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THE QUANTUM SABINE LAW FOR RESONANCES IN TRANSMISSION PROBLEMS

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We prove a quantum version of the Sabine law from acoustics describing the location of resonances in transmission problems. This work extends the work of the author to a broader class of systems. Our main applications are to scattering by transparent obstacles, scattering by highly frequency-dependent delta potentials, and boundary stabilized wave equations. We give a sharp characterization of the resonance-free regions in terms of dynamical quantities. In particular, we relate the imaginary part of resonances, or generalized eigenvalues, to the chord lengths and reflectivity coefficients for the ray dynamics, thus proving a quantum version of the Sabine law.

1. Introduction

In this paper we study scattering in systems where the metric or potential has a singularity along an interface. Metric examples include scattering in media having sharp changes of index of refraction [Cardoso et al. 1999; 2001; Popov and Vodev 1999a], in dielectric microcavities [Cao and Wiersig 2015] and in fiber optic cables [Elliott and Gilmore 2002]. Schrödinger operators with a distributional potential along a hypersurface can be used to model quantum corrals, concert halls, and other thin barriers [Barr et al. 2010; Crommie et al. 1995]. Such potentials are also used to understand leaky quantum graphs [Exner 2008].

MSC2010: 35P20, 35P25.
Keywords: transmission, resonances, boundary integral operators, transparent, scattering.

Mathematically, an abrupt change in the index of refraction corresponds to a discontinuity in the metric along a hypersurface. Scattering in such situations has been studied in [Bellassoued 2003; Cardoso et al. 1999; 2001; Popov and Vodev 1999a; 1999b], while scattering by certain distributional potentials has been studied in [Galkowski 2014; 2016; Galkowski and Smith 2015]. These types of problems have also been studied from the point of view of propagation of singularities [Melrose and Taylor 2010; Miller 2000; Weiss and Hagedorn 1985] and quantum chaos [Jakobson et al. 2015].

For a Schrödinger operator, P, on $L^2(\mathbb{R}^d)$ (d odd) it is often possible to prove that solutions, u, to

$$(\partial_t^2 + P)u = 0$$

have expansions roughly of the form

$$u \sim \sum_{\lambda\in\mathrm{Res}} e^{-it\lambda} u_\lambda, \tag{1}$$

where Res is the set of *scattering resonances* of P. Thus, the real and (negative) imaginary parts of a scattering resonance correspond respectively to the frequency and decay rate of the associated resonance state, $e^{-it\lambda}u_\lambda$. This expression is similar to the expansion in terms of eigenvalues that one obtains when solving the wave equation on a compact manifold. Hence, for leaky systems, scattering resonances play the role of eigenvalues in the closed setting.

To get a quantitative heuristic for the decay of waves (the imaginary part of resonances), we imagine that the interface for our problem occurs at $\partial\Omega$ for some $\Omega \Subset \mathbb{R}^d$. We then think of solving the wave equation

$$(\partial_t^2 + P)u = 0, \quad u|_{t=0} = u_0, \;\; u_t|_{t=0} = 0,$$

with initial data u_0 a wave packet (that is, a function localized in frequency and space up to the scale allowed by the uncertainty principle) localized at position $x_0 \in \Omega$ and frequency $\xi_0 \in S^{d-1}$. We also assume that P creates waves with speed c. The solution, u, then propagates along the billiard flow starting from (x_0, ξ_0). At each intersection of the billiard flow with the boundary, the amplitude inside of Ω will decay by a factor, R, depending on the point and direction of intersection. Suppose that the billiard flow from (x_0, ξ_0) intersects the boundary at $(x_n, \xi_n) \in \partial\Omega \times S^{d-1}$, $n > 0$. Let $l_n = |x_{n+1} - x_n|$ be the distance between two consecutive intersections with the boundary (see Figure 1). Then the amplitude of the wave decays by a factor $\prod_{i=1}^n R_i$ in time $\sum_{i=1}^n c^{-1} l_i$, where $R_i = R(x_i, \xi_i)$. The energy scales as amplitude squared and since the imaginary part of a resonance gives the exponential decay rate of L^2 norm, this leads us to the heuristic that resonances should occur at

$$\operatorname{Im}\lambda = \frac{\overline{\log|R|^2}}{2c^{-1}\bar{l}}, \tag{2}$$

where the map $\bar{\cdot}$ is defined by $\bar{f} = \frac{1}{N}\sum_{i=1}^N f_i$. In the early 1900s, Sabine [1922] postulated that the decay rate of acoustic waves in a region with leaky walls is determined by the average decay over billiards trajectories. The expression (2) provides a precise statement of Sabine's idea and, because resonances are a spectral quantity, we refer to such an expression as a quantum Sabine law. We will show in Theorem 1.11 that such a Sabine law holds for many different types of transmission problems.

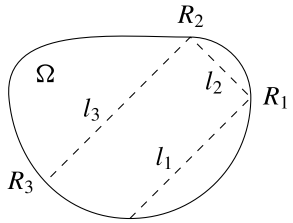


Figure 1. Path of a wave packet along with the lengths between each intersection (l_i) and the reflection coefficient at each point of intersection with the boundary (R_i). After each reflection with the boundary, the amplitude of the wave packet inside Ω decays by a factor of R_i. If the speed of the wave is c, the time between reflections is given by $c^{-1}l_i$.

Although the appearance of scattering resonances in (1) is intuitive, a more mathematically useful definition of a scattering resonance is as a pole of the meromorphic continuation of

$$(P-\lambda^2)^{-1}$$

from $\operatorname{Im}\lambda \gg 1$. This description allows us to show that the existence of a scattering resonance at λ corresponds to the existence of a nonzero λ-outgoing solution to

$$(P-\lambda^2)u = 0.$$

By λ-outgoing we mean that there exists $g \in L^2_{\text{comp}}(\mathbb{R}^d)$ and $M \geq 0$ such that

$$u(x) = (R_0(\lambda)g)(x), \quad |x| \geq M.$$

Here, $R_0(\lambda)$ is the meromorphic continuation of $(-\Delta-\lambda^2)^{-1}$ from $\operatorname{Im}\lambda \gg 1$ as an operator $R_0(\lambda) : L^2_{\text{comp}}(\mathbb{R}^d) \to L^2_{\text{loc}}(\mathbb{R}^d)$. (For a more complete description of mathematical scattering and further references, see [Dyatlov and Zworski 2018].)

We start by considering a few applications of our main theorem (see Theorem 1.11).

1A. ***Transparent obstacles.*** Our first application is to scattering by a transparent obstacle, that is, an obstacle with different refractive index than the ambient medium. In particular, let $\Omega \Subset \mathbb{R}^d$ be strictly convex with smooth boundary, $c \in \mathbb{R}_+ \setminus \{1\}$ be the speed of light in Ω, and $\aleph > 0$ be a coupling parameter. In [Cardoso et al. 1999], Cardoso, Popov, and Vodev show that the set of scattering resonances in this setting is given by λ such that there is a nonzero solution to

$$\begin{cases} (-c^2\Delta-\lambda^2)u_1 = 0 & \text{in } \Omega, \\ (-\Delta-\lambda^2)u_2 = 0 & \text{in } \mathbb{R}^d \setminus \bar{\Omega}, \\ u_1 = u_2 & \text{on } \partial\Omega, \\ \partial_\nu u_1 - \aleph\partial_\nu u_2 = 0 & \text{on } \partial\Omega, \\ u_2 \text{ is } \lambda\text{-outgoing.} \end{cases} \tag{3}$$

We denote the set of such λ by Λ. Here, ν denotes the outward unit normal to $\partial\Omega$.

Let $T^*\partial\Omega$ be the cotangent bundle to $\partial\Omega$ and $B^*\partial\Omega$ denote the coball bundle of $\partial\Omega$. Let $\pi_x : T^*\partial\Omega \to \partial\Omega$ be the projection to the base. Then define $r, l_N, r_N \in C^\infty(B^*\partial\Omega)$ and

$$l \in C^\infty\big(T^*\partial\Omega \times T^*\partial\Omega \setminus \{(x,\xi',x,\eta') \in T^*\partial\Omega \times T^*\partial\Omega\}\big) \cap C(T^*\partial\Omega \times T^*\partial\Omega)$$

by

$$\begin{aligned} r(x',\xi') &:= \frac{\sqrt{1-|\xi'|_g^2} - \aleph\sqrt{c^2-|\xi'|_g^2}}{\aleph\sqrt{c^2-|\xi'|_g^2}+\sqrt{1-|\xi'|_g^2}}, & r_N(q) &:= \frac{\sum_{j=1}^N \log|r(\beta^j(q))|^2}{N}, \\ l(q_1,q_2) &:= |\pi_x(q_1)-\pi_x(q_2)|, & l_N(q) &:= \frac{\sum_{j=1}^N l(\beta^{j-1}(q),\beta^j(q))}{N}, \end{aligned} \tag{4}$$

where $\beta : B^*\partial\Omega \to B^*\partial\Omega$ denotes the billiard ball map (see Section 5) and $|\xi'|_g$ denotes the norm induced on the fibers of $T^*\partial\Omega$ by the metric on $\mathbb{R}^d$. Then r is the reflectivity for the transparent obstacle problem. Note that we take the branch of the square root such that $\sqrt{-1} = i$ and place the branch cut on the negative imaginary axis.

Remark 1.1. • We will use ξ' to denote coordinates in the fiber of $T^*\partial\Omega$ and q to denote points in $T^*\partial\Omega$ throughout this paper.

• Note that the log in the definition of r_N appears because we measure exponential rates of decay and the reflection coefficient acts by multiplication.

Theorem 1.2. *Let* $\Omega \Subset \mathbb{R}^d$ *be strictly convex with smooth boundary and suppose that* $0 < c \neq 1$, $\aleph > 0$. *Then for all* $M, \epsilon > 0$ *there exists* $\lambda_0 > 0$ *such that for* $\lambda \in \Lambda$ *with* $\operatorname{Re}\lambda \geq \lambda_0$ *and* $\operatorname{Im}\lambda \geq -M\log\operatorname{Re}\lambda$,

$$\sup_{N>0}\inf_{|\xi'|_g\leq 1}\frac{r_N}{2c^{-1}l_N} - \epsilon \leq \operatorname{Im}\lambda \leq \inf_{N>0}\sup_{|\xi'|_g\leq 1}\frac{r_N}{2c^{-1}l_N} + \epsilon.$$

Moreover, for every $\aleph$, c *as above, and* $K > 0$, *this bound is sharp in the region* $\operatorname{Im}\lambda \geq -K$ *when* $\Omega = B(0,1) \subset \mathbb{R}^2$.

Remark 1.3. • The lower bound in Theorem 1.2 is nontrivial, i.e., $|r(x',\xi')| > 0$, if either $c < 1$ and $\aleph < c^{-1}$, or $c > 1$ and $\aleph > c^{-1}$. This corresponds to *transverse electric waves* (TE). The opposite case, when there is no lower bound, corresponds to *transverse magnetic waves* (TM). In the TM case, the angle at which $r(x',\xi') = 0$ is called the *Brewster angle* [Ida 2000, Chapter 13]. At this angle, there is complete transmission of the wave in the ray dynamics picture.

• The upper bound in Theorem 1.2 is nontrivial if $c > 1$. When $c < 1$, Popov and Vodev [1999b] show that the presence of total internal reflection (see Figure 2) produces resonances $\{\lambda_n\}_{n=1}^\infty$ with $\operatorname{Re}\lambda_n \to \infty$ and $\operatorname{Im}\lambda_n = O((\operatorname{Re}\lambda_n)^{-\infty})$.

• The bounds for resonances given in Theorem 1.2 match our prediction (2). (See also Figure 3 for numerically computed resonances in the transmission problem.)

Theorem 1.2 improves upon the results of Cardoso, Popov and Vodev [Cardoso et al. 1999; 2001] by giving sharp estimates on the sizes of the resonance-free regions, as well as expanding the range of parameters, $\aleph$, for which we have only a band of resonances.

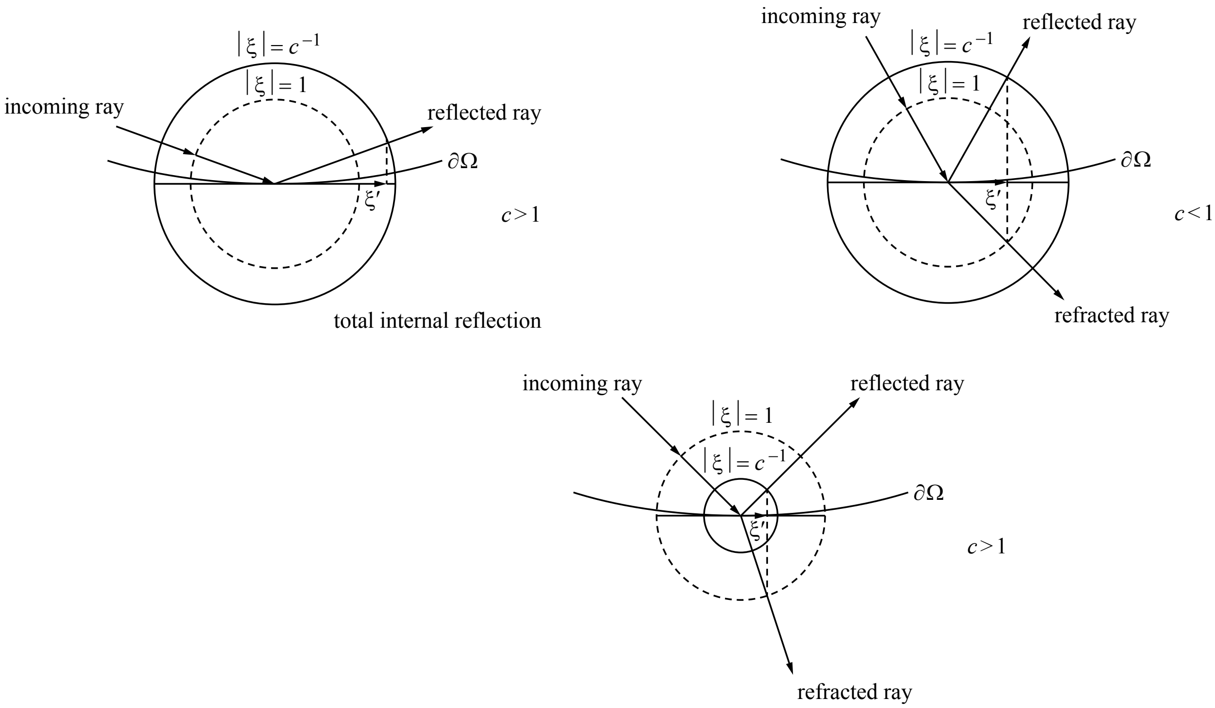


Figure 2. Geometry of reflection and refraction at the boundary of an interface between a medium with speed of light c and one with speed of light 1. Total internal reflection occurs when the incoming ray does not project onto the ball of radius 1 in the ξ'-variable.

1B. ***Highly frequency-dependent Delta potentials.*** Let $\Psi^\infty(\partial\Omega)$ denote the set of semiclassical pseudodifferential operators of all orders whose seminorms are bounded by a constant independent of h so that $h^{-N}\Psi^\infty(\partial\Omega)$ denotes those whose seminorms are bounded by h^{-N} (see Section 2 for more details).

We next consider operators of the form

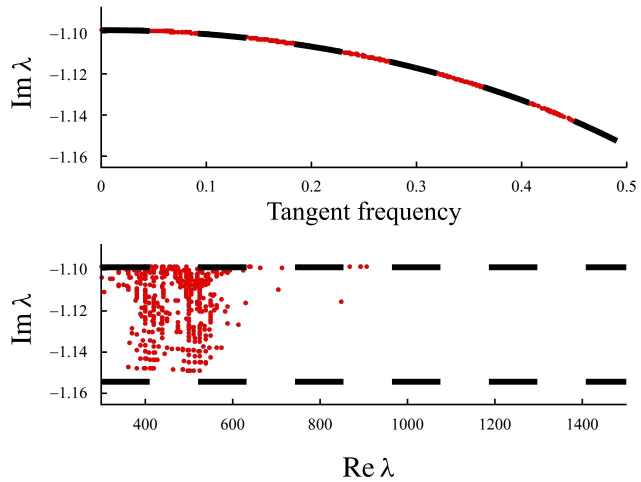
$$-h^2\Delta + h(h\delta_{\partial\Omega} \otimes V) =: -h^2\Delta_{\partial\Omega,\delta}, \tag{5}$$

where $h \in (0, 1]$ is a semiclassical parameter that should be thought of as the wavenumber (i.e., the inverse of the frequency), $V \in h^{-N}\Psi^\infty(\partial\Omega)$, and for $u, w \in C_c^\infty(\mathbb{R}^d)$

$$\langle(\delta_{\partial\Omega} \otimes V)u, w\rangle := \int_{\partial\Omega} (Vu)(x)w(x)\, d\sigma(x) \tag{6}$$

and σ is the surface measure of $\partial\Omega$. (See [Galkowski and Smith 2015, Section 2.1] for the formal definition of this operator.) These operators are used as models for quantum corrals [Barr et al. 2010; Crommie et al. 1995], as well as concert halls, leaky quantum graphs [Exner 2008] and other thin barriers.

In a typical physical system, the interaction between a potential and a wave depends on the frequency of the interacting wave. Therefore, we are motivated to consider h-dependent potentials V. Moreover,



	decay rate	tangential frequency
quantum quantity	$\operatorname{Im}\lambda$	$n/(\operatorname{Re}\lambda)$
classical quantity	$cr_1/(2l_1)$	$c\xi'$

Figure 3. Numerically computed resonances for the transparent obstacle problem with $c = 2$ and $\aleph = 1$ when $\Omega = B(0, 1) \subset \mathbb{R}^2$. (See Figure 10 for other values of c and $\aleph$.) In this case, we expand the solutions to (3) as $u_i(r, \theta) = \sum_n u_{i,n}(r)e^{in\theta}$ and solve for some of the resonances with $\operatorname{Re}\lambda \sim 500$. In the lower graph, the red circles show $\operatorname{Im}\lambda$ vs. $\operatorname{Re}\lambda$. The dashed black lines show the upper and lower bounds for $\operatorname{Im}\lambda$ (since $\aleph$ is in the TE range with have both an upper and lower bound) from Theorem 1.2. Notice that by orthogonality of $e^{in\theta}$ and $e^{im\theta}$ for $m \neq n$, the pair $(u_{1,n}e^{in\theta}, u_{2,n}e^{in\theta})$ satisfies (3). In the top graph, the red circles show $\operatorname{Im}\lambda$ vs. $n/\operatorname{Re}\lambda$ for such pairs. The dashed curve shows a plot of $(cr_1/(2l_1))(c\xi')$, the decay rate predicted for a billiard's trajectory traveling with scaled tangent frequency $c\xi'$. See the table for the relationship between the points $(\operatorname{Im}\lambda, n/\operatorname{Re}\lambda)$ and $(cr_1/(2l_1)(c\xi'), c\xi')$ predicted by the quantum Sabine law.

if one considers the delta interaction in one dimension

$$-\Delta + \delta(x_1) \otimes 1$$

and rescales to $y = hx$, we obtain

$$-h^2\Delta_y^2 + \delta(y_1/h) \otimes 1 = -h^2\partial_y^2 + h\delta(y_1) \otimes 1, \tag{7}$$

which corresponds to $V = h^{-1}$ in (5). The operator (7) describes the *quantum point* interaction [Miller 2000].

Another motivation for highly frequency-dependent delta potentials is the wave equation

$$\begin{cases} (\partial_t^2 - \Delta + i(\delta_{\partial\Omega} \otimes (\langle a(x), \partial_x\rangle + a_0(x)\partial_t))u = F & \text{in } \mathbb{R}^d, \\ F \in L^2_{\text{comp}}((0,\infty)_t \times \mathbb{R}^d), \qquad u = 0 & \text{on } t < 0, \end{cases}$$

where $a, a_0 \in C^\infty(\partial\Omega; \mathbb{R})$, and the tensor product acts as in (6). Then, taking the time Fourier transform

$$\mathcal{F}_{t\to\lambda}u(x,\lambda) := \int_0^\infty e^{it\lambda}u(x,t)\,dt,$$

gives with $\lambda = z/h$

$$(-h^2\Delta - z^2 + z(h\delta_{\partial\Omega}\otimes(\langle z^{-1}a, hD_x\rangle + a_0))\mathcal{F}_{t\to z/h}u = \mathcal{F}_{t\to z/h}F.$$

Remark 1.4. Note that we have switched the usual convention for the Fourier transform in our definition of $\mathcal{F}_{t\to\lambda}$ so that the integral converges absolutely for $\operatorname{Im}\lambda > 0$.

In [Galkowski and Smith 2015], Smith and the author show that the set of scattering resonances, $\Lambda(h)$, is equal to the set of z such that there is a nonzero solution to

$$\begin{cases} (-h^2\Delta - z^2)u_1 = 0 & \text{in } \Omega,\\ (-h^2\Delta - z^2)u_2 = 0 & \text{in } \mathbb{R}^d \setminus \overline{\Omega},\\ u_1 = u_2 & \text{on } \partial\Omega,\\ \partial_\nu u_1 - \partial_\nu u_2 + Vu_1 = 0 & \text{on } \partial\Omega,\\ u_2 \text{ is } z/h \text{ outgoing.} \end{cases}$$

Define

$$\Lambda_{\log}(h) := \{z \in \Lambda(h) : z \in [1 - Ch, 1 + Ch] + i[-Mh\log h^{-1}, 0]\}. \tag{8}$$

For $V \in h^{-N}\Psi^\infty(\partial\Omega)$ with real-valued symbol, $\sigma(V)$, the reflectivity, $r \in C^\infty(B^*\partial\Omega)$, is given by

$$r(x', \xi') := \frac{h\sigma(V)}{2i\sqrt{1 - |\xi'|_g^2} - h\sigma(V)},$$

with $r_N(q)$ and $l_N(q)$ as in (4). For a more general definition of r see (18) and for r_N see (21).

Let $\Psi^m(\partial\Omega)$ denote the set of semiclassical pseudodifferential operators of order m (see Section 2) and $\Lambda_{\log}(h)$ be as in (8). Next, let

$$\begin{gathered} Ai(s) := \frac{1}{2\pi}\int e^{i(st+t^3/3)}\,dt, \quad A_-(s) := Ai(e^{2\pi i/3}s), \quad \Phi_-(s) := A_-'(s)/A_-(s), \\ 0 > \zeta_1 > \zeta_2 > \dots \text{ be the zeros of } Ai(s). \end{gathered} \tag{9}$$

Finally, let $Q(x, \xi') \in C^\infty(T^*\partial\Omega)$ be the symbol of the second fundamental form to $\partial\Omega$. Then we have:

Theorem 1.5. *Let $\Omega \Subset \mathbb{R}^d$ be strictly convex with smooth boundary, $\alpha \geq -1$, and suppose that $V \in h^\alpha\Psi^1(\partial\Omega)$ is self-adjoint with $\sigma(V) \geq 0$ and $\sigma(V) > c > 0$ in a neighborhood of $\{|\xi'|_g = 1\}$.*

(1) *Suppose that $\alpha > -\frac{5}{6}$. Then for all $\epsilon > 0$, $N_1 > 0$ there exist $\epsilon_1 > 0$, $h_0 > 0$ such that for $0 < h < h_0$*

$$\Lambda_{\log}(h) \subset \left\{\frac{\operatorname{Im} z}{h} \leq \inf_{N\leq N_1}\sup_{|\xi'|<1-\epsilon_1}\frac{r_N}{2l_N} + \epsilon\right\}.$$

(2) *Suppose that $-\frac{5}{6} \geq \alpha \geq -1$. Then for all $\epsilon > 0$, $M > 0$, there exists $h_0 > 0$ such that for $0 < h < h_0$*

$$\Lambda_{\log}(h) \subset \bigcup_{j=1}^{M}\left\{B_{\min} - \epsilon \leq \frac{h^{1/2}\operatorname{Im} z}{\operatorname{Im}\Phi_-(\zeta_j)} \leq B_{\max} + \epsilon\right\} \cup \left\{\frac{h^{1/2}\operatorname{Im} z}{\operatorname{Im}\Phi_-(\zeta_{M+1})} \geq B_{\min} - \epsilon\right\},$$

where

$$B_{\max} := \sup_{|\xi'|_g=1} \frac{2^{1/2} Q(x,\xi')^{1/2}}{|\sigma(V)(x,\xi')|^2}, \quad B_{\min} := \inf_{|\xi'|_g=1} \frac{2^{1/2} Q(x,\xi')^{1/2}}{|\sigma(V)(x,\xi')|^2}.$$

Moreover, these estimates are sharp in the case of $\Omega = B(0,1) \subset \mathbb{R}^2$ *with* $V \equiv 1$.

Theorem 1.5 verifies several conjectures from [Galkowski 2014] and generalizes the results from [Galkowski 2016] to arbitrary convex domains. It also provides a second general class of examples that may have resonances with $-\operatorname{Im} z/h \sim ch^{\gamma}$ for some $\gamma > 0$, that is, resonances converging to the real axis at a fixed polynomial rate, but no faster. Compared to the work in [Galkowski 2014, Theorem 5.4], Theorem 1.5 allows for potentials that depend more strongly on frequency. When the dependence is strong enough $\left(\alpha \le -\frac{5}{6}\right)$, the new phenomenon of a band structure appears. (See Figure 4 for a schematic of the results of the theorem.)

Remark 1.6. • Under the pinching condition,

$$\frac{B_{\min}}{B_{\max}} > \frac{\operatorname{Im} \Phi_-(\zeta_j)}{\operatorname{Im} \Phi_-(\zeta_{j+1})},$$

there is a gap between the j-th and $(j+1)$-th band of resonances given by Theorem 1.5 for $\alpha \le -\frac{5}{6}$. For

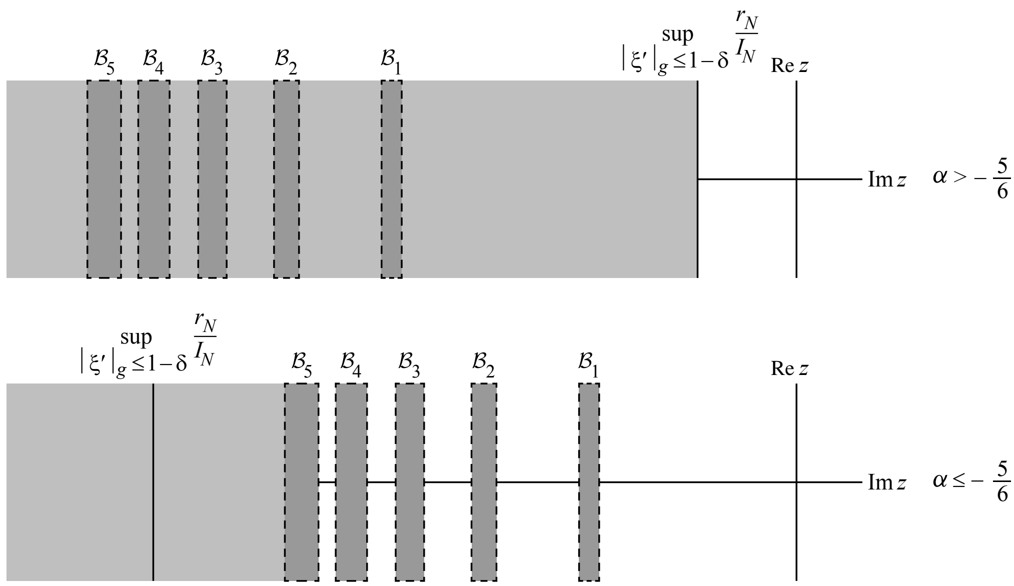


Figure 4. Schematic representation of the resonance-free regions from Theorem 1.5 for $\alpha > -\frac{5}{6}$ on the top and $\alpha \le -\frac{5}{6}$ on the bottom. Resonances lie in the dark gray bands, $\mathcal{B}_j := \{B_{\min} - \epsilon \le (h^{2/3} \operatorname{Im} z)/(\operatorname{Im} \Phi_-(\zeta_j)) \le B_{\max} + \epsilon\}$, or the light gray shaded region, but not in the white regions. Note that the bands start to group closer together as they go deeper into the complex plane. Thus, there will be only a finite number of bands if $B_{\max}/B_{\min} \ne 1$. See also Figures 6 and 7 for numerically computed resonances in the case of the disk where $B_{\max}/B_{\min} = 1$ when $V \equiv h^{\alpha}$.

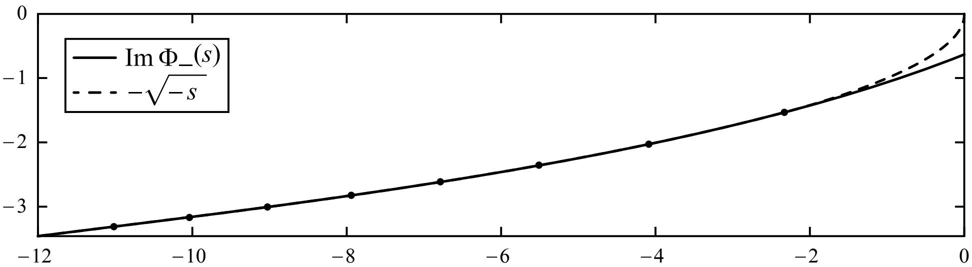


Figure 5. Graphs of $\operatorname{Im}\Phi_-$ (solid) and $-\sqrt{-s}$ (dashed). The black dots are placed at $(\zeta_j, \operatorname{Im}\Phi_-(\zeta_j))$.

a plot of $\operatorname{Im}\Phi_-(s)$, see Figure 5.

- To see that the resonance bands in Theorem 1.5 for $\alpha \le -\frac{1}{2}$ agree with those in [Galkowski 2016], observe that

$$\operatorname{Im}\frac{A'_-(\zeta_j)}{A_-(\zeta_j)} = \operatorname{Im}\frac{2\pi Ai'(\zeta_j)A'_-(\zeta_j)}{e^{5\pi i/6}} = -\frac{2\pi Ai'(\zeta_j)Ai'(\zeta_j)}{2|A_-(\zeta_j)Ai'(\zeta_j)|^3(2\pi)^3} = -\frac{1}{8\pi^2|A_-(\zeta_j)^3 Ai'(\zeta_j)|}.$$

1C. ***Boundary stabilization problem.*** Our final application of Theorem 1.11 is to a boundary stabilized wave equation

$$\begin{cases} (\partial_t^2 - \Delta)u = F & \text{in } \Omega, \\ \partial_\nu u + a(x)\partial_t u = 0 & \text{on } \partial\Omega, \\ u \equiv 0 & \text{on } t < -1, \\ F \in L^2_{\text{comp}}((0,\infty)_t \times \Omega), \end{cases} \tag{10}$$

with $0 \le a(x) \in C^\infty(\partial\Omega;\mathbb{R})$. It is not hard to see that the energy

$$E(t) := \tfrac{1}{2}(\|\partial_t u\|^2 + \|\nabla u\|^2)$$

for the corresponding initial value problem is nonincreasing. The study of (10) has a long history; for instance, see [Bardos et al. 1992], where Bardos, Lebeau, and Rauch give nearly sharp conditions on a to guarantee exponential decay of the energy.

Here, we impose the strongly dissipative condition $0 < a_0 \le a$ and study the asymptotic ($|\operatorname{Re}\lambda| \gg 1$) spectral gap for the corresponding stationary problem. That is, taking the Fourier transform in time, we study

$$\begin{cases} (-\Delta - \lambda^2)\mathcal{F}_{t\to\lambda}u = \mathcal{F}_{t\to\lambda}F & \text{in } \Omega, \\ (\partial_\nu - i\lambda a(x))\mathcal{F}_{t\to\lambda}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{11}$$

Cardoso and Vodev [2010] showed the existence of a spectral gap in a much more general, but still strongly dissipative, situation. Here, we give estimates on the size of the gap. Let Λ denote the set of λ such that (11) has a nonzero solution. The reflectivity, $r \in C^\infty(B^*\partial\Omega)$, for this problem is given by

$$r(x',\xi') := \frac{\sqrt{1-|\xi'|_g^2} - a(x')}{a(x') + \sqrt{1-|\xi'|_g^2}}$$

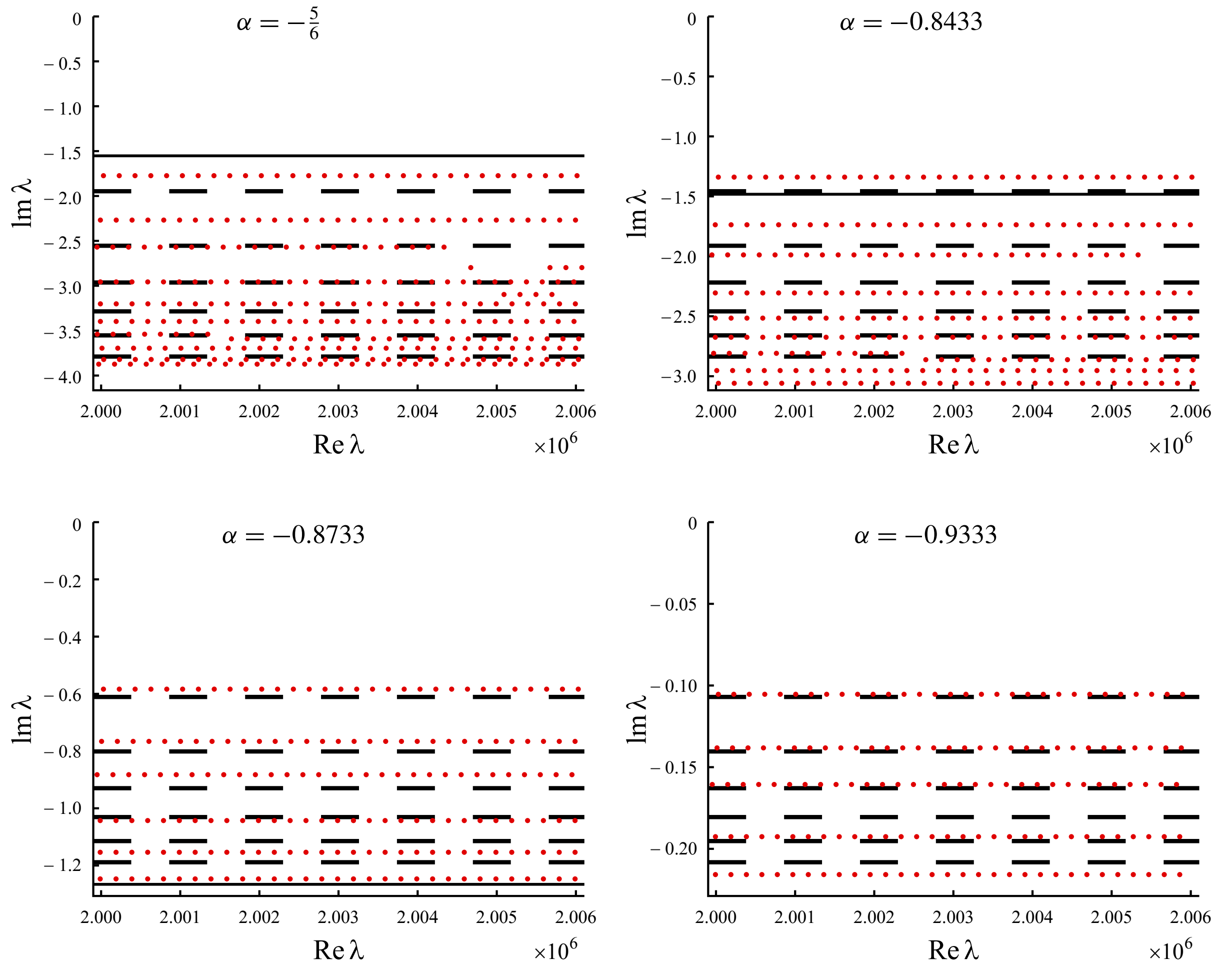


Figure 6. Resonances for the delta potential on the circle with $\operatorname{Re}\lambda \sim 10^6$, $V \equiv (\operatorname{Re}\lambda)^{-\alpha}$ and several α. The plots show $\operatorname{Im}\lambda$ vs. $\operatorname{Re}\lambda$ in each case. The solid black line shows the (logarithmic) bound for resonances coming from nonglancing trajectories and the dashed black lines show the first few (polynomial) bands of resonances from near glancing trajectories. Since the solid black line is above the dashed black lines at $\alpha = -\frac{5}{6}$, it is necessary to go to still larger $\operatorname{Re}\lambda$ to see the transition to resonances with fixed-size imaginary parts. However, at $\alpha < -\frac{5}{6}$, we start to see better agreement with the bands of resonances predicted in Theorem 1.5.

and l_N, r_N as in (4).

