On the 16-rank of class groups of $\mathbb{Q}(\sqrt{-2p})$ for primes $p \equiv 1 \bmod 4$

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Abstract

We use Vinogradov's method to prove equidistribution of a spin symbol governing the 16-rank of class groups of quadratic number fields $\mathbb{Q}(\sqrt{-2p})$, where $p \equiv 1 \bmod 4$ is a prime.

1 Introduction

Recently, the authors have used Vinogradov's method to prove density results about elements of order 16 in class groups in certain *thin* families of quadratic number fields parametrized by a single prime number, namely the families $\{\mathbb{Q}(\sqrt{-2p})\}_{p\equiv -1 \bmod 4}$ and $\{\mathbb{Q}(\sqrt{-p})\}_p$ [17, 13]. In this paper, we establish a density result for the family $\{\mathbb{Q}(\sqrt{-2p})\}_{p\equiv 1 \bmod 4}$, thereby completing the picture for the 16-rank in families of imaginary quadratic fields with cyclic 2-class groups and even discriminant. Although our overarching methods are similar to those originally developed in the work of Friedlander et al. [10], the technical difficulties in the present case are different and require a more careful study of the spin symbols governing the 16-rank. The main distinguishing feature of the present work is that this careful study allows us to avoid relying on a conjecture about short character sums appearing in [10, 13], thus making our results unconditional.

More generally, given a sequence of complex numbers $\{a_n\}_n$ indexed by natural numbers, a problem of interest in analytic number theory is to prove an asymptotic formula for the sum over primes

$$S(X) := \sum_{\substack{p \text{ prime} \\ p \leq X}} a_p$$

as $X \to \infty$. Many sequences $\{a_n\}_n$ admit asymptotic formulas for $S(X)$ via various generalizations of the Prime Number Theorem, with essentially the best known error terms coming from ideas of de la Valée Poussin already in 1899 [4]. In 1947, Vinogradov [25, 26] invented another method to treat certain sequences which could not be handled with a variant of the Prime Number Theorem. His method has since been clarified and made easier to apply,

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most notably by Vaughan [24] and, for applications relating to more general number fields, by Friedlander et al. [10]. Nonetheless, there is a relative paucity of interesting sequences $\{a_n\}_n$ that admit an asymptotic formula for $S(X)$ via Vinogradov's method. The purpose of this paper is to present yet another such sequence, of a similar nature as those appearing in [10, 13]; similarly as in [13], the asymptotics we obtain have implications in the arithmetic statistics of class groups of number fields.

Let $p \equiv 1 \bmod 4$ be a prime number, and let $\mathrm{Cl}(-8p)$ denote the class group of the quadratic number field $\mathbb{Q}(\sqrt{-2p})$ of discriminant $-8p$. The finite abelian group $\mathrm{Cl}(-8p)$ measures the failure of unique factorization in the ring $\mathbb{Z}[\sqrt{-2p}]$. By Gauss's genus theory [12], the 2-part of $\mathrm{Cl}(-8p)$ is cyclic and non-trivial, and hence determined by the largest power of 2 dividing the order of $\mathrm{Cl}(-8p)$. For each integer $k \geq 1$, we define a density $\delta(2^k)$, if it exists, as

$$\delta(2^k) := \lim_{X\to\infty} \frac{\#\{p \leq X : \ p \equiv 1 \bmod 4, \ 2^k | \#\mathrm{Cl}(-8p)\}}{\#\{p \leq X : \ p \equiv 1 \bmod 4\}}.$$

As stated above, the 2-part of $\mathrm{Cl}(-8p)$ is cyclic and non-trivial, so $\delta(2) = 1$. It follows from the Chebotarev Density Theorem (a generalization of the Prime Number Theorem) that $\delta(4) = \frac{1}{2}$ and $\delta(8) = \frac{1}{4}$; indeed, Rédei [19] proved that $4|\#\mathrm{Cl}(-8p)$ if and only if p splits completely in $\mathbb{Q}(\zeta_8)$, and Stevenhagen [23] proved that $8|\#\mathrm{Cl}(-8p)$ if and only if p splits completely in $\mathbb{Q}(\zeta_8, \sqrt[4]{2})$, where ζ_8 denotes a primitive 8th root of unity. The qualitative behavior of divisibility by 16 departs from that of divisibility by lower 2-powers in that it can no longer be proved by a simple application of the Chebotarev Density Theorem. We instead use Vinogradov's method to prove

Theorem 1. *For a prime number $p \equiv 1 \bmod 4$, let $e_p = 0$ if $\mathrm{Cl}(-8p)$ does not have an element of order 8, let $e_p = 1$ if $\mathrm{Cl}(-8p)$ has an element of order 16, and let $e_p = -1$ otherwise. Then for all $X > 0$, we have*

$$\sum_{\substack{p \leq X \\ p\equiv 1 \bmod 4}} e_p \ll X^{1-\frac{1}{3200}},$$

where the implied constant is absolute. In particular, $\delta(16) = \frac{1}{8}$.

In combination with [17], we get

Corollary 2. *For a prime number p, let $h_2(-2p)$ denote the cardinality of the 2-part of the class group $\mathrm{Cl}(-8p)$. For an integer $k \geq 0$, let $\delta'(2^k)$ denote the natural density (in the set of all primes) of primes p such that $h_2(-2p) = 2^k$, if it exists. Then $\delta'(1) = 0$, $\delta'(2) = \frac{1}{2}$, $\delta'(4) = \frac{1}{4}$, and $\delta'(8) = \frac{1}{8}$.*

The power-saving bound in Theorem 1, similarly to the main results in [17] and [13], is another piece of evidence that *governing fields* for the 16-rank do *not* exist. For a sampling on previous work about governing fields, see [2], [3], [18], and [22].

The strategy to prove Theorem 1 is to construct a sequence $\{a_n\}_n$ which simultaneously carries arithmetic information about divisibility by 16 when n is a prime number congruent to 1 modulo 4 and is conducive to Vinogradov's method. On one hand, the criterion for divisibility by 16 cannot be stated naturally over the rational numbers $\mathbb{Q}$. For instance, even the criterion for divisibility by 8 is most naturally stated over a field of degree 8 over $\mathbb{Q}$. On the other hand, proving analytic estimates in a number field generally becomes more difficult as the degree of the number field increases, as exemplified by the reliance on a conjecture on short character sums in [10]. We manage to work over $\mathbb{Q}(\zeta_8)$, a field of degree 4. Although the methods of Friedlander et al. [10] narrowly miss the mark of being unconditional for number fields of degree 4, we manage to exploit the arithmetic structure of our sequence to ensure that Theorem 1 is unconditional.

Lastly, for work concerning the average behavior of the 2-parts of class groups of quadratic number fields in families that are *not* thin, i.e., for which the average number of primes dividing the discriminant grows as the discriminant grows, we point the reader to the extensive work of Fouvry and Klüners [5, 6, 7, 8] on the 4-rank and certain cases of the 8-rank and more recently to the work of Smith on the 8- and higher 2-power-ranks [20, 21]. While Smith's methods in [21] appear to be very powerful, the authors believe that they are unlikely to be applicable to thin families of the type appearing in this paper.

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2 Encoding the 16-rank of $\mathrm{Cl}(-8p)$ into sequences $\{a_\mathfrak{n}\}_\mathfrak{n}$

Given an integer $k \geq 1$, the 2^k-rank of a finite abelian group G, denoted by $\mathrm{rk}_{2^k} G$, is defined as the dimension of the $\mathbb{F}_2$-vector space $2^{k-1}G/2^kG$. If the 2-part of G is cyclic, then $\mathrm{rk}_{2^k} G \in \{0, 1\}$, and $\mathrm{rk}_{2^k} G = 1$ if and only if $2^k | \#G$. The order of a class group is called the class number, and we denote the class number of $\mathrm{Cl}(-8p)$ by $h(-8p)$.

