

	[3], [7], [18]	[13]	This paper	
Theory	Graphical Linear Algebra		Graphical Affine Algebra	
Syntax	$\text{Circ}_{\mathbb{K}}$	$\text{Circ}_{\mathbb{N}}$	$\text{ACirc}_{\mathbb{K}}$, Sec. II	$\text{ACirc}_{\mathbb{N}}$, Sec. II
Semantics	$\text{LinRel}_{\mathbb{K}}$	AddRel	$\text{AffRel}_{\mathbb{K}}$, Sec. III	$\text{AffRel}_{\mathbb{N}}$, Sec. III
Axioms	$\text{IH}_{\mathbb{K}}$	RC	$\text{AIH}_{\mathbb{K}}$, Sec. IV	ARC, Sec. IV
Embeds	Signal Flow Graphs	Petri Nets	Electrical circuits Sec. VI	Stateless connectors Sec. V

Fig. 1. Overview on GLA and GAA, in the notation of this paper.

in Section I, for different \mathbb{R} s, $\text{Circ}_{\mathbb{R}}$ is able to model linear dynamical systems [7], [19], phase-free quantum processes [5], Petri nets [13], and more. The focus of this paper is exploring the expressivity and the equational theories supported by the extended language $\text{ACirc}_{\mathbb{R}}$.

Symbols of $\text{ACirc}_{\mathbb{R}}$ are rendered pictorially, as we will treat them formally as *string diagrams* [20] in due course (Section II-B). This also explains the use of two binary operations, *sequential* ($c; d$) and *parallel* ($c \oplus d$) composition: they are those of monoidal categories.

The diagrammatic syntax is variable-free, but requires a simple sorting discipline. A sort is a pair (k, l) , with $k, l \in \mathbb{N}$; intuitively, k and l are the number of dangling wires on each side of a $\text{ACirc}_{\mathbb{R}}$ -diagram. We shall only consider terms that are sortable, according to the following rules.

$$\begin{array}{l}
\bullet \text{---} : (1, 0) \quad \text{---} \bullet : (1, 2) \quad \text{---} \cup \text{---} : (2, 1) \quad \text{---} \circ \text{---} : (0, 1) \\
\bullet \text{---} : (0, 1) \quad \text{---} \cup \text{---} : (2, 1) \quad \text{---} \vdash : (0, 1) \quad \text{---} \boxed{k} \text{---} : (1, 1) \\
\text{---} : (1, 1) \quad \text{---} \boxed{\cdot} : (0, 0) \quad \text{---} \times : (2, 2) \\
\frac{c : (k_1, k_2) \quad d : (k_2, k_3)}{c; d : (k_1, k_3)} \quad \frac{c : (k_1, l_1) \quad d : (k_2, l_2)}{c \oplus d : (k_1 + k_2, l_1 + l_2)}
\end{array}$$

An easy induction confirms uniqueness of sorting: if $c : (k, l)$ and $c : (k', l')$, then $k = k'$ and $l = l'$.

The semantics $\langle \cdot \rangle_{\mathbb{R}}$ of $\text{ACirc}_{\mathbb{R}}$ is defined inductively by the clauses in Figure 2, where we write \bullet for the unique \mathbb{R} -vector of length zero.

Intuitively, $\text{---} \cup \text{---}$ *duplicates*, $\text{---} \bullet$ *discards* and $\text{---} \cup \text{---}$ *sums* values, whereas $\text{---} \circ \text{---}$ produces *zero* values, $\text{---} \vdash$ produces *one* value, and $\text{---} \boxed{k} \text{---}$ multiplies by k a given value. The mirror images $\text{---} \cup \text{---}$, $\text{---} \bullet$ have behaviour defined symmetrically with respect to $\text{---} \cup \text{---}$ and $\text{---} \bullet$. Finally, behaviours combine sequentially, where values synchronise along the common boundary of diagrams, or in parallel, where values are simply stacked. Note that $\langle \cdot \rangle_{\mathbb{R}}$ is defined in terms of relations rather than functions; thus it is neutral with respect to flow directionality. For further discussion on this point, see [8].

B. From Terms to String Diagrams

Our goal is to characterise semantic equivalence in $\text{ACirc}_{\mathbb{R}}$ equationally, for different choices of \mathbb{R} . In each case, these

equations contain the laws of symmetric monoidal categories (SMCs). Thus it makes sense to move from raw terms, as in (7), to string diagrams: this is the remit of the subsection.

First, we enhance our graphical notation by depicting $c : (k, l)$ as $\overset{k}{\text{---}} \boxed{c} \text{---}^l$, $c; d$ as $\overset{k_1}{\text{---}} \boxed{c} \text{---}^k \overset{k_2}{\text{---}} \boxed{d} \text{---}^{k_3}$ and $c \oplus d$ as $\overset{k_1}{\text{---}} \boxed{c} \text{---}^{l_1} \overset{k_2}{\text{---}} \boxed{d} \text{---}^{l_2}$,

where the labelled wire ---^k stands for a stack of k wires. We often omit wire labels when it does not lead to confusion.

The laws of SMCs are given in Figure (3) in the graphical notation. They yields a structural equivalence on $\text{ACirc}_{\mathbb{R}}$ -terms, which is preserved by the semantics. More precisely, writing \equiv for the smallest congruence over $\text{ACirc}_{\mathbb{R}}$ -terms generated by the equations in Figure 3, we have that $c \equiv d$ implies $\langle c \rangle_{\mathbb{R}} = \langle d \rangle_{\mathbb{R}}$. Because of this observation, henceforth we shall consider the terms of $\text{ACirc}_{\mathbb{R}}$ as arrows of an SMC, which by a mild abuse of notation we will also denote $\text{ACirc}_{\mathbb{R}}$. In fact, $\text{ACirc}_{\mathbb{R}}$ is a specific kind of SMC, known as a *prop*.

Definition 1. A *prop* is a symmetric strict monoidal category with objects the natural numbers, with $k \oplus l$ defined by the addition $k + l$. A *prop morphism* $\mathcal{F} : \mathbb{C} \rightarrow \mathbb{D}$ is a symmetric monoidal functor from \mathbb{C} to \mathbb{D} that is identity on objects.

$\text{ACirc}_{\mathbb{R}}$ is defined as a prop with arrows $k \rightarrow l$ sorted terms $c : (k, l)$ of the corresponding syntax modulo \equiv , with sequential composition $c; d$, monoidal product $c \oplus d$, and symmetries defined by the corresponding operations in (7) (c.f. [19, Definition 2.3] for the details of the free construction of a prop from a syntax).

For uniformity, we shall also model the semantic domains of our calculus as props. This is based on the definition below.

Definition 2 ($\text{Rel}_{\mathbb{R}}$). Given a semiring \mathbb{R} , let $\text{Rel}_{\mathbb{R}}$ be the prop with arrows $k \rightarrow l$ relations R from X^k to X^l , i.e. $R \subseteq X^k \times X^l$. Given $R : k_1 \rightarrow k_2$ and $S : k_2 \rightarrow k_3$, their composition $R; S : k_1 \rightarrow k_3$ is

$$\{(\mathbf{x}, \mathbf{z}) : \mathbf{x} \in X^{k_1}, \mathbf{z} \in X^{k_3} \text{ and there exists } \mathbf{z} \in X^{k_2} \text{ such that } (\mathbf{x}, \mathbf{y}) \in R \text{ and } (\mathbf{y}, \mathbf{z}) \in S\}. \quad (7)$$

Given $R : k_1 \rightarrow l_1$ and $S : k_2 \rightarrow l_2$ their monoidal product is obtained by taking their cartesian product, i.e. $R \oplus S : k_1 + k_2 \rightarrow l_1 + l_2$ is the relation

$$\{((\mathbf{x}_1, \mathbf{x}_2), (\mathbf{y}_1, \mathbf{y}_2)) : \mathbf{x}_1 \in X^{k_1}, \mathbf{x}_2 \in X^{k_2}, \mathbf{y}_1 \in X^{l_1}, \mathbf{y}_2 \in X^{l_2} \text{ such that } (\mathbf{x}_1, \mathbf{y}_1) \in R \text{ and } (\mathbf{x}_2, \mathbf{y}_2) \in S\}. \quad (8)$$

Identities and symmetries are defined in the obvious way.

Note that our inductive definition of Figure 2 yields a morphism of props $\langle \cdot \rangle_{\mathbb{R}} : \text{ACirc}_{\mathbb{R}} \rightarrow \text{Rel}_{\mathbb{R}}$. Functoriality here means that the interpretation is *compositional* with respect to the operations $;$ and \oplus .

C. Compact closed structure

String diagrams $\text{---} \cup \bullet$, called *cup* and $\bullet \text{---} \cup$, *cap*, play a special role. Their behaviour, according to the semantics $\langle \cdot \rangle_{\mathbb{R}}$,

$$\begin{aligned}
\langle \dashv \bullet \rangle_{\mathbb{R}} &:= \{(x, \bullet) \mid x \in \mathbb{R}\} & \langle \dashv \frown \rangle_{\mathbb{R}} &:= \{(x, (\frac{x}{\bullet})) \mid x \in \mathbb{R}\} & \langle \bullet \dashv \rangle_{\mathbb{R}} &:= \{(\bullet, x) \mid x \in \mathbb{R}\} & \langle \frown \bullet \rangle_{\mathbb{R}} &:= \{((\frac{x}{\bullet}), x) \mid x \in \mathbb{R}\} \\
\langle \dashv \circ \rangle_{\mathbb{R}} &:= \{((\frac{x}{\bullet}), x + y) \mid x, y \in \mathbb{R}\} & \langle \circ \dashv \rangle_{\mathbb{R}} &:= \{(\bullet, 0)\} & \langle \dashv \overline{k} \rangle_{\mathbb{R}} &:= \{(x, k \cdot x) \mid x \in \mathbb{R}\} & \langle \dashv \vdash \rangle_{\mathbb{R}} &:= \{(\bullet, 1)\} \\
\langle \dashv \lrcorner \rangle_{\mathbb{R}} &:= \{(\bullet, \bullet)\} & \langle \dashv \rangle_{\mathbb{R}} &:= \{(x, x) \mid x \in \mathbb{R}\} & \langle \times \rangle_{\mathbb{R}} &:= \{((\frac{x}{y}), (\frac{y}{x})) \mid x, y \in \mathbb{R}\} \\
\langle c; d \rangle_{\mathbb{R}} &:= \{(\mathbf{a}, \mathbf{b}) \mid \exists \mathbf{w}. (\mathbf{a}, \mathbf{w}) \in \langle c \rangle_{\mathbb{R}}, (\mathbf{w}, \mathbf{b}) \in \langle d \rangle_{\mathbb{R}}\} & \langle c \oplus d \rangle_{\mathbb{R}} &:= \{((\mathbf{a}_1, \mathbf{b}_1), (\mathbf{a}_2, \mathbf{b}_2)) \mid (\mathbf{a}_1, \mathbf{a}_2) \in \langle c \rangle_{\mathbb{R}}, (\mathbf{b}_1, \mathbf{b}_2) \in \langle d \rangle_{\mathbb{R}}\}
\end{aligned}$$

Fig. 2. Semantics of ACirc_R.