Theorem 1.7. *Let $\Omega \Subset \mathbb{R}^d$ be strictly convex with smooth boundary and $a(x) \geq a_0 > 0$. Then for all $\epsilon, M > 0$ there exist $\lambda_0 > 0$ such that for $\lambda \in \Lambda$ with $|\operatorname{Re}\lambda| > \lambda_0$ and $\operatorname{Im}\lambda \geq -M\log|\operatorname{Re}\lambda|$,*

$$\sup_{N>0}\inf_{|\xi'|_g\leq 1}\frac{r_N}{2l_N} - \epsilon \leq \operatorname{Im}\lambda \leq \inf_{N>0}\sup_{|\xi'|_g\leq 1}\frac{r_N}{2l_N} + \epsilon. \tag{12}$$

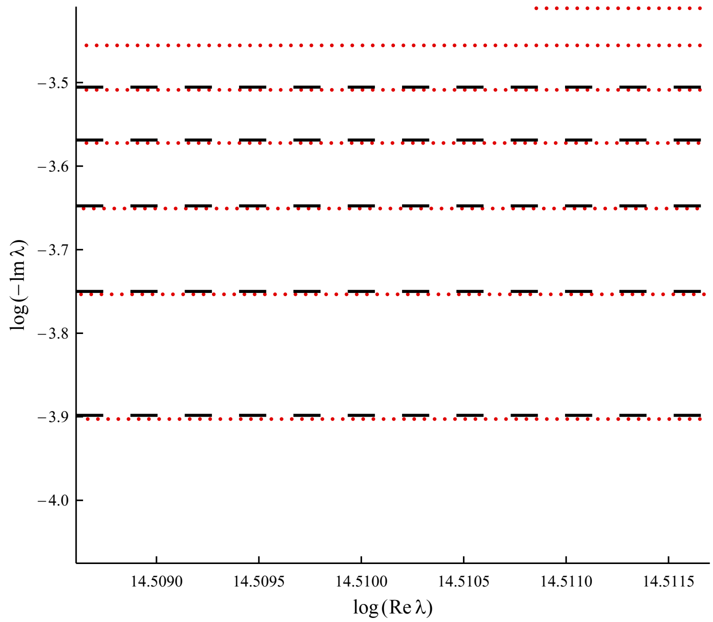


Figure 7. Plot of resonances for the delta potential on the disk with $V \equiv \operatorname{Re}\lambda$. In particular, we show $\log(\operatorname{Re}\lambda)$ vs. $\log(-\operatorname{Im}\lambda)$ for $\operatorname{Re}\lambda \sim 10^6$. The bands predicted by Theorem 1.5 are shown by the black dashed lines.

Theorem 1.7 can also be obtained from the results of [Koch and Tataru 1995]. Indeed, the result contained there actually implies a stronger estimate than (12) in the case of (11). We include this application to give a new proof of those results in this special case and to show that our analysis may be applied even to nontransmission problems. Moreover, note that the operator $a\partial_t$ can be replaced by a much more general pseudodifferential operator and our methods still apply.

1D. ***The general setup: a generalized boundary damped wave equation.*** Theorems 1.2, 1.5, and 1.7 are a consequence of analysis of the boundary damped problem

$$\begin{cases} (-h^2\Delta - z^2)u = w & \text{in } \Omega, \\ h\partial_\nu u + Bu = hv & \text{on } \partial\Omega, \end{cases} \tag{13}$$

with $\operatorname{Re} z \sim 1$. Here, the operator B plays the role of damping waves upon interaction with the boundary and encodes the interaction with the exterior of Ω in the case of scattering problems.

Let $N_2(z/h)$ denote the outgoing Dirichlet-to-Neumann map for $\mathbb{R}^d \setminus \overline{\Omega}$, that is, the map given by $C^\infty(\Omega) \ni f \mapsto -\partial_\nu u$, where u solves

$$\begin{cases} (-h^2\Delta - z^2)u = 0 & \text{in } \mathbb{R}^d \setminus \overline{\Omega}, \\ u|_{\partial\Omega} = f, \\ u \text{ is } z/h \text{ outgoing.} \end{cases}$$

We assume that $B = hN_2(z/h) + hV(z)$, where V is in a certain second microlocal class of pseudodifferential operators which we specify later.

Remark 1.8. By replacing $\tilde{h} = hE$, $B(h) = EB(\tilde{h}/E)$, and $\tilde{z} = Ez$ we may work with $\operatorname{Re} z \sim E$. Notice that $z/h = \tilde{z}/\tilde{h}$ so operators that are functions of z/h do not change under this rescaling.

We first introduce some notation. Let

$$D_M(h) := [1-h, 1+h] + i[-Mh \log h^{-1}, Mh \log h^{-1}].$$

Let $\gamma : H^s(\mathbb{R}^d) \to H^{s-1/2}(\partial\Omega),\ s > \frac{1}{2}$, be the restriction operator. Then the *single-layer operator* is given by

$$G(z/h) := \gamma R_0(z/h)\gamma^*.$$

Recall that $R_0(\lambda)$ is the meromorphic continuation of $(-\Delta - \lambda^2)^{-1}$. From [Galkowski 2014, Lemma 4.25; Hassell and Zelditch 2004, Proposition 4.1] (see also Lemma 7.3), we have

$$G(z/h) = G_\Delta(z/h) + G_B(z/h) + G_g(z/h) + O_{\mathcal{D}'(\partial\Omega)\to C^\infty(\partial\Omega)}(h^\infty),$$

where G_Δ is pseudodifferential, G_B is a semiclassical Fourier integral operator associated to the billiard ball map (see Section 2 for the definition of semiclassical Fourier integral operators), and G_g is microlocalized near $|\xi'|_g = 1$. Let $m \geq 0$ and $\Psi^{0,m}_{2/3}(|\xi'|_g = E')$ denote the set of pseudodifferential operators that are second microlocalized near $|\xi'|_g = E$ (see Section 4).

We now introduce assumptions on V. Suppose $a_1 \in \mathbb{R},\ \alpha \geq -1,\ E' \in \mathbb{R} \setminus \{1\},\ \delta > 0,\ M, M_1, M_2 > 0,$ $0 < \epsilon < \frac{1}{2}$. Let $\langle \cdot \rangle \in C^\infty(T^*\partial\Omega)$ be given by $\langle \xi' \rangle := (1 + |\xi'|_g^2)^{1/2}$. We assume that

$$V = a_1 N_2(z/h) + V_1, \quad V_1 \in h^\alpha \Psi^{0,m}_{2/3}(|\xi'|_g = E'), \quad V \text{ is elliptic on } ||\xi'|_g - 1| < \delta, \tag{14}$$

$$\begin{aligned} \left|1 + \frac{h\sigma(V)}{2\sqrt{|\xi'|_g^2 - 1}}\right| &\geq \delta\left(\left\langle \frac{h^{1+\alpha}}{\sqrt{|\xi'|_g^2 - 1}} \right\rangle + \langle \xi' \rangle^{m-1}\right), && |\xi'|_g > 1 + M_1 h^{2/3}, \\ \left|1 + \frac{hi\sigma(V)}{2\sqrt{1 - |\xi'|_g^2}}\right| &\geq \delta\left\langle \frac{h^{1+\alpha}}{\sqrt{1 - |\xi'|_g^2}} \right\rangle, && |\xi'|_g \leq 1 - h^\epsilon, \end{aligned} \tag{15}$$

$$V(z) \text{ is an analytic family of operators for } z \in D_M(h), \tag{16}$$

$$\log\left(1 + \frac{h\sigma(V)}{\sqrt{|\xi'|_g^2 - 1}}\right) \text{ exists and is smooth on } T^*\partial\Omega \setminus \{|\xi'|_g \leq M_2\}. \tag{17}$$

We say that $\mathcal{AV}(a_1, \alpha, E', m, \delta, M, M_1, M_2, \epsilon)$ holds when (14)–(17) hold.

We now give a heuristic understanding of (14)–(17). The assumption (14) describes the structure of the operator V in particular, allowing us to include copies of $N_2(z/(hE'))$, which encodes the exterior behavior of waves at speed $\sqrt{E}^{-1}$. We assume that V is elliptic on $|\xi'|_g = 1$, the glancing set for the problem inside Ω, to simplify some of our analysis and guarantee that glancing effects play a nontrivial role in the analysis. Notice in particular that if $\mathrm{WF_h}(V) \cap \{|\xi'|_g = 1\} = \varnothing$, then waves near glancing escape Ω essentially without reflection. This ellipticity assumption is not necessary for our analysis, but since the main advantage of the present paper over [Galkowski 2014] is the analysis near glancing, we include it to simplify our presentation.

Next, (15) guarantees that the problem is locally elliptic in the sense that if a singularity emerges from $(x', \xi') \in T^*\partial\Omega$, then there must be a singularity coming into (x', ξ'). That is, the boundary cannot produce singularities spontaneously. Furthermore, this guarantees that there are no solutions microlocalized in the elliptic region $|\xi'|_g > 1$.

Finally, (16) and (17) are used to guarantee that the resolvent operator corresponding to (13) is meromorphic and hence that it makes sense to discuss its poles.

Remark 1.9. • For the definition of ellipticity of V, see Sections 2C2 and 4A1.

• These are not quite the most general assumptions we can make on V, but in practice all situations we have in mind fall into this category. For the most general assumptions, see (65) and for the statement of the theorem in that case, see Theorem 9.11.

• We make the assumption that V is elliptic near glancing so there is no rapid loss of energy near glancing. We could remove this assumption, but there would be no new phenomena and the analysis near glancing would be more complicated.

• The final assumption (17) (used to prove that the underlying problem is Fredholm) is satisfied for example when $m < 1$, or when $m \geq 1$ and for some θ_0 fixed and $\sigma(\widetilde{V})$ real-valued

$$\sigma(V) = e^{i\theta_0}\sigma(\widetilde{V}), \quad |\xi'|_g \geq M_2.$$

Lt $\chi \in C_c^\infty(\mathbb{R})$ with $\chi \equiv 1$ near 0, and define

$$\chi_\epsilon \in C^\infty(T^*\partial\Omega), \quad \chi_\epsilon(x, \xi') := 1 - \chi\left(\frac{1 - |\xi'|_g}{h^\epsilon}\right),$$

$$R := -(I + G_\Delta^{1/2} V G_\Delta^{1/2})^{-1} G_\Delta^{1/2} V G_\Delta^{1/2} \operatorname{Op_h}(\chi_\epsilon), \tag{18}$$

$$T(z) := G_\Delta^{-1/2}(z) G_B(z) G_\Delta^{-1/2}(z), \tag{19}$$

where G_B is the Fourier integral operator component of $G(z)$. (See Section 2 for an explanation of the quantization procedure $\operatorname{Op_h}$.) Note also that the inverse $(I + G_\Delta^{1/2} V G_\Delta^{1/2})^{-1}$ makes sense microlocally on $\operatorname{supp} \chi_\epsilon$ by (15).

Let $\tilde{\sigma}$ denote the compressed shymbol (see [Galkowski 2014, Section 2.3] or Section 3). Then let $l_N,\ r_N(z) : B^*\partial\Omega \to \mathbb{R}$ be

$$l(q, q') := |\pi_x(q) - \pi_x(q')|, \quad l_N(q) := \frac{1}{N}\sum_{k=1}^{N} l(\beta^{k-1}(q), \beta^k(q)), \tag{20}$$

$$r_N(q) := \frac{\operatorname{Im} z}{h} l_N(q) + \frac{1}{2N}\log\tilde{\sigma}(((RT(z))^*)^N (RT(z))^N)(q) \tag{21}$$

(recall that $B^*\partial\Omega$ is the coball bundle of the boundary and β is the billiard ball map). The term $(\operatorname{Im} z/h) l_N$ in (21) serves to cancel the growth of $T(z)$ in the definition of r_N.

Remark 1.10. Note that we use the notion of the compressed shymbol instead of a variable order symbol since we do not wish to make any a priori assumption on how the symbol of V varies from point to point. Moreover, the order of the symbol will vary also as a function of $\operatorname{Im} z$.

In fact, for $0 < N$ independent of h we have

$$r_N(q) = \frac{1}{2N}\sum_{n=1}^{N} \log\left|(\tilde{\sigma}(R)\circ\beta^n(q) + O(h^{I_R(q)+1-2\epsilon}))\right|^2, \tag{22}$$

where $I_R(q)$ is the local order of R at q (see [Galkowski 2014, Section 2.3] or Section 3). The expression (22) illustrates that r_N is the logarithmic average reflectivity over N iterations of the billiard ball map.

Let

$$\mathcal{P}(z) := \begin{pmatrix} -h^2\Delta - z^2 \\ \partial_\nu + B \end{pmatrix} : H_{\mathrm{h}}^{s+2}(\Omega) \to H_{\mathrm{h}}^{s}(\Omega) \oplus H_{\mathrm{h}}^{s+1/2-\max(m-1,0)}(\partial\Omega), \tag{23}$$

where H_{h}^m denotes the semiclassical Sobolev space with norm

$$\|u\|_{H_{\mathrm{h}}^m} := \|\langle hD\rangle^m u\|_{L^2}. \tag{24}$$

(See [Zworski 2012, Section 7.1] for a more precise definition.) Let $\Phi_-(s)$ and ζ_j be as in (9) and $Q \in C^\infty(T^*\partial\Omega)$ be the symbol of the second fundamental form to $\partial\Omega$ (as in Theorem 1.5), and define $f_j(\,\cdot\,; h) \in C^\infty(T^*\partial\Omega)$ for $j = 1, 2, \dots$ by

$$f_j(q;h) := \frac{Q(q)\big((2hQ(q))^{1/3}(1+a_1)\operatorname{Im}\Phi_-(\zeta_j) + \sigma(h\operatorname{Im}V_1)(q)\big)}{|\sigma(hV)(q)|^2}.$$

Let $S^*\partial\Omega$ denote the cosphere bundle of $\partial\Omega$ and

$$\mathcal{B}_{j,\pm}(\epsilon, C; h) := \left\{ z \in D_M(h) \,:\, \inf_{S^*\partial\Omega}(f_j(q;h) - Ch^{-\alpha})(1\mp\epsilon) \le \frac{\operatorname{Im} z}{h} \le \sup_{S^*\partial\Omega}(f_j(q;h) + Ch^{-\alpha})(1\pm\epsilon) \right\}.$$

Then Theorems 1.2, 1.5, and 1.7 are a consequence of the following:

Theorem 1.11. *Let $\Omega \Subset \mathbb{R}^d$ be strictly convex with smooth boundary. Fix $\epsilon > 0$, $M > 0$, $N_1, N_2 > 0$, $m \ge 0$ and suppose that $\mathcal{AV}(a_1, \alpha, E', m, \delta_0, M, M_1, M_2, \epsilon_0)$ holds. Then there exist $h_0 > 0$, $C, c, N > 0$, so that if $0 < h < h_0$, $z \in D_M(h)$,*

$$\frac{\operatorname{Im} z}{h} \le \sup_{N\le N_1}\inf_{|\xi'|_g\le 1-h^{\epsilon_0}} \frac{r_N}{2l_N}(1-\epsilon) \quad \text{or} \quad \frac{\operatorname{Im} z}{h} \ge \inf_{N\le N_1}\sup_{|\xi'|_g\le 1-h^{\epsilon_0}} \frac{r_N}{2l_N}(1+\epsilon), \tag{25}$$

$\pm\operatorname{Im} z \ge 0$, $z \notin \bigcup_{j=1}^{N_2}\mathcal{B}_{j,\pm}(\epsilon, C; h)$, and

$$\frac{\operatorname{Im} z}{h} \ge \sup_{S^*\partial\Omega}(f_{N_2+1}(q;h) + Ch^{-\alpha})(1\pm\epsilon),$$

then $\mathcal{P}(z)$ is invertible, and if $\mathcal{P}(z)u = \begin{pmatrix} 0 \\ v \end{pmatrix}$, then

$$\|u|_{\partial\Omega}\|_{H_{\mathrm{h}}^m} \le ch^{-N}\|v\|_{L^2}. \tag{26}$$

Observe that Theorem 1.11 (in particular, (25)) takes the same form as (2). Thus, the poles of $\mathcal{P}(z)^{-1}$ are controlled by the average reflectivity in the hyperbolic region. To see that this continues up to the glancing set and hence that Theorem 1.11 is a quantum version of the Sabine law, observe that (2) matches (25). Moreover, using Lemma 5.3, that V is elliptic near $|\xi'|_g = 1$ and

$$\sigma(R) = \frac{\sigma(hV)}{2i\sqrt{1-|\xi'|^2} - hV},$$

we have that for $q = (x, \xi') \in B^*\partial\Omega$ with $\sqrt{1-|\xi'|_g^2} \ll h^{1+\alpha}$

$$\frac{\log|R(\beta(q))|^2}{2l(q,\beta(q))} = \frac{-Q(x,\xi')(\sqrt{1-|\xi'|^2} - \operatorname{Im} hV)}{|\sigma(hV)|^2} + O(h^{-\alpha-1}\sqrt{1-|\xi'|^2}), \tag{27}$$

where, as above, $Q(x, \xi')$ is the symbol of the second fundamental form to $\partial\Omega$. Now,

$$\operatorname{Im}\Phi_-(s) \sim -\sqrt{-s}, \quad s \to -\infty \quad \text{(see Figure 5)}.$$

Therefore (27) matches the bounds in Theorem 1.11 modulo:

(1) Modes cannot concentrate closer than $h^{2/3}$ to $\{|\xi'|_g = 1\}$ (the glancing set).

(2) A quantization involving the zeros of the Airy function happens at scale $h^{2/3}$ near glancing.

(3) $-\sqrt{-s}$ is replaced by $\operatorname{Im}\Phi_-(s)$.

1E. ***Outline of the proof.*** Proving Theorem 1.11 amounts to understanding the location of resonances, which correspond to z such that $\mathcal{P}(z)$ is not invertible. We proceed by proving the estimate (26) on solutions to (13), which implies an estimate on $\mathcal{P}(z)^{-1}$.

To avoid analyzing the microlocally complicated interior Dirichlet-to-Neumann map, we change the boundary condition. In particular, we have

$$(I + GV)\psi = Gv. \tag{28}$$

We then proceed similarly to [Galkowski 2014] and decompose the boundary microlocally into the hyperbolic, glancing, and elliptic regions given respectively by

$$\mathcal{H} = \{(x,\xi') \in T^*\partial\Omega : |\xi'|_g \le 1 - h^\epsilon\},$$
$$\mathcal{G} = \{(x,\xi') \in T^*\partial\Omega : ||\xi'|_g - 1| \le h^\epsilon\},$$
$$\mathcal{E} = \{(x,\xi') \in T^*\partial\Omega : |\xi'|_g \ge 1 + h^\epsilon\}.$$

Then, letting $\mathbb{1}_U$ be an operator microlocally equal to the identity on U and U' be a slight enlargement of U, we have

$$(I - \mathbb{1}_{U'})G\mathbb{1}_U = O_{\Psi^{-\infty}}(h^\infty),$$

where U is any of $\mathcal{H}$, $\mathcal{G}$, or $\mathcal{E}$. This allows us to work with each region separately.

For notational convenience, let $\psi = u|_{\partial\Omega}$ and recall that

$$\mathcal{P}(z)u = \begin{pmatrix} 0 \\ v \end{pmatrix}, \tag{29}$$

where $\mathcal{P}(z)$ is as in (23). We first consider $\mathcal{E}$. Here, G is a pseudodifferential operator and our assumptions on V allow us to prove estimates on $\mathbb{1}_{\mathcal{E}}\psi$ in terms of v. We then consider the hyperbolic region, $\mathcal{H}$. Here the situation is more complicated because G consists of two pieces: G_B, a Fourier integral operator (FIO) associated to the billiard ball map, and G_Δ, a pseudodifferential operator. Using the calculus of FIO's, we are able to reduce estimating solutions to (28) microlocally in $\mathcal{H}$ to estimating solutions to

$$(I-(RT)^N)u = Av$$

for some A. Then, again using the calculus of FIOs, we see that $I-(RT)^N$ is microlocally invertible under the conditions given in (25).

Up to this point, the analysis in the present paper requires only minor changes from that in [Galkowski 2014]. However, the analysis near glancing is substantially different and heavily uses the microlocal model for G and $\mathcal{S}\ell := 1_\Omega R_0(z/h)\gamma^*$ near glancing given in [loc. cit., Section 4.5]. The analysis in [loc. cit., Chapter 5] uses only the microlocal model for G and does so simply to obtain a norm bound on G near glancing. Here we use the precise microlocal properties of G and $\mathcal{S}\ell$ near glancing.

We start by analyzing $I+GV$ as a second microlocal pseudodifferential operator on

$$\mathcal{G}_+ := \{(x,\xi')\in T^*\partial\Omega : 1-Mh^{2/3}\le |\xi'|_g \le 1+h^\epsilon\},$$

which is the microlocal region closest to glancing. When α is sufficiently small $\left(\alpha < -\frac{2}{3}\right)$, we see that $I+GV$ is elliptic on $\mathcal{G}_+$ outside of a union of $h^{2/3}$ thickened hypersurfaces given by

$$\mathcal{G}_N := \bigcup_{j=1}^N \mathcal{G}_j,\quad \mathcal{G}_j := \left\{(x,\xi')\in T^*\partial\Omega : \left|\frac{|\xi'|_g^2-1}{(2Q(x,\xi'))^{2/3}} - h^{2/3}\zeta_j\right| \le \delta h^{2/3}\right\}.$$

Since we have microlocal invertibility on $\mathcal{G}_+$ off of $\mathcal{G}_N$, resonance states must concentrate on $\mathcal{G}_N$. This is the quantization condition which occurs at scale $h^{2/3}$.

To get this quantization property, we have used the microlocal structure of G. To obtain estimates the remaining part of ψ, i.e., on $\psi_g := (\mathbb{1}_{\mathcal{G}_N}+\mathbb{1}_{\mathcal{G}_-})\psi$, where

$$\mathcal{G}_- := \{(x,\xi')\in T^*\partial\Omega : |\xi'|_g \le 1-Mh^{2/3}\},$$

we will use the microlocal structure of $\mathcal{S}\ell$.

We have that u solves (29). Integrating by parts in Ω, we have

$$\left(\frac{2\operatorname{Re} z \operatorname{Im} z}{h}\|u\|_{L^2}^2 - \operatorname{Im}\langle B\psi,\psi\rangle\right) = -\operatorname{Im}\langle hv,\psi\rangle. \tag{30}$$

Then, letting $\mathcal{D}\ell$ denote the double layer potential and using a classical boundary layer formula, together with the boundary condition from (13), we have

$$u = h^{-1}\mathcal{S}\ell\, h\,\partial_\nu u - \mathcal{D}\ell\, u = -(h^{-1}\mathcal{S}\ell\, B + \mathcal{D}\ell)\psi + \mathcal{S}\ell v = -\mathcal{S}\ell\, V\psi + \mathcal{S}\ell\, v.$$

So, we can write u in terms of ψ via the boundary layer potential, $\mathcal{S}\ell$. Another technical innovation in our proof is to use the model for $\mathcal{S}\ell$ near glancing to identify $\mathcal{S}\ell^*\mathcal{S}\ell$ as a second microlocal pseudodifferential operator on $\mathcal{G}$. We are then able to apply the sharp Gårding inequality to obtain upper and lower bounds on

$$\left(\frac{2\operatorname{Re} z \operatorname{Im} z}{h}\|u_g\|_{L^2}^2 - \operatorname{Im}\langle B\psi_g, \psi_g\rangle\right),$$

where $u_g = -h^{-1}\mathcal{S}\ell V\psi_g$. Together with (30), this allows us to estimate ψ_g in terms of v.

Combining the estimates on $\mathcal{E}$, $\mathcal{G}$, and $\mathcal{H}$, we are able to estimate ψ in terms of v. In order to prove that condition (3) of Theorem 1.11, together with (25), implies (26), we refine our estimates on $\mathcal{G}$ when $|\operatorname{Im} z| \geq ch^N$ for some $N > 0$.

Because we have polynomial bounds on the interior Dirichlet-to-Neumann map, $N_1(z/h)$, in this region and since $(N_1 + N_2)G = I = G(N_1 + N_2)$, we are able to show that if

$$(I + GV)\tilde{\psi} = w,$$

then there exists $v = (N_1 + N_2)w$ such that $(I + GV)\tilde{\psi} = Gv$ and hence there exists $\tilde{u}$ solving (13) with v replaced by $(N_1 + N_2)w = O_{L^2\to L^2}(h^{-N})w$.

Returning to the original problem, $(I + GV)\psi = Gv$, we see that for δ small enough, $\mathcal{G}_j$ are separated by $\delta h^{2/3}$. Hence, we can find ψ_j microlocalized $\delta h^{2/3}$-close to $\mathcal{G}_j$ such that

$$(I + GV)\psi_j = w_j, \quad \|w_j\| \leq Ch^{-M}\|v\|.$$

Therefore, we can find u_j solving (13) with $u_j|_{\partial\Omega} = \psi_j$ and $v = v_j = h(N_1 + N_2)w_j$ and, repeating the analysis above using boundary layer operators, we can obtain estimates on ψ_j. Together with knowledge of the symbol of N_2 and of $\mathcal{S}\ell^*\mathcal{S}\ell$, this finishes the proof of Theorem 1.11.

1F. ***Organization of the paper.*** The paper is organized as follows. We start by introducing the necessary standard semiclassical tools as well as the shymbol from [Galkowski 2014] in Sections 2 and 3. Then in Section 4, we introduce the second microlocal calculus from [Sjöstrand and Zworski 1999]. We conclude the preliminary material with Section 5 where we introduce the billiard ball flow and map.

As a guide for the general case, Section 6 analyzes the single- and double-layer potentials, respectively

$$\begin{aligned}\mathcal{S}\ell(\lambda)f(x) &:= \int_{\partial\Omega} R_0(\lambda)(x,y)f(y)\,dS(y), && x \in \Omega,\\ \mathcal{D}\ell(\lambda)f(x) &:= \int_{\partial\Omega} \partial_{\nu_y}R_0(\lambda)(x,y)f(y)\,dS(y), && x \in \Omega,\end{aligned} \tag{31}$$

and operators, respectively

$$\begin{aligned}G(\lambda)f(x) &:= \int_{\partial\Omega} R_0(\lambda)(x,y)f(y)\,dS(y), && x \in \partial\Omega,\\ N(\lambda)f(x) &:= \int_{\partial\Omega} \partial_{\nu_y}R_0(\lambda)(x,y)f(y)\,dS(y), && x \in \partial\Omega,\end{aligned}$$

in the special case of the Friedlander model. Section 7 contains the analysis of the boundary layer potentials and operators in the general strictly convex case. Next, Section 8 gives the proof of Theorem 1.11 including the Fredholm property and meromorphy of the resolvent for $\mathcal{P}$. Sections 10, 11, and 12 respectively contain the necessary material to deduce Theorems 1.2, 1.5, and 1.7 from Theorem 1.11. Finally, Section 13 gives the proof that Theorem 1.2 is sharp in the case of the unit disk.

2. Semiclassical preliminaries

In this section, we review the methods of semiclassical analysis which are needed throughout the rest of our work. The theories of pseudodifferential operators, wavefront sets, and the local theory of Fourier integral operators are standard and our treatment follows that in [Dyatlov and Guillarmou 2014; Zworski 2012]. We introduce the notion *shymbol* from [Galkowski 2014], which is a notion of sheaf-valued symbol that is sensitive to local changes in the semiclassical order of a symbol.

2A. ***Notation.*** We review the relevant notation from semiclassical analysis in this section. For more details, see [Dimassi and Sjöstrand 1999; Zworski 2012].

2A1. *Big O notation.* The $O(\cdot)$ and $o(\cdot)$ notations are used in this paper in the following ways: We write $u = O_{\mathcal{X}}(F)$ if the norm of u in the functional space $\mathcal{X}$ is bounded by the expression F times a constant. We write $u = o_{\mathcal{X}}(F)$ if the norm of u has

$$\lim_{s\to s_0} \frac{\|u(s)\|_{\mathcal{X}}}{F(s)} = 0,$$

where s is the relevant parameter. If no space $\mathcal{X}$ is specified, then $u = O(F)$ and $u = o(F)$ mean

$$|u(s)| \le C|F(s)| \quad \text{and} \quad \lim_{s\to s_0} \frac{|u(s)|}{F(s)} = 0 \tag{32}$$

respectively.

2A2. *Phase space.* Let M be a d-dimensional manifold without boundary. Then we denote an element of the cotangent bundle to M by (x,ξ), where $\xi \in T_x^*M$.

2B. ***Symbols and quantization.*** We start by defining the exotic symbol class $f(h)S^m_\delta(M)$.

Definition 2.1. Let $a(x,\xi;h) \in C^\infty(T^*M \times [0,h_0))$, $f \in C^\infty((0,h_0))$, $m \in \mathbb{R}$, and $\delta \in \left[0,\frac{1}{2}\right)$. Then, we say that $a \in f(h)S^m_\delta(T^*M)$ if for every $K \Subset M$ and ς, ϖ multi-indices, there exists $C_{\varsigma\varpi K}$ such that

$$|\partial_x^\varsigma \partial_\xi^\varpi a(x,\xi;h)| \le C_{\varsigma\varpi K} f(h) h^{-\delta(|\varsigma|+|\varpi|)} \langle\xi\rangle^{m-|\varpi|}. \tag{33}$$

We define $S^\infty_\delta := \bigcup_m S^m_\delta$, $S^{-\infty}_\delta := \bigcap_m S^m_\delta$ and when one of the parameters δ or m is 0, we suppress it in the notation.

We say that $a(x,\xi;h) \in S^{\mathrm{comp}}_\delta(M)$ if $a \in S_\delta(M)$ and a is supported in some h-independent compact set.

This definition of a symbol is invariant under changes of variables (see for example [Zworski 2012, Theorem 9.4] or more precisely, the arguments therein).

2C. *Pseudodifferential operators.* We follow [Zworski 2012, Section 14.2] to define the algebra $\Psi^m_\delta(M)$ of pseudodifferential operators with symbols in $S^m_\delta(M)$. (For the details of the construction of these operators, see for example [loc. cit., Sections 4.4, 14.12]. See also [Hörmander 1985a, Chapter 18; Grigis and Sjöstrand 1994, Chapter 3].) Since we have made no assumption on the behavior of our symbols as $x \to \infty$, we do not have control over the behavior of $\Psi^k_\delta(M)$ near infinity in M. However, we do require that all operators $A \in \Psi^m_\delta(M)$ are *properly supported*. That is, the restriction of each projection map $\pi_x, \pi_{x'} : M \times M \to M$ to the support of $K_A(x, x'; h)$, the Schwartz kernel of A, is a proper map. For the construction of such a quantization procedure, see for example [Hörmander 1985a, Proposition 18.1.22]. An element in $A \in \Psi^m_\delta(M)$ acts $H^s_{\mathrm{h,loc}}(M) \to H^{s-m}_{\mathrm{h,loc}}(M)$, where $H^s_{\mathrm{h,loc}}(M)$ denotes the space of distributions locally in the semiclassical Sobolev space $H^s_{\mathrm{h}}(M)$. The definitions of these spaces can be found for example in [Zworski 2012, Section 7.1]. Finally, we say that a properly supported operator, A, with

$$A : \mathcal{D}'(M) \to C^\infty(M)$$

and each seminorm $O(h^\infty)$ is $O_{\Psi^{-\infty}}(h^\infty)$. We include operators that are $O_{\Psi^{-\infty}}(h^\infty)$ in all pseudodifferential classes.

With this definition, we have the semiclassical principal symbol map

$$\sigma : \Psi^m_\delta(M) \to S^m_\delta(M)/h^{1-2\delta} S^{m-1}_\delta(M) \tag{34}$$

and a noncanonical quantization map

$$\mathrm{Op_h} : S^m_\delta(M) \to \Psi^m_\delta(M)$$

with the property that $\sigma \circ \mathrm{Op_h}$ is the natural projection map onto $S^m_\delta(M)/h^{1-2\delta} S^{m-1}_\delta(M)$.

Henceforward, we will take $\sigma(A)$ to be any representative of the corresponding equivalence class in the right-hand side of (34). We do not include the subprincipal symbol because then the calculus of pseudodifferential operators would be more complicated. With this in mind, the standard calculus of pseudodifferential operators with symbols in S^m_δ gives for $A \in \Psi^{m_1}_\delta(M)$ and $B \in \Psi^{m_2}_\delta(M)$,

$$\begin{aligned}
\sigma(A^*) &= \overline{\sigma(A)} + O_{S^{m_1-1}_\delta(M)}(h^{1-2\delta}),\\
\sigma(AB) &= \sigma(A)\sigma(B) + O_{S^{m_1+m_2-1}_\delta(M)}(h^{1-2\delta}),\\
\sigma([A,B]) &= -ih\{\sigma(A), \sigma(B)\} + O_{S^{m_1+m_2-2}(M)}(h^{2(1-2\delta)}).
\end{aligned}$$

Here $\{\cdot,\cdot\}$ denotes the Poisson bracket and we take adjoints with respect to $L^2(M)$.

2C1. *Wavefront sets and microsupport of pseudodifferential operators.* In order to define a notion of wavefront set that captures both h-microlocal and C^∞ behavior, we define the *fiber radially compactified cotangent bundle*, $\overline{T}^*M$, by $\overline{T}^*M = T^*M \sqcup S^*M$, where

$$S^*M := (T^*M \setminus \{M \times 0\})/\mathbb{R}_+$$

and the $\mathbb{R}_+$ action is given by $(t, (x, \xi)) \mapsto (x, t\xi)$. Let $|\cdot|_g$ denote the norm induced on T^*M by the

Riemannian metric g. Then a neighborhood of a point $(x_0,\xi_0)\in S^*M$ is given by $V\cap\{|\xi|_g\geq K\}$, where V is an open conic neighborhood of $(x_0,\xi_0)\in T^*M$.

For each $A\in\Psi^m_\delta(M)$ there exists $a\in S^m_\delta(M)$ with $A=\mathrm{Op_h}(a)+O_{\Psi^{-\infty}}(h^\infty)$. Then the *semiclassical wavefront set of* A, $\mathrm{WF}_{\mathrm{h},\Psi}(A)\subset\overline{T}^*M$, is defined as follows. A point $(x,\xi)\in\overline{T}^*M$ does not lie in $\mathrm{WF}_{\mathrm{h},\Psi}(A)$ if there exists a neighborhood U of (x,ξ) such that each (x,ξ) derivative of a is $O(h^\infty\langle\xi\rangle^{-\infty})$ in U. As in [Alexandrova 2008], we write

$$\mathrm{WF}_{\mathrm{h},\Psi}(A)=:\mathrm{WF}^{\mathrm{f}}_{\mathrm{h},\Psi}(A)\sqcup\mathrm{WF}^{\mathrm{i}}_{\mathrm{h},\Psi}(A),$$

where $\mathrm{WF}^{\mathrm{f}}_{\mathrm{h},\Psi}(A)=\mathrm{WF_h}(A)\cap T^*M$ and $\mathrm{WF}^{\mathrm{i}}_{\mathrm{h},\Psi}(A)=\mathrm{WF_h}(A)\cap S^*M$.

Operators with compact wavefront sets in T^*M are called *compactly microlocalized*. These are operators of the form

$$\mathrm{Op_h}(a)+O_{\Psi^{-\infty}}(h^\infty)$$

for some $a\in S^{\mathrm{comp}}_\delta(M)$. The class of all compactly microlocalized operators in $\Psi^m_\delta(M)$ is denoted by $\Psi^{\mathrm{comp}}_\delta(M)$.

We will also need a finer notion of microsupport on h-dependent sets.

Definition 2.2. An operator $A\in\Psi^{\mathrm{comp}}_\delta(M)$ is *microsupported* on an h-dependent family of sets $V(h)\subset T^*M$ if we can write $A=\mathrm{Op_h}(a)+O_{\Psi^{-\infty}}(h^\infty)$, where for each compact set $K\subset T^*M$, each differential operator ∂^ς on T^*M, and each N, there exists a constant $C_{\varsigma NK}$ such that for h small enough,

$$\sup_{(x,\xi)\in K\setminus V(h)}|\partial^\varsigma a(x,\xi;h)|\leq C_{\varsigma NK}h^N.$$

We then write

$$\mathrm{MS}_{\mathrm{h},\Psi}(A)\subset V(h).$$

The change of variables formula for the full symbol of a pseudodifferential operator [Zworski 2012, Theorem 9.10] contains an asymptotic expansion in powers of h consisting of derivatives of the original symbol. Thus Definition 2.2 does not depend on the choice of the quantization procedure $\mathrm{Op_h}$. Moreover, since we take $\delta<\frac{1}{2}$, if $A\in\Psi^{\mathrm{comp}}_\delta$ is microsupported inside some $V(h)$ and $B\in\Psi^m_\delta$, then AB, BA, and A^* are also microsupported inside $V(h)$. This implies the following.

Lemma 2.3. *Suppose that* $A,B\in\Psi^{\mathrm{comp}}_\delta$ *and* $\mathrm{MS}_{\mathrm{h},\Psi}(A)\cap\mathrm{MS}_{\mathrm{h},\Psi}(B)=\varnothing$. *Then*

$$\mathrm{WF}_{\mathrm{h},\Psi}(AB)=\varnothing.$$

For $A\in\Psi^{\mathrm{comp}}_\delta(M)$, we know $(x,\xi)\notin\mathrm{WF_h}(A)$ if and only if there exists an h-independent neighborhood u of (x,ξ) such that A is microsupported on the complement of U. However, A need only be microsupported on any h-independent neighborhood of $\mathrm{WF}_{\mathrm{h},\Psi}(A)$, not on $\mathrm{WF}_{\mathrm{h},\Psi}(A)$ itself. Also, notice that by Taylor's formula if $A\in\Psi^{\mathrm{comp}}_\delta(M)$ is microsupported in $V(h)$ and $\delta'>\delta$, then A is also microsupported on the set of all points in $V(h)$ which are at least $h^{\delta'}$ away from the complement of $V(h)$.

Remark 2.4. Notice that since we are working with $A\in\Psi^{\mathrm{comp}}_\delta(M)$ for $0\leq\delta<\frac{1}{2}$ we have $a\in S^{\mathrm{comp}}_\delta(T^*M)$ and a can only vary on a scale $\sim h^{-\delta}$. This implies that the set $\mathrm{MS}_{\mathrm{h},\Psi}(A)$ will respect the uncertainty principle.

