_\alpha[u] \le C \int |\boldsymbol\xi|^\gamma |\hat{u}(\boldsymbol\xi)|^2 d\boldsymbol\xi, \ S_\alpha[u] \le C \int |\mathbf{x}|^\beta |u(\mathbf{x})|^2 d\mathbf{x}, \tag{2.9}$$

with a constant C independent of u. Moreover, for any $u \in D[T]$ we also have

$$\lim_{\alpha \to 0} K_\alpha[u] = \int \Psi_\gamma(\boldsymbol\xi) |\hat{u}(\boldsymbol\xi)|^2 d\boldsymbol\xi, \tag{2.10}$$

and

$$\lim_{\alpha \to 0} S_\alpha[u] = 2 \int \Phi_\beta(\mathbf{x}) |u(\mathbf{x})|^2 d\mathbf{x}, \tag{2.11}$$

by the Dominated Convergence Theorem.

Lemma 2.2. *For any $u \in D[T]$ and $w_\alpha = W_\alpha u$, we have*

$$K_\alpha[w_\alpha - u] \to 0, \ \alpha \to 0. \tag{2.12}$$

Also, for any $u, v \in D[T]$ we have

$$\alpha^{-\sigma} |R_\alpha[u, v]| \to 0, \alpha \to 0. \tag{2.13}$$

Proof. Proof of (2.12). Estimate:

$$\begin{aligned} K_\alpha[w_\alpha - u] &\le C\alpha^{-\sigma} \int |w_\alpha(\mathbf{x}) - u(\mathbf{x})|^2 d\mathbf{x} \\ &= C\alpha^{-\sigma} \int \big(1 - W_\alpha(\mathbf{x})\big)^2 |u(\mathbf{x})|^2 d\mathbf{x}. \end{aligned}$$

Here we used the fact that $0 \le 1 - b_\alpha \le C$ with some constant C. The right-hand side tends to zero by (2.3).

It suffices to prove (2.13) for $u = v$. Consider separately the terms in the representation (2.8). Write:

$$K_\alpha[w_\alpha] = K_\alpha[u] + 2\,\mathrm{Re}\,K_\alpha[u, w_\alpha - u] + K_\alpha[w_\alpha - u].$$

The last term tends to zero by (2.12). Now estimate the second term:

$$|K_\alpha[u, w_\alpha - u]|^2 \le K_\alpha[u] K_\alpha[w_\alpha - u].$$

In view of (2.9), the first factor is uniformly bounded, and the second one tends to zero. Thus

$$K_\alpha[w_\alpha] - K_\alpha[u] \to 0, \ \alpha \to 0.$$

Together with (2.10) and (2.11) this implies that

$$\lim_{\alpha\to 0}\big(K_\alpha[w_\alpha] + S_\alpha[u]\big) = T[u],$$

see (1.8). Due to (2.8) this implies (2.13). □

The lower bound for the eigenvalues $\lambda_\alpha^{(n)}$, i.e. the upper bound for the left-hand side of (1.9), is rather straightforward.

Lemma 2.3. *For all $n = 1, 2, \dots,$ we have*

$$\limsup_{\alpha\to\infty} \alpha^{-\sigma}(1 - \lambda_\alpha^{(n)}) \le \mu^{(n)}. \tag{2.14}$$

Proof. Let $\mathcal{K}_n \subset \mathsf{L}^2(\mathbb{R}^d)$, $n \ge 1$, be the span of the eigenfunctions $\phi^{(1)}, \phi^{(2)}, \dots, \phi^{(n)}$, so $\dim \mathcal{K}_n = n$. By the max-min principle (see e.g. [1, p. 212, Theorem 5]),

$$\lambda_\alpha^{(n)} \ge \min(B_\alpha u, u),$$

where the minimum is taken over all functions $u \in \mathcal{K}_n$ such that $\|u\| = 1$. Thus by definition (2.4)

$$\lambda_\alpha^{(n)} \ge 1 - \alpha^\sigma \max_{u\in\mathcal{K}_n, \|u\|=1} T[u] - n \max_{1\le j,k\le n} |R_\alpha[\phi^{(j)}, \phi^{(k)}]|.$$