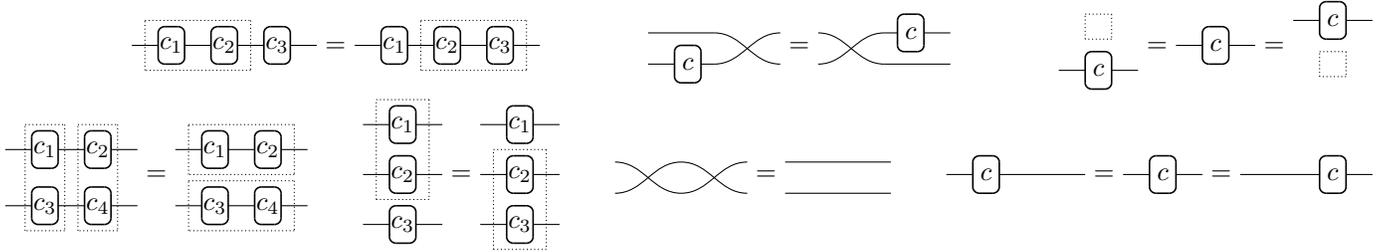


Fig. 3. Laws of Symmetric Monoidal Categories. Sort labels are omitted for readability.

intuitively forces the two ports on the left (respectively right) to carry the same resources, thus acting as left (right) feedback:

$$\langle \dashv \bullet \bullet \rangle_{\mathbb{R}} = \{((\frac{x}{\bullet}), \bullet) \mid x \in X\} \quad \langle \bullet \bullet \dashv \rangle_{\mathbb{R}} = \{(\bullet, (\frac{x}{\bullet})) \mid x \in X\}$$

Using these diagrams (along with \dashv and \times) as building blocks, is it possible to define for each $k \in \mathbb{N}$, diagrams $\dashv \overline{k} \bullet \bullet$: $k + k \rightarrow 0$ and $\bullet \bullet \dashv \overline{k}$: $0 \rightarrow k + k$ with semantics

$$\langle \dashv \overline{k} \bullet \bullet \rangle_{\mathbb{R}} = \{((\frac{\mathbf{a}}{\bullet}), \bullet) \mid \mathbf{a} \in X^k\} \quad \langle \bullet \bullet \dashv \overline{k} \rangle_{\mathbb{R}} = \{(\bullet, (\frac{\mathbf{a}}{\bullet})) \mid \mathbf{a} \in X^k\}$$

See e.g. [19, §5.1] for details. These arrows give rise to a (self-dual) *compact closed structure* [21], that is, they satisfy

$$\begin{array}{c} \bullet \bullet \dashv \bullet \bullet \\ \bullet \bullet \dashv \bullet \bullet \end{array} = \text{---} = \begin{array}{c} \bullet \bullet \dashv \bullet \bullet \\ \bullet \bullet \dashv \bullet \bullet \end{array}$$

As for identities and symmetries, we will often omit the label k on cups and caps for readability. The graphical language of compact closed props allows us to bend wires at will, treating them as unoriented edges between the connection points of individual components. It also allows the introduction of “right-to-left” versions of each generator in our diagrammatic syntax. We explicitly introduce these counterparts as syntactic sugar, since they will be used in subsequent sections.

$$\begin{array}{ccc}
\begin{array}{c} \bullet \bullet \dashv \bullet \bullet \\ \bullet \bullet \dashv \bullet \bullet \end{array} & := & \begin{array}{c} \bullet \bullet \dashv \bullet \bullet \\ \bullet \bullet \dashv \bullet \bullet \end{array} & \begin{array}{c} \bullet \bullet \dashv \bullet \bullet \\ \bullet \bullet \dashv \bullet \bullet \end{array} & := & \begin{array}{c} \bullet \bullet \dashv \bullet \bullet \\ \bullet \bullet \dashv \bullet \bullet \end{array} \\
\begin{array}{c} \bullet \bullet \dashv \bullet \bullet \\ \bullet \bullet \dashv \bullet \bullet \end{array} & := & \begin{array}{c} \bullet \bullet \dashv \bullet \bullet \\ \bullet \bullet \dashv \bullet \bullet \end{array} & \begin{array}{c} \bullet \bullet \dashv \bullet \bullet \\ \bullet \bullet \dashv \bullet \bullet \end{array} & := & \begin{array}{c} \bullet \bullet \dashv \bullet \bullet \\ \bullet \bullet \dashv \bullet \bullet \end{array}
\end{array} \quad (9)$$

The associated behaviour, in each case, is the opposite relation: for example, $\langle \dashv \circ \rangle_{\mathbb{R}} = \{(n + m, (\frac{n}{m})) \mid n, m \in \mathbb{N}\}$.

D. Examples

We conclude this section by showcasing the expressivity of ACirc_R. The first two examples are captured by the fragment Circ_R, whereas the last two are genuinely affine (they make use of $\dashv \vdash$), and anticipate the case studies of Section V and VI.

a) *Lowpass filter*: Take \mathbb{R} to be $\mathbb{R}(x)$. The idea is that elements of $\mathbb{R}(x)$ express Laplace-transformed signals in the frequency domain. In other words, multiplication by x corresponds to differentiation w.r.t. the time variable. The Circ _{$\mathbb{R}(x)$} -diagram on the left below denotes a *low-pass filter*: a filter that attenuates signals with frequencies higher than a certain cutoff ($\leq \frac{1}{2\pi\tau}$) and lets through those that are lower. The semantics of the diagram, on the right below, allows us to retrieve the transfer function (the ratio of the left and right boundary signals) of the filter.

$$\begin{array}{c} \bullet \bullet \dashv \bullet \bullet \\ \bullet \bullet \dashv \bullet \bullet \end{array} \begin{array}{c} \tau \\ \tau \\ x \end{array} = \left\{ (\phi, \phi') \mid \exists \phi_1, \phi_2, \phi_2 = \phi' \text{ and } \phi - \phi_2 \tau x = \phi_2 \right\} \\
\left\langle \begin{array}{c} \bullet \bullet \dashv \bullet \bullet \\ \bullet \bullet \dashv \bullet \bullet \end{array} \right\rangle_{\mathbb{R}(x)} = \left\{ (\phi, \phi') \mid \phi' = \frac{1}{1 + \tau x} \phi \right\}$$

b) *Less than or equal to*: Again for $\mathbb{R} = \mathbb{N}$, consider the diagram $\dashv \circ$ of Circ _{\mathbb{N}} . One can check that

$$\begin{array}{c} \bullet \bullet \dashv \bullet \bullet \\ \bullet \bullet \dashv \bullet \bullet \end{array} = \{(m, n) \mid \exists m'. m + m' = n\} \\
\left\langle \begin{array}{c} \bullet \bullet \dashv \bullet \bullet \\ \bullet \bullet \dashv \bullet \bullet \end{array} \right\rangle_{\mathbb{N}} = \{(m, n) \mid m \leq n\}$$

Observe that if, instead of \mathbb{N} , we parametrise the theory over a semiring with additive inverses, such as \mathbb{R} , then the same diagram collapses to the total relation $\langle \dashv \bullet \bullet \rangle_{\mathbb{R}} = \mathbb{R} \times \mathbb{R}$.

c) *Mutual exclusion*: For the diagram $\dashv \vdash$ of Circ _{\mathbb{N}} from (4), first notice that $\langle \dashv \vdash \rangle_{\mathbb{N}}$ is by definition equal to

$$\begin{array}{c} \bullet \bullet \dashv \bullet \bullet \\ \bullet \bullet \dashv \bullet \bullet \end{array} = \{(n, n) \mid \exists m \in \mathbb{N}, n + m = 1\} \\
= \{(0, 0), (1, 1)\}$$

Then we can derive that $\left(\binom{m}{n}, z\right) \in \langle \bigcirc \dashv \bigcirc \rangle_{\mathbb{N}}$ whenever $z = m + n \leq 1$. We have, therefore

$$\langle \bigcirc \dashv \bigcirc \rangle_{\mathbb{N}} = \left\{ \left(\binom{1}{0}, 1 \right), \left(\binom{0}{1}, 1 \right), \left(\binom{0}{0}, 0 \right) \right\},$$

that is, at most one of the upper and lower port on the left may synchronise with the port on the right. This diagram thus implements *mutual exclusion*.

d) *Ideal voltage source*: The behaviour of electrical components is expressed as a voltage-current relationship between their terminals. As was done in [15] we will use one wire to denote voltage and another to denote the current, each modelled as a real number. With this interpretation, a simple voltage source with fixed voltage k is denoted by the same $\text{ACirc}_{\mathbb{R}}$ diagram as in (5) and we can see that its denotation coincides with (3):

$$\left\langle \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \right\rangle_{\mathbb{R}} = \left\{ \left(\binom{\phi_1}{i}, \binom{\phi_2}{i} \right) \mid \phi_1 + k = \phi_2 \right\}$$

III. CATEGORIES OF AFFINE RELATIONS

In the remainder of the paper we focus on two instances of \mathbb{R} , motivated by applications: the case when \mathbb{R} is a field \mathbb{K} , and the case when \mathbb{R} is the semiring \mathbb{N} of natural numbers.

Our overall goal is to equip $\text{ACirc}_{\mathbb{K}}$ and $\text{ACirc}_{\mathbb{N}}$ with sound and fully complete axiomatisations with respect to suitable classes of relations in the semantic domain. This is articulated in two steps. In this section, we identify these classes as \mathbb{K} -*affine relations* and \mathbb{N} -*affine relations*, showing that they form props $\text{AffRel}_{\mathbb{K}}$ and $\text{AffRel}_{\mathbb{N}}$ respectively. The next section will conclude the characterisation by introducing equational theories for $\text{ACirc}_{\mathbb{K}}$ and $\text{ACirc}_{\mathbb{N}}$ which are sound and fully complete with respect to $\text{AffRel}_{\mathbb{K}}$ and $\text{AffRel}_{\mathbb{N}}$.

A. Affine relations over a field

Fix a field \mathbb{K} . We recall the concept of linear relation.

Definition 3 ([3], [7]). A \mathbb{K} -linear relation of type $k \rightarrow l$ is a linear subspace of $\mathbb{K}^k \times \mathbb{K}^l$, considered as a \mathbb{K} -vector space. \mathbb{K} -linear relations form a sub-prop $\text{LinRel}_{\mathbb{K}}$ of $\text{Rel}_{\mathbb{K}}$, with composition defined by (7)-(8).