2C2. *Ellipticity and L^2 operator norm.* For $A \in \Psi^m_\delta(M)$, define its *elliptic set* $\mathrm{ell}(A) \subset T^*M$ as follows: $(x, \xi) \in \mathrm{ell}(A)$ if and only if there exists a neighborhood U of (x, ξ) in $\overline{T}^*M$ and a constant C such that $|\sigma(A)| \geq C^{-1}\langle\xi\rangle^m$ in $U \cap T^*M$. The following statement is the standard semiclassical elliptic estimate; see [Hörmander 1985a, Theorem 18.1.24′] for the closely related microlocal case and for example [Dyatlov 2012, Section 2.2] for the semiclassical case.

Lemma 2.5. *Suppose that $P \in \Psi^m_\delta(M)$ and $A \in \Psi^{m'}_\delta(M)$ with* $\mathrm{WF}_{\mathrm{h},\Psi}(A) \subset \mathrm{ell}(P)$. *Then for each $\chi \in C^\infty_c(M)$, there exist $Q_i \in \Psi^{m'-m}_\delta(M)$ such that*

$$\chi A = \chi Q_1 P + O_{\Psi^{-\infty}_\delta}(h^\infty) = \chi P Q_2 + O_{\Psi^{-\infty}}(h^\infty).$$

In particular, for each $s \in \mathbb{R}$ and $u \in H^{s+m'}_{\mathrm{h}}$ there exists $C > 0$ such that for all $N > 0$, and $\chi_1 \in C^\infty(M)$ with $\chi_1 \equiv 1$ on $\operatorname{supp}\chi$,

$$\|\chi A u\|_{H^s_{\mathrm{h}}} \leq C\|\chi P u\|_{H^{s+m'-m}_{\mathrm{h}}} + O(h^\infty)\|\chi_1 u\|_{H^{-N}_{\mathrm{h}}}.$$

We also recall the estimate for the $L^2 \to L^2$ norm of a pseudodifferential operator (see for example [Zworski 2012, Chapter 13]).

Lemma 2.6. *Suppose that $A \in \Psi_\delta(M)$. Then there exists $C > 0$ such that*

$$\|A\|_{L^2\to L^2} \leq \sup_{T^*M} |\sigma(A)| + Ch^{1-2\delta}.$$

2D. ***Semiclassical microlocalization of distributions and operators.***

2D1. *Semiclassical wavefront sets and microsupport for distributions.* An h-dependent family $u(h) : (0, h_0) \to \mathcal{D}'(M)$ is called *h-tempered* if for each open $U \Subset M$, there exist constants C and N such that

$$\|u(h)\|_{H^{-N}_{\mathrm{h}}(U)} \leq Ch^{-N}. \tag{35}$$

For a tempered distribution u, we say that $(x_0, \xi_0) \in \overline{T}^*M$ does not lie in the wavefront set $\mathrm{WF}_{\mathrm{h}}(u)$ if there exists a neighborhood V of (x_0, ξ_0) such that for each $A \in \Psi(M)$ with $\mathrm{WF}_{\mathrm{h},\Psi}(A) \subset V$, we have $Au = O_{C^\infty}(h^\infty)$. As above, we write

$$\mathrm{WF}_{\mathrm{h}}(u) = \mathrm{WF}^{\mathrm{f}}_{\mathrm{h}}(u) \sqcup \mathrm{WF}^{\mathrm{i}}_{\mathrm{h}}(u)$$

where $\mathrm{WF}^{\mathrm{i}}_{\mathrm{h}}(u) = \mathrm{WF}_{\mathrm{h}}(u) \cap S^*M$. By Lemma 2.5, $(x_0, \xi_0) \notin \mathrm{WF}_{\mathrm{h}}(u)$ if and only if there exists compactly supported $A \in \Psi(M)$ elliptic at (x_0, ξ_0) such that $Au = O_{C^\infty}(h^\infty)$. The wavefront set of u is a closed subset of $\overline{T}^*M$. It is empty if and only if $u = O_{C^\infty(M)}(h^\infty)$. We can also verify that for u tempered and $A \in \Psi^m_\delta(M)$, we have $\mathrm{WF}_{\mathrm{h}}(Au) \subset \mathrm{WF}_{\mathrm{h},\Psi}(A) \cap \mathrm{WF}_{\mathrm{h}}(u)$.

Definition 2.7. A tempered distribution u is said to be *microsupported* on an h-dependent family of sets $V(h) \subset T^*M$ if for some $\delta \in \left[0, \tfrac{1}{2}\right)$, for all $A \in \Psi_\delta(M)$ with $\mathrm{MS}_{\mathrm{h},\Psi}(A) \cap V = \varnothing$ we have $\mathrm{WF}_{\mathrm{h}}(Au) = \varnothing$.

2D2. *Semiclassical wavefront sets of tempered operators.* An h-dependent family of operators $A(h):\mathcal{S}(M)\to\mathcal{S}'(M')$ is called *h-tempered* if for each $U\Subset M$, there exists $N\geq 0$ and $k\in\mathbb{Z}^+$ such that

$$\|A(h)\|_{H_{\mathrm{h}}^k(U)\to H_{\mathrm{h,loc}}^{-k}(M')}\leq Ch^{-N}. \tag{36}$$

For an h-tempered family of operators, we write that the wavefront set of A is given by

$$\mathrm{WF}_{\mathrm{h}}'(A):=\{(x,\xi,y,\eta):(x,\xi,y,-\eta)\in\mathrm{WF}_{\mathrm{h}}(K_A)\},$$

where K_A is the Schwartz kernel of A.

Definition 2.8. A tempered operator A is said to be *microsupported* on an h-dependent family of sets $V(h)\subset T^*M\times T^*M'$ if for all $\delta\in\left[0,\frac{1}{2}\right)$ and each $B_1\in\Psi_\delta(M')$ and $B_2\in\Psi_\delta(M)$ with

$$(\mathrm{MS}_{\mathrm{h},\Psi}(B_1)\times\mathrm{MS}_{\mathrm{h},\Psi}(B_2))\cap V=\varnothing,$$

we have $\mathrm{WF}_{\mathrm{h}}(B_1AB_2)=\varnothing$. We then write

$$\mathrm{MS}_{\mathrm{h}}{}'(A)\subset V(h).$$

Remark 2.9. With the definitions above, we have for $A\in\Psi_\delta^m(M)$,

$$\mathrm{WF}_{\mathrm{h}}'(A)=\{(x,\xi,x,\xi)\;:\;(x,\xi)\in\mathrm{WF}_{\mathrm{h},\Psi}(A)\}.$$

In addition, we have that if $A\in\Psi_\delta^{\mathrm{comp}}$, then $\mathrm{MS}_{\mathrm{h},\Psi}(A)\subset V(h)$ if and only if

$$\mathrm{MS}_{\mathrm{h}}'(A)\subset\{(x,\xi,x,\xi)\;:\;(x,\xi)\in V(h)\}.$$

Since there is a simple relationship between $\mathrm{WF}_{\mathrm{h},\Psi}$ and WF_{h}, as well as $\mathrm{MS}_{\mathrm{h},\Psi}$ and MS_{h}, we will only use the notation without Ψ from this point forward and the correct object will be understood from context.

2E. *Semiclassical Lagrangian distributions.* In this subsection, we review some facts from the theory of semiclassical Lagrangian distributions. See [Guillemin and Sternberg 1977, Chapter 6; Vũ Ngọc 2006, Section 2.3] for a detailed account, and [Hörmander 1985b, Section 25.1; Grigis and Sjöstrand 1994, Chapter 11] for the microlocal case. We do not attempt to define the principal symbol as a globally invariant object. Indeed, it is not always possible to do so in the semiclassical setting. When it is possible to do so, i.e., when the Lagrangian is exact, we define the symbol modulo the Maslov bundle. Taking symbols modulo the Maslov bundle makes the theory considerably simpler. We can make this simplification since for all of our symbolic computations, we work only in a single coordinate chart and, moreover, we always work with exact Lagrangians.

2E1. *Phase functions.* Let M be a manifold without boundary. We denote its dimension by d. Let $\varphi(x,\theta)$ be a smooth real-valued function on some open subset U_φ of $M\times\mathbb{R}^L$ for some L; we call x the *base variable* and θ the *oscillatory variable*. As in [Hörmander 1985a, Section 21.2], we say that φ is a *phase function* if the differentials $(\partial_{\theta_1}\varphi),\dots,d(\partial_{\theta_L}\varphi)$ on the *critical set*

$$C_\varphi:=\{(x,\theta):\partial_\theta\varphi=0\}\subset U_\varphi \tag{37}$$

are independent. Note that

$$\Lambda_\varphi := \{(x, \partial_x \varphi(x,\theta)) : (x,\theta) \in C_\varphi\} \subset T^*M$$

is an immersed Lagrangian submanifold (we will shrink the domain of φ to make it embedded).

2E2. *Symbols.* Let $\delta \in \left[0, \frac{1}{2}\right)$. A smooth function $a(x,\theta;h)$ is called a compactly supported symbol of type δ on U_φ if it is supported in some compact h-independent subset of U_φ, and for each differential operator ∂^ς on $M \times \mathbb{R}^L$, there exists a constant C_ς such that

$$\sup_{U_\varphi} |\partial^\varsigma a| \le C_\varsigma h^{-\delta|\varsigma|}.$$

As above, we write $a \in S^{\text{comp}}_\delta(U_\varphi)$ and set $S^{\text{comp}} := S^{\text{comp}}_0$.

2E3. *Lagrangian distributions.* Given a phase function φ and a symbol $a \in S^{\text{comp}}_\delta(U_\varphi)$, consider the h-dependent family of functions

$$u(x;h) = (2\pi h)^{-(d+2L)/4} \int_{\mathbb{R}^L} e^{i\varphi(x,\theta)/h} a(x,\theta;h)\, d\theta. \tag{38}$$

We call u a *Lagrangian distribution* of type δ generated by φ and denote this by $u \in I^{\text{comp}}_\delta(\Lambda_\varphi)$.

By the method of nonstationary phase, if $\operatorname{supp} a$ is contained in some h-dependent compact set $K(h) \subset U_\varphi$, then

$$\mathrm{MS_h}(u) \subset \{(x, \partial_x\varphi(x,\theta)) : (x,\theta) \in C_\varphi \cap K(h)\} \subset \Lambda_\varphi. \tag{39}$$

Remark 2.10. We are using the fact that $a \in S_\delta(U_\varphi)$ for some $\delta < \frac{1}{2}$ here.

2E4. *Principal symbols.* We define the principal symbol of a Lagrangian distribution independently of the choice of φ. To do this, we will need to use half-densities on Λ_φ (see, for example [Zworski 2012, Chapter 9] for a definition).

Following [Hörmander 1985b, Section 25.1], let

$$\Phi = \begin{pmatrix} \varphi''_{xx} & \varphi''_{x\theta} \\ \varphi''_{\theta x} & \varphi''_{\theta\theta} \end{pmatrix}.$$

Lemma 2.11. *Modulo Maslov factors, and a factor $e^{iA/h}$ for some constant $A \in \mathbb{R}$ depending on φ, the* principal symbol

$$\sigma(u) \in S^{\text{comp}}_\delta(\Lambda_\varphi; \Omega^{1/2}) / h^{1-2\delta} S^{\text{comp}}_\delta(\Lambda_\varphi; \Omega^{1/2})$$

is a half density given by

$$\sigma(u)(x,\xi) = |d\xi|^{1/2} a(x,\theta) e^{i\pi/4 \operatorname{sgn}\Phi} |\det \Phi|^{-1/2}.$$

Remark 2.12. In the case that Λ_φ is exact, the factor $e^{iA/h}$ can be removed.

Definition 2.13. Let $\Lambda \subset T^*M$ be an embedded Lagrangian submanifold. We say that an h-dependent family of functions $u(x;h) \in C^\infty_c(M)$ is a (compactly supported and compactly microlocalized) Lagrangian distribution of type δ associated to Λ if it can be written as a sum of finitely many functions of the form (38), for different phase functions φ parametrizing open subsets of Λ, plus an $O_{C_c^\infty}(h^\infty)$ remainder. Denote by $I_\delta^{\mathrm{comp}}(\Lambda)$ the space of all such distributions, and put $I^{\mathrm{comp}}(\Lambda) := I_0^{\mathrm{comp}}(\Lambda)$.

The action of a pseudodifferential operator on a Lagrangian distribution is given by the following lemma, following from the method of stationary phase:

Lemma 2.14. *Let $u \in I_\delta^{\mathrm{comp}}(\Lambda)$ and $P \in \Psi_\delta^m(M)$. Then $Pu \in I_\delta^{\mathrm{comp}}(\Lambda)$ and*

$$\sigma(Pu) = \sigma(P)|_\Lambda \cdot \sigma(u) + O(h^{1-2\delta})_{S_\delta^{\mathrm{comp}}(\Lambda)}.$$

2F. *Fourier integral operators.* A special case of Lagrangian distributions are Fourier integral operators associated to canonical graphs. Let M be a manifolds of dimension d. Consider a Lagrangian submanifold $\Lambda \subset T^*M \times T^*M$ given by

$$\Lambda = \{(\kappa(y,\eta), y, -\eta)\},$$

where κ is a symplectomorphism.

A compactly supported operator $U : \mathcal{D}'(M') \to C_c^\infty(M)$ is called a (semiclassical) *Fourier integral operator* of type δ associated to κ if its Schwartz kernel $K_U(x, x')$ lies in $I_\delta^{\mathrm{comp}}(\Lambda)$. We write $U \in I_\delta^{\mathrm{comp}}(C)$, where

$$C = \{(x, \xi, y, \eta) : (x, \xi, y, -\eta) \in \Lambda\}.$$

The numerology $h^{-(d+2L)/4}$ in (38) is explained by the fact that the normalization for Fourier integral operators is chosen so that

$$\|U\|_{L^2(M)\to L^2(M)} \sim 1$$

when C is generated by a symplectomorphism.

We will need the following lemma from the calculus of Fourier integral operators.

Lemma 2.15. *Let $A \in I_\delta^{\mathrm{comp}}(M \times M, C)$ and $P \in \Psi_\delta^{\mathrm{comp}}(M)$. Then, $A^*PA \in \Psi_\delta^{\mathrm{comp}}(M)$ and*

$$\sigma(A^*PA)(q) = |\sigma(A)(q, \kappa(q))|^2 \sigma(P)(\kappa(q)).$$

3. The shymbol

It will be useful to calculate symbols of operators whose semiclassical order may vary from point to point in T^*M. One can often handle this type of behavior by using weights to compensate for the growth. However, this requires some a priori knowledge of how the order changes and limits the allowable size in the change of order. In this section, we will develop a notion of a sheaf-valued symbol, the *shymbol*, that can be used to work in this setting without such a priori knowledge.

Let M be a compact manifold. Let $\mathcal{T}(T^*M)$ be the topology on T^*M. For $s \in \mathbb{R}$, define the symbol map

$$\sigma_s : h^s \Psi_\delta^{\mathrm{comp}} \to h^s S_\delta^{\mathrm{comp}} / h^{s+1-2\delta} S_\delta^{\mathrm{comp}}.$$

Suppose that for some $N > 0$ and $\delta \in \left[0, \frac{1}{2}\right)$, we have $A \in h^{-N}\Psi_\delta^{\mathrm{comp}}(M)$. We define a finer notion of symbol for such a pseudodifferential operator. Fix $0 < \epsilon \ll 1 - 2\delta$. For each open set $U \in \mathcal{T}(T^*M)$, define *the ϵ-order of A on U*

$$I_A^\epsilon(U) := \sup_{s\in\mathcal{S}_\epsilon} s + 1 - 2\delta,$$

where

$$\mathcal{S}_\epsilon := \left\{s \in \epsilon\mathbb{Z} : \text{there exists } \chi \in C_c^\infty(T^*M),\ \chi|_U = 1,\ \sigma_s(\mathrm{Op_h}(\chi) A\, \mathrm{Op_h}(\chi))|_U \equiv 0\right\}.$$

Then it is clear that for any $V \Subset U$ there exists $\chi \in C_c^\infty(U)$ with $\chi = 1$ on V such that $\mathrm{Op_h}(\chi) A\, \mathrm{Op_h}(\chi) \in h^{I_A^\epsilon(U)}\Psi_\delta^{\mathrm{comp}}(M)$.

Give $\mathcal{T}(T^*M)$ the ordering that $U \le V$ if $V \subset U$ with morphisms $U \to V$ if $U \le V$. Notice that $U \le V$ implies $I_A^\epsilon(U) \le I_A^\epsilon(V)$. Then define the functor $F_A^\epsilon : \mathcal{T}(T^*M) \to$ **Comm** (the category of commutative rings) by

$$F_A^\epsilon(U) = \begin{cases} h^{I_A^\epsilon(U)} S_\delta^{\mathrm{comp}}(M)|_U \,/\, h^{I_A^\epsilon(U)+1-2\delta} S_\delta^{\mathrm{comp}}(M)|_U, & I_A^\epsilon(U) \ne \infty, \\ \{0\}, & I_A^\epsilon(U) = \infty, \end{cases}$$

$$F_A^\epsilon(U \to V) = \begin{cases} h^{I_A^\epsilon(V) - I_A^\epsilon(U)}|_V, & I_A^\epsilon(V) \ne \infty, \\ 0, & I_A^\epsilon(V) = \infty. \end{cases}$$

Then F_A^ϵ is a presheaf on T^*M. We sheafify F_A^ϵ, still denoting the resulting sheaf by F_A^ϵ, and say that *A is of ϵ-class F_A^ϵ*. We define the *stalk* of the sheaf at q by $F_A^\epsilon(q) := \varinjlim_{q\in U} F_A^\epsilon(U)$.

Now, for every $U \subset \mathcal{T}(T^*M)$, $I_A^\epsilon(U) \ne \infty$, there exists $\chi_U \in C_c^\infty(T^*M)$ with $\chi_U \equiv 1$ on U such that

$$\sigma_{I_A^\epsilon(U)}(\mathrm{Op_h}(\chi_U) A\, \mathrm{Op_h}(\chi_U))|_U \ne 0.$$

Then we define the *ϵ-shymbol of A* to be the section of F_A^ϵ, $\tilde{\sigma}_{(\cdot)}^\epsilon(A) : \mathcal{T}(T^*M) \to F_A^\epsilon(\,\cdot\,)$, given by

$$\tilde{\sigma}_U^\epsilon(A) := \begin{cases} \sigma_{I_A^\epsilon(U)}(\mathrm{Op_h}(\chi_U) A\, \mathrm{Op_h}(\chi_U))|_U, & I_A^\epsilon(U) \ne \infty, \\ 0, & I_A^\epsilon(U) = \infty. \end{cases}$$

Define also the *ϵ-stalk shymbol*, $\tilde{\sigma}^\epsilon(A)_q$ to be the germ of $\tilde{\sigma}^\epsilon(A)$ at q as a section of F_A^ϵ.

Now, define

$$I_A^\epsilon(q) := \sup\{I_A^\epsilon(U) : q \in U\}.$$

Let $U_n \downarrow \{q\}$ be a sequence of open sets. We then define the simpler *compressed shymbol* by

$$\begin{gathered} \tilde{\sigma}^\epsilon(A) : T^*M \to \bigsqcup_q h^{I_A^\epsilon(q)}\mathbb{C}/h^{I_A^\epsilon(q)+1-2\delta}\mathbb{C}, \\ \tilde{\sigma}^\epsilon(A)(q) := \begin{cases} 0, & I_A^\epsilon(q) = \infty, \\ \lim_n \tilde{\sigma}_{U_n}^\epsilon(A)(q), & I_A^\epsilon(q) < \infty. \end{cases} \end{gathered} \tag{40}$$

The limit in (40) exists since if $I_A^\epsilon(q) < \infty$, then there exists $U \ni q$ such that we have $I_A^\epsilon(V) = I_A^\epsilon(U)$ for all $V \subset U$. This also shows that the limit is independent of the choice of sequence of $U_n \downarrow q$. It is easy to see from standard composition formulae that the compressed shymbol has

$$\tilde{\sigma}^\epsilon(AB)(q) = \tilde{\sigma}^\epsilon(A)(q)\tilde{\sigma}(B)(q), \quad A \in h^{-N}\Psi_\delta^{\mathrm{comp}} \text{ and } B \in h^{-M}\Psi_\delta^{\mathrm{comp}}.$$

Moreover,

$$\tilde{\sigma}^{\epsilon}([A,B])(q) = -ih\{\tilde{\sigma}^{\epsilon}(A)(q), \tilde{\sigma}^{\epsilon}(B)(q)\}.$$

The following lemma follows from standard formulas for the composition of FIOs combined with the definitions above:

Lemma 3.1. *Suppose that $A \in \Psi_{\delta}^{\mathrm{comp}}$ and let T be a semiclassical FIO associated to the symplectomorphism κ with elliptic symbol $t \in S_{\delta}$. Then for $N > 0$ independent of h,*

$$(AT)_N := (T^*A^*)^N (AT)^N$$

has

$$\tilde{\sigma}^{\epsilon}((AT)_N)(q) = \prod_{i=1}^{N}\big(|\tilde{\sigma}^{\epsilon}(A)t|^2 \circ \kappa^i(q) + O(h^{I^{\epsilon}_{A_i}(\beta^k(q))+1-2\delta})\big).$$

Proof. Fix $q \in T^*M$. Let $\chi_k \in C_c^{\infty}(T^*M)$ have $\chi_k = 1$ on $B\big(q, \frac{1}{k}\big)$, the open ball of radius k^{-1} around q, and $\operatorname{supp} \chi_k \subset B\big(q, \frac{2}{k}\big)$. Then let $D := \mathrm{Op_h}(\chi_k)(AT)_N \,\mathrm{Op_h}(\chi_k)$. We have

$$D = \mathrm{Op_h}(\chi_k)(A_N T A_{N-1} T \cdots A_1 T)^*(A_N T A_{N-1} T \cdots A_1 T)\,\mathrm{Op_h}(\chi_k) + O_{\Psi_{\delta}^{\mathrm{comp}}}(h^{\infty}),$$

where $A_i = \mathrm{Op_h}(\psi_{k,i}) A\, \mathrm{Op_h}(\psi_{k,i})$ with $C_c^{\infty}(T^*M) \ni \psi_{k,i} = 1$ in some neighborhood of $\beta^i(q)$ and is supported inside a neighborhood $U_{k,i}$ of $\beta^i(q)$ such that $U_{k,i} \downarrow q$. Then the result follows from standard composition formulae in Lemma 2.15. □

Now, since $\epsilon > 0$ is arbitrary, we define the *semiclassical order of A at q* by $I_A(q) := \sup_{\epsilon>0} I_A^{\epsilon}(q)$ with the understanding that $f = O(h^{I_A(q)})$ means that for any $\epsilon > 0$,

$$|f(q)| \leq C_{\epsilon} h^{I_A(q)-\epsilon}.$$

Furthermore, we suppress the ϵ in the notation $\tilde{\sigma}^{\epsilon}(A)(q)$ and denote the *compressed shymbol* by $\tilde{\sigma}(A)(q)$, again with the understanding that for any $\epsilon > 0$,

$$\tilde{\sigma}(A)(q) \in h^{I_A(q)-\epsilon}\mathbb{C}/h^{I_A(q)+1-2\delta-\epsilon}\mathbb{C}.$$

4. A second microlocal calculus

In the present work, it will be necessary to localize $h^{2/3}$ near the glancing submanifold in $T^*\partial\Omega$. In order to do this, we present the second microlocal calculus from [Sjöstrand and Zworski 1999].

4A. ***The local model.*** We start by considering the model case of $\Sigma_0 = \{\xi_1 = 0\} \subset T^*\mathbb{R}^d$. Suppose that U is a neighborhood of $(0,0)$ and $a \in C_c^{\infty}(U)$. In that case, we write $a = a(x,\xi,\lambda;h)$ with $\lambda = h^{-\delta}\xi_1$. Suppose that $\epsilon < \min\big(\frac{1}{2},\delta\big)$, and $\epsilon + \delta \leq 1$. We say that $a \in S^{k_1}_{\delta,\epsilon}(\Sigma_0)$ if and only if

$$\partial_x^{\varsigma}\partial_{\xi}^{\varpi}\partial_{\lambda}^{k} a(x,\xi,\lambda;h) = O(h^{-\epsilon(|\varsigma|+|\varpi|)}\langle h^{\epsilon}\lambda\rangle^{k_1-k}). \tag{41}$$

We will write

$$a = \widetilde{O}_{\epsilon}(\langle h^{\epsilon}\lambda\rangle^{k_1}) \quad \text{if and only if} \quad (41) \text{ holds.}$$

For such a, we define the exact quantization

$$\widetilde{\mathrm{Op}}_{\mathrm{h}}(a)u = \frac{1}{(2\pi h)^d}\int a\left(\frac{x+y}{2}, \xi, h^{-\delta}\xi_1; h\right)e^{(i/h)\langle x-y,\xi\rangle}u(y)\,dy\,d\xi.$$

Lemma 4.1. *Suppose that* $a = \widetilde{O}_\epsilon(\langle h^\epsilon\lambda\rangle^{k_1})$ *and* $b = \widetilde{O}_\epsilon(\langle h^\epsilon\lambda\rangle^{k_2})$. *Then*

$$\widetilde{\mathrm{Op}}_{\mathrm{h}}(a)\circ\widetilde{\mathrm{Op}}_{\mathrm{h}}(b) = \widetilde{\mathrm{Op}}_{\mathrm{h}}(a \sharp b),$$

where

$$a\sharp b = e^{ihA(D)}(a|_{\lambda=h^{-\delta}\xi_1}b|_{\mu=h^{-\delta}\eta_1})\Big|_{\substack{y=x\\ \xi=\eta}} = \widetilde{O}_\epsilon(\langle h^\epsilon\lambda\rangle^{k_1+k_2}),$$

where

$$A(D) = \tfrac{1}{2}\sigma((D_x, D_\xi), (D_y, D_\eta)) =: \tfrac{1}{2}\langle QD, D\rangle.$$

Moreover if $\epsilon + \delta < 1$,

$$a\sharp b = \sum_{k=0}^{\infty}\frac{i^k h^k}{k!}A(D)^k(a|_{\lambda=h^{-\delta}\xi_1}b|_{\mu=h^{-\delta}\eta_1})\Big|_{\substack{y=x\\ \eta=\xi}} \quad \mathrm{mod}\ h^\infty\Psi^{-\infty}.$$

We say that $a(x,\xi,y,\eta,\lambda,\mu) = \widetilde{O}_\epsilon(\langle h^\epsilon\lambda\rangle^{k_1}\langle h^\epsilon\mu\rangle^{k_2})$ if

$$|\partial_x^{\varsigma_1}\partial_y^{\varsigma_2}\partial_\xi^{\varpi_1}\partial_\eta^{\varpi_2}\partial_\lambda^{m_1}\partial_\mu^{m_2}a| \le C_{\varsigma\varpi m}h^{-\epsilon(|\varsigma|+|\varpi|)}\langle h^\epsilon\lambda\rangle^{k_1-m_1}\langle h^\epsilon\mu\rangle^{k_2-m_2}.$$

The only part of this lemma that is nonstandard is the following. The rest follows from applying stationary phase.

Lemma 4.2. $$e^{ihA(D)}: \widetilde{O}_\epsilon(\langle h^\epsilon\lambda\rangle^{k_1}\langle h^\epsilon\mu\rangle^{k_2}) \to \widetilde{O}_\epsilon(\langle h^\epsilon\lambda\rangle^{k_1}\langle h^\epsilon\mu\rangle^{k_2}).$$

Proof. We start by considering the case of one dimension. Let $w_1 = (x_1', \xi_1', y_1', \eta_1')$ and

$$\varphi_1(w_1) = \tfrac{1}{2}(\langle\xi_1', y_1'\rangle - \langle\eta_1', x_1'\rangle).$$

Then, with $z = (x_1, \xi_1, y_1, \eta_1)$,

$$c := (e^{ihA(D)}a)(z,\mu) = Ch^{-2}\int e^{-(i/h)\varphi_1(w)}a(w-z, \lambda - h^{-\delta}\xi_1, \mu - h^{-\delta}\eta_1)\,dw.$$

Then, rescale $(x_1', y_1') = (\tilde{x}_1, \tilde{y}_1)h^{-(1-\delta)}$ and $(\xi_1', \eta_1') = (\tilde{\xi}_1, \tilde{\eta}_1)h^{-\delta}$. We have that with $\tilde{w} = (\tilde{x}_1, \tilde{\xi}_1, \tilde{y}_1, \tilde{\eta}_1)$,

$$\begin{aligned} c = C\int e^{-i\varphi_1(\tilde{w})}\big(&\chi(\tilde{w})a(x_1 - h^{1-\delta}\tilde{x}_1, \xi_1 - h^\delta\tilde{\xi}_1, y_1 - h^{1-\delta}\tilde{y}_1, \eta_1 - h^\delta\tilde{\eta}_1, \lambda - \tilde{\xi}_1, \mu - \tilde{\eta}_1)\\ &+ (1-\chi(\tilde{w}))a(x_1 - h^{1-\delta}\tilde{x}_1, \xi_1 - h^\delta\tilde{\xi}_1, y_1 - h^{1-\delta}\tilde{y}_1, \eta_1 - h^\delta\tilde{\eta}_1, \lambda - \tilde{\xi}_1, \mu - \tilde{\eta}_1)\big)\,d\tilde{w}\\ =:{}& A + B, \end{aligned}$$

where $\chi \in C_c^\infty(\mathbb{R}^4)$ has $\chi \equiv 1$ on $B(0,1)$ and $\operatorname{supp}\chi \subset B(0,2)$. Then

$$|\partial^\varsigma A(z,\lambda,\mu)| \le C\sup_{|\tilde{w}|\le 2}|\partial^\varsigma a(x_1 - h^{1-\delta}\tilde{x}_1, \xi_1 - h^\delta\tilde{\xi}_1, y_1 - h^{1-\delta}\tilde{y}_1, \eta_1 - h^\delta\tilde{\eta}_1, \lambda - \tilde{\xi}_1, \mu - \tilde{\eta}_1)|$$

and hence $A = \widetilde{O}_\epsilon(\langle h^\epsilon\lambda\rangle^{k_1}\langle h^\epsilon\mu\rangle^{k_2})$. Letting

$$L := \frac{-\langle \partial\varphi(\tilde{w}), D_{\tilde{w}}\rangle}{|\partial\varphi(\tilde{w})|^2}$$

and integrating by parts sufficiently many times shows also that $B = \widetilde{O}_\epsilon(\langle h^\epsilon\lambda\rangle^{k_1}\langle h^\epsilon\mu\rangle^{k_2})$.

To obtain the general case, we simply observe that

$$e^{ihA(D_x,D_\xi,D_y,D_\eta)} = e^{ihA(D_{x'},D_{\xi'},D_{y'},D_{\eta'})}e^{ihA(D_{x_1},D_{\xi_1},D_{y_1}D_{\eta_1})}$$

and use that

$$e^{ihA(D_{x'},D_{\xi'},D_{y'},D_{\eta'})} : S_\epsilon \to S_\epsilon. \qquad \square$$

Now, rewriting the asymptotic expansion, and assuming that $|\xi_1| \le C$ so that

$$h^{1-2\epsilon} \le Ch^{1-\delta-\epsilon}\langle h^\epsilon\lambda\rangle^{-1},$$

we have if $\epsilon+\delta<1$, taking $p_1 > (1-2\epsilon)/(1-\delta-\epsilon)$,

$$\begin{aligned} a \sharp b(x,\xi,\lambda;h) &= \sum_{k=0}^{\infty}\frac{i^kh^k}{2^kk!}(\sigma(D_x, D_{\xi_1}+h^{-\delta}D_\lambda, D_{\xi'}, D_y, D_{\eta_1}+h^{-\delta}D_\mu, D_{\eta'}))^k ab\big|_{\substack{y=x,\,\eta=\xi\\ \lambda=\mu}} \\ &= ab + \frac{1}{2i}h^{1-\delta}(\partial_\lambda b\partial_{x_1}a - \partial_\lambda a\partial_{x_1}b) + \frac{h}{2i}\{a,b\} \\ &\quad + \sum_{k=2}^{p_1}\frac{i^kh^{k(1-\delta)}}{2^kk!}(\sigma(D_{x_1}, D_\lambda, D_{y_1}, D_\mu))^k ab\big|_{\substack{y=x,\,\eta=\xi\\ \lambda=\mu}} + \widetilde{O}_\epsilon(h^{2-3\epsilon-\delta}\langle h^\epsilon\lambda\rangle^{k_1+k_2-1}). \end{aligned}$$

4A1. *Ellipticity and boundedness in the local model.* We now present the analogs of microlocal elliptic estimates and the sharp Gårding inequalities in the second microlocal setting. Suppose that $\epsilon+\delta<1$ and $a = \widetilde{O}_\epsilon(\langle h^\epsilon\lambda\rangle^{k_1})$. We define the *elliptic set* of a, $\mathrm{ell}(a)$, by $(x,\xi,\lambda)\in\mathrm{ell}(a)$ if there exists a neighborhood, U, of (x,ξ,λ) and $c>0$ so that $|a| > c\langle h^\epsilon\lambda\rangle^{k_1}$ on U.

Lemma 4.3. *Suppose that* $p = \widetilde{O}_\epsilon(\langle h^\epsilon\lambda\rangle^{k_1})$, $b = \widetilde{O}_\epsilon(\langle h^\epsilon\lambda\rangle^{k_2})$ *and* $\operatorname{supp} b \subset \mathrm{ell}(p)$. *Then there exists* $a_i = O_\epsilon(\langle h^\epsilon\lambda\rangle^{k_2-k_1})$, $i=1,2$, *so that*

$$\widetilde{\mathrm{Op}}_{\mathrm{h}}(a_1)\,\widetilde{\mathrm{Op}}_{\mathrm{h}}(p) = \widetilde{\mathrm{Op}}_{\mathrm{h}}(p)\,\widetilde{\mathrm{Op}}_{\mathrm{h}}(a_2) + O_{\Psi^{-\infty}}(h^\infty) = \widetilde{\mathrm{Op}}_{\mathrm{h}}(b) + O_{\Psi^{-\infty}}(h^\infty).$$

Proof. By elementary analysis, one sees that

$$\partial^\varsigma p^{-1} = p^{-1}\sum_{k=1}^{|\varsigma|}\sum_{\substack{\varsigma=\varpi^1+\cdots+\varpi^k\\ |\varpi^j|\ge 1}} C_{\varpi^1,\dots,\varpi^k}\prod_{j=1}^{k}(p^{-1}\partial^{\varpi_j}p)$$

(see for example the proof of [Zworski 2012, Theorem 4.32]). Thus, since $|p|\ge c\langle h^\epsilon\lambda\rangle^{k_1}$ on $\operatorname{supp} b$,

$$q_0 : bp^{-1} = \widetilde{O}_\epsilon(\langle h^\epsilon\lambda\rangle^{k_2-k_1}).$$

So,

$$\widetilde{\mathrm{Op}}_{\mathrm{h}}(q_0)\,\widetilde{\mathrm{Op}}_{\mathrm{h}}(p) = \widetilde{\mathrm{Op}}_{\mathrm{h}}(b) + h^{1-\delta-\epsilon}\,\widetilde{\mathrm{Op}}_{\mathrm{h}}(e_1) + O_{\Psi^{-\infty}}(h^\infty),$$

where $e_1 = \widetilde{O}_\epsilon(\langle h^\epsilon\lambda\rangle^{k_2-1})$ with $\operatorname{supp} e_1 \subset \operatorname{ell}(p)$. Thus, setting $r_1 = h^{1-\delta-\epsilon}e_1$, letting $q_1 = -r_1 p^{-1} = h^{1-\delta-\epsilon}\widetilde{O}_\epsilon(\langle h^\epsilon\lambda\rangle^{k_2-k_1-1})$, and continuing in this way, we obtain

$$q_n = h^{n(1-\delta-\epsilon)}\widetilde{O}_\epsilon(\langle h^\epsilon\lambda\rangle^{k_2-k_1-n})$$

so that with $a_1 \sim \sum q_i$,

$$\widetilde{\operatorname{Op}}_{\mathrm{h}}(a_1)\,\widetilde{\operatorname{Op}}_{\mathrm{h}}(p) = \widetilde{\operatorname{Op}}_{\mathrm{h}}(b) + O_{\Psi^{-\infty}}(h^\infty).$$

A similar argument, yields a_2. □

Lemma 4.4. *Suppose that $a = \widetilde{O}_\epsilon(\langle h^\epsilon\lambda\rangle^{k_1})$. Then*

$$\|\widetilde{\operatorname{Op}}_{\mathrm{h}}(a)\|_{L^2\to L^2} \le C\langle h^{(\epsilon-\delta)k_1}\rangle.$$

Proof. The proof for $h = 1$ follows from for example [Zworski 2012, Theorem 4.23]. Suppose that $u(x) \in \mathcal{S}$. The proof follows that in [loc. cit., Theorem 5.1]. We have

$$\|\widetilde{\operatorname{Op}}(a)\|_{L^2\to L^2} \le C \sup_{|\varsigma|\le Md} |\partial^\varsigma a|.$$

So, we rescale $\tilde{\xi} = h^{-(1+\delta)/2}\xi$, $\tilde{x} = h^{-(1-\delta)/2}x$ and $\tilde{u}(\tilde{x}) = h^{(1-\delta)d/4}u(h^{(1-\delta)/2}\tilde{x})$. Then,

$$\widetilde{\operatorname{Op}}_{\mathrm{h}}(a)u(x) = h^{-(d(1-\delta))/4}\,\widetilde{\operatorname{Op}}(a_h)\tilde{u}(\tilde{x}),$$

where

$$a_h(\tilde{x},\tilde{\xi}) := a(h^{(1-\delta)/2}\tilde{x}, h^{(1+\delta)/2}\tilde{\xi}, h^{(1-\delta)/2}\tilde{\xi}_1).$$

Therefore,

$$\begin{aligned}
\|\widetilde{\operatorname{Op}}_{\mathrm{h}}(a)u\|_{L^2_x} &= \|\widetilde{\operatorname{Op}}(a_h)\tilde{u}(\tilde{x})\|_{L^2_{\tilde{x}}} \le \|\widetilde{\operatorname{Op}}(a_h)\|_{L^2\to L^2}\|\tilde{u}\|_{L^2_{\tilde{x}}} \\
&\le C \sup_{|\varsigma|\le Md} |\partial^\varsigma a_h|\,\|u\|_{L^2_x} \\
&\le C \sup_{|(\varsigma,\varpi,k)|\le Md} h^{(|\varsigma|+|\varpi|+k)(1-\delta)/2}|\partial_x^\varsigma\partial_\xi^\varpi\partial_\lambda^k a| \\
&\le C \sup_{|(\varsigma,\varpi,k)|\le Md} h^{(|\varsigma|+|\varpi|+k)(1-\delta)/2}(1+\langle h^{\epsilon-\delta}\rangle^{k_1-k}).
\end{aligned}$$ □

We now prove an analog of the sharp Gårding inequality for the second microlocal operators.

