DOTTY TRIANGLES

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Two years ago we became interested in Papic, Mulligan and Mitchelmore’s (2011) findings that developing Australian pre-school children’s pattern awareness had a significant impact on their mathematics. We decided to try something similar, with English three to five year olds and their teachers, using activities based on this research. We were particularly interested in the potential benefits for disadvantaged children, which were found by Papic (2013) and Kidd *et al.* (2014). We were also interested in pattern as a cross-curricular mathematical topic likely to appeal to early years teachers.

With a group of nursery and reception teachers we developed a pattern intervention based on Papic *et al.* (2011). In order to evaluate the effects of the intervention, we used several pre- and post-assessment tasks derived from Papic *et al.* and Mulligan, Mitchelmore and Stephanou (2015). One of these tasks involved showing children a 6-dot triangle (see Figure 1) and asking them to copy it. In this article we discuss the children’s responses to this task, which we analysed using two different frameworks. This was because we felt that the initial framework did not discriminate sufficiently between responses and therefore did not recognise aspects of children’s thinking that we thought were important. While there is much we might discuss about the socio-cultural aspects involved, here we consider their responses from a cognitive viewpoint.

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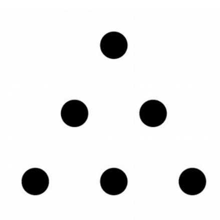

Figure 1. 6-dot triangle.

Patterns and structures

Mulligan *et al.* (2015) assert that an awareness of pattern and structure is fundamental to developing mathematical concepts. They have found that children in the first years of school tend to demonstrate consistent degrees of awareness across a range of mathematical structures, such as sequences, shape and alignment. They have established criteria for standardised levels of 'Awareness of Mathematical Pattern and Structure' (AMPS) which indicate "the ability of a child to recognise and use pattern and structure across mathematics concepts" (p. 10). They define these terms as follows:

> A **pattern** is some regularity observed in a mathematical context and the description of this regularity is its structure. A **structure** is a defining rule and/or relationships between parts of the pattern that lead to generalisations. (p. 1)

Mulligan *et al.* (2015) propose five levels of AMPS: 'Prestructural', 'Emergent', 'Partial Structural', 'Structural' and 'Advanced Structural', which they assessed through a range of tasks requiring children to reproduce or extend patterns and structures of different kinds. At the 'Prestructural' level, children might show awareness of limited and disjointed aspects of the pattern, whereas children's responses at the 'Emergent' level identify some pattern features but not their organisation or underlying structure. 'Partial Structural' responses show most features of the pattern but have inaccurate or incomplete organisation, compared with those at a 'Structural' level, which are completely accurate. Children who give 'Advanced Structural' responses can extend or generalise the pattern structure. We saw the value of these levels for teachers as focusing attention on children's trajectories towards identifying relationships and then generalising these. We therefore selected some assessments, of which 'dotty triangles' was one, and analysed children's responses using the criteria for the AMPS levels.

The other framework which we later used to analyse the children's responses was the Structure of Observed Learning Outcome (SOLO) taxonomy (Biggs & Collis, 1982), which can be applied to any area of learning, not just mathematics. Mulligan and Mitchelmore (2009, p. 38) originally drew on the SOLO taxonomy to develop the AMPS framework. They describe how the application of this to young children's concepts of multiplication and division drew their attention to "ikonic features and structural characteristics of mathematical responses". However they found it was not suitable for analysing specific areas of mathematics. The SOLO taxonomy proposes to be a general measure of learning quality, analysing a child's response to an individual task. The criteria focus on the number of elements that children identify in a task. Like AMPS, the grading system has five levels, 'Prestructural', 'Unistructural', 'Multistructural', 'Relational' and 'Extended Abstract'. Similarly to AMPS, it distinguishes responses which use the interrelations between the elements to make a coherent whole, classifying these as 'Relational', with 'Extended Abstract' responses involving generalisation to a novel context. Where it differs from AMPS is that responses identifying one feature of the pattern are classified as 'Unistructural' rather than 'Prestructural', which is applied only to responses with no relevant features. Children's responses are 'Multistructural' when they identify several elements but do not connect them as a whole.