By analogy, we define affine relations from affine subspaces.

Definition 4. An affine subspace of \mathbb{K}^d is a subset $V \subseteq \mathbb{K}^d$ that is either empty or for which there exists a vector $\mathbf{a} \in \mathbb{K}^d$ and a linear subspace L of \mathbb{K}^d such that $V = \mathbf{a} + L := \{\mathbf{a} + \mathbf{x} \mid \mathbf{x} \in L\}$.

A \mathbb{K} -affine relation of type $k \rightarrow l$ is an affine subspace of $\mathbb{K}^k \times \mathbb{K}^l$, considered as a \mathbb{K} -vector space.

Our next goal is to show that \mathbb{K} -affine relations form a prop. To this aim, it is useful to give a construction that allows us to relate linear and affine relations systematically, using *homogenisation*, a well-known technique of affine and convex geometry. To every affine relation we associate a linear relation in a unique way, by embedding it into a space with an additional dimension. First, for a set $X \subseteq \mathbb{K}^k \times \mathbb{K}^l$, let

$${}^0X = \left\{ \left(\binom{0}{\mathbf{a}}, \mathbf{b} \right) \mid (\mathbf{a}, \mathbf{b}) \in X \right\} \subseteq \mathbb{K}^{k+1} \times \mathbb{K}^l \quad (10)$$

Definition 5. Let $R: k \rightarrow l$ be an affine relation given by $(\mathbf{a}, \mathbf{b}) + L$. Its homogenisation is the linear relation $\widehat{R}: k+1 \rightarrow l$ defined as $\widehat{R} = \left(\binom{1}{\mathbf{a}}, \mathbf{b} \right) + {}^0L$.

Homogenisation satisfies an useful property:

$$(\mathbf{a}, \mathbf{b}) \in R \quad \text{if and only if} \quad \left(\binom{1}{\mathbf{a}}, \mathbf{b} \right) \in \widehat{R}. \quad (11)$$

We will now use homogenisation to prove affine relations close under relational composition. If $R: k \rightarrow l$ and $S: l \rightarrow p$ are affine relations, we cannot directly compose their homogenisations $\widehat{R}: k+1 \rightarrow l$ and $\widehat{S}: l+1 \rightarrow p$ as linear relations, as the types do not match.

We can, however, interpret the homogenisation construction as embedding affine relations into a subcategory of the co-Kleisli category of the $(-)+1$ comonad, with structural natural transformations $\mu: (-)+1 \rightarrow (-)+2$ and $\epsilon: (-)+1 \rightarrow \text{id}$:

$$\mu_k = \left\langle \begin{array}{c} \bullet \\ \text{---} \\ \text{---} \end{array} \right\rangle_{\mathbb{K}} \quad \text{and} \quad \epsilon_k = \left\langle \begin{array}{c} \bullet \\ \text{---} \\ \text{---} \end{array} \right\rangle_{\mathbb{K}} \quad (12)$$

The comonad laws are a consequence of the comonoid law for \dashv and are easy to check. For $U: k+1 \rightarrow l$ and $V: l+1 \rightarrow p$ two linear relations, their coKleisli composition $U \widehat{\circ} V$ is:

$$\left\langle \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \right\rangle_{\mathbb{K}} \quad (13)$$

We are now able to prove the following closure property.

Proposition 6. The composite (7) of two \mathbb{K} -affine relations is a \mathbb{K} -affine relation.

Proof. Let $R: k \rightarrow l$ and $S: l \rightarrow p$ be two \mathbb{K} -affine relations. If $(\mathbf{x}, \mathbf{z}) \in R; S$ there exists $\mathbf{y} \in \mathbb{K}^k$ such that $(\mathbf{x}, \mathbf{y}) \in R$ and $(\mathbf{y}, \mathbf{z}) \in S$. So $\left(\binom{1}{\mathbf{x}}, \mathbf{y} \right) \in \widehat{R}$ and $\left(\binom{1}{\mathbf{y}}, \mathbf{z} \right) \in \widehat{S}$ and therefore $\left(\binom{1}{\mathbf{x}}, \mathbf{z} \right) \in \widehat{R} \widehat{\circ} \widehat{S}$. As long as $R; S$ is nonempty (in which case we are done), we can always find a basis of $\widehat{R} \widehat{\circ} \widehat{S}$ of the form $\left\{ \left(\binom{1}{\mathbf{e}}, \mathbf{f} \right), \left(\binom{0}{\mathbf{e}_1}, \mathbf{f}_1 \right), \dots, \left(\binom{0}{\mathbf{e}_n}, \mathbf{f}_n \right) \right\}$. Then, clearly $R; S = (\mathbf{e}, \mathbf{f}) + L$ where L is the linear subspace spanned by $\{(\mathbf{e}_1, \mathbf{f}_1), \dots, (\mathbf{e}_n, \mathbf{f}_n)\}$. \square

As a result, we can define the prop $\text{AffRel}_{\mathbb{K}}$ of \mathbb{K} -affine relations, as a sub-prop of $\text{Rel}_{\mathbb{K}}$.

B. Affine relations over the semiring \mathbb{N}

When the semiring is \mathbb{N} , the absence of additive and multiplicative inverses yields a different semantics, which requires slightly more work than the \mathbb{K} -affine case. Our departure point is the notion of additive relation.

Definition 7 ([13]). An additive relation of type $k \rightarrow l$ is a subset $R \subseteq \mathbb{N}^k \times \mathbb{N}^l$ such that (i) $(\mathbf{0}, \mathbf{0}) \in R$ and (ii) if $(\mathbf{a}, \mathbf{b}), (\mathbf{a}', \mathbf{b}') \in R$ then $(\mathbf{a} + \mathbf{a}', \mathbf{b} + \mathbf{b}') \in R$.

Every pair $(\mathbf{a}, \mathbf{b}) \in \mathbb{N}^k \times \mathbb{N}^l$ generates an additive relation $\langle (\mathbf{a}, \mathbf{b}) \rangle = \{(n\mathbf{a}, n\mathbf{b}) \mid n \in \mathbb{N}\}$. More generally, for a finite set $G = \{(\mathbf{a}_1, \mathbf{b}_1), \dots, (\mathbf{a}_p, \mathbf{b}_p)\}$ of points in $\mathbb{N}^k \times \mathbb{N}^l$, we write $\langle G \rangle$ for the additive relation

$$\langle G \rangle = \left\{ \sum_{i=1}^p n_i (\mathbf{a}_i, \mathbf{b}_i) \mid n_1, \dots, n_p \in \mathbb{N} \right\}. \quad (14)$$

For our characterisation only finitely generated additive relations are relevant.

Definition 8. An additive relation $R: k \rightarrow l$ is finitely generated (f.g.) if there exists a finite set of vectors $G = \{(\mathbf{a}_1, \mathbf{b}_1), \dots, (\mathbf{a}_p, \mathbf{b}_p)\}$ such that $R = \langle G \rangle$. F.g. \mathbb{N} -additive relations form a sub-prop AddRel of $\text{Rel}_{\mathbb{N}}$.

Example 9. The two pictures in the left of Figure 4 represents the additive relations of type $1 \rightarrow 1$ generated by $\{(1, 2), (3, 1)\}$ and $\{(1, 3), (2, 2), (4, 1)\}$ respectively.

Henceforward, when we say ‘‘additive relation’’, we mean f.g. additive relation. Differently from the linear case, closure under composition of additive relations is a nontrivial fact [13]. The following is another important difference between the linear and additive worlds.

Proposition 10 ([13, Proposition 23]). *Every additive relation has a unique minimal (for inclusion) generating set, called its Hilbert basis.*

We now move to defining \mathbb{N} -affine relations. Recall that the Minkowski sum of sets $C, D \subseteq \mathbb{N}^d$ is $C + D = \{\mathbf{c} + \mathbf{d} \mid \mathbf{c} \in C, \mathbf{d} \in D\}$. If one of these is a singleton, e.g. $C = \{\mathbf{c}\}$, we will abuse notation and write $\mathbf{c} + D$. Note that $\emptyset + D = \emptyset$.

Definition 11. A \mathbb{N} -affine relation $R: k \rightarrow l$ is a set $R \subseteq \mathbb{N}^k \times \mathbb{N}^l$ for which there exists finite $B, D \subseteq \mathbb{N}^k \times \mathbb{N}^l$ such that $R = \bigcup_{(\mathbf{a}, \mathbf{b}) \in B} \{(\mathbf{a}, \mathbf{b}) + \langle D \rangle\}$. Elements of B are called base points and those of D directions.

The first noticeable difference with affine relations over a field is that \mathbb{N} -affine relations can have more than one base point. If $B = \{(\mathbf{a}, \mathbf{b})\}$ is a singleton, $R = (\mathbf{a}, \mathbf{b}) + D$ is the translation of an additive relation by (\mathbf{a}, \mathbf{b}) . If $B = \{(\mathbf{0}, \mathbf{0})\}$, R is an additive relation. Thus every additive relation R is \mathbb{N} -affine: take $B = \{(\mathbf{0}, \mathbf{0})\}$ and D to be a generating set of R . An \mathbb{N} -affine relation is, therefore, a finite union of translated additive relations.

Example 12. The two pictures on the right of Figure 4 represent the two \mathbb{N} -affine relations of type $1 \rightarrow 1$ with respective bases $\{(2, 2), (4, 2)\}$ and $\{(1, 2)\}$, and respective directions $\{(1, 2), (3, 1)\}$ and $\{(1, 3), (2, 2), (4, 1)\}$.

Note that every finite subset S of \mathbb{N}^{k+l} is also \mathbb{N} -affine (by setting $B = S$ and $D = \{(\mathbf{0}, \mathbf{0})\}$). Finally, \emptyset is \mathbb{N} -affine (e.g. by taking $B = \emptyset$) but not additive.

As for \mathbb{K} -affine relations, we can define the homogenisation of an \mathbb{N} -affine relation. For this we will also need the following notation, for $X \subseteq \mathbb{N}^k \times \mathbb{N}^l$:

$${}^1X = \left\{ \left(\begin{pmatrix} 1 \\ \mathbf{a} \end{pmatrix}, \mathbf{b} \right) \mid (\mathbf{a}, \mathbf{b}) \in X \right\} \subseteq \mathbb{N}^{k+1} \times \mathbb{N}^l.$$

Definition 13. Let $R: k \rightarrow l$ be an \mathbb{N} -affine relation with base points B and directions D . Its homogenisation is the additive relation $\widehat{R}: k+1 \rightarrow l$ defined as $\widehat{R} = \langle {}^1B \cup {}^0D \rangle$.

As for \mathbb{K} -affine relations, the homogenisation of \mathbb{N} -affine relations satisfies the crucial property of equation (11). We can now prove that \mathbb{N} -affine relations form a category.

Proposition 14. *The composite (7) of two \mathbb{N} -affine relations is an \mathbb{N} -affine relation.*

Proof. Let $R: k \rightarrow l$ and $S: l \rightarrow p$ be \mathbb{N} -affine relations and $R; S$ their composite. We can obtain the base points and directions of $R; S$ from the Hilbert basis H of the additive relation $\widehat{R}; \widehat{S}$. If $(\mathbf{a}, \mathbf{c}) \in R; S$, there exists $\mathbf{b} \in \mathbb{N}^k$ such that $(\mathbf{a}, \mathbf{b}) \in R$ and $(\mathbf{b}, \mathbf{c}) \in S$. So $\left(\begin{pmatrix} 1 \\ \mathbf{a} \end{pmatrix}, \mathbf{b}\right) \in \widehat{R}$ and $\left(\begin{pmatrix} 1 \\ \mathbf{b} \end{pmatrix}, \mathbf{c}\right) \in \widehat{S}$ and therefore $\left(\begin{pmatrix} 1 \\ \mathbf{a} \end{pmatrix}, \mathbf{c}\right) \in \widehat{R}; \widehat{S}$. We can decompose this last pair into a weighted sum of H -elements $\left(\begin{pmatrix} 1 \\ \mathbf{a} \end{pmatrix}, \mathbf{c}\right) = \left(\begin{pmatrix} 1 \\ \mathbf{f} \end{pmatrix}, \mathbf{g}\right) + \sum_{i=1}^m n_i \left(\begin{pmatrix} 0 \\ \mathbf{d}_i \end{pmatrix}, \mathbf{e}_i\right)$, where $\left(\begin{pmatrix} 1 \\ \mathbf{f} \end{pmatrix}, \mathbf{g}\right), \left(\begin{pmatrix} 0 \\ \mathbf{d}_i \end{pmatrix}, \mathbf{e}_i\right) \in H$ and $n_i \in \mathbb{N}$ for $1 \leq i \leq m$. Then, $(\mathbf{a}, \mathbf{c}) = (\mathbf{f}, \mathbf{g}) + \sum_{i=1}^m n_i (\mathbf{d}_i, \mathbf{e}_i)$. Therefore, since (\mathbf{a}, \mathbf{c}) was arbitrary, we conclude that $R; S = \bigcup_{(\mathbf{x}, \mathbf{y}) \in B} \{(\mathbf{x}, \mathbf{y}) + \langle D \rangle\}$, with $B = \{(\mathbf{f}, \mathbf{g}) \mid \left(\begin{pmatrix} 1 \\ \mathbf{f} \end{pmatrix}, \mathbf{g}\right) \in H\}$ and $D = \{(\mathbf{d}, \mathbf{e}) \mid \left(\begin{pmatrix} 0 \\ \mathbf{d} \end{pmatrix}, \mathbf{e}\right) \in H\}$. \square

As a result, we can define the prop $\text{AffRel}_{\mathbb{N}}$ of \mathbb{N} -affine relations, as a sub-prop of $\text{Rel}_{\mathbb{N}}$.

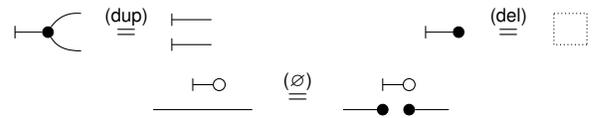
IV. AXIOMATISING AFFINE RELATIONS

This section contains the main technical contribution of the paper, namely the introduction of equational theories for $\text{ACirc}_{\mathbb{K}}$ and $\text{ACirc}_{\mathbb{N}}$ that are sound and fully complete for their affine relation semantics. In each of the two cases, we first introduce the equational theory and its properties. Then we show that there is a prop isomorphism between the syntax modulo the equations, and affine relations. The full completeness result follows from the isomorphism.

A. Axiomatising \mathbb{K} -affine relations

As in the previous section, we start with the (non-affine) linear case. It is proven in [3], [7] that the quotient of $\text{Circ}_{\mathbb{K}}$ by the equations of *interacting Hopf algebras* (reported in Fig. 6, Appendix A), which we call $\text{IH}_{\mathbb{K}}$, is isomorphic to $\text{LinRel}_{\mathbb{K}}$.

Our characterisation will extend this isomorphism to affine systems. The challenge is to identify equations that govern the behaviour of the new generator \dashv . We claim that adding the following three equations is sufficient:



First, let us explain the new axioms. The first two say that \dashv can be deleted and copied by the comonoid structure, just like \circ . This has the effect of constraining the interpretation of \dashv as a 0-ary functional relation, i.e. it is a constant.

More interestingly, the third equation is justified by the possibility of expressing the empty set, by, for example,

$$\langle \dashv \circ \rangle_{\mathbb{K}} = \{(\bullet, 1)\}; \{(0, \bullet)\} = \emptyset. \quad (15)$$

As mentioned previously, \emptyset is an affine relation that is not linear. Since for any R and S in $\text{Rel}_{\mathbb{R}}$, $\emptyset \oplus R = \emptyset \oplus S =$

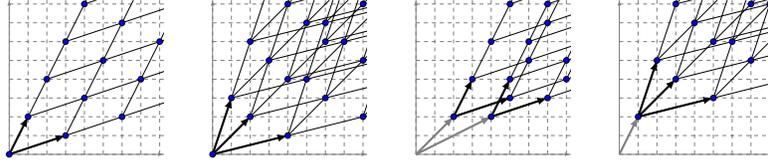


Fig. 4. Two additive and two \mathbb{N} -affine relations.

\emptyset , composing or taking the monoidal product of \emptyset with any relation results in \emptyset . Thus \emptyset is analogous to logical false.

Definition 15. The prop $\text{AIH}_{\mathbb{K}}$ (affine interacting Hopf algebras) is the quotient of $\text{ACirc}_{\mathbb{K}}$ by the equations of $\text{IH}_{\mathbb{K}}$ (Fig. 6, Appendix A) plus (dup), (del) and (\emptyset).

We are going to show that $\text{AIH}_{\mathbb{K}}$ is isomorphic to $\text{AffRel}_{\mathbb{K}}$. First, the following lemma formalises the preceding discussion about equation (\emptyset).

Lemma 16. For any two arrows $c, d : k \rightarrow l$ of $\text{AIH}_{\mathbb{K}}$,

$$k \begin{array}{c} \text{---} \circ \text{---} \\ | \\ \text{---} \text{---} c \text{---} \text{---} \\ | \\ \text{---} \text{---} \end{array} l = k \begin{array}{c} \text{---} \circ \text{---} \\ | \\ \text{---} \text{---} d \text{---} \text{---} \\ | \\ \text{---} \text{---} \end{array} l$$

Proof. In Appendix C-A. \square

We are now ready to prove our characterisation result. Because all the equations of $\text{AIH}_{\mathbb{K}}$ are sound in $\text{LinRel}_{\mathbb{K}}$, we can define a prop morphism $\llbracket - \rrbracket_{\mathbb{K}} : \text{AIH}_{\mathbb{K}} \rightarrow \text{LinRel}_{\mathbb{K}}$ inductively by the same clauses (Figure 2) of $\langle \cdot \rangle_{\mathbb{K}}$.

Theorem 17. $\llbracket \cdot \rrbracket_{\mathbb{K}} : \text{AIH}_{\mathbb{K}} \rightarrow \text{LinRel}_{\mathbb{K}}$ is a prop isomorphism.

Proof. First, we show that $\llbracket \cdot \rrbracket_{\mathbb{K}}$ is full. Diagrammatically, homogenisation means that \mathbb{K} -affine relations can be thought of as \mathbb{K} -linear relations with an extra dangling wire for the additional dimension. Because the restriction of $\llbracket - \rrbracket_{\mathbb{K}}$ to a functor $\text{IH}_{\mathbb{K}} \rightarrow \text{LinRel}_{\mathbb{K}}$ is well-defined and an isomorphism (thus, also full) [3, Theorem 6.4], we can always obtain a string diagram in $\text{IH}_{\mathbb{K}}$ for the homogenisation \hat{R} of an affine relation R . Then, we can use generator \vdash to plug this wire, obtaining a string diagram

$$k \begin{array}{c} \text{---} \\ | \\ \text{---} \text{---} \hat{R} \text{---} \text{---} \\ | \\ \text{---} \text{---} \end{array} l .$$

Finally, equation (11) implies that

$$\left[\left[k \begin{array}{c} \text{---} \\ | \\ \text{---} \text{---} \hat{R} \text{---} \text{---} \\ | \\ \text{---} \text{---} \end{array} l \right] \right]_{\mathbb{K}} = R \quad (16)$$

proving that $\llbracket - \rrbracket_{\mathbb{K}}$ is full. It remains to show that $\llbracket - \rrbracket_{\mathbb{K}}$ is faithful. We will use a normal form argument, which relies the isomorphism of $\text{IH}_{\mathbb{K}}$ and $\text{LinRel}_{\mathbb{K}}$ [3, Theorem 6.4]. Let $d : k \rightarrow l$ be a diagram in $\text{AIH}_{\mathbb{K}}$. By naturality of the symmetry we may write d as follows:

$$k \begin{array}{c} \text{---} \\ | \\ \text{---} \text{---} d \text{---} \text{---} \\ | \\ \text{---} \text{---} \end{array} l = k \begin{array}{c} \text{---} \\ | \\ \text{---} \text{---} c \text{---} \text{---} \\ | \\ \text{---} \text{---} \end{array} l \quad (17)$$

for some diagram c , in the image of the embedding $\text{IH}_{\mathbb{K}} \hookrightarrow \text{AIH}_{\mathbb{K}}$. In graphical terms, we have pulled all copies of \vdash up and down, past the rest of the diagram which represents some linear relation c . We may now simplify (17):

$$k \begin{array}{c} \text{---} \\ | \\ \text{---} \text{---} c \text{---} \text{---} \\ | \\ \text{---} \text{---} \end{array} l \stackrel{(\text{dup})}{=} k \begin{array}{c} \text{---} \\ | \\ \text{---} \text{---} c \text{---} \text{---} \\ | \\ \text{---} \text{---} \end{array} l \quad (18)$$

$$= k \begin{array}{c} \text{---} \\ | \\ \text{---} \text{---} c' \text{---} \text{---} \\ | \\ \text{---} \text{---} \end{array} l \quad (19)$$

where c' is the diagram enclosed in the dotted box. Finally, from the span normal form for linear relations [3, Theorem 6.2], we can find a diagram $e : p \rightarrow l + 1 + k$ representing some matrix \mathcal{M}_e (i.e. $\llbracket e \rrbracket_{\mathbb{K}} = \{(\mathbf{a}, \mathcal{M}_e \mathbf{a}) \mid \mathbf{a} \in \mathbb{K}^p\}$), such that