Lemma 4.5. *Suppose that $a = \widetilde{O}_0(\langle\lambda\rangle^0)$ and $a \ge 0$. Then*

$$\langle\widetilde{\operatorname{Op}}_{\mathrm{h}}(a)u, u\rangle \ge -Ch^{1-\delta}\|u\|^2_{L^2}.$$

Proof. We again follow the proof in the classical case. (See for example [Zworski 2012, Theorem 4.32]). Fix $\tilde{h}$ sufficiently small and let $\gamma = h^\epsilon/\tilde{h}$. We will show that $q = (a+\gamma)^{-1}$ satisfies

$$\partial_x^\varsigma\partial_\xi^\varpi\partial_\lambda^k q = O(h^{-\epsilon}\tilde{h}(\tilde{h}h)^{-(\epsilon/2)(|\varsigma|+|\varpi|+k)}\langle\lambda\rangle^{-k}). \tag{42}$$

That is $q \in h^{-\epsilon}\tilde{h}S^0_{\delta+\epsilon/2,\epsilon/2}(\Sigma_0)$. We will then be able to invert $a+\gamma$ when $\epsilon \le 1-\delta$.

First, since $a \ge 0$ and $a = \widetilde{O}_0(\langle\lambda\rangle^0)$, we have $|\partial_\lambda a| \le C\langle\lambda\rangle^{-1}a^{1/2}$. (See for example [loc. cit., Lemma 4.31].) Moreover, $|\partial_x a| + |\partial_\xi a| \le Ca^{1/2}$. Then recall that

$$\partial^{\varsigma}(a+\gamma)^{-1}=(a+\gamma)^{-1}\sum_{k=1}^{|\varsigma|}\sum_{\substack{\varsigma=\varpi^1+\cdots+\varpi^k\\|\varpi^j|\geq 1}}C_{\varpi^1,\dots,\varpi^k}\prod_{j=1}^{k}((a+\gamma)^{-1}\partial_x^{\varpi_{j,1}}\partial_\xi^{\varpi_{j,2}}\partial_\lambda^{\varpi_{j,3}}a). \tag{43}$$

Now,

$$|\partial_\lambda a|(a+\gamma)^{-1}\leq C\gamma^{-1/2}\langle\lambda\rangle^{-1}$$

and for $|\varpi|=1$

$$(|\partial_x^{\varpi}a|+|\partial_\xi^{\varpi}a|)(a+\gamma)^{-1}\leq C\gamma^{-1/2}.$$

Moreover, for $|(\varsigma,\varpi,k)|\geq 2$,

$$|\partial_x^{\varsigma}\partial_\xi^{\varpi}\partial_\lambda^k a|(a+\gamma)^{-1}\leq C\gamma^{-1}\langle\lambda\rangle^{-k}.$$

So,

$$\left|\prod_{j=1}^{k}(a+\gamma)^{-1}\partial_x^{\varpi_{j,1}}\partial_\xi^{\varpi_{j,2}}\partial_\lambda^{\varpi_{j,3}}a\right|\leq C\prod_{|\varpi|\geq 2}\gamma^{-1}\langle\lambda\rangle^{-\varpi_{j,3}}\prod_{|\varpi|=1}\gamma^{-1/2}\langle\lambda\rangle^{-\varpi_{j,3}}\leq C\langle\lambda\rangle^{-\varsigma_3}\gamma^{-|\varsigma|/2}.$$

Plugging this into (43) gives (42).

We now choose $\epsilon=1-\delta$. So, $a+\gamma\in S^0_{\delta,0}(\Sigma_0)\subset S^0_{\delta+\epsilon/2,\epsilon/2}(\Sigma_0)$. Then, write $a_1(x,\xi,\lambda_1)$ for the function such that

$$\underset{\delta}{\widetilde{\mathrm{Op}}_{\mathrm{h}}}(a)=\underset{\delta+\epsilon/2}{\widetilde{\mathrm{Op}}_{\mathrm{h}}}(a_1).$$

Write also $q_1=(a_1+\gamma)^{-1}$. So we can define $(a_1+\gamma)\,\sharp\,q_1$ Then, using Taylor's formula and letting $w=(x,\xi)$, $z=(y,\eta)$,

$$\begin{aligned}(a_1+\gamma)\,\sharp\,q_1&=e^{ihA(D)}(a_1+\gamma)|_{\lambda=h^{-\delta-\epsilon/2}\xi_1}q_1|_{\mu=h^{-\delta-\epsilon/2}\xi_1}\big|_{w=z}\\&=1+\int_0^1(1-t)e^{ithA(D)}(ihA(D))^2(a_1(w,h^{-\delta-\epsilon/2}\xi_1)q_1(z,h^{-\delta-\epsilon/2}\eta_1))\big|_{w=z}\,dt\\&=:1+r(z).\end{aligned}$$

Note that we have used $\{a_1+\gamma,(a_1+\gamma)^{-1}\}=0$. Now, $(ihA(D))^2(a_1+\gamma)\,\sharp\,q_1\in\tilde{h}S^0_{\delta+\epsilon/2,\epsilon/2}(\Sigma_0)$. So,

$$\|\underset{\delta+\epsilon/2}{\widetilde{\mathrm{Op}}_{\mathrm{h}}}(r)\|_{L^2\to L^2}\leq C\tilde{h}\leq\tfrac{1}{2}$$

for $\tilde{h}$ small enough. Thus, $\widetilde{\mathrm{Op}}_{\mathrm{h}\,\delta+\epsilon/2}(q)$ is an approximate right (and similarly left) inverse for $\widetilde{\mathrm{Op}}_{\mathrm{h}\,\delta}(a)+\gamma$. This implies that $(\widetilde{\mathrm{Op}}_{\mathrm{h}\,\delta}(a)+\gamma+\gamma_1)^{-1}$ exists for any $\gamma_1\geq 0$ Therefore,

$$\operatorname{Spec}(\underset{\delta}{\widetilde{\mathrm{Op}}_{\mathrm{h}}}(a))\subset[-\gamma,\infty).$$

Thus, by [Zworski 2012, Theorem C.8]

$$\langle\widetilde{\mathrm{Op}}_{\mathrm{h}}(a)u,u\rangle\geq-\gamma\|u\|^2_{L^2}.\qquad\square$$

Using the sharp Gårding inequality, it is not hard to prove:

Lemma 4.6. *Suppose* $a=\widetilde{O}(\langle\lambda\rangle^0)$. *Then,*

$$\langle\widetilde{\mathrm{Op}}_{\mathrm{h}}(a)^*\,\widetilde{\mathrm{Op}}_{\mathrm{h}}(a)u,u\rangle\leq(\sup|a|+Ch^{1-\delta})\|u\|^2_{L^2}.$$

4B. ***The global second microlocal calculus.*** Let $\Sigma \subset T^*M$ be a smooth compact hypersurface. Let V_i denote vector fields tangent to Σ and W_i denote any vector fields. Let $0 \le \delta < 1$. We define the symbol class $S_\delta^{k_1,k_2}(M;\Sigma)$ by $a \in S_\delta^{k_1,k_2}(M;\Sigma)$ if and only if

$$\begin{aligned} V_1\cdots V_{l_1}W_1\cdots W_{l_2}a &= O(h^{-\delta l_2}\langle h^{-\delta}d(\Sigma,\cdot)\rangle^{k_1}) && \text{near } \Sigma,\\ \partial_x^\varsigma\partial_\xi^\varpi a(x,\xi;h) &= O(h^{-\delta k_1}\langle\xi\rangle^{k_2-|\varpi|}) && \text{away from } \Sigma, \end{aligned} \tag{44}$$

where $d(\Sigma,\cdot)$ denotes the absolute value of any defining function of Σ that behaves like $\langle\xi\rangle$ near fiber infinity. Then we have the following.

Lemma 4.7. *For $0 \le \delta < 1$, there exists a class of operators, $\Psi_\delta^{k_1,k_2}(M;\Sigma)$, acting on $C^\infty(M)$ and maps*

$$\begin{aligned} \mathrm{Op}_{\mathrm{h},\Sigma} &: S_\delta^{k_1,k_2}(T^*M;\Sigma) \to \Psi_\delta^{k_1,k_2}(M;\Sigma),\\ \sigma_\Sigma &: \Psi_\delta^{k_1,k_2}(M;\Sigma) \to S_\delta^{k_1,k_2}(T^*M;\Sigma)/h^{1-\delta}S_\delta^{k_1-1,k_2-1}(T^*M;\Sigma) \end{aligned}$$

such that

$$\sigma_\Sigma(A\circ B) = \sigma_\Sigma(A)\sigma_\Sigma(B),$$

the sequence

$$0 \to h^{1-\delta}\Psi_\delta^{k_1-1,k_2-1}(M;\Sigma) \to \Psi_\delta^{k_1,k_2}(M;\Sigma) \xrightarrow{\sigma_\Sigma} S_\delta^{k_1,k_2}(T^*M;\Sigma)/h^{1-\delta}S_\delta^{k_1-1,k_2-1}(T^*M;\Sigma) \to 0$$

is a short exact sequence, and

$$\sigma_\Sigma\circ\mathrm{Op}_{\mathrm{h},\Sigma} : S_\delta^{k_1,k_2}(T^*M;\Sigma) \to S_\delta^{k_1,k_2}(T^*X;\Sigma)/h^{1-\delta}S_\delta^{k_1-1,k_2-1}(T^*M;\Sigma)$$

is the natural projection map.

As in, [Sjöstrand and Zworski 1999] near Σ it is possible to reduce all computations to the case where $\Sigma = \Sigma_0 := \{\xi_1 = 0\}$. We then have analogs of all the properties from the model case for the global calculus. We sometimes suppress M and T^*M in our notation, writing only $S_\delta^{k_1,k_2}(\Sigma)$ and $\Psi_\delta^{k_1,k_2}(\Sigma)$. We also sometimes suppress the Σ in $\mathrm{Op}_{\mathrm{h},\Sigma}$ to simplify notation.

5. The billiard ball flow and map

Recall that $\Omega \Subset \mathbb{R}^d$ is an open set with smooth boundary $\partial\Omega$. We need notation for the billiard ball flow and billiard ball map. Write ν for the outward-pointing unit normal to $\partial\Omega$. Then

$$S^*\mathbb{R}^d|_{\partial\Omega} = \partial\Omega_+ \sqcup \partial\Omega_- \sqcup \partial\Omega_0,$$

where $(x,\xi) \in \partial\Omega_+$ if ξ is pointing out of Ω (i.e., $\nu(\xi) > 0$), $(x,\xi) \in \partial\Omega_-$ if it points inward (i.e., $\nu(\xi) < 0$), and $(x,\xi) \in \partial\Omega_0$ if $(x,\xi) \in S^*\partial\Omega$. The points $(x,\xi) \in \partial\Omega_0$ are called *glancing* points. Let $B^*\partial\Omega$ be the unit coball bundle of $\partial\Omega$ and denote by $\pi_\pm : \partial\Omega_\pm \to B^*\partial\Omega$ and $\pi : S^*\mathbb{R}^d|_{\partial\Omega} \to \overline{B^*\partial\Omega}$ the canonical projections onto $\overline{B^*\partial\Omega}$. Then the maps $\pi_\pm$ are invertible. Finally, write

$$t_0(x,\xi) = \inf\{t > 0 : \exp_t(x,\xi) \in T^*\mathbb{R}^d|_{\partial\Omega}\},$$

where $\exp_t(x,\xi)$ denotes the lift of the geodesic flow to the cotangent bundle. That is, t_0 is the first positive time at which the geodesic starting at (x,ξ) intersects $\partial\Omega$.

We define the broken geodesic flow as in [Dyatlov and Zworski 2013, Appendix A]. Without loss of generality, we assume $t_0 > 0$. Fix $(x, \xi) \in S^*\mathbb{R}^d$ and set $t_0 = t_0(x, \xi)$. If $\exp_{t_0}(x, \xi) \in \partial\Omega_0$, then the billiard flow cannot be continued past t_0. Otherwise there are two cases: $\exp_{t_0}(x, \xi) \in \partial\Omega_+$ or $\exp_{t_0}(x, \xi) \in \partial\Omega_-$. We let

$$(x_0, \xi_0) = \begin{cases} \pi_-^{-1}(\pi_+(\exp_{t_0}(x, \xi))) \in \partial\Omega_-, & \text{if } \exp_{t_0}(x, \xi) \in \partial\Omega_+, \\ \pi_+^{-1}(\pi_-(\exp_{t_0}(x, \xi))) \in \partial\Omega_+, & \text{if } \exp_{t_0}(x, \xi) \in \partial\Omega_-. \end{cases}$$

We then define $\varphi_t(x, \xi)$, the *broken geodesic flow*, inductively by putting

$$\varphi_t(x, \xi) = \begin{cases} \exp_t(x, \xi), & 0 \le t < t_0, \\ \varphi_{t-t_0}(x_0, \xi_0), & t \ge t_0. \end{cases}$$

We introduce notation from [Safarov 1987] for the billiard flow. Let K be the set of ternary fractions of the form $0.k_1k_2, \dots,$ where $k_j = 0$ or 1 and S denote the left shift operator

$$S(0.k_1k_2\dots) = 0.k_2k_3\dots.$$

For $k \in K$, we define the billiard flow of type k, $G_k^t : S^*\mathbb{R}^d \to S^*\mathbb{R}^d$, as follows. For $0 \le t \le t_0$,

$$G_k^t(x, \xi) = \begin{cases} \varphi_t(x, \xi) & \text{if } k_1 = 0, \\ \exp_t(x, \xi) & \text{if } k_1 = 1. \end{cases} \tag{45}$$

Then, we define G_k^t inductively for $t > t_0$ by

$$G_k^t(x, \xi) = G_{Sk}^{t-t_0}(G_k^{t_0}(x, \xi)). \tag{46}$$

We call G_k^t the billiard flow of type k. By [Safarov 1987, Proposition 2.1], G_k^t is measure-preserving.

Remark 5.1. • In [Safarov 1987], geodesics could be of multiple types when total internal reflection occurred. However, in our situation, the metrics on either side of the boundary match, so there is no total internal reflection and geodesics are uniquely identified by their starting points and $k \in K$.

• In general, there exist situations where G_k^t intersects the boundary infinitely many times in finite time. However, since we work in convex domains, we need not consider this situation. For a proof of this fact, see the proof of Lemma 5.3. Note of course that the number of possible reflections in a given time T grows as one approaches glancing points.

Now, for $k \in K$ and $T > 0$, we define the set $\mathcal{O}_{T,k} \subset S^*\mathbb{R}^d$ to be the complement of the set of (x, ξ) such that one can define the flow G_k^t for $t \in [0, T]$. That is, $\mathcal{O}_{T,k}$ is the set for which the billiard flow of type k is glancing in time $0 \le t \le T$. Last, define the set

$$\mathcal{O}_T = \bigcup_{k \in K} \mathcal{O}_{k,T}. \tag{47}$$

The billiard ball map reduces the dynamics of G_0^k to the boundary. We define the billiard ball map as in [Guillemin and Uhlmann 1981]. Let $(x, \xi') \in B^*\partial\Omega$ and let $(x, \xi) = \pi_-^{-1}(x, \xi') \in \partial\Omega_-$ be the unique

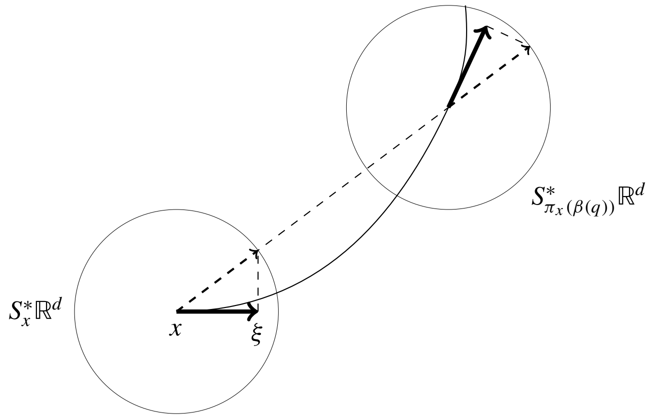


Figure 8. How the billiard ball map is constructed. Let $q=(x,\xi)\in B^*\partial\Omega$. The solid black arrow on the left denotes the covector $\xi\in B^*_x\partial\Omega$ and that on the right $\xi(\beta(q))\in B^*_{\pi_x(\beta(q))}\partial\Omega$. The center of the left circle is x and that of the right is $\pi_x(\beta(q))$.

inward-pointing covector with $\pi(x,\xi)=(x,\xi')$. Then, the billiard ball map $\beta : B^*\partial\Omega \to \overline{B^*\partial\Omega}$ maps (x,ξ') to the projection onto $T^*\partial\Omega$ of the first intersection of the billiard flow with the boundary. That is,

$$\beta : (x,\xi') \mapsto \pi(\exp_{t_0(x,\xi)}(x,\xi)). \tag{48}$$

Remark 5.2. • Just like the billiard flow, the billiard ball map is not defined for $(x,\xi')\in\pi(\partial\Omega_0)=S^*\partial\Omega$. However, since we consider convex domains, $\beta : B^*\Omega \to B^*\Omega$ and β^n is well-defined on $B^*\partial\Omega$.

• Figure 8 shows the process by which the billiard ball map is defined.

The billiard ball map is symplectic. This follows from the fact that the Euclidean distance function $|x-x'|$ is locally a generating function for β; that is, the graph of β in a neighborhood of (x_0,ξ_0,y_0,η_0) is given by

$$\{(x, -d_x|x-y|, y, d_y|x-y|) : (x,y)\in\partial\Omega\times\partial\Omega\}. \tag{49}$$

We denote the graph of β by C_b. For strictly convex Ω, C_b is given globally by (49).

We also write

$$\beta_E := (x(\beta(x,\xi/\sqrt{E})), \sqrt{E}\xi(\beta(x,\xi/\sqrt{E}))) : B^*_E\partial\Omega \to B^*_E\partial\Omega,$$

where $B^*_E\partial\Omega$ is the coball bundle of radius $\sqrt{E}$.

5A. ***Dynamics in strictly convex domains.*** We are interested in the behavior of the billiard ball map, $\beta(q)$, when $|\xi'(q)|_g$ is close to 1. Our interest in this region comes from a desire to understand how the reflection coefficients R from (18) behave when a wave travels nearly tangent to a strictly convex boundary.

Fix $q=(x_0,\xi_0)\in B^*\partial\Omega$ so that $\partial\Omega$ is strictly convex near x_0 and $|\xi_0|^2_g$ is sufficiently close to 1. Let $\gamma : [0,\delta) \to \partial\Omega$ be the unique length-minimizing geodesic connecting x_0 and $\pi_x(\beta(q))$. The existence and uniqueness of such a geodesic is guaranteed for $|\xi_0|^2_g$ close enough to 1 by the strict convexity of $\partial\Omega$.

Indeed, this follows from the fact that $l(q,\beta(q))\to 0$ as $|\xi_0|_g^2\to 1$ and the fact that the exponential map is a diffeomorphism for small times.

Let $s\in[0,\delta)$ have $\gamma(s)=\pi_x(\beta(q))$. We first examine how the normal component to $\partial\Omega$ changes under the billiard ball map. Let Δ_{ξ_d} denote the change in the normal component under β. Then

$$\Delta_{\xi_d}=\frac{((\gamma(s)-\gamma(0))\cdot\nu(0)-(\gamma(0)-\gamma(s))\cdot\nu(s))}{|\gamma(s)-\gamma(0)|}=\frac{(\gamma(s)-\gamma(0))\cdot(\nu(0)+\nu(s))}{|\gamma(s)-\gamma(0)|}.$$

Here $|\cdot|$ is the euclidean norm in $\mathbb{R}^d$ and ν is the inward-pointing unit normal.

First, note that

$$\begin{aligned}\gamma''(s)&=\kappa(s)\nu(s), & \nu'(s)\cdot\gamma'(s)&=-\kappa(s),\\ \gamma'(s)\cdot\nu(s)&=0, & \|\gamma'(s)\|&=\|\nu(s)\|=1,\end{aligned}$$

where $\kappa(s)$ is the curvature of the geodesic γ as a curve in $\mathbb{R}^d$. Then, expanding in Taylor series gives

$$\begin{aligned}\Delta_{\xi_d}[s+O(s^2)]&=[\gamma'(0)s+\gamma''(0)\tfrac{1}{2}s^2+\gamma^{(3)}(0)\tfrac{1}{6}s^3+O(s^4)]\cdot[2\nu(0)+\nu'(0)s+\nu''(0)\tfrac{1}{2}s^2+O(s^3)],\\ \Delta_{\xi_d}[1+O(s)]&=2\gamma'(0)\cdot\nu(0)+(\gamma'\cdot\nu)'(0)s+(2\gamma^{(3)}(0)\cdot\nu(0)+3(\gamma'\cdot\nu')'(0))\tfrac{1}{6}s^2+O(s^3)\end{aligned}$$

and

$$\begin{aligned}\Delta_{\xi_d}&=[2(\kappa'(0)\nu(0)-\kappa(0)\nu'(0))\cdot\nu(0)-3\kappa'(0)]\tfrac{1}{6}s^2+O(s^3)\\ &=(2\kappa'(0)-3\kappa'(0))\tfrac{1}{6}s^2+O(s^3)=-\kappa'(0)\tfrac{1}{6}s^2+O(s^3).\end{aligned}\tag{50}$$

Next observe that

$$\sqrt{1-|\xi'(q)|_g^2}=\frac{\gamma(s)-\gamma(0)}{|\gamma(s)-\gamma(0)|}\cdot\nu(0)=\tfrac{1}{2}\kappa(0)s+O(s^2).$$

Now, using $\kappa(0)>c>0$ for Ω strictly convex this implies

$$s=\frac{2\sqrt{1-|\xi'(q)|_g^2}}{\kappa(0)}+O((1-|\xi'|_g^2))$$

and therefore,

$$l(q,\beta(q))=|\gamma(s)-\gamma(0)|=s+O(s^2)=\frac{2}{\kappa(0)}\sqrt{1-|\xi'|_g^2}+O(1-|\xi'|_g^2).$$

Summarizing, we have:

Lemma 5.3. *Let $\Omega\subset\mathbb{R}^d$ be strictly convex. Then, for $q\in B^*\partial\Omega$ sufficiently close to $S^*\partial\Omega$*

$$\begin{aligned}\sqrt{1-|\xi'(\beta(q))|_g^2}&=\sqrt{1-|\xi'(q)|_g^2}+O(1-|\xi'(q)|_g^2),\\ l(q,\beta(q))&=\frac{2}{\kappa(0)}\sqrt{1-|\xi'|_g^2}+O(1-|\xi'|_g^2).\end{aligned}$$

This implies that set of $O(h^\epsilon)$ near glancing points is stable under the billiard ball map. This also follows from the equivalence of glancing hypersurfaces [Melrose 1976].

6. Boundary layer operators and potentials in the nonhomogeneous Friedlander model

Our goal is to give microlocal descriptions of the boundary layer operators and potentials near a glancing point. We start by considering the nonhomogeneous Friedlander model problem

$$\begin{gathered}((hD_{x_d})^2-\mu x_d+hD_{y_1})u=0,\quad u(0,y)=f(y),\\ u|_{x_d>0}\text{ outgoing},\quad \|u\|_{L^2((-\infty,0]\times\mathbb{R}^{d-1})}<\infty.\end{gathered}\tag{51}$$

Then, let $\mathcal{F}_h(u)$ denote the semiclassical Fourier transform in y,

$$\mathcal{F}_hu(x_d,\eta):=\frac{1}{(2\pi h)^{d-1}}\int u(x_d,y)e^{-(i/h)\langle y,\eta\rangle}\,dy.$$

Rescaling $w=h^{-2/3}\mu^{1/3}x_d$ gives

$$h^{2/3}\mu^{-1/3}(D_w^2-w+h^{-2/3}\mu^{-2/3}\eta_1)\mathcal{F}_h(u)(w,\eta)=0,\quad \mathcal{F}_h(u)(0,\eta)=\mathcal{F}_h(f)(\eta).$$

Hence, using (51)

$$\mathcal{F}_h(u)(x_d,\eta)=\begin{cases}\dfrac{Ai(-h^{-2/3}\mu^{1/3}x_d+h^{-2/3}\mu^{-2/3}\eta_1)}{Ai(h^{-2/3}\mu^{-2/3}\eta_1)}\mathcal{F}_h(f)(\eta), & x_d<0,\\ \dfrac{A_-(-h^{-2/3}\mu^{1/3}x_d+h^{-2/3}\mu^{-2/3}\eta_1)}{A_-(h^{-2/3}\mu^{-2/3}\eta_1)}\mathcal{F}_h(f)(\eta), & x_d>0.\end{cases}$$

So, the Dirichlet-to-Neumann map for the interior problem ($x_d<0$) is given by

$$\mathcal{F}_h(N_1f)(\eta)=-h^{-2/3}\mu^{1/3}\frac{Ai'(h^{-2/3}\mu^{-2/3}\eta_1)}{Ai(h^{-2/3}\mu^{-2/3}\eta_1)}\mathcal{F}_h(f)(\eta)$$

and that for the exterior problem ($x_d>0$) by

$$\mathcal{F}_h(N_2f)(\eta)=h^{-2/3}\mu^{1/3}\frac{A_-'(h^{-2/3}\mu^{-2/3}\eta_1)}{A_-(h^{-2/3}\mu^{-2/3}\eta_1)}\mathcal{F}_h(f)(\eta).$$

Remark 6.1. Since the goal of this section is only to present a simple model where the calculations are exact, we ignore the poles in N_1. It is possible to find the single- and double-layer operators and potentials without using the Dirichlet-to-Neumann map N_1 (see [Galkowski 2014, Section 4.5]; see also [Taylor 2011, Section 7.11] for a general introduction to layer potential methods), but it simplifies the presentation to do so here.

So, letting $\Theta_h(\eta)=h^{-2/3}\mu^{-2/3}\eta_1$, the single-layer operator is given by

$$\begin{aligned}\mathcal{F}_h(Gf)(\eta)=\mathcal{F}_h((N_1+N_2)^{-1}f)(\eta)&=h^{2/3}\mu^{-1/3}\frac{Ai(\Theta_h)A_-(\Theta_h)}{A_-'(\Theta_h)Ai(\Theta_h)-Ai'(\Theta_h)A_-(\Theta_h)}\mathcal{F}_h(f)(\eta)\\ &=h^{2/3}\mu^{-1/3}2\pi e^{\pi i/6}Ai(\Theta_h)A_-(\Theta_h)\mathcal{F}_h(f)(\eta)\end{aligned}$$

and the double-layer operator is given by

$$\mathcal{F}_h(Nf)(\eta)=\tfrac{1}{2}\mathcal{F}_h(f)(\eta)-\mathcal{F}_h(GN_2f)(\eta)=\big(\tfrac{1}{2}-2\pi e^{\pi i/6}Ai(\Theta_h)A_-'(\Theta_h)\big)\mathcal{F}_h(f)(\eta).$$

Therefore, since

$$\gamma^+\mathcal{S}\ell = G, \quad \gamma^+\mathcal{D}\ell = -\tfrac{1}{2}I + N,$$

and both solve the Friedlander model equation away from $x_d = 0$,

$$\mathcal{F}_h(\mathcal{S}\ell f) = h^{2/3}\mu^{-1/3}2\pi e^{\pi i/6}Ai(-h^{-2/3}\mu^{1/3}x_d + \Theta_h)A_-(\Theta_h)\mathcal{F}_h(f)(\eta),$$
$$\mathcal{F}_h(\mathcal{D}\ell f) = -2\pi e^{\pi i/6}Ai(-h^{-2/3}\mu^{1/3}x_d + \Theta_h)A_-'(\Theta_h)\mathcal{F}_h(f)(\eta).$$

Now, consider the kernel of $\mathcal{S}\ell^*\mathcal{S}\ell$,

$$\begin{aligned}
&\mathcal{S}\ell^*\mathcal{S}\ell(x', y') \\
&= \frac{4\pi^2\mu^{-2/3}h^{4/3}}{(2\pi h)^{2d-2}} \iint_{-\infty}^{0} \overline{Ai(-h^{-2/3}\mu^{1/3}w_1 + \Theta_h(\eta))}\,\overline{A_-(\Theta_h(\eta))} \\
&\qquad\qquad \times Ai(-h^{-2/3}\mu^{1/3}w_1 + \Theta_h(\xi))A_-(\Theta_h(\xi))e^{(i/h)(\langle x'-w',\eta\rangle + \langle w'-y',\xi\rangle)}\,dw_1\,d\xi\,dw'\,d\eta \\
&= \frac{4\pi^2\mu^{-1}h^2}{(2\pi h)^{d-1}} \iint_{\Theta_h(\xi)}^{\infty} |Ai(s)|^2|A_-(\Theta_h(\xi))|^2 e^{(i/h)\langle x'-y',\xi\rangle}\,d\xi \\
&= \frac{h^2}{\mu}\frac{1}{(2\pi h)^{d-1}} \int \Psi_{\mathcal{S}\ell}(\Theta_h(\xi))e^{(i/h)\langle x'-y',\xi\rangle}\,ds\,d\xi.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathcal{S}\ell^*\mathcal{D}\ell(x', y') &= -\frac{h^{4/3}}{\mu^{2/3}}\frac{1}{(2\pi h)^{d-1}} \int \overline{\Psi_{\mathcal{D}\ell\mathcal{S}\ell}(\Theta_h(\xi))}e^{(i/h)\langle x'-y',\xi\rangle}\,d\xi, \\
\mathcal{D}\ell^*\mathcal{S}\ell(x', y') &= -\frac{h^{4/3}}{\mu^{2/3}}\frac{1}{(2\pi h)^{d-1}} \int \Psi_{\mathcal{D}\ell\mathcal{S}\ell}(\Theta_h(\xi))e^{(i/h)\langle x'-y',\xi\rangle}\,d\xi, \\
\mathcal{D}\ell^*\mathcal{D}\ell(x', y') &= \frac{h^{2/3}}{\mu^{1/3}}\frac{1}{(2\pi h)^{d-1}} \int \Psi_{\mathcal{D}\ell}(\Theta_h(\xi))e^{(i/h)\langle x'-y',\xi\rangle}\,d\xi,
\end{aligned}$$

where

$$\begin{aligned}
\Psi_{\mathcal{S}\ell}(x) &:= 4\pi^2\int_x^{\infty} |Ai(s)|^2|A_-(x)|^2\,ds = 4\pi^2|A_-(x)|^2[(Ai'(x))^2 - x(Ai(x))^2], \\
\Psi_{\mathcal{D}\ell\mathcal{S}\ell}(x) &:= 4\pi^2\int_x^{\infty} |Ai(s)|^2A_-(x)\overline{A_-'(x)}\,ds = 4\pi^2A_-(x)\overline{A_-'(x)}[(Ai'(x))^2 - x(Ai(x))^2], \qquad (52) \\
\Psi_{\mathcal{D}\ell}(x) &:= 4\pi^2\int_x^{\infty} |Ai(s)|^2|A_-'(x)|^2\,ds = 4\pi^2|A_-'(x)|^2[(Ai'(x))^2 - x(Ai(x))^2],
\end{aligned}$$

since

$$\int_x^{\infty}(Ai(s))^2\,ds = (Ai'(x))^2 - x(Ai(x))^2.$$

Using the Wronskian we have $\Psi_{\mathcal{S}\ell}(\zeta_j) = 1$, where ζ_j is a zero of the Airy function, i.e., $Ai(\zeta_j) = 0$. Moreover, using asymptotics for the Airy function, as $x \to -\infty$,

$$\Psi_{\mathcal{S}\ell}(x) \sim 1, \quad \Psi_{\mathcal{D}\ell}(x) \sim -x, \quad \Psi_{\mathcal{D}\ell\mathcal{S}\ell} \sim i(-x)^{1/2}.$$

See Figure 9.

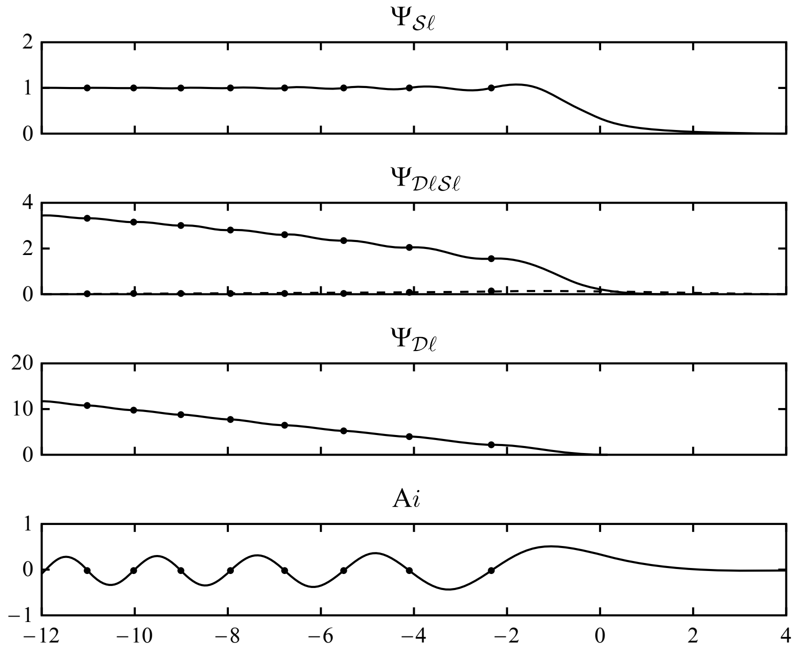


Figure 9. We plot the symbols of $\mathcal{S}\ell^*\mathcal{S}\ell$, $\mathcal{D}\ell^*\mathcal{S}\ell$ and $\mathcal{D}\ell^*\mathcal{D}\ell$. From top to bottom, the graphs show $\Psi_{\mathcal{S}\ell}$, $\Psi_{\mathcal{D}\ell\mathcal{S}\ell}$, $\Psi_{\mathcal{D}\ell}$. The bottom graph shows Ai for reference. In the graph of $\Psi_{\mathcal{D}\ell\mathcal{S}\ell}$, the imaginary part is shown in the solid line, and the real part in the dashed line. The black dots in each graph show $(\zeta_j, f(\zeta_j))$, where ζ_j are the zeros of $Ai(s)$ and f is one of $\Psi_{\mathcal{S}\ell}$, $\Psi_{\mathcal{D}\ell\mathcal{S}\ell}$, $\Psi\mathcal{D}\ell$ or Ai as described at the top of each graph.

7. Analysis of the boundary layer operators and potentials near glancing

Our next task is to show that analogs of all of the formulas for the boundary layer operators and potentials from Section 6 hold in the general case.

7A. ***Preliminaries for the general case.*** In order to make an analysis similar to that for the model case, we use the microlocal models for G, N, $\mathcal{S}\ell$, and $\mathcal{D}\ell$ developed in [Galkowski 2014, Section 4.5]. We recall the results here. The idea is to write a parametrix for the solution to the problem

$$(-h^2\Delta - z^2)u = L^*\delta_{\partial\Omega}\otimes g_1 + \delta_{\partial\Omega}\otimes g_2,$$

where f_i are microlocalized near glancing and $\delta_{\partial\Omega}$ denotes the surface measure on $\partial\Omega$. The parametrix for the problem will be a sum of oscillatory integrals of the form

$$\begin{aligned} H_1F &= (2\pi h)^{-d+1}\int (f_0 Ai(h^{-2/3}\rho) + ih^{1/3} f_1 Ai'(h^{-2/3}\rho))A_-(h^{-2/3}\Theta)e^{(i/h)\theta}\mathcal{F}_h(F)(\xi')\,d\xi', \\ H_2F &= (2\pi h)^{-d+1}\int (f_0 Ai(h^{-2/3}\rho) + ih^{1/3} f_1 Ai'(h^{-2/3}\rho))A'_-(h^{-2/3}\Theta)e^{(i/h)\theta}\mathcal{F}_h(F)(\xi')\,d\xi', \end{aligned} \tag{53}$$

where f_i solve certain transport equations and ρ, θ certain eikonal equations. The boundary values of f_0 and f_1 are determined by the limiting behavior of $\mathcal{D}\ell g_1$ and $\mathcal{S}\ell g_0$ at $\partial\Omega$.