The SOLO taxonomy was extended by Fujita and Yamamoto (2011), who classified children's generalised or 'Extended abstract' responses as 'Unistructural-2', 'Multistructural-2' and 'Relational-2', according to the number of elements and sophistication involved. (They renamed responses at the previous levels as 'Unistructural-1', 'Multistructural-1' and 'Relational-1'.) Their levelling system might therefore be developed into a spiral of increasing complexity. We found that both these versions of the SOLO taxonomy were useful in providing finer distinctions between young children's responses as well as more positive labels for these.

Dotty patterns

For two years we worked with teachers of nursery and reception classes from six London schools with children from diverse social and ethnic backgrounds. (In nursery classes in England children are usually three to four years old and in reception they are four to five.) We asked the teachers to each identify a group of focus children, including number 'experts' and number 'novices'. These terms referred to children who were either more or less confident in aspects of number such as counting and the teachers decided for themselves which children to identify. As we were particularly interested in the potential benefits of the project for disadvantaged children, we encouraged teachers to identify more number novices than experts in the focus groups, as well as children with special needs, including several from a class of children assessed with autistic spectrum disorder. Their ages during the project ranged from three and a half for the nursery children to five and a half for children at the end of reception. As part of pre- and post-assessments, in autumn and summer, we asked children to copy the 6-dot triangle, collecting 69 drawings in all. This was one of several tasks derived from Papic *et al.* (2011). Others included subitising dot arrays as well as copying and continuing repeating patterns. While we knew that assigning levels on the basis of a few 'single shot' tasks would not take account of young children's fleeting preoccupations or the social context, Mulligan *et al.*'s (2015) protocols, criteria and exemplars provided a standardised approach to generating and analysing responses as well as an attempt at measuring progress. However, due possibly to the characteristics of our sample in terms of age, attainment and special needs, we found that AMPS classified most of the focus children as 'Prestructural' or 'Emergent', despite wide differences in responses. We realised that we could analyse the data according to the number of pattern elements the children represented, using the SOLO taxonomy, which might provide greater discrimination between responses. The results of these two analyses are presented below.

We also assessed the children's number skills and understanding, in order to monitor correlations in number and pattern progress. While this is not our focus here, the children's responses in the number assessment tasks do give clues about their number expertise. The tasks included reciting numbers, giving a teddy bear a number of jewels and showing him the correct numeral, as well as working out the number of bears hidden in a tent when some had been added or removed. These tasks showed us something about the children's numerical knowledge and their ability to mentally solve problems with small numbers, and generally confirmed the teachers' identification of them as number experts or novices. A flavour of the children's responses are given in this article, alongside their pattern drawings.

Comparison of two frameworks

Although the children were copying the image in front of them, their responses were very varied, with drawings ranging mainly from lots of dots (see Figure 2) to accurate 6 dot triangles. We found some children's responses very surprising, considering that they had the 6 dot triangle in front of them. One child drew 6 lines (see Figure 3), three children just wrote numerals, such as 123456 or 6 (see Figure 4), and three children drew multiple triangles (see Figure 5). We were also surprised to find how similar the range of 'copied' responses were to those from Mulligan *et al.* (2015), where children were asked to draw the pattern from memory. Following the AMPS criteria, which involved considering the degree to which children drew the triangle in rows, we assigned levels ranging from 'Prestructural' to 'Structural'.

Insert Thouless Figure 2 about here

Figure 2. Maya (all names are pseudonyms) was four years old, an older number novice at the end of the nursery year who could recite numbers to 10, and give two but not three jewels to the bear. She drew lots of dots.

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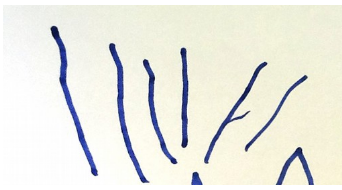

Figure 3. Enzo, a three and a half year old number expert at the beginning of nursery, recited to 15 and counted out 9

jewels. He seemed to focus on number by drawing 6 lines, ignoring both dots and the triangle shape.

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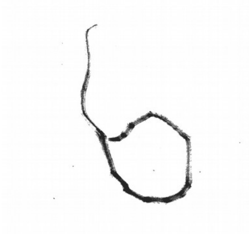

Figure 4. Jarred, a young four year old at the end of nursery, drew a wobbly six. He was a number expert who recited numbers to 30 and counted out 13 jewels. He jokingly showed the numeral 9 to the teddy and voiced the teddy's response as 'seven', thereby proving his superior knowledge of numerals.