The criterion for divisibility of $h(-8p)$ by 16 that we will use is due to Leonard and Williams [16, Theorem 2, p. 204]. Given a prime number $p \equiv 1 \bmod 8$ (so that $4|h(-8p)$), there exist integers u and v such that

$$p = u^2 - 2v^2, \qquad u > 0. \tag{2.1}$$

The integers u and v are *not* uniquely determined by p; nevertheless, if (u_0, v_0) is one such pair, then, every such pair (u, v) is of the form $u + v\sqrt{2} = \varepsilon^{2m}(u_0 \pm v_0\sqrt{2})$ for some $m \in \mathbb{Z}$, where $\varepsilon = 1 + \sqrt{2}$. The criterion for divisibility by 8 can be restated in terms of a quadratic residue symbol; one has

$$8|h(-8p) \Longleftrightarrow \left(\frac{u}{p}\right)_2 = 1.$$

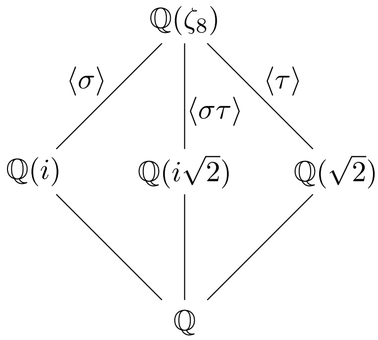
Note that $1 = (u/p)_2 = (p/u)_2 = (-2/u)_2$, so that $8|h(-8p)$ if and only if $u \equiv 1, 3 \bmod 8$. As $\varepsilon^2(u + v\sqrt{2}) = (3u + 4v) + (2u + 3v)\sqrt{2}$ and v is even, we can always choose u and v in (2.1) so that $u \equiv 1 \bmod 8$. The criterion for divisibility of $h(-8p)$ by 16 states that if u and v are integers satisfying (2.1) and $u \equiv 1 \bmod 8$, then

$$16|h(-8p) \Longleftrightarrow \left(\frac{u}{p}\right)_4 = 1,$$

where $(u/p)_4$ is equal to 1 or -1 depending on whether or not u is a fourth power modulo p. To take advantage of the multiplicative properties of the fourth-power residue symbol, one has to work over a field containing $i = \sqrt{-1}$, a primitive fourth root of unity. Since u appears naturally via the splitting of p in $\mathbb{Q}(\sqrt{2})$, we see that the natural setting for the criterion above is the number field

$$M := \mathbb{Q}(\sqrt{2}, i) = \mathbb{Q}(\zeta_8),$$

of degree 4 over $\mathbb{Q}$. It is straightforward to check that the class number of M and each of its subfields is 1, that 2 is totally ramified in M, and that the unit group of its ring of integers $\mathcal{O}_M = \mathbb{Z}[\zeta_8]$ is generated by ζ_8 and $\varepsilon = 1 + \sqrt{2}$. Note that $M/\mathbb{Q}$ is a normal extension with Galois group isomorphic to the Klein four group, say $\{1, \sigma, \tau, \sigma\tau\}$, where σ fixes $\mathbb{Q}(i)$ and τ fixes $\mathbb{Q}(\sqrt{2})$.



Let $p \equiv 1 \bmod 8$ be a prime, so that p splits completely in M. Then there exists $w \in \mathcal{O}_M$ such that $\mathrm{N}(w) = p$, i.e., such that $p = w\sigma(w)\tau(w)\sigma\tau(w)$. Note that the inclusion $\mathbb{Z} \hookrightarrow \mathcal{O}_M$ induces an isomorphism $\mathbb{Z}/(p) \cong \mathcal{O}_M/(w)$, so that an integer n is a fourth power modulo p exactly when it is a fourth power modulo w. As $w\tau(w) \in \mathbb{Z}[\sqrt{2}]$, there exist integers u and v such that $w\tau(w) = u + v\sqrt{2}$. Then $u = (w\tau(w) + \sigma(w)\sigma\tau(w))/2$. With this in mind, we define, for any $\alpha \in \mathbb{Z}[\sqrt{2}]$,

$$\mathrm{r}(\alpha) = \frac{1}{2}\left(\alpha + \sigma(\alpha)\right)$$

and, for *any odd* (i.e., coprime to 2) $w \in \mathcal{O}_M$, not necessarily prime,

$$[w] := \left(\frac{\mathrm{r}(w\tau(w))}{w}\right)_4,$$

where $(\cdot/\cdot)_4$ is the quartic residue symbol in M; we recall the definition of $(\cdot/\cdot)_4$ in the next section. A simple computation shows that $\mathrm{r}(w\tau(w)) > 0$ for any non-zero $w \in \mathcal{O}_M$. Hence $16|h(-8p)$ if and only if $[w] = 1$, where w is any element of $\mathcal{O}_M$ such that $\mathrm{N}(w) = p$ and $\mathrm{r}(w) \equiv 1 \bmod 8$.

Given a Dirichlet character χ modulo 8, we define, for any odd $w \in \mathcal{O}_M$,

$$[w]_\chi := [w] \cdot \chi(\mathrm{r}(w\tau(w))).$$

Then

$$\frac{1}{4}\sum_{\chi \bmod 8} [w]_\chi = \begin{cases} [w] & \text{if } \mathrm{r}(w\tau(w)) \equiv 1 \bmod 8, \\ 0 & \text{otherwise,} \end{cases}$$

where the sum is over all Dirichlet characters modulo 8. Another simple computation shows that, for all odd $w \in \mathcal{O}_M$, we have $[\zeta_8 w] = [w]$. We note that $\mathrm{r}(\varepsilon^2\alpha) \equiv 3 \cdot \mathrm{r}(\alpha) \bmod 8$ for any

$\alpha \in \mathbb{Z}[\sqrt{2}]$, so that $\chi(\mathrm{r}(\varepsilon^2 w\tau(\varepsilon^2 w))) = \chi(\mathrm{r}(w\tau(w)))$ for every Dirichlet character χ modulo 8. Finally, we note that

$$[w] = \left(\frac{16\mathrm{r}(w\tau(w))}{w}\right)_4 = \left(\frac{8\sigma(w)\sigma\tau(w)}{w}\right)_4, \tag{2.2}$$

so that

$$[\varepsilon w] = \left(\frac{\sigma(\varepsilon)}{w}\right)_2 [w],$$

and hence $[\varepsilon^2 w] = [w]$. Having determined the action of the units $\mathcal{O}_M^\times$ on $[\cdot]_\chi$, we can define, for each Dirichlet character χ modulo 8, a sequence $\{a(\chi)_\mathfrak{n}\}_\mathfrak{n}$ indexed by *ideals* of $\mathcal{O}_M$ by setting $a(\chi)_\mathfrak{n} = 0$ if $\mathfrak{n}$ is even, and otherwise

$$a(\chi)_\mathfrak{n} := [w]_\chi + [\varepsilon w]_\chi, \tag{2.3}$$

where w is any generator of the odd ideal $\mathfrak{n}$. Again because $\mathrm{r}(\varepsilon^2\alpha) \equiv 3 \cdot \mathrm{r}(\alpha) \bmod 8$ for any $\alpha \in \mathbb{Z}[\sqrt{2}]$, we see that if $8|h(-8p)$, then exactly one of $\mathrm{r}(w\tau(w))$ and $\mathrm{r}(\varepsilon w\tau(\varepsilon w))$ is 1 mod 8, and if $8 \nmid h(-8p)$, then neither is 1 mod 8. We have proved

Proposition 2.1. *Let $p \equiv 1 \bmod 8$ be a prime, and let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_M$ lying above p. Then*

$$\frac{1}{4}\sum_{\chi \bmod 8} a(\chi)_\mathfrak{p} = \begin{cases} 1 & \text{if } 16|h(-8p), \\ -1 & \text{if } 8|h(-8p) \text{ but } 16 \nmid h(-8p), \\ 0 & \text{otherwise,} \end{cases}$$

where the sum is over Dirichlet characters modulo 8.

3 Prerequisites

We now collect some definitions and facts that we will use in our proof of Theorem 1.

3.1 Quartic residue symbols and quartic reciprocity

Let L be a number field with ring of integers $\mathcal{O}_L$. Let $\mathfrak{p}$ be an odd prime ideal of $\mathcal{O}_L$ and let $\alpha \in \mathcal{O}_L$. One defines the *quadratic residue symbol* $(\alpha/\mathfrak{p})_{L,2}$ by setting

$$\left(\frac{\alpha}{\mathfrak{p}}\right)_{L,2} := \begin{cases} 0 & \text{if } \alpha \in \mathfrak{p} \\ 1 & \text{if } \alpha \notin \mathfrak{p} \text{ and } \alpha \equiv \beta^2 \bmod \mathfrak{p} \text{ for some } \beta \in \mathcal{O}_L \\ -1 & \text{otherwise.} \end{cases}$$

Then we have $(\alpha/\mathfrak{p})_{L,2} \equiv \alpha^{\frac{\mathrm{N}_{L/\mathbb{Q}}(\mathfrak{p})-1}{2}} \bmod \mathfrak{p}$. The quadratic residue symbol is then extended multiplicatively to all odd ideals $\mathfrak{n}$, and then also to all odd elements β in $\mathcal{O}_L$ by setting $(\alpha/\beta)_{L,2} = (\alpha/\beta\mathcal{O}_L)_{L,2}$. To define the quartic residue symbol, we assume that L contains $\mathbb{Q}(i)$. Then one can define the *quartic residue symbol* $(\alpha/\mathfrak{p})_{L,4}$ as the element of $\{\pm 1, \pm i, 0\}$ such that

$$\left(\frac{\alpha}{\mathfrak{p}}\right)_{L,4} \equiv \alpha^{\frac{\mathrm{N}_{L/\mathbb{Q}}(\mathfrak{p})-1}{4}} \bmod \mathfrak{p},$$

and extend this to all odd ideals $\mathfrak{n}$ and odd elements β in the same way as the quadratic residue symbol. A key property of the quartic residue symbol that we will use extensively is the following weak version of quartic reciprocity in $M := \mathbb{Q}(\zeta_8)$.