$$k \begin{array}{c} \text{---} \\ | \\ \text{---} \text{---} d \text{---} \text{---} \\ | \\ \text{---} \text{---} \end{array} l = \begin{array}{c} p \\ \bullet \\ \text{---} \text{---} e \text{---} \text{---} \\ | \\ k \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \text{---} \end{array} l \quad (20)$$

$$= \begin{array}{c} p \\ \bullet \\ \text{---} \text{---} e \text{---} \text{---} \\ | \\ k \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \text{---} \end{array} l \quad (21)$$

and the columns of \mathcal{M}_a generate $\llbracket c' \rrbracket_{\mathbb{K}}$ (for the definition of \dashv , see (9)). By equation (11) and the isomorphism $\text{IH}_{\mathbb{K}} \cong \text{LinRel}_{\mathbb{K}}$, the affine relation $\llbracket d \rrbracket_{\mathbb{K}}$ is uniquely characterised by this decomposition if it is nonempty. If it empty however, it means that \dashv is plugged to a wire of e representing some row of \mathcal{M}_e (in the sense of Appendix B) which is uniformly zero. Indeed, if $(a_1 \dots a_n)$ is this row, $\llbracket d \rrbracket_{\mathbb{K}}$ is empty precisely when there is no $x_1, \dots, x_n \in \mathbb{K}$ such that $a_1 x_1 + \dots + a_n x_n = 1$. This happens only when $a_i = 0$ for all $1 \leq i \leq n$ or, in diagrammatic form, when:

$$k \begin{array}{c} \text{---} \\ | \\ \text{---} \text{---} d \text{---} \text{---} \\ | \\ \text{---} \text{---} \end{array} l = \begin{array}{c} p \\ \bullet \\ \text{---} \text{---} f \text{---} \text{---} \\ | \\ k \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \text{---} \end{array} l \quad (22)$$

for some diagram f representing a matrix. By Lemma 39 all diagrams of this form are equal, which concludes the proof. \square

$2 \subseteq \mathbb{N}$, interpreting a relation over 2 as a relation over \mathbb{N} .²
For all c in SCCirc , the diagram below commutes.

$$\begin{array}{ccc}
\text{SCCirc} & \xrightarrow{\mathcal{E}(-)} & \text{ARC} \\
\downarrow \langle\langle - \rangle\rangle & & \cong \downarrow [\cdot]_{\mathbb{N}} \\
& & \text{AffRel}_{\mathbb{N}} \\
& & \downarrow \iota_1 \\
\text{Rel}_2 & \xrightarrow{\iota_2} & \text{Rel}_{\mathbb{N}}
\end{array}$$

Proof. By induction on SCCirc . \square

As a consequence of Theorem 27 we obtain a sound and complete axiomatisation for equivalence of stateless connectors by means of the axioms of ARC , in Figure 7.

Corollary 28. For any two stateless connectors c and d in SCCirc ,

$$\langle\langle c \rangle\rangle = \langle\langle d \rangle\rangle \quad \text{iff} \quad \mathcal{E}(c) = \mathcal{E}(d)$$

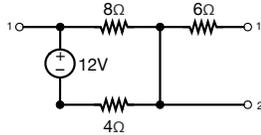
Remark 29. Theorem 26 in [14] states that the connectors in SCCirc can denote exactly those relations in Rel_2 that contain the vector 0. ACirc_2 can express more relations of Rel_2 , for instance the not relation denoted by the following diagram:

$$\left\langle \begin{array}{c} \bullet \\ \text{---} \\ \circ \end{array} \right\rangle_{\mathbb{N}} = \{(0, 1), (1, 0)\}$$

In fact, all relations in Rel_2 can be expressed by ACirc_2 since every finite subset of $\mathbb{N}^k \times \mathbb{N}^l$ is an \mathbb{N} -affine relation $k \rightarrow l$, so in particular every subset containing only 0s and 1s is in $\text{AffRel}_{\mathbb{N}}$.

VI. CASE STUDY II: ELECTRIC CIRCUITS

Elementary electrical engineering focusses on open linear circuit analysis. An example is illustrated below.



Such circuits may include voltage $\left(\begin{array}{c} k \\ \ominus \\ \oplus \end{array}\right)$ and current sources $\left(\begin{array}{c} k \\ \ominus \\ \oplus \end{array}\right)$, resistors $\left(\begin{array}{c} k \\ \text{---} \\ \text{---} \end{array}\right)$, junctions (filled nodes) and open terminals (unfilled nodes).

The section is structured as follows. We begin by making these open circuits formal as combinatorial structures. We then present open circuits as algebraic structures, and give a compositional semantics in terms of \mathbb{K} -affine relations. In Subsection VI-A, we use the axiomatisation of Section IV to give a sound and complete calculus for the analysis of open linear circuits. We end in Subsection VI-B by showing how to handle circuits with time-dependent currents and voltages, which also feature inductors $\left(\begin{array}{c} k \\ \text{---} \\ \text{---} \end{array}\right)$ and capacitors $\left(\begin{array}{c} k \\ \text{---} \\ \text{---} \end{array}\right)$.

We can make *closed* (i.e. those without open terminals) circuits precise as combinatorial structures by considering them as multigraphs with a mixture of directed and undirected edges. Directed edges are either voltage and current sources, while undirected edges are resistors. Finally, every edge is labelled by a non-negative real, denoting either voltage (in volts), current (in amperes) or resistance (in ohms). Formally, then, a closed circuit is

$$\{X, V, C, R, vs, vt : V \rightarrow X, cs, ct : C \rightarrow X, \\
rc : R \rightarrow \mathcal{P}_2(X), q : V + C + R \rightarrow \mathbb{R}_+\}$$

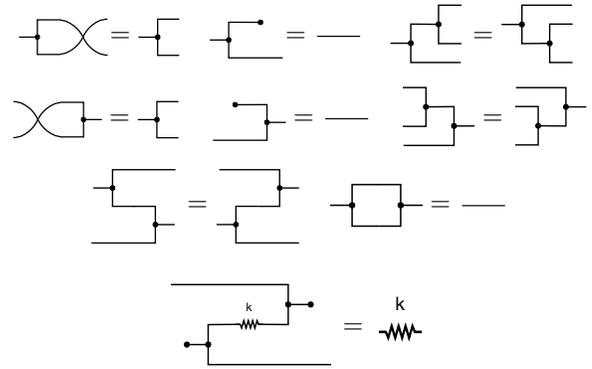
where X, V, C, R are, correspondingly, finite sets of nodes, voltage sources, current sources and resistors, vs, vt, cs, ct, rc give the connectivity of the edges, and q the labels.

To consider open circuits, we consider a certain category of *cospans*. First, the category CCirc of closed circuits and their obvious choice of morphism has pushouts. Next, any finite ordinal can be considered as a discrete closed circuit, with the ordinal serving as its set of nodes. We therefore consider the full subcategory OCirc of the category of cospans $\text{Cospan}(\text{CCirc})$ with objects finite ordinals. Having ordinals as objects reflects the numbering the left and right open terminals, as we have done in the example diagram above. It is straightforward to verify that OCirc is a prop.

We now give a straightforward algebraic characterisation of OCirc . The prop ECirc has signature

$$\left\{ \begin{array}{c} k \\ \text{---} \\ \text{---} \end{array}, \begin{array}{c} k \\ \ominus \\ \oplus \end{array}, \begin{array}{c} k \\ \oplus \\ \ominus \end{array} \right\}_{k \in \mathbb{R}_+} \cup \left\{ \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array}, \begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array}, \begin{array}{c} \text{---} \\ \text{---} \\ \bullet \end{array}, \begin{array}{c} \text{---} \\ \text{---} \\ \bullet \end{array} \right\} \quad (28)$$

where the parameter k ranges over the non-negative reals. Arrows $m \rightarrow n$ of ECirc represent open linear electrical circuits with m open terminals on the left and n open terminals on the right. The following are the equations:



The equations, apart from the last, are those of special Frobenius monoids [24]. The final equation reflects the fact that resistors are bidirectional. We state the following without proof, which is similar to [25, Proposition 3.2] and [26, Theorem 3.3].

Proposition 30. As props, $\text{OCirc} \cong \text{ECirc}$. \square

Having established open circuits as both combinatorial (OCirc) and algebraic (ECirc) structures, we can now give

²Note that ι_2 is *not* a functor since it does not preserve identities.

$$\begin{aligned}
\mathcal{I}\left(\begin{array}{c} k \\ \text{---} \end{array}\right) &= \text{---} \xrightarrow{\llbracket \cdot \rrbracket_{\mathbb{R}}} \left\{ \left((\phi_1), (\phi_2) \mid \phi_2 - \phi_1 = ki \right) \right\} \\
\mathcal{I}\left(\begin{array}{c} k \\ \text{---} \\ \text{---} \end{array}\right) &= \text{---} \xrightarrow{\llbracket \cdot \rrbracket_{\mathbb{R}}} \left\{ \left((\phi_1), (\phi_2) \mid \phi_2 - \phi_1 = k \right) \right\} \\
\mathcal{I}\left(\begin{array}{c} k \\ \text{---} \\ \text{---} \\ \text{---} \end{array}\right) &= \text{---} \xrightarrow{\llbracket \cdot \rrbracket_{\mathbb{R}}} \left\{ \left((\phi_k), (\phi_k) \right) \right\} \\
\mathcal{I}\left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array}\right) &= \text{---} \xrightarrow{\llbracket \cdot \rrbracket_{\mathbb{R}}} \left\{ \left(\begin{array}{c} \phi \\ i_1 \\ \phi \\ i_2 \\ \phi \\ i_3 \end{array}, \left(\begin{array}{c} \phi \\ i_2 \\ \phi \\ i_3 \end{array} \mid i_1 + i_2 + i_3 = 0 \right) \right) \right\} \\
\mathcal{I}\left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array}\right) &= \text{---} \xrightarrow{\llbracket \cdot \rrbracket_{\mathbb{R}}} \left\{ \left(\begin{array}{c} \phi \\ i_2 \\ \phi \\ i_3 \end{array}, \left(\phi, i_1 \mid i_1 + i_2 + i_3 = 0 \right) \right) \right\} \\
\mathcal{I}\left(\begin{array}{c} \bullet \\ \text{---} \\ \text{---} \end{array}\right) &= \text{---} \xrightarrow{\llbracket \cdot \rrbracket_{\mathbb{R}}} \left\{ \bullet, (\phi) \right\} \quad \mathcal{I}\left(\begin{array}{c} \text{---} \\ \text{---} \\ \bullet \end{array}\right) = \text{---} \xrightarrow{\llbracket \cdot \rrbracket_{\mathbb{R}}} \left\{ (\phi), \bullet \right\}
\end{aligned}$$

Fig. 5. Compositional semantics of electrical circuits.

a compositional semantics in terms of affine relations in Fig. VI. For each generator we give its translation as $\text{AIH}_{\mathbb{R}}$ -diagram and the associated \mathbb{R} -affine relation in symbolic notation. A similar semantics was given by Baez and Coya [15], [27], building on the work of Baez, Erbele and Fong [7], [28], and Rosebrugh, Sabadini and Walters [25]. Components of an electrical circuit denote a relationship between current and voltage, traditionally modelled as real values. The semantics of an open circuit of type $m \rightarrow n$ is thus a relation from \mathbb{R}^{2m} to \mathbb{R}^{2n} , with the behaviour of the individual elements given by Kirchoff's laws. We use $\phi \in \mathbb{R}$ to range over voltages and $i \in \mathbb{R}$ to range over currents.