Let $z = 1 + i\mu$ with $|\mu| \le Mh \log h^{-1}$. Then let $\epsilon(h) := \max(h, |\mu|)$. Let $(x_0, \xi_0) \in S^*\partial\Omega$ and suppose that in coordinates (x', x_d) near x_0, with $\partial\Omega = \{x_d = 0\}$ and $x_d > 0$ in Ω,

$$-h^2\Delta = \sum_{ij} a^{ij} hD_{x_i} hD_{x_j} + h\Big(\sum_i b^i hD_{x_i} + c\Big).$$

Then there exist

$$\rho(x, \xi'; h) = \rho_0 + \sum_j \rho_j \epsilon(h)^j, \quad \theta(x, \xi'; h) = \theta_0 + \sum_j \theta_j \epsilon(h)^j$$

solving the eikonal equations

$$\begin{cases} z^2 + O(h^\infty) = \langle ad\theta, d\theta\rangle - \rho\langle ad\rho, d\rho\rangle, \\ O(h^\infty) = 2\langle d\theta, d\rho\rangle \end{cases}$$

on $\rho_0 \le 0$ and in Taylor series at $\rho_0 = 0$, $x_d = 0$. Here, ρ_0, θ_0 are real-valued solving

$$\begin{cases} 1 = \langle ad\theta_0, d\theta_0\rangle - \rho\langle ad\rho_0, d\rho_0\rangle, \\ 0 = 2\langle d\theta_0, d\rho_0\rangle \end{cases}$$

on $\rho_0 \le 0$ and in Taylor series at $\rho_0 = 0$, $x_d = 0$. We need a few additional properties of ρ and θ. In particular,

$$\rho_0|_{\partial\Omega} = \xi_1, \quad \partial_{x_d}\rho_0|_{\partial\Omega} > 0, \quad \partial^2_{x'\xi'}\theta_0|_{\partial\Omega} \ne 0 \tag{54}$$

and $\theta_{0b} := \theta_0|_{\partial\Omega}$ has that

$$\kappa : (\partial_{\xi'}\theta_{0b}(x', \xi'), \xi') \mapsto (x', \partial_{x'}\theta_{0b}(x', \xi')) \tag{55}$$

is a symplectomorphism reducing the billiard ball map for the Friedlander-model case to that for Ω. We also write $\theta_b = \theta|_{\partial\Omega}$. Next, let

$$\Theta := \rho|_{\partial\Omega} = \xi_1 + i\epsilon(h), \quad \Theta_0 := \rho_0|_{\partial\Omega} = \xi_1.$$

Finally, there exist

$$f_i \sim \sum_{j=0}^{\infty} f_{i,j} h^j, \quad i = 0, 1,$$

with $f_{0b} := f_0|_{\partial\Omega}$ having $|f_{0b}| > c > 0$ and $g_1|_{\partial\Omega} = 0$ solving

$$\begin{cases} 2\langle ad\theta_0, df_{0,n}\rangle + 2\rho_0\langle ad\rho_0, df_{1,n}\rangle + \langle b, df_{0,n}\rangle + \langle ad\rho_0, d\rho_0\rangle f_{1,n} \\ \qquad\qquad - P_2\theta_0 f_{1,n} - \rho_0(P_2\rho_0) f_{1,n} = F_{1,n}(\theta, \rho, f_{i,m<n}, \mu), \\ 2\langle ad\rho_0, df_{0,n}\rangle - 2\langle ad\theta_0, df_{1,n} - \langle b, df_{1,n}\rangle\rangle - (P_2\rho_0) f_{0,n} + (P_2\theta_0) f_{1,n} = F_{2,n}(\theta, \rho, f_{i,m<n}, \mu) \end{cases} \tag{56}$$

on $\rho_0 \le 0$ and in Taylor series at $\rho_0 = 0$, $x_d = 0$ so that for H_i as in (53) $(-h^2\Delta - z^2)H_i F = O_{\Psi^{-\infty}}(h^\infty)F$ whenever F is supported h^ϵ-close to $\xi_1 = 0$. If $|\mu| \le Ch$, then this also holds when F is supported δ-close to $\xi_1 = 0$ for δ small enough.

7A1. *Identification of* $\partial_{x_d}\rho_0|_{x_d=0}$ *and* $|\partial_{y'}\theta_{0b}|_g^2$. It will be useful to have the values of $\partial_{x_d}\rho_0|_{x_d=0}$ and $|\partial_{y'}\theta_{0b}|_g^2$. To obtain these, we simply write the eikonal equations in normal geodesic coordinates. Recall that in normal geodesic coordinates (y', x_d) with $x_d > 0$ in Ω,

$$-h^2\Delta = (hD_{x_d})^2 + R(y', hD_{y'}) + 2x_d Q(x_d, y', hD_{y'}) + hF(x_d, y')hD_{x_d},$$

where

$$R(y', D_{y'}) = -\Delta_{\partial\Omega} = \bar{g}^{-1/2}\sum_{ij} D_{y_i}\bar{g}^{1/2}g^{ij}D_{y_j}, \quad \bar{g} = (\det(g^{ij}))^{-1/2},$$
$$Q(0, y', D_{y'}) = \sum_{ij} D_{y_j}\bar{g}^{1/2}a_{ij}D_{y_i},$$

where $Q(y', \xi') = \sum_{ij} a_{ij}(y')\xi_i\xi_j$ is the second fundamental form of $\partial\Omega$ lifted to $T^*\partial\Omega$, $g^{ij} = g^{ij}(y')$ is the metric on $T^*\partial\Omega$, and $R(y', \xi') = \sum_{ij} g^{ij}\xi_i\xi_j$ is the symbol of $-h^2\Delta_{\partial\Omega}$.

Using the eikonal equations for ρ_0 and θ_0 in these coordinates,

$$\begin{cases} 1 = (\partial_{x_d}\theta_0)^2 + R(y', \partial_{y'}\theta_0) + 2x_d(Q(y', \partial_{y'}\theta_0) + O(x_d))] \\ \qquad\qquad - \rho_0[(\partial_{x_d}\rho_0)^2 + R(y', \partial_{y'}\rho_0) + 2x_d(Q(y', \partial_{y'}\rho_0) + O(x_d))], \\ 0 = 2(\partial_{x_d}\theta_0\partial_{x_d}\rho_0 + g^{ij}\partial_{y_i}\theta_0\partial_{y_j}\rho_0 + 2x_d(a_{ij}\partial_{y_i}\theta_0\partial_{y_j}\rho_0 + O(x_d)). \end{cases}$$

Now, we know that $\rho_0|_{x_d=0} = \xi_1$ and $\partial_{x_d}\rho_0|_{x_d=0} > 0$. So, evaluation at $x_d = 0$ shows

$$1 = (\partial_{x_d}\theta_0)^2 + R(y', \partial_{y'}\theta_0) - \xi_1(\partial_{x_d}\rho_0)^2 = R(y', \partial_{y'}\theta_0) - \xi_1(\partial_{x_d}\rho_0)^2,$$
$$0 = \partial_{x_d}\theta_0.$$

Moreover, differentiating the first equation in x_d and the second in y' and evaluating at $x_d = 0$ shows

$$0 = 2g^{ij}\partial^2_{x_d y_j}\theta_0\partial_{y_i}\theta_0 + 2Q(y', \partial_{y'}\theta_0) - (\partial_{x_d}\rho_0)^3 - 2\xi_1\partial^2_{x_d}\rho_0\partial_{x_d}\rho_0,$$
$$0 = 2(\partial^2_{y'x_d}\theta_0\partial_{x_d}\rho_0).$$

Hence,

$$(\partial_{x_d}\rho_0)^3|_{x_d=0} = 2Q(y', \partial_{y'}\theta_0) - 2(\xi_1\partial^2_{x_d}\rho_0\partial_{x_d}\rho_0)|_{x_d=0} = 2Q(y; \partial_{y'}\theta_{0b}) + O(\xi_1),$$
$$R(y', \partial_{y'}\theta_{0b}) = |\partial_{y'}\theta_{0b}|_g^2 = 1 + \xi_1(\partial_{x_d}\rho_0)^2|_{x_d=0}.$$

The implicit function theorem then implies that with $\xi' = \partial_{y'}\theta_{0b}$,

$$\xi_1 = \frac{|\xi'|_g^2 - 1}{(2Q(y, \xi'))^{2/3}} + O((|\xi'|_g^2 - 1)^2), \quad \partial_{x_d}\rho = 2Q(y, \xi') + O(|\xi'|_g^2 - 1). \tag{57}$$

Now, in coordinates $(x, \xi) = \kappa^{-1}(y, \eta)$, where κ is as in (55), we have

$$\beta(x, \xi) = (x_1 - 2\sqrt{-\xi_1}, x', \xi)$$

since κ reduces the Friedlander model to the billiard ball map for Ω. Let φ_i be a partition of unity on $1 - \epsilon \le |\xi'|_g \le 1 + \epsilon$ for some $\epsilon > 0$ small enough so that on $\operatorname{supp}\varphi_i$ κ_i^{-1} is defined, with κ_i given by

(55). Let

$$\Xi := \sum_i \varphi_i \xi_1(\kappa_i^{-1}(x,\xi)). \tag{58}$$

Then we have the following lemma given the existence of an *approximate interpolating Hamiltonian* for the billiard ball map. In particular, the lemma follows from the equivalence of glancing hypersurfaces [Melrose 1976] (see [Kovachev and Popov 1990, Proposition 3.1] for a proof; see also [Marvizi and Melrose 1982]).

Lemma 7.1. *Let* Ξ *be as in* (58)*. Then at* $S^*\partial\Omega$, $\Xi = 0$, $|d\Xi| > 0$ *and* $\Xi < 0$ *in* $B^*\partial\Omega$*. Moreover*,

$$\Xi \circ \beta(q) - \Xi(q) = O((|\xi'|_g^2 - 1)^\infty),$$

$$\beta(q) - \exp(-2\sqrt{-\Xi}H_\Xi)(q) = O((|\xi'|_g^2 - 1)^\infty),$$

$$\Xi(x', \xi') = \frac{|\xi'|_g^2 - 1}{(2Q(x',\xi'))^{2/3}} + O((|\xi'|_g^2 - 1)^2).$$

7A2. *Microlocal description of the boundary layer potentials and operators.* We now recall the microlocal descriptions of the boundary layer potentials and operators near glancing from [Galkowski 2014, Section 4.5]. Let $\mathcal{A}i$, $\mathcal{A}i'$, $\mathcal{A}_-$, and $\mathcal{A}_-'$ denote the Fourier multiplier with multiplier $Ai(\Theta_h)$, $Ai'(\Theta_h)$, $A_-(\Theta_h)$, and $A_-'(\Theta_h)$, where for convenience we define

$$\Theta_h := h^{-2/3}\Theta, \quad \Theta_{0h} := h^{-2/3}\Theta_0, \quad \rho_h := h^{-2/3}\rho, \quad \rho_{0h} := h^{-2/3}\rho.$$

Next, let

$$Jf := (2\pi h)^{-d+1} \int f_{0b} e^{(i/h)(\theta_0 + \langle x'-y', \xi'\rangle)} f(y')\, dy'\, d\xi',$$

$$JCf := (2\pi h)^{-d+1} \int f_{0b} (\partial_{x_d}\rho + ih\partial_{x_d} g_1)|_{x_d=0} e^{(i/h)(\theta_0 + \langle x'-y', \xi'\rangle)} f(y')\, dy'\, d\xi',$$

$$JBf := (2\pi h)^{-d+1} \int f_{0b} \partial_{x_d} g_0|_{x_d=0} e^{(i/h)(\theta_0 + \langle x'-y', \xi'\rangle)} f(y')\, dy'\, d\xi'.$$

Then J is an elliptic semiclassical Fourier integral operator quantizing the reduction of the Friedlander glancing pair to the glancing pair $\partial\Omega$, $S^*\mathbb{R}^d$ and it is not hard to check that $B, C \in \Psi(\partial\Omega)$ so that for any $\delta > 0$,

$$\sigma(JCJ^{-1}) = (2Q(x,\xi'))^{1/3} + O_{S_\delta}(h^{1-2\delta}),$$

where Q is the second fundamental form lifted to the cotangent bundle, $T^*\partial\Omega$. Thus C is elliptic.

Lemma 7.2. *Suppose that* $(x_0, \xi_0) \in S^*\partial\Omega$ *and* $\zeta \in C_c^\infty(\mathbb{R}^d)$ *have* $\zeta \equiv 1$ *on* $[-1,1]$ *with* $\operatorname{supp}\zeta \subset [-2,2]$. *Then there exists* $\delta > 0$ *such that for any* $M, \epsilon > 0$ *if* $|\operatorname{Im} z| \le Mh\log h^{-1}$,

$$GXg = h^{2/3}\omega^{-1}\chi J\mathcal{A}i\mathcal{A}_- C^{-1}J^{-1}Xg + O_{\Psi^{-\infty}}(h^\infty)g,$$

$$NXg = \left(\tfrac{1}{2}\mathrm{Id} - \omega^{-1}\chi J(\mathcal{A}i\mathcal{A}_-' + h^{2/3}\mathcal{A}_-\mathcal{A}iC^{-1}B)J^{-1}\right)Xg + O_{\Psi^{-\infty}}(h^\infty)g,$$

$$(\mathcal{S}\ell Xg)|_\Omega = \omega^{-1}h^{2/3}A_{1,g}JC^{-1}J^{-1}Xg + O_{\mathcal{D}'\to C^\infty}(h^\infty)g,$$

$$(\mathcal{D}\ell Xg)|_\Omega = -\omega^{-1}A_{2,g}Xg - h^{2/3}\omega^{-1}A_{1,g}JC^{-1}BJ^{-1}Xg + O_{\mathcal{D}'\to C^\infty}(h^\infty)g,$$

where $\omega = e^{-\pi i/6}/(2\pi)$,

$$\chi := \zeta((3\delta)^{-1}|x-x_0|), \quad A_{1,g} := \chi H_1 J^{-1}, \quad A_{2,g} := \chi H_2 J^{-1},$$
$$X := \mathrm{Op_h}[\zeta(\delta^{-1}(|x-x_0| + |\xi' - \xi_0|_g))\zeta(h^{-\epsilon}\delta^{-1}||\xi'|_g - 1|)].$$

If we only allow $|\operatorname{Im} z| \le Mh$, *then we can set* $\epsilon = 0$.

A simple calculation shows that on $-Mh^{2/3} \le \xi_1$,

$$Ai A_-(\Theta_h) \in \Psi^{-1/2,-1/2}_{2/3}(\xi_1 = 0),$$

and on $-Ch^{\epsilon} \le \xi_1 \le -Mh^{2/3}$,

$$Ai A_-(\Theta_h) \in h^{-1/4+\epsilon/4}\Psi^{-1/2,-1/2}_{1-\epsilon/2}(\xi_1 = 0).$$

Moreover, for $\xi_1 \ge Mh^{2/3}$,

$$2\pi e^{\pi i/6} Ai A_-(\Theta_h) = \frac{h^{1/3}}{2\sqrt{\xi_1}}(1 + O(h(\xi_1)^{-3/2})).$$

So, using (57)

$$\begin{aligned}
&\sigma(Jh^{2/3}\omega^{-1}\chi \mathcal{A}i\mathcal{A}_- C^{-1}J^{-1}X)\\
&= \frac{h}{2\sqrt{\xi_1(\kappa^{-1}(q))}}(1+O(h(\xi_1)^{-3/2}))\frac{1}{\partial_{x_d}\rho(\kappa^{-1}(q))}\zeta(\delta^{-1}(|x-x_0|+|\xi'-\xi_0|_g))\zeta(h^{-\epsilon}\delta^{-1}||\xi'|_g-1|)\\
&= \frac{h}{2\sqrt{|\xi'|_g^2-1}}(1+O(h(|\xi'|_g^2-1)^{-3/2}))\ \zeta(\delta^{-1}(|x-x_0|+|\xi'-\xi_0|_g))\zeta(h^{-\epsilon}\delta^{-1}||\xi'|_g-1|). \qquad (59)
\end{aligned}$$

Finally, we recall the decomposition of the boundary layer operators away from glancing from [Galkowski 2014, Lemma 4.27]. For a similar decomposition when $\operatorname{Im} z = 0$ see [Hassell and Zelditch 2004, Proposition 4.1].

Lemma 7.3. *Let* $\Omega \subset \mathbb{R}^d$ *be strictly convex with* $\partial\Omega \in C^\infty$. *Then for all* $\frac{1}{2} > \epsilon,\ \gamma > 0$, *and* $z = E + O(h^{1-\gamma})$ *with* $\operatorname{Im} z \ge -Ch\log h^{-1}$. *Then*

$$G(z/h) := G_\Delta(z) + G_B(z) + G_g(z) + O_{\mathcal{D}'\to C^\infty}(h^\infty),$$
$$N(z/h) := N_\Delta(z) + N_B(z) + N_g(z) + O_{\mathcal{D}'\to C^\infty}(h^\infty),$$
$$\partial_\nu \mathcal{D}\ell(z/h) := \partial_\nu \mathcal{D}\ell_\Delta(z) + \partial_\nu \mathcal{D}\ell_B(z) + \partial_\nu \mathcal{D}\ell_g(z) + O_{\mathcal{D}'\to C^\infty}(h^\infty),$$

where $G_\Delta \in h^{1-\epsilon/2}\Psi^{-1}_\epsilon,\ N_\Delta \in h^{1-2\epsilon}\Psi^{-1}_\epsilon,\ \partial_\nu\mathcal{D}\ell_\Delta \in h^{-1}\Psi^1_\epsilon$, *and*

$$G_B \in h^{1-\epsilon/2} e^{(\operatorname{Im} z)_- d_\Omega/h} I^{\mathrm{comp}}_\delta(C_b),$$
$$N_B \in e^{(\operatorname{Im} z)_- d_\Omega/h} I^{\mathrm{comp}}_\delta(C_b),$$
$$\partial_\nu \mathcal{D}\ell_B \in h^{-1} e^{(\operatorname{Im} z)_- d_\Omega/h} I^{\mathrm{comp}}_\delta(C_b)$$

are FIOs associated to β_E, *where* $\delta = \max(\epsilon, \gamma)$. *Moreover,*

$$\mathrm{MS_h}'((\,\cdot\,)_B) \subset \big\{(q,p) \in B_E^*\partial\Omega \times B_E^*\partial\Omega : \min(E - |\xi'(q)|_g,\ E - |\xi'(q)|_g,\ l(q,p)) > ch^{\epsilon}\big\},$$

$$\mathrm{MS_h}'((\,\cdot\,)_g) \subset \big\{(q,p) \in T^*\partial\Omega \times T^*\partial\Omega : \max(|E - |\xi'(q)|_g|,\ |E - |\xi'(p)|_g|, l(q,p)) < ch^{\epsilon}\big\},$$

$$\sigma(G_\Delta) = \frac{ih}{2\sqrt{E^2 - |\xi'|_g^2}}, \quad \sigma(\partial_\nu \mathcal{D}\ell_\Delta) = \frac{ih^{-1}\sqrt{E^2-|\xi'|_g^2}}{2},$$

$$\sigma(G_B e^{(\operatorname{Im} z/h)\operatorname{Op_h}(l(q,\beta_E(q)))}) = \frac{he^{(i/h)\operatorname{Re}\omega_0 l(q,\beta_E(q))}}{2(E^2-|\xi'(\beta_E(q))|_g^2)^{1/4}(E^2-|\xi'(q)|_g^2)^{1/4}}dq^{1/2},$$

$$\sigma(N_B e^{(\operatorname{Im} z/h)\operatorname{Op_h}(l(q,\beta_E(q)))}) = \frac{-ie^{(i/h)\operatorname{Re}\omega_0 l(q,\beta_E(q))}(E^2-|\xi'(q)|_g^2)^{1/4}}{2(E^2-|\xi'(\beta_E(q))|_g^2)^{1/4}}dq^{1/2},$$

$$\sigma(\partial_\nu \mathcal{D}\ell_B e^{(\operatorname{Im} z/h)\operatorname{Op_h}(l(q,\beta_E(q)))}) = \frac{h^{-1}e^{(i/h)\operatorname{Re}\omega_0 l(q,\beta_E(q))}(E^2-|\xi'(\beta_E(q))|_g^2)^{1/4}(E^2-|\xi'(q)|_g^2)^{1/4}}{2}dq^{1/2},$$

where we take $\sqrt{z} = \sqrt{|z|}e^{(1/2)\operatorname{Arg}(z)}$ *for* $-\frac{\pi}{2} < \operatorname{Arg}(z) < \frac{3\pi}{2}$.

Remark 7.4. The decomposition in [Hassell and Zelditch 2004] is slightly less precise than that in [Galkowski 2014] because the glancing pieces are microlocalized to a neighborhood of $S^*\partial\Omega \times S^*\partial\Omega$ rather than to a neighborhood of $S^*\partial\Omega \times S^*\partial\Omega \cap \Delta(T^*\partial\Omega)$, where $\Delta(T^*\partial\Omega)$ denotes the diagonal.

In particular, Lemma 7.3, together with (59), implies that there exists $M > 0$ such that for $\chi = \chi(|\xi'|_g) \in \Psi^{0,0}_{2/3}(|\xi'|_g = 1)$ with $\operatorname{supp}\chi \subset \{|\xi'|_g \geq 1 + Mh^{2/3}\}$, we have $G\operatorname{Op_h}(\chi) \in h^{2/3}\Psi^{-1/2,-1/2}_{2/3}(|\xi'|_g = 1)$ with

$$\sigma(G\operatorname{Op_h}(\chi)) = \frac{h\chi(|\xi'|_g)}{2\sqrt{|\xi'|_g^2 - 1}}(1 + O(h(|\xi'|_g^2 - 1)^{-3/2})). \tag{60}$$

7B. *Analysis of $\mathcal{S}\ell^*\mathcal{S}\ell$, $\mathcal{D}\ell^*\mathcal{D}\ell$, and $\mathcal{D}\ell^*\mathcal{S}\ell$ near glancing.* Our next goal is to understand $\mathcal{S}\ell^*\mathcal{S}\ell$, $\mathcal{D}\ell^*\mathcal{D}\ell$, and $\mathcal{D}\ell^*\mathcal{S}\ell$ microlocally near glancing points. To do this, we will use the microlocal description of $\mathcal{S}\ell$ and $\mathcal{D}\ell$ from Lemma 7.2. In particular, let J_1 be a microlocally unitary FIO quantizing κ, where κ is as in (55). Then we prove:

Lemma 7.5. *Fix* $z = 1 + i\mu$ *with* $|\mu| \leq Mh\log h^{-1}$. *Then for any* $\epsilon > 0$ *and* $\delta \leq \frac{2}{3}$, *for* $\chi \in \Psi^{0,0}_{2/3}(|\xi'|_g = 1)$ *self-adjoint with* $\operatorname{WF_h}(\chi) \subset \{||\xi'|_g - 1| \leq h^{\delta}\}$,

$$\chi\,\mathcal{S}\ell^*\mathcal{S}\ell\,\chi \in h^{2-\epsilon}\Psi^{0,0}_{1-\delta/2}(\{|\xi'|_g = 1\}), \quad \chi\,\mathcal{D}\ell^*\mathcal{S}\ell\,\chi,\ \chi\,\mathcal{S}\ell^*\mathcal{D}\ell\,\chi \in h^{3/2-\delta/4-\epsilon}\Psi^{0,1/2}_{1-\delta/2}(\{|\xi'|_g = 1\}),$$
$$\chi\,\mathcal{D}\ell^*\mathcal{D}\ell\,\chi \in h^{1-\delta/2-\epsilon}\Psi^{0,1}_{1-\delta/2}(\{|\xi'|_g = 1\}).$$

Moreover,

$$\sigma(J_1^*\chi\,\mathcal{S}\ell^*\mathcal{S}\ell\,\chi J_1) = \frac{h^2\Psi_{\mathcal{S}\ell}(h^{-2/3}\Theta_0(\xi'))\chi^2(\kappa(x',\xi'))}{2Q(\kappa(x',\xi'))},$$

$$\sigma(J_1^*\chi\,\mathcal{D}\ell^*\mathcal{S}\ell\,\chi J_1) = \frac{h^{4/3}\Psi_{\mathcal{D}\ell\mathcal{S}\ell}(h^{-2/3}\Theta_0(\xi'))\chi^2(\kappa(x',\xi'))}{(2Q(\kappa(x',\xi')))^{2/3}},$$

$$\sigma(J_1^*\chi\,\mathcal{S}\ell^*\mathcal{D}\ell\,\chi\,J_1)=\frac{h^{4/3}\overline{\Psi_{\mathcal{D}\ell\mathcal{S}\ell}(h^{-2/3}\Theta_0(\xi'))}\chi^2(\kappa(x',\xi'))}{(2Q(\kappa(x',\xi')))^{2/3}},$$

$$\sigma(J_1^*\chi\,\mathcal{D}\ell^*\mathcal{D}\ell\,\chi\,J_1)=\frac{h^{2/3}\Psi_{\mathcal{D}\ell}(h^{-2/3}\Theta_0(\xi'))\chi^2(\kappa(x',\xi'))}{(2Q(\kappa(x',\xi')))^{1/3}}.$$

We prove this lemma using Lemma 7.2 to write a parametrix for $\mathcal{S}\ell^*\mathcal{S}\ell$. We then Taylor expand the Airy functions around their values at the boundary of Ω and estimate each of the terms. The higher-order terms in the expansion will turn out to be lower-order in h and the symbols will be found by computing the first term. The operators $\mathcal{D}\ell^*\mathcal{S}\ell$, $\mathcal{S}\ell^*\mathcal{D}\ell$ and $\mathcal{D}\ell^*\mathcal{D}\ell$ are handled similarly.

7B1. *Estimates on the remainder terms.* We first give estimates on the size of terms that will be lower-order. These terms arise from a Taylor expansion of the integrand when computing $\mathcal{S}\ell^*\mathcal{S}\ell$ using the microlocal model from Lemma 7.2. In particular, consider an operator with kernel given by

$$\begin{aligned}R_{ijklmno}=(2\pi h)^{-2d+2}\iint_0^\infty & b(w,x',y',\eta,\xi')h^{-2/3(j+k)}(\rho(w,\eta')-\rho(w,\xi'))^k(\Theta(\eta')-\Theta(\xi'))^j w_d^n\\ &\times Ai^{(l)}(\rho_h(w,\xi'))A_-^{(m)}(\Theta_h(\xi'))\overline{Ai^{(o)}(\rho_h(w,\xi'))A_-^{(i)}(\Theta_h(\xi'))}\\ &\times e^{(i/h)(\theta(w,\xi')-\theta(w,\eta')-\theta_b(y',\xi')+\theta_b(x',\eta'))}\,dw_d\,d\xi'\,d\eta'\,dw',\end{aligned}$$

where $b\in S_\delta(\xi_1=0)$ is supported in $|\Theta(\xi')|,|\Theta(\eta')|\le Ch^\delta$. First, observe that since $\partial_{x_d}\rho_0>0$ and

$$Ai(t)\le Ce^{-t^{3/2}}\quad\text{for } t\gg 1,$$

we may assume that b is supported on $w_d<\epsilon$ for any $\epsilon>0$ by introducing an $O(e^{-C/h})$ error. Next, notice that

$$\theta(w,\xi')-\theta(w,\eta')=\theta_b(w',\xi')-\theta_b(w',\eta')+w_d^2\langle\xi'-\eta',r(w,\xi',\eta')\rangle.$$

So,

$$\partial_{w'}\theta(w,\xi')-\partial_{w'}\theta(w,\eta')=(\partial^2_{x'\xi'}\theta_b(w',\eta')+w_d^2\partial_{w'}r)(\xi'-\eta')$$

and, using that $\partial^2_{x'\xi'}\theta_b\neq 0$, for w_d small enough, the phase is stationary precisely at $\xi'=\eta'$.

We first change variables so that $W_d=h^{-2/3}\rho_0(w,\xi')$. Then, $w_d=h^{2/3}e(W_d,w',\xi')(W_d-h^{-2/3}\Theta_0(\xi'))$, where e is elliptic. So, the kernel takes the form

$$\begin{aligned}&R_{ijklmno}\\ &\quad=(2\pi h)^{-2d+2}h^{2/3}\iint_{h^{-2/3}\Theta_0(\xi')}^\infty b_1(h^{2/3}W_d,w',x',y',\eta,\xi')h^{-2/3(j+k-n)}\\ &\qquad\qquad\times\big(\rho(w_d(W_d,w',\xi'),\eta')-h^{2/3}W_d-\epsilon(h)\rho_1\big)^k\\ &\qquad\qquad\times(\Theta(\eta')-\Theta(\xi'))^j(W_d-h^{-2/3}\Theta_0(\xi'))^n\\ &\qquad\qquad\times Ai^{(l)}(W_d+h^{-2/3}\epsilon(h)\rho_1(w,\xi'))A_-^{(m)}(\Theta_h(\xi'))\\ &\qquad\qquad\times\overline{Ai^{(o)}(W_d+h^{-2/3}\epsilon(h)\rho_1(w,\xi'))A_-^{(i)}(\Theta_h(\xi'))}\,e^{\frac{i}{h}\Gamma}dW_d\,d\xi'\,d\eta'\,dw',\end{aligned}$$

where

$$\Gamma=\theta_b(w',\xi')-\theta_b(w',\eta')-\theta_b(y',\xi')+\theta_b(x',\eta')+h^{4/3}(W_d-\Theta_0(\xi'))^2\langle\xi'-\eta',r(W_d,w',x',\xi',\eta')\rangle.$$

Now, the integrand vanishes to order $|\xi'-\eta'|^{j+k}$, and the phase is stationary in w' precisely at $\xi'=\eta'$. Hence, integrating by parts $j+k$ times in w' and then applying stationary phase in the w', η'variables gives a finite sum of terms (possibly with additional positive powers of h) of the form

$$\begin{aligned}\frac{h^{2/3}}{(2\pi h)^{d-1}}\iint_{h^{-2/3}\Theta_0(\xi')}^{\infty} & b_2(h^{2/3}W_d, x'+O((W_d-h^{-2/3}\Theta_0(\xi'))^2h^{4/3}), x', y', \xi', \xi')\\ & \times h^{1/3(j+k+2(n-p-q))}\epsilon(h)^{p+q}(W_d-h^{-2/3}\Theta_0(\xi'))^n\\ & \times Ai^{(l+p)}(W_d+h^{-2/3}\epsilon(h)\rho_1)A_-^{(m)}(\Theta_h(\xi'))\\ & \times \overline{Ai^{(o+q)}(W_d+h^{-2/3}\epsilon(h)\rho_1)A_-^{(i)}(\Theta_h(\xi'))}e^{(i/h)(\theta_b(x',\xi')-\theta_b(y',\xi'))}\,dW_d\,d\xi'.\end{aligned}$$

Note that we can apply stationary phase in the w', η'variables since $\partial^2_{x'\xi'}\theta_{0b}\neq 0$. Next, change variables $\xi'\mapsto \Xi'(x',y',\xi')$ so that

$$\theta_{0b}(x',\xi')-\theta_{0b}(y',\eta')=\langle x'-y', \Xi'(x',y',\xi')\rangle.$$

To find such a change of variables, observe that $\Xi(x',x',\xi')=\partial_{x'}\theta_{0b}$ and hence $\partial_{\xi'}\Xi=\partial^2_{\xi'x'}\theta_{0b}\neq 0$ so we can apply the implicit function theorem. Then, integrating in W_d and using the fact that on $\operatorname{supp} b_2$, $|\Theta(\xi')|\leq Ch^{\delta}$, we obtain

$$R_{ijklmno}=(2\pi h)^{-d+1}\int b_3(x',y',\xi';h)e^{(i/h)\langle x'-y',\xi'\rangle}\,d\xi',$$

where, letting

$$\begin{aligned} r&=\tfrac{1}{2}(2n+l+p+o+q+m+i-4)+\tfrac{1}{4}(\delta^0_{o+q}+\delta^0_{l+p}+\delta^0_i+\delta^0_m),\\ b_3&\in h^{2/3+(1/3)(j+k+p+q+2n)}(\log h^{-1})^{p+q}h^{\max(r,0)(1/3-\delta/2)}S^{0,r}_{1-\delta/2}(\mathbb{R}^{d-1};\{\xi_1=0\}).\end{aligned}$$

Hence, the operator $R^{ijklmno}$ with kernel $R_{ijklmno}$ has for any $\epsilon>0$,

$$\begin{aligned} R^{ijklmno}&\in h^{1/3(j+k-l-m-i-o+2)+\delta/2(l+m+i+2n+o)-\epsilon}\Psi^{0,0}_{1-\delta/2}(\{\xi_1=0\}),\\ R^{ijklmno}&\in h^{2/3+(1/3)(j+k+2n)}h^{\max(r,0)(1/3-\delta/2)}\Psi^{0,r}_{1-\delta/2}(\mathbb{R}^{d-1};\{\xi_1=0\}).\end{aligned}$$

7B2. *The principal part.* By the analysis above, we see that when microlocalized near glancing points $\mathcal{S}\ell^*\mathcal{S}\ell$, $\mathcal{D}\ell^*\mathcal{D}\ell$, and $\mathcal{D}\ell^*\mathcal{S}\ell$ are pseudodifferential in a second microlocal class. We just need to compute the principal symbol of these operators. The symbols will turn out to be $\Psi_{\mathcal{S}\ell}$, $\Psi_{\mathcal{D}\ell}$, and $\Psi_{\mathcal{D}\ell\mathcal{S}\ell}$, respectively.