Lemma 3.1. *Let $\alpha, \beta \in \mathcal{O}_M$ with β odd. Then $(\alpha/\beta)_{M,4}$ depends only on the congruence class of β modulo $16\alpha\mathcal{O}_M$. Moreover, if α is also odd, then*

$$\left(\frac{\alpha}{\beta}\right)_{M,4} = \mu \cdot \left(\frac{\beta}{\alpha}\right)_{M,4},$$

where $\mu \in \{\pm 1, \pm i\}$ depends only on the congruence classes of α and β modulo $16\mathcal{O}_M$.

Proof. This follows from [15, Proposition 6.11, p. 199]. □

3.2 Field lowering

A key feature of our proof is the reduction of quartic residue symbols in a quartic number field to quadratic residue symbols in a quadratic field. We do this by using the following three lemmas.

Lemma 3.2. *Let K be a number field and let $\mathfrak{p}$ be an odd prime ideal of K. Suppose that L is a quadratic extension of K such that L contains $\mathbb{Q}(i)$ and $\mathfrak{p}$ splits in L. Denote by ψ the non-trivial element in Gal(L/K). Then if ψ fixes $\mathbb{Q}(i)$ we have for all $\alpha \in \mathcal{O}_K$*

$$\left(\frac{\alpha}{\mathfrak{p}\mathcal{O}_L}\right)_{L,4} = \left(\frac{\alpha}{\mathfrak{p}\mathcal{O}_K}\right)_{K,2}$$

and if ψ does not fix $\mathbb{Q}(i)$ we have for all $\alpha \in \mathcal{O}_K$ with $\mathfrak{p} \nmid \alpha$

$$\left(\frac{\alpha}{\mathfrak{p}\mathcal{O}_L}\right)_{L,4} = 1$$

Proof. Since $\mathfrak{p}$ splits in L, we can write $\mathfrak{p} = \mathfrak{q}\psi(\mathfrak{q})$ for some prime ideal $\mathfrak{q}$ of L. Hence we have

$$\left(\frac{\alpha}{\mathfrak{p}\mathcal{O}_L}\right)_{L,4} = \left(\frac{\alpha}{\mathfrak{q}}\right)_{L,4} \left(\frac{\alpha}{\psi(\mathfrak{q})}\right)_{L,4}.$$

If ψ fixes i we find that

$$\left(\frac{\alpha}{\mathfrak{q}}\right)_{L,4} = \psi\left(\left(\frac{\alpha}{\mathfrak{q}}\right)_{L,4}\right) = \left(\frac{\psi(\alpha)}{\psi(\mathfrak{q})}\right)_{L,4} = \left(\frac{\alpha}{\psi(\mathfrak{q})}\right)_{L,4}.$$

Combining this with the previous identity gives

$$\left(\frac{\alpha}{\mathfrak{p}\mathcal{O}_L}\right)_{L,4} = \left(\frac{\alpha}{\mathfrak{q}}\right)_{L,4}^2 = \left(\frac{\alpha}{\mathfrak{q}}\right)_{L,2} = \left(\frac{\alpha}{\mathfrak{p}\mathcal{O}_K}\right)_{K,2},$$

establishing the first part of the lemma. If ψ does not fix i we find that

$$\left(\frac{\alpha}{\mathfrak{p}\mathcal{O}_L}\right)_{L,4} = \left(\frac{\alpha}{\mathfrak{q}}\right)_{L,4} \left(\frac{\alpha}{\psi(\mathfrak{q})}\right)_{L,4} = \left(\frac{\alpha}{\mathfrak{q}}\right)_{L,4} \psi\left(\left(\frac{\alpha}{\mathfrak{q}}\right)_{L,4}\right) = 1$$

by checking this for all values of $(\alpha/\mathfrak{q})_{L,4} \in \{\pm 1, \pm i\}$. This completes the proof. □

Lemma 3.3. *Let K be a number field and let $\mathfrak{p}$ be an odd prime ideal of K of degree 1 lying above p. Suppose that L is a quadratic extension of K such that L contains $\mathbb{Q}(i)$ and $\mathfrak{p}$ stays inert in L. Then we have for all $\alpha \in \mathcal{O}_K$*

$$\left(\frac{\alpha}{\mathfrak{p}\mathcal{O}_L}\right)_{L,4} = \left(\frac{\alpha}{\mathfrak{p}\mathcal{O}_K}\right)_{K,2}^{\frac{p+1}{2}}.$$

Proof. We have

$$\left(\frac{\alpha}{\mathfrak{p}\mathcal{O}_L}\right)_{L,4} \equiv \alpha^{\frac{\mathrm{N}_L(\mathfrak{p})-1}{4}} \equiv \alpha^{\frac{p^2-1}{4}} \equiv \left(\alpha^{\frac{p-1}{2}}\right)^{\frac{p+1}{2}} \equiv \left(\alpha^{\frac{\mathrm{N}_K(\mathfrak{p})-1}{2}}\right)^{\frac{p+1}{2}} \equiv \left(\frac{\alpha}{\mathfrak{p}\mathcal{O}_K}\right)_{K,2}^{\frac{p+1}{2}} \bmod \mathfrak{p},$$

which immediately implies the lemma. □

Note that the previous lemmas only work if $\alpha \in \mathcal{O}_K$. Our last lemma gives a way to ensure that $\alpha \in \mathcal{O}_K$.

Lemma 3.4. *Let K be a number field and let L be a quadratic extension of K. Denote by ψ the non-trivial element in Gal(L/K). Suppose that $\mathfrak{p}$ is a prime ideal of K that does not ramify in L and further suppose that $\beta \in \mathcal{O}_L$ satisfies $\beta \equiv \psi(\beta) \bmod \mathfrak{p}\mathcal{O}_L$. Then there is $\beta' \in \mathcal{O}_K$ such that $\beta' \equiv \beta \bmod \mathfrak{p}\mathcal{O}_L$.*

Proof. Since by assumption $\mathfrak{p}$ does not ramify in L, we may assume that $\mathfrak{p}$ splits or stays inert in L. Let us first do the case that $\mathfrak{p}$ stays inert, which means precisely that $\psi(\mathfrak{p}) = \mathfrak{p}$. We conclude that ψ is in the decomposition group of $\mathfrak{p}$. Furthermore, the inertia group of $\mathfrak{p}$ is trivial by the assumption that $\mathfrak{p}$ does not ramify. Since ψ is not the identity, it follows that ψ must become the Frobenius map of the finite field extension $\mathcal{O}_K/\mathfrak{p} \hookrightarrow \mathcal{O}_L/\mathfrak{p}$. Then $\beta \equiv \psi(\beta) \bmod \mathfrak{p}\mathcal{O}_L$ means that β is fixed by the Frobenius map. We conclude that β comes from $\mathcal{O}_K/\mathfrak{p}$, which we had to prove.

