Definition and analysis of the allocation algorithm

Problem formulation: Let *n* be the number of units and $\mathbf{y} = \{y_i\}$, $i = 1,2, \dots, n$ the given numbers of beds in the respective units, where $Y = \sum_{i=1}^{n} y_i$. The different y_i are not necessarily integers, but *Y* is. We want to find an *n*-vector of positive integers $\mathbf{x} = \{x_i\}$, interpreted as the number of beds allocated to unit *i*, such that $\sum_{i=1}^{n} x_i = Y$ and **x** is as close as possible to **y**, where the distance is defined as the sum of the squared differences: $C(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} (x_i - y_i)^2$.

Solution: For this, we implement a so-called "greedy algorithm", which first allocates one bed to each unit, and thereafter allocates the remaining beds one by one, minimizing the resulting cost $C(\boldsymbol{x}, \boldsymbol{y})$ at each step. This algorithm gives an optimal solution.

Proof: Let **x** be the calculated allocation, and assume that that it is not optimal. This means that there exist indices *i* and *j*, such that $C(\mathbf{x}, \mathbf{y})$ can be reduced by decreasing x_i by 1 and increasing x_j by 1 (i.e. moving a bed from unit *i* to *j*). This gives $(x_i - 1 - y_i)^2 - (x_i - y_i)^2 < \left(x_j - y_j\right)^2 - \left(x_j + 1 - y_j\right)^2$, where the left hand side is the change in *C*-value from decreasing x_i and the right hand side is the change from increasing x_j. The inequality can be simplified to $x_j - y_j < (x_i - 1) - y_i$. Consider the step in the algorithm immediately after the last bed is assigned to unit *i* and let **z** be the state of the **x** vector at that point. Then $z_i = x_i$. Since the algorithm only increases the **x**-values, we have $z_j \leq x_j$. Therefore, the equation $z_j - y_j < (z_i - 1) - y_i$ is satisfied, which gives a contradiction since this inequality implies that the *C*-value would have been lower if the latest bed were allocated to unit *j*. QED.

Corrollary: From the proof we see that it is unnecessary to compute the full $C(\mathbf{x}, \mathbf{y})$ for each unit at each step, as the optimization step is equivalent to finding the index *i* that minimizes $x_i - y_i$. This is reasonable, since this is the unit where the current bed allocation is furthest away from the target (given number of beds) .