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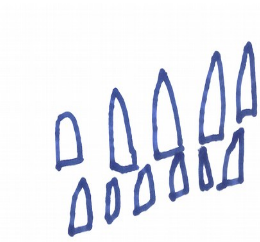

Figure 5. Latoya, an older 5 year old at the beginning of the reception year, was a number expert who could count beyond 100 and mentally solve simple addition and subtraction problems. She chose to draw two rows of six triangles.

However, we were not happy about classifying the responses which involved only lines or numerals as 'Prestructural'. It seemed the children were responding to one aspect of the pattern structure. This reminded us of the SOLO taxonomy, which classifies task responses by the number of elements identified. According to this, responses which identified one relevant attribute would be ‘Unistructural’. We thought again about the 22 children who drew lots of dots, as in Figure 2. They had recognised the dottiness of the image (an element of pattern structure we had not previously considered) so their responses would also be considered ‘Unistructural’.

We then considered other responses. Three children drew a line of six dots (see Figure 6). According to AMPS criteria, these responses were also ‘Prestructural’, as there was no triangularity. However, the children had identified at least three relevant elements of the pattern, in number, dots and rows, some with equal spacing. Latoya, whose drawing of many triangles (Figure 5) seemed initially to focus just on shape, on closer inspection showed two rows of six, so might also be seen as recognising number and ‘rowness’ as well as triangularity. The SOLO taxonomy would give more credit for these, classifying them as ‘Multistructural’, allowing us to further discriminate the children’s responses. When we reconsidered the 6-dot triangle in these terms, we identified five pattern elements which children had variously reproduced or not. These were the dot shape, the numerosity of six, the triangular shape, the rows and equal spacing. According to Sarama and Clements (2009) young children tend to perceive shapes as independent elements, which they only later integrate into a whole. This developmental trajectory in processing complex images would be recognised in the SOLO taxonomy.

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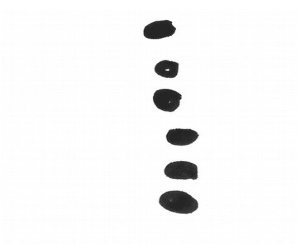

Figure 6. Clementine, a younger 5 year old child at the end of the reception year, was a number expert who could count past 120 and mentally solve simple subtraction problems. She seemed to focus on the number of dots and ignore the shape.

We then reconsidered the most common kind of response, where children drew triangles with an outline of multiple dots (see Figure 7). In the SOLO taxonomy this would also be classified as 'Multistructural', because the children had identified more than one attribute (dottiness and triangularity). According to the AMPS criteria these were classified as 'Emergent' because they had a triangular shape but the dots were not drawn in rows.

Insert Thouless Figure 7 about here

Figure 7. Sophie, a young five year old at the end of reception, was a number expert who could mentally solve simple addition and subtraction problems. After counting to 110, she said the next number was 'a thousand and one'. Her dotty triangle is an example of the most common response.

We questioned the privileging of the 'rowness' of the triangle in the AMPs criteria, whereby the higher level, 'Partial structural', was assigned to drawings where children had constructed a triangle by rows, even with the wrong numbers of dots (see Figure 8). In the SOLO taxonomy Figure 8 would still be considered 'Multistructural', because the child had recognized more than one element (dottiness, triangularity, and rows) but had not yet integrated all the elements into one coherent shape. However, since the 6-dot image can validly be seen as an 'empty' triangle, and such young children were unlikely to be familiar with triangular number images, we wondered why they would notice rows at all. According to Sarama and Clements (2009) young children may also look for a 'contour' in an image, which would explain why most children drew a triangular outline of dots.

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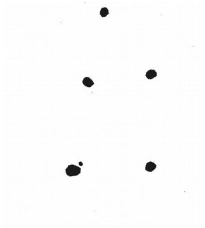

Figure 8. Eliza, a very young four year old at the end of reception, was a number novice who recited numbers to 29, but did not recognise any numerals. She drew a triangle of five dots well spaced in rows.