We still have to prove the lemma if $\mathfrak{p}$ splits. In this case we can write $\mathfrak{p} = \mathfrak{q}\psi(\mathfrak{q})$ for some prime ideal $\mathfrak{q}$ of L. Note that

$$\mathcal{O}_K/\mathfrak{p} \hookrightarrow \mathcal{O}_L/\mathfrak{p}\mathcal{O}_L \cong \mathcal{O}_L/\mathfrak{q} \times \mathcal{O}_L/\psi(\mathfrak{q}). \tag{3.1}$$

One checks that ψ is the automorphism of $\mathcal{O}_L/\mathfrak{q} \times \mathcal{O}_L/\psi(\mathfrak{q})$ that maps (x, y) to $(\psi(y), \psi(x))$. Hence $\beta \equiv \psi(\beta) \bmod \mathfrak{p}\mathcal{O}_L$ implies that there is some $x \in \mathcal{O}_L/\mathfrak{q}$ such that $\beta = (x, \psi(x))$ as an element of $\mathcal{O}_L/\mathfrak{q} \times \mathcal{O}_L/\psi(\mathfrak{q})$. Since $\mathcal{O}_K/\mathfrak{p} \cong \mathcal{O}_L/\mathfrak{q}$, we can pick $\beta' \in \mathcal{O}_K$ such that β' maps to x under the natural inclusion $\mathcal{O}_K/\mathfrak{p} \hookrightarrow \mathcal{O}_L/\mathfrak{q}$. Then it follows that β maps to $(\beta', \psi(\beta'))$ under the maps given as in (3.1). This implies that $\beta' \equiv \beta \bmod \mathfrak{p}\mathcal{O}_L$ as desired. □

3.3 A fundamental domain for the action of $\mathcal{O}_M^\times$

In defining $a(\chi)_\mathfrak{n}$ for odd ideals $\mathfrak{n}$ of $\mathcal{O}_M$, we had to choose a generator w for the ideal $\mathfrak{n}$. There are many such choices, since the group of units of $\mathcal{O}_M$ is quite large, i.e.,

$$\mathcal{O}_M^\times = \langle \zeta_8 \rangle \times \langle \varepsilon \rangle,$$

where $\varepsilon = 1 + \sqrt{2}$ as before. It will be important to us that we can choose generators that are in some sense as small as possible. We will do so by constructing a fundamental domain for the action (by multiplication) of $\mathcal{O}_M^\times$ on $\mathcal{O}_M$. The lemma that follows is usually implicitly proved in most number theory textbooks, but we have not been able to find a reference stating exactly the somewhat peculiar version that we will need. Below we deduce this version from [14, Lemma 1, p. 131].

More generally, let F be a number field of degree n over $\mathbb{Q}$ with ring of integers $\mathcal{O}_F$. Let $\sigma_1, \ldots, \sigma_r : F \hookrightarrow \mathbb{R}$ be the real embeddings of F and let $\tau_1, \bar{\tau}_1, \ldots, \tau_s, \bar{\tau}_s : F \hookrightarrow \mathbb{C}$ be the pairs of non-real complex conjugate embeddings of F (so that $r + 2s = n$). Let T be the subgroup of the unit group $\mathcal{O}_F^\times$ consisting of units of finite order. By Dirichlet's Unit Theorem, there exists a free abelian subgroup $V \subset \mathcal{O}_F^\times$ of rank $r + s - 1$ such that $\mathcal{O}_F^\times = T \times V$; fix one such V.

Let $\eta = \{\eta_1, \ldots, \eta_n\}$ be an integral basis for $\mathcal{O}_F$; it defines an isomorphism $i_\eta : \mathbb{Q}^n \to F$ via the map $(a_1, \ldots, a_n) \mapsto a_1\eta_1 + \cdots a_n\eta_n$. For a subset $S \subset \mathbb{R}^n$ and an element $\alpha = a_1\eta_1 + \cdots + a_n\eta_n \in F$, we will say that α is in S (or $\alpha \in S$) to mean that $(a_1, \ldots, a_n) \in S$. Let $f_\eta \in \mathbb{Z}[x_1, \ldots, x_n]$ be the homogeneous polynomial of degree n in n variables defined by $f_\eta(x_1, \ldots, x_n) = \mathrm{N}(x_1\eta_1 + \cdots + x_n\eta_n)$. For a subset $S \subset \mathbb{R}^n$ and a real number $X > 0$, let $S(X)$ be the set of all $(s_1, \ldots, s_n) \in S$ such that $|f_\eta(s_1, \ldots, s_n)| \leq X$.

Lemma 3.5. *There exists a subset $\mathcal{D} \subset \mathbb{R}^n$ such that:*

(1) for all $\alpha \in \mathcal{O}_F \setminus \{0\}$, there exists a unique $v \in V$ such that $v\alpha \in \mathcal{D}$; moreover, the complete set of $u \in \mathcal{O}_F^\times$ such that $u\alpha \in \mathcal{D}$ is $\{\mu v : \mu \in T\}$;

(2) $\mathcal{D}(1)$ has an $(n-1)$-Lipschitz parametrizable boundary; and

(3) there exists a constant $C_\eta > 0$ such that for all $\alpha = a_1\eta_1 + \cdots + a_n\eta_n \in \mathcal{D}$ (with $a_i \in \mathbb{Z}$), we have $|a_i| \leq C_\eta \cdot |\mathrm{N}(\alpha)|^{\frac{1}{n}}$.

Proof. Let $J = \mathbb{R}^r \times \mathbb{C}^s$. Then $j = (\sigma_1, \ldots, \sigma_r, \tau_1, \ldots, \tau_s)$ defines an embedding $j : F \hookrightarrow J$. Moreover, $j \circ i_\eta : \mathbb{Q}^n \to J$ is a *linear* map of $\mathbb{Q}$-vector spaces. By extension of scalars, we extend this to a linear map

$$\bar{j} : \mathbb{R}^n \to J.$$

It follows from [14, Lemma 1, p. 131] and its proof that there is a subset $D \subset J^\times$ such that:

(1') for all $\alpha \in J^\times$, there exists a unique $v \in V$ such that $v\alpha \in D$; moreover, the complete set of $u \in \mathcal{O}_F^\times$ such that $u\alpha \in D$ is $\{\mu v : \mu \in T\}$; and

(2') $D(1) = \{(\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s) \in D : \prod_{i=1}^r |\alpha_i| \prod_{j=1}^s |\beta_j|^2 \leq 1\}$ has an $(n-1)$-Lipschitz parametrizable boundary.

(3') for all non-zero $t \in \mathbb{R}$, we have $tD = D$.

Let $\mathcal{D} = \bar{j}^{-1}(D)$. Then (1) follows immediately from (1'). Since $\bar{j}$ is linear and hence Lipschitz continuous, (2') immediately implies (2) (after also taking into account the definitions of $D(1)$, f_η, and $\mathcal{D}(1)$). By (2), the set $\mathcal{D}(1) \subset \mathbb{R}^n$ is bounded, so we can set

$$C_\eta = \sup\{|a_i| : (a_1, \ldots, a_n) \in \mathcal{D}(1)\}.$$

Finally, again because $\bar{j}$ is linear, (3') implies that $t\mathcal{D} = \mathcal{D}$ for all non-zero $t \in \mathbb{R}$, so that $\mathcal{D}(t) = t^{1/n}\mathcal{D}(1)$. This proves (3). □

3.4 General bilinear sum estimates

Let F, n, η, and V be as in Section 3.3. Fix a fundamental domain $\mathcal{D}$ for the action of V on $\mathcal{O}_F$ as in Lemma 3.5. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be a pair of translates of $\mathcal{D}$, i.e., $\mathcal{D}_i = v_i\mathcal{D}$ for some

$v_i \in V$. Let $\mathfrak{f}$ be a non-zero ideal in $\mathcal{O}_F$, and let $S_{\mathfrak{f}}$ be the set of elements in $\mathcal{O}_F$ coprime to $\mathfrak{f}$. Suppose γ is a map

$$\gamma : S_{\mathfrak{f}} \times \mathcal{O}_F \to \{-1, 0, 1\}$$

satisfying the following properties:

(P1) for every pair of invertible congruence classes ω and ζ modulo $\mathfrak{f}$, there exists $\mu(\omega, \zeta) \in \{\pm 1\}$ such that $\gamma(w, z) = \mu(\omega, \zeta)\gamma(z, w)$ whenever $w \equiv \omega \bmod \mathfrak{f}$ and $z \equiv \zeta \bmod \mathfrak{f}$;

(P2) for all $z_1, z_2 \in \mathcal{O}_F$ and all $w \in S_{\mathfrak{f}}$, we have $\gamma(w, z_1z_2) = \gamma(w, z_1)\gamma(w, z_2)$; similarly, for all $w_1, w_2 \in S_{\mathfrak{f}}$ and all $z \in \mathcal{O}_F$, we have $\gamma(w_1w_2, z) = \gamma(w_1, z)\gamma(w_2, z)$; and

(P3) for all $w \in S_{\mathfrak{f}}$, if $z_1, z_2 \in \mathcal{O}_F$ satisfy $z_1 \equiv z_2 \bmod \mathrm{N}(w)$ (i.e., $z_1 - z_2 = N(w)\alpha$ for some $\alpha \in \mathcal{O}_F$), then we have $\gamma(w, z_1) = \gamma(w, z_2)$; moreover, if $|\mathrm{N}(w)|$ is not a square in $\mathbb{Z}$, we have $\sum_{\xi \bmod \mathrm{N}(w)} \gamma(w, \xi) = 0$.

