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ON STRONG DYNAMICS OF COMPRESSIBLE TWO-COMPONENT MIXTURE FLOW*

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Abstract. We investigate a system describing the flow of a compressible two-component mixture. The system is composed of the compressible Navier–Stokes equations coupled with nonsymmetric reaction-diffusion equations describing the evolution of fractional masses. We show the local existence and, under certain smallness assumptions, also the global existence of unique strong solutions in an L_p-L_q framework. Our approach is based on so-called entropic variables which enable us to rewrite the system in a symmetric form. Then, applying Lagrangian coordinates, we show the local existence of solutions applying the L_p-L_q maximal regularity estimate. Next, applying an exponential decay estimate we show that the solution exists globally in time provided the initial data is sufficiently close to some constants. The nonlinear estimates impose restrictions $2 < p < \infty$, $3 < q < \infty$. However, for the purpose of generality, we show the linear estimates for a wider range of p and q.

Key words. compressible Navier–Stokes equations, Maxwell–Stefan equations, gaseous mixtures, regular solutions, maximal regularity, decay estimates

AMS subject classifications. 76N10, 35Q30

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1. Introduction.

The Navier–Stokes–Maxwell–Stefan equations provide a description of the multicomponent reactive flows. The system consists of compressible Navier–Stokes equations for the barycentric velocity and total density as well as the convection-diffusion equations for the constituents of the mixture. The two subsystems are coupled by the form of the pressure in the momentum equation and the form of the fluxes in the species equations. The relation between the diffusion driving forces for the constituents and the diffusion fluxes is called the Maxwell–Stefan equations.

In this paper we are interested in analysis of a simple two-component mixture model with neglect of the heat-conduction and reactivity. The associated system of PDEs reads as follