arXiv:1903.11281v2 [math.AP] 13 Sep 2019

On the maximal L_p-L_q regularity of solutions to a general linear parabolic system

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Abstract

We show the existence of solution in the maximal $L_p - L_q$ regularity framework to a class of symmetric parabolic problems on a uniformly C^2 domain in $\mathbb{R}^n$. Our approach consist in showing $\mathcal{R}$ - boundedness of families of solution operators to corresponding resolvent problems first in the whole space, then in half-space, perturbed half-space and finally, using localization arguments, on the domain. Assuming additionally boudedness of the domain we also show exponential decay of the solution. In particular, our approach does not require assuming a priori the uniform Lopatinskii - Shapiro condition.

MSC Classification: 35K40, 35K51
Keywords: *linear parabolic system, maximal regularity, $\mathcal{R}$-boundedness*

1 Introduction

In this paper we consider the following initial-boundary value problem:

$$\begin{cases} \displaystyle\sum_{\ell=1}^{n} R_{k\ell}(x)\partial_t u_\ell(x,t) - \mathrm{div}\left(\sum_{\ell=1}^{n} B_{k\ell}(x)\nabla u_\ell(x,t)\right) = F_k(x,t) & \text{in } \Omega\times(0,T), \\ \displaystyle\sum_{\ell=1}^{n} B_{k\ell}(x)\nabla u_\ell(x,t)\cdot\mathbf{n}(x) = G_k(x,t) & \text{on } \Gamma\times(0,T), \\ u_k|_{t=0}(x) = u_{0k}(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where n is an arbitrary large natural number, $k\in\{1,\dots,n\}$, Ω a uniformly C^2 domain in $\mathbb{R}^N$ $(N\geq 2)$, Γ is the boundary of Ω, $\mathbf{n}$ is the unit outer normal vector to Γ, $x=(x_1,\dots,x_N)$ is a point of Ω, and $t\in(0,T)$ is a time variable.

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The n-vector of unknown functions is denoted by $\mathbf{u} = (u_1, \dots, u_n)^\top$ where $(\cdot)^\top$ denotes the transposed $(\cdot)$. Similarly, $\mathbf{F} = (F_1, \dots, F_n)^\top$, $\mathbf{G} = (G_1, \dots, G_n)^\top$, and $\mathbf{u}_0 = (u_{01}, \dots, u_{0n})^\top$ denote given n-vectors of functions prescribing the right hand side of the equations, the boundary and the initial conditions, respectively.

The $n \times n$ matrices $B = [B_{k\ell}(x)]$ and $R = [R_{k\ell}(x)]$ are given and we assume that all their components $B_{k\ell}(x)$ and $R_{k\ell}(x)$ are uniformly Hölder continuous functions of order $\sigma > 0$ and that $\nabla B_{k\ell}$ and $\nabla R_{k\ell}$ are integrable with some exponent $r \in (N, \infty)$, i.e. we have

$$\begin{aligned} &|B(x)|, |R(x)| \leq M_0 \quad \text{for any } x \in \Omega, \quad \|\nabla(B, R)\|_{L_r(\Omega)} \leq M_0, \\ &|B(x) - B(y)| \leq M_0|x - y|^\sigma, \quad |R(x) - R(y)| \leq M_0|x - y|^\sigma \quad \text{for any } x, y \in \Omega. \end{aligned} \tag{1.2}$$

for some positive constant M_0.

Moreover, we assume that the matrices B and R are positive and symmetric, and that there exists constant $m_1 > 0$ for which

$$\langle B(x)\mathbf{v}, \overline{\mathbf{v}} \rangle \geq m_1|\mathbf{v}|^2, \quad \langle R(x)\mathbf{v}, \overline{\mathbf{v}} \rangle \geq m_1|\mathbf{v}|^2 \tag{1.3}$$

for any complex n-vector $\mathbf{v}$ and any $x \in \Omega$. Here and in the following, $\overline{\mathbf{v}}$ denotes the complex conjugate of $\mathbf{v}$ and $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^n$.

In the rest of this paper we will rather use the following more compact matrix formulation of the system (1.1):

$$R\partial_t\mathbf{u} - \operatorname{div}(B\nabla\mathbf{u}) = \mathbf{F} \quad \text{in } \Omega \times (0, T), \quad B(\nabla\mathbf{u} \cdot \mathbf{n}) = \mathbf{G} \quad \text{on } \Gamma \times (0, T), \tag{1.4}$$

subject to the initial condition: $\mathbf{u}|_{t=0} = \mathbf{u}_0$ in Ω, where we follow the convention:

$$\nabla\mathbf{u} = [\partial_1\mathbf{u}, \dots, \partial_n\mathbf{u}], \quad \nabla\mathbf{u} \cdot \mathbf{n} = \sum_{j=1}^N n^j \partial_j \mathbf{u},$$

and divergence of a $n \times n$ matrix A is understood as a vector

$$\operatorname{div} A = [\operatorname{div}[A]_{1,\cdot}, \dots, \operatorname{div}[A]_{n,\cdot}]^\top,$$

where $[A]_{k,\cdot}$ is the k-th row of A, $\mathbf{n} = (n_1, \dots, n_N)^\top$, $\nabla u_\ell = (\partial_1 u_\ell, \dots, \partial_N u_\ell)^\top$, $\partial_i = \partial/\partial x_i$.

The issue of maximal regularity for linear parabolic problems is nowadays well investigated area. The development of the theory dates back to papers of Lopatinskii [24] and Shapiro [32] from the early fifties, where certain algebraic condition was introduced that guarantees the well posedness for a class of parabolic problems. This condition, referred to as Lopatinskii-Shapiro condition (LS), corresponds to uniform, with respect to the parameter, solvability of the family of elliptic problems on a half space. The LS condition has been ever since assumed in many well-posedness results for parabolic problems as it provides resolvent estimates allowing to show maximal regularity for corresponding parabolic problems. The earliest results concerning the resolvent estimates for elliptic operators satisfying this condition have been shown by Agmon [2], and by Arganovich and Vishik [4].

As far as the Cauchy problems are concerned, the maximal regularity in $L_p(X)$, where X is a Banach space with the Unconditional Martingale Difference property (UMD property) has been shown by Da Prato and Gisvard [10], Dore and Venni [13], and Prüss [31] and Giga and Sohr [17], among others. For a summary of these results we refer the reader to the monograph of Amman [5, Theorem 4.10.7]. One should also mention a different approach based on potential theory applied by Ladyzhenskaya, Solonnikov and Uraltseva in [23] to prove the maximal regularity in $L_p((0,T), L_p(G))$ for G bounded and $1 < p < \infty$.

The concept of $\mathcal{R}$-sectorial operators and operator-valued Fourier multipliers, essential from the point of view of the present paper, originates from the work of Weis [38]. In this paper a characterization of the class of operators with maximal regularity was given in terms of $\mathcal{R}$-boundedness of family of associated resolvent operators. This approach has been applied for the first time to show maximal L_p regularity for the Cauchy problem by Kalton and Weis in [22]. Further results in this spirit have been shown by Denk, Hieber and Prüss in [11]. In particular, Theorem 8.2 from this work concerns the maximal L_p-regularity for a class of parabolic initial-boundary problems. We also recommend it as a collection of auxiliary results and for extensive list of references on the subject.

The above overview is obviously far from complete, but it should be emphasized that all above mentioned results assume a certain version of LS condition. However, for some problems this condition could be rather difficult to check. A classical way around this obstacle consist in applying energy estimates to show the existence of weak solutions and regularizing it using a priori estimates in the maximal regularity setting, see for example [25], [26]. Another way to solve the problem directly, without assuming the LS condition, consists in solving the problem first on the whole space, then on a half-space, further its perturbation and finally, with a standard localization procedure, on a domain. This idea has been used, for example, in the work of Enomoto and Shibata [14], where the maximal $L_p - L_q$ regularity of solutions was proven first for the Stokes operator and then for the compressible Navier-Stokes equations. This has been then extended in [15] to the case of some free boundary problem. Our strategy relies very much on the technique developed in these two papers. Let us also mention that a similar idea in critical regularity Besov space framework has been developed in [8], [9], [7].

All of above mentioned results deal with a single equation or a system of two-three equations. The main contribution of our paper is that it provides the maximal $L_p - L_q$ regularity result for arbitrary large and more general system without the LS condition.

Symmetric parabolic systems of type (1.1) arise in particular in mathematical description of multicomponent systems with complex diffusion. Equations (1.1) can be regarded as linearization of complex systems that model, for example, the motion of multicomponent mixture, transport of ions, or the evolution of densities of interacting species. Although in above described models the original problem is often non-symmetric and only positive semidefinite, it reveals entropy structure which allows to rewrite the problem in the so-called entropic variables and to reduce the problem by one equation. The resulting system is then symmetric and it is reasonable to assume or even in certain cases it is possible to show that the system is strictly parabolic. An overview of such models together with a self contained description of entropy-based approach is presented in monograph [20]. In this context the present result has been already used in a very recent work of the authors [29], where we proved the existence and maximal regularity of solutions to the Navier-Stokes type of system of $(n+1)$- component mixture. We used the main result of this paper, Theorem 6, to generate stability and maximal $L_p - L_q$ regularity result for linearization of the species subsystem. In particular, as we are interested in short time existence, linearizing around the initial conditions we obtain time-indepentent coefficients. Earlier, in [28] we also considered a simplified version of this system modelling the two component compressible mixture. In that case the linearized system was reduced to a single equation, and therefore much more straightforward to deal with. Up to our knowledge, the only other result for such type of systems, is due to Herberg, Meyries, Prüss and Wilke [18], and it is restricted to the incompressible, isothermal and isobaric multicomponent flows. Rather than eliminating one equation from the system of reaction-diffusion equations and symmetrizing it using the entropy normal form, the authors work with the whole system of $(n+1)$ equations. Its principal part is only normally elliptic on the space $\mathbb{E} = \{\mathrm{e}\}^\top$, where e is a $(n+1)$-vector of all entries equal to 1. However, it allows for verification of the LS condition at the linear level, which we do not require here.

1.1 Preliminaries

Here we recall some definitions and auxiliary results which are used in the paper.

Definition 1. We say that Ω is a uniform C^2 domain, if there exist positive constants K, L_1, and L_2 such that the following assertion holds: For any $x_0 = (x_{01}, \ldots, x_{0N}) \in \Gamma$ there exist a coordinate number j and a C^2 function $h(x')$ defined on $B'_{L_1}(x'_0)$ such that $\|h\|_{H^k_\infty(B'_{a_1}(x'_0))} \le K$ and

$$\Omega \cap B_{L_2}(x_0) = \{x \in \mathbb{R}^N \mid x_j > h(x') \ \ (x' \in B'_{L_1}(x'_0))\} \cap B_{L_2}(x_0),$$
$$\Gamma \cap B_{L_2}(x_0) = \{x \in \mathbb{R}^N \mid x_j = h(x') \ \ (x' \in B'_{L_1}(x'_0))\} \cap B_{L_2}(x_0).$$

Here, we have set

$$y' = (y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_N) \ \ (y \in \{x, x_0\}),$$
$$B'_{L_1}(x'_0) = \{x' \in \mathbb{R}^{N-1} \mid |x' - x'_0| < L_1\},$$
$$B_{L_2}(x_0) = \{x \in \mathbb{R}^N \mid |x - x_0| < L_2\}.$$

Let us also recall the definition of the Fourier transform and its inverse

$$\mathcal{F}[f](\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} f(x)\, dx, \quad \mathcal{F}^{-1}_\xi[g](x) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{i\xi\cdot x} g(\xi)\, d\xi. \tag{1.5}$$

Analogously we introduce the partial Fourier transform $\mathcal{F}_{x'}$ and its inverse transform $\mathcal{F}^{-1}_{\xi'}$ by setting

$$\begin{aligned} \mathcal{F}_{x'}[f](\xi', x_N) &= \int_{\mathbb{R}^{N-1}} e^{-ix'\cdot\xi'} f(x', x_N)\, dx', \\ \mathcal{F}^{-1}_{\xi'}[g](x) &= \frac{1}{(2\pi)^{N-1}} \int_{\mathbb{R}^{N-1}} e^{i\xi'\cdot x'} g(\xi', x_N)\, d\xi', \end{aligned} \tag{1.6}$$

where $x' = (x_1, \ldots, x_{N-1})$ and $\xi' = (\xi_1, \ldots, \xi_{N-1})$. Next, we recall the definition of $\mathcal{R}$ boundedness of a family of operators

Definition 2. Let X and Y be two Banach spaces. A family of operators $\mathcal{T} \subset \mathcal{L}(X, Y)$ is called $\mathcal{R}$ bounded on $\mathcal{L}(X, Y)$, if there exist constants $C > 0$ and $p \in [1, \infty)$ such that for each $m \in \mathbb{N}$, $\{f_j\}_{j=1}^m \subset X^m$, and $\{T_j\}_{j=1}^m \subset \mathcal{T}^m$, we have

$$\|\sum_{k=1}^m r_k T_k f_k\|_{L_p((0,1),Y)} \le C \|\sum_{k=1}^m r_k f_k\|_{L_p((0,1),X)}.$$

Here, $\mathcal{L}(X, Y)$ denotes the set of all bounded linear functions from X into Y and the Rademacher functions r_k, $k \in \mathbb{N}$, are given by $r_k : [0, 1] \to \{-1, 1\}$; $t \mapsto \operatorname{sign}(\sin 2^k \pi t)$. The smallest such C is called $\mathcal{R}$ bound of $\mathcal{T}$ on $\mathcal{L}(X, Y)$, which is denoted by $\mathcal{R}_{\mathcal{L}(X,Y)}\mathcal{T}$.

Finally we recall

Definition 3. For any Banach space X, $H^{1/2}_p(\mathbb{R}, X)$ denotes the set of all X valued Bessel potential functions, f, satisfying

$$\|f\|_{H^{1/2}_p(\mathbb{R},X)} = \Big(\int_{\mathbb{R}} \|\mathcal{F}^{-1}[(1+\tau^2)^{1/4}\mathcal{F}[f](\tau)]\|^p \,\mathrm{d}\tau\Big)^{1/p} < \infty, \tag{1.7}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and the inverse Fourier transform, respectively.

To end this subsection, we introduce some fundamental properties of $\mathcal{R}$-bounded operators and Bourgain's results concerning Fourier multiplier theorems with scalar multiplier. (see, e.g., [11, Remarks 3.2 and Proposition 3.4] and [6]).

Proposition 4. *a) Let X and Y be Banach spaces, and let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families in $\mathcal{L}(X, Y)$. Then, $\mathcal{T} + \mathcal{S} = \{T + S \mid T \in \mathcal{T},\ S \in \mathcal{S}\}$ is also an $\mathcal{R}$-bounded family in $\mathcal{L}(X, Y)$ and*

$$\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T} + \mathcal{S}) \le \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T}) + \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{S}).$$

b) Let X, Y and Z be Banach spaces, and let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families in $\mathcal{L}(X, Y)$ and $\mathcal{L}(Y, Z)$, respectively. Then, $\mathcal{S}\mathcal{T} = \{ST \mid T \in \mathcal{T},\ S \in \mathcal{S}\}$ also an $\mathcal{R}$-bounded family in $\mathcal{L}(X, Z)$ and

$$\mathcal{R}_{\mathcal{L}(X,Z)}(\mathcal{S}\mathcal{T}) \le \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})\mathcal{R}_{\mathcal{L}(Y,Z)}(\mathcal{S}).$$

c) Let $1 < p,\, q < \infty$ and let D be a domain in $\mathbb{R}^N$. Let $m = m(\lambda)$ be a bounded function defined on a subset Λ in $\mathbb{C}$ and let $M_m(\lambda)$ be a map defined by $M_m(\lambda)f = m(\lambda)f$ for any $f \in L_q(D)$. Then, $\mathcal{R}_{\mathcal{L}(L_q(D))}(\{M_m(\lambda) \mid \lambda \in \Lambda\}) \le C_{N,q,D}\|m\|_{L_\infty(\Lambda)}$.

d) Let $n = n(\tau)$ be a C^1-function defined on $\mathbb{R} \setminus \{0\}$ that satisfies the conditions $|n(\tau)| \le \gamma$ and $|\tau n'(\tau)| \le \gamma$ with some constant $\gamma > 0$ for any $\tau \in \mathbb{R} \setminus \{0\}$. Let T_n be the operator-valued Fourier multiplier defined by $T_n f = \mathcal{F}^{-1}(n\mathcal{F}[f])$ for any f with $\mathcal{F}[f] \in \mathcal{D}(\mathbb{R}, L_q(D))$. Then, T_n can be extended to a bounded linear operator from $L_p(\mathbb{R}, L_q(D))$ into itself. Moreover, denoting this extension also by T_n, we have

$$\|T_n\|_{\mathcal{L}(L_p(\mathbb{R},L_q(D)))} \le C_{p,q,D}\gamma.$$

Here, $\mathcal{D}(\mathbb{R}, L_q(D))$ denotes the set of all $L_q(D)$-valued C^∞-functions on $\mathbb{R}$ with compact support.

We finish this section with showing

Lemma 5. *Let $N < q \le r \le \infty$. Then*

$$\|\nabla(fg)\|_{L_q(D)} \le C_D\{(\|g\|_{L_\infty(D)}\|\nabla f\|_{L_q(D)} + \|\nabla g\|_{L_r(D)}(\alpha\|\nabla f\|_{L_q(D)} + C_\alpha\|f\|_{L_q(D)})\} \tag{1.8}$$

for any $\alpha \in (0, 1)$ with some constant C_α depending on α, where D is any domain in $\mathbb{R}^N$ with uniform C^2 boundary.

Proof. When $r = q$, we have

$$\|\nabla(fg)\|_{L_q(D)} \le \|\nabla f\|_{L_q(D)}\|g\|_{L_\infty(D)} + \|f\|_{L_\infty(D)}\|\nabla g\|_{L_q(D)}.$$

Since $N < q = r < \infty$, by Sobolev's imbedding theorem, we have

$$\|\nabla(fg))|_{L_q(D)} \le C_D\{\|g\|_{L_\infty(D)}\|\nabla f\|_{L_q(D)} + C_{q,\tau}\|\nabla g\|_{L_r(D)}\|f\|_{W^{N/q+\tau}_q(D)}\} \tag{1.9}$$

with some small number $\tau > 0$ for which $N/q + \tau < 1$, where $C_{q,\tau}$ is a constant depending on q and τ essentially. When $1 < q < r$, let s be a number for which $1/q = 1/r + 1/s$, and then by Hölder's inequality, we have

$$\|\nabla(fg)\|_{L_q(D)} \leq C_D\{\|g\|_{L_\infty(D)}\|\nabla f\|_{L_q(D)} + \|\nabla g\|_{L_r(D)}\|f\|_{L_s(D)}\}.$$

Since $N(1/q - 1/s) = N/r < 1$, by Sobolev's imbedding theorem, we have (1.9).