We will consider bilinear sums of the type

$$B(M, N; \omega, \zeta) := \sum_{\substack{w \in \mathcal{D}_1(M) \\ w \equiv \omega \bmod \mathfrak{f}}} \sum_{\substack{z \in \mathcal{D}_2(N) \\ z \equiv \zeta \bmod \mathfrak{f}}} \alpha_w \beta_z \gamma(w, z), \tag{3.2}$$

where $\{\alpha_w\}_w$ and $\{\beta_z\}_z$ are bounded sequences of complex numbers, ω and ζ are invertible congruence classes modulo $\mathfrak{f}$, and M and N are positive real numbers. Recall that $w \in \mathcal{D}_1(M)$ if and only if $w \in \mathcal{D}_1$ and $|\mathrm{N}(w)| \leq M$, and similarly for $\mathcal{D}_2(N)$. Also recall that n is the degree of $F/\mathbb{Q}$. The following proposition is analogous to the bilinear sum estimates in [9, 10].

Proposition 3.6. *We have*

$$B(M, N; \omega, \zeta) \ll_\epsilon \left(M^{-\frac{1}{6n}} + N^{-\frac{1}{6n}}\right)(MN)^{1+\epsilon},$$

where the implied constant depends on ϵ, on the units v_1 and v_2, on the supremum norms of $\{\alpha_w\}_w$ and $\{\beta_z\}_z$, and the congruence classes ω and ζ modulo $\mathfrak{f}$.

Proof. We will prove that

$$B(M, N; \omega, \zeta) \ll_\epsilon M^{-\frac{1}{6n}}(MN)^{1+\epsilon} \tag{3.3}$$

whenever $N \geq M$; the proposition then immediately follows from the symmetry of the sum $B(M, N; \omega, \zeta)$ coming from property (P1). So suppose that $N \geq M$. We fix an integer $k \geq 2n$, and we apply Hölder's inequality (with $1 = \frac{k-1}{k} + \frac{1}{k}$) to the w variable to get

$$|B(M, N; \omega, \zeta)|^k \leq \left(\sum_w |\alpha_w|^{\frac{k}{k-1}}\right)^{k-1} \sum_w \left|\sum_z \beta_z \gamma(w, z)\right|^k,$$

where the summations over w and z are as above in (3.2). The first factor above is bounded trivially by $\ll M^{k-1}$, where the implied constant depends on the supremum norm of the sequence $\{\alpha_w\}_w$, on the fixed unit v_1, and on the constant C_η from part (3) of Lemma 3.5.

We use property (P2), as well as the identity $|\alpha|^k = \alpha^k \cdot (|\alpha|/\alpha)^k$, to expand the inner sum in the second factor above, getting

$$|B(M,N;\omega,\zeta)|^k \ll M^{k-1} \sum_w \varepsilon(w) \sum_z \beta'_z \gamma(w,z),$$

where

$$\beta'_z = \sum_{\substack{z=z_1\cdots z_k \\ z_1,\ldots,z_k \in \mathcal{D}_2(N) \\ z_1 \equiv \cdots \equiv z_k \equiv \zeta \bmod \mathfrak{f}}} \beta_{z_1} \cdots \beta_{z_k},$$

where $\varepsilon(w) = (|\sum_z \beta_z \gamma(w,z)|/\sum_z \beta_z \gamma(w,z))^k$, and where once again the summation conditions for w are as in (3.2). Since an ideal $\mathfrak{n}$ in $\mathcal{O}_F$ can be written as a product of k ideals in at most $\ll_\epsilon \mathrm{N}(\mathfrak{n})^\epsilon$ ways, and since $\mathcal{D}_2$ contains at most one generator of any principal ideal, we see that $\beta'_z \ll_\epsilon N^\epsilon$. Moreover, the coordinates of each $z_i \in \mathcal{D}_2$ $(1 \leq i \leq k)$ of absolute norm at most N in the basis η are bounded by $N^{\frac{1}{n}}$ times a constant depending on the unit v_2 and on C_η from Lemma 3.5. Hence we may assume that the sum $\sum_z \beta'_z \gamma(w,z)$ above is over $z = a_1\eta_1 + \cdots + a_n\eta_n$ in a box $\mathcal{B}$ defined by $|a_j| \ll N^{\frac{k}{n}}$ $(1 \leq j \leq n)$, with the implied constant depending on v_2 and on the integral basis η. Next, we apply the Cauchy-Schwarz inequality to the z variable above and use property (P2) to get

$$\left| \sum_w \varepsilon(w) \sum_z \beta'_z \gamma(w,z) \right|^2 \ll_\epsilon N^{k+\epsilon} \sum_{w_1} \sum_{w_2} \varepsilon(w_1)\overline{\varepsilon(w_2)} \sum_z \gamma(w_1 w_2, z),$$

where the summation conditions for w_1 and w_2 are as those for w in (3.2), while the inner sum is over $z \in \mathcal{B}$. We break up the sum over z into congruence classes ξ modulo $\mathrm{N}(w_1w_2)$ and note that, by property (P3),

$$\sum_{\xi \bmod \mathrm{N}(w_1w_2)} \gamma(w_1w_2, z) = 0$$

unless $|\mathrm{N}(w_1w_2)|$ is a square. By counting points z in the box $\mathcal{B}$ and noting that $|\mathrm{N}(w_1w_2)| \leq M^2$, this gives

$$\sum_z \gamma(w_1w_2,z) \ll \begin{cases} N^k & \text{if } |\mathrm{N}(w_1w_2)| = \square \\ \sum_{i=1}^n M^{2i} N^{k(1-\frac{i}{n})} & \text{otherwise.} \end{cases}$$

Since we took $k \geq 2n$ and since $N \geq M$, we have $N^{\frac{k}{n}} \geq M^2$, so the last bound can be simplified to $M^2 N^{k(1-\frac{1}{n})}$. Hence, putting together all of the bounds above, we get

$$\begin{aligned} |B(M,N;\omega,\zeta)|^{2k} &\ll_\epsilon M^{2k-2}N^k \left(M \cdot N^k + M^2 \cdot M^2 N^{k(1-\frac{1}{n})} \right) (MN)^\epsilon \\ &\ll_\epsilon \left(M^{2k-1}N^{2k} + M^{2k+2} N^{2k(1-\frac{1}{2n})} \right) (MN)^\epsilon. \end{aligned}$$

Since $N \geq M$, if we take $k = 3n$, we get that $N^{2k\frac{1}{2n}} \geq M^3$, so that the first term above dominates the second term. With this choice of k, we get

$$|B(M,N;\omega,\zeta)| \ll_\epsilon M^{-\frac{1}{6n}} (MN)^{1+\epsilon},$$

and this finishes the proof of (3.3). $\square$

3.5 The sieve

We will prove Theorem 1 by a sieve of Friedlander et al. [10] that generalizes the ideas of Vinogradov [25, 26] to the setting of number fields. Let χ be a Dirichlet character modulo 8, and let $a(\chi)_{\mathfrak{n}}$ be defined as in (2.3). We will prove the following two propositions.

Proposition 3.7. *For every $\epsilon > 0$, we have*

$$\sum_{\mathrm{N}(\mathfrak{n})\leq X,\ \mathfrak{m}|\mathfrak{n}} a(\chi)_{\mathfrak{n}} \ll_\epsilon X^{1-\frac{1}{64}+\epsilon}$$

uniformly for all non-zero ideals $\mathfrak{m}$ of $\mathcal{O}_M$ and all $X \geq 2$.

Proposition 3.8. *For every $\epsilon > 0$, we have*

$$\sum_{\mathrm{N}(\mathfrak{m})\leq M}\sum_{\mathrm{N}(\mathfrak{n})\leq N} \alpha_{\mathfrak{m}}\beta_{\mathfrak{n}}a(\chi)_{\mathfrak{mn}} \ll_\epsilon (M+N)^{\frac{1}{24}}(MN)^{1-\frac{1}{24}+\epsilon}$$

uniformly for all $M, N \geq 2$ and sequences of complex numbers $\{\alpha_{\mathfrak{m}}\}$ and $\{\beta_{\mathfrak{n}}\}$ satisfying $|\alpha_{\mathfrak{m}}|, |\beta_{\mathfrak{n}}| \leq 1$.

From these two propositions we can apply [10, Proposition 5.2, p. 722] with $\theta_1 = \frac{1}{64}$ and $\theta_2 = \frac{1}{24}$ to prove

$$\sum_{\mathrm{N}(\mathfrak{n})\leq X} a(\chi)_{\mathfrak{n}}\Lambda(\mathfrak{n}) \ll_\theta X^{1-\theta}$$

for all $\theta < 1/(49\cdot 64) = 1/3136$. By partial summation, it follows that, say,

$$\sum_{\mathrm{N}(\mathfrak{p})\leq X} a(\chi)_{\mathfrak{p}} \ll X^{1-\frac{1}{3200}}. \tag{3.4}$$

As

$$\sum_{\substack{\mathrm{N}(\mathfrak{p})\leq X \\ \mathfrak{p}\text{ lies over } p\not\equiv 1 \bmod 8}} 1 \ll X^{\frac{1}{2}},$$

Theorem 1 follows from (3.4) and Proposition 2.1. It now remains to prove Propositions 3.7 and 3.8.

4 Proof of Proposition 3.7

Let χ be a Dirichlet character modulo 8. Let $\mathfrak{m}$ be an odd ideal of $\mathcal{O}_M$. In view of Proposition 2.1 we must bound the following sum

$$A(x) = A(x;\chi,\mathfrak{m}) := \sum_{\substack{\mathrm{N}(\mathfrak{a})\leq x \\ (\mathfrak{a},2)=1,\ \mathfrak{m}|\mathfrak{a}}} \left([\alpha]_\chi + [\varepsilon\alpha]_\chi\right),$$

where α is chosen to be any generator of $\mathfrak{a}$. Our proof is based on the argument in [13, Section 3, p. 12-19], which is in turn based on [10, Section 6, p. 722-733]. Let $\mathcal{D}$ be a fundamental domain for the action of $\mathcal{O}_M^\times$ on $\mathcal{O}_M \setminus \{0\}$ as in Lemma 3.5, with respect to the integral basis $\eta = \{1, \zeta_8, \zeta_8^2, \zeta_8^3\}$. Each non-zero ideal $\mathfrak{a}$ has exactly 8 generators $\alpha \in \mathcal{D}$. Set $u_1 = 1$

and $u_2 = \varepsilon$. Set $F = 16$. Note that $\chi(\mathrm{r}(\alpha\tau(\alpha)))$ depends only on the congruence class of α modulo 8. After splitting the above sum into congruence classes modulo F, and using (2.2) and Lemma 3.1, we find that

$$A(x) = \frac{1}{8}\sum_{i=1}^{2}\sum_{\substack{\rho \bmod F\\(\rho,F)=1}}\mu(\rho,u_i)A(x;\rho,u_i),$$

where $\mu(\rho,u_i) \in \{\pm1,\pm i\}$ depends only on ρ and u_i and where

$$A(x;\rho,u_i) := \sum_{\substack{\alpha\in u_i\mathcal{D},\ \mathrm{N}(\alpha)\le x\\ \alpha\equiv\rho \bmod F\\ \alpha\equiv 0 \bmod \mathfrak{m}}}\left(\frac{\sigma(\alpha)}{\alpha}\right)_{M,4}\left(\frac{\sigma\tau(\alpha)}{\alpha}\right)_{M,4}.$$

Our goal is to estimate $A(x;\rho,u_i)$ separately for each congruence class $\rho \bmod F$, $(\rho,F)=1$ and unit u_i. We view $\mathcal{O}_M$ as a $\mathbb{Z}$-module of rank 4 and decompose it as $\mathcal{O}_M = \mathbb{Z}\oplus\mathbb{M}$, where $\mathbb{M} = \mathbb{Z}\zeta_8\oplus\mathbb{Z}\zeta_8^2\oplus\mathbb{Z}\zeta_8^3$ is a free $\mathbb{Z}$-module of rank 3. We can write α uniquely as

$$\alpha = a+\beta, \text{ with } a\in\mathbb{Z}, \beta\in\mathbb{M},$$

so that the summation conditions above are equivalent to

$$a+\beta\in u_i\mathcal{D},\quad \mathrm{N}(a+\beta)\le x,\quad a+\beta\equiv\rho \bmod F,\quad a+\beta\equiv 0 \bmod \mathfrak{m}. \tag{$*$}$$

We may assume that $\sigma(\beta)\neq\beta$ and $\sigma\tau(\beta)\neq\beta$. Indeed, if $\sigma(\beta)=\beta$ or $\sigma\tau(\beta)=\beta$, the residue symbol in $A(x;\rho,u_i)$ is zero. We are now going to rewrite $(\sigma(\alpha)/\alpha)_{M,4}$ and $(\sigma\tau(\alpha)/\alpha)_{M,4}$ by using the same trick as in [10, p. 725]. Put

$$\sigma(\beta)-\beta = \eta^2c_0c \quad\text{ and }\quad \sigma\tau(\beta)-\beta = \eta'^2c_0'c'$$

with $c_0,c_0',c,c',\eta,\eta'\in\mathcal{O}_M$, $c_0,c_0' \mid F$ squarefree, $\eta,\eta' \mid F^\infty$ and $(c,F)=(c',F)=1$. By multiplying with an appropriate unit we can even ensure that $c\in\mathbb{Z}[i]$ and $c'\in\mathbb{Z}[\sqrt{-2}]$. Indeed, observe that

$$\alpha' := \frac{\sigma(\alpha)-\alpha}{\zeta_8} = \frac{\sigma(\beta)-\beta}{\zeta_8}\in\mathbb{Z}[i], \tag{4.1}$$

and we have a similar identity for $\sigma\tau(\beta)-\beta$. Then we obtain, just as in [13, p. 14], by Lemma 3.1,

$$\left(\frac{\sigma(\alpha)}{\alpha}\right)_{M,4} = \mu_1\cdot\left(\frac{a+\beta}{c\mathcal{O}_M}\right)_{M,4} \quad\text{ and }\quad \left(\frac{\sigma\tau(\alpha)}{\alpha}\right)_{M,4} = \mu_2\cdot\left(\frac{a+\beta}{c'\mathcal{O}_M}\right)_{M,4},$$

where $\mu_1,\mu_2\in\{\pm1,\pm i\}$ depend only on ρ and β. Hence

$$A(x;\rho,u_i)\le\sum_{\beta\in\mathbb{M}}|T(x;\beta,\rho,u_i)|,$$

where

$$T(x;\beta,\rho,u_i) := \sum_{\substack{a\in\mathbb{Z}\\ a+\beta \text{ sat. } (*)}}\left(\frac{a+\beta}{c\mathcal{O}_M}\right)_{M,4}\left(\frac{a+\beta}{c'\mathcal{O}_M}\right)_{M,4}.$$

From now on we treat β as fixed and estimate $T(x;\beta,\rho,u_i)$. It is here that we deviate from [10] and [13]. Since we chose $c' \in \mathbb{Z}[\sqrt{-2}]$, we can factor the principal ideal $(c') \subset \mathbb{Z}[\sqrt{-2}]$ into prime ideals in $\mathbb{Z}[\sqrt{-2}]$ that do not ramify in M, say, $(c') = \prod_{i=1}^k \mathfrak{p}_i^{e_i}$, so that

$$\left(\frac{a+\beta}{c'\mathcal{O}_M}\right)_{M,4} = \prod_{i=1}^k \left(\frac{a+\beta}{\mathfrak{p}_i\mathcal{O}_M}\right)_{M,4}^{e_i}.$$

We claim that $((a+\beta)/\mathfrak{p}\mathcal{O}_M)_{M,4} = 1$ if $\mathfrak{p} \nmid a+\beta$. As a first step we can replace β by some $\beta' \in \mathbb{Z}[\sqrt{-2}]$ due to Lemma 3.4. Then Lemma 3.2 proves the claim if $\mathfrak{p}$ splits in M. Finally suppose that $\mathfrak{p}$ stays inert in M. If we define $p := \mathfrak{p} \cap \mathbb{Z}$, we find that $p \equiv 3 \bmod 8$. Hence Lemma 3.3 finishes the proof of the claim.

The factor $((a+\beta)/c\mathcal{O}_M)_{M,4}$ is handled more similarly to [10, (6.21), p. 727]. Since we chose $c \in \mathbb{Z}[i]$, we factor $(c) \subset \mathbb{Z}[i]$ in $\mathbb{Z}[i]$ as $(c) = \mathfrak{g}\mathfrak{q}$ in the unique way so that $q := N_{\mathbb{Q}(i)/\mathbb{Q}}(\mathfrak{q})$ is a squarefree odd integer and $g := N_{\mathbb{Q}(i)/\mathbb{Q}}(\mathfrak{g})$ is an odd squarefull integer coprime with q.

Lemma 3.4 and the Chinese remainder theorem imply that there exists $\beta' \in \mathbb{Z}[i]$ such that $\beta \equiv \beta' \bmod \mathfrak{q}\mathcal{O}_M$. Next, Lemma 3.2 and Lemma 3.3 imply that $((a+\beta')/\mathfrak{q}\mathcal{O}_M)_{M,4} = ((a+\beta')/\mathfrak{q})_{\mathbb{Q}(i),2}$. Finally, as q is squarefree, the Chinese remainder theorem guarantees the existence of a rational integer b such that $\beta' \equiv b \bmod \mathfrak{q}$. Combining all of this gives

$$\left(\frac{a+\beta}{c\mathcal{O}_M}\right)_{M,4} = \left(\frac{a+\beta}{\mathfrak{g}\mathcal{O}_M}\right)_{M,4} \left(\frac{a+b}{\mathfrak{q}}\right)_{\mathbb{Q}(i),2}.$$

Since c depends on β and not on a, we find that b depends on β and not on a. Now define g_0 as the radical of g, i.e., $g_0 := \prod_{p|g} p$. We observe that the quartic residue symbol $(\alpha/\mathfrak{g}\mathcal{O}_M)_{M,4}$ is periodic in α modulo $\mathfrak{g}^* := \prod_{\mathfrak{p}|\mathfrak{g}} \mathfrak{p}$. But clearly $\mathfrak{g}^*$ divides g_0, and hence we conclude that $((a+\beta)/\mathfrak{g}\mathcal{O}_M)_{M,4}$ is periodic of period g_0 when viewed as a function of $a \in \mathbb{Z}$. So we split $T(x;\beta,\rho,u_i)$ into congruence classes modulo g_0, giving

$$|T(x;\beta,\rho,u_i)| \leq \sum_{a_0 \bmod g_0} |T(x;\beta,\rho,u_i,a_0)|,$$

where

$$T(x;\beta,\rho,u_i,a_0) = \sum_{\substack{a\in\mathbb{Z} \\ a+\beta \text{ sat. } (*) \\ a\equiv a_0 \bmod g_0}} \left(\frac{a+b}{\mathfrak{q}}\right)_{\mathbb{Q}(i),2} \left(\frac{a+\beta}{c'\mathcal{O}_M}\right)_{M,4}.$$

We have already proven that $((a+\beta)/c'\mathcal{O}_M)_{M,4} = 1$ unless $\gcd(a+\beta, c') \neq (1)$ and in this case we have $((a+\beta)/c'\mathcal{O}_M)_{M,4} = 0$. An application of inclusion-exclusion gives

$$|T(x;\beta,\rho,u_i,a_0)| \leq \sum_{\substack{\mathfrak{d}|c'\mathcal{O}_M \\ \mathfrak{d} \text{ square-free}}} |T(x;\beta,\rho,u_i,a_0,\mathfrak{d})|,$$

where

$$T(x;\beta,\rho,u_i,a_0,\mathfrak{d}) := \sum_{\substack{a\in\mathbb{Z} \\ a+\beta \text{ sat. } (*) \\ a\equiv a_0 \bmod g_0 \\ a+\beta\equiv 0 \bmod \mathfrak{d}}} \left(\frac{a+b}{\mathfrak{q}}\right)_{\mathbb{Q}(i),2}. \tag{4.2}$$

We unwrap the summation conditions above similarly as in [10, p. 728]. Certainly $a+\beta \in u_i\mathcal{D}$ implies that $|a| \leq Cx^{\frac{1}{4}}$, where $C > 0$ depends only on one of the two fixed units u_i. The condition $\mathrm{N}_{M/\mathbb{Q}}(a+\beta) \leq x$ is for fixed β and x a polynomial inequality of degree 4 in a. Hence the summation variable $a \in \mathbb{Z}$ runs over at most 4 intervals of length $\leq Cx^{1/4}$ with endpoints depending on β and x.

Next, the congruence conditions $a+\beta \equiv \rho \bmod F$, $a+\beta \equiv 0 \bmod \mathfrak{m}$, $a \equiv a_0 \bmod g_0$ and $a+\beta \equiv 0 \bmod \mathfrak{d}$ imply that a runs over some arithmetic progression of modulus k dividing g_0mdF, where we define $m := \mathrm{N}_{M/\mathbb{Q}}(\mathfrak{m})$ and $d := \mathrm{N}_{M/\mathbb{Q}}(\mathfrak{d})$. Moreover, as $q = \mathrm{N}_{\mathbb{Q}(i)/\mathbb{Q}}(\mathfrak{q})$ is squarefree, $(\cdot/\mathfrak{q})_{\mathbb{Q}(i),2} : \mathbb{Z} \to \{\pm 1, 0\}$ is the real primitive Dirichlet character of modulus q.

All in all, the sum in (4.2) can be rewritten as at most 4 incomplete real character sums of length $\ll x^{\frac{1}{4}}$ and modulus $q \ll x^{\frac{1}{2}}$, each of which runs over an arithmetic progression of modulus k. When the modulus q of the Dirichlet character divides the modulus k of the arithmetic progression, one does not get the desired cancellation. So for now we assume that $q \nmid k$, and we will handle the case $q \mid k$ later. As has been explained in [11, 7., p. 924-925], Burgess's bound for short character sums [1] implies that for each integer $r \geq 2$, we have

$$T(x;\beta,\rho,u_i,a_0,\mathfrak{d}) \ll_{\epsilon,r} x^{\frac{1}{4}\left(1-\frac{1}{r}\right)} \cdot x^{\frac{1}{2}\left(\frac{r+1}{4r^2}+\epsilon\right)},$$

so that on taking $r = 2$, we obtain

$$T(x;\beta,\rho,u_i) \ll_\epsilon g_0 x^{\frac{1}{4}-\frac{1}{32}+\epsilon}. \tag{4.3}$$

It remains to do the case $q \mid k$. Certainly, this implies $q \mid md$. So (4.3) holds if $q \nmid md$. Recall that $(c) = \mathfrak{g}\mathfrak{q}$, hence we have (4.3) unless

$$p \mid \mathrm{N}_{\mathbb{Q}(i)/\mathbb{Q}}(\alpha') \implies p^2 \mid mdF\mathrm{N}_{\mathbb{Q}(i)/\mathbb{Q}}(\alpha') \tag{4.4}$$

for all primes p, where α' is defined as in (4.1). Define $A_\square(x;\rho,u_i)$ as the contribution to $A(x;\rho,u_i)$ from β satisfying (4.4). Then we get

$$A_\square(x;\rho,u_i) \leq |\{\alpha \in u_i\mathcal{D} : N_{M/\mathbb{Q}}(\alpha) \leq x,\ p \mid \mathrm{N}_{\mathbb{Q}(i)/\mathbb{Q}}(\alpha') \implies p^2 \mid mdF\mathrm{N}_{\mathbb{Q}(i)/\mathbb{Q}}(\alpha')\}|.$$

We decompose $\mathcal{O}_M$ as $\mathcal{O}_M = \mathbb{Z}[i] \oplus \mathbb{M}'$, where $\mathbb{M}' = \mathbb{Z}\zeta_8 \oplus \mathbb{Z}\zeta_8^3 = \mathbb{Z}[i] \cdot \zeta_8$ is a free $\mathbb{Z}$-module of rank 2. The linear map $\mathbb{M}' \to \mathbb{Z}[i]$ given by $\alpha \mapsto \alpha'$ is injective. Now suppose $\alpha \in u_i\mathcal{D}$ and $\mathrm{N}_{M/\mathbb{Q}}(\alpha) \leq x$. Then by Lemma 3.5, if we write $\alpha = a_1 + a_2 i + (a_3 + a_4 i)\zeta_8$, we have $a_j \ll x^{\frac{1}{4}}$ for $1 \leq j \leq 4$. Hence the norm $\mathrm{N}_{\mathbb{Q}(i)/\mathbb{Q}}(\cdot)$ of $\alpha' = -2(a_3 + a_4 i)$ is $\ll x^{\frac{1}{2}}$, and so

$$A_\square(x;\rho,u_i) \ll x^{\frac{1}{2}} |\{\alpha' \in \mathbb{Z}[i] : \mathrm{N}_{\mathbb{Q}(i)/\mathbb{Q}}(\alpha') \ll x^{\frac{1}{2}},\ p \mid \mathrm{N}_{\mathbb{Q}(i)/\mathbb{Q}}(\alpha') \implies p^2 \mid mdF\mathrm{N}_{\mathbb{Q}(i)/\mathbb{Q}}(\alpha')\}|.$$

Note that there are at most $\ll_\epsilon b^\epsilon$ elements $\alpha' \in \mathbb{Z}[i]$ such that $\mathrm{N}_{\mathbb{Q}(i)/\mathbb{Q}}(\alpha') = b$. This gives

$$A_\square(x;\rho,u_i) \ll_\epsilon x^{\frac{1}{2}+\epsilon} \sum_{\substack{b \ll x^{\frac{1}{2}};\\ p\mid b \implies p^2 \mid mdFb}} 1,$$

where b runs over the positive rational integers. We assume that $m \leq x$ because otherwise $A(x)$ is the empty sum. This shows that $md \ll x^2$ and we conclude that

$$A_\square(x;\rho,u_i) \ll_\epsilon x^{\frac{3}{4}+\epsilon}.$$

Let $A_0(x;\rho,u_i)$ be the contribution to $A(x;\rho,u_i)$ of the terms $\alpha = a+\beta$ not satisfying (4.4). Then we can split $A(x;\rho,u_i)$ as

$$A(x;\rho,u_i) = A_\square(x;\rho,u_i) + A_0(x;\rho,u_i).$$

To estimate $A_0(x;\rho,u_i)$ we can try to use our bound (4.3) for every relevant β, but for this we need g_0 to be small. Hence we make the further partition

$$A_0(x;\rho,u_i) = A_1(x;\rho,u_i) + A_2(x;\rho,u_i),$$

where β satisfies the additional constraint

$$\begin{aligned} g_0 &\le Z \text{ in the sum } A_1(x;\rho,u_i), \\ g_0 &> Z \text{ in the sum } A_2(x;\rho,u_i). \end{aligned}$$

Here Z is at our disposal, and we choose it later. We estimate $A_1(x;\rho,u_i)$ as in [10] by using (4.3) and summing over $\beta = b_1\zeta_8 + b_2\zeta_8^2 + b_3\zeta_8^3 \in \mathbb{M}$ satisfying $b_i \ll x^{\frac{1}{4}}$ for $1 \le i \le 3$ to obtain

$$A_1(x;\rho,u_i) \ll_\epsilon Zx^{1-\frac{1}{32}+\epsilon}.$$

To finish the proof of Proposition 3.7 it remains to estimate $A_2(x;\rho,u_i)$. Note that $g_0 \le \sqrt{g}$ and $g \le \mathrm{N}_{\mathbb{Q}(i)/\mathbb{Q}}(c) \le \mathrm{N}_{\mathbb{Q}(i)/\mathbb{Q}}(\alpha') \ll x^{\frac{1}{2}}$. Hence, similarly as for $A_\square(x;\rho,u_i)$, with $b = \mathrm{N}_{\mathbb{Q}(i)/\mathbb{Q}}(\alpha')$, we have

$$A_2(x;\rho,u_i) \ll_\epsilon x^{\frac{1}{2}+\epsilon} \sum_{Z<g_0\ll x^{\frac{1}{4}}} \sum_{\substack{b\ll x^{\frac{1}{2}} \\ g_0^2 \mid b}} 1 \ll_\epsilon Z^{-1}x^{1+\epsilon}.$$

Picking $Z = x^{\frac{1}{64}}$ finishes the proof of Proposition 3.7.

5 Proof of Proposition 3.8

Let w and z be odd elements in $\mathcal{O}_M$. All quadratic and quartic residue symbols that follow are over M. By (2.2), we have

$$[wz] = \left(\frac{8\sigma(wz)\sigma\tau(wz)}{wz}\right)_4 = [w][z]\left(\frac{\sigma(w)}{z}\right)_4\left(\frac{\sigma\tau(w)}{z}\right)_4\left(\frac{\sigma(z)}{w}\right)_4\left(\frac{\sigma\tau(z)}{w}\right)_4.$$

By Lemma 3.1, we have, for some $\mu_1 \in \{\pm1,\pm i\}$ that depends only on the congruence classes of w and z modulo 16,

$$\left(\frac{\sigma(w)}{z}\right)_4\left(\frac{\sigma(z)}{w}\right)_4 = \mu_1\left(\frac{z}{\sigma(w)}\right)_4\left(\frac{\sigma(z)}{w}\right)_4 = \mu_1\left(\frac{z}{\sigma(w)}\right)_4\sigma\left(\frac{z}{\sigma(w)}\right)_4 = \mu_1\left(\frac{z}{\sigma(w)}\right)_2,$$

because $\sigma(i) = i$. Similarly, for some $\mu_2 \in \{\pm1,\pm i\}$ that depends only on the congruence classes of w and z modulo 16,

$$\left(\frac{\sigma\tau(w)}{z}\right)_4\left(\frac{\sigma\tau(z)}{w}\right)_4 = \mu_2\left(\frac{z}{\sigma\tau(w)}\right)_4\sigma\tau\left(\frac{z}{\sigma\tau(w)}\right)_4 = \mu_2,$$

because $\sigma\tau(i) = -i$. Hence we get, for $\mu_3 = \mu_1\mu_2$,

$$[wz] = \mu_3[w][z]\left(\frac{z}{\sigma(w)}\right)_2. \tag{5.1}$$

This twisted multiplicativity formula for the symbol $[\cdot]$ is what makes the estimate in Proposition 3.8 possible; it is analogous to [9, Lemma 20.1, p. 1021], [10, (3.8), p. 708], [17, Proposition 8, p. 1010], and [13, (4.1), p. 19].

Let χ be a Dirichlet character modulo 8, and let $\{a(\chi)_{\mathfrak{n}}\}_{\mathfrak{n}}$ be the sequence defined in (2.3). Let $\{\alpha_{\mathfrak{m}}\}_{\mathfrak{m}}$ and $\{\beta_{\mathfrak{n}}\}_{\mathfrak{n}}$ be any two bounded sequences of complex numbers. Since each ideal of $\mathcal{O}_M$ has 8 different generators in $\mathcal{D}$, we have

$$\sum_{\mathrm{N}(\mathfrak{m})\leq M}\sum_{\mathrm{N}(\mathfrak{n})\leq N}\alpha_{\mathfrak{m}}\beta_{\mathfrak{n}}a(\chi)_{\mathfrak{mn}} = \frac{1}{8^2}\sum_{w\in\mathcal{D}(M)}\sum_{z\in\mathcal{D}(N)}\alpha_w\beta_z([wz]_\chi + [\varepsilon wz]_\chi).$$

Here $\varepsilon = 1+\sqrt{2}$, $\alpha_w := \alpha_{(w)}$ and $\beta_z := \beta_{(z)}$. Note that for any odd element $\alpha \in \mathcal{O}_M$, we have $[\alpha]_\chi = \mu_4 \cdot [\alpha]$ for some $\mu_4 \in \{\pm 1, \pm i\}$ that depends only on the congruence class of α modulo 8 (and so also modulo 16). Also note that (5.1) implies that $[\varepsilon wz] = \mu_5[wz]$ for some $\mu_5 \in \{\pm 1, \pm i\}$ that depends only on the congruence class of wz modulo 16. Hence, by restricting w and z to congruence classes modulo 16, we may break up the sum above into $2\cdot 16^2$ sums of the shape

$$\mu_6 \sum_{\substack{w\in\mathcal{D}(M)\\ w\equiv\omega \bmod 16}}\sum_{\substack{z\in\mathcal{D}(N)\\ z\equiv\zeta \bmod 16}}\alpha_w\beta_z[wz],$$

where $\mu_6 \in \{\pm 1, \pm i\}$ depends only on the congruence classes ω and ζ modulo 16. Again by (5.1), we can replace α_w and β_z by $\alpha_w[w]$ and $\beta_z[z]$ to arrive at the sum

$$\mu_7 \sum_{\substack{w\in\mathcal{D}(M)\\ w\equiv\omega \bmod 16}}\sum_{\substack{z\in\mathcal{D}(N)\\ z\equiv\zeta \bmod 16}}\alpha_w\beta_z\left(\frac{z}{\sigma(w)}\right)_2,$$

where $\mu_7 \in \{\pm 1, \pm i\}$ depends only on ω and ζ. One can now apply Proposition 3.6 with $\gamma(w,z) = \left(\frac{z}{\sigma(w)}\right)_2$ (and $F = \mathbb{Q}(\zeta_8)$, $n = 4$, $\mathfrak{f} = (16)$). Indeed, property (P1) follows from Lemma 3.1, while property (P2) and the first part of property (P3) follow from basic properties of the quadratic residue symbol in $\mathbb{Q}(\zeta_8)$. For the second part of property (P3), we note that if $|\mathrm{N}(w)|$ is not a square in $\mathbb{Z}$, then the ideal generated by $\sigma(w)$ cannot be the square of an ideal in $\mathbb{Q}(\zeta_8)$, and the rest follows. This finishes the proof of Proposition 3.8.

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