Since $\{\phi^{(j)}\}$ are eigenfunctions of T,

$$\max_{u\in\mathcal{K}_n, \|u\|=1} T[u] = \mu^{(n)},$$

and the required result now follows from (2.13). □

Now we can establish the uniform localization of the eigenfunctions $\psi_\alpha^{(n)}$, $n = 1, 2, \dots$. Denote

$$\theta_\alpha^{(n)}(\mathbf{x}) = W_\alpha(\mathbf{x})\psi_\alpha^{(n)}(\mathbf{x}).$$

Below we use the notation $\chi_R = \chi_R(\mathbf{t})$ for the indicator function of the ball $\{\mathbf{t} \in \mathbb{R}^d : |\mathbf{t}| < R\}$.

Lemma 2.4. *For all $n = 1, 2, \dots$, the forms $K_\alpha[\theta_\alpha^{(n)}]$ and $S_\alpha[\psi_\alpha^{(n)}]$ are bounded uniformly in α:*

$$\limsup_{\alpha\to 0}\big(K_\alpha[\theta_\alpha^{(n)}] + S_\alpha[\psi_\alpha^{(n)}]\big) \le \mu^{(n)}, \tag{2.15}$$

and

$$\|\theta_\alpha^{(n)} - \psi_\alpha^{(n)}\| \to 0,\ \ \alpha \to 0. \tag{2.16}$$

Moreover, for all $R > 0$ we have

$$\liminf_{\alpha\to 0}\ \|\widehat{\psi_\alpha^{(n)}}\chi_R\|^2 \ge 1 - C\mu^{(n)} R^{-\gamma}, \tag{2.17}$$

and

$$\liminf_{\alpha\to 0}\ \|\psi_\alpha^{(n)}\chi_R\|^2 \ge 1 - C\mu^{(n)} R^{-\beta}. \tag{2.18}$$

with some constant C, independent of n and R.

Proof. We drop the superscript "n" for brevity. According to (2.7),

$$\alpha^{-\sigma}(1 - \lambda_\alpha) = K_\alpha[\theta_\alpha] + S_\alpha[\psi_\alpha].$$

Now (2.15) follows from (2.14). Now write

$$\|\theta_\alpha - \psi_\alpha\|^2 = \int \big(1 - W_\alpha(\mathbf{x})\big)^2 |\psi_\alpha(\mathbf{x})|^2 d\mathbf{x}.$$

The straightforward estimate

$$\frac{1}{2}(1 - W_\alpha)^2 \le 1 - W_\alpha = \frac{1 - W_\alpha^2}{1 + W_\alpha} \le C(1 - W_\alpha^2),$$

by the definition (2.6), implies that

$$\|\theta_\alpha - \psi_\alpha\|^2 \le C\alpha^\sigma S_\alpha[\psi_\alpha],$$

which leads to the convergence $\|\theta_\alpha - \psi_\alpha\| \to 0$, $\alpha \to 0$, in view of (2.15).

Proof of (2.17). By Condition 1.1(2), the point $\boldsymbol{\xi} = \mathbf{0}$ is the global maximum of $b_\alpha(\boldsymbol{\xi})$, so in view of (1.6), for all $|\boldsymbol{\xi}| > R, R > 0$ and all sufficiently small α we have

$$b_\alpha(\boldsymbol{\xi}) = a\big(\alpha^{\frac{\beta}{\gamma+\beta}}\boldsymbol{\xi}\big) \le 1 - CR^\gamma\alpha^\sigma,$$

with some constant C. Thus $\alpha^{-\sigma}(1 - b_\alpha(\boldsymbol{\xi})) \ge CR^\gamma$, and hence

$$K_\alpha[\theta_\alpha] \ge CR^\gamma \int\limits_{|\boldsymbol{\xi}|>R} |\hat{\theta}_\alpha(\boldsymbol{\xi})|^2 d\boldsymbol{\xi},$$

so that, by (2.15), $\|\hat{\theta}_\alpha(1-\chi_R)\|^2 \le C\mu R^{-\gamma}$. Together with (2.16) this leads to (2.17).