Finally, by real interpolation theory,

$$\|f\|_{W_q^{N/q+\tau}(D)} \leq C\|f\|_{L_q(D)}^{1-(N/q+\tau)}\|f\|_{H_q^1(D)}^{(N/q+\tau)},$$

and therefore we have (1.8). □

1.2 Main results

In this paper, we shall prove the maximal L_p-L_q regularity theorem for Eq. (1.1):

Theorem 6. *Let $1 < p, q < \infty$ and $T > 0$. Assume that $2/p + 1/q \neq 1$ and that Ω is a uniformly C^2 domain in $\mathbb{R}^N$ $(N \geq 2)$.*

Existence. *Let $\mathbf{u}_0 = (u_{01}, \ldots, u_{0n})^\top \in B_{q,p}^{2(1-1/p)}(\Omega)^n$, $\mathbf{F} \in L_p((0,T), L_q(\Omega)^n)$ and $\mathbf{G} \in L_p(\mathbb{R}, H_q^1(\Omega)^n) \cap H_p^{1/2}(\mathbb{R}, L_q(\Omega)^n)$ be given functions satisfying the compatibility conditions:*

$$B(\nabla \mathbf{u}_0 \cdot \mathbf{n}) = \mathbf{G}(\cdot, 0) \quad \text{on } \Gamma \tag{1.10}$$

provided $2/p + 1/q < 1$. Then, problem (1.1) admits a unique solution $\mathbf{u} = (u_1, \ldots, u_n)^\top$ with

$$\mathbf{u} \in L_p((0,T), H_q^2(\Omega)^n) \cap H_p^1((0,T), L_q(\Omega)^n) \tag{1.11}$$

possessing the estimate:

$$\begin{aligned} &\|\mathbf{u}\|_{L_p((0,T),H_q^2(\Omega))} + \|\partial_t \mathbf{u}\|_{L_p((0,T),L_q(\Omega))} \leq Ce^{\gamma T}(\|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \\ &\quad + \|\mathbf{F}\|_{L_p((0,T),L_q(\Omega))} + \|e^{-\gamma t}\mathbf{G}\|_{L_p(\mathbb{R},H_q^1(\Omega))} + (1+\gamma^{1/2})\|e^{-\gamma t}\mathbf{G}\|_{H_p^{1/2}(\mathbb{R},L_q(\Omega))}) \end{aligned} \tag{1.12}$$

for any $\gamma \geq \gamma_0 > 0$ with some constants C and γ_0, where C is independent of γ.

Uniqueness. *Let $\mathbf{u}$ be a n-vector of functions satisfying the regularity condition (1.11) and the homogeneous equations:*

$$R\partial_t \mathbf{u} - \operatorname{div}(B\nabla \mathbf{u}) = 0 \quad \text{in } \Omega \times (0,T), \quad B(\nabla \mathbf{u} \cdot \mathbf{n})|_\Gamma = 0, \quad \mathbf{u}|_{t=0} = 0, \tag{1.13}$$

then $\mathbf{u} = 0$.

To prove Theorem 6, our approach is to use the $\mathcal{R}$ bounded solution operator for the corresponding generalized resolvent problem and Weis's operator valued Fourier multiplier theorem [38]. Below we state the existence theorem of such operators.

We consider the generalized resolvent problem corresponding to Eq. (1.4):

$$\lambda R\mathbf{v} - \operatorname{div}(B\nabla \mathbf{v}) = \mathbf{f} \quad \text{in } \Omega, \quad B(\nabla \mathbf{v} \cdot \mathbf{n}) = \mathbf{g} \quad \text{on } \Gamma, \tag{1.14}$$

where $\mathbf{v} = (v_1, \ldots, v_n)^\top$, $\mathbf{f} = (f_1, \ldots, f_n)^\top$ and $\mathbf{g} = (g_1, \ldots, g_n)^\top$. We shall prove the following theorem.

Theorem 7. *Let $1 < q < \infty$ and $0 < \epsilon < \pi/2$. Assume that Ω is a unformly C^2 domain in $\mathbb{R}^N$.*
Existence. *Let*

$$
\begin{aligned}
X_q(\Omega) &= \{(\mathbf{f}, \mathbf{g}) \mid \mathbf{f} = (f_1, \ldots, f_n) \in L_q(\Omega)^n, \quad \mathbf{g} = (g_1, \ldots, g_n)^\top \in H_q^1(\Omega)^n\}, \\
\mathcal{X}_q(\Omega) &= \{(F_1, F_2, F_3) \mid F_1, F_2 \in L_q(\Omega)^n, \quad F_3 \in H_q^1(\Omega)^n\},
\end{aligned}
\tag{1.15}
$$

with the norms

$$
\begin{aligned}
\|(\mathbf{f}, \mathbf{g})\|_{X_q(\Omega)} &= \|\mathbf{f}\|_{L_q(\Omega)} + \|\mathbf{g}\|_{H_q^1(\Omega)}, \\
\|(F_1, F_2, F_3)\|_{\mathcal{X}_q(\Omega)} &= \|(F_1, F_2)\|_{L_q(\Omega)} + \|F_3\|_{H_q^1(\Omega)},
\end{aligned}
\tag{1.16}
$$

and

$$
\Sigma_\epsilon = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| \le \pi - \epsilon\}, \quad \Sigma_{\epsilon,\lambda_0} = \{\lambda \in \Sigma_\epsilon \mid |\lambda| \ge \lambda_0\}. \tag{1.17}
$$

Then, there exist a constant $\lambda_0 > 0$ and an operator family $\mathcal{S}(\lambda) \in \mathrm{Hol}(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}(\Omega), H_q^2(\Omega)^n))$ (holomorphic on $\Sigma_{\epsilon,\lambda_0}$) such that for any $(\mathbf{f}, \mathbf{g}) \in X_q(\Omega)$ and $\lambda \in \Sigma_{\epsilon,\lambda_0}$, $\mathbf{v} = (v_1, \ldots, v_n)^\top = \mathcal{S}(\lambda) H_\lambda(\mathbf{f}, \mathbf{g})$ with $H_\lambda(\mathbf{f}, \mathbf{g}) = (\mathbf{f}, \lambda^{1/2}\mathbf{g}, \mathbf{g})$ is a solution of Eq. (1.14).

Moreover, we have

$$
\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), H_q^{2-k}(\Omega)^n)}(\{(\tau\partial_\tau)^\ell(\lambda^{k/2}\mathcal{S}(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le r_b \tag{1.18}
$$

for $k = 0, 1, 2$ and $\ell = 0, 1$ with some constant r_b, where $\lambda = \gamma + i\tau \in \mathbb{C}$.
Uniqueness. *Let $\mathbf{v} \in H_q^2(\Omega)^n$ satisfy the homogeneous equations:*

$$
\lambda R\mathbf{v} - \mathrm{div}\,(B\nabla\mathbf{v}) = 0 \quad \text{in } \Omega, \quad B(\nabla\mathbf{v} \cdot \mathbf{n})|_\Gamma = 0,
$$

then $\mathbf{v} = 0$.

Remark 8. The constant γ_0 from Theorem 6 can be chosen the same as the constant λ_0 from Theorem 7.

The second main result of our paper extends Theorem 6 giving a time-independent estimate provided boundary of the domain and zero mean assumptions on the data.

Theorem 9. *Let $1 < p, q < \infty$ and $T = \infty$ in Theorem 6. Assume that $2/p + 1/q \ne 1$ and that Ω is a bounded domain, whose boundary, Γ, is a compact C^2 hypersurface. Then, there exists a $\gamma_0 > 0$ for which the following assertion holds: Let $\mathbf{u}_0$, $\mathbf{F}$ and $\mathbf{G}$ be functions given in Theorem 6. Moreover, we assume that*

$$
\int_\Omega \mathbf{F}(x,t)\,dx + \int_\Gamma \mathbf{G}(x,t)\,d\sigma = 0 \quad \text{for any } t > 0 \quad \text{and} \quad \int_\Omega R\mathbf{u}_0\,dx = 0, \tag{1.19}
$$

$$
\|e^{\gamma t}\mathbf{F}\|_{L_p((0,\infty),L_q(\Omega))} + \|e^{\gamma t}\mathbf{G}\|_{L_p(\mathbb{R},H_q^1(\Omega)^n)} + (1+\gamma^{1/2})\|e^{\gamma t}\mathbf{G}\|_{H_p^{1/2}(\mathbb{R},L_q(\Omega))} < \infty \tag{1.20}
$$

for any $\gamma \le \gamma_0$, where $d\sigma$ is the surface element of Γ. Then, the solution $\mathbf{u}$ obtained in Theorem 6 decays exponentially, that is $\mathbf{u}$ satisfies the estimate:

$$
\begin{aligned}
&\|e^{\gamma t}\mathbf{u}\|_{L_p((0,\infty),H_q^2(\Omega))} + \|e^{\gamma t}\partial_t\mathbf{u}\|_{L_p((0,\infty),L_q(\Omega))} \\
&\quad \le C(\|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} + \|e^{\gamma t}\mathbf{F}\|_{L_p((0,\infty),L_q(\Omega))} + \|e^{\gamma t}\mathbf{G}\|_{L_p(\mathbb{R},H_q^1(\Omega))} + \|e^{\gamma t}\mathbf{G}\|_{H_p^{1/2}(\mathbb{R},L_q(\Omega))})
\end{aligned}
$$

for any $\gamma \le \gamma_0$ with some constant C.

Theorem 6 can be proved by applying Weis' theorem [38] to the representation formula of solutions to (1.1) given by Theorem 7. Thus, this paper is devoted to the proof of Theorem 7 mainly. In Section 2 we solve the problem in the whole space. Section 3 is dedicated to problem in a halfspace. This is the most technical part of the proof because of complexity of the solution formula. In Section 4 we consider a result in a perturbed halfspace and finally, in Section 5, we use the properties of a uniform C^2 domains to prove Theorem 7. The two concluding sections are then dedicated to the proofs of Theorem 6 in Section 6, and Theorem 9 in Section 7.

2 Analysis in the whole space

2.1 Constant coefficients case

Let x_0 be any point of Ω and set $B^0 = B(x_0)$ and $R^0 = R(x_0)$. In this subsection, we consider the constant coefficients system

$$\lambda R^0 \mathbf{v} - B^0 \Delta \mathbf{v} = \mathbf{f} \quad \text{in } \mathbb{R}^N. \tag{2.1}$$

By assumptions (1.2) and (1.3), R^0 and B^0 are symmetric matrices and satisfy the following conditions:

$$|R^0|, |B^0| \leq M_0, \quad \langle R^0\mathbf{a}, \overline{\mathbf{a}}\rangle \geq m_1|\mathbf{a}|^2, \quad \langle B^0\mathbf{a}, \overline{\mathbf{a}}\rangle \geq m_1|\mathbf{a}|^2 \tag{2.2}$$

for any $\mathbf{a} \in \mathbb{C}^n$. Applying the Fourier transform to Eq. (2.1) gives

$$(R^0\lambda + B^0|\xi^2|)\mathcal{F}[\mathbf{v}] = \mathcal{F}[\mathbf{f}] \quad \text{in } \mathbb{R}^N. \tag{2.3}$$

Lemma 10. *Let $0 < \epsilon < \pi/2$. The matrix $R^0\lambda + B^0|\xi^2|$ is invertible at least for $(\lambda, \xi) \in \Sigma_\epsilon \times (\mathbb{R}^N \setminus \{0\})$ and there exists a constant $m_2 > 0$ depending on M_0, m_1 and ϵ, but independent of $x_0 \in \Omega$, for which*

$$|(R^0\lambda + B^0|\xi|^2)^{-1}| \leq m_2(|\lambda| + |\xi^2|)^{-1} \tag{2.4}$$

for any $(\lambda, \xi) \in \Sigma_\epsilon \times (\mathbb{R}^N \setminus \{0\})$.

Proof. Let $(\lambda, \xi) \in \Sigma_\epsilon \times (\mathbb{R}^N \setminus \{0\})$. We take $\lambda = |\lambda|(\cos\theta + i\sin\theta)$ and we compute

$$\begin{aligned}|\lambda\langle R^0\mathbf{a}, \overline{\mathbf{a}}\rangle + |\xi|^2\langle B^0\mathbf{a}, \overline{\mathbf{a}}\rangle|^2 &= (\langle R^0\mathbf{a}, \overline{\mathbf{a}}\rangle|\lambda|\cos\theta + |\xi|^2\langle B^0\mathbf{a}, \overline{\mathbf{a}}\rangle)^2 + (\langle R^0\mathbf{a}, \overline{\mathbf{a}}\rangle|\lambda|\sin\theta)^2 \\ &|\langle R^0\mathbf{a}, \overline{\mathbf{a}}\rangle|^2|\lambda|^2 + 2|\lambda||\xi|^2\langle R^0\mathbf{a}, \overline{\mathbf{a}}\rangle\langle B^0\mathbf{a}, \overline{\mathbf{a}}\rangle\cos\theta + |\xi|^4|\langle B^0\mathbf{a}, \overline{\mathbf{a}}\rangle|^2.\end{aligned} \tag{2.5}$$

Because $|\theta| \leq \pi - \epsilon$ thus $\cos\theta \geq \cos(\pi - \epsilon) > -1$ and so

$$\begin{aligned}&|\lambda\langle R^0\mathbf{a}, \overline{\mathbf{a}}\rangle + |\xi|^2\langle B^0\mathbf{a}, \overline{\mathbf{a}}\rangle|^2 \\ &\quad \geq |\langle R^0\mathbf{a}, \overline{\mathbf{a}}\rangle|^2|\lambda|^2 - 2|\lambda||\xi|^2\langle R^0\mathbf{a}, \overline{\mathbf{a}}\rangle\langle B^0\mathbf{a}, \overline{\mathbf{a}}\rangle|\cos(\pi - \epsilon)| + |\xi|^4|\langle B^0\mathbf{a}, \overline{\mathbf{a}}\rangle|^2 \\ &\quad = |\cos(\pi - \epsilon)|(|\lambda|\langle R^0\mathbf{a}, \overline{\mathbf{a}}\rangle - |\xi|^2\langle B^0\mathbf{a}, \overline{\mathbf{a}}\rangle)^2 \\ &\qquad + (1 - |\cos(\pi - \epsilon)|)[(|\lambda|\langle R^0\mathbf{a}, \overline{\mathbf{a}}\rangle)^2 + (|\xi|^2\langle B^0\mathbf{a}, \overline{\mathbf{a}}\rangle)^2] \\ &\quad \geq (1 - |\cos(\pi - \epsilon)|)m_1^2|\mathbf{a}|^4(|\lambda|^2 + |\xi|^4).\end{aligned} \tag{2.6}$$

Note that $|\cos(\pi - \epsilon)| = |\cos\epsilon|$, and

$$1 - |\cos\epsilon| = \frac{|\sin\epsilon|^2}{1 + |\cos\epsilon|} \geq \frac{1}{2}|\sin\epsilon|^2,$$

therefore

$$|\langle (R^0\lambda + B^0|\xi|^2)\mathbf{a}, \overline{\mathbf{a}}\rangle| \geq C|\sin(\epsilon)|\sqrt{|\lambda|^2 + |\xi|^4}|\mathbf{a}|^2. \tag{2.7}$$

Thus, if $(R^0\lambda+B^0|\xi|^2)\mathbf{a} = 0$, then $\mathbf{a} = 0$, which means that the matrix $R^0\lambda+B^0|\xi|^2$ is injection, and so $\det(R^0\lambda + B^0|\xi|^2) \neq 0$. Thus,

$$(R^0\lambda + B^0|\xi|^2)^{-1} = [\det(R^0\lambda + B^0|\xi|^2)]^{-1}\mathrm{cof}(R^0\lambda + B^0|\xi|^2) \tag{2.8}$$

exists. We now prove (2.4). Let

$$\tilde{\lambda} = \frac{\lambda}{|\lambda| + |\xi|^2}, \quad \tilde{\xi}_j = \frac{\xi_j}{\sqrt{|\lambda| + |\xi|^2}},$$

and then $\det(R^0\lambda + B^0|\xi|^2) = (|\lambda| + |\xi|^2)^n \det(R^0\tilde{\lambda} + B^0|\tilde{\xi}|^2)$. Since $(\tilde{\lambda}, \tilde{\xi})$ ranges on some compact set in $\mathbb{C} \times \mathbb{R}^N$ as $|\tilde{\lambda}| + |\tilde{\xi}|^2 = 1$ for $(\lambda, \xi) \in \Sigma_\epsilon \times \mathbb{R}^N \setminus \{0\}$, there exists $\tilde{m}_2$ such that

$$|\det(R^0\tilde{\lambda} + B^0|\tilde{\xi}|^2)| \geq \tilde{m}_2.$$

This $\tilde{m}_2$ depends also on ϵ and M_0, but is independent of $x_0 \in \Omega$ due to (1.3). Thus, we have

$$|\det(R^0\lambda + B^0|\xi|^2)| \geq \tilde{m}_2(|\lambda| + |\xi|^2)^n.$$

Since the cofactor matrix of $R^0\lambda+B^0|\xi|^2$ is bounded by some constant independent of x_0 times $(|\lambda| + |\xi|^2)^{n-1}$, we have (2.4). This completes the proof of Lemma 10. □

One of the main tools in proving the existence of $\mathcal{R}$ bounded solution operators in $\mathbb{R}^N$ is the following lemma due to Denk and Schnaubelt [12, Lemma 2.1] and Enomoto and Shibata [14, Theorem 3.3].

Lemma 11. *Let $1 < q < \infty$ and let Λ be a set in $\mathbb{C}$. Let $m = m(\lambda, \xi)$ be a function defined on $\Lambda \times (\mathbb{R}^N \setminus \{0\})$ which is infinitely differentiable with respect to $\xi \in \mathbb{R}^N \setminus \{0\}$ for each $\lambda \in \Lambda$. Assume that for any multi-index $\alpha \in \mathbb{N}_0^N$ there exists a constant C_α depending on α and Λ such that*

$$|\partial_\xi^\alpha m(\lambda, \xi)| \leq C_\alpha|\xi|^{-|\alpha|} \tag{2.9}$$

for any $(\lambda, \xi) \in \Lambda \times (\mathbb{R}^N \setminus \{0\})$. Let K_λ be an operator defined by $K_\lambda f = \mathcal{F}_\xi^{-1}[m(\lambda, \xi)\mathcal{F}f(\xi)]$. Then, the family of operators $\{K_\lambda \mid \lambda \in \Lambda\}$ is $\mathcal{R}$-bounded on $\mathcal{L}(L_q(\mathbb{R}^N))$ and

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N))}(\{K_\lambda \mid \lambda \in \Lambda\}) \leq C_{q,N} \max_{|\alpha| \leq N+1} C_\alpha \tag{2.10}$$

with some constant $C_{q,N}$ depending only on q and N.

By Lemma 10, we can define a solution $\mathbf{v}$ of Eq. (2.1) by

$$\mathbf{v} = \mathcal{F}^{-1}[(R^0\lambda + B^0|\xi|^2)^{-1}\mathcal{F}[\mathbf{f}](\xi)], \tag{2.11}$$

and so for any multi-index $\alpha \in \mathbb{N}_0^N$ we have

$$\partial_\xi^\alpha \mathbf{v} = \mathcal{F}^{-1}[(i\xi)^\alpha(R^0\lambda + B^0|\xi|^2)^{-1}\mathcal{F}[\mathbf{f}](\xi)]. \tag{2.12}$$

Differentiating $(R^0\lambda + B^0|\xi|^2)^{-1}$ expressed by the formula (2.8) w.r.t. $\xi = (\xi^1, \ldots, \xi^N)$, and τ, respectively and using (2.4) we can estimate

$$\begin{aligned} |\partial_\xi^\alpha(R^0\lambda + B^0|\xi|^2)^{-1}| &\leq C_\alpha(|\lambda| + |\xi|^2)^{-1}|\xi|^{-|\alpha|}, \\ |\partial_\xi^\alpha((\tau\partial_\tau)(R^0\lambda + B^0|\xi|^2)^{-1})| &\leq C_\alpha(|\lambda| + |\xi|^2)^{-1}|\xi|^{-|\alpha|} \end{aligned} \tag{2.13}$$

for any multi-index $\alpha \in \mathbb{N}_0^N$, $\lambda = \gamma + i\tau \in \Sigma_\epsilon$ and $\xi \in \mathbb{R}^N \setminus \{0\}$. Applying Lemma 11 to the solution operator defined by (2.11) and (2.12) for $\alpha = 1, 2$, we have the following theorem, which is the main result of this subsection.

Theorem 12. *Let $1 < q < \infty$ and $0 < \epsilon < \pi/2$. Then, there exists an operator family $\mathcal{T}_0(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(L_q(\mathbb{R}^N)^n, H_q^2(\mathbb{R}^N)^n))$ such that for any $\lambda \in \Sigma_{\epsilon,\lambda_0}$ and $\mathbf{f} \in L_q(\mathbb{R}^N)^n$, $\mathbf{v} = \mathcal{T}_0(\lambda)\mathbf{f}$ is a unique solution of Eq. (2.1).*

Moreover, for any $\lambda_0 > 0$ there exists a constant r_b independent of $x_0 \in \Omega$ for which

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N)^n, H_q^{2-k}(\mathbb{R}^n))}(\{(\tau\partial_\tau)^\ell(\lambda^{k/2}\mathcal{T}_0(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le r_b \tag{2.14}$$

for $k = 0, 1, 2$ and $\ell = 0, 1$.

2.2 Perturbed problem in $\mathbb{R}^N$

In this subsection, we consider the case where the coefficients of the matrices R and B depend on x variable. Let us fix $x_0 \in \Omega$. Let M_1 be a small positive number to be determined later. Let $d_0 > 0$ be a positive number such that

$$|R(x) - R(x_0)| \le M_1, \quad |B(x) - B(x_0)| \le M_1 \tag{2.15}$$

for $x \in B_{d_0}(x_0)$, Let φ be a function in $C_0^\infty(\mathbb{R}^N)$ which equals one for $x \in B_{d_0/2}(x_0)$ and zero for $x \notin B_{2d_0/3}(x_0)$. Let

$$\tilde{R}(x) = \varphi(x)R(x) + (1 - \varphi(x))R(x_0),$$
$$\tilde{B}(x) = \varphi(x)B(x) + (1 - \varphi(x))B(x_0),$$

where $B(x)$ and $R(x)$ denote the functions extended to the whole space, we consider a perturbed problem:

$$\lambda\tilde{R}\mathbf{v} - \mathrm{div}\,(\tilde{B}\nabla\mathbf{v}) = \mathbf{f} \quad \text{in } \mathbb{R}^N. \tag{2.16}$$

In this subsection, we shall prove the following theorem.

Theorem 13. *Assume that the coefficient matrices R and B satisfy the conditions in (1.2) with some exponent $r \in (N, \infty)$. Let $1 < q \le r$ and $0 < \epsilon < \pi/2$. Then, there exist $M_1 > 0$, $\lambda_0 > 0$ and an operator family $\mathcal{T}_1(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, L(L_q(\mathbb{R}^N)^n, H_q^2(\mathbb{R}^N)^n))$ such that for any $\lambda \in \Sigma_{\epsilon,\lambda_0}$ and $\mathbf{f} \in L_q(\mathbb{R}^N)^n$, $\mathbf{v} = \mathcal{T}_1(\lambda)\mathbf{f}$ is a unique solution of Eq. (2.16) and*

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N)^n, H_q^{2-j}(\mathbb{R}^N)^n)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}\mathcal{T}_1(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\} \le 2r_b$$

for $\ell = 0, 1$ and $j = 0, 1, 2$ with some constant r_b independent of $x_0 \in \Omega$. Here, λ_0 and r_b are the same constants as in Theorem 12.

Proof. To construct an $\mathcal{R}$-bounded solution operator for Eq. (2.16), we consider the equation:

$$\lambda R(x_0)\mathbf{v} - B(x_0)\Delta\mathbf{v} + \mathbf{R}\mathbf{v} = \mathbf{f} \quad \text{in } \mathbb{R}^N. \tag{2.17}$$

Above we have set

$$\mathbf{R}\mathbf{v} = \lambda\varphi(x)(R(x) - R(x_0))\mathbf{v} - \mathrm{div}\,(\varphi(x)(B(x) - B(x_0))\nabla\mathbf{v}).$$

Let $\mathcal{T}_0(\lambda)$ be the $\mathcal{R}$-bounded solution operator given in Theorem 12, and we set $\mathbf{v} = \mathcal{T}_0(\lambda)\mathbf{f}$ in (2.17). Then, we have

$$\lambda R(x_0)\mathcal{T}_0(\lambda)\mathbf{f} - B(x_0)\Delta\mathcal{T}_0(\lambda)\mathbf{f} + \mathbf{R}\mathcal{T}_0(\lambda)\mathbf{f} = (\mathbf{I} + \mathcal{R}(\lambda))\mathbf{f} \quad \text{in } \mathbb{R}^N, \tag{2.18}$$

where

$$\mathcal{R}(\lambda)\mathbf{f} = \lambda\varphi(x)(R(x) - R(x_0))\mathcal{T}_0(\lambda)\mathbf{f} - \mathrm{div}\,(\varphi(x)(B(x) - B(x_0))\nabla\mathcal{T}_0(\lambda)\mathbf{f}).$$

Applying (1.8) and using the conditions (1.2), we have

$$\begin{aligned}&\|\mathrm{div}\,(\varphi(\cdot)(B(\cdot) - B(x_0))\nabla\mathcal{T}_0(\lambda)\mathbf{f})\|_{L_q(\mathbb{R}^N)} \\ &\quad \le CM_0(M_1 + \alpha)\|\nabla^2\mathcal{T}_0(\lambda)\mathbf{f}\|_{L_q(\mathbb{R}^N)} + C_\alpha M_0\|\nabla\mathcal{T}_0(\lambda)\mathbf{f}\|_{L_q(\mathbb{R}^N)}.\end{aligned}$$

By (1.2), we also have

$$\|\lambda\varphi(\cdot)(R(\cdot) - R(x_0))\mathcal{T}_0(\lambda)\mathbf{f}\|_{L_q(\mathbb{R}^N)} \le CM_0M_1|\lambda|\|\mathcal{T}_0(\lambda)\mathbf{f}\|_{L_q(\mathbb{R}^N)}.$$

Using Theorem 12 and Proposition 4, we have

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N)^n)}(\{(\tau\partial_\tau)^\ell\mathcal{R}(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_1}\}) \le C\{M_0(M_1 + \alpha) + C_\alpha M_0\lambda_1^{-1/2}\}r_b.$$

for any $\lambda_1 \ge \lambda_0$. Thus, choosing M_1 and α so small that $CM_0r_bM_1 < 1/8$, $CM_0r_b\alpha < 1/8$ and choosing $\lambda_0 > 0$ so large that $CC_\alpha M_0r_b\lambda_0^{-1/2} < 1/4$, we have

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N)^n)}(\{(\tau\partial_\tau)^\ell\mathcal{R}(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le 1/2.$$

Thus, we can construct the inverse operator $(\mathbf{I} + \mathcal{R}(\lambda))^{-1} = \sum_{j=0}^\infty[-\mathcal{R}(\lambda)]^j$. Then, taking $\tilde{\mathbf{f}} = (\mathbf{I} + \mathcal{R}(\lambda))\mathbf{f}$ in (2.18) we see that

$$\mathbf{v} = \mathcal{T}_1(\lambda)\mathbf{f} = \mathcal{T}_0(\lambda)(\mathbf{I} + \mathcal{R}(\lambda))^{-1}\mathbf{f}$$

is a required $\mathcal{R}$ bounded solution operator with $\mathcal{R}$ bound:

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N)^n, H_q^{2-j}(\mathbb{R}^N)^n)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}\mathcal{T}_1(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le 2r_b$$

for $\ell = 0, 1$ and $j = 0, 1, 2$. The uniqueness of solutions follows from the existence of solutions of the dual problem. This completes the proof of Theorem 13.

□

3 Model problem in the half-space

Let x_0 be any point on Γ and set $R^1 = R(x_0)$ and $B^1 = B(x_0)$. In this section, we consider problem:

$$\lambda R^1\mathbf{v} - \mathrm{div}\,(B^1\nabla\mathbf{v}) = \mathbf{f} \quad \text{in } \mathbb{R}^N_+, \quad B^1(\nabla\mathbf{v}\cdot\mathbf{n}_0) = \mathbf{g} \quad \text{on } \mathbb{R}^N_0, \tag{3.1}$$

where

$$\mathbb{R}^N_+ = \{x = (x_1, \ldots, x_N) \mid x_N > 0\}, \;\; \mathbb{R}^N_0 = \{x = (x_1, \ldots, x_N) \mid x_N = 0\},$$

and $\mathbf{n}_0 = (0, \ldots, 0, -1)^\top$. First, we consider the case where $\mathbf{g} = 0$.

Theorem 14. *Let $1 < q < \infty$ and $0 < \epsilon < \pi/2$. Then, there exists an operator family $\mathcal{T}_2(\lambda) \in \mathrm{Hol}\,(\Sigma_\epsilon, \mathcal{L}(L_q(\mathbb{R}^N)^n, H^2_q(\mathbb{R}^N)^n))$ such that for any $\lambda \in \Sigma_\epsilon$ and $\mathbf{f} \in L_q(\mathbb{R}^N_+)^n$, $\mathbf{v} = \mathcal{T}_2(\lambda)\mathbf{f}$ is a unique solution of Eq. (3.1) with $g_k = 0$ $(k = 1, \ldots, n)$.*

Moreover, for any $\lambda_0 > 0$ there exists a constant r_b independent of $x_0 \in \Gamma$ for which

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+)^n, H^{2-k}_q(\mathbb{R}^N_+)^n)}(\{(\tau\partial_\tau)^\ell(\lambda^{k/2}\mathcal{T}_2(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \leq r_b \tag{3.2}$$

for $k = 0, 1, 2$ and $\ell = 0, 1$.

Proof. Given $\mathbf{f} = (f_1, \ldots, f_n)^\top$ in the right side of Eq. (3.1), let f^e_j be an even extension of f_j to $x_N < 0$ defined by letting

$$f^e_j(x) = \begin{cases} f(x', x_N) & \text{for } x_N > 0, \\ f(x', -x_N) & \text{for } x_N < 0, \end{cases}$$

where $x' = (x_1, \ldots, x_{N-1})$. Set $\mathbf{F}^e = (f^e_1, \ldots, f^e_n)^\top$ and we consider the whole space problem:

$$\lambda R^1\mathbf{U} - \mathrm{div}\,(B^1\nabla\mathbf{U}) = \mathbf{F}^e \quad \text{in } \mathbb{R}^N. \tag{3.3}$$

Let

$$\mathcal{T}_2(\lambda)\mathbf{F}^e(x) = \mathcal{F}^{-1}[(R^1\lambda + B^1|\xi|^2)^{-1}\hat{\mathbf{F}}^e(\xi)](x).$$

Obviously, $\mathbf{U} = \mathcal{T}_2(\lambda)\mathbf{F}^e$ satisfies Eq. (3.2), and so in particular

$$\lambda R^1\mathbf{U} - \mathrm{div}\,(B^1\nabla\mathbf{U}) = \mathbf{f} \quad \text{in } \mathbb{R}^N_+.$$

Moreover, by Theorem 12, $\mathcal{T}_2(\lambda)$ has the same $\mathcal{R}$-bound as in (2.14). Thus, our task is to prove that

$$\frac{\partial}{\partial x_N}\mathbf{U}|_{x_N=0} = 0. \tag{3.4}$$

Each term of $\mathcal{T}_2(\lambda)\mathbf{F}^e$ has a form:

$$I_{kl}(x) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{ix\cdot\xi}\frac{\lambda^{n-1-\ell}|\xi|^{2\ell}}{\det(R^1\lambda + B^1|\xi|^2)}\hat{f}^e_k(\xi)\,d\xi$$

for some $k \in \{1 \ldots n\}$, $\ell \in \{1 \ldots n-1\}$. Thus,

$$\partial_N I_{kl}|_{x_N=0} = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{ix'\cdot\xi'}\frac{\lambda^{n-1-\ell}|\xi|^{2\ell}i\xi_N\hat{f}^e_k(\xi)}{\det(R^1\lambda + B^1|\xi|^2)}\,d\xi.$$

Applying the Fourier transform with respect to x', we have

$$\begin{aligned}
\mathcal{F}^{-1}_{x'}(\partial_N I_{kl}|_{x_N=0})(\xi') &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{F}^{-1}_{x'}\left[\int_{\mathbb{R}^{N-1}} e^{ix'\cdot\xi'}\frac{\lambda^{n-1-\ell}|\xi|^{2\ell}i\xi_N\hat{f}^e_k(\xi)}{\det(R^1\lambda + B^1|\xi|^2)}\,d\xi'\right] \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\lambda^{n-1-\ell}|\xi|^{2\ell}i\xi_N\hat{f}^e(\xi)}{\det(R^1\lambda + B^2|\xi|^2)}\,d\xi_N \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\lambda^{n-1-\ell}|\xi|^{2\ell}i\xi_N}{\det(R^1\lambda + B^2|\xi|^2)}\int_0^\infty (e^{-iy_N\xi_N} + e^{iy_N\xi_N})\hat{f}(\xi', y_N)\,dy_N \\
&= \lambda^{n-1-\ell}\int_0^\infty \hat{f}(\xi', y_N)\,dy_N\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|\xi|^{2\ell}i\xi_N(e^{-iy_N\xi_N} + e^{iy_N\xi_N})}{\det(R^1\lambda + B^1|\xi|^2)}\,d\xi_N.
\end{aligned}$$

Thus, in order to show (3.4) it is enough to prove that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|\xi|^{2\ell}i\xi_N(e^{-iy_N\xi_N}+e^{iy_N\xi_N})}{\det(R^1\lambda+B^1|\xi|^2)}\,d\xi_N=0. \tag{3.5}$$

We can write

$$\det(R^1\lambda+B^1|\xi|^2)=a_0|\xi|^{2n}+\sum_{j=1}^{n}a_j\lambda^j|\xi|^{2(n-j)}. \tag{3.6}$$

Let $t=|\xi|^2$, then (3.6) rewrites as

$$a_0t^n+\sum_{j=1}^{n}a_j\lambda^jt^{n-j}=a_0\prod_{j=1}^{m}(t+k_j|\lambda|)^{n_j},$$

where m and n_j are constants depending on λ for which $n=\sum_{j=1}^m n_j$ and k_j are functions with respect to $\lambda/|\lambda|$ such that $k_j\neq k_\ell$ for $j\neq\ell$. In view of (2.4), $a_0\prod_{j=1}^m(t+k_j|\lambda|)^{n_j}\neq 0$ for $t\geq 0$ and $\lambda\in\Sigma_\epsilon$, and so $k_j\notin(-\infty,0)$ for $\lambda\in\Sigma_\epsilon$. Thus, we have

$$\det(R^1\lambda+B^1|\xi|^2)=a_0\prod_{j=1}^{n}(\xi_n^2+|\xi'|^2+k_j|\lambda|)^{n_j}=a_0\prod_{j=1}^{m}(\xi_n+i\omega_j)^{n_j}(\xi_n-i\omega_j)^{n_j} \tag{3.7}$$

with $\omega_j=\sqrt{|\xi'|^2+k_j|\lambda|}$ where we take $\mathrm{Re}\,\omega_j>0$. We rewrite the lhs of (3.5):

Lemma 15. *We have*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|\xi|^{2\ell}i\xi_N(e^{-iy_N\xi_N}+e^{iy_N\xi_N})}{\det(R^1\lambda+B^1|\xi|^2)}\,d\xi_N=\sum_{j=1}^{m}\frac{1}{(n_j-1)!}J_j \tag{3.8}$$

with

$$\begin{aligned}J_j&=\Big(\frac{\partial}{\partial\xi_N}\Big)^{n_j-1}\frac{f_j(\xi_N^2)i\xi_Ne^{iy_N\xi_N}}{(\xi_N+i\omega_j)^{n_j}}\Big|_{\xi_N=i\omega_j}-\Big(\frac{\partial}{\partial\xi_N}\Big)^{n_j-1}\frac{f_j(\xi_N^2)i\xi_Ne^{-iy_N\xi_N}}{(\xi_N-i\omega_j)^{n_j}}\Big|_{\xi_N=-i\omega_j}\\&:=J_j^+-J_j^-,\end{aligned} \tag{3.9}$$

where we have set

$$f_j(y)=\frac{[|\xi'|^2+y]^l}{\prod_{\ell\neq j}(y+|\xi'|^2+k_\ell|\lambda|)^{n_\ell}}.$$

Proof. The proof follows by direct computation of the integral on the l.h.s. of (3.8) as a limit of curve integrals of a complex function which are computed using residue theorem. Denoting the integrand by $f(\xi_N)$ we have

$$\int_{-\infty}^{\infty}f(\xi_N)\,d\xi_N=\lim_{R\to\infty}\int_{\gamma_R^+}f(\xi_N)\,d\xi_N-\lim_{R\to\infty}\int_{L_R^+}f(\xi_N)\,d\xi_N \tag{3.10}$$

where

$$L_R^+=\{z\in\mathbb{C}:\mathrm{Im}z>0,\ |z|=R\},\quad \gamma_R^+=[-R,R]\times\{\mathrm{Im}z=0\}\cup L_R^+.$$

Writing $\xi_N=a+bi$ we easily verify that the integral over L_R^+ vanishes as $R\to\infty$, and therefore by residue theorem the integral in the lhs of (3.8) will be equal to sum of residua of the integrand on the upper complex halfplane. In order to compute the residua notice that by (3.7) we have

$$f(\xi_N)=\frac{|\xi|^2i\xi_Ne^{iy_N}}{\Pi_{j=1}^m(\xi_N-i\omega_j)^{n_j}},$$

therefore in a neighbourhood of $\xi_N = i\omega_j$ we have

$$f(\xi_N) = \frac{g_j(\xi_N)}{(\xi_N - i\omega_j)^{n_j}},$$

where

$$g_j(\xi_N) = \frac{f_j(\xi_N^2) i\xi_N e^{iy_N\xi_N}}{(\xi_N + i\omega_j)^{n_j}}$$

is holomorphic, which implies the form of J_j^+ in (3.9). The part with $e^{-iy_N\xi_N}$ is calculated in the same way extending the integral to a curve contained in lower complex hyperplane leading to the form of J_j^-. □

It is easy to observe that

Lemma 16. *We have the following identities*

$$\begin{aligned} \partial_N^{2\ell-1} f_j(\xi_N^2) &= a_0^{(2\ell-1)} f_j^{(2\ell-1)}(\xi_N^2)\xi_N^{2\ell-1} + a_1^{(2\ell-1)} f_j^{(2\ell-2)}(\xi_N^2)\xi_N^{2\ell-3} + \cdots \\ &\quad + a_{\ell-2}^{(2\ell-1)} f_j^{(\ell+1)}(\xi_N^2)\xi_N^3 + a_{\ell-1}^{(2\ell-1)} f_j^{(\ell)}(\xi_N^2)\xi_N, \end{aligned} \tag{3.11}$$

$$\begin{aligned} \partial_N^{2\ell} f_j(\xi_N^2) &= a_0^{(2\ell)} f_j^{(2\ell)}(\xi_N^2)\xi_N^{2\ell} + a_1^{(2\ell)} f_j^{(2\ell-1)}(\xi_N^2)\xi_N^{2\ell-2} + \cdots \\ &\quad + a_{\ell-1}^{(2\ell)} f_j^{(\ell+1)}(\xi_N^2)\xi_N^2 + a_\ell^{(2\ell)} f_j^{(\ell)}(\xi_N^2), \end{aligned} \tag{3.12}$$

with some coefficients $a_m^{(k)}$, *where* $f_j^{(\ell)} = \partial^\ell f_j/\partial y_j$.

Proof. It is enough to observe that

$$\partial_{\xi^N}[f^{2l-k}(\xi_N^2)\xi_N^{2(l-k)-1}] = a_{kl}\xi_N^{2(l-k-1)} f^{(2l-k)}(\xi_N^2) + 2f^{(2l-k+1)}(\xi_N^2)\xi_N^{2(l-k)}$$

and

$$\partial_{\xi^N}[f^{2l-k}(\xi_N^2)\xi_N^{2(l-k)}] = b_{kl}\xi_N^{2(l-k)-1} f^{(2l-k)}(\xi_N^2) + 2f^{(2l-k+1)}(\xi_N^2)\xi_N^{2(l-k)+1}$$

for some coefficients a_{kl}, b_{kl}, therefore (3.11) follows by induction. □

By Lemma 16, there exist some functions $g_j^{(\ell)}$ for which

$$\partial_N^{2\ell-1} f_j(\xi_N^2)|_{\xi_N=\pm i\omega_j} = \pm i g_j^{(2\ell-1)}(\omega_j^2)\omega_j, \quad \partial_N^{2\ell} f_j(\xi_N^2)|_{\xi_N=\pm i\omega_j} = g_j^{(2\ell)}(\omega_j^2). \tag{3.13}$$

Notice that the uppercase index $g^{(l)}$ does not denote differentiation contrarily to $f^{(l)}$. By Leibniz rule, we have

$$\begin{aligned} &\left(\frac{\partial}{\partial\xi_N}\right)^{n_j-1} \frac{f_j(\xi_N^2) i\xi_N e^{\pm iy_N\xi_N}}{(\xi_N \pm i\omega_j)^{n_j}}\Big|_{\xi_N=\pm i\omega_j} = \\ &\quad = \sum_{k=0}^{n_j-1} \partial_{\xi_N}^k (i\xi_N) \partial_{\xi_N}^{n_j-1-k} \frac{f_j(\xi_N^2) i\xi_N e^{\pm iy_N\xi_N}}{(\xi_N \pm i\omega_j)^{n_j}}\Big|_{\xi_N=\pm i\omega_j} \\ &\quad = i\xi_N \left(\frac{\partial}{\partial\xi_N}\right)^{n_j-1} \frac{f_j(\xi_N^2) e^{\pm iy_N\xi_N}}{(\xi_N \pm i\omega_j)^{n_j}}\Big|_{\xi_N=\pm i\omega_j} + i\left(\frac{\partial}{\partial\xi_N}\right)^{n_j-2} \frac{f_j(\xi_N^2) e^{\pm iy_N\xi_N}}{(\xi_N \pm i\omega_j)^{n_j}}\Big|_{\xi_N=\pm i\omega_j} \\ &\quad = \sum_{k_1+k_2+k_3=n_j-1} C_{k_1,k_2,k_3}^{(n_j-1)} L_1^\pm + \sum_{k_1+k_2+k_3=n_j-2} C_{k_1,k_2,k_3}^{(n_j-2)} i L_2^\pm, \end{aligned}$$

where

$$L_1^{\pm} = i\xi_N\Big(\frac{\partial}{\partial\xi_N}\Big)^{k_1} f_j(\xi_N^2)\Big(\frac{\partial}{\partial\xi_N}\Big)^{k_2} e^{\pm i\xi_N y_N}\Big(\frac{\partial}{\partial\xi_N}\Big)^{k_3}(\xi_N \pm i\omega_j)^{-n_j}\Big|_{\xi_N=\pm i\omega_j},$$
$$L_2^{\pm} = \Big(\frac{\partial}{\partial\xi_N}\Big)^{k_1} f_j(\xi_N^2)\Big(\frac{\partial}{\partial\xi_N}\Big)^{k_2} e^{\pm i\xi_N y_N}\Big(\frac{\partial}{\partial\xi_N}\Big)^{k_3}(\xi_N \pm i\omega_j)^{-n_j}\Big|_{\xi_N=\pm i\omega_j}$$

with some permutation numbers $C^{(n_j-1)}_{k_1,k_2,k_3}$ and $C^{(n_j-2)}_{k_1,k_2,k_3}$. Now our goal is to show

$$L_i^+ = L_i^-, \quad i = 1, 2. \tag{3.14}$$

Then by (3.9) we have $J_j^+ = J_j^-$, and therefore (3.5) holds due to (3.8). Let us start with observing that

$$\partial_{\xi_N}^{k_2} e^{\pm i\xi_N y_N}\Big|_{\xi_N=\pm i\omega_j} = (\pm i y_N)^{k_2} e^{\pm i\xi_N y_N}\Big|_{\xi_N=\pm i\omega_j} = (\pm i y_N)^{k_2} e^{-\omega_j y_N} \tag{3.15}$$

and

$$\partial_{\xi_N}^{k_3}(\xi_N \pm i\omega_j)^{-n_j} = d_{k3}(\xi_N \pm i\omega_j)^{-n_j-k_3}, \tag{3.16}$$

where $d_{k3} = (-n_j)(-n_j-1)\cdots(-n_j-k_3+1)$. In order to show that $L_1^+ = L_1^-$ we assume $k_1+k_2+k_3 = n_j - 1$ and consider first the case when k_1 is odd. Using (3.13),(3.15) and (3.16) we get

$$\begin{aligned} L_1^{\pm} &= (\mp\omega_j)(\pm i g_j^{(k_1)}(\omega_j^2)\omega_j)(\pm i y_N)^{k_2} e^{-\omega_j y_N} 2^{-n_j-k_3} d_{k_3}(\pm i\omega_j)^{-n_j-k_3} \\ &= (\pm i)^{k_2}(\pm i)^{-k_1-k_2-2k_3-1}(-i) g_j^{(k_1)}(\omega_j^2)\omega_j^2 e^{-\omega_j y_N}(2\omega_j)^{-n_j-k_3} d_{k_3} y_N^{k_2} \\ &= -i^{-2k_3-k_1} g_j^{(k_1)}(\omega_j^2)\omega_j^2 e^{-\omega_j y_N}(2\omega_j)^{-n_j-k_3} d_{k_3} y_N^{k_2} \end{aligned} \tag{3.17}$$

where we have used that $(\pm 1)^{2k_3+k_1+1} = 1$ because k_1 is odd. In the same manner, when k_1 is even, we have

$$\begin{aligned} L_1^{\pm} &= \mp\omega_j g_j^{(k_1)}(\omega_j^2)(\pm i y_N)^{k_2} e^{-\omega_j y_N} 2^{-n_j-k_3} d_{k_3}(\pm i\omega_j)^{-n_j-k_3} \\ &= (\pm i)^{k_2}(\pm i)^{-k_1-k_2-2k_3}(\mp\omega_j)(\pm\omega_j)^{-1} g_j^{(2l)}(\omega_j^2)\omega_j y_N^{k_2} e^{-\omega_j y_N} 2^{-n_j-k_3} d_{k_3}\omega_j^{-k_1-k_2-2k_3} \\ &= -i^{-k_1-2k_3}\omega_j g_j^{(2l)}(\omega_j^2)\omega_j y_N^{k_2} e^{-\omega_j y_N} 2^{-n_j-k_3} d_{k_3}\omega_j^{-k_1-k_2-2k_3}, \end{aligned} \tag{3.18}$$

since this time $(\pm 1)^{k_1+2k_3} = 1$ because k_1 is even.

In order to show that $L_2^+ = L_2^-$ we assume $n_j = k_1+k_2+k_3+2$ and again consider first k_1 odd. Then

$$\begin{aligned} L_2^{\pm} &= \pm i g_j^{(k_1)}(\omega_j^2)\omega_j(\pm i y_N)^{k_2} e^{-\omega_j y_N} d_{k_3} 2^{-n_j-k_3}(\pm i\omega_j)^{-n_j-k_3} \\ &= (\pm i)^{1+k_2}(\pm i)^{-k_1-k_2-2k_3-2} g_j^{(k_1)}(\omega_j^2)\omega_j y_N^{k_2} e^{-\omega_j y_N} d_{k_3}(2\omega_j)^{-n_j-k_3} \\ &= i^{-k_1-2k_3-1} g_j^{(k_1)}(\omega_j^2)\omega_j y_N^{k_2} e^{-\omega_j y_N} d_{k_3}(2\omega_j)^{-n_j-k_3}, \end{aligned} \tag{3.19}$$

where we have used $(\pm 1)^{-k_1-2k_3-1} = 1$ because k_1 is odd. When k_1 is even, we have

$$\begin{aligned} L_2^{\pm} &= g_j^{(k_1)}(\omega_j^2)(\pm i y_N)^{k_2} e^{-\omega_j y_N} d_{k_3} 2^{-n_j-k_3}(\pm i\omega_j)^{-n_j-k_3} \\ &= (\pm i)^{k_2}(\pm i)^{-k_1-k_2-2k_3-2} g_j^{(k_1)}(\omega_j^2) y_N^{k_2} e^{-\omega_j y_N} d_{k_3}(2\omega_j)^{-n_j-k_3} \\ &= i^{-k_1-2k_3-2} g_j^{(2l)}(\omega_j^2) y_N^{k_2} e^{-\omega_j y_N} d_{k_3}(2\omega_j)^{-n_j-k_3}, \end{aligned} \tag{3.20}$$

since $(\pm 1)^{-k_1-2k_3-2} = 1$ as k_1 is even. From (3.17)-(3.20) we conclude (3.14), which leads to (3.5) as explained above. This completes the proof of Theorem 14. □

We now prove the existence of an $\mathcal{R}$ bounded solution operator for Eq. (3.1).

Corollary 17. *Let $1 < q < \infty$ and $0 < \epsilon < \pi/2$. Let $X_q(\mathbb{R}^N_+)$ and $\mathcal{X}_q(\mathbb{R}^N_+)$ be spaces defined by replacing Ω by $\mathbb{R}^N_+$ in Theorem 7. Then, there exists an operator family $\mathcal{T}_3(\lambda) \in \mathrm{Hol}\,(\Sigma_\epsilon, \mathcal{L}(\mathcal{X}_q(\mathbb{R}^N_+), H^2_q(\mathbb{R}^N)^n))$ such that $\mathbf{v} = \mathcal{T}_3(\lambda)(\mathbf{f}, \lambda^{1/2}\mathbf{g}, \mathbf{g})$ is a unique solution of Eq. (3.1) for any $\lambda \in \Sigma_\epsilon$ and $(\mathbf{f}, \mathbf{g}) \in X_q(\mathbb{R}^N_+)^n$.*

Moreover, for any $\lambda_0 > 0$ there exists a constant r_b independent of $x_0 \in \Gamma$ for which

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\mathbb{R}^N_+), H^{2-k}_q(\mathbb{R}^N+))}(\{(\tau\partial_\tau)^\ell(\lambda^{k/2}\mathcal{T}_3(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le r_b \tag{3.21}$$

for $k = 0, 1, 2$ and $\ell = 0, 1$.

Proof. Notice that $\nabla\mathbf{v} \cdot \mathbf{n}_0 = -\partial_N\mathbf{v}$. Let $\mathbf{h} = (B^1)^{-1}\mathbf{g}$, and consider the boundary value problem:

$$\lambda\mathbf{w} - \Delta\mathbf{w} = 0 \quad \text{in } \mathbb{R}^N_+, \quad \partial_N\mathbf{w} = -\mathbf{h}. \quad \text{on } \mathbb{R}^N_0. \tag{3.22}$$

To define a solution operator of Eq. (3.22), we use the partial Fourier transform $\mathcal{F}_{x'}$ and its inverse transform $\mathcal{F}^{-1}_{\xi'}$ defined in (1.6). Applying the partial Fourier transform to Eq. (3.22), we have

$$((\lambda + |\xi'|^2) - \partial_N^2)\mathcal{F}_{x'}[\mathbf{w}](\xi', x_N) = 0 \quad \text{on } (0,\infty), \quad \partial_N\mathcal{F}_{x'}[\mathbf{w}](\xi', x_N)|_{x_N=0} = -\mathcal{F}_{x'}[\mathbf{h}](\xi', 0).$$

Thus, we have

$$\mathcal{F}_{x'}[\mathbf{w}](\xi', x_N) = \frac{e^{-\sqrt{\lambda+|\xi'|^2}x_N}}{\sqrt{\lambda + |\xi'|^2}}\mathcal{F}_{x'}[\mathbf{h}](\xi', 0).$$

And so, we define a solution operator $U(\lambda)$ by setting

$$U(\lambda)\mathbf{h} = \mathcal{F}^{-1}_{\xi'}\Big[\frac{e^{-\sqrt{\lambda+|\xi'|^2}x_N}}{\sqrt{\lambda + |\xi'|^2}}\mathcal{F}_{x'}[\mathbf{h}](\xi', 0)\Big](x').$$

By the Volevich trick:

$$f(x_N)g(0) = -\int_0^\infty \partial_N f\big((x_N + y_N)g(y_N)\big)\,dy_n,$$

with

$$f(x_N) = \frac{e^{-\sqrt{\lambda+|\xi'|^2}x_N}}{\sqrt{\lambda + |\xi'|^2}}, \quad g(y_N) = \mathcal{F}_{x'}[\mathbf{h}](\xi', y_N)$$

we write $U(\lambda)\mathbf{h}$ as

$$\begin{aligned}
&U(\lambda)\mathbf{h} \\
&= -\int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\frac{e^{-\sqrt{\lambda+|\xi'|^2}(x_N+y_N)}}{\sqrt{\lambda + |\xi'|^2}}(\partial_N\mathcal{F}_{x'}[\mathbf{h}](\xi', y_N) - \sqrt{\lambda + |\xi'|^2}\mathcal{F}_{x'}[\mathbf{h}](\xi', y_N))\Big](x')\,dy_N \\
&= -\int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\frac{e^{-\sqrt{\lambda+|\xi'|^2}(x_N+y_N)}}{\sqrt{\lambda + |\xi'|^2}}\Big(\mathcal{F}_{x'}[\partial_N\mathbf{h}](\xi', y_N) - \frac{\lambda^{1/2}}{\sqrt{\lambda + |\xi'|^2}}\mathcal{F}_{x'}[\lambda^{1/2}\mathbf{h}](\xi', y_N) \\
&\qquad + \sum_{j=1}^{N-1}\frac{i\xi_j}{\sqrt{\lambda + |\xi'|^2}}\mathcal{F}_{x'}[\partial_j\mathbf{h}](\xi', y_N)\Big)\Big](x')\,dy_N.
\end{aligned}$$

Let $\mathcal{Y}_q(\mathbb{R}^N_+) = \{(F_2, F_3) \mid F_2 \in L_q(\mathbb{R}^N_+)^N,\ F_3 \in H^1_q(\mathbb{R}^N_+)^N\}$. And then, we define an operator $\mathcal{U}(\lambda)$ acting on $(F_2, F_3) \in \mathcal{Y}_q(\mathbb{R}^N_+)$ by letting

$$
\begin{aligned}
&\mathcal{U}(\lambda)(F_2, F_3) \\
&= -\int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\frac{e^{-\sqrt{\lambda+|\xi'|^2}(x_N+y_N)}}{\sqrt{\lambda+|\xi'|^2}}\Big(\mathcal{F}_{x'}[\partial_N F_3](\xi', y_N) - \frac{\lambda^{1/2}}{\sqrt{\lambda+|\xi'|^2}}\mathcal{F}_{x'}[F_2](\xi', y_N) \\
&\qquad + \sum_{j=1}^{N-1}\frac{i\xi_j}{\sqrt{\lambda+|\xi'|^2}}\mathcal{F}_{x'}[\partial_j F_3](\xi', y_N)\Big)\Big](x')\,dy_N,
\end{aligned}
$$

and then we have

$$U(\lambda)\mathbf{h} = \mathcal{U}(\lambda)(\lambda^{1/2}\mathbf{h}, \mathbf{h}).$$

Moreover, using the same argument as in [34, Sect. 5], we see that

$$\mathcal{R}_{(\mathcal{L}_q(\mathbb{R}^N_+), H^{2-j}_q(\mathbb{R}^N)^n)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}\mathcal{U}(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \leq r_b \tag{3.23}$$

for $\ell = 0, 1$ and $j = 0, 1, 2$, where r_b is a constant depending on ϵ, $\lambda_0 > 0$, M_0 and m_1, but independent of $x_0 \in \Gamma$.

Let $\mathcal{T}_2(\lambda)$ be the operator given in Theorem 12. Letting $\mathbf{F} = \lambda R^1 U(\lambda)\mathbf{h} - \mathrm{div}\,(B^1\nabla U(\lambda)\mathbf{h})$, and setting $\mathbf{v} = \mathcal{T}_2(\lambda)(\mathbf{f} - \mathbf{F}) + U_0(\lambda)\mathbf{h}$ with $\mathbf{h} = (B^1)^{-1}\mathbf{g}$, we see that $\mathbf{v}$ is a unique solution of Eq. (3.1). The uniqueness follows from the existence of solutions of the dual problem. Thus, combining Theorem 12 and (3.23), we have Corollary 17. This completes the proof. □

4 Analysis in a bent half-space

Let Φ be a diffeomorphism of C^1 class on $\mathbb{R}^N$ and Φ^{-1} the inverse of Φ. We assume that $\nabla\Phi = \mathcal{A} + \mathcal{B}(x)$ and $\nabla\Phi^{-1} = \mathcal{A}^{-1} + \mathcal{B}_{-1}(y)$, where $\mathcal{A}$ is an orthogonal matrix with constant coefficients, $\mathcal{A}^{-1}$ is the inverse matrix of $\mathcal{A}$, and $\mathcal{B}(x)$ and $\mathcal{B}_{-1}(y)$ are matrices of $C^0(\mathbb{R}^N)$ functions satisfying the conditions:

$$\|(\mathcal{B}, \mathcal{B}_{-1})\|_{L_\infty(\mathbb{R}^N)} \leq M_1, \quad \|\nabla(\mathcal{B}, \mathcal{B}_{-1})\|_{L_r(\mathbb{R}^N)} \leq C_K. \tag{4.1}$$

In the above formula r is an exponent such that $N < r < \infty$ and C_K is a constant depending on the constants K, L_1 and L_2 appearing in Definition 1. We choose M_1 small enough eventually, and so we may assume that $0 < M_1 \leq 1 \leq C_K$ without loss of generality. Let

$$\Omega_+ = \Phi(\mathbb{R}^N_+) = \{y = \Phi(x) \mid x \in \mathbb{R}^N_+\}, \quad \Gamma_+ = \Phi(\mathbb{R}^N_0) = \{y = \Phi(x) \mid x \in \mathbb{R}^N_0\}.$$

Let $\mathbf{n}_+$ be the unit outer normal to Γ_+ and let $\partial_{\mathbf{n}_+} = \mathbf{n}_+ \cdot \nabla$. Let y_0 be any point of Γ_+ and we fix it. We assume in this section that there exist a positive number d_0 for which

$$|R(y) - R(y_0)| \leq M_1, \quad |B(y) - B(y_0)| \leq M_1 \tag{4.2}$$

for any $y \in B_{d_0}(y_0)$. Moreover, let M_2 be a number for which

$$\|\nabla(R, B)\|_{L_r(\mathbb{R}^N)} \leq M_2. \tag{4.3}$$

Note that since R and B are the extensions of functions defined on Ω, due to (1.2), we may take $M_2 = M_2(M_0)$. We may assume that

$$C_K \leq M_2. \tag{4.4}$$

Let $\varphi(y)$ be a function in $C^\infty(\mathbb{R}^N)$ such that

$$\varphi(y) = \begin{cases} 1, & y \in B_{d_0/3}(y_0), \\ 0, & y \notin B_{2d_0/3}(y_0). \end{cases} \tag{4.5}$$

We define

$$\tilde R(y) = \varphi(y)R(y) + (1-\varphi(y))R(y_0), \quad \tilde B(y) = \varphi(y)B(y) + (1-\varphi(y))B(y_0).$$

In this section, we consider the following resolvent problem:

$$\lambda \tilde R \mathbf{v} - \mathrm{div}\,(\tilde B \nabla \mathbf{v}) = \mathbf{f} \quad \text{in } \Omega_+, \quad \tilde B(\nabla \mathbf{v} \cdot \mathbf{n}_+) = \mathbf{g} \quad \text{on } \Gamma_+. \tag{4.6}$$

We shall prove the following theorem.

Theorem 18. *Let $1 < q \le r$. Let $X_q(\Omega_+)$ and $\mathcal{X}_q(\Omega_+)$ be the spaces defined by replacing Ω by Ω_+ in Theorem 7. Then, there exist a small number $M_1 > 0$, a constant $\lambda_0 > 0$ and an operator family $\mathcal{T}_+(\lambda)$ with*

$$\mathcal{T}_+(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_q(\Omega_+), H^2_q(\Omega)^n))$$

such that such that if (4.2) is satisfied then for any $\lambda \in \Sigma_{\epsilon,\lambda_0}$ and $(\mathbf{f}, \mathbf{g}) \in X_q(\Omega_+)$, $\mathbf{v} = \mathcal{T}_+(\lambda)(\mathbf{f}, \lambda^{1/2}\mathbf{g}, \mathbf{g})$ is a unique solution of Eq. (4.6), and

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega_+), H^{2-j}_q(\Omega_+)^n)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}\mathcal{T}_+(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le r_b$$

for $\ell = 0, 1$ and $j = 0, 1, 2$ with some constant r_b independent of M_1 and M_2, where M_2 is from (4.3).

Proof. The uniqueness of solutions follows from the existence of solutions to the dual problem, and so we only prove the existence of $\mathcal{R}$ bounded solution operator $\mathcal{T}_+(\lambda)$. We use the change of variables: $y = \Phi(x)$ to transform Eq. (4.6) to the equations in the half-space. We have

$$\Big(\frac{\partial x_j}{\partial y_k}\Big)(\Phi(x)) = a_{jk} + b_{jk}(x), \tag{4.7}$$

where a_{jk} and $b_{jk}(x)$ are the $(j,k)^{\mathrm{th}}$ components of $\mathcal{A}^{-1}$ and $\mathcal{B}_{-1}(\Phi(x))$, respectively. Since $\mathcal{A}^{-1}$ is an orthogonal matrix and thanks to (4.1), we have

$$\sum_{j=1}^N a_{jk}a_{j\ell} = \sum_{j=1}^N a_{kj}a_{\ell j} = \delta_{k\ell}, \quad \|b_{jk}\|_{L_\infty(\mathbb{R}^N)} \le M_1, \quad \|\nabla b_{jk}\|_{L_r(\mathbb{R}^N)} \le C_K. \tag{4.8}$$

By (4.7), we derive the formula for change of variables from y to x, namely

$$\frac{\partial}{\partial y_j} = \sum_{k=1}^N (a_{kj} + b_{kj}(x))\frac{\partial}{\partial x_k}. \tag{4.9}$$

Applying this formula we get that

$$\begin{aligned} \frac{\partial^2}{\partial^2 y_j} &= \sum_{k,\ell=1}^N (a_{\ell j} + b_{\ell j}(x))\frac{\partial}{\partial x_\ell}\left((a_{kj} + b_{kj}(x))\frac{\partial}{\partial x_k}\right) \\ &= \sum_{k,\ell=1}^N (a_{\ell j}a_{kj} + b_{\ell j}(x)a_{kj})\frac{\partial^2}{\partial x_\ell \partial x_k} + \sum_{k,\ell=1}^N (a_{\ell j}b_{kj}(x) + b_{\ell j}(x)b_{kj}(x))\frac{\partial^2}{\partial x_\ell \partial x_k} \\ &\quad + \sum_{k,\ell=1}^N (a_{\ell j} + b_{\ell j}(x))\frac{\partial b_{kj}(x)}{\partial x_\ell}\frac{\partial}{\partial x_k}. \end{aligned} \tag{4.10}$$

Note that by (4.8), we have

$$\sum_{k,\ell=1}^{N}\sum_{j=1}^{N} a_{\ell j}a_{kj}\frac{\partial^2}{\partial x_\ell \partial x_k} = \sum_{k=1}^{N}\frac{\partial^2}{\partial^2 x_k},$$

therefore

$$\Delta_y = \sum_{j=1}^{N}\frac{\partial^2}{\partial^2 y_j} = \Delta_x + D_2\nabla_x^2 + D_1\nabla_x \tag{4.11}$$

where

$$\Delta_x = \sum_{k=1}^{N}\frac{\partial^2}{\partial^2 x_k}, \quad D_2\nabla_x^2 = \sum_{j,k,\ell=1}^{N}(a_{kj}b_{\ell j}(x) + b_{kj}(x)(a_{\ell j} + b_{\ell j}(x)))\frac{\partial^2}{\partial x_\ell \partial x_k},$$
$$D_1\nabla_x = \sum_{j,k,\ell=1}^{N}(a_{\ell j} + b_{\ell j}(x))\frac{\partial b_{kj}(x)}{\partial x_j}\frac{\partial}{\partial x_k}.$$

We now transform the form of the outer normal vector $\mathbf{n}_+(y)$ to Γ_+ at point $y = \Phi(x)$. Since Γ_+ is represented by $x_N = \Phi_N^{-1}(y) = 0$, the gradient of function $\Phi_N^{-1}(y)$ will indicate the normal direction, therefore after normalization, we obtain

$$\mathbf{n}_+(y) = \mathbf{n}_+(\Phi(x)) = -\frac{(\frac{\partial x_N}{\partial y_1}, \dots, \frac{\partial x_N}{\partial y_N})^\top}{|(\frac{\partial x_N}{\partial y_1}, \dots, \frac{\partial x_N}{\partial y_N})|} = -\frac{(a_{N1} + b_{N1}(x), \dots, a_{NN} + b_{NN}(x))^\top}{d(x)}, \tag{4.12}$$

where for the second equality we used (4.7). Having this we note that

$$\begin{aligned}\nabla_y v^i(y)\cdot \mathbf{n}_+(y) &= \sum_{j=1}^{N}\frac{\partial v^i(y)}{\partial y_j}n_+^j(y) = -\sum_{j,k=1}^{N}(a_{kj} + b_{kj}(x))\frac{a_{Nj} + b_{Nj}(x)}{d(x)}\frac{\partial u^i(x)}{\partial x_k} \\ &= -d^{-1}(x)\left[\frac{\partial u^i(x)}{\partial x_N} + \sum_{j,k=1}^{N}\{(a_{kj} + b_{kj}(x))b_{Nj}(x) + a_{Nj}b_{kj}(x)\}\frac{\partial u^i(x)}{\partial x_k}\right],\end{aligned} \tag{4.13}$$

where we denoted $u^i(x) = v^i \circ \Phi(x)$. Note that by (4.8) we have

$$d(x) = \sqrt{\sum_{j=1}^{N}(a_{Nj} + b_{Nj}(x))^2} = \sqrt{1 + \sum_{j=1}^{N}(2a_{Nj}b_{Nj}(x) + b_{Nj}(x)^2)}.$$

Therefore, choosing $M_1 > 0$ sufficiently small, we have

$$d^{-1}(x) = 1 + \tilde{d}(x) \tag{4.14}$$

with $|\tilde{d}(x)| \le C|\sum_{j=1}^{N}(2a_{Nj}b_{Nj}(x) + b_{Nj}(x)^2)| \le CM_1$ and

$$\|\nabla\tilde{d}\|_{L_r(\mathbb{R}^N)} \le C\sum_{j=1}^{N}(\|a_{Nj}\|_{L_\infty(\mathbb{R}^n)} + \|b_{Nj}\|_{L_\infty(\mathbb{R}^n)})\|\nabla b_{Nj}\|_{L_r(\mathbb{R}^n)} \le CC_k \le CM_2,$$

where in the last inequality we have used (4.4).

Finally, by (4.9), (4.11), (4.12), (4.13) and (4.14), the system (4.6) is transformed to

$$\lambda R(y_0)\mathbf{u} - B(y_0)\Delta_x \mathbf{u} + \mathbf{F}(\mathbf{u}) = \tilde{\mathbf{f}} \quad \text{in } \mathbb{R}^N_+, \quad B(y_0)(\nabla_x \mathbf{u}\cdot \mathbf{n}_0(x)) + \mathbf{G}(\mathbf{u}) = \tilde{\mathbf{g}} \quad \text{on } \mathbb{R}^N_0, \tag{4.15}$$

where $\mathbf{n}_0 = (0, \dots, 0, -1)$, and

$$\mathbf{u}(x) = \mathbf{v}\circ\Phi(x), \quad \tilde{\mathbf{f}}(x) = \mathbf{f}\circ\Phi(x), \quad \tilde{\mathbf{g}}(x) = \mathbf{g}\circ\Phi(x),$$

and, by (4.5) and (4.12)-(4.14),

$$\begin{aligned}
\mathbf{F}(\mathbf{u}) &= \{\lambda[\varphi(\cdot)(R(\cdot) - R(y_0))\mathbf{v}] - [\operatorname{div}_y(\varphi(y)(B(y) - B(y_0))\nabla_y \mathbf{v})]\}\circ\Phi \\
&\quad - B(y_0)(D_2\nabla_x^2\mathbf{u} + D_1\nabla_x\mathbf{u}), \\
\mathbf{G}(\mathbf{u}) &= B(y_0)\tilde{d}\nabla\mathbf{u}\cdot\mathbf{n}_0 + \{\phi(y)(B(y) - B(y_0))\nabla_y\mathbf{v}\cdot\mathbf{n}_+\}\circ\Phi \\
&\quad - \frac{B(y_0)}{d}\sum_{j,k=1}^N (a_{kj}b_{Nj} + b_{kj}(a_{Nj} + b_{Nj}))\frac{\partial \mathbf{u}}{\partial x_k}.
\end{aligned}$$

Using (1.8), (4.2), (4.3), and (4.14), we have

$$\begin{aligned}
\|\mathbf{F}(\mathbf{u})\|_{L_q(\mathbb{R}^N_+)} &\le C(|\lambda| M_1\|\mathbf{u}\|_{L_q(\mathbb{R}^N_+)} + (M_1+\alpha)\|\mathbf{u}\|_{H^2_q(\mathbb{R}^N_+)}) + C_{\alpha,M_2}\|\mathbf{u}\|_{H^1_q(\mathbb{R}^N_+)}, \\
|\lambda|^{1/2}\|\mathbf{G}(\mathbf{u})\|_{L_q(\mathbb{R}^N_+)} &\le CM_1|\lambda|^{1/2}\|\mathbf{u}\|_{H^1_q(\mathbb{R}^N_+)}, \\
\|\mathbf{G}(\mathbf{u})\|_{H^1_q(\mathbb{R}^N_+)} &\le C(M_1+\alpha)\|\mathbf{u}\|_{H^2_q(\mathbb{R}^N_+)} + C_{\alpha,M_2}\|\mathbf{u}\|_{H^1_q(\mathbb{R}^N_+)}
\end{aligned} \tag{4.16}$$

for any $\alpha > 0$, where C is a constant independent of α, M_1, λ_1 and C_{α,M_2} is a constant depending on α and M_2.

Let $\mathcal{T}_3(\lambda)$ be the $\mathcal{R}$-bounded solution operator for Eq. (3.1) given in Corollary 17. Taking $\mathbf{u} = \mathcal{T}_3(\lambda)(\tilde{\mathbf{f}}, \lambda^{1/2}\tilde{\mathbf{g}}, \mathbf{g})$ in (4.15), we get

$$\begin{aligned}
&\lambda R(y_0)\mathcal{T}_3(\lambda)(\tilde{\mathbf{f}}, \lambda^{1/2}\tilde{\mathbf{g}}, \mathbf{g}) - B(y_0)\Delta_x\mathcal{T}_3(\lambda)(\tilde{\mathbf{f}}, \lambda^{1/2}\tilde{\mathbf{g}}, \tilde{\mathbf{g}}) + \mathbf{F}(\mathcal{T}_3(\lambda)(\tilde{\mathbf{f}}, \lambda^{1/2}\tilde{\mathbf{g}}, \tilde{\mathbf{g}})) \\
&\qquad = \tilde{\mathbf{f}} + \mathbf{F}(\mathcal{T}_3(\lambda)(\tilde{\mathbf{f}}, \lambda^{1/2}\tilde{\mathbf{g}}, \tilde{\mathbf{g}})) \quad \text{in } \mathbb{R}^N_+, \\
&B(y_0)(\nabla_x\mathcal{T}_3(\lambda)(\tilde{\mathbf{f}}, \lambda^{1/2}\tilde{\mathbf{g}}, \tilde{\mathbf{g}})\cdot\mathbf{n}_0(x)) + \mathbf{G}(\mathcal{T}_3(\lambda)(\tilde{\mathbf{f}}, \lambda^{1/2}\tilde{\mathbf{g}}, \tilde{\mathbf{g}})) \\
&\qquad = \tilde{\mathbf{g}} + \mathbf{G}(\mathcal{T}_3(\lambda)(\tilde{\mathbf{f}}, \lambda^{1/2}\tilde{\mathbf{g}}, \tilde{\mathbf{g}})) \quad \text{on } \mathbb{R}^N_0.
\end{aligned} \tag{4.17}$$

Let us now denote

$$\mathcal{R}_+(\lambda)H_\lambda(\tilde{\mathbf{f}}, \tilde{\mathbf{g}}) = (\mathbf{F}(\mathcal{T}_3(\lambda)H_\lambda(\tilde{\mathbf{f}}, \tilde{\mathbf{g}})), \mathbf{G}(\mathcal{T}_3(\lambda)H_\lambda(\tilde{\mathbf{f}}, \tilde{\mathbf{g}}))), \tag{4.18}$$

where $H_\lambda(\tilde{\mathbf{f}}, \tilde{\mathbf{g}}) = (\tilde{\mathbf{f}}, \lambda^{1/2}\tilde{\mathbf{g}}, \tilde{\mathbf{g}})$.

By (4.16), Corollary 17 and Proposition 4, we have

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\mathbb{R}^N_+))}(\{(\tau\partial_\tau)^\ell H_\lambda\mathcal{R}_+(\lambda) \mid \lambda\in\Sigma_{\epsilon,\lambda_1}\}) \le \{C(M_1+\alpha) + C_{\alpha,M_2}\lambda_1^{-1/2}\}r_b$$

for $\ell = 0, 1$. Thus, choosing α and M_1 so small that $C\alpha r_b < 1/8$, $CM_1 r_b < 1/8$ and choosing $\lambda_1 \ge \lambda_0$ so large that $C_{\alpha,M_2}\lambda_1^{-1/2} r_b \le 1/4$, we have

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\mathbb{R}^N_+))}(\{(\tau\partial_\tau)^\ell H_\lambda\mathcal{R}_+(\lambda) \mid \lambda\in\Sigma_{\epsilon,\lambda_0}\}) \le 1/2 \tag{4.19}$$

for $\ell = 0, 1$. Next, let us denote

$$\mathbf{R}_+(\lambda)(\tilde{\mathbf{f}}, \tilde{\mathbf{g}}) = \mathcal{R}_+(\lambda)H_\lambda(\tilde{\mathbf{f}}, \tilde{\mathbf{g}}). \tag{4.20}$$

Since for any $\lambda \neq 0$ the norm $\|\tilde{\mathbf{f}}, \tilde{\mathbf{g}}\|_{X_q(\mathbb{R}^N_+)}$ is equivalent to $\|H_\lambda(\tilde{\mathbf{f}}, \tilde{\mathbf{g}})\|_{\mathcal{X}_q(\mathbb{R}^N_+)}$ (according to definition (1.16)), we can construct an operator

$$(\mathbf{I} + \mathbf{R}_+(\lambda))^{-1} = \sum_{m=0}^{\infty} (-\mathbf{R}_+(\lambda))^m \quad \text{in} \quad X_q(\mathbb{R}^N_+).$$

Rewriting now (4.17) as

$$L(y_0)\mathcal{T}_3(\lambda)H_\lambda(\tilde{\mathbf{f}}, \tilde{\mathbf{g}}) = [\mathbf{I} + \mathbf{R}_+(\lambda)](\tilde{\mathbf{f}}, \tilde{\mathbf{g}}), \tag{4.21}$$

with

$$L(y_0)(\cdot) = \left[\begin{array}{c} \lambda R(y_0)(\cdot) - B(y_0)\Delta(\cdot) + \mathbf{F}(\cdot) \\ B(y_0)(\nabla(\cdot) \cdot \mathbf{n}_0(x)) + \mathbf{G}(\cdot) \end{array} \right] \tag{4.22}$$

and taking

$$(\bar{\mathbf{f}}, \bar{\mathbf{g}}) = [\mathbf{I} + \mathbf{R}_+(\lambda)](\tilde{\mathbf{f}}, \tilde{\mathbf{g}})$$

in (4.21) we see that

$$\mathbf{u} = \mathcal{T}_3(\lambda)H_\lambda(\mathbf{I} - \mathbf{R}_+(\lambda))^{-1}(\tilde{\mathbf{f}}, \tilde{\mathbf{g}})$$

$\mathbf{u} \in H^2_q(\mathbb{R}^N_+)^n$ is a unique solution of Eq. (4.15).

As for the $\mathcal{R}$ bounded operator, the estimate (4.19) implies the existence of

$$(\mathbf{I} + H_\lambda \mathcal{R}(\lambda))^{-1} = \sum_{m=0}^{\infty} (-H_\lambda \mathcal{R}(\lambda))^m.$$

By (4.20) we have

$$H_\lambda(\mathbf{I} - \mathbf{R}_+(\lambda))^{-1} = (\mathbf{I} - \mathcal{R}(\lambda))^{-1} H_\lambda,$$

and so we have

$$\mathbf{u} = \mathcal{T}_3(\lambda)(\mathbf{I} - \mathcal{R}(\lambda))^{-1} H_\lambda(\tilde{\mathbf{f}}, \tilde{\mathbf{g}}). \tag{4.23}$$

Thus, setting

$$\mathcal{T}_4(\lambda) = \mathcal{T}_3(\lambda)(\mathbf{I} - \mathcal{R}(\lambda))^{-1},$$

by (4.23), (4.19), and Corollary 17 we see that $\mathbf{u} = \mathcal{T}_4(\lambda)H_\lambda(\tilde{\mathbf{f}}, \tilde{\mathbf{g}})$ is a solution of Eq. (4.15), and

$$\mathcal{R}_{\mathcal{X}_q(\mathbb{R}^N_+), H^{2-j}_q(\mathbb{R}^N_+))}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}\mathcal{T}_4(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_1}\}) \leq 2r_b$$

for $\ell = 0, 1$ and $j = 0, 1, 2$. If we set

$$\mathcal{T}_+(\lambda)F = [\mathcal{T}_4(\lambda)F \circ \Phi] \circ \Phi^{-1},$$

for $F = (F_1, F_2, F_3) \in \mathcal{X}_q(\Omega_+)$, then, $\mathcal{T}_+$ is a required $\mathcal{R}$ bounded solution operator for Eq. (4.6), which completes the proof of Theorem 18. □

5 Proof of Theorem 7

To prove Theorem 7, we need to use several properties of uniform C^2 domain, which are stated in the following proposition. For the proof of this result we refer for example to [14], Proposition 6.1.

Proposition 19. *Let Ω be a uniform C^2-domain in $\mathbb{R}^N$ with boundary Γ. Then, for any positive constant M_1, there exist a constant $d \in (0,1)$, at most countably many functions $\Phi_j \in C^2(\mathbb{R}^N)$, and points $x_j^1 \in \Omega$ and $x_j^2 \in \Gamma$ $(j \in \mathbb{N})$ such that the following assertions hold:*

(1) *For every $j \in \mathbb{N}$, the map $\mathbb{R}^N \ni x \to \Phi_j(x) \in \mathbb{R}^N$ is bijective.*

(2) $\Omega = (\bigcup_{j=1}^\infty B_d(x_j^1)) \cup (\bigcup_{j=1}^\infty (\Phi_j(\mathbb{R}^N_+) \cap B_d(x_j^2)))$, $B_d(x_j^1) \subset \Omega$, $\Phi_j(\mathbb{R}^N_+) \cap B_d(x_j^2) = \Omega \cap B_d(x_j^2)$, *and* $\Phi_j(\mathbb{R}^N_0) \cap B_d(x_j^2) = \Gamma \cap B_d(x_j^2)$.

(3) *There exist C^∞ functions ζ_j^i, $\tilde\zeta_j^i$ $(i=1,2, j \in \mathbb{N})$ such that*

$$\mathrm{supp}\, \zeta_j^i,\, \mathrm{supp}\, \tilde\zeta_j^i \subset B_d(x_j^i), \qquad \|\zeta_j^i\|_{H^2_\infty(\mathbb{R}^N)} \le c_0, \quad \|\tilde\zeta_j^i\|_{H^2_\infty(\mathbb{R}^N)} \le c_0,$$

$$\tilde\zeta_j^i = 1 \quad on \quad \mathrm{supp}\, \zeta_j^i, \qquad \sum_{i=1,2}\sum_{j=1}^\infty \zeta_j^i = 1 \quad on \quad \overline{\Omega}, \qquad \sum_{j=1}^\infty \zeta_j^2 = 1 \quad on \quad \Gamma.$$

Here, c_0 is a constant which depends on d, N, q, q' and r, but is independent of $j \in \mathbb{N}$.

(4) $\nabla\Phi_j = \mathcal{R}_j + R_i, \nabla(\Phi_j)^{-1} = \mathcal{R}_j^- + R_j^-$, *where $\mathcal{R}_j$ and $\mathcal{R}_j^-$ are $N \times N$ constant orthogonal matrices, and R_j and R_j^- are $N \times N$ matrices of H^1_∞ functions defined on $\mathbb{R}^N$ which satisfy the conditions:*

$$\|R_j\|_{L_\infty(\mathbb{R}^N)} \le M_1, \quad \|R_j^-\|_{L_\infty(\mathbb{R}^N)} \le M_1,$$

and

$$\|\nabla R_j\|_{L_\infty(\mathbb{R}^N)} \le C_K, \quad \|\nabla R_j^-\|_{L_\infty(\mathbb{R}^N)} \le C_K$$

for any $j \in \mathbb{N}$. Here, C_K is a constant depending only on constants K, L_1 and L_2 appearing in Definition 1.

(5) *There exist a natural number $L > 2$ such that any $L+1$ distinct sets of $\{B_d(x_j^i) \mid i = 1,2,\ j \in \mathbb{N}\}$ have an empty intersection.*

By the finite intersection property stated in point (5) of Proposition 19, we have

$$\Big(\sum_{i=1,2}\sum_{j=1}^\infty \|f\|^q_{L_q(B_j^i\cap\Omega)}\Big)^{1/q} \le C_q\|f\|_{L_q(\Omega)} \tag{5.1}$$

for any $f \in L_q(\Omega)$ and $1 \le q < \infty$. In particular, by (5.1) we have

Corollary 20. *Let $i = 1,2$ and $1 < q < \infty$. Let $\{f_j\}_{j=0}^\infty$ be a sequence of functions in $L_q(\Omega)$ such that $\sum_{j=0}^\infty \|f_j\|^q_{L_q(\Omega)} < \infty$, and $\mathrm{supp}\, f_j \subset B_d(x_j^i)$ $(j \in \mathbb{N})$. Then, $\sum_{j=0}^\infty f_j \in L_q(\Omega)$ and $\|\sum_{j=1}^\infty f_j\|_{L_q(\Omega)} \le (\sum_{j=1}^\infty \|f_j\|^q_{L_q(\Omega)})^{1/q}$.*

In what follows, we write $\Omega_j = \Phi_j(\mathbb{R}^N_+)$ and $\Gamma_j = \Phi_j(\mathbb{R}^N_0)$ for $j \in \mathbb{N}$.
Moreover, we denote the unit outer normal to Γ_j by $\mathbf{n}_j$. Notice that $\mathbf{n}_j = \mathbf{n}$ on Γ_j.
By (1.2), choosing d smaller if necessary, we may assume that

$$|R(x) - R(x_j^i)| \le M_1, \quad |B(x) - B(x_j^i)| \le M_1 \quad \text{for } x \in B_d(x_j^i) \cap \overline{\Omega}. \tag{5.2}$$

Let ζ_j^i and $\tilde\zeta_j^i$ be functions given in Proposition 19 and set

$$R^{ij}(x) = \tilde\zeta_j^i(x)R(x) + (1-\tilde\zeta_j^i(x))R(x_j^i), \quad B^{ij}(x) = \tilde\zeta_j^i(x)B(x) + (1-\tilde\zeta_j^i(x))B(x_j^i)$$

Notice that

$$\zeta_j^i(x)R^{ij}(x) = \zeta_j^i(x)R(x), \quad \zeta_j^i(x)B^{ij}(x) = \zeta_j^i(x)B(x), \tag{5.3}$$

because $\tilde\zeta_j^i = 1$ on $\operatorname{supp}\zeta_j^i$. To construct a parametrix for Eq. (1.14), given $(\mathbf{f}, \mathbf{g}) \in X_q(\Omega)$, we consider the following equations:

$$\lambda R^{1j}\mathbf{v}_j^1 - \operatorname{div}(B^{1j}\nabla\mathbf{v}_j^1) = \tilde\zeta_j^1\mathbf{f} \quad \text{in } \mathbb{R}^N, \tag{5.4}$$

$$\lambda R^{2j}\mathbf{v}_j^2 - \operatorname{div}(B^{2j}\nabla\mathbf{v}_1^2) = \tilde\zeta_j^2\mathbf{f} \quad \text{in } \Omega_j, \quad B^{2j}(\nabla\mathbf{v}_j^2\cdot\mathbf{n}_j) = \tilde\zeta_j^2\mathbf{g} \quad \text{on } \Gamma_j. \tag{5.5}$$

By Theorem 13 and Theorem 18, there exist $\mathcal{R}$ bounded solution operators $\mathcal{D}_j^i(\lambda)$ for Eq. (5.4) and Eq. (5.5) with

$$\mathcal{D}_j^1(\lambda) \in \operatorname{Hol}(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(L_q(\mathbb{R}^N)^n, H_q^2(\mathbb{R}^N)^n)), \quad \mathcal{D}_j^2(\lambda) \in \operatorname{Hol}(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_q(\Omega_j)^n, H_q^2(\Omega_j)^n)) \tag{5.6}$$

such that for any $(\mathbf{f}, \mathbf{g}) \in X_q(\Omega)^n$, $\mathbf{v}_j^1 = \mathcal{D}_j^1(\lambda)\tilde\zeta_j^1\mathbf{f}$ is a unique solution of Eq. (5.4) and $\mathbf{v}_j^2 = \mathcal{D}_j^2(\lambda)H_\lambda(\tilde\zeta_j^2\mathbf{f}, \tilde\zeta_j^2\mathbf{g})$ is a unique solution of Eq. (5.5), respectively. Moreover, we have

$$\begin{aligned} &\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N)^n, H_q^{2-k}(\mathbb{R}^N)^n)}(\{(\tau\partial_\tau)^\ell(\lambda^{k/2}\mathcal{D}_j^1(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le r_b, \\ &\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega_j)^n, H_q^{2-k}(\Omega_j)^n)}(\{(\tau\partial_\tau)^\ell(\lambda^{k/2}\mathcal{D}_j^2(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le r_b \end{aligned} \tag{5.7}$$

for $\ell = 0, 1$ and $k = 0, 1, 2$, where λ_0 and r_b are independent of $j \in \mathbb{N}$. In particular, by (5.7), we have

$$\begin{aligned} &\sum_{k=0}^2 |\lambda|^{k/2}\|\mathbf{v}_j^1\|_{H_q^{2-k}(\mathbb{R}^N)} \le r_b\|\tilde\zeta_j^1\mathbf{f}\|_{L_q(\mathbb{R}^N)}, \\ &\sum_{k=0}^2 |\lambda|^{k/2}\|\mathbf{v}_j^2\|_{H_q^{2-k}(\Omega_j)} \le r_b\{\|\tilde\zeta_j^2\mathbf{f}\|_{L_q(\Omega_j)} + |\lambda|^{1/2}\|\tilde\zeta_j^2\mathbf{g}\|_{L_q(\Omega_j)} + \|\tilde\zeta_j^2\mathbf{g}\|_{H_q^1(\Omega_j)}\}. \end{aligned} \tag{5.8}$$

Let us now introduce the notation

$$\mathbf{U}(\lambda)(\mathbf{f}, \mathbf{g}) = \sum_{i=1,2}\sum_{j=1}^\infty \zeta_j^i\mathbf{v}_j^i, \quad \mathcal{U}(\lambda)F = \sum_{j=1}^\infty \zeta_j^1\mathcal{D}_j^1(\lambda)F_1 + \sum_{j=1}^\infty \zeta_j^2\mathcal{D}_j^2(\lambda)F$$

for $(\mathbf{f}, \mathbf{g}) \in X_q(\Omega)$ and $F = (F_1, F_2, F_3) \in \mathcal{X}_q(\Omega)$. By (5.1), Corollary 20, (5.6) and (5.8), we have $\mathbf{U}(\lambda)(\mathbf{f}, \mathbf{g}) \in H_q^2(\Omega)^N$, $\mathcal{U}(\lambda) \in \operatorname{Hol}(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_q(\Omega), H_q^2(\Omega)^n))$,

$$\begin{aligned} &\sum_{k=0}^2 |\lambda|^{k/2}\|\mathbf{U}(\lambda)(\mathbf{f}, \mathbf{g})\|_{H_q^{2-k}(\Omega)} \le C_q r_b(\|\mathbf{f}\|_{L_q(\Omega)} + |\lambda|^{1/2}\|\mathbf{g}\|_{L_q(\Omega)} + \|\mathbf{g}\|_{H_q^1(\Omega)}), \\ &\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), H_q^{2-k}(\Omega)^n)}(\{(\tau\partial_\tau)^\ell(\lambda^{k/2}\mathcal{U}(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le C_q r_b. \end{aligned} \tag{5.9}$$

Obviously, we have

$$\mathbf{U}(\lambda)(\mathbf{f}, \mathbf{g}) = \mathcal{U}(\lambda)H_\lambda(\mathbf{f}, \mathbf{g}). \tag{5.10}$$

Moreover, noting (5.3) and using (5.4) and (5.5), we have

$$\begin{cases} \lambda R\mathbf{U}(\lambda)(\mathbf{f},\mathbf{g}) - \operatorname{div}(B\nabla\mathbf{U}(\lambda)(\mathbf{f},\mathbf{g})) = \mathbf{f} - \mathbf{V}_0(\lambda)(\mathbf{f},\mathbf{g}) & \text{in } \Omega, \\ B(\nabla\mathbf{U}(\lambda)(\mathbf{f},\mathbf{g})\cdot\mathbf{n}) = \mathbf{g} - \mathbf{V}_b(\lambda)(\mathbf{f},\mathbf{g}) & \text{on } \Gamma. \end{cases} \tag{5.11}$$

In the above we used the fact that $\tilde\zeta^i_j\zeta^i_j = \zeta^i_j$, $\sum_{i=1,2}\sum_{j=1}^\infty \zeta^i_j = 1$, and so

$$\sum_{i=1,2}\sum_{j=1}^\infty \tilde\zeta^i_j\zeta^i_j\mathbf{f} = \sum_{i=1,2}\sum_{j=1}^\infty \zeta^i_j\mathbf{f} = \mathbf{f},$$

and we denoted

$$\begin{aligned} \mathbf{V}_0(\lambda)(\mathbf{f},\mathbf{g}) &= \sum_{i=1,2}\sum_{j=1}^\infty \operatorname{div}(B^{ij}(\nabla\zeta^i_j)\mathbf{v}^i_j) + \sum_{i=1,2}\sum_{j=1}^\infty (\nabla\zeta^i_j)\cdot(B^{ij}\nabla\mathbf{v}^i_j), \\ \mathbf{V}_b(\lambda)(\mathbf{f},\mathbf{g}) &= \sum_{j=1}^\infty B^{2j}(\nabla\zeta^2_j\cdot\mathbf{n}_j)\mathbf{v}^2_j. \end{aligned}$$

Let us also denote

$$\begin{aligned} \mathcal{V}_0(\lambda)F &= \sum_{j=1}^\infty \operatorname{div}(B^{1j}(\nabla\zeta^1_j)\mathcal{D}^1_j(\lambda)(\tilde\zeta^1_jF_1)) + \sum_{j=1}^\infty B^{2j}(\nabla\zeta^2_j)\mathcal{D}^2_j(\lambda)(\tilde\zeta^2_jF) \\ &\quad + \sum_{j=1}^\infty (\nabla\zeta^1_j)\cdot(B^{1j}\nabla\mathcal{D}^1_j(\lambda)(\tilde\zeta^1_jF_1)) + \sum_{j=1}^\infty (\nabla\zeta^2_j)\cdot(B^{2j}\nabla\mathcal{D}^2_j(\lambda)(\tilde\zeta^2_jF)), \\ \mathcal{V}_b(\lambda)F &= \sum_{j=1}^\infty B^{2j}(\nabla\zeta^2_j\cdot\mathbf{n}_j)\mathcal{D}^2_j(\lambda)(\tilde\zeta^2_jF), \end{aligned}$$

for $F = (F_1, F_2, F_3) \in \mathcal{X}_q(\Omega)$. Moreover, we set

$$\mathbf{V}(\lambda)(\mathbf{f},\mathbf{g}) = (\mathbf{V}_0(\lambda)(\mathbf{f},\mathbf{g}), \mathbf{V}_b(\lambda)(\mathbf{f},\mathbf{g})), \quad \mathcal{V}(\lambda)F = (\mathcal{V}_0(\lambda)F, \mathcal{V}_b(\lambda)F).$$

In particular, we have

$$\mathbf{V}(\lambda)(\mathbf{f},\mathbf{g}) = \mathcal{V}(\lambda)H_\lambda(\mathbf{f},\mathbf{g}) \tag{5.12}$$

for any $(\mathbf{f},\mathbf{g}) \in X_q(\Omega)$. By Proposition 4, (5.7), (1.8) and (1.2), we have

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega))}(\{(\tau\partial_\tau)^\ell(H_\lambda\mathcal{V}(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_1}\}) \le CM_0r_b\lambda_1^{-1/2}$$

for $\ell = 0, 1$ and $\lambda_1 \ge \lambda_0$. Thus, choosing λ_0 so large that $CM_0r_b\lambda_1^{-1/2} \le 1/2$, we have

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega))}(\{(\tau\partial_\tau)^\ell(H_\lambda\mathcal{V}(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le 1/2 \tag{5.13}$$

for $\ell = 0, 1$. By (5.12) and (5.13), we have

$$\|H_\lambda\mathbf{V}(\lambda)(\mathbf{f},\mathbf{g})\|_{\mathcal{X}_q(\Omega)} \le (1/2)\|H_\lambda(\mathbf{f},\mathbf{g})\|_{\mathcal{X}_q(\Omega)}. \tag{5.14}$$

The $\|H_\lambda(\mathbf{f},\mathbf{g})\|_{\mathcal{X}_q(\Omega)}$ is equivalent norm to $\|(\mathbf{f},\mathbf{g})\|_{X_q(\Omega)}$ for $\lambda \ne 0$, and therefore, it follows from (5.14) that the inverse operator $(\mathbf{I} - \mathbf{V}(\lambda))^{-1} = \sum_{j=0}^\infty \mathbf{V}(\lambda)^j$ exists in $X_q(\Omega)$. Moreover, by (5.13), the inverse operator $(\mathbf{I} - H_\lambda\mathcal{V}(\lambda))^{-1} = \sum_{j=0}^\infty (H_\lambda\mathcal{V}(\lambda))^j$ exists in $\mathcal{X}_q(\Omega)$. By (5.12),

$$H_\lambda(\mathbf{I} - \mathbf{V}(\lambda))^{-1} = (\mathbf{I} - H_\lambda\mathcal{V}(\lambda))^{-1}H_\lambda. \tag{5.15}$$

In view of (5.11) and (5.10), $\mathbf{v} = \mathbf{U}(\lambda)(\mathbf{I} - \mathbf{V}(\lambda))^{-1}(\mathbf{f}, \mathbf{g})$ is a unique solution of Eq. (1.4) or (1.1). The uniqueness follows from the existence of the dual problem. By (5.10) and (5.15), this $\mathbf{v}$ is represented by $\mathbf{v} = \mathcal{U}(\lambda)(\mathbf{I} - \mathcal{H}\mathcal{V}(\lambda))^{-1}H_\lambda(\mathbf{f}, \mathbf{g})$. Thus, setting $\mathcal{S}(\lambda) = \mathcal{U}(\lambda)(\mathbf{I} - H_\lambda\mathcal{V}(\lambda))^{-1}$, by (5.10), (5.11) and Proposition 4, we see that $\mathbf{v} = \mathcal{S}(\lambda)H_\lambda(\mathbf{f}, \mathbf{g})$ is a unique solution of Eq. (1.1) and

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), H^{2-k}_q(\Omega)^n)}(\{(\tau\partial_\tau)^\ell(\lambda^{k/2}\mathcal{S}(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \leq 2r_b$$

for $\ell = 0, 1$ and $k = 0, 1, 2$. This completes the proof of Theorem 7. □

6 Proof of Theorem 6

To prove the existence part of Theorem 6, we first consider an artificial initial-boundary problem:

$$\partial_t\mathbf{u} - R^{-1}\mathrm{div}\,(B\nabla\mathbf{u}) = 0 \quad \text{in } \Omega \times (0, T), \quad B(\nabla\mathbf{u}\cdot\mathbf{n})|_\Gamma = 0, \quad \mathbf{u}|_{t=0} = \mathbf{u}_0. \tag{6.1}$$

The corresponding resolvent problem of Eq. (6.1) is the following system:

$$\lambda\mathbf{v} - R^{-1}\mathrm{div}\,(B\nabla\mathbf{v}) = \mathbf{f} \quad \text{in } \Omega, \quad B(\nabla\mathbf{v}\cdot\mathbf{n})|_\Gamma = 0. \tag{6.2}$$

If we set

$$\mathbf{D}_q(\Omega) = \{\mathbf{v} \in H^2_q(\Omega)^n \mid B(\nabla\mathbf{v}\cdot\mathbf{n}) = 0 \quad \text{on } \Gamma\},$$
$$\mathbf{A}\mathbf{v} = R^{-1}\mathrm{div}\,(B\nabla\mathbf{v}) \quad \text{for } \mathbf{v} \in \mathbf{D}_q(\Omega),$$

then Eq. (6.2) is written in the form:

$$(\lambda - \mathbf{A})\mathbf{v} = \mathbf{f}. \tag{6.3}$$

Let $\mathcal{S}(\lambda)$ be the $\mathcal{R}$-bounded solution operator given in Theorem 7, then a unique solution of (6.3) is given by $\mathbf{v} = \mathcal{S}(\lambda)(R\mathbf{f}, 0)$. Therefore, by Theorem 7 and (1.3), we have

$$\sum_{k=0}^{2} |\lambda|^{k/2}\|\mathbf{v}\|_{H^{2-k}_q(\Omega)} \leq C_{m_1}r_b\|\mathbf{f}\|_{L_q(\Omega)},$$

for any $\lambda \in \Sigma_{\epsilon,\lambda_0}$ and $\mathbf{f} \in L_q(\Omega)^n$. By the semi-group theory, the operator $\mathbf{A}$ generates an C_0 analytic semigroup $\{T(t)\}_{t\geq 0}$ possessing the estimate:

$$\|T(t)\mathbf{u}_0\|_{L_q(\Omega)} + t\|\partial_t T(t)\mathbf{u}_0\|_{L_q(\Omega)} \leq Ce^{\gamma t}\|\mathbf{u}_0\|_{L_q(\Omega)},$$
$$\|\partial_t T(t)\mathbf{u}_0\|_{L_q(\Omega)} \leq Ce^{\gamma t}\|\mathbf{u}_0\|_{H^2_q(\Omega)},$$

for any $t > 0$ with some constants $\gamma \in \mathbb{R}$ and $C > 0$. Using the real interpolation theorem (cf. Tanabe [37, Subsec. 1.4]) we can prove:

Theorem 21. *Let $1 < p, q < \infty$. Assume that Ω is a uniformly C^2 domain. Let*

$$\mathcal{D}_{q,p}(\Omega) = (L_q(\Omega)^n, \mathbf{D}_q(\Omega))_{1-1/p,p},$$

where $(\cdot,\cdot)_{1-1/p,p}$ is a real interpolation functor ([1, Chapter 7]). Then, for any $\mathbf{u}_0 \in \mathcal{D}_{p,q}(\Omega)$, problem (6.1) admits a unique solution $\mathbf{u}$ with

$$e^{-\gamma t}\mathbf{u} \in H^1_p((0,\infty), L_q(\Omega)^n) \cap L_p((0,\infty), H^2_q(\Omega)^n)$$

possessing the esitmate:

$$\|e^{-\gamma t}\mathbf{u}\|_{L_p((0,\infty),H^2_q(\Omega))} + \|e^{-\gamma t}\partial_t\mathbf{u}\|_{L_p((0,\infty),L_q(\Omega))} \leq C\|\mathbf{u}_0\|_{B^{(2-1/p)}_{q,p}(\Omega)}$$

for any $\gamma > \lambda_0$ with some constant C depending on λ_0 that is the same as in Theorem 7.

Proof. The proof of Theorem 21 follows the same lines as the Theorem 3.9 in [33], so we skip it. ☐

Remark 22. Note that $\mathbf{u}_0 \in B^{2(1-1/p)}_{q,p}(\Omega)^n$ satisfies the condition:

$$B(\nabla \mathbf{u}_0 \cdot \mathbf{n}) = 0 \quad \text{on } \Gamma,$$

then $\mathbf{u}_0 \in \mathcal{D}_{q,p}(\Omega)$ when $2/p + 1/q < 1$. Moreover, when $2/p + 1/q > 1$, than any $\mathbf{u}_0 \in B^{2(1-1/p)}_{q,p}(\Omega)$ belongs to $\mathcal{D}_{q,p}(\Omega)^n$.

We now proceed the existence part of Theorem 6. Let $\mathcal{S}(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_q(\Omega), H^2_q(\Omega)^n))$ be a solution operator of problem (1.14) that exists due to Theorem 7. Let

$$\mathbf{F} \in L_p((0,T), L_q(\Omega)^n), \quad e^{-\gamma t}\mathbf{G} \in L_p(\mathbb{R}, H^1_q(\Omega)^n) \cap H^{1/2}_p(\mathbb{R}, L_q(\Omega)^n).$$

for any $\gamma > \lambda_0$. Let $\mathbf{F}_0$ be the zero extension of $\mathbf{F}$ outside of $(0,T)$, that is $\mathbf{F}_0(\cdot,t) = \mathbf{F}(\cdot,t)$ for $t \in (0,T)$ and $\mathbf{F}_0(\cdot,t) = 0$ for $t \notin (0,T)$. We consider the following time-dependent problem:

$$R\partial_t \mathbf{v} - \mathrm{div}\,(B\nabla \mathbf{v}) = \mathbf{F}_0 \quad \text{in } \Omega \times \mathbb{R}, \quad B(\nabla \mathbf{v} \cdot \mathbf{n}) = \mathbf{G} \quad \text{on } \Gamma \times \mathbb{R}. \tag{6.4}$$

Let $\mathcal{L}$ and $\mathcal{L}^{-1}$ be the Laplace transform and the Laplace inverse transform, that is

$$\begin{aligned}\mathcal{L}[f](\lambda) &= \int_{-\infty}^{\infty} e^{-(\gamma+i\tau)t} f(t)\,dt = \mathcal{F}[e^{-\gamma t} f](\tau) \quad (\lambda = \gamma + i\tau),\\ \mathcal{L}^{-1}[g](t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\gamma + i\tau} t g(\gamma + i\tau)\,d\tau = e^{\gamma t}\mathcal{F}^{-1}_\tau[g(\gamma + i\tau)](t).\end{aligned}$$

Applying Laplace transformation to (6.4), we have

$$\lambda R\mathcal{L}[\mathbf{v}] - \mathrm{div}\,(B\nabla\mathcal{L}[\mathbf{v}]) = \mathcal{L}[\mathbf{F}_0] \quad \text{in } \Omega, \quad B(\nabla\mathcal{L}[\mathbf{v}]\cdot\mathbf{n}) = \mathcal{L}[\mathbf{G}] \quad \text{on } \Gamma.$$

In view of Theorem 7, we have

$$\mathcal{L}[\mathbf{v}] = \mathcal{S}(\lambda)(\mathcal{L}[\mathbf{F}_0](\lambda), \lambda^{1/2}\mathcal{L}[\mathbf{G}](\lambda), \mathcal{L}[\mathbf{G}](\lambda))$$

for $\gamma > \lambda_0$ with $\lambda = \gamma + i\tau \in \mathbb{C}$. Thus, a solution $\mathbf{v}$ of Eq. (6.4) is given by

$$\begin{aligned}\mathbf{v} &= \mathcal{L}^{-1}[\mathcal{S}(\lambda)(\mathcal{L}[\mathbf{F}_0](\lambda), \lambda^{1/2}\mathcal{L}[\mathbf{G}](\lambda), \mathcal{L}[\mathbf{G}](\lambda))](t)\\ &= e^{\gamma t}\mathcal{F}^{-1}_\tau[\mathcal{S}(\gamma + i\tau)\mathcal{F}[e^{-\gamma t}(\mathbf{F}_0, \Lambda^{1/2}_\gamma \mathbf{G}, \mathbf{G})](\tau)](t)\end{aligned}$$

for any $\gamma > \lambda_0$. Here, $\Lambda^{1/2}_\gamma$ is the operator defined by setting

$$\Lambda^{1/2}_\gamma g = \mathcal{L}^{-1}[\lambda^{1/2}\mathcal{L}[g](\lambda)].$$

By the Cauchy theorem in theory of functions of one complex variable, the value of $\mathbf{v}$ is independent of choice of $\gamma > \lambda_0$. By Theorem 7 and Weis's operator valued Fourier multiplier theorem [38], we have

$$\begin{aligned}&\|e^{-\gamma t}\mathbf{v}\|_{L_p((\mathbb{R}, H^2_q(\Omega))} + \|e^{-\gamma t}\partial_t\mathbf{v}\|_{L_p(\mathbb{R}, L_q(\Omega))}\\ &\quad \le C(\|e^{-\gamma t}\mathbf{F}_0\|_{L_p(\mathbb{R}, L_q(\Omega))} + \|e^{-\gamma t}\Lambda^{1/2}_\gamma \mathbf{G}\|_{L_p(\mathbb{R}, L_q(\Omega))} + \|e^{-\gamma t}\mathbf{G}\|_{L_p(\mathbb{R}, H^1_q(\Omega))})\end{aligned}$$

for any $\gamma > \lambda_0$ with some constant C depending on λ_0. Since $|(\tau\partial_\tau)\lambda^{1/2}(1+\tau^2)^{-1/4}| \le C(1+\gamma^{1/2}$ for any $\lambda = \gamma + i\tau \in \mathbb{C}$ with $\gamma > \lambda_0$, by Proposition 4 we have

$$\|e^{-\gamma t}\Lambda_\gamma^{1/2}\mathbf{G}\|_{L_p(\mathbb{R},L_q(\Omega))} \le C(1+\gamma^{1/2})\|e^{-\gamma t}\mathbf{G}\|_{H_p^{1/2}(\mathbb{R},L_q(\Omega))}.$$

Summing up, we have proved that $\mathbf{v}$ satisfies Eq. (6.4) and the estimate:

$$\begin{aligned}&\|\mathbf{v}\|_{L_p(((0,T),H_q^2(\Omega))} + \|\partial_t\mathbf{v}\|_{L_p((0,T),L_q(\Omega))}\\ &\quad\le Ce^{\gamma T}(\|\mathbf{F}\|_{L_p((0,T),L_q(\Omega))} + (1+\gamma^{1/2})\|e^{-\gamma t}\mathbf{G}\|_{H_p^{1/2}(\mathbb{R},L_q(\Omega))} + \|e^{-\gamma t}\mathbf{G}\|_{L_p(\mathbb{R},H_q^1(\Omega))})\end{aligned}$$

for any $\gamma > \lambda_0$ with some constants C depending on λ_0.

Next, to compensate for the lack of the initial condition, we consider the following initial problem:

$$R\partial_t\mathbf{w} - \mathrm{div}\,(B\nabla\mathbf{w}) = 0 \quad \text{in } \Omega\times(0,\infty), \quad B(\nabla\mathbf{w}\cdot\mathbf{n})|_\Gamma = 0, \quad \mathbf{w}|_{t=0} = \mathbf{u}_0 - \mathbf{v}|_{t=0}. \tag{6.5}$$

By (1.10), we see that $\mathbf{u}_0 - \mathbf{v}|_{t=0} \in \mathcal{D}_{q,p}(\Omega)$ when $2/p+1/q \neq 1$, and so, by Theorem 21, problem (6.5) admits a unique solution $\mathbf{w}$ with

$$e^{-\gamma t}\mathbf{w} \in H_p^1((0,\infty),L_q(\Omega)^n) \cap L_p((0,\infty),H_q^2(\Omega)^n)$$

possessing the estimate:

$$\|e^{-\gamma t}\mathbf{w}\|_{L_p((0,\infty),H_q^2(\Omega))} + \|e^{-\gamma t}\partial_t\mathbf{w}\|_{L_p((0,\infty),L_q(\Omega))} \le C\|\mathbf{u}_0 - \mathbf{v}|_{t=0}\|_{B_{q,p}^{2(1-1/p)}(\Omega))},$$

for any $\gamma > \lambda_0$. Again, by the real interpolation theorem we have

$$\|\mathbf{v}|_{t=0}\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \le C(\|e^{-\gamma t}\mathbf{v}\|_{L_p((0,\infty),H_q^2(\Omega))} + \|e^{-\gamma t}\partial_t\mathbf{v}\|_{L_p((0,\infty),L_q(\Omega))})$$

for some $\gamma > \lambda_0$, because $e^{-\gamma t}\mathbf{v}|_{t=0} = \mathbf{v}|_{t=0}$.

Summing up, we have proved that $\mathbf{u} = \mathbf{v}+\mathbf{w}$ is a required solution of Eq. (1.1) or equivalently of (1.4) possessing the estimate (1.12). This completes the proof of of the first part of Theorem 6 devoted to the existence of a solution.

In order to prove the uniqueness of solutions of Eq. (1.1) we now consider $\mathbf{u}$ satisfying the regularity condition (1.11) and the homogeneous system of equations (1.13). Let $\mathbf{u}_0$ be the zero extension of $\mathbf{u}$ to $t < 0$, that is $\mathbf{u}_0(\cdot,t) = \mathbf{u}(\cdot,t)$ for $t \in (0,T)$ and $\mathbf{u}_0(\cdot,t) = 0$ for $t < 0$. We define $\mathbf{v}$ by letting

$$\mathbf{v}(\cdot,t) = \begin{cases}\mathbf{u}_0(\cdot,t) & \text{for } t < T\\ \mathbf{u}_0(\cdot,2T-t) & \text{for } t \ge T.\end{cases}$$

Since $\mathbf{u}|_{t=0} = 0$, we see that

$$\mathbf{v} \in H_p^1(\mathbb{R},L_q(\Omega)^n) \cap L_p(\mathbb{R},H_q^2(\Omega)^n),$$

that $\mathbf{v}$ vanishes for $t \notin (0,2T)$, and that $\mathbf{v}$ satisfies the homogeneous equations:

$$R\partial_t\mathbf{v} - \mathrm{div}\,(B\nabla\mathbf{v}) = 0 \quad \text{in } \Omega\times\mathbf{R}, \quad B(\nabla\mathbf{v}\cdot\mathbf{n})|_\Gamma = 0. \tag{6.6}$$

Applying the Laplace transform to (6.6) yields that

$$\lambda R\mathcal{L}[\mathbf{v}] - \mathrm{div}\,(B\nabla\mathcal{L}[\mathbf{v}]) = 0 \quad \text{in } \Omega, \quad B(\nabla\mathcal{L}[\mathbf{v}]\cdot\mathbf{n})|_\Gamma = 0.$$

Since

$$\|\mathcal{L}[\mathbf{v}](\gamma+i\tau)\|_{H^2_q(\Omega)} \leq \int_0^{2T} e^{\gamma t}\|\mathbf{v}(\cdot,t)\|_{H^2_q(\Omega)}\,dt \leq e^{2\gamma T}(2T)^{1/p'}\|\mathbf{v}\|_{L_p((0,2T),H^2_q(\Omega))}$$
$$\leq 2e^{\gamma T}(2T)^{1/p'}\|\mathbf{u}\|_{L_p((0,T),H^2_q(\Omega))} < \infty,$$

the uniqueness stated in Theorem 7 yields that $\mathcal{L}[\mathbf{v}](\lambda) = 0$ for $\lambda \in \Sigma_{\epsilon,\lambda_0}$. But, $\mathcal{L}[\mathbf{v}](\lambda)$ is holomorphic in $\mathbb{C}$, because $\mathbf{v}$ vanishes for $t \notin (0, 2T)$. Thus, $\mathcal{L}[\mathbf{v}]$ is identically zero, which yields that $\mathbf{v} = 0$. Thus, $\mathbf{u} = 0$. This completes the proof of uniqueness of solutions from Theorem 6. □

7 Proof of Theorem 9

We follow an argument from Section 3 of [35]. First we prove the exponential stability of semigroup $\{T(t)\}_{t\geq 0}$ associated with the problem

$$R\partial_t\mathbf{u} - \mathrm{div}\,(B\nabla\mathbf{u}) = 0 \quad \text{in } \Omega\times(0,\infty), \quad B(\nabla\mathbf{u}\cdot\mathbf{n})|_\Gamma = 0, \quad \mathbf{u}|_{t=0} = \mathbf{u}_0. \tag{7.1}$$

For this purpose, we consider the resolvent problem:

$$\lambda R\mathbf{v} - \mathrm{div}\,(B\nabla\mathbf{v}) = \mathbf{f} \quad \text{in } \Omega, \quad B(\nabla\mathbf{v}\cdot\mathbf{n})|_\Gamma = 0. \tag{7.2}$$

Let us define:

$$\hat{L}_q(\Omega)^n = \{\mathbf{f}\in L_q(\Omega)^n \mid \int_\Omega \mathbf{f}\,dx = 0\},$$
$$\hat{H}^2_q(\Omega)^n = \{\mathbf{v}\in H^2_q(\Omega)^n \mid B(\nabla\mathbf{v}\cdot\mathbf{n})|_\Gamma = 0, \quad \int_\Omega R\mathbf{v}\,dx = 0\}.$$

By Theorem 7, there exists a $\lambda_0 > 0$ such that for any $\lambda\in\Sigma_{\epsilon,\lambda_0}$ and $\mathbf{f}\in L_q(\Omega)^n$, problem (7.2) admits a unique solution $\mathbf{v}\in H^2_q(\Omega)^n$ satisfying:

$$|\lambda|\|\mathbf{v}\|_{L_q(\Omega)} + \|\mathbf{v}\|_{H^2_q(\Omega)} \leq C\|\mathbf{f}\|_{L_q(\Omega)} \tag{7.3}$$

for some constant $C > 0$. In addition, if $\mathbf{f}\in\hat{L}_q(\Omega)^n$, then $\mathbf{v}\in\hat{H}^2_q(\Omega)^n$ when $\lambda\neq 0$. In fact, integrating (7.2) and using the Gauss divergence theorem leads to

$$\lambda\int_\Omega R\mathbf{v}\,dx = 0,$$

which, combined with $\lambda\neq 0$, yields that

$$\int_\Omega R\mathbf{v}\,dx = 0. \tag{7.4}$$

Let $\mathcal{B}$ be an operator acting on $\mathbf{v}\in\hat{H}^2_q(\Omega)^n$ defined by setting $\mathcal{B}\mathbf{v} = \mathrm{div}\,(B\nabla\mathbf{v})$ for $\mathbf{v}\in\hat{H}^2_q(\Omega)^n$. Then $(\lambda R-\mathcal{B})$ is a bijective map from $\hat{H}^2_q(\Omega)^n$ onto $\hat{L}_q(\Omega)^n$ when $\lambda\in\Sigma_{\epsilon,\lambda_0}$. Since Ω is bounded, by the Rellich compactness theorem $(\lambda R-\mathcal{B})^{-1}$ is a compact operator from $L_q(\Omega)^n$ into itself. Thus, by Riesz-Schauder theory, especially Fredholm alternative principle, the injectiveness of $\lambda R-\mathcal{B}$ implies the bijectiveness. Let $\lambda\notin(-\infty,0)$ and let $\mathbf{v}\in\hat{H}^2_q(\Omega)^n$ satisfy the homogeneous equations:

$$\lambda R\mathbf{v} - \mathrm{div}\,(B\nabla\mathbf{v}) = 0 \quad \text{in } \Omega, \quad B(\nabla\mathbf{v}\cdot\mathbf{n})|_\Gamma = 0.$$

Let $2 \leq q < \infty$, and then $\hat{H}^2_q(\Omega) \subset \hat{H}^2_2(\Omega)$. Multiplying the above equation by $\overline{\mathbf{v}}$, with $\overline{\mathbf{v}}$ being the complex conjugate of $\mathbf{v}$, integrating the resulting formula over Ω, and using the Gauss divergence theorem leads to

$$\lambda(R\mathbf{v}, \overline{\mathbf{v}})_\Omega + (B\nabla\mathbf{v}, \overline{\nabla\mathbf{v}})_\Omega = 0, \tag{7.5}$$

where

$$(B\nabla\mathbf{v}, \overline{\nabla\mathbf{v}}) := \sum_{k,l=1}^n B_{kl}(\nabla v_l, \nabla\overline{v_k})_\Omega = \sum_{j,k,l=1}^n (B_k l\partial_{x_j} v_l, \partial_{x_j}\overline{v_k})_\Omega = \sum_{j=1}^n (B\partial_{x_j}\mathbf{v}, \overline{\partial_{x_j}\mathbf{v}})_\Omega.$$

In particular, $(R\mathbf{v}, \overline{\mathbf{v}})_\Omega$ and $(B\nabla\mathbf{v}, \overline{\nabla\mathbf{v}})_\Omega$ are real numbers. Therefore, if $\operatorname{Im}\lambda \neq 0$, taking the imaginary part of (7.5) we have $(R\mathbf{v}, \overline{\mathbf{v}})_\Omega = 0$ which yields that $\|\mathbf{v}\|^2_{L_2(\Omega)} = 0$. Thus, we have $\mathbf{v} = 0$, that is the uniqueness holds. In $\operatorname{Im}\lambda = 0$ then $\operatorname{Re}\lambda \geq 0$ since $\lambda \notin (-\infty, 0)$. Now in order to show uniqueness we take the real part of (7.5) which implies

$$m_1\|v\|^2_{L_2(\Omega)} + \|\nabla\mathbf{v}\|^2_{L_2(\Omega)} \leq 0.$$

Thus, again, $\mathbf{v} = 0$. From these considerations, for $\lambda \notin (-\infty, 0)$, $(\lambda R - \mathcal{B})$ is a bijective map from $\hat{H}^2_q(\Omega)^n$ onto $\hat{L}_q(\Omega)^n$ provided $2 \leq q < \infty$. In the case where $1 < q < 2$, the uniqueness follows from the bijectiveness of the operator $\bar{\lambda}R - \mathcal{B}$ for $2 \leq q < \infty$, and so the operator $(\lambda R - \mathcal{B})$ is also a bijective map from $\hat{H}^2_q(\Omega)^n$ onto $\hat{L}_q(\Omega)^n$. From the standard argument in the theory of C_0 analytic semigroups, we see that for any $\epsilon \in (0, \pi/2)$ the resolvent estimate (7.3) holds for any $\lambda \in \Sigma_\epsilon \cup \{0\}$ with some uniform constant C depending solely on ϵ. From this it follows that there exists a C_0 analytic semigroup $\{T(t)\}_{t\geq 0}$ associated with problem (7.1) possessing the estimate:

$$\|T(t)\mathbf{u}_0\|_{L_q(\Omega)} \leq Me^{-\delta t}\|\mathbf{u}_0\|_{L_q(\Omega)}, \tag{7.6}$$

for any $t > 0$ and $\mathbf{u}_0 \in \hat{L}_q(\Omega)^n$ with some positive constants M and δ.

We now prove Theorem 9. For this purpose, we first consider the shifted equations:

$$\begin{aligned} R(\partial_t\mathbf{w} + \eta\mathbf{w}) - \operatorname{div} B(\nabla\mathbf{w}) &= \mathbf{F} && \text{in } \Omega \times (0, \infty), \\ B(\nabla\mathbf{w} \cdot \mathbf{n}) &= \mathbf{G} && \text{on } \Gamma \times (0, \infty), \\ \mathbf{w}|_{t=0} &= \mathbf{u}_0 && \text{in } \Omega. \end{aligned} \tag{7.7}$$

In view of Theorem 7, there exist a large positive constant η and a positive constant γ_0 such that any solution $\mathbf{w}$ of equations (7.7) satisfies the exponential decay property:

$$\|e^{\gamma t}\mathbf{w}\|_{L_p((0,\infty),H^2_q(\Omega))} + \|e^{\gamma t}\partial_t\mathbf{w}\|_{L_p((0,\infty),L_q(\Omega))} \leq C\mathcal{F}_\gamma \tag{7.8}$$

for any $\gamma \leq \gamma_0$ with some positive constants $C > 0$ and γ_0, where we have set

$$\mathcal{F}_\gamma = \|\mathbf{u}_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)} + \|e^{\gamma t}\mathbf{F}\|_{L_p((0,\infty),L_q(\Omega))} + \|e^{\gamma t}\mathbf{G}\|_{L_p(\mathbb{R},H^1_q(\Omega))} + (1 + \gamma^{1/2})\|e^{\gamma t}\mathbf{G}\|_{H^{1/2}_p(\mathbb{R},L_q(\Omega))}.$$

In fact, $\Sigma_\epsilon + \eta = \{\lambda + \eta \mid \lambda \in \Sigma_\epsilon\} \subset \Sigma_{\epsilon,\lambda_0}$ for any large $\eta > 0$. Repeating the proof of Theorem 6 gives the assertion above.

In particular, conditions (1.19) and (1.20) give that

$$\int_\Omega R(x)\mathbf{w}(x,t)\,dx = 0 \quad \text{for any } t > 0. \tag{7.9}$$

In fact, integrating (7.7) over Ω and using the Gauss divergence theorem implies that

$$\frac{d}{dt}\int_\Omega R\mathbf{w}\,dx + \eta\int_\Omega R\mathbf{w}\,dx = \int_\Omega \mathbf{F}(x,t)\,dx + \int_\Gamma \mathbf{G}(x,t)\,d\sigma = 0$$

for any $t > 0$ because of (1.19). Integrating this formula over $(0,t)$ and using (1.19) give that

$$\int_\Omega R(x)\mathbf{w}(x,t)\,dx = \int_\Omega R(x)\mathbf{u}_0(x)\,dx = 0 \quad \text{for any } t > 0.$$

We now consider the compensation equation:

$$R\partial_t\mathbf{v} - \mathrm{div}\,(B\nabla\mathbf{v}) = -\eta R\mathbf{w} \quad \text{in } \Omega\times(0,\infty), \quad B(\nabla\mathbf{v}\cdot\mathbf{n})|_\Gamma = 0, \quad \mathbf{v}|_{t=0}.$$

Since $\mathbf{w}(x,t) \in \hat{L}_q(\Omega)^n$ for any $t > 0$ as follows from (7.9), by the Duhamel principle, we have

$$\mathbf{v}(\cdot,t) = -\eta\int_0^t T(t-s)(R\mathbf{w})(\cdot,s)\,ds.$$

Choosing γ_0 smaller if necessary, we may assume that $\delta > \gamma_0$, and so by (7.6)

$$\begin{aligned}
\|e^{\gamma t}\mathbf{v}(\cdot,t)\|_{L_q(\Omega)} &\leq M\int_0^t e^{-\delta(t-s)}e^{\gamma(t-s)}e^{\gamma s}\|(R\mathbf{w})(\cdot,s)\|_{L_q(\Omega)}\,ds \\
&\leq M\int_0^t [e^{-(\delta-\gamma_0)(t-s)}]^{1/p'+1/p}e^{\gamma(t-s)}e^{\gamma s}\|(R\mathbf{w})(\cdot,s)\|_{L_q(\Omega)}\,ds \\
&\leq M\Big(\int_0^t e^{-(\delta-\gamma_0)(t-s)}\,ds\Big)^{1/p'}\Big(\int_0^t e^{-(\delta-\gamma_0)(t-s)}(e^{\gamma s}\|(R\mathbf{w})(\cdot,s)\|_{L_q(\Omega)})^p\,ds\Big)^{1/p},
\end{aligned}$$

which, combined with (7.8), yields that

$$\|e^{\gamma t}\mathbf{v}\|_{L_p((0,\infty),L_q(\Omega))} \leq C\mathcal{F}_\gamma \tag{7.10}$$

for any $\gamma \leq \gamma_0$.

Since $\mathbf{v}$ satisfies the shifted equations:

$$R(\partial_t\mathbf{v} + \eta\mathbf{v}) - \mathrm{div}\,(B\nabla\mathbf{v}) = -\eta R\mathbf{w} + \eta R\mathbf{v} \quad \text{in } \Omega\times(0,\infty), \quad B(\nabla\mathbf{v}\cdot\mathbf{n})|_\Gamma = 0, \quad \mathbf{v}|_{t=0} = 0,$$

we have, analogously to (7.8),

$$\|e^{\gamma t}\mathbf{v}\|_{L_p((0,\infty),H^2_q(\Omega))} + \|e^{\gamma t}\partial_t\mathbf{v}\|_{L_p((0,\infty),L_q(\Omega))} \leq C\|e^{\gamma t}(\mathbf{w},\mathbf{v})\|_{L_p((0,\infty),L_q(\Omega))},$$

which, combined with (7.10) and (7.8), yields that

$$\|e^{\gamma t}(\mathbf{v}+\mathbf{w})\|_{L_p((0,\infty),H^2_q(\Omega))} + \|e^{\gamma t}\partial_t(\mathbf{v}+\mathbf{w})\|_{L_p((0,\infty),L_q(\Omega))} \leq C_\gamma$$

for any $\gamma \leq \gamma_0$. Therefore, $\mathbf{u} = \mathbf{v} + \mathbf{w}$ is a required solution, which completes the proof of Theorem 9. □.

Acknowledgement. The authors would like to thank the anonymous referee for suggesting including the exponential decay result (Theorem 9).

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