First, using the principle of stationary phase, we compute

$$\begin{aligned} J^{-1}f&=(2\pi h)^{-d+1}\int b_0(y',\xi')e^{(i/h)(\langle x',\xi'\rangle-\theta_b(y',\xi'))}f(y')\,dy'\,d\xi',\\ C^{-1}J^{-1}f&=(2\pi h)^{-d+1}\int b_1(y',\xi')e^{(i/h)(\langle x',\xi'\rangle-\theta_b(y',\xi'))}f(y')\,dy'\,d\xi',\end{aligned}$$

where

$$b_0=\frac{|\det\partial^2_{x'\xi'}\theta_b(y',\xi')|}{g_{0b}(y',\xi')}+O_S(h)\quad\text{and}\quad b_1=\frac{b_0(y',\xi')}{\partial_{x_d}\rho(y',\xi')}+O_S(h).$$

Denote the kernels of $\mathcal{S}\ell^*\mathcal{S}\ell$, $\mathcal{D}\ell^*\mathcal{D}\ell$, and $\mathcal{D}\ell^*\mathcal{S}\ell$ by $K_{\mathcal{S}\ell}$, $K_{\mathcal{D}\ell}$, and $K_{\mathcal{D}\ell\mathcal{S}\ell}$ respectively. We explicitly consider $\mathcal{S}\ell^*\mathcal{S}\ell$ and we record the end result for the others. The kernel of $\mathcal{S}\ell$ is given by

$$\mathcal{S}\ell(x,y)=\frac{2\pi e^{\pi i/6}h^{2/3}}{(2\pi h)^{d-1}}\int\big(g_0(x,\xi')Ai(\rho_h(x,\xi'))+ih^{1/3}g_1(x,\xi')Ai'(\rho_h(x,\xi'))\big)\\ \times A_-(\Theta_h(\xi'))b_1(y',\xi')e^{(i/h)(\theta(x,\xi')-\theta_b(y',\xi'))}\,d\xi'.$$

The kernel of $\mathcal{S}\ell^*\mathcal{S}\ell$ is given by

$$K_{\mathcal{S}\ell}=\frac{4\pi^2h^{4/3}}{(2\pi h)^{2d-2}}\iint_0^\infty\big(g_0(w,\xi')Ai(\rho_h(w,\xi'))+ih^{1/3}g_1(w,\xi')Ai'(\rho_h(w,\xi'))\big)\\ \times\big(\overline{g_0(w,\eta')Ai(\rho_h(w,\eta'))}-ih^{1/3}\overline{g_1(w,\eta')Ai'(\rho_h(w,\eta'))}\big)\\ \times A_-(\Theta_h(\xi'))\overline{A_-(\Theta_h(\eta'))}b_1(y',\xi')\overline{b_1(x',\eta')}\\ \times e^{(i/h)(\theta_b(x',\eta')-\theta(w,\eta')+\theta(w,\xi')-\theta_b(y',\xi'))}\,dw_d\,dw'\,d\xi'\,d\eta'.$$

Taylor expanding the Airy functions around $\rho_h(w,\xi')$ and $\Theta_h(\xi')$ produces lower-order terms of the form $h^{4/3}R_{0jkjk0}$, $(j,k)\neq(0,0)$, $h^{5/3}R_{0jk(j+1)k10}$, $h^{5/3}R_{0jkjk11}$ and $h^2R_{0jk(j+1)k21}$. In particular, $\mathcal{S}\ell^*\mathcal{S}\ell=A+O_{\Psi_{1-\delta/2}(\{\xi_1=0\})}(h^{2+\delta/2-\epsilon})$, where A has kernel

$$A(x,y)=\frac{4\pi^2h^{4/3}}{(2\pi h)^{2d-2}}\iint_0^\infty g_0(w,\xi')Ai(\rho_h(w,\xi'))\overline{g_0(w,\eta')Ai(\rho_h(w,\xi'))}A_-(\Theta_h(\xi'))\overline{A_-(\Theta_h(\xi'))}\\ \times b_1(y',\xi')\overline{b_1(x',\eta')}e^{(i/h)(\theta_b(x',\eta')-\theta(w,\eta')+\theta(w,\xi')-\theta_b(y',\xi'))}\,dw_d\,dw'\,d\xi'\,d\eta'.$$

Then, changing variables $W_d\mapsto h^{-2/3}\rho_0(w,\xi')$ and performing stationary phase as in the analysis of R_{jklmno} gives

$$A(x,y)=\frac{4\pi^2h^2}{(2\pi h)^{d-1}}\iint_{h^{-2/3}\Theta_0(\xi')}^\infty\frac{a_0(x',\xi')\overline{a_0(x',\xi')}+O_{S_{1-\delta/2}}(h^{1/3+\delta/2-\epsilon})}{|\det\partial^2_{x'\xi'}\theta(x'\xi')|\partial_{x_d}\rho(x',\xi')}\\ \times|Ai(W_d+h^{-2/3}\epsilon(h)\rho_1)|^2|A_-(\Theta_h(\xi'))|^2b_1(y',\xi')\overline{b_1(x',\xi')}\\ \times e^{(i/h)(\theta_b(x',\xi')-\theta_b(y',\xi'))}\,dW_d\,dw'\,d\xi'\,d\eta'.$$

Using that the phase is stationary at $x'=y'$ to integrate by parts in ξ' when terms of size $|x'-y'|$ appear, using that for any $\epsilon>0$,

$$Ai(W_d+h^{-2/3}\epsilon(h)\rho_1)=\begin{cases}Ai(W_d)+O_S(h^{-2/3}\epsilon(h)\langle W_d\rangle^{1/4}), & W_d\le C,\\ Ai(W_d)+O_S(h^{-2/3}\epsilon(h)\langle W_d\rangle^{1/4}e^{-2/3W_d^{3/2}}), & W_d\ge C,\end{cases}$$

and using the definition of $\Psi_{\mathcal{S}\ell}$, gives for any $\epsilon>0$

$$A(x,y)=\frac{h^2}{(2\pi h)^{d-1}}\int\frac{|a_0(x',\xi')|^2|b_1(x',\xi')|^2\Psi_{\mathcal{S}\ell}(\Theta_{0h}(\xi'))+O_{S_{1-\delta/2}}(h^{\delta/2-\epsilon})}{|\det\partial^2_{x'\xi'}\theta_{0b}(x',\xi')|\partial_{x_d}\rho(x',\xi')}e^{(i/h)(\theta_b(x',\xi')-\theta_b(y',\xi'))}\,d\xi'.$$

Now, let J_1 be a microlocally unitary semiclassical FIO quantizing κ; i.e.,

$$J_1f=(2\pi h)^{-d+1}\int c(x',\xi')e^{(i/h)(\theta_{0b}(x',\xi')-\langle y',\xi'\rangle)}\,d\xi',$$

where

$$c = |\det \partial^2_{x'\xi'}\theta_{0b}(x',\xi')|^{1/2} + O(h).$$

Applying stationary phase gives

$$\begin{aligned} &J_1^* A J_1(x,y) \\ &= \frac{h^2}{(2\pi h)^{d-1}} \int \frac{\bar{c}(w',\xi')c(z',\xi')|a_0(w',\xi')|^2|b_1(w',\xi')|^2\Psi_{\mathcal{S}\ell}(\Theta_{0h}(\xi')) + O_{S_{1-\delta/2}}(h^{\delta/2-\epsilon})}{|\det \partial^2_{x'\xi'}\theta_{0b}(w',\xi')|^2 \partial_{x_d}\rho(w',\xi')|\det \partial^2_{x'\xi'}\theta_{0b}(z',\xi')|}\Bigg|_{\substack{y'=\partial_{\xi'}\theta_{0b}(z',\xi') \\ x'=\partial_{\xi'}\theta_{0b}(w',\xi')}} \\ &\qquad \times e^{(i/h)(\langle x'-y',\xi'\rangle + \theta_{1b}(w',\xi') - \theta_{1b}(z',\xi'))}\, d\xi'. \end{aligned}$$

Again, using integration by parts on terms that are $O(|x'-y'|)$, we can assume that $x'=y'$ in the amplitude and hence have

$$\begin{aligned} &J_1^* A J_1(x,y) \\ &= \frac{h^2}{(2\pi h)^{d-1}} \int \frac{\bar{c}(w',\xi')c(w',\xi')|a_0(w',\xi')|^2|b_1(w',\xi')|^2\Psi_{\mathcal{S}\ell}(\Theta_{0h}(\xi')) + O_{S_{1-\delta/2}}(h^{\delta/2-\epsilon})}{|\det \partial^2_{x'\xi'}\theta_{0b}(w',\xi')|^2 \partial_{x_d}\rho(w',\xi')|\det \partial^2_{x'\xi'}\theta_{0b}(w',\xi')|}\Bigg|_{x'=\partial_{\xi'}\theta_{0b}(w',\xi')} \\ &\qquad \times e^{(i/h)\langle x'-y',\xi'\rangle}\, d\xi'. \end{aligned}$$

So, plugging in the definitions of c and b_1, we have

$$J_1^* \mathcal{S}\ell^* \mathcal{S}\ell J_1 = \frac{h^2}{(2\pi h)^{d-1}} \int \frac{\Psi_{\mathcal{S}\ell}(\Theta_{0h}(\xi')) + O_{S_{1-\delta/2}}(h^{\delta/2-\epsilon})}{|\partial_{x_d}\rho(w',\xi')|^2 \partial_{x_d}\rho(w',\xi')}\Bigg|_{x'=\partial_{\xi'}\theta_{0b}(w',\xi')} e^{(i/h)\langle x'-y',\xi'\rangle}\, d\xi'. \tag{61}$$

Similar computations give

$$\begin{aligned} J_1^* \mathcal{D}\ell^* \mathcal{S}\ell J_1 &= \frac{h^{4/3}}{(2\pi h)^{d-1}} \int \frac{\Psi_{\mathcal{D}\ell\mathcal{S}\ell}(\Theta_{0h}(\xi')) + O_{S^{0,1/2}_{1-\delta/2}}(h^{1/6+\delta/4-\epsilon})}{(\partial_{x_d}\rho(w',\xi'))^2}\Bigg|_{x'=\partial_{\xi'}\theta_{0b}(w',\xi')} e^{(i/h)\langle x'-y',\xi'\rangle}\, d\xi', \\ J_1^* \mathcal{S}\ell^* \mathcal{D}\ell J_1 &= \frac{h^{4/3}}{(2\pi h)^{d-1}} \int \frac{\overline{\Psi_{\mathcal{D}\ell\mathcal{S}\ell}(\Theta_{0h}(\xi'))} + O_{S^{0,1/2}_{1-\delta/2}}(h^{1/6+\delta/4-\epsilon})}{\overline{(\partial_{x_d}\rho(w',\xi'))^2}}\Bigg|_{x'=\partial_{\xi'}\theta_{0b}(w',\xi')} e^{(i/h)\langle x'-y',\xi'\rangle}\, d\xi', \\ J_1^* \mathcal{D}\ell^* \mathcal{D}\ell J_1 &= \frac{h^{2/3}}{(2\pi h)^{d-1}} \int \frac{\Psi_{\mathcal{D}\ell}(\Theta_{0h}(\xi')) + O_{S^{0,1}_{1-\delta/2}}(h^{1/3-\epsilon})}{\partial_{x_d}\rho(w',\xi')}\Bigg|_{x'=\partial_{\xi'}\theta_{0b}(w',\xi')} e^{(i/h)\langle x'-y',\xi'\rangle}\, d\xi'. \end{aligned} \tag{62}$$

Hence, all of the above operators are second microlocal pseudodifferential operators with respect to the glancing surface $\{|\xi'|_g = 1\}$.

Plugging (57) into (61) and (62) gives

$$\sigma(J_1^* \mathcal{S}\ell^* \mathcal{S}\ell J_1) = \frac{h^2 \Psi_{\mathcal{S}\ell}(h^{-2/3}\Theta_0(\xi'))}{2Q(\kappa(x',\xi'))}, \qquad \sigma(J_1^* \mathcal{D}\ell^* \mathcal{S}\ell J_1) = \frac{h^{4/3} \Psi_{\mathcal{D}\ell\mathcal{S}\ell}(h^{-2/3}\Theta_0(\xi'))}{(2Q(\kappa(x',\xi')))^{2/3}},$$

$$\sigma(J_1^* \mathcal{S}\ell^* \mathcal{D}\ell J_1) = \frac{h^{4/3} \overline{\Psi_{\mathcal{D}\ell\mathcal{S}\ell}(h^{-2/3}\Theta_0(\xi'))}}{(2Q(\kappa(x',\xi')))^{2/3}}, \quad \sigma(J_1^* \mathcal{D}\ell^* \mathcal{D}\ell J_1) = \frac{h^{2/3} \Psi_{\mathcal{D}\ell}(h^{-2/3}\Theta_0(\xi'))}{(2Q(\kappa(x',\xi')))^{1/3}},$$

where κ is as in (55).

8. Preliminary analysis of the generalized boundary damped equation

We examine problems of the form

$$\begin{cases} (-h^2\Delta - z^2)u = w & \text{in } \Omega, \\ h\partial_\nu u + Bu = hv & \text{on } \partial\Omega, \\ u|_{\partial\Omega} = \psi, \end{cases} \tag{63}$$

$$z \in [1-ch, 1+ch] + i[-Mh\log h^{-1}, Mh\log h^{-1}]. \tag{64}$$

We then assume that $B = hN_2(z/h) + hV(z)$, with V analytic for z as in (64),

$$V \in h^\alpha(\Psi^{0,m}_{2/3}\{|\xi'|_g = E'\} \cup \Psi^{0,m}_{2/3}\{|\xi'|_g = 1\})$$

for some $\alpha \geq -1$ and $m \in \mathbb{R}$.

Furthermore, suppose that for some $\delta > 0$, $M, M_1 > 0$, and $0 < \epsilon < \frac{1}{2}$

$$\begin{gathered} V \text{ is elliptic, on } ||\xi'|_g - 1| < \delta, \\ \left|1 + \frac{h\sigma(V)}{2\sqrt{|\xi'|_g^2 - 1}}\right| \geq \delta\left(\left\langle \frac{h^{1+\alpha}}{\sqrt{|\xi'|_g^2-1}}\right\rangle + \langle \xi' \rangle^{m-1}\right), \qquad |\xi'|_g > 1 + Mh^{2/3}, \\ \left|1 + \frac{ih\sigma(V)}{2\sqrt{1-|\xi'|_g^2}}\right| \geq \delta\left\langle \frac{h^{1+\alpha}}{\sqrt{1-|\xi'|_g^2}}\right\rangle, \qquad |\xi'|_g \leq 1 - h^\epsilon, \\ \log\left(1 + \frac{h\sigma(V)}{\sqrt{|\xi'|_g^2-1}}\right) \text{ exists and is smooth on } T^*\partial\Omega \setminus \{|\xi'|_g \leq M_1\}. \end{gathered} \tag{65}$$

The problem (63) is a highly generalized version of a standard boundary damped equation which was studied in the seminal work of Bardos, Lebeau and Rauch [Bardos et al. 1992]; see also [Koch and Tataru 1995]. In order to study this problem from the spectral point of view, we must see that the inverse operator is meromorphic with finite rank poles. This is similar to the analysis in the case of the standard damped wave equation (see for example [Zworski 2012, Chapter 5]).

8A. *Meromorphy of the resolvent.* For $s > -\frac{1}{2}$, let

$$\mathcal{P}(z) := \begin{pmatrix} -h^2\Delta - z^2 \\ \gamma\partial_\nu + h^{-1}B(z)\gamma \end{pmatrix} : H^{s+2}(\Omega) \to H^s(\Omega) \oplus H^{s+1/2-\max(m-1,0)}(\partial\Omega).$$

We will show that $\mathcal{P}(z)^{-1}$ is a meromorphic family of operators with finite-rank poles. Our analysis is similar in spirit to that for potential and black box scattering; see for example [Dyatlov and Zworski 2018, Chapters 2,3,4].

Then, when $(I + VG)^{-1} : H^s(\partial\Omega) \to H^{s+\max(m-1,0)}(\partial\Omega)$ exists,

$$(\mathcal{P}^{-1})^t = \begin{pmatrix} [I - \mathcal{S}\ell(I+VG)^{-1}(\gamma\partial_\nu + h^{-1}B\gamma)]h^{-2}1_\Omega R_0(z/h)1_\Omega \\ \mathcal{S}\ell(I+VG)^{-1} \end{pmatrix}$$

$$: H^s(\Omega) \oplus H^{s+1/2-\max(m-1,0)}(\partial\Omega) \to H^{s+2}(\Omega).$$

To check that this is the inverse, we simply apply the jumps formulas from for example [Galkowski 2014, Lemma 4.1 and Proposition 4.1.1]. For the Sobolev mapping properties of $1_\Omega R_0 1_\Omega$, $\mathcal{S}\ell$, $\mathcal{D}\ell$, see for example [Epstein 2007, Theorems 9, 10]. Now,

$$(I+VG)^{-1} = I - V(I+GV)^{-1}G, \quad (I+GV)^{-1} = I - G(I+VG)^{-1}V;$$

therefore, $I+GV$ is invertible if and only if $I+VG$ is invertible. Thus, to check that $\mathcal{P}^{-1}$ has a meromorphic continuation from $\operatorname{Im} z > 0$, it is enough to check that $(I+GV)^{-1}$ does. To see this, we first show that $I+GV$ is a holomorphic family of Fredholm operators with index 0 on the domain of R_0. The condition (65) and Lemma 7.3 imply that for M sufficiently large and $0 \le \chi_0 \in C_c^\infty(\mathbb{R})$ with $\chi_0 \equiv 1$ on $|x| \le M$ and $\operatorname{supp} \chi_0 \subset \{|x| \le M+1\}$, $(I+GV)(1-\chi(|hD'|_g)) \in \Psi^{\max(m-1),0)}(\partial\Omega)$ is elliptic on $|\xi'|_g \ge M+1$ with symbol

$$f := \sigma((I+GV)(1-\chi_0(|hD'|_g))) = \left(1 + \frac{h\sigma(V)}{2\sqrt{|\xi'|_g^2 - 1}}\right)(1-\chi_0(|\xi'|_g)).$$

Then, for $k = 1, 2$, let $0 \le \chi_k \in C_c^\infty(\mathbb{R})$ with $\chi_k \equiv 1$ on $|x| \le M+1$ and $\operatorname{supp} \chi_k \subset \operatorname{supp} \chi_{k+1}$ with $\operatorname{supp} \chi_2 \subset \{|x| \le M+2\}$. Then, by assumption, $\log(f/|f|)$ is well-defined on $\operatorname{supp} \chi_2(|\xi'|_g)$ and hence for $K > 0$ large enough

$$q = f + K\chi_2(|\xi'|_g)\left(\frac{f}{|f|}\right)^{1-\chi_1(|\xi'|_g)} \in S^{m-1} \text{ has } |q| \ge c\langle\xi'\rangle^{m-1}.$$

Now, $\operatorname{Op_h}(q) : H_{\mathrm{h}}^{s+\max(m-1,0)}(\partial\Omega) \to H_h^s(\partial\Omega)$ is invertible for h small enough and

$$\operatorname{Op_h}(q)(I+GV) = I + A_1, \quad (I+GV)\operatorname{Op_h}(q) = I + A_2,$$
$$\operatorname{Op_h}(q)(I+VG) = I + A_3, \quad (I+VG)\operatorname{Op_h}(q) = I + A_4,$$

with $A_i : H_{\mathrm{h}}^s(\partial\Omega) \to H_{\mathrm{h}}^{s-1}(\partial\Omega)$. Therefore, both $I+GV$ and $I+VG$ are Fredholm with index 0. The analysis below will show that there exists z_0 with $\operatorname{Im} z > 0$ so that $I+GV$ is injective. Therefore, $(I+GV)^{-1}$ exists at z_0 and by the analytic Fredholm theorem has a meromorphic continuation to $\mathbb{C}$ when d is odd and to the logarithmic cover of $\mathbb{C} \setminus \{0\}$ when d is even.

Write

$$(I+VG)\varphi = v. \tag{66}$$

Note that if φ has (66), then $u = \mathcal{S}\ell\varphi$ solves (67) with $w = 0$ and $\psi = G\varphi$. That is,

$$\begin{cases} (-h^2\Delta - z^2)u = 0 & \text{in } \Omega, \\ h\partial_\nu u + Bu = hv & \text{on } \partial\Omega, \\ u|_{\partial\Omega} = \psi. \end{cases} \tag{67}$$

Similarly, if

$$(I+GV)\psi = Gv, \tag{68}$$

then

$$u = -\mathcal{S}\ell V\psi + \mathcal{S}\ell v$$

solves (67). Now, suppose that u solves (67). Then

$$u = h^{-1}\mathcal{S}\ell h\,\partial_\nu u - \mathcal{D}\ell u|_{\partial\Omega} = -h^{-1}\mathcal{S}\ell\, Bu|_{\partial\Omega} - \mathcal{D}\ell u|_{\partial\Omega} + \mathcal{S}\ell\, v = -\mathcal{S}\ell\, Vu|_{\partial\Omega} + \mathcal{S}\ell\, v,$$

where we have used that in Ω,

$$\mathcal{S}\ell N_2 + \mathcal{D}\ell = 0 \quad \text{and hence} \quad (h^{-1}\mathcal{S}\ell\, B + \mathcal{D}\ell) = \mathcal{S}\ell\, V.$$

Therefore, taking $x \to \partial\Omega$ gives

$$u|_{\partial\Omega} = -GVu|_{\partial\Omega} + Gv \quad \Longrightarrow \quad (I + GV)u|_{\partial\Omega} = Gv.$$

That is, $\psi := u|_{\partial\Omega}$ solves (68). Finally, if ψ solves (68), then $\varphi := v - V\psi$ solves (66).

Lemma 8.1. *The following are equivalent*:

(1) *u solves* (67).

(2) $u = \mathcal{S}\ell(v - Vu|_{\partial\Omega})$.

(3) $u|_{\partial\Omega} = \psi$ *solves* (68).

(4) $v - Vu|_{\partial\Omega} = \varphi$ *solves* (66).

Note also that since $I + VG$ is Fredholm with index 0, it is not invertible if and only if there exists a nonzero solution ψ to $(I + VG)\psi = 0$. Hence, together with Lemma 8.1, we have proved the following.

Lemma 8.2. *The operator $\mathcal{P}^{-1}$ is meromorphic on the domain of $R_0(\lambda)$ and the following are equivalent*:

(1) *$\mathcal{P}^{-1}(z)$ has a pole at z_0.*

(2) *There exists a nonzero solution ψ to $(I + G(z_0)V(z_0)\psi = 0$.*

(3) *There exists a nonzero solution φ to $(I + V(z_0)G(z_0)\varphi = 0$.*

(4) *There exists a nonzero solution u to* (67) *with $v = 0$.*

9. Microlocal analysis of the generalized boundary damped wave equation

We now proceed to study the poles of $\mathcal{P}(z)^{-1}$. It is convenient to study (68) because then the solution to (67) has $u|_{\partial\Omega} = \psi$. From now on, we do so without comment.

9A. ***Brief outline of the computations.*** The analysis in the next few sections proceeds as follows. We first study the elliptic region where there is no propagation and hence the analysis is relatively simple. Then, we study the hyperbolic region where standard propagation occurs. In this case, we use the decomposition of G (Lemma 7.3) to rewrite (68) in terms of the reflectivity operator, R from (18) and transition operator T from (19). We use the symbolic calculus of FIO's to show that this new operator has a microlocal inverse on the hyperbolic set. However, we must show that this inverse preserves the hyperbolic set up to a small remainder. This is done in Lemma 9.3.

Putting these two regions together leaves the glancing region to be analyzed. Here, we apply the microlocal models of G and $\mathcal{S}\ell$ near glancing from Lemmas 7.2 and 7.5. We start by using (68), together with the model for G near glancing, to further localize ψ near certain "almost glancing hypersurfaces". Using that $\mathcal{S}\ell V\psi$ solves (67) with $v=0$, we obtain estimates on $\operatorname{Im} z$ from the description of $\mathcal{S}\ell^*\mathcal{S}\ell$ near glancing.

9B. *Elliptic region.* Fix $0<\epsilon<\frac{1}{2}$ and $0<c_1<c_2<c$. We first estimate solutions to (68) in the elliptic region $\mathcal{E}:=\{|\xi'|_g\geq 1+ch^\epsilon\}$.

Let $\chi_1\in S_\epsilon(|\xi'|_g=1)$ have $\chi_1\equiv 1$ on $|\xi'|_g\geq\{1+c_2h^\epsilon\}$ and $\operatorname{supp}\chi_1\subset\{|\xi'|_g\geq 1+c_1h^\epsilon\}$. Also, let $\chi_2\in S_\epsilon(|\xi'|_g=1)$ have $\operatorname{supp}\chi_2\subset\{|\xi'|_g\geq 1+c_2h^\epsilon\}$ and $\chi_2\equiv 1$ on $\{|\xi'|_g\geq 1+ch^\epsilon\}$. Let $X_1=\operatorname{Op_h}(\chi_1)$ and $X_2=\operatorname{Op_h}(\chi_2)$.

Let ψ solve (68). Then, we have

$$(I+GV)X_1\psi=[GV,X_1]\psi+X_1Gv.$$

Now, by Lemma 7.3, $GVX_1=G_\Delta VX_1+O_{\Psi^{-\infty}}(h^\infty)$, where $G_\Delta\in h^{2/3}\Psi_{2/3}^{-1/2,-1}(|\xi'|_g=1)$. By our assumptions on V and Lemma 4.3, there exists

$$A\in h^{\max(-2/3-\alpha,0)}\Psi_{2/3}^{1/2,\min(0,1-m)}(|\xi'|_g=1)\cup\Psi_{2/3}^{0,\min(0,1-m)}(|\xi'|_g=E')$$

so that $A(I+G_\Delta V)=X_2$ and $\operatorname{MS_h}(A)\subset\{\chi_1\equiv 1\}$. So,

$$X_2\psi=A[G_\Delta V,X_1]\psi+AX_1Gv+O_{\Psi^{-\infty}}(h^\infty)(\psi+v)$$

and hence,

$$\begin{aligned}\|X_2\psi\|_{H_h^m}&\leq C(\|A[G_\Delta V,X_1]\psi\|_{L^2}+\|AX_1G_\Delta v\|_{H_h^m}+O(h^\infty)(\|\psi\|_{H_h^{-N}}+\|v\|_{H_h^{-N}})\\&\leq C(h^{1-\epsilon/2}\|v\|_{L^2}+O(h^\infty)\|\psi\|).\end{aligned}$$

Summarizing:

Lemma 9.1. *For all* $0<\epsilon<\frac{1}{2}$, $c>0$, *and* $N>0$, *there exists* $h_0=h_0(\epsilon,c)>0$ *such that for* $0<h<h_0$, $\chi\in S_\epsilon^{0,0}(|\xi'|_g=1)$ *with* $\operatorname{supp}\chi\subset\{|\xi'|_g\geq 1+ch^\epsilon\}$, *and* ψ *solving* (68)

$$\|\operatorname{Op_h}(\chi)\psi\|_{H_h^m}\leq C(h^{1-\epsilon/2}\|v\|_{L^2}+O(h^\infty)\|\psi\|_{H_h^{-N}}).$$

9C. *Hyperbolic region.* Recall from Lemma 7.3 that

$$G=G_\Delta+G_B+G_g+O_{L^2\to C^\infty}(h^\infty).$$

First suppose that $\operatorname{MS_h}(X)\subset\{|\xi'|_g\leq 1-ch^\epsilon\}$ for some $0<\epsilon<\frac{1}{2}$. Then, suppose that

$$(I+GV)X\psi=f$$

and let $G_\Delta^{-1/2}$ be a microlocal inverse for $G_\Delta^{1/2}$ on

$$\mathcal{H}:=\{|\xi'|_g\leq 1-r_{\mathcal{H}}h^\epsilon\},$$

where $r_{\mathcal{H}}\ll c$. Then

$$\begin{aligned}(I+GV)X_1\psi&=(I+(G_\Delta+G_B)V)X_1\psi+O(h^\infty)\psi\\&=(I+G_\Delta^{1/2}(I+G_\Delta^{-1/2}G_BG_\Delta^{-1/2})G_\Delta^{1/2}V)X_1\psi+O(h^\infty)\psi=f.\end{aligned}$$

Thus, f is microlocalized on $\mathcal{H}$ and, following the formal algebra in [Zaletel 2010, Section 2] multiplying by $G_\Delta^{1/2}V$, we have

$$G_\Delta^{1/2}VX_1\psi = -G_\Delta^{1/2}VG_\Delta^{1/2}(I+G_\Delta^{-1/2}G_BG_\Delta^{-1/2})G_\Delta^{1/2}VX_1\psi + O(h^\infty)\psi + G_\Delta^{1/2}Vf.$$

Remark 9.2. By Lemma 5.3, a microlocal inverse on $\mathcal{H}$ will be a microlocal inverse on $\mathrm{MS_h}(G_BX_1)$.

Writing $\varphi = G_\Delta^{1/2}VX_1\psi$ and $T = G_\Delta^{-1/2}G_BG_\Delta^{-1/2}$, we have

$$(I+G_\Delta^{1/2}VG_\Delta^{1/2})\varphi = -G_\Delta^{1/2}VG_\Delta^{1/2}T\varphi + O(h^\infty)\psi + G_\Delta^{1/2}Vf.$$

Hence, letting

$$R := -(I+G_\Delta^{1/2}VG_\Delta^{1/2})^{-1}G_\Delta^{1/2}VG_\Delta^{1/2},$$

we have

$$\varphi = RT\varphi + O(h^\infty)\psi - RG_\Delta^{-1/2}f.$$

Here, T is an FIO associated to the billiard map such that

$$\sigma\left(\exp\left(\frac{\operatorname{Im} z}{h}\operatorname{Op_h}(l(q,\beta(q)))\right)T\right)(\beta(q),q) = \exp\left(\frac{i\operatorname{Re}\omega_0 l(\beta(q),q)}{h}\right)e^{-i\pi/4}dq^{1/2} \in S_\epsilon$$

and $R \in \Psi_\epsilon \cup \Psi^{0,0}_{2/3}(|\xi'|_g = E')$ is as in (18).

Thus by the wavefront set calculus we have for $N > 0$ independent of h,

$$(I-(RT)^N)\varphi = O(h^\infty)\psi - \sum_{m=0}^{N-1}(RT)^m RG_\Delta^{-1/2}f \tag{69}$$

and by Egorov's theorem (Lemma 2.15), we have

$$(RT)_N := ((RT)^*)^N(RT)^N = \operatorname{Op_h}(a_N) + O_{\Psi^{-\infty}}(h^\infty), \tag{70}$$

where $a_N \in S_\epsilon \cup S^{0,0}_{2/3}(|\xi'|_g = E')$. Moreover, with $\delta = \max\big(2\epsilon, \frac{2}{3}\big)$ for u with $\mathrm{MS_h}(u) \subset \mathcal{H}$, by the sharp Gårding inequality, Lemma 4.5, and Lemma 4.6,

$$\begin{gathered}\inf_{\mathcal{H}}\big(|\tilde\sigma((RT)_N)(q)| + O(h^{I_{(RT)_N}(q)+1-\delta})\big)\|u\|_{L^2} \le \|(RT)^N u\|^2_{L^2},\\ \|(RT)^N u\|^2 \le \sup_{\mathcal{H}}\big(|\tilde\sigma((RT)_N)(q)| + O(h^{I_{(RT)_N}(q)+1-\delta})\big)\|u\|_{L^2}.\end{gathered}$$

Let

$$\varkappa_1 := 1 - \sqrt{\sup_{\mathcal{H}}\tilde\sigma((RT)_N)}, \qquad \varkappa_2 := \sqrt{\inf_{\mathcal{H}}\tilde\sigma((RT)_N)} - 1.$$

Finally, let $\varkappa = \max(\varkappa_1, \varkappa_2)$. Then, we have:

Lemma 9.3. *Suppose that $\varkappa > h^{\gamma_1}$, where $\gamma_1 < \min\big(\frac{1}{2}-\epsilon, \frac{1}{6}\big)$. Let $c > r_{\mathcal{H}}$ and $g \in L^2$ have $\mathrm{MS_h}(g) \subset \{1 - Ch^\epsilon \le |\xi'|_g \le 1 - ch^\epsilon\}$. If*

$$(I-(RT)^N)u = g,$$

then for any $\delta > 0$,

$$\mathrm{MS_h}(u) \subset \{1-(C+\delta)h^\epsilon \le |\xi'|_g \le 1-(c-\delta)h^\epsilon\}.$$

In particular, there exists an operator A *with* $\|A\|_{L^2\to L^2} \le 2\varkappa^{-1}$,

$$A(I-(RT)^N) = I \quad \textit{microlocally on } \mathcal{H},$$

and if $\mathrm{MS_h}(g) \subset \{1 - Ch^\epsilon \le |\xi'|_g \le 1 - ch^\epsilon\}$, *then*

$$\mathrm{MS_h}(Ag) \subset \{1-(C+\delta)h^\epsilon \le |\xi'|_g \le 1-(c-\delta)h^\epsilon\}.$$

Proof. In the case that $\varkappa_2 > h^{\gamma_1}$, we write

$$(I-(RT)^N) = -(RT)^N(I-(RT)^{-N})$$

microlocally on $\mathcal{H}$ and invert by Neumann series to see that for any g, $(I-(RT)^N)u = g$ has a unique solution modulo h^∞ with $\|u\| \le \varkappa^{-1}\|g\|$. On the other hand, if $\varkappa_1 > h^{\gamma_1}$, then $\|(RT)^N\| \le 1-\varkappa_1$, and we have that for any g, $(I-(RT)^N)u = g$ has a unique solution with $\|u\| \le \varkappa_1^{-1}\|g\|$.

We will consider the case of $\varkappa_1 > h^{\gamma_1}$, the case of $\varkappa_2 < h^{\gamma_1}$ being similar with $(RT)^N$ replaced by $(RT)^{-N}$. Inversion by Neumann series already shows that we can solve $(I-(RT)^N)u_1 = g$ with $\|u_1\| \le \varkappa^{-1}\|g\|$. To complete the proof of the lemma, we need to show that this inverse has the required microsupport property. For this, we need a fine almost invariance result near the glancing set. In particular, by Lemma 7.1, that there exists an approximate first integral $\Xi(x,\xi) \in C^\infty(\overline{B^*\partial\Omega})$ so that $\Xi = 0$, $|d\Xi| > 0$ on $S^*\partial\Omega$, $\Xi < 0$ in $B^*\partial\Omega$ and

$$\Xi(\beta(q)) - \Xi(q) = r(q), \tag{71}$$

with $r(q) \in C^\infty(B^*\partial\Omega)$ vanishing to infinite order at $S^*\partial\Omega$. (See also [Kovachev and Popov 1990; Marvizi and Melrose 1982; Popov and Vodev 1999b].) In particular, we have that in neighborhood of $S^*\partial\Omega$,

$$\Xi(x',\xi') = e(x',\xi)(|\xi'|_g^2 - 1),$$

with $e > c > 0$.

For $k \ge 1$, let $\chi_k = \chi_k(\zeta)$ with $\chi_{k+1} \equiv 1$ on $\operatorname{supp}\chi_k$ and $\chi_1 \equiv 1$ on $\mathrm{MS_h}(g)$ so that

$$\operatorname{supp}\chi_k \subset \{1-(C+\delta)h^\epsilon \le |\xi'|_g \le 1-(c-\delta)h^\epsilon\}.$$

Let $X_k = \mathrm{Op_h}(\chi_k)$. Finally, let $\chi_\infty \in S_\epsilon$ with $\chi_\infty \equiv 1$ on $\bigcup_k \operatorname{supp}\chi_k$ and

$$\operatorname{supp}\chi_\infty \subset \{1-(C+2\delta)h^\epsilon \le |\xi'|_g \le 1-(c-2\delta)h^\epsilon\}.$$

Then (71) implies

$$|\chi_k(\beta(q)) - \chi_k(q)| = O(h^\infty).$$

Suppose that u is the unique solution of

$$(I-(RT)^N)u = g.$$

We will show that u is microlocalized as described in the lemma. Letting $u_1 = u$, we have

$$(I-(RT)^N)X_1u_1 = g + O(h^\infty)g + [X_1,(RT)^N]X_\infty u_1 =: g + g_1.$$

Let $\delta = \max\big(2\epsilon, \frac{2}{3}\big)$. Then
$$[X_1, T] = T(T^{-1}X_1T - X_1) = Th^{1-\delta}B,$$
with $B \in \Psi_\epsilon$. In fact,
$$T^{-1}X_1T = \mathrm{Op_h}(\chi_1(\beta(q)) + O_{\Psi_\epsilon}(h^{1-2\epsilon}). \tag{72}$$
Hence, since $X_\infty u$ is microlocalized h^ϵ-close to glancing,
$$\mathrm{MS_h}([X_1, (RT)^N]X_\infty u_1) \subset \{\chi_2 \equiv 1\},$$
and $g_1 := [X_1, (RT)^N]X_\infty u_1$ has
$$\|g_1\| \le Ch^{1-\delta}\varkappa^{-1}\|g\|_{L^2}.$$
Now, let u_2 have
$$(I - (RT)^N)u_2 = -g_1, \quad \|u_2\| \le \varkappa^{-1}\|g_1\| \le Ch^{1-\delta}\varkappa^{-2}\|g\|.$$
So,
$$(I - (RT)^N)(X_1u + u_2) = g + O(h^\infty)g.$$
Continuing in this way, let
$$(I - (RT)^N)u_k = -g_{k-1}, \quad g_{k-1} = [X_{k-1}, (RT)^N]X_\infty u_{k-1}.$$
Then,
$$\|u_k\| \le \varkappa^{-2k}(h^{k(1-\delta)})\|g\|_{L^2}.$$
Moreover, letting $\tilde{u} \sim \sum_k X_k u_k$, we have $X_\infty \tilde{u} = \tilde{u} + O(h^\infty)\tilde{u}$ and
$$(I - (RT)^N)\tilde{u} = g + O(h^\infty)g,$$
which implies $\tilde{u} - u = O(h^\infty)$ and hence that $(I - (RT)^N)$ has a microlocal inverse, A, with the properties claimed in the lemma. □

We now suppose that ψ solves (68) and use (69) to obtain estimates on ψ. Let $\chi_k \in S_\epsilon$ with $\chi_k \equiv 1$ on $\{|\xi'|_g \le 1 - 2kch^\epsilon\}$ and $\operatorname{supp} \chi_k \subset \{|\xi'|_g \le 1 - (2k-1)ch^\epsilon\}$. Then
$$(I + GV)X_1\psi = -[X_1, GV]\psi + X_1Gv =: \psi_1 + \tilde{v},$$
where $\mathrm{MS_h}(\psi_1) \subset \mathcal{H} \cap \{|\xi'|_g \ge 1 - 3c/2h^\epsilon\}$. Then with $\varphi = G_\Delta^{1/2}VX_1\psi$,
$$(I - (RT)^N)\varphi = O(h^\infty)\psi - \sum_{m=0}^{N-1}(RT)^m RG_\Delta^{-1/2}(\psi_1 + \tilde{v}),$$
and hence by Lemma 9.3, when $\varkappa \ge h^{\gamma_1}$ for $\gamma_1 < \min\big(\frac{1}{2} - \epsilon, \frac{1}{6}\big)$,
$$\varphi = O(h^\infty)\psi - \sum_{m=0}^{N_1} A(RT)^m RG_\Delta^{-1/2}(\psi_1 + \tilde{v})$$

and, using the microsupport statement from Lemma 9.3,

$$X_2\varphi = -\sum_{m=0}^{N-1} A(RT)^m RG_\Delta^{-1/2}\tilde{v} + O_{\Psi^{-\infty}}(h^\infty)(\psi + v).$$

Hence,

$$\begin{aligned}\|X_2\varphi\|_{L^2} &\le \varkappa^{-1}\left\|\sum_{m=0}^{N-1}(RT)^m RG_\Delta^{-1/2}X_1Gv\right\| + O(h^\infty)(\|\psi\|+\|v\|)\\ &\le C\varkappa^{-1}e^{ND_\Omega(\operatorname{Im} z)_-/h}h^{1/2-\epsilon/2}\|v\| + O(h^\infty)\|\psi\|.\end{aligned}$$

Then, since $\varphi = G_\Delta^{1/2}VX_1\psi$, we know $VX_1\psi = G_\Delta^{-1/2}\varphi + O(h^\infty)\psi$ and

$$\begin{aligned}X_3\psi &= -X_3GV\psi + X_3Gv = -X_3GVX_1\psi + X_3Gv + O(h^\infty)\psi\\ &= -X_3GG_\Delta^{-1/2}\varphi + X_3Gv + O(h^\infty)\psi = -X_3GG_\Delta^{-1/2}X_2\varphi + X_3Gv + O(h^\infty)\psi.\end{aligned}$$

Hence,

$$\begin{aligned}\|X_3\psi\| &\le \|X_3GG_\Delta^{-1/2}X_2\varphi\| + \|X_3Gv\| + O(h^\infty)\|\psi\|\\ &\le C(\varkappa^{-1}h^{1-\epsilon}e^{(N+1)D_\Omega(\operatorname{Im} z)_-/h}\|v\| + O(h^\infty)\|\psi\|).\end{aligned}$$

Next, we examine when $\varkappa \ge ch^{\gamma_1}$. If this is not the case, then

$$\liminf_{h\to 0}\frac{\inf||\tilde{\sigma}((RT)_N)(q)|-1|}{h^{\gamma_1}} = 0.$$

So, let

$$|\tilde{\sigma}(RT)_N(q)| = e^{e(q)}.$$

Taking logs and renormalizing we have

$$\frac{2\operatorname{Im} z}{h}Nl_N(q) - \frac{2\operatorname{Im} z}{h}Nl_N(q) + \log|\tilde{\sigma}((RT)_N)(q)| = e(q).$$

This implies

$$-\frac{\operatorname{Im} z}{h} = -l_N^{-1}(q)\left[\frac{\operatorname{Im} z}{h}l_N(q) + \frac{1}{2N}\log|\tilde{\sigma}((RT)_N)(q)| + e(q)\right] = -l_N^{-1}(q)(r_N(q)+e(q)),$$

where r_N as in (21). Thus, if $\varkappa \le ch^{\gamma_1}$, for any $c>0$,

$$\inf_{\mathcal{H}} -l_N^{-1}(r_N + ch^{\gamma_1}) \le -\frac{\operatorname{Im} z}{h} \le \sup_{\mathcal{H}} -l_N^{-1}(r_N - ch^{\gamma_1}).$$

Now, writing

$$RT = \left[R\exp\left(-\frac{\operatorname{Im} z}{h}\operatorname{Op_h}(l(q),\beta(q))\right)\right]\left[\exp\left(\frac{\operatorname{Im} z}{h}\operatorname{Op_h}(l(q),\beta(q))\right)T\right]$$

and applying Lemma 3.1 shows that

$$\begin{aligned}r_N(q) &:= \tilde{\sigma}((RT)_N)(q)\\ &= \exp\left(-\frac{2\operatorname{Im} z}{h}\sum_{n=0}^{N-1} l(\beta^n(q),\beta^{n+1}(q))\right)\prod_{i=1}^{N}\left(|\tilde{\sigma}(R)(\beta^i(q))|^2 + O(h^{I_R(\beta^i(q))+1-2\epsilon})\right).\end{aligned}$$

Summarizing the discussion, we have:

Lemma 9.4. *Let* $0<\epsilon<\frac{1}{2}$, $\gamma_1<\min\left(\frac{1}{2}-\epsilon,\frac{1}{6}\right)$, $c>0$, $M>0$ *and suppose that* $\chi\equiv 1$ *on* $\{|\xi'|_g\le 1-Ch^\epsilon\}$ *and* $\operatorname{supp}\chi\subset\{|\xi'|\le 1-ch^\epsilon\}$. *Suppose further that* ψ *solves* (68). *Then there exists* $h_0>0$ *small enough so that if* $0<h<h_0$ *and*

$$-\frac{\operatorname{Im} z}{h}<\inf_{\mathcal{H}}-l_N^{-1}(r_N+ch^{\gamma_1})\quad\text{or}\quad-\frac{\operatorname{Im} z}{h}>\sup_{\mathcal{H}}-l_N^{-1}(r_N-ch^{\gamma_1}),\tag{73}$$

where l_N *and* r_N *are as in* (20) *and* (21) *respectively, then*

$$\|\operatorname{Op_h}(\chi)\psi\|_{L^2}\le C\big(h^{1-\epsilon-\gamma_1}e^{(N+1)D_\Omega(\operatorname{Im} z)_-/h}\|v\|_{L^2}+O(h^\infty)\|\psi\|_{H_{\mathrm{h}}^{-M}}\big).\tag{74}$$

9D. ***Glancing region.*** Let $\chi\in S_\epsilon(|\xi'|_g=1)$ with $\chi\equiv 1$ on $\{||\xi'|_g-1|\le ch^\epsilon\}$ and $\operatorname{supp}\chi\subset\{||\xi'|_g-1|\le Ch^\epsilon\}$. Then

$$(I+GV)\operatorname{Op_h}(\chi)\psi=[GV,\operatorname{Op_h}(\chi)]\psi+\operatorname{Op_h}(\chi)Gv.$$

Let φ_i be a partition of unity on $S^*\partial\Omega$. We then use the microlocal model for G near glancing:

$$\sum_i(I+h^{2/3}J_i\omega^{-1}\mathcal{A}_-\mathcal{A}iC^{-1}J_i^{-1}V)\varphi_i\operatorname{Op_h}(\chi)\psi=\operatorname{Op_h}(\chi)Gv+[GV,\operatorname{Op_h}(\chi)]\psi+O(h^\infty)(\psi).$$

First, observe that if $\alpha>-\frac{2}{3}$, then our model shows that $(I+GV)$ is an elliptic pseudodifferential operator on $\operatorname{supp}\chi$ and hence:

Lemma 9.5. *Suppose* $\alpha>-\frac{2}{3}$. *Then under the assumptions of Lemma 9.4, there exists* $N>0$ *such that*

$$\|\psi\|_{L^2}\le Ch^{-N}\|v\|_{L^2}.$$

Throughout the rest of our analysis near glancing, it will be convenient to use Ξ from Lemma 7.1. Then

$$\Xi(x',\xi'):=(|\xi'|_g^2-1)(2Q(x',\xi'))^{-2/3}+O((|\xi'|_g^2-1)^2).$$

Moreover, $\xi_1(\kappa^{-1}(x',\xi'))=\Xi(x',\xi')+O((|\xi'|_g^2-1)^\infty)$, where κ is the symplectomorphism (55) reducing the billiard ball map for the Friedlander model to that for Ω near $(x',\xi')\in S^*\partial\Omega$. In particular, notice that if $\chi\in S_\epsilon^{0,0}(\xi_1=1)$ with $\operatorname{supp}\chi\subset\{ah_1^\epsilon\le 1-|\xi'|_g^2\le bh^\epsilon\}$, then

$$\sigma(J_i\operatorname{Op_h}(\chi(\Xi))J_i^{-1})=\chi(\xi_1),$$
$$\operatorname{MS_h}({}_i\operatorname{Op_h}(\chi(\Xi))J_i^{-1})\subset\{ah_1^\epsilon\le\xi_1\le bh^{\epsilon_2}\}.$$

Now, the assumption that on $|\xi'|_g-1>Mh^{2/3}$

$$\left|1+\frac{h\sigma(V)}{2\sqrt{|\xi'|_g^2-1}}\right|\ge\delta\left\langle\frac{h^{1+\alpha}}{\sqrt{|\xi'|_g^2-1}}\right\rangle$$

(see (65)), together with Lemma 4.3 and (60), implies that $I+GV$ is microlocally invertible on $|\xi'|_g\ge 1+Mh^{2/3}$.