Proposition 31. $\mathcal{I}(-) : \text{ECirc} \rightarrow \text{AffRel}_{\mathbb{R}}$ is a symmetric strict monoidal functor.

Proof. See Appendix C-B. \square

Note that $\mathcal{I}(-)$ fails to be a morphism of props for the simple reason that it is not identity on objects: a single wire of an electrical circuit maps to two wires of $\text{AIH}_{\mathbb{R}}$, with these used to keep track of the voltage and the current. Indeed, on objects $\mathcal{I}(1) = 2$.

A. $\text{AIH}_{\mathbb{R}}$ as a Calculus of Electrical Circuits

Below we give a few examples of how $\text{AIH}_{\mathbb{R}}$ can be used to derive well-known properties of circuits.

Lemma 32 (Properties of resistors).

$$\mathcal{I}\left(\begin{array}{c} a \\ \text{---} \\ \text{---} \\ \text{---} \\ b \end{array}\right) = \mathcal{I}\left(\begin{array}{c} a+b \\ \text{---} \end{array}\right) \quad \mathcal{I}\left(\begin{array}{c} a \\ \text{---} \\ \text{---} \\ \text{---} \\ b \end{array}\right) = \mathcal{I}\left(\begin{array}{c} ab/(a+b) \\ \text{---} \end{array}\right)$$

Proof. See Appendix C-B. \square

Lemma 33 (Properties of voltage sources).

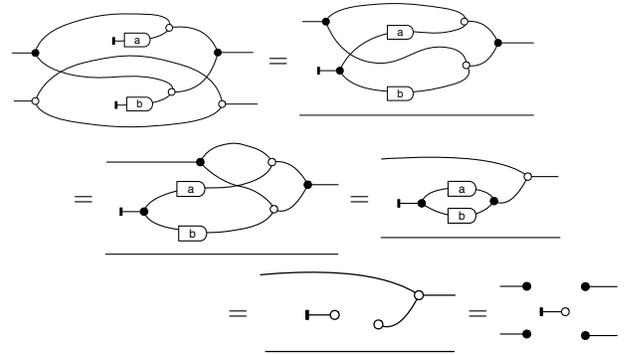
$$\mathcal{I}\left(\begin{array}{c} a \\ \text{---} \\ \text{---} \\ \text{---} \\ b \end{array}\right) = \mathcal{I}\left(\begin{array}{c} a+b \\ \text{---} \end{array}\right) \quad \mathcal{I}\left(\begin{array}{c} a \\ \text{---} \\ \text{---} \\ \text{---} \\ a \end{array}\right) = \mathcal{I}\left(\begin{array}{c} a \\ \text{---} \end{array}\right)$$

Proof. We only prove the first equality.

$$\begin{aligned}
\mathcal{I}\left(\begin{array}{c} a \\ \text{---} \\ \text{---} \\ \text{---} \\ b \end{array}\right) &= \text{---} \xrightarrow{\llbracket \cdot \rrbracket_{\mathbb{R}}} \left\{ \left((\phi_1), (\phi_2) \mid \phi_2 - \phi_1 = a \right) \right\} \\
&= \text{---} \xrightarrow{\llbracket \cdot \rrbracket_{\mathbb{R}}} \left\{ \left((\phi_1), (\phi_2) \mid \phi_2 - \phi_1 = a+b \right) \right\} \\
&= \mathcal{I}\left(\begin{array}{c} a+b \\ \text{---} \end{array}\right)
\end{aligned}$$

\square

Remark 34. In engineering literature, parallel voltage sources of different voltages are disallowed. It is nonetheless interesting to see what happens in the semantics.



This, as we have seen, is the way of expressing the empty relation in graphical affine algebra.

Lemma 35 (Properties of current sources).

$$\mathcal{I}\left(\begin{array}{c} a \\ \text{---} \\ \text{---} \\ \text{---} \\ a \end{array}\right) = \mathcal{I}\left(\begin{array}{c} a \\ \text{---} \end{array}\right) \quad \mathcal{I}\left(\begin{array}{c} a \\ \text{---} \\ \text{---} \\ \text{---} \\ b \end{array}\right) = \mathcal{I}\left(\begin{array}{c} a+b \\ \text{---} \end{array}\right)$$

Proof. We prove only the second equality.

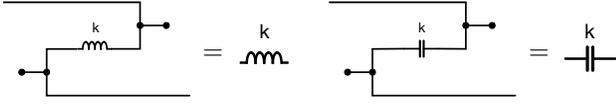
$$\begin{aligned}
\mathcal{I}\left(\begin{array}{c} a \\ \text{---} \\ \text{---} \\ \text{---} \\ b \end{array}\right) &= \text{---} \xrightarrow{\llbracket \cdot \rrbracket_{\mathbb{R}}} \left\{ \left((\phi_1), (\phi_2) \mid \phi_2 - \phi_1 = a \right) \right\} \\
&= \text{---} \xrightarrow{\llbracket \cdot \rrbracket_{\mathbb{R}}} \left\{ \left((\phi_1), (\phi_2) \mid \phi_2 - \phi_1 = a+b \right) \right\} \\
&= \mathcal{I}\left(\begin{array}{c} a+b \\ \text{---} \end{array}\right)
\end{aligned}$$

\square

Remark 36. Just as different voltage sources cannot be put in parallel (Remark 34), different current sources cannot be put in series: a similar graphical calculation as in Remark 34 yields the empty relation.

To capture time-dependent currents and voltages, we extend circuits with two additional kinds of undirected edges, *inductors* $\overset{k}{\text{---}}\text{---}\text{---}$ and *capacitors* $\overset{k}{\text{---}}\text{---}\text{---}$, each with labels from \mathbb{R}_+ , signifying inductance and capacitance. Omitting the details of the straightforward formalisation, we obtain \mathbf{CCirc}_s by considering the category of such extended circuits and, via cospans, the corresponding category of open circuits \mathbf{OCirc}_s .

To obtain \mathbf{ECirc}_s we extend the signature (28) with inductors and capacitors $\left\{ \overset{k}{\text{---}}\text{---}\text{---}, \overset{k}{\text{---}}\text{---}\text{---} \right\}_{k \in \mathbb{R}_+}$ and extend the set equations of \mathbf{ECirc} with those that indicate that these additional elements are undirected



The following is a simple extension of the correspondence shown in Proposition 30.

Proposition 37. *As props, $\mathbf{OCirc}_s \cong \mathbf{ECirc}_s$.* \square

By moving from the reals \mathbb{R} to the field of polynomial fractions $\mathbb{R}(x)$, or equivalently, rational functions in one variable, we can give a compositional semantics of circuits with time-dependent currents and voltages. The idea is to let multiplication by x express differentiation by the time variable, as usually done in engineering via Laplace transforms. We extend the mapping of Figure 5 as follows:

$$\mathcal{I}\left(\overset{k}{\text{---}}\text{---}\text{---}\right) = \left[\text{Circuit Diagram} \right] \xrightarrow{\llbracket \cdot \rrbracket_{\mathbb{R}(x)}} \left\{ \left(\binom{\phi_1}{i}, \binom{\phi_2}{i} \right) \mid \phi_2 - \phi_1 = kxi \right\}$$

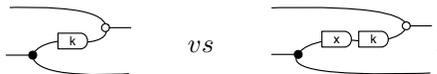
$$\mathcal{I}\left(\overset{k}{\text{---}}\text{---}\text{---}\right) = \left[\text{Circuit Diagram} \right] \xrightarrow{\llbracket \cdot \rrbracket_{\mathbb{R}(x)}} \left\{ \left(\binom{\phi_1}{i}, \binom{\phi_2}{i} \right) \mid i = kx(\phi_2 - \phi_1) \right\}$$

We can then show that also the extended semantics is functorial.

Proposition 38. $\mathcal{I}(-) : \mathbf{ECirc}_s \rightarrow \mathbf{AffRel}_{\mathbb{R}(x)}$ is a symmetric strict monoidal functor.

Proof. Follows from the proof of Proposition 44. It suffices to check that the undirectedness of $\overset{k}{\text{---}}\text{---}\text{---}$ and $\overset{k}{\text{---}}\text{---}\text{---}$ is respected, but the derivation given in the proof of Proposition 44 can easily be adapted in each case. \square

Exploiting the isomorphism $\mathbf{AIH}_{\mathbb{R}(x)} \cong \mathbf{AffRel}_{\mathbb{R}(x)}$, we can reason equationally also on this extended class of circuits. For instance, one can show that inductors behave analogously to resistors when put in series and in parallel—*cf.* Lemma 32 and Appendix C-B, Lemma 46. This is mostly evident when observing the structural similarity of their $\mathbf{AIH}_{\mathbb{R}(x)}$ -interpretation:



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APPENDIX A

AXIOMATISATIONS FOR NON-AFFINE CIRCUIT DIAGRAMS

In this appendix we report from [3] and [13] the equational theories axiomatising (non-affine) linear relations and additive relations, called the theory of interacting Hopf algebras ($\text{IH}_{\mathbb{K}}$) and the algebra of resources (RC) respectively. For each theory, we provide a brief description of its equations.

A. Interacting Hopf algebras

The following description refers to Figure 6.

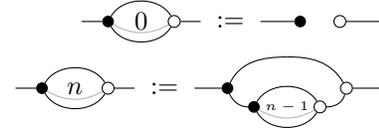
- In the first block, both the black and white structures are commutative monoids and comonoids, expressing fundamental properties of addition and copying.
- In the second block, the white monoid and black comonoid interact as a bimonoid. Bimonoids are one of two canonical ways that monoids and comonoids interact, as shown in [29].
- In the third and fourth block, both the black and the white monoid/comonoid pair form an extraspecial Frobenius monoid. The Frobenius equations (fr 1) and (fr 2) are a famous algebraic pattern which establishes a bridge between algebraic and topological phenomena, see [24], [30], [31]. The “extraspecial” refers to the two additional equations, the *special* equation ($\bullet\text{-sp}$) and the *bone* equation ($\bullet\text{-bo}$). The Frobenius equations, together with the special equation, are the another canonical pattern of interaction between monoids and comonoids identified in [29]. Together with the bone equation, the set of four equations characterises *corelations*, see [32]–[34].
- The equations in the fifth block are parametrised over $r \in \mathbb{K}$ and describe commutativity of $\text{---}\boxed{k}\text{---}$ with respect to the other operations, as well as multiplication and addition of scalars.
- Finally, the last block describes the interpretation of $\text{---}\boxed{k}\text{---}$ as division by k .