In the second year of the project, we prompted children who correctly drew the triangle in rows to extend it, by pointing below the bottom line and asking what would come next, as in Mulligan *et al.*'s (2015) assessment. This question was not asked of pre-school children In Papic *et al.*'s (2011) assessment, but we decided to include it to identify children with a higher AMPS level. One child drew a line of appropriate length below the triangle (see Figure 9) and six children drew a fourth row with multiple dots (see Figure 10).

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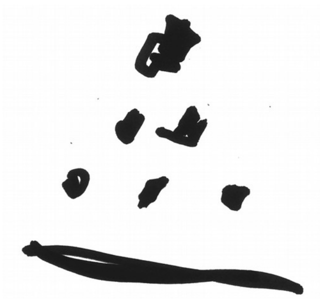

Figure 9. Aisha, a young five year old number expert assessed with autistic spectrum disorder, at the end of a second nursery year recited numbers to 100 by 10s and then counted by 100s to 1000. She drew the pattern correctly and extended it by drawing a line of appropriate length.

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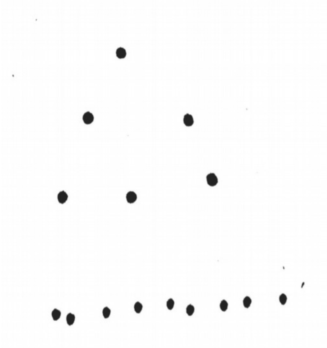

Figure 10. *Fatima, an older five year old at the end of reception, was a number novice who recited numbers only to 12, but seemed to have benefitted from the class's experience in looking at patterns, by copying the pattern and attempting a dotty extension.*

These responses were assigned the AMPS 'Structural' level, as the original triangle was correct but the extension was not. However, they could be classified further using the SOLO taxonomy, as shown by Fujita and Yamamoto (2011). Extensions involving one element, such as the line extension (see Figure 9) would be at the 'Unistructural-2' level because the child maintained the 'rowness' of the drawing in her extension but did not include any other elements. Those drawings where extensions included dottiness, rowness and an approximate triangularity, as in Figure 10, would be at the 'Multistructural-2' level because extensions showed several elements.

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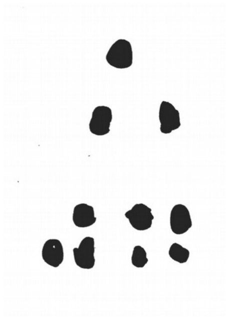

Figure 11. *Harry, a younger five year old at the end of a reception, was a number expert; he counted to 110, then said the next number was 'a thousand'. He correctly extended the pattern by drawing four dots.*

The six responses which extended the triangle with a row of four dots (see Figure 11) would be at the 'Advanced Structural ' AMPS level, which is equivalent to the 'Relational-2' SOLO level, as they used the interrelations between elements to extend the whole. Both approaches thereby concurred in recognising a generalising process, but the SOLO taxonomy provided a more fine-grained analysis of children's responses overall. It also resulted in only three children's responses being assigned a Pre-structural level. These were scribbles or images which contained no recognisable elements of the dotty triangle (see Figure 12). Therefore most responses were classified more positively, as identifying some pattern elements and involving some degree of mathematical thinking.

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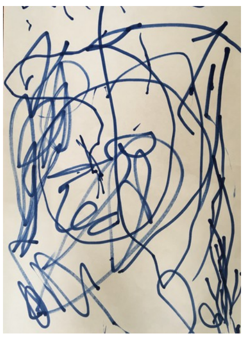

Figure 12. *A 'Prestructural' drawing. Ezra was a three year old number novice at the beginning of his nursery year who*

counted '1, 22, 100'. He could give two jewels but not three. His response contained no recognisable elements of the task.

While we know that what children draw is not necessarily what they 'see', these drawings seemed to indicate what children were paying attention to. Analysing these responses made us aware that simple images could provide complex processing challenges for young children.