Proof of (2.18) is similar. By Condition 1.1(2) and by (1.5), for all $|\mathbf{x}| > R$, $R > 0$, we have $|W_\alpha(\mathbf{x})|^2 \le 1 - CR^\beta\alpha^\sigma$, and hence

$$S_\alpha[\psi_\alpha] \ge CR^\beta \int\limits_{|\mathbf{x}|>R} |\psi_\alpha(\mathbf{x})|^2 d\mathbf{x},$$

so that by (2.15) again, $\|\psi_\alpha(1 - \chi_R)\|^2 \le C\mu R^{-\beta}$. This leads to (2.18). □

With the help of Lemma 2.4, in the proof of Theorem 1.2 we show that any weakly convergent sequence of the eigenfunctions $\psi_\alpha^{(n)}$ in fact converges in norm. For this we rely on the following result:

Proposition 2.5. *(See [6, Lemma 12]) Let $f_j \in \mathsf{L}^2(\mathbb{R}^d)$ be a sequence such that $\|f_j\| \le C$ uniformly in $j = 1, 2, \dots$, and $f_j(\mathbf{x}) = 0$ for all $|\mathbf{x}| \ge \rho > 0$ and all $j = 1, 2, \dots$. Suppose that f_j converges weakly to $f \in \mathsf{L}^2(\mathbb{R}^d)$ as $j \to \infty$, and that for some constant $A > 0$, and all $R \ge R_0 > 0$,*

$$\liminf_{j\to\infty} \|\hat{f}_j\chi_R\| \ge A - CR^{-\varkappa}, \ \varkappa > 0, \tag{2.19}$$

with some constant C independent of j, R. Then $\|f\| \ge A$.

3. Proof of Theorem 1.2

As before, we assume that a and V satisfy Condition 1.1, and that $A_0 = V_0 = 1$.

The next lemma is the last step towards the proof of Theorem 1.2.

Lemma 3.1. *Suppose that for some sequence $\alpha_k > 0$, convergent to zero as $k \to \infty$, the sequence of eigenfunctions $\psi_{\alpha_k}^{(n)}$ converges weakly to $\psi^{(n)}$. Then*

1. *The sequence $\psi_{\alpha_k}^{(n)}$ converges to $\psi^{(n)}$ in norm as $k \to \infty$,*
2. *The norm limit $\psi^{(n)}$ belongs to $D[T]$, and*

$$\lim_{k\to\infty} \alpha_k{}^{-\sigma}\big((\psi_{\alpha_k}^{(n)}, g) - B_{\alpha_k}[\psi_{\alpha_k}^{(n)}, g]\big) = T[\psi^{(n)}, g], \tag{3.1}$$

for any $g \in D[T]$.

Proof. As before, we omit the superscript "n". Also for brevity we write α instead of α_k.

Proof of (1). Due to the formula

$$\|\psi - \psi_\alpha\|^2 = 1 + \|\psi\|^2 - 2\,\mathrm{Re}(\psi_\alpha, \psi) \to 1 - \|\psi\|^2,\ \alpha \to 0,$$

it suffices to show that $\|\psi\| = 1$.

For a number $\rho \ge 1$ denote $w_{\alpha,\rho} = \psi_\alpha \chi_\rho$, $y_{\alpha,\rho} = \psi_\alpha(1 - \chi_\rho)$. Thus, by (2.17) and (2.18), for any $R \ge 1$ we have

$$\|\widehat{w_{\alpha,\rho}}\chi_R\| \ge \|\widehat{\psi_\alpha}\chi_R\| - \|y_{\alpha,\rho}\| \ge 1 - C\mu R^{-\gamma} - C(\mu\rho^{-\beta})^{\frac{1}{2}}.$$

Since $\psi_\alpha \to \psi$ weakly, then for any $\rho > 0$ the family $w_{\alpha,\rho}$ converges to $\psi\chi_\rho$ weakly. Using Proposition 2.5 for the sequence $w_{\alpha,\rho}$ we conclude that

$$\|\psi\chi_\rho\| \ge 1 - C(\mu\rho^{-\beta})^{\frac{1}{2}}.$$

Since ρ is arbitrary, this means that $\|\psi\| = 1$, which implies the norm convergence $\psi_\alpha \to \psi$, $\alpha \to 0$, as claimed.