B. The algebra of resources

The following description refers to Figure 7.

- The first three blocks are the same as in $\text{IH}_{\mathbb{K}}$, cf. Figure 6.
- In the fourth block, deviating from the equational theory of $\text{IH}_{\mathbb{K}}$, the white monoid-comonoid pair forms a special bimonoid, not a Frobenius monoid. In fact, the Frobenius structure is equivalent to the presence of additive inverses [35], [36] and it is incompatible with the non-negativity of the natural numbers: $a + b = 0 \implies a = b = 0$, reflected by ($\circ\text{-bi}(\text{co})\text{un}$).
- In the fifth block, two more equations capture key properties of addition in \mathbb{N} . Note that the second can be seen as a unary version of the first. In particular, (can) is one of the two equations that axiomatises the notion of complementary observables in categorical approaches to quantum mechanics [37].
- Finally, the last equation is an axiom scheme, parametrised over $n \in \mathbb{N}$. It uses the following syntactic

sugar, along with the obvious mirror image versions, defined recursively:



to represent the additive relations of the form $\langle (1, n) \rangle$. Equations ($n\text{-inv}$) are one half of the equations that concerns such sugars in $\text{IH}_{\mathbb{K}}$. Their symmetric variant is *not* present since they are not sound for AddRel and rely on the presence of multiplicative inverses, which \mathbb{N} does not have.

APPENDIX B

PICTURING MATRICES

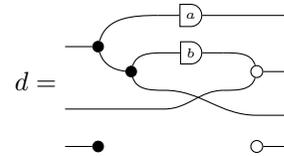
The proofs of completeness of $\text{AIH}_{\mathbb{K}}$ (Theorem 17) and ARC (Theorem 22) exploit the ability to represent matrices in the graphical syntax. Details can be found in [4, Sec. 3.2] but we recall the basics below.

Roughly speaking, the theory of matrices lives inside both $\text{AIH}_{\mathbb{K}}$ and ARC as the theory of the bimonoid $\text{---}\bullet\text{---}$, $\text{---}\circ\text{---}$, $\text{---}\circ\text{---}$, $\text{---}\bullet\text{---}$ along with that of the scalars $\text{---}\boxed{k}\text{---}$. It means that, using only these we can represent any matrix with coefficients in \mathbb{K} or \mathbb{N} . And, moreover, reasoning about them can be done entirely graphically, as the corresponding equational theories are complete.

To develop some intuition for this correspondence, let us demonstrate how matrices are represented diagrammatically. An $l \times k$ matrix \mathcal{M}_d corresponds to a diagram d with k wires on the left and l wires on the right—the left ports can be interpreted as the columns and the right ports as the rows of \mathcal{M}_d . The left j th port is connected to the i th port on the right through an r -weighted wire whenever coefficient $(\mathcal{M}_d)_{ij}$ is a nonzero scalar $r \in R$. When the $(\mathcal{M}_d)_{ij}$ entry is 0, they are disconnected. Since composition along a wire carries the multiplicative structure of R , we can simply draw the connection as a plain wire if $(\mathcal{M}_d)_{ij} = 1$. For example,

$$\text{The matrix } \mathcal{M}_d = \begin{pmatrix} a & 0 & 0 \\ b & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is represented by the following diagram:



Conversely, given a diagram, we recover the matrix by counting weighted paths from left to right ports. In terms of the isomorphisms $\llbracket \cdot \rrbracket_{\mathbb{R}}$ of Section IV, we have that $\llbracket d \rrbracket_{\mathbb{R}} = \{(\mathbf{a}, \mathcal{M}_d \mathbf{a} \mid \mathbf{a} \in \mathbb{R}^k)\}$.

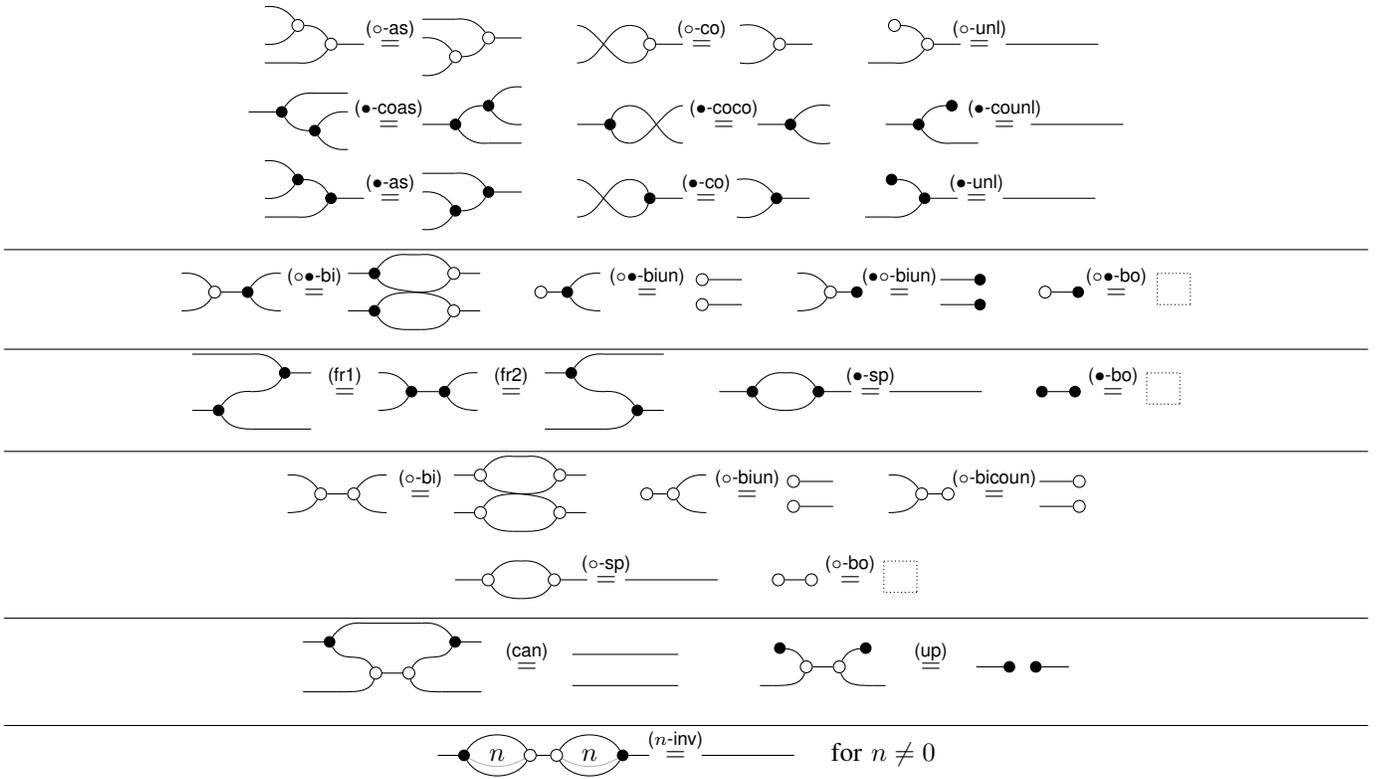
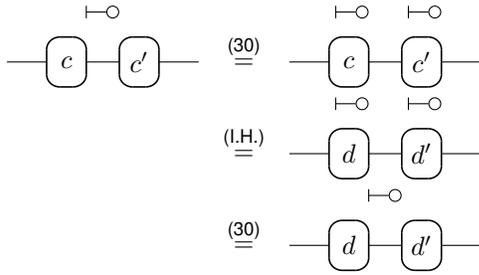
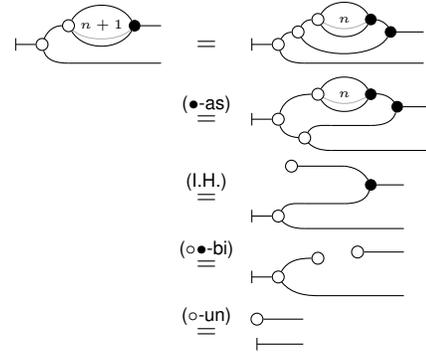


Fig. 7. Axioms of the algebra of resources (RC).

so that their composition satisfies the expected property:



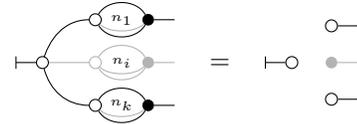
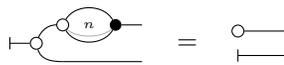
integer $n > 1$. Then,



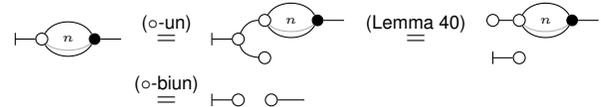
The case of the monoidal product is entirely analogous. \square

Lemma 41. For n_1, \dots, n_k such that $n_i > 1$ for all $i \leq k$:

Lemma 40. For all $n > 1$,

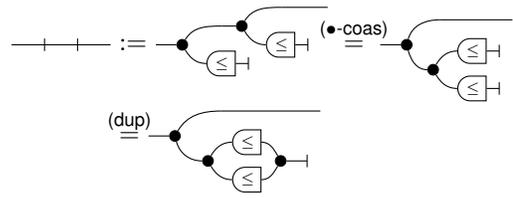
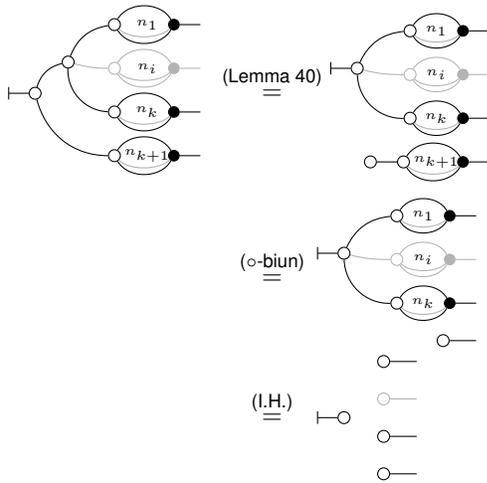


Proof. By induction on k . For $k = 1$ it is a direct consequence of Lemma 40:

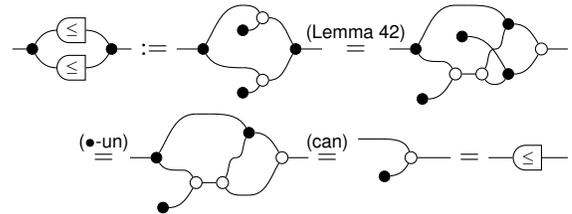


Proof. By induction on n . The base case is the axiom (cons). Assume that the statement of the lemma is true for some

Assume that the statement of the lemma holds for some k . *Proof.*
Then,



It is now enough to show that

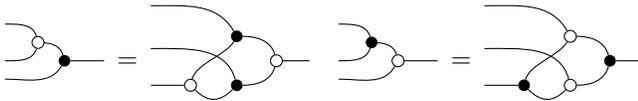


□

B. Missing Proofs for the Case Studies

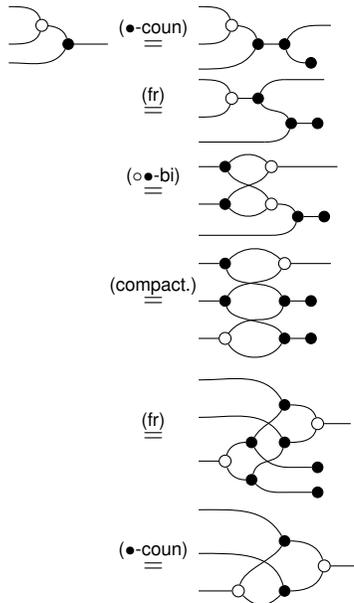
The following is a useful graphical result.