A dynamic viewpoint

And then came Rosie, who was four and a half years old at the end of nursery, could recite numbers to 39 and counted out 5 jewels for the teddy. She had last seen this type of pattern nine months before, when she had responded by drawing a triangle with a dotty outline. Rosie started by drawing a row of six dots, then another one above it, followed by a single dot above that (see Figure 13). She then said, "What it looks like is three at the bottom, then two, then one" and drew the triangular 6 dot pattern (see Figure 14). She followed this immediately with a 10 dot triangle, drawing 4 dots on the bottom row, then 3, then 2, then 1. When asked, "What were you thinking?" she replied, "This" and drew lines across the rows of 3, 2, and 1 dots, thereby obscuring the structure (see Figure 15). My direct questioning failed to prompt her to explain her thinking further. The end result was a group of three drawings (see Figure 16) that would have been impossible to interpret had we not been observing her as she completed it.

Insert Thouless Figure 13 about here

Figure 13. Rosie's first drawing, 2 rows of 6 dots, plus one.

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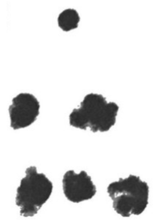

Figure 14. Rosie's second drawing, the correctly copied triangle.

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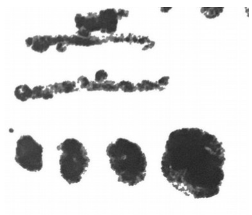

Figure 15. Rosie's third drawing, the 10 dot triangle, with lines drawn through the top three rows afterwards.

Insert Thouless Figure 16 about here

Figure 16. Rosie's three drawings.

Rosie appeared to have gone through several levels of pattern awareness in a couple of minutes. Her first version (Figure 13) recognised several independent elements: number, dots, equal spacing and possibly rows (and perhaps she had added the top dot to give a somewhat triangular shape). She then described the structure and constructed the triangle by rows (Figure 14), before generalising the structure to the next triangular number image (Figure 15). She had rapidly progressed from ‘Multistructural-1’ to ‘Relational-2’, according to Fujita and Yamamoto’s levels. This made us realise that the task of ‘copying’ could involve a dynamic analytic process, rather than the recording of a static mental image.

As Rosie was exceptional in her confidence to draw several versions of the image and in articulating her thinking through speech and gesture, we wondered if we had overlooked evidence of other children going through a similar process. We found that Sarah, at the end of the previous year, might also have demonstrated this. She had first copied the image by drawing an inverted V, then placed a dot on each corner of the triangle, followed by dots at the mid-point of each side, thereby constructing an ‘empty triangle’ image (see Figure 17).

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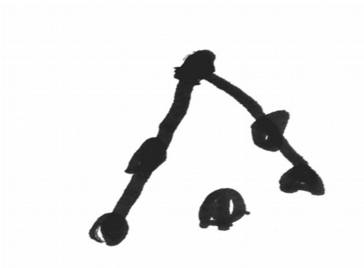

Figure 17. Sarah was an older four year old number novice at the end of nursery, who recited numbers to 14 and counted out 5 jewels. She first constructed a triangle shape then added midpoints on each side.

It seemed that Sarah had first recognised the triangle’s contour, then constructed it with dots, positioning them in stages to ensure numerosity and equal spacing. Sarah had progressed through three levels in both systems, from ‘Emergent’ to ‘Structural’ (AMPS) and ‘Unistructural-1’ to ‘Relational-1’ (SOLO). Both children seemed to have gone through a process of decomposing and then recomposing the pattern structure in different ways. We realised that the levels could be used to follow a child’s train of thought while analysing a mathematical structure. We had not recognised this previously in Sarah’s response, because the stages were overlaid, whereas Rosie had helpfully provided three separate drawings and provided a commentary on her thinking as she drew. We are not sure how many other children might have gone through similar processes. Perhaps, if we had observed other children more closely while they were drawing, we might have identified similar thinking processes.

Discussion

This experience provided us with insights related to the mathematical demands of a copying task, the strengths and weaknesses of the two frameworks we used, the children's thinking processes, and the dangers of 'snapshot' assessements.

Copying as mathematically demanding task

We realised that copying an image or a model should not be dismissed as a trivial closed task. As Papic *et al.* (2011) pointed out, it requires children to pay close attention, to compare their copy to the original and to notice what is the same and what is different. Papic *et al.* suggest that teachers' support for this process, in focusing children's attention and discussing children's versions in comparison with the original pattern, might teach them to analyse patterns and contribute to their later raised mathematical achievement. Therefore copying can be an engaging and mathematically demanding task; however, the level of challenge may depend on the number of elements involved. Analysing these responses made us aware that simple images can provide complex processing challenges for young children, which we as teachers might not be sufficiently aware of.

Two frameworks

Using the criteria from two frameworks was useful in highlighting different but complementary aspects of the children's responses. The AMPS framework revealed the variation in children's responses to pattern structure, whereas the SOLO taxonomy highlighted children's identification of different pattern elements. This more detailed analysis allowed us to acknowledge children's thinking more positively and to recognise the complex demands of an apparently simple task for young children. Teachers to whom we have shown these two analyses appreciate the SOLO taxonomy's positive recognition of children's thinking even when focusing on just one or two elements and the finer discrimination between responses of very young children. It also seems easier to count the elements involved, rather than trying to identify a more elusive 'structure', but with the same benefits of analysing children's difficulties or ease in processing relationships between elements. The use of two systems which assign different levels to the same child's work reminds us of the need to be wary of any levelling system. Criticisms may be raised against both frameworks. For instance the AMPS identification of 'structure' in terms of rows in the 6 dot triangle raises questions about level validity. (However, the number of tasks required to allocate a 'best fit' level of AMPS might mitigate this overall.) The SOLO taxonomy might be criticised in assigning one 'multistructural' level to responses of varying complexity. Such criticisms raise the possibility of further perspectives, such as considering responses in terms of dispositions towards number or shape recognition.

Children's thinking processes

Drawing the pattern allowed children to express what they had noticed with individuality and detail. For Rosie and Sarah, drawing enabled them to record successive stages of thought, suggesting that some children's responses might represent only an initial impression or first stage in a dynamic process of image analysis. It may be that children notice one or two features initially, then notice more and eventually work out how the elements are related in the complete pattern. However, it seems likely that this process will take varying amounts of time for different children. Given more time and invitations to make subsequent attempts on following days, other children's responses might have shown further analysis and thinking. Rosie also volunteered a commentary while drawing, on which she expanded with prompting, suggesting that some children might have expressed themselves further in words and gestures, if encouraged to do so. However, few four year olds are as articulate or confident as Rosie. While it is important to pay attention to young children's words and actions, questioning them directly about their thinking is likely to be unproductive, as it proved with Rosie. Perhaps if we had asked children to copy the pattern with counters, some might have explained their thinking more readily. Our future research will include video, which would also enable us to show recordings to the children to gain their reflections on their thinking processes and thereby gain further insights. The time taken by different children to process images seems especially relevant to the current emphasis on subitising 'quick images'. These may be confusing for children, and also for teachers, if children do not focus on the expected elements.

The dangers of single shot assessment and levels

We know that individual young children's responses are highly variable according to context, and of course what they draw is not a replication of what they 'see'. Rosie's drawing, which did not fit in any box, underlined this. A young child's response in a moment in time is not evidence of a static state of understanding. Even though we used a composite score from a varied range of tasks (and Mulligan *et al.*'s (2015) assessment levels are standardised across a wider task battery than ours) assigning young children static levels of awareness or understanding is a dubious practice, especially if these are communicated and used as a basis for action, with all the dangers of self-fulfilling prophecies. However, tasks which provide such a range of responses are useful for revealing the ways that different individuals think and provoke teaching strategies to build on these. Instead of using taxonomies to categorise children or their responses, their criteria may be used to follow a child's train of thought and to provide insights into the complex stages children go through in activities which involve processing images and analysing mathematical structures. A more useful analytical tool might attempt to map dynamic thinking processes by identifying task elements in finer detail, while also considering alternative interpretations of regularity and structure.

So, although young children's drawings may not fit into the tidy boxes of taxonomies, it seems that the process of trying to fit them in helped in understanding children's thinking which was outside the boxes. This attempt at classifying children's responses underlined the benefits of using more than one theoretical perspective to provide a more positive and complex analysis. Finally, Rosie reminded us not to underestimate the capacity of four year olds to analyse and think mathematically.

Acknowledgement

We are indebted to Jo Mulligan for generously sharing her work with us and to the committed and creative teachers and children who have worked with the Pattern Project.

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