Proof of (2). By Part (1) above, and by (2.16), we have

$$\|\hat{\theta}_\alpha - \hat{\psi}\| \le \|\theta_\alpha - \psi_\alpha\| + \|\psi_\alpha - \psi\| \to 0,\ \alpha \to 0.$$

Thus for a subsequence $\hat{\theta}_\alpha$, there is a pointwise convergence $\hat{\theta}_\alpha \to \hat{\psi}$, $\alpha \to 0$. By (2.1), the integrand in $K_\alpha[\theta_\alpha]$ converges pointwise to $\Psi_\gamma(\boldsymbol{\xi})|\hat{\psi}(\boldsymbol{\xi})|^2$. By (2.15), $K_\alpha[\theta_\alpha]$ is uniformly bounded, so by Fatou's Lemma, $|\boldsymbol{\xi}|^{\gamma/2}\hat{\psi} \in \mathsf{L}^2(\mathbb{R}^d)$.

By (2.2), the integrand in $S_\alpha[\psi_\alpha]$ converges pointwise to $2\Phi_\beta(\mathbf{x})|\psi(\mathbf{x})|^2$. By (2.15), $S_\alpha[\psi_\alpha]$ is uniformly bounded, so by Fatou's Lemma again, $|\mathbf{x}|^{\beta/2}\psi \in \mathsf{L}^2(\mathbb{R}^d)$. Together with the previously obtained property $|\boldsymbol{\xi}|^{\gamma/2}\hat{\psi} \in \mathsf{L}^2(\mathbb{R}^d)$, this means that $\psi \in D[T]$.

Proof of (3.1) is similar to that of (2.13), but is somewhat more complicated since it involves functions ψ_α depending on the parameter α. By (2.7),

$$\alpha^{-\sigma}\big((\psi_\alpha, g) - B_\alpha[\psi_\alpha, g]\big) = K_\alpha[\theta_\alpha, y_\alpha] + S_\alpha[\psi_\alpha, g],$$

where $y_\alpha = W_\alpha g$. We prove that

$$\lim_{\alpha \to 0} K_\alpha[\theta_\alpha, y_\alpha] = \int \Psi_\gamma(\boldsymbol{\xi})\hat{\psi}(\boldsymbol{\xi})\overline{\hat{g}(\boldsymbol{\xi})}d\boldsymbol{\xi}, \tag{3.2}$$

and

$$\lim_{\alpha\to 0} S_\alpha[\psi_\alpha, g] = 2\int \Phi_\beta(\mathbf{x})\psi(\mathbf{x})\overline{g(\mathbf{x})}d\mathbf{x}. \tag{3.3}$$

Estimate:

$$\big|K_\alpha[\theta_\alpha, y_\alpha] - K_\alpha[\theta_\alpha, g]\big|^2 \le K_\alpha[\theta_\alpha]K_\alpha[y_\alpha - g].$$

The first factor is bounded uniformly in α by (2.15), and the second one tends to zero due to (2.12). This shows that

$$K_\alpha[\theta_\alpha, y_\alpha] - K_\alpha[\theta_\alpha, g] \to 0,\ \alpha \to 0. \tag{3.4}$$

Because of this property, and because of (2.9), in the proof of (3.2) we may assume that $\hat{g}$ is compactly supported, i.e. $\hat{g}(\boldsymbol{\xi}) = 0$ for all $|\boldsymbol{\xi}| > R$ with some $R > 0$. The convergence (2.1) is uniform in $\boldsymbol{\xi} : |\boldsymbol{\xi}| \le R$ for any R. At the same time, as shown earlier, $\|\hat{\theta}_\alpha - \hat{\psi}\| \to 0, \alpha \to 0$, so that

$$K_\alpha[\theta_\alpha, g] \to \int \Psi_\gamma(\boldsymbol{\xi})\hat{\psi}(\boldsymbol{\xi})\overline{\hat{g}(\boldsymbol{\xi})}d\boldsymbol{\xi},\ \alpha \to 0.$$

Together with (3.4) this gives (3.2).

Proof of (3.3) is simpler. Because of (2.9), we may assume that g is compactly supported. The convergence (2.2) is uniform in $\mathbf{x} : |\mathbf{x}| \le R$ for any $R > 0$. Using the property $\|\psi_\alpha - \psi\| \to 0, \alpha \to 0$, established in Part 1, we obtain

$$S_\alpha[\psi_\alpha, g] \to \int 2\Phi_\beta(\mathbf{x})\psi(\mathbf{x})\overline{g(\mathbf{x})}d\mathbf{x},\ \alpha \to 0,$$

so that (3.3) holds.

Put together (3.2) and (3.3) to conclude that

$$\alpha^{-\sigma}\big((\psi_\alpha, g) - B_\alpha[\psi_\alpha, g]\big) \to T[\psi, g],\ \alpha \to 0,$$

as required. □

Proof of Theorem 1.2. The proof essentially follows the plan of [13]. It suffices to show that for any sequence $\alpha_k \to 0, k \to \infty$, one can find a subsequence $\alpha_{k_l} \to 0$, $l \to \infty$, such that

$$\lim_{l\to\infty} \alpha_{k_l}^{-\sigma}(1 - \lambda^{(n)}_{\alpha_{k_l}}) = \mu^{(n)}. \tag{3.5}$$

Since $\|\psi^{(n)}_{\alpha_k}\| = 1$, one can extract a subsequence $\alpha_{k_l} \to 0$ such that $\psi^{(n)}_{\alpha_{k_l}}$ converges weekly as $l \to \infty$. By Lemma 3.1 $\psi^{(n)}_{\alpha_{k_l}}$ converges in norm as $l \to \infty$. Denote by $\psi^{(n)}$ its limit, so $\|\psi^{(n)}\| = 1$. Further for simplicity we write $\psi^{(n)}_\alpha$ and $\lambda^{(n)}_\alpha$ instead of $\psi^{(n)}_{\alpha_{k_l}}$ and $\lambda^{(n)}_{\alpha_{k_l}}$. As

$\psi_\alpha^{(n)}$, $n = 1, 2, \dots$, are pair-wise orthogonal, so are their limits $\psi^{(n)}$, $n = 1, 2, \dots$.

Fix a number $n = 1, 2, \dots$. For an arbitrary function $f \in D[T]$ write

$$\alpha^{-\sigma}(1 - \lambda_\alpha^{(n)})(\psi_\alpha^{(n)}, f) = \alpha^{-\sigma}\big((\psi_\alpha^{(n)}, f) - B_\alpha[\psi_\alpha^{(n)}, f]\big).$$

Suppose that f is such that $(\psi^{(n)}, f) \neq 0$. Then, in view of (3.1),

$$\lim_{\alpha \to 0} \alpha^{-\sigma}(1 - \lambda_\alpha^{(n)}) = \frac{T[\psi^{(n)}, f]}{(\psi^{(n)}, f)}.$$

Let $f = \phi^{(j)}$, where $\phi^{(j)}$ is chosen in such a way that $(\phi^{(j)}, \psi^{(n)}) \neq 0$. This is possible due to the completeness of the family $\phi^{(k)}$, $k = 1, 2, \dots$. Thus

$$\lim_{\alpha \to 0} \alpha^{-\sigma}(1 - \lambda_\alpha^{(n)}) = \mu^{(j)}.$$

By the uniqueness of the above limit, $(\psi^{(j)}, \phi^{(s)}) = 0$ for all s's such that $\mu^{(s)} \neq \mu^{(j)}$. Thus, by completeness of the system $\{\phi^{(k)}\}$, the function $\psi^{(n)}$ is an eigenfunction of T with the eigenvalue $\mu^{(j)}$, i.e. $T[\psi^{(n)}] = \mu^{(j)}$.

Further proof is by induction. Let $n = 1$, so that by (2.14), $\mu^{(j)} \leq \mu^{(1)}$, and hence $j = 1$, and $\psi^{(1)}$ is the eigenfunction of T with eigenvalue $\mu^{(1)}$. Suppose that for some n, the collection $\psi^{(1)}, \psi^{(2)}, \dots, \psi^{(n-1)}$ are eigenfunctions of T with eigenvalues $\mu^{(1)}, \mu^{(2)}, \dots, \mu^{(n-1)}$. Since $\psi^{(n)}$ is orthogonal to each $\psi^{(k)}$, $k = 1, 2, \dots, n-1$, by the standard min-max (or, more precisely, max-min) principle for operators semi-bounded from below, we have $T[\psi^{(n)}] \geq \mu^{(n)}$, which means that $\mu^{(j)} \geq \mu^{(n)}$. On the other hand, by (2.14),

$$\lim_{\alpha \to 0} \alpha^{-\sigma}(1 - \lambda_\alpha^{(n)}) \leq \mu^{(n)},$$

and hence $\mu^{(j)} \leq \mu^{(n)}$. Therefore $\mu^{(j)} = \mu^{(n)}$, and $\psi^{(n)}$ is the eigenfunction of T with eigenvalue $\mu^{(n)}$. By induction, the formula (3.5) is proved for all n, which entails (1.9), and hence proves Theorem 1.2. $\square$

Acknowledgements. The author is grateful to R. Romanov for bringing the problem studied in this paper to his attention, for pointing out references [2] and [5], and for many stimulating discussions. The author is also grateful to G. Raikov for suggesting references concerning low-lying eigenvalues of Schrödinger operators.

Thanks also go to the anonymous referees for useful comments.

The author was supported by EPSRC grant EP/J016829/1.

References

[1] M.Š. Birman and M. Z. Solomyak, *Spectral theory of self-adjoint operators in Hilbert space*, Reidel, 1987.

[2] R. L. Bowden, C.D. Williams, *Solution of the Initial-Value Transport Problem for Monoenergetic Neutrons in Slab Geometry*, J. Math. Phys. **5**(1964), 1527–1540.

[3] B. Helffer, Y. A. Kordyukov, *Accurate Semiclassical Spectral Asymptotics for a Two-Dimensional Magnetic Schrödinger Operator*, Ann. Henri Poincaré **16** (2015), 1651-1688.

[4] B. Helffer, J. Sjstrand, *Puits multiples en mécanique semi-classique. VI. Cas des puits sous-variétés*, Ann. Inst. H. Poincaré Phys. Théor. **46** (1987), no. 4, 353–372.

[5] Y.A. Kuperin, S.Naboko, R. Romanov, *Spectral Analysis of the Transport Operator: A Functional Model Approach*, Indiana Univ. Math. J., **51**(2002), No. 6, 1389–1425.

[6] B. Mityagin, A. V. Sobolev, *A family of anisotropic integral operators and behavior of its maximal eigenvalue*, J. Spectral Theory **1**, Issue 4 (2011), 443–460.

[7] R. Romanov, *in preparation.*

[8] M. A. Shubin, *Pseudodifferential Operators and Spectral Theory*, Springer, 2001.

[9] B. Simon, *Semiclassical analysis of low lying eigenvalues. I. Nondegenerate minima: asymptotic expansions*, Ann. Inst. H. Poincaré Sect. A (N.S.) **38** (1983), no. 3, 295–308.

[10] B. Simon, *Semiclassical analysis of low lying eigenvalues. II. Tunneling*, Ann. of Math. (2) **120** (1984), no. 1, 89–118.

[11] D. Slepian, H. O. Pollak, *Prolate Spheroidal Wave Functions, Fourier Analysis and Uncertainty I*, Bell System Technical Journal **40**, Issue 1 (1961), 43–63.

[12] D. Slepian, *Prolate Spheroidal Wave Functions, Fourier Analysis and Uncertainty IV: Extensions to Many Dimensions; Generalized Prolate Spheroidal Functions*, Bell System Technical Journal **43**, Issue 6 (1964), 3009–3057.

[13] H. Widom, *Extreme eigenvalues of translation kernels*, Trans. Amer. Math. Soc. **100** 1961, 252–262.

[14] H. Widom, *Extreme eigenvalues of N-dimensional convolution operators*, Trans. Amer. Math. Soc. **106**(1963), 391–414.

Department of Mathematics, University College London, Gower Street, London, WC1E 6BT UK

E-mail address: a.sobolev@ucl.ac.uk