Lemma 42. In $\text{IH}_{\mathbb{R}}$ and ARC , we have



□

Proof. We prove only one of the equalities.



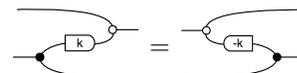
Proposition 44. $\mathcal{I}(-) : \text{ECirc} \rightarrow \text{AffRel}_{\mathbb{R}}$ is a symmetric strict monoidal functor.

Proof. It suffices to show that the mapping is well-defined, i.e. that the equations of ECirc are respected by the semantics: whenever $t = u$ in ECirc then $\mathcal{I}(t) = \mathcal{I}(u)$. The equations of special Frobenius algebras obviously hold, since we are working over $\text{AIH}_{\mathbb{R}}$. The final equation, expressing the bidirectionality of resistors,

$$\mathcal{I} \left(\begin{array}{c} \text{---} \\ | \\ \text{---} \text{---} \text{---} \\ | \\ \text{---} \\ \text{---} \end{array} \right) = \mathcal{I} \left(\begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} \right) \quad (31)$$

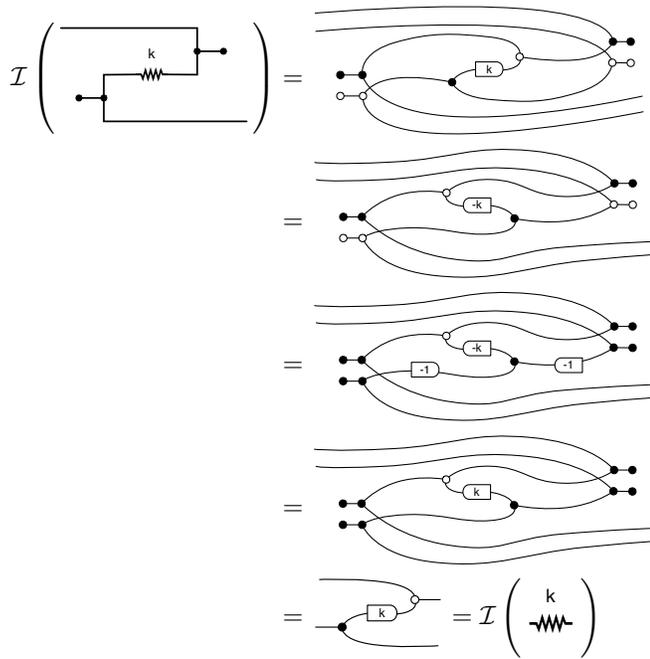
can be checked as follows. First, it is an easy exercise of graphical linear algebra to show that

□



Proposition 43. $\text{---} \text{---} \text{---} = \text{---} \text{---} \text{---}$ in ARC .

Then, we have

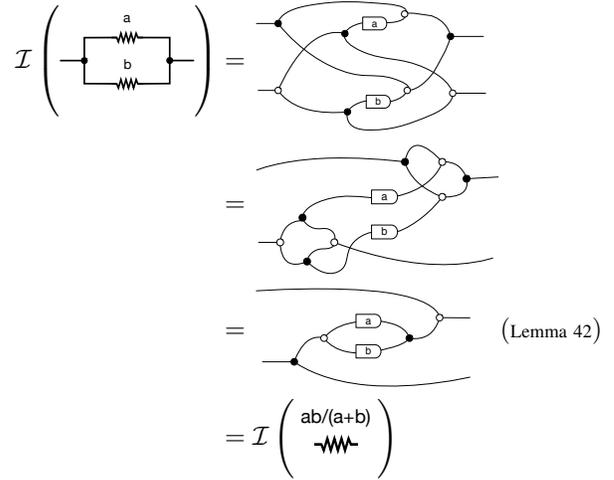
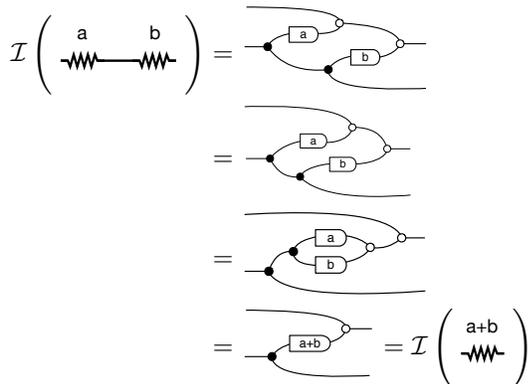


where the penultimate step follows from compactness (see Section II-C). \square

Lemma 45 (Properties of resistors).

$$\mathcal{I}\left(\begin{array}{c} a \\ \text{---} \\ b \end{array}\right) = \mathcal{I}\left(\begin{array}{c} a+b \\ \text{---} \end{array}\right) \quad \mathcal{I}\left(\begin{array}{c} a \\ \text{---} \\ b \end{array}\right) = \mathcal{I}\left(\begin{array}{c} ab/(a+b) \\ \text{---} \end{array}\right)$$

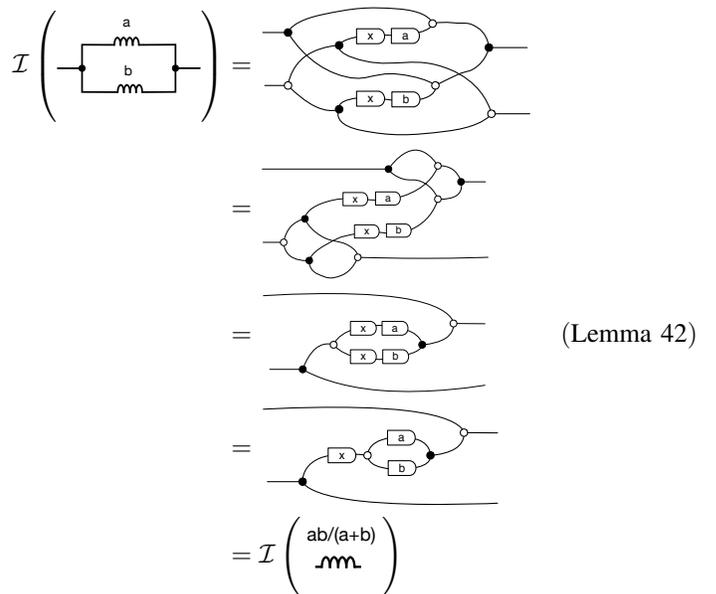
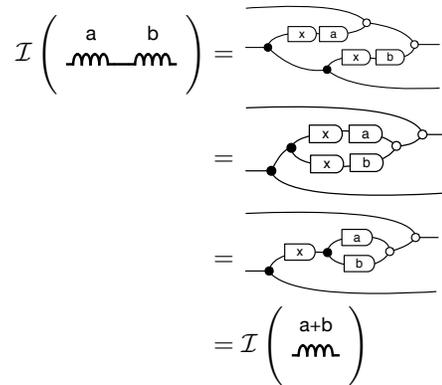
Proof.



Lemma 46. [Properties of inductors]

$$\mathcal{I}\left(\begin{array}{c} a \\ \text{---} \\ b \end{array}\right) = \mathcal{I}\left(\begin{array}{c} a+b \\ \text{---} \end{array}\right) \quad \mathcal{I}\left(\begin{array}{c} a \\ \text{---} \\ b \end{array}\right) = \mathcal{I}\left(\begin{array}{c} ab/(a+b) \\ \text{---} \end{array}\right)$$

Proof.



\square

\square

Lemma 47. [Properties of capacitors]

$$\mathcal{I}\left(\begin{array}{c} a \\ \parallel \\ b \end{array}\right) = \mathcal{I}\left(\begin{array}{c} ab/(a+b) \\ \parallel \end{array}\right) \quad \mathcal{I}\left(\begin{array}{c} a \\ \parallel \\ b \end{array}\right) = \mathcal{I}\left(\begin{array}{c} a+b \\ \parallel \end{array}\right)$$

Proof.

$$\begin{aligned} \mathcal{I}\left(\begin{array}{c} a \\ \parallel \\ b \end{array}\right) &= \begin{array}{c} \text{---} \bullet \text{---} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \text{---} \bullet \text{---} \end{array} \\ &= \begin{array}{c} \text{---} \bullet \text{---} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \text{---} \bullet \text{---} \end{array} \\ &= \begin{array}{c} \text{---} \bullet \text{---} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \text{---} \bullet \text{---} \end{array} \\ &= \mathcal{I}\left(\begin{array}{c} ab/(a+b) \\ \parallel \end{array}\right) \end{aligned}$$

$$\begin{aligned} \mathcal{I}\left(\begin{array}{c} a \\ \parallel \\ b \end{array}\right) &= \begin{array}{c} \text{---} \bullet \text{---} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \text{---} \bullet \text{---} \end{array} \\ &= \begin{array}{c} \text{---} \bullet \text{---} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \text{---} \bullet \text{---} \end{array} \\ &= \begin{array}{c} \text{---} \bullet \text{---} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \text{---} \bullet \text{---} \end{array} \quad (\text{Lemma 42}) \\ &= \begin{array}{c} \text{---} \bullet \text{---} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \text{---} \bullet \text{---} \end{array} \\ &= \mathcal{I}\left(\begin{array}{c} a+b \\ \parallel \end{array}\right) \end{aligned}$$

□