When $\alpha < -\frac{2}{3}$, we can localize further. In particular, fix $M_1 > 0$. Then since V is elliptic and $\alpha < -\frac{2}{3}$, $I + GV$ is an elliptic pseudodifferential operator when for some $\delta > 0$ and all $1 \le j \le M_1$,

$$|h^{-2/3}\Xi(x',\xi') + \zeta_j| \ge \delta, \quad h^{-2/3}\Xi(x',\xi') + \zeta_{M_1+1} \ge \delta.$$

So, there exists $C > 0$ such that, letting $\chi_2 \in S_{2/3}(|\xi'|_g = 0)$ have $\operatorname{supp}\chi_2 \subset |\xi_1| \le Ch^{\epsilon}$ and

$$\chi_2 \equiv 1 \quad \text{on} \begin{cases} |\xi_1| \le CMh^{2/3}, & \alpha = -\frac{2}{3}, \\ |\xi_1 h^{-2/3} + \zeta_j| \le \delta,\ Ch^{\epsilon} \le \xi_1 \le h^{2/3}\zeta_{M_1} + \delta h^{2/3}, & \alpha < -\frac{2}{3}, \end{cases} \tag{75}$$

we have:

Lemma 9.6. *Let χ_2 be as in* (75). *Then*

$$\|(1 - \operatorname{Op_h}(\chi_2(\Xi)))\operatorname{Op_h}(\chi_1)\psi\| \le Ch^{-1/3+\epsilon/2-\alpha}(\|\operatorname{Op_h}(\chi)Gv\| + \|[GV, \operatorname{Op_h}(\chi)]\psi\| + O(h^{\infty})\|\psi\|,$$

and hence, under the assumptions of Lemma 9.4,

$$\|(1 - \operatorname{Op_h}(\chi_2(\Xi)))\operatorname{Op_h}(\chi_1)\psi\| \le Ch^{-1/3+\epsilon/2}(h^{2/3} + h^{1-\epsilon-\gamma_1}e^{(N+1)D_\Omega(\operatorname{Im} z)_-/h})\|v\| + O(h^{\infty})\|\psi\|.$$

9D1. *Flux formula.* With χ_2 as in (75), define

$$\psi_{ng} := (1 - \operatorname{Op_h}(\chi_2(\Xi))\operatorname{Op_h}(\chi_1))\psi$$

and $\psi_g := \psi - \psi_{ng}$.

By an integration by parts, we have for a solution u to (67),

$$\left(\frac{2\operatorname{Re} z \operatorname{Im} z}{h}\|u\|_{L^2}^2 - \operatorname{Im}\langle B\psi, \psi\rangle\right) = -\operatorname{Im}\langle hv, \psi\rangle. \tag{76}$$

On the other hand,

$$u = h^{-1}\mathcal{S}\ell h\,\partial_\nu u - \mathcal{D}\ell u = -(h^{-1}\mathcal{S}\ell B + \mathcal{D}\ell)\psi + \mathcal{S}\ell v = -\mathcal{S}\ell V\psi + \mathcal{S}\ell v. \tag{77}$$

Since we already have estimates for ψ_{ng}, we write

$$u = (-\mathcal{S}\ell V\psi_g) + (\mathcal{S}\ell(v - V\psi_{ng})) =: u_g + u_{ng}.$$

Now, [Han and Tacy 2015, Theorem 1.1], together with an application of the Phragmén–Lindelöf principle, implies

$$\begin{aligned} \|\mathcal{S}\ell(v - V\psi_{ng})\| = \|u_{ng}\| &\le h^{5/6}e^{D_\Omega(\operatorname{Im} z)_-/h}(\|v\| + h^{\alpha}\|\psi_{ng}\|_{H_h^m}), \\ \|\mathcal{S}\ell V\psi_g\| = \|u_g\| &\le Ch^{5/6+\alpha}e^{D_\Omega(\operatorname{Im} z)_-/h}\|\psi_g\|. \end{aligned}$$

Then,

$$\begin{aligned} \|u\|^2 - \|u_g\|^2 &= 2\operatorname{Re}\langle u_g, u_{ng}\rangle + \|u_{ng}\|^2 \\ &\le \delta\|u_g\|^2 + (1 + 2\delta^{-1})\|u_{ng}\|^2 \\ &\le C\delta h^{5/3+2\alpha}e^{2D_\Omega(\operatorname{Im} z)_-/h}\|\psi_g\|^2 + (1 + 2\delta^{-1})\|u_{ng}\|^2 \end{aligned}$$

and

$$\left|\langle B\psi,\psi\rangle-\langle B\psi_g,\psi_g\rangle\right| = |\langle B\psi_g,\psi_{ng}\rangle+\langle B\psi_{ng},\psi_g\rangle+\langle B\psi_{ng},\psi_{ng}\rangle| \\ \le C(\delta\|\psi_g\|^2+C(1+\delta^{-1}))\|\psi_{ng}\|^2_{H^m_{\mathrm{h}}}.$$

Now, rewrite (76) as

$$\frac{2\operatorname{Re} z\operatorname{Im} z}{h}\|u_g\|^2-\operatorname{Im}\langle B\psi_g,\psi_g\rangle=\operatorname{Im}\langle hv,\psi\rangle+\frac{2\operatorname{Re} z\operatorname{Im} z}{h}(\|u_g\|^2-\|u\|^2)+\operatorname{Im}(\langle B\psi_g,\psi_g\rangle-\langle B\psi,\psi\rangle).$$

Plugging our estimates in together gives

$$\begin{aligned}\left|\frac{2\operatorname{Re} z\operatorname{Im} z}{h}\|u_g\|^2+\operatorname{Im}\langle -hN_2\psi_g,\psi_g\rangle+\langle -h\operatorname{Im} V\psi_g,\psi_g\rangle\right|& \\ \le Ch(\delta_1^{-1}\|v\|^2+\delta_1\|\psi\|^2)+C|\operatorname{Im} z|h^{-1}(\delta_2 h^{5/3+2\alpha}e^{2D_\Omega(\operatorname{Im} z)_-/h}\|\psi_g\|^2+(1+\delta_2^{-1})\|u_{ng}\|^2)& \\ +C(\delta_3\|\psi_g\|^2+(1+\delta_3^{-1})\|\psi_{ng}\|^2_{H^m_{\mathrm{h}}})& \\ \le C(\delta_1 h+|\operatorname{Im} z|h^{2/3+2\alpha}e^{2D_\Omega(\operatorname{Im} z)_-/h}\delta_2+\delta_3)\|\psi_g\|^2_{L^2}& \\ +C(h\delta_1+\delta_3^{-1}+(1+\delta_2^{-1})|\operatorname{Im} z|h^{2/3+2\alpha}e^{2D_\Omega(\operatorname{Im} z)_-/h})\|\psi_{ng}\|^2_{H^m_{\mathrm{h}}}& \\ +C(h\delta_1^{-1}+\delta_2^{-1}|\operatorname{Im} z|h^{2/3+2\alpha}e^{2D_\Omega(\operatorname{Im} z)_-/h})\|v\|^2_{L^2}.& \quad (78)\end{aligned}$$

In particular, we have:

Lemma 9.7. *For all* $\gamma_1\in\mathbb{R}$, $c>0$, *there exists* $C>0$ *so that if*

$$\left|\frac{2\operatorname{Re} z\operatorname{Im} z}{h}\|u_g\|^2+\operatorname{Im}\langle -hN_2\psi_g,\psi_g\rangle+\langle -h\operatorname{Im} V\psi_g,\psi_g\rangle\right|\ge ch^{\gamma_1}\|\psi_g\|^2, \quad (79)$$

then

$$\|\psi_g\|^2\le C(h^{\gamma_1}+h^{-\gamma_1}+(1+|\operatorname{Im} z|h^{2/3+2\alpha-\gamma_1}e^{2D_\Omega(\operatorname{Im} z)_-/h})|\operatorname{Im} z|h^{2/3+2\alpha}e^{2D_\Omega(\operatorname{Im} z)_-/h})\|\psi_{ng}\|^2_{H^m_{\mathrm{h}}} \\ +C(h^{2-\gamma_1}+|\operatorname{Im} z|^2h^{4/3+2\alpha-\gamma_1}e^{4D_\Omega(\operatorname{Im} z)_-/h})\|v\|^2_{L^2}.$$

9D2. *Estimates on the glancing set.* We now obtain estimates of the form (79) using the description of the single- and double-layer potentials from Section 7. First, observe that

$$\|u_g\|^2_{L^2(\Omega)}=\langle\mathcal{B}\psi_g,\psi_g\rangle_{L^2(\partial\Omega)},$$

where by Lemma 7.5

$$\mathcal{B}:=V^*\mathcal{S}\ell^*\mathcal{S}\ell V\in h^{2+2\alpha}\Psi_{1-\epsilon/2}(|\xi'|_g=1)$$

is elliptic and has symbol given by

$$\sigma(\mathcal{B})=\frac{|\sigma(hV)|^2}{2Q}(\Psi_{\mathcal{S}\ell}(\alpha_{0h})\circ\kappa^{-1}).$$

Take $\epsilon,\epsilon_1>0$ small enough and let

$$\mathcal{L}_\alpha:=\begin{cases}\{||\xi'|_g-1|\le h^\epsilon,\ |\Xi+h^{2/3}\zeta_j|<\epsilon_1h^{2/3}\text{ or }\Xi\le -M_1h^{2/3}\}, & \alpha<-\frac{2}{3},\\ \{||\xi|'_g-1|\le CMh^{2/3}, & \alpha\ge-\frac{2}{3},\end{cases} \quad (80)$$

where C and M are as in (75).

Now, define

$$\frac{2\operatorname{Re} z \operatorname{Im} z}{h}\|u_g\|^2 + \operatorname{Im}\langle -hN_2\psi_g, \psi_g\rangle + \langle -h \operatorname{Im} V\psi_g, \psi_g\rangle = \langle A\psi_g, \psi_g\rangle,$$

where

$$A := \frac{2\operatorname{Re} z \operatorname{Im} z}{h}\mathcal{B} - \operatorname{Im}(hN_2 + hV_2).$$

Then, applying the sharp Gårding inequality (see Lemma 4.5) along with bounds on the norm of pseudodifferential operators (see Lemma 4.6), we obtain

$$\inf_{\mathcal{L}_\alpha}\biggl(\frac{2\operatorname{Re} z \operatorname{Im} z}{h}\frac{|\sigma(hV)|^2}{2Q}\Psi_{\mathcal{S}\ell}(h^{-2/3}\Xi)(1+O(h^{\epsilon/2})) \\ - h(\operatorname{Im}\sigma(N_2) + \sigma(\operatorname{Im} V)) - ch^{1/3+\epsilon/2} - ch^{4/3+\alpha}\biggr)\|\psi_g\|^2 \le \langle A\psi_g, \psi_g\rangle \quad (81)$$

and

$$\langle A\psi_g, \psi_g\rangle \le \sup_{\mathcal{L}_\alpha}\biggl(\frac{2\operatorname{Re} z \operatorname{Im} z}{h}\frac{|\sigma(hV)|^2}{2Q}\Psi_{\mathcal{S}\ell}(h^{-2/3}\Xi)(1+O(h^{\epsilon/2})) \\ - h(\operatorname{Im}\sigma(N_2) + \sigma(\operatorname{Im} V)) + ch^{1/3+\epsilon/2} + ch^{4/3+\alpha}\biggr)\|\psi_g\|^2. \quad (82)$$

Notice that for all $\delta > 0$, there exists M_1 large enough and ϵ_1 small enough so that

$$1-\delta \le \Psi_{\mathcal{S}\ell}(h^{-2/3}\Xi) \le 1+\delta, \quad (x, \xi) \in \mathcal{L}_\alpha \quad \bigl(\alpha < -\tfrac{2}{3}\bigr).$$

So, we have:

Lemma 9.8. *For all $\delta > 0$ there exists $h_0 > 0,\ N, M > 0,\ C, c > 0$ such that for $0 < h < h_0$ if $\pm\operatorname{Im} z \ge 0$ and one of*

$$\begin{aligned} \frac{-\operatorname{Im} z}{h} &\le \inf_{\mathcal{L}_\alpha} -\frac{h(\operatorname{Im}\sigma(N_2) + \sigma(\operatorname{Im} V) + c(h^{1/3+\alpha} + h^{-1/3+\epsilon/2}))Q}{|\sigma(hV)|^2\Psi_{\mathcal{S}\ell}(h^{-2/3}\Xi)}(1\pm\delta), \\ \frac{-\operatorname{Im} z}{h} &\ge \sup_{\mathcal{L}_\alpha} -\frac{h(\operatorname{Im}\sigma(N_2) + \sigma(\operatorname{Im} V) - c(h^{1/3+\alpha} + h^{-1/3+\epsilon/2}))Q}{|\sigma(hV)|^2\Psi_{\mathcal{S}\ell}(h^{-2/3}\Xi)}(1\mp\delta), \end{aligned} \quad (83)$$

holds, then

$$\|\psi_g\|_{L^2} \le Ch^{-N}(\|v\|_{L^2} + \|\psi_{ng}\|_{H_h^m}) + O(h^\infty)\|\psi\|_{H_h^{-M}}. \quad (84)$$

If $\alpha < -\frac{2}{3}$, we can replace the conditions (83) *with*

$$\begin{aligned} \frac{-\operatorname{Im} z}{h} &\le \inf_{\mathcal{L}_\alpha} -\frac{h(\operatorname{Im}\sigma(N_2) + \sigma(\operatorname{Im} V) + c(h^{1/3+\alpha} + h^{-1/3+\epsilon/2})Q}{|\sigma(hV)|^2}(1\pm\delta), \\ \frac{-\operatorname{Im} z}{h} &\ge \sup_{\mathcal{L}_\alpha} -\frac{h(\operatorname{Im}\sigma(N_2) + \sigma(\operatorname{Im} V) - c(h^{1/3+\alpha} + h^{-1/3+\epsilon/2}))Q}{|\sigma(hV)|^2}(1\mp\delta). \end{aligned}$$

9E. ***Further localization away from the real axis when $\alpha < -\frac{2}{3}$.*** We now focus our attention on the region $|\operatorname{Im} z| \geq ch^N$ for some $N > 0$ and $\alpha < -\frac{2}{3}$. In this region, we are able to decompose $\psi = u|_{\partial\Omega}$ into pieces, ψ_j, concentrating at $\Xi = \zeta_j h^{2/3}$, that still have

$$(I + GV)\psi_j = Gv_j,$$

with the norm of v_j controlled by the norm of v.

We again use the representation of G near glancing. With χ and φ_i as above

$$\Big(I + \sum_i h^{2/3} J_i \omega^{-1} \mathcal{A}_i \mathcal{A}i C^{-1} J_i^{-1} V \varphi_i\Big) \operatorname{Op_h}(\chi)\psi = \operatorname{Op_h}(\chi)Gv + [GV, \operatorname{Op_h}(\chi)]\psi + O(h^\infty)\psi.$$

Fix $\epsilon_1 > 0$ small enough and let $\chi_j \equiv 1$ on $|\xi_1 + \zeta_j| \leq \epsilon_1 h^{2/3}$ with $\operatorname{supp} \chi_j \subset |\xi_1 + \zeta_j| \leq 2\epsilon_1 h^{2/3}$ and let $L_j = \operatorname{Op_h}(\chi_j(\Xi))$. Then

$$\sum_i (I + h^{2/3} J_i \omega^{-1} \mathcal{A}_i \mathcal{A}i C^{-1} J_i^{-1} V)\varphi_i L_j \operatorname{Op_h}(\chi)\psi \\ = L_j \operatorname{Op_h}(\chi)Gv + L_j[GV, \operatorname{Op_h}(\chi)]\psi + [GV, L_j] \operatorname{Op_h}(\chi)\psi + O(h^\infty)\psi.$$

Now, $[GV, L_j]$ is a pseudodifferential operator with support on the complement of $\mathcal{L}_\alpha$. Therefore by Lemma 9.6 there exists $M > 0$ so that

$$\|[GV, L_j] \operatorname{Op_h}(\chi)\psi\| \leq h^{-M}\|v\| + O(h^\infty)\|\psi\|.$$

So,

$$(I + GV)L_j \operatorname{Op_h}(\chi)\psi = w,$$

with

$$\|w\| \leq h^{-M}\|v\| + O(h^\infty)\|\psi\|.$$

Now, $G^{-1} = N_1 + N_2$ and since $|\operatorname{Im} z| \leq Mh \log h^{-1}$,

$$\|h(N_1 + N_2)\|_{H_h^1 \to L^2} \leq \frac{C}{|\operatorname{Im} z|}.$$

Hence, using that $|\operatorname{Im} z| \geq ch^N$, we have

$$(I + GV)L_j \operatorname{Op_h}(\chi)\psi = GG^{-1}w = G(N_1 + N_2)w =: Gv_j$$

so that for some $M > 0$,

$$\|v_j\| \leq h^{-M}\|v\| + O(h^\infty)\|\psi\|.$$

So, formulas (76) and (77) hold with ψ replaced by $L_j \operatorname{Op_h}(\chi)\psi$ and v replaced by v_j. Let $\psi_j = L_j \operatorname{Op_h}(\chi)\psi$,

$$\mathcal{L}_j := \{|\Xi(x', \xi) + h^{2/3}\zeta_j| < 2\epsilon_1 h^{2/3},$$

and u_j be the solution to

$$\begin{cases} (-h^2\Delta - z^2)u_j = 0 & \text{in } \Omega, \\ (h\partial_\nu + B)u_j = v_j & \text{on } \partial\Omega, \\ u_j|_{\partial\Omega} = \psi_j. \end{cases}$$

Next, fix $\delta > 0$, take ϵ_1 small enough, and let $\pm \operatorname{Im} z \geq 0$. Then following the arguments above,

$$\inf_{\mathcal{L}_j}\left(\frac{2\operatorname{Im} z}{h(1\pm\delta)}\frac{|\sigma(hV)|^2}{2Q} - h(\operatorname{Im}\sigma(N_2)+\sigma(\operatorname{Im} V)) - ch^{1/3+\epsilon/2} - ch^{4/3+\alpha}\right)\|\psi_j\|^2 \leq \langle A\psi_j, \psi_j\rangle, \quad (85)$$

$$\langle A\psi_j, \psi_j\rangle \leq \sup_{\mathcal{L}_j}\left(\frac{2\operatorname{Im} z}{h(1\mp\delta)}\frac{|\sigma(hV)|^2}{2Q} - h(\operatorname{Im}\sigma(N_2)+\sigma(\operatorname{Im} V)) + ch^{1/3+\epsilon/2} + ch^{4/3+\alpha}\right)\|\psi_j\|^2 \quad (86)$$

and

$$|\langle A\psi_j, \psi_j\rangle| \leq C(\delta^{-1}\|v_j\|^2 + \delta\|\psi_j\|^2). \quad (87)$$

In particular, using that

$$\sigma(hN_2) = (2hQ)^{1/3}\frac{A_-'(h^{-2/3}\Xi)}{A_-(h^{-2/3}\Xi)},$$

we have:

Lemma 9.9. *Suppose that $\pm \operatorname{Im} z \geq ch^M$, $\alpha < -\frac{2}{3}$. Fix $j > 0$. Then there exist $h_0 > 0$, $N, C > 0$ such that if one of*

$$\frac{-\operatorname{Im} z}{h} \leq \inf_{\mathcal{L}_j} - \frac{h(h^{-2/3}(2Q)^{1/3}\operatorname{Im}(A_-'(-\zeta_j)/A_-(-\zeta_j)) + \sigma(\operatorname{Im} V) + ch^{1/3+\alpha})Q}{|\sigma(hV)|^2}(1\pm\delta),$$

$$\frac{-\operatorname{Im} z}{h} \geq \sup_{\mathcal{L}_j} - \frac{h(h^{-2/3}(2Q)^{1/3}\operatorname{Im}(A_-'(-\zeta_j)/A_-(-\zeta_j)) + \sigma(\operatorname{Im} V) - ch^{1/3+\alpha})Q}{|\sigma(hV)|^2}(1\mp\delta)$$

holds then

$$\|\psi_j\| \leq Ch^{-N}\|v\| + O(h^\infty)\|\psi\|.$$

With these estimates in hand, for any $M > 0$, let

$$\mathcal{L}_M' := \{-2h^\epsilon \leq \Xi \leq (-\zeta_{M+1} + 2\epsilon)h^{2/3}\} \quad (88)$$

and let $\chi_2' = \chi_2'(\xi_1) \in S_{2/3}$ have $\chi_2 \equiv 1$ on

$$\{-h^\epsilon \leq \xi_1 \leq (-\zeta_{M+1} + \epsilon)h^{2/3}\}$$

and $\operatorname{supp}\chi_2 \subset \mathcal{L}_M'$. Then define

$$\psi_g' = \operatorname{Op_h}(\chi_2(\Xi))\operatorname{Op_h}(\chi_1)\psi$$

and $\psi_{ng}' = \psi - \psi_g'$. Thus (81) and (82) still hold with $\mathcal{L}$ replaced by $\mathcal{L}_M'$ and we have:

Lemma 9.10. *For all $\delta > 0$ there exist $h_0 > 0$, $N, M > 0$, $C > 0$ such that for $0 < h < h_0$ if $\pm \operatorname{Im} z \geq 0$ and one of*

$$\frac{-\operatorname{Im} z}{h} \leq \inf_{\mathcal{L}_M'} - \frac{h(\operatorname{Im}\sigma(N_2) + \sigma(\operatorname{Im} V) + ch^{1/3+\alpha})Q}{|\sigma(hV)|^2}(1\pm\delta),$$

$$\frac{-\operatorname{Im} z}{h} \geq \sup_{\mathcal{L}_M'} - \frac{h(\operatorname{Im}\sigma(N_2) + \sigma(\operatorname{Im} V) - ch^{1/3+\alpha})Q}{|\sigma(hV)|^2}(1\mp\delta)$$

holds then

$$\|\psi_g'\|_{L^2} \leq Ch^{-N}(\|v\|_{L^2} + \|\psi_{ng}'\|_{H_{\mathrm{h}}^m}) + O(h^\infty)\|\psi\|_{H_{\mathrm{h}}^{-N}}.$$

So, combining Lemmas 9.1, 9.4, 9.5, 9.6, 9.8, 9.9, and 9.10 gives:

Theorem 9.11. *Let ψ be a solution to* (68). *Fix $\delta > 0$, $0 < \epsilon < \frac{1}{2}$, $\gamma_1 < \min\left(\frac{1}{2} - \epsilon, \frac{1}{6}\right)$, $M_1, M_2 > 0$. Then there exists $h_0 > 0$ and $N > 0$ such that for $0 < h < h_0$ if*

$$-\frac{\operatorname{Im} z}{h} < \inf_{\mathcal{H}} -l_N^{-1}(r_N + ch^{\gamma_1}) \quad \textit{or} \quad -\frac{\operatorname{Im} z}{h} > \sup_{\mathcal{H}} -l_N^{-1}(r_N - ch^{\gamma_1}),$$

$\pm \operatorname{Im} z \geq 0$, and one of

$$\begin{aligned} \frac{-\operatorname{Im} z}{h} &\leq \inf_{\mathcal{L}_\alpha} -\frac{h(\operatorname{Im}\sigma(N_2) + \sigma(\operatorname{Im} V) + c(h^{1/3+\alpha} + h^{-1/3+\epsilon/2}))Q}{|\sigma(hV)|^2 \Psi_{\mathcal{S}\ell}(h^{-2/3}\Xi)}(1 \pm \delta), \\ \frac{-\operatorname{Im} z}{h} &\geq \sup_{\mathcal{L}_\alpha} -\frac{h(\operatorname{Im}\sigma(N_2) + \sigma(\operatorname{Im} V) - c(h^{1/3+\alpha} + h^{-1/3+\epsilon/2}))Q}{|\sigma(hV)|^2 \Psi_{\mathcal{S}\ell}(h^{-2/3}\Xi)}(1 \mp \delta) \end{aligned} \tag{89}$$

holds then

$$\|\psi\|_{L^2} \leq Ch^{-N}\|v\|_{L^2} \tag{90}$$

and $\mathcal{P}(z)$ is invertible. Moreover, if $\alpha < -\frac{2}{3}$ then (89) *can be replaced by*

$$\begin{aligned} \frac{-\operatorname{Im} z}{h} &\leq \inf_{\mathcal{L}_\alpha} -\frac{h(\operatorname{Im}\sigma(N_2) + \sigma(\operatorname{Im} V) + c(h^{1/3+\alpha} + h^{-1/3+\epsilon/2})Q}{|\sigma(hV)|^2}(1 \pm \delta), \\ \frac{-\operatorname{Im} z}{h} &\geq \sup_{\mathcal{L}_\alpha} -\frac{h(\operatorname{Im}\sigma(N_2) + \sigma(\operatorname{Im} V) - c(h^{1/3+\alpha} + h^{-1/3+\epsilon/2}))Q}{|\sigma(hV)|^2}(1 \mp \delta). \end{aligned} \tag{91}$$

Finally, if $\pm \operatorname{Im} z \geq ch^{M_1}$ and $\alpha < -\frac{2}{3}$, then (90) *holds and $\mathcal{P}(z)$ is invertible if*

$$\frac{-\operatorname{Im} z}{h} \leq \inf_{\mathcal{L}'_{M_2}} -\frac{h(\operatorname{Im}\sigma(N_2) + \sigma(\operatorname{Im} V) + ch^{1/3+\alpha})Q}{|\sigma(hV)|^2}(1 \pm \delta),$$

$$\frac{-\operatorname{Im} z}{h} \geq \sup_{\mathcal{L}'_{M_2}} -\frac{h(\operatorname{Im}\sigma(N_2) + \sigma(\operatorname{Im} V) - ch^{1/3+\alpha})Q}{|\sigma(hV)|^2}(1 \mp \delta),$$

and one of the following holds for $1 \leq j \leq M_2$:

$$\begin{aligned} \frac{-\operatorname{Im} z}{h} &\leq \inf_{\mathcal{L}_j} -\frac{h(h^{-2/3}(2Q)^{1/3}\operatorname{Im}(A'_-(-\zeta_j)/A_-(-\zeta_j)) + \sigma(\operatorname{Im} V) + ch^{1/3+\alpha})Q}{|\sigma(hV)|^2}(1 \pm \delta), \\ \frac{-\operatorname{Im} z}{h} &\geq \sup_{\mathcal{L}_j} -\frac{h(h^{-2/3}(2Q)^{1/3}\operatorname{Im}(A'_-(-\zeta_j)/A_-(-\zeta_j)) + \sigma(\operatorname{Im} V) - ch^{1/3+\alpha})Q}{|\sigma(hV)|^2}(1 \mp \delta). \end{aligned}$$

In particular, this implies Theorem 1.11.

10. Application to transparent obstacles

In the case of transparent obstacles, we want to consider (3), repeated here for the reader's convenience,

$$\begin{cases}(-c^2\Delta-\lambda^2)u_1=0 & \text{in } \Omega,\\ (-\Delta-\lambda^2)u_2=0 & \text{in } \mathbb{R}^d\setminus\bar{\Omega},\\ u_1=u_2 & \text{on } \partial\Omega,\\ \partial_\nu u_1-\aleph\partial_\nu u_2=0 & \text{on } \partial\Omega,\\ u_2 \text{ is } \lambda\text{-outgoing.}\end{cases}$$

Thus, writing $\lambda=cz/h$, in the language of (67),

$$B=hN_2(z/h)+\aleph hN_2(cz/h)-hN_2(z/h),$$

where N_2 is the outgoing Dirichlet-to-Neumann map for the exterior problem (see Section 1D).

Thus, $V=\aleph N_2(cz/h)-N_2(z/h)$ has

$$V\in h^{-2/3}\big(\Psi^{1/2,1}_{2/3}(|\xi'|_g=c)\cup\Psi^{1/2,1}_{2/3}(|\xi'|_g=1)\big)\subset h^{-1}\big(\Psi^{0,1}_{2/3}(|\xi'|_g=c)\cup\Psi^{0,1}_{2/3}(|\xi'|_g=1)\big).$$

In order to fit the transparent obstacle problem into the framework of Theorem 9.11 with $\alpha=-1$, we only need to check that V is elliptic near $|\xi'|_g=1$ and that $1+h\sigma(V)/(2\sqrt{|\xi'|_g^2-1})$ has the required properties. We start by calculating the symbols of $\mathcal{B}$, B, and V. Let Ξ^E be the function given by Lemma 7.1 when we replace 1 by E in the eikonal equation for ρ_0 and θ_0. Set

$$g_E(x,\xi'):=(2Q(x,\xi'))^{1/3}\frac{A_-'(h^{-2/3}\Xi^E)}{A_-(h^{-2/3}\Xi^E)}.$$

Then,

$$\sigma(B)=\sigma(h\aleph N_2(cz/h))=\begin{cases}-i\aleph\sqrt{c^2-|\xi'|_g^2}, & |\xi'|_g\le c-h^\epsilon,\\ \aleph h^{1/3}g_c(x,\xi'), & ||\xi'|_g-c|\le h^\epsilon,\\ \aleph\sqrt{|\xi'|_g^2-c^2}, & |\xi'|_g\ge c+h^\epsilon,\end{cases}$$

$$\sigma(hV)=\begin{cases} i(\sqrt{1-|\xi'|_g^2}-\aleph\sqrt{c^2-|\xi'|_g^2}), & |\xi'|_g\le \min(1,c)-h^\epsilon,\\ i\sqrt{1-|\xi'|_g^2}+\aleph\sqrt{|\xi'|_g^2-c^2}, & c+h^\epsilon\le|\xi'|_g\le 1-h^\epsilon,\\ -i\aleph\sqrt{c^2-|\xi'|_g^2}-\sqrt{|\xi'|_g^2-1}, & 1+h^\epsilon\le|\xi'|_g\le c-h^\epsilon,\\ \aleph\sqrt{|\xi'|_g^2-c^2}-\sqrt{|\xi'|_g^2-1}, & |\xi'|_g\ge\max(1,c)+h^\epsilon,\\ h^{1/3}\aleph g_c+i\sqrt{1-|\xi'|_g^2}, & ||\xi'|_g-c|\le h^\epsilon,\ |\xi'|_g\le 1-h^\epsilon,\\ h^{1/3}\aleph g_c-\sqrt{|\xi'|_g^2-1}, & ||\xi'|_g-c|\le h^\epsilon,\ |\xi'|_g\ge 1+h^\epsilon,\\ -i\aleph\sqrt{c^2-|\xi'|_g^2}-h^{1/3}g_1, & ||\xi'|_g-1|\le h^\epsilon,\ |\xi'|_g\le c-h^\epsilon,\\ \aleph\sqrt{|\xi'|_g^2-c^2}-h^{1/3}g_1, & ||\xi'|_g-1|\le h^\epsilon,\ |\xi'|_g\ge c+h^\epsilon,\end{cases}$$

$$\sigma(\mathcal{B})=\frac{\aleph^2|c^2-|\xi'|_g^2|}{2Q}\Psi_{\mathcal{S}\ell}(h^{-2/3}\Xi)(1+o(1)),\quad ||\xi'|_g-1|\le h^\epsilon.$$

Now, we compute

$$1+\frac{h\sigma(V)}{2\sqrt{|\xi'|_g^2-1}}=\begin{cases}\frac{1}{2}+\frac{1}{2}\aleph\sqrt{c^2-|\xi'|_g^2}/\sqrt{1-|\xi'|_g^2}, & |\xi'|_g\le\min(1,c)-h^\epsilon,\\ \frac{1}{2}+i\frac{1}{2}\aleph\sqrt{|\xi'|_g^2-c^2}/\sqrt{1-|\xi'|_g^2}, & c+h^\epsilon\le|\xi'|_g\le 1-h^\epsilon,\\ \frac{1}{2}-i\frac{1}{2}\aleph\sqrt{c^2-|\xi'|_g^2}/\sqrt{|\xi'|_g^2-1}, & 1+Mh^{2/3}\le|\xi'|_g\le c-h^\epsilon,\\ \frac{1}{2}+\frac{1}{2}\aleph\sqrt{|\xi'|_g^2-c^2}/\sqrt{|\xi'|_g^2-1}, & \max(c+h^\epsilon,1+Mh^{2/3})\le|\xi'|_g,\\ \frac{1}{2}+i\frac{1}{2}\aleph h^{1/3}g_c/\sqrt{1-|\xi'|_g^2}, & |c-|\xi'|_g|\le h^\epsilon,\ |\xi'|_g\le 1-h^\epsilon,\\ \frac{1}{2}+\frac{1}{2}\aleph h^{1/3}g_c/\sqrt{|\xi'|_g^2-1}, & |c-|\xi'|_g|\le h^\epsilon,\ |\xi'|_g\ge 1+Mh^{2/3}.\end{cases}$$

Thus, we can see that V is elliptic near $|\xi'|_g=1$ and the transparent obstacle problem fits into the framework of Theorem 9.11.

In order to finish the proof of Theorem 1.2, we just need to check a few symbolic properties. First, notice $V=\aleph N_2(cz/h)-N_2(z/h)$. Thus,

$$\sigma(N_2(z/h)+V)=\aleph\sigma(N_2(cz/h))=-ih^{-1}\aleph\sqrt{c^2-|\xi'|_g^2},$$

where we take $\sqrt{-1}=i$. Putting this in (83) gives that (84) holds when $c>1$ and

$$\frac{-\operatorname{Im} z}{h}\le\inf_{|\xi'(q)|_g=1}-\frac{Q}{\aleph\sqrt{c^2-1}}(1\pm\delta)\quad\text{or}\quad\frac{-\operatorname{Im} z}{h}\ge\sup_{|\xi'(q)|_g=1}-\frac{Q}{\aleph\sqrt{c^2-1}}(1\mp\delta)$$

or when $c<1$ and

$$\frac{-\operatorname{Im} z}{h}\ge\delta.$$

Next, observe that

$$\sigma(R)=\begin{cases}(-\sqrt{1-|\xi'|_g^2}+\aleph\sqrt{c^2-|\xi'|_g^2})/(\sqrt{1-|\xi'|_g^2}+\aleph\sqrt{c^2-|\xi'|_g^2}), & |\xi'|_g\le\min(1,c)-h^\epsilon,\\ (i\aleph h^{1/3}g_c-\sqrt{1-|\xi'|_g^2})/(\sqrt{1-|\xi'|_g^2}+i\aleph h^{1/3}g_c), & |c-|\xi'|_g|\le h^\epsilon,\ |\xi'|_g\le 1-h^\epsilon,\\ (-\sqrt{1-|\xi'|_g^2}+i\aleph\sqrt{|\xi'|_g^2-c^2})/(\sqrt{1-|\xi'|_g^2}+i\aleph\sqrt{|\xi'|_g^2-c^2}), & c+h^\epsilon\le|\xi'|_g\le 1-h^\epsilon.\end{cases}$$

The following geometric lemma completes the proof of Theorem 1.2.

Lemma 10.1. *Fix $N>0$ and let $(x_0,\xi_0)\in S^*\partial\Omega$ and suppose that $\{(x_n,\xi_n)\}\subset B^*\partial\Omega$ has $(x_n,\xi_n)\to(x_0,\xi_0)$. Then*

$$l_N^{-1}r_N\to\begin{cases}Q(x_0,\xi_0)/(\aleph\sqrt{c^2-1}), & c>1,\\ 0, & c<1.\end{cases}$$

Proof. The conclusion for $c<1$ is clear since for $|\xi'|_g>c$, we have $\log|\sigma(R)|^2=0$. So, we need only consider the case $c>1$. First, write

$$|\sigma(R)|^2(x,\xi')=1-\frac{4\sqrt{1-|\xi'|_g^2}}{\sqrt{1-|\xi'|_g^2}+\aleph\sqrt{c^2-|\xi'|_g^2}}+O(1-|\xi'|_g^2).$$

So,

$$\log|\sigma(R)|^2(x,\xi')=-\frac{4\sqrt{1-|\xi'|_g^2}}{\sqrt{1-|\xi'|_g^2}+\aleph\sqrt{c^2-|\xi'|_g^2}}+O(1-|\xi'|_g^2). \tag{92}$$

Now, by Lemma 5.3

$$\sqrt{1-|\xi'(\beta(q))|_g^2}=\sqrt{1-|\xi'(q)|_g^2}+O(1-|\xi'|_g^2),\quad l(q,\beta(q))=\frac{2}{\kappa(0)}\sqrt{1-|\xi'|_g^2}+O(1-|\xi'|_g^2),$$

where $\kappa(s)$ is the curvature of the unique length-minimizing geodesic, γ, in $\partial\Omega$ connecting $\pi_x(q)$ and $\pi_x(\beta(q))$ at the point $\gamma(s)$. Thus, we have that for q sufficiently close to glancing,

$$\frac{\log|\sigma(R)(\beta(q))|^2}{2l(q,\beta(q))}=-\frac{\kappa(0)}{\aleph\sqrt{c^2-|\xi'|_g^2}}+O(\sqrt{1-|\xi'|_g^2}).$$

Moreover, since $\sqrt{1-|\xi'(\beta(q))|_g^2}=\sqrt{1-|\xi'|_g^2}+O(1-|\xi'|_g^2)$ and $\kappa(s)=\kappa(0)+O(s)=\kappa(0)+O(\sqrt{1-|\xi'|_g^2})$, we have

$$\frac{r_N}{l_N}=-\frac{\kappa(0)}{\aleph\sqrt{c^2-|\xi'|_g^2}}+O(1-|\xi'|_g^2).$$

All that remains to prove is that $\kappa(0)=Q(x,\xi')+o(1)$ as $|\xi'|_g\to 1$. This follows from the fact that the curvature of the geodesic on $\partial\Omega$ passing through x in the direction ξ' is $Q(x,\xi')$ together with the fact that

$$\gamma'(0)-\frac{\xi'}{|\xi'|_g}=o(1).$$

To see this we simply use the fact that a billiards trajectory approaches a geodesic as $|\xi'|_g\to 1$ (see for example [Petkov and Stoyanov 1992]). □

Together, this discussion proves Theorem 1.2.

11. Application to δ potentials

For the application to δ potentials, we consider

$$(-h^2\Delta+h^2V\otimes\delta_{\partial\Omega}-z^2)u=0,\quad u \text{ is } z/h \text{ outgoing}.$$

It is shown in [Galkowski and Smith 2015] that this is equivalent to $u=u_1\oplus u_2$, where $u_1=u1_\Omega$ and $u_2=u1_{\mathbb{R}^d\setminus\bar\Omega}$, solving

$$\begin{cases}(-h^2\Delta-z^2)u=0 & \text{in } \mathbb{R}^d\setminus\partial\Omega,\\ u_1-u_2=0 & \text{on } \partial\Omega,\\ h\partial_\nu u_1-h\partial_\nu u_2+hVu_1=0 & \text{on } \partial\Omega,\\ u_2 \text{ is } z/h\text{-outgoing}.\end{cases}\tag{93}$$

In this case, $V=V$ (indeed this is the motivation for our notation). For our purposes, we will assume that $V\in h^\alpha\Psi^1$ is self-adjoint and hence $\operatorname{Im}V=0$. Moreover, we assume that $\alpha\geq -1$ and $\sigma(V)\geq ch^\alpha$ on $|\xi'|_g=1$ and for any $\delta>0$, there exists $c>0$ so that $h\sigma(V)/(2\sqrt{|\xi'|^2-1})>-1+c$ on $|\xi'|_g\geq 1+\delta$. This clearly implies all of the assumptions (65). Theorem 9.11 then yields Theorem 1.5 as a corollary.

12. Application to boundary stabilization

The application to the boundary stabilization problem (11) is similar to that for the transmission problem. In particular, note that

$$1+\frac{i\sigma(hV)}{2\sqrt{1-|\xi'|_g^2}}=-\frac{1}{2}+\frac{a}{2\sqrt{1-|\xi'|_g^2}}$$

and the fact that $a \geq a_0 > 0$ imply the ellipticity of V. Finally, an argument identical to that in Lemma 10.1, together with Theorem 9.11, gives Theorem 1.7.

13. Optimality for the transparent obstacle problem on the circle

For the optimality of Theorem 1.5, see [Galkowski 2016]. We now show that Theorem 1.2 is optimal in the case of the unit disk in $\mathbb{R}^2$. In this case, (3) reads

$$\begin{cases}(-c^2\Delta-\lambda^2)u_1=0 & \text{in } B(0,1),\\ (-\Delta-\lambda^2)u_2=0 & \text{in } \mathbb{R}^d\setminus\overline{B(0,1)},\\ u_1=u_2 & \text{on } |x|=1,\\ \partial_r u_1-\aleph\partial_r u_2=0 & \text{on } |x|=1,\\ u_2 \text{ is } \lambda\text{−outgoing.}\end{cases}$$

We now expand u_i in Fourier series, writing

$$u_i(r,\theta)=\sum_n u_{i,n}(r)e^{in\theta}.$$

Then,

$$\left(-c^2\partial_r^2-\frac{c^2}{r}\partial_r+\frac{c^2n^2}{r}-\lambda^2\right)u_{1,n}(r)=0,\qquad \left(-\partial_r^2-\frac{1}{r}\partial_r+\frac{n^2}{r}-\lambda^2\right)u_{2,n}(r)=0.$$

Multiplying by r^2 and rescaling by $x_1=\lambda c^{-1}r$ for $u_{1,n}$ and $x_2=\lambda r$ for u_2, we see that $u_{i,n}(x_i)$ solves Bessel's equation. Together with the outgoing condition for u_2 and the fact that u_1 is in L^2, this implies

$$u_{1,n}=K_nJ_n(\lambda c^{-1}r),\qquad u_{2,n}=C_nH_n^{(1)}(\lambda r).$$

Then, the boundary conditions imply that either $K_n=C_n=0$ or $C_n\neq 0$ and

$$\frac{K_n}{C_n}=\frac{H_n^{(1)}(\lambda)}{J_n(c^{-1}\lambda)},\qquad K_nc^{-1}\lambda J_n'(\lambda c^{-1})-C_n\aleph\lambda H_n^{(1)'}(\lambda)=0.$$

Rewriting this (and assuming $\lambda\neq 0$) we have

$$f(\lambda):=c^{-1}J_n'(c^{-1}\lambda)H_n^{(1)}(\lambda)-\aleph H_n^{(1)'}(\lambda)J_n(c^{-1}\lambda)=0. \tag{94}$$

See Figure 10 for numerically computed resonances, i.e., numerically computed solutions to (94).

Throughout this section we will refer to microlocalization of the Fourier modes $e^{in\theta}$. Notice that for a Fourier mode $u_n=(u_{1,n}(r)\oplus u_{2,n})e^{in\theta}$, the component of the frequency tangent to $\partial B(0,1)$ is given by n and the rest of the oscillations are normal to the boundary. Naively taking the Fourier transform, we see that if $(-\Delta-\lambda^2)u=0$, then the Fourier support of u is contained in $|\xi|^2=\lambda^2$. Therefore, since $|\operatorname{Im}\lambda|\ll|\operatorname{Re}\lambda|$ the total frequency of the mode is given by $|\operatorname{Re}\lambda|$ and the fraction of frequency tangent to the boundary is given by $n/\operatorname{Re}\lambda$. This can be reinterpreted in terms of the semiclassical wavefront set (with $\operatorname{Re}\lambda = h^{-1}$) of the mode as saying that

$$\mathrm{WF_h}(u_n|_{\partial\Omega}) \subset \{|\xi'|_g = hn\}.$$

For this reason, we refer to modes with $n \ll |\operatorname{Re}\lambda|$ as normal to the boundary, those with $\epsilon|\operatorname{Re}\lambda| < n < (c^{-1}-\epsilon)|\operatorname{Re}\lambda|$ as transverse, and $(c^{-1}-\epsilon)|\operatorname{Re}\lambda| < n$ as glancing.

13A. ***Asymptotics of Bessel and Hankel functions.*** We collect here some properties of the Airy and Bessel functions that are used in the analysis for the unit disk. These formulae can be found in, for example, [Olver et al. 2010, Chapters 9, 10].

Recall that the Bessel functions of order n are solutions to

$$z^2y'' + zy' + (z^2 - n^2)y = 0.$$

We consider the two independent solutions $H_n^{(1)}(z)$ and $J_n(z)$.

We now record some asymptotic properties of Bessel functions. Consider n fixed and $z \to \infty$:

$$J_n(z) = \left(\frac{1}{2\pi z}\right)^{1/2}\left(e^{i(z-n\pi/2-\pi/4)} + e^{-i(z-n\pi/2-\pi/4)} + O(|z|^{-1}e^{|\operatorname{Im} z|})\right),$$

$$H_n^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{1/2}\left(e^{i(z-n\pi/2-\pi/4)} + O(|z|^{-1}e^{|\operatorname{Im} z|})\right),$$

$$J_n'(z) = i\left(\frac{1}{2\pi z}\right)^{1/2}\left(e^{i(z-n\pi/2-\pi/4)} - e^{-i(z-n\pi/2-\pi/4)} + O(|z|^{-1}e^{|\operatorname{Im} z|})\right),$$

$$H_n^{(1)\prime}(z) = i\left(\frac{2}{\pi z}\right)^{1/2}\left(e^{i(z-n\pi/2-\pi/4)} + O(|z|^{-1}e^{|\operatorname{Im} z|})\right),$$

$$J_n'(c^{-1}z)H_n^{(1)}(z) = \frac{i\sqrt{c}}{\pi z}\left(e^{i((c^{-1}+1)z-n\pi-\pi/2)} - e^{-i(c^{-1}-1)z} + O(|z|^{-1}e^{(c^{-1}+1)|\operatorname{Im} z|})\right), \tag{95}$$

$$J_n(c^{-1}z)H_n^{(1)\prime}(z) = \frac{i\sqrt{c}}{\pi z}\left(e^{i((c^{-1}+1)z-n\pi-\pi/2)} + e^{-i(c^{-1}-1)z} + O(|z|^{-1}e^{(c^{-1}+1)|\operatorname{Im} z|})\right). \tag{96}$$

Next, we record asymptotics that are uniform in n and z as $n \to \infty$. Let $\zeta = \zeta(z)$ be the unique smooth solution on $0 < z < \infty$ to

$$\left(\frac{d\zeta}{dz}\right)^2 = \frac{1-z^2}{\zeta z^2}, \tag{97}$$

with

$$\lim_{z\to 0} \zeta = \infty, \quad \lim_{z\to 1} \zeta = 0, \quad \lim_{z\to\infty} \zeta = -\infty.$$

Then

$$\tfrac{2}{3}(-\zeta)^{3/2} = \sqrt{z^2-1} - \operatorname{arcsec}(z), \qquad 1 < z < \infty, \tag{98}$$

$$\tfrac{2}{3}(\zeta)^{3/2} = \log\left(\frac{1+\sqrt{1-z^2}}{z}\right) - \sqrt{1-z^2}, \quad 0 < z < 1,$$

$$\frac{1-z^2}{\zeta z^2} \to \sqrt[3]{2}, \qquad z \to 0. \tag{99}$$

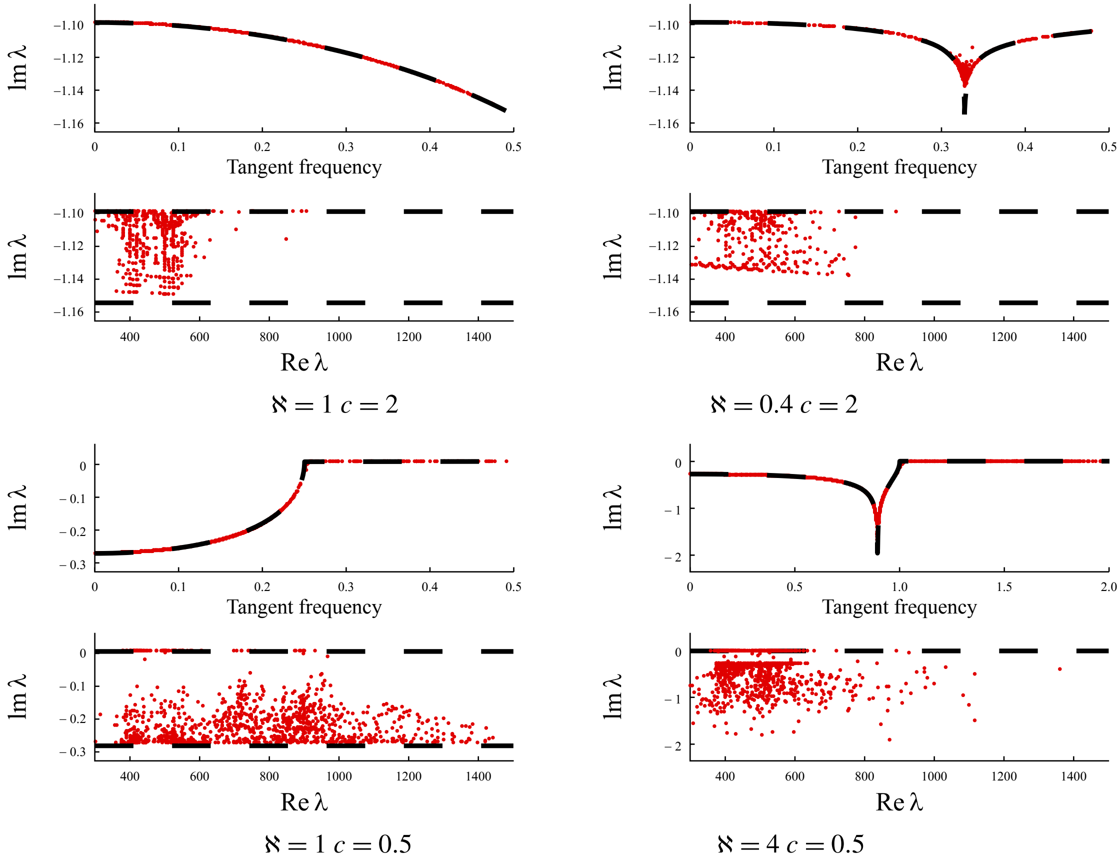


Figure 10. Numerically computed resonances for the transparent obstacle problem with various c and $\aleph$ when $\Omega = B(0, 1) \subset \mathbb{R}^2$. In this case, we expand the solutions to (3) as $u_i(r, \theta) = \sum_n u_{i,n}(r)e^{in\theta}$ and solve for some of the resonances with $\operatorname{Re}\lambda \sim 800$. In the lower graphs of each of the four subfigures, the red circles show $\operatorname{Im}\lambda$ vs. $\operatorname{Re}\lambda$. The dashed black lines show the upper and lower bounds for $\operatorname{Im}\lambda$ when $\aleph$ corresponds to TE waves and the upper bounds on $\operatorname{Im}\lambda$ when $\aleph$ corresponds to TM waves from Theorem 1.2. Notice that by orthogonality of $e^{in\theta}$ and $e^{im\theta}$ for $m \neq n$, the pair $(u_{1,n}e^{in\theta}, u_{2,n}e^{in\theta})$ satisfies (3). In the top graph of each subfigure, the red circles show $\operatorname{Im}\lambda$ vs. $n/\operatorname{Re}\lambda$ for such pairs. That is, we plot $\operatorname{Im}\lambda$ vs. the scaled tangent frequency of the resonance state. The dashed curve shows a plot of $(cr_1/(2l_1))(c\xi)$, the decay rate predicted for a billiards trajectory traveling with scaled tangent frequency $c\xi$. The large spikes in the top graphs occur at the Brewster angle when $\aleph$ corresponds to TM waves.

Let

$$Ai(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i((1/3)t^3+st)}\,dt$$

for $s \in \mathbb{R}$ be the Airy function solving $Ai''(z) - zAi(z) = 0$. Then, $A_-(z) = Ai(e^{2\pi i/3}z)$ is another solution of the Airy equation.

For z fixed as $n \to \infty$

$$\begin{aligned}
J_n(nz) &= \left(\frac{4\zeta}{1-z^2}\right)^{1/4}\left(\frac{Ai(n^{2/3}\zeta)}{n^{1/3}} + O(Ei(5/3,7/3))\right),\\
H_n^{(1)}(nz) &= 2e^{-\pi i/3}\left(\frac{4\zeta}{1-z^2}\right)^{1/4}\left(\frac{A_-(n^{2/3}\zeta)}{n^{1/3}} + O(E_-(5/3,7/3))\right),\\
J_n'(nz) &= -\frac{2}{z}\left(\frac{1-z^2}{4\zeta}\right)^{1/4}\left(\frac{Ai'(n^{2/3}\zeta)}{n^{2/3}} + O(Ei(8/3,4/3))\right),\\
H_n^{(1)'}(nz) &= \frac{4e^{2\pi i/3}}{z}\left(\frac{1-z^2}{4\zeta}\right)^{1/4}\left(\frac{A_-'(n^{2/3}\zeta)}{n^{2/3}} + O(E_-(8/3,4/3))\right),\\
J_n'(c^{-1}nz)H_n^{(1)}(nz) &= \frac{4e^{2\pi i/3}c}{z}\left(\frac{(1-c^{-2}z^2)\zeta(z)}{\zeta(c^{-1}z)(1-z^2)}\right)^{1/4}\left(\frac{Ai'(n^{2/3}\zeta(c^{-1}z))}{n^{2/3}} + O(Ei(8/3,4/3)(c^{-1}z))\right)\\
&\qquad\times\left(\frac{A_-(n^{2/3}\zeta)}{n^{1/3}} + O(E_-(5/3,7/3)(z))\right)\\
J_n(c^{-1}nz)H_n^{(1)'}(nz) &= \frac{4e^{2\pi i/3}}{z}\left(\frac{(1-z^2)\zeta(c^{-1}z)}{\zeta(z)(1-c^{-2}z^2)}\right)^{1/4}\left(\frac{Ai(n^{2/3}\zeta(c^{-1}z))}{n^{1/3}} + O(Ei(5/3,7/3)(c^{-1}z))\right),\\
&\qquad\times\left(\frac{A_-'(n^{2/3}\zeta(z))}{n^{2/3}} + O(E_-(8/3,4/3)(z))\right),
\end{aligned}$$

where

$$\begin{aligned}
E_-(s,t) &= |A_-'(n^{2/3}\zeta)|n^{-s} + |A_-(n^{2/3}\zeta)|n^{-t},\\
Ei(s,t) &= |Ai'(n^{2/3}\zeta)|n^{-s} + |Ai(n^{-2/3}\zeta)|n^{-t}.
\end{aligned} \tag{100}$$

We now record some facts about the Airy functions Ai and A_-. For $s \in \mathbb{R}$,

$$Ai(s) = e^{-\pi i/3}A_-(s) + e^{\pi i/3}\overline{A_-(s)}$$

and hence

$$\operatorname{Im}(e^{-5\pi i/6}A_-(s)) = -\tfrac{1}{2}Ai(s). \tag{101}$$

Next, we record asymptotics for Airy functions as $z \to \infty$ in the sector $|\operatorname{Arg} z| < \frac{\pi}{3} - \delta$. Many of these asymptotic formulae hold in larger regions, but we restrict our attention to this sector. Let $\eta = 2/3z^{3/2}$, where we take principal branch of the square root. Then

$$\begin{aligned}
A_-(z) &= \frac{e^{-\pi i/6}e^{\eta}}{2\sqrt{\pi}z^{1/4}}(1+O(|z|^{-3/2})), & A_-(-z) &= \frac{e^{\pi i/12}e^{i\eta}}{2\sqrt{\pi}z^{1/4}},\\
A_-'(z) &= \frac{e^{-\pi i/6}z^{1/4}e^{\eta}}{2\sqrt{\pi}}(1+O(|z|^{-3/2})), & A_-'(-z) &= \frac{e^{-5\pi i/12}z^{1/4}e^{i\eta}}{2\sqrt{\pi}},\\
Ai(z) &= \frac{z^{-1/4}e^{-\eta}}{2\sqrt{\pi}}(1+O(|z|^{-3/2})), & Ai(-z) &= \frac{z^{-1/4}}{2\sqrt{\pi}}(e^{i\eta-i\pi/4}+e^{-i\eta+i\pi/4}+O(|z|^{-3/2}e^{|\operatorname{Im}\eta|})),\\
Ai'(z) &= -\frac{z^{1/4}e^{-\eta}}{2\sqrt{\pi}}(1+O(|z|^{-3/2})), & Ai'(-z) &= \frac{z^{1/4}}{2i\sqrt{\pi}}(e^{i\eta-i\pi/4}-e^{-i\eta+i\pi/4}+O(|z|^{-3/2}e^{|\operatorname{Im}\eta|})).
\end{aligned} \tag{102}$$

13B. ***Resonances normal to the boundary (fixed n).*** First, we fix $n \geq 0$ and examine solutions with $\operatorname{Re}\lambda \to \infty$. We assume that $\aleph \neq c^{-1}$. Consider (94) and apply the asymptotics (95) and (96) with $\operatorname{Im}\lambda \leq 0$

$$(c^{-1} - \aleph)e^{i((c^{-1}+1)\lambda - n\pi - \pi/2)} - (c^{-1} + \aleph)e^{-i(c^{-1}-1)\lambda} + O(|z|^{-1}e^{(c^{-1}+1)|\operatorname{Im} z|}) = 0.$$

So, ignoring the error term for now, we have

$$\frac{1 - \aleph c}{1 + \aleph c} e^{i(2c^{-1}\lambda_0 - n\pi - \pi/2)} = 1.$$

So,

$$c^{-1}\operatorname{Im}\lambda_0 = \frac{1}{2}\log\left|\frac{1-\aleph c}{1+\aleph c}\right|, \qquad c^{-1}\operatorname{Re}\lambda_0 = \frac{2 - \operatorname{sgn}(1-\aleph c) + 2n + 4k}{4}\pi.$$

Taking λ_0 as above, we have $f(\lambda_0) = O(|\operatorname{Re}\lambda_0|^{-1})$, $|f'(\lambda_0)| \geq c$, and $|f''(\lambda)| \leq C$ for $|\lambda - \lambda_0| < \delta$ for some $\delta > 0$. We now recall Newton's method (see for example [Galkowski 2016, Lemma 4.1]).

Lemma 13.1. *Suppose that $z_0 \in \mathbb{C}$. Let $\Omega := \{z \in \mathbb{C} : |z - z_0| \leq \epsilon\}$ and suppose $f : \Omega \to \mathbb{C}$ is analytic. Suppose that*

$$|f(z_0)| \leq a, \quad |\partial_z f(z_0)| \geq b, \quad \sup_{z\in\Omega} |\partial_z^2 f(z)| \leq d.$$

Then if

$$a + d\epsilon^2 < \epsilon b < c < 1, \tag{103}$$

there is a unique solution z to $f(z) = 0$ in Ω.

Using this, we have that there exists a unique solution λ_1 to $f(\lambda_1) = 0$ with $|\lambda_1 - \lambda_0| = O(|\operatorname{Re}\lambda_0|^{-1})$.

13C. ***Resonances with nonzero tangent frequency (ϵ Re $\lambda \leq n \leq$ (min(1, c) $-\ \epsilon$) Re λ).*** In this case, we write

$$f(\lambda) := c^{-1} J_n'\left(n\frac{c^{-1}\lambda}{n}\right) H_n^{(1)}(\lambda) - \aleph H_n^{(1)\prime}(\lambda) J_n\left(n\frac{c^{-1}\lambda}{n}\right) = 0.$$

Write $z = \lambda/n$. Then taking $n \leq (c - \epsilon)\operatorname{Re}\lambda$ and ignoring error terms, $f(\lambda_0) = 0$ implies

$$\left[\left(\frac{1-c^{-2}z_0^2}{1-z_0^2}\right)^{1/2} - \aleph\right] e^{(4in/3)((-\zeta(c^{-1}z_0))^{3/2} - i\pi/2} = \left[\left(\frac{1-c^{-2}z_0^2}{1-z_0^2}\right)^{1/2} + \aleph\right] - i\frac{\sqrt{c^{-2}z_0^2 - 1} - \aleph\sqrt{z_0^2 - 1}}{\sqrt{c^{-2}z_0^2 - 1} + \aleph\sqrt{z_0^2-1}}$$
$$= e^{-(4in/3)((-\zeta(c^{-1}z_0))^{3/2}}. \tag{104}$$

Fix $\max(c, 1) + \delta < r < \infty$ with $\delta < c^2$ so that

$$\sqrt{c^{-2}r^2 - 1} - \aleph\sqrt{r^2 - 1} \neq 0.$$

Let

$$g(s, n, k) := \sqrt{c^{-2}s^2 - 1} - \operatorname{arcsec}(c^{-1}s) + \frac{4k - \operatorname{sgn}(\sqrt{c^{-2}s^2 - 1} - \aleph\sqrt{s^2 - 1})}{4n}\pi.$$

Then, fix $q \in \mathbb{Z}_+,\ p \in \mathbb{Z}$ and let $n = qm$ and $k = pm$ so that

$$g(s, qm, pm) = \sqrt{c^{-2}s^2 - 1} - \operatorname{arcsec}(c^{-1}s) + \frac{p}{q}\pi - \frac{\operatorname{sgn}(\sqrt{c^{-2}s^2-1} - \aleph\sqrt{s^2-1})}{4mq}\pi.$$

Then, for any $\epsilon > 0$ small enough, there exist p_ϵ, q_ϵ so that

$$|g(r, qm, pm)| < \epsilon + O(m^{-1}),$$
$$\partial_s g(r, q_\epsilon m, p_\epsilon m) = \frac{\sqrt{c^{-2}r^2-1}}{r} \geq C\sqrt{\delta}, \quad \partial_s^2 g(r, q_\epsilon m, p_\epsilon m) = -\frac{r^{-3}}{\sqrt{c^{-2}-r^{-2}}} \leq \frac{C}{\sqrt{\delta}}.$$

Therefore, taking ϵ small enough and m large enough (depending on $r - c$), there is a solution r_m to $g(r_m, q_\epsilon m, p_\epsilon m) = 0$ with $|r - r_m| < C\epsilon$.

With this r_m, let

$$\lambda_0 = mqr_m + i\frac{r_m}{2\sqrt{c^{-2}r_m^2-1}}\log\left|\frac{\sqrt{c^{-2}r_m^2-1} - \aleph\sqrt{r_m^2-1}}{\sqrt{c^{-2}r_m^2-1} + \aleph\sqrt{r_m^2-1}}\right|$$

and $z_0 = \lambda_0/mq$. Let

$$H(z, n) = \exp\left(-\tfrac{4}{3}in(-\zeta(c^{-1}z))^{3/2}\right) + i\frac{\sqrt{c^{-2}z^2-1} - \aleph\sqrt{z^2-1}}{\sqrt{c^{-2}z^2-1} + \aleph\sqrt{z^2-1}}.$$

Then, accounting for the errors omitted to obtain (104) there is a function $a(z, n) = O(n^{-2/3})$, analytic in z, such that nz is a resonance if and only if

$$H(z, n) = a(z, n)\left[1 + \exp\left(-\tfrac{4}{3}in(-\zeta(c^{-1}z))^{3/2}\right)\right].$$

Now, using (98)

$$\begin{aligned}
-\tfrac{4}{3}&imq(-\zeta(c^{-1}z_0))^{3/2} \\
&= -2imq(\sqrt{c^{-2}z_0^2-1} - \operatorname{arcsec}(c^{-1}z_0)) \\
&= -2imq\left(\sqrt{c^{-2}r_m^2-1} - \operatorname{arcsec}(c^{-1}r_m) + i\frac{\sqrt{c^{-2}r_m^2-1}}{r_m}\operatorname{Im} z_0 + O((\operatorname{Im} z_0)^2)\right) \\
&= i\left(2mpi - \tfrac{1}{2}\operatorname{sgn}(\sqrt{c^{-2}r_m^2-1} - \aleph\sqrt{r_m^2-1})\right)\pi + \log\left|\frac{\sqrt{c^{-2}r_m^2-1} - \aleph\sqrt{r_m^2-1}}{\sqrt{c^{-2}r_m^2-1} + \aleph\sqrt{r_m^2-1}}\right| + O((mq)^{-1}).
\end{aligned}$$

So,

$$\begin{aligned}
&\exp\left(-\tfrac{4}{3}imq(-\zeta(c^{-1}z_0))^{3/2}\right) \\
&\qquad = -\operatorname{sgn}(\sqrt{c^{-2}r_m^2-1} - \aleph\sqrt{c^{-2}r_m^2-1})i\left|\frac{\sqrt{c^{-2}r_m^2-1} - \aleph\sqrt{r_m^2-1}}{\sqrt{c^{-2}r_m^2-1} + \aleph\sqrt{r_m^2-1}}\right|(1 + O((mq)^{-1})).
\end{aligned}$$

So, $H(z_0, mq) = O((mq)^{-1})$. Moreover, $|z_0 - z| \leq 1$,

$$|\partial_z H(z, mq)| \geq cmq.$$

Hence, by the implicit function theorem, there exists a resonance z_1 with

$$z_1 = z_0 + O\left(\frac{\sup_{|z-z_0|\le 1}\left|a(z,mq)[1+\exp(-(4imq/3)(-\zeta(c^{-1}z))^{3/2})]\right|}{\inf_{|z-z_0|\le 1}|\partial_z H(z,mq)|}\right) = z_0 + O((mq)^{-5/3}).$$

Thus, there is a resonance, λ_1 with

$$\lambda_1 = mqr_m + i\frac{r_m}{2\sqrt{c^{-2}r_m^2-1}}\log\left|\frac{\sqrt{c^{-2}r_m^2-1}-\aleph\sqrt{r_m^2-1}}{\sqrt{c^{-2}r_m^2-1}+\aleph\sqrt{r_m^2-1}}\right| + O((mq)^{-2/3}).$$

Now, notice that if $|\xi'|_g^{-1}c = r$, then on $B(0,1)$, we have $l((x,\xi'),\beta(x,\xi')) = 2\sqrt{1-r^{-2}c^2}$. So,

$$\begin{aligned} l_N^{-1}r_N(x,\xi') &= \frac{1}{4\sqrt{1-r^{-2}c^2}}\log\left|\frac{\sqrt{1-r^{-2}c^2}-\aleph\sqrt{c^2-r^{-2}c^2}}{\sqrt{1-r^{-2}c^2}+\aleph\sqrt{c^2-c^2r^{-2}}}\right|^2 \\ &= \frac{c^{-1}r}{2\sqrt{c^{-2}r^2-1}}\log\left|\frac{\sqrt{c^{-2}r^2-1}-\aleph\sqrt{r^2-1}}{\sqrt{c^{-2}r^2-1}+\aleph\sqrt{r^2-1}}\right|. \end{aligned}$$

Now, by construction for any r with $\max(1,c) < r < \infty$ such that $\sqrt{c^{-2}r^2-1}-\aleph\sqrt{r^2-1} \neq 0$ and δ small enough, we have $|r - r_m| < \delta$ so, taking m large enough,

$$|c^{-1}\operatorname{Im}\lambda - l_N^{-1}r_N(x,\xi')| \le C\delta.$$

This shows that Theorem 1.2 is sharp. Moreover, when $c < 1$, [Popov and Vodev 1999b] shows that there are sequences of resonances converging to the real axis that have $n \approx c^{-1}\operatorname{Re}\lambda$.

Remark 13.2. Notice also that

$$\frac{mq}{c^{-1}\operatorname{Re}\lambda} = cr_m^{-1} = |\xi'|_g.$$

Thus, since (94) with parameter n corresponds to a resonant state with $u|_{\partial\Omega} = Ae^{in\theta}$, the semiclassical tangent frequency of the resonance state is $cn/\operatorname{Re}\lambda$ when we take $\operatorname{Re} z \sim c$. Plugging this into $cl_N^{-1}r_N(x,\xi')$ gives the decay rate of the resonance state. See also Figures 3 and 10 for numerically computed resonances in this case.

Appendix: List of notation

For the convenience of the reader, we include a list of some of the notation used in this paper.

- Ω: strictly convex domain with smooth boundary — Section 1A.
- $l(q_1, q_2)$: chord length — (20).
- $l_N(q)$: average chord length — (20).
- $|\xi'|_g$ metric induced on $T^*\partial\Omega$ — Section 1A.
- $\beta : B^*\partial\Omega \to B^*\partial\Omega$: the billiard ball map — Section 5.
- $\Psi_\delta^m(M)$: semiclassical pseudifferential operator classes — Section 2.
- $S_\delta^m(T^*M)$: symbol classes — (33).

- $\sigma : \Psi^m_\delta(M) \to S^m_\delta(T^*M)$: the symbol map — (34).
- Ai, A_i, Φ_-, ζ_i: Airy-related functions — Section 1B, (9).
- $Q(x', \xi') \in C^\infty(T^*\partial\Omega)$: the symbol of the second fundamental form — Section 1B.
- $N_2(z/h)$: the outgoing Dirichlet-to-Neumann Map — Section 1D.
- $G(z/h)$: the single-layer operator — Section 1D.
- G_B, G_Δ: decomposition of G — Lemma 7.3.
- $\Psi^{k_1,k_2}_\delta(M;\Sigma)$, $S^{k_1,k_2}_\delta(M;\Sigma)$: second microlocal operators and symbols — Section 4.
- R: the reflection operator — (18).
- T: the transition operator — (19).
- $\mathrm{Op_h}$: quantization operator — Section 2.
- r_N: the average reflectivity — (21).
- $\tilde{\sigma}$: the compressed shymbol — Section 3.
- $I_A(q)$: the order of A at q — Section 3.
- H^m_h : semiclassical Sobolev spaces — (24).
- $\mathcal{S}\ell$, $\mathcal{D}\ell$, respectively the single and double-layer operators — (31).
- $\mathcal{O}(\cdot)$ and $o(\cdot)$ — (32).
- $\mathrm{WF_h}$ the semiclassical wavefront set — Definition 2.7.
- $\Psi_{\mathcal{S}\ell}$, $\Psi_{\mathcal{D}\ell\mathcal{S}\ell}$, $\Psi_{\mathcal{D}\ell}$ symbols of layer potentials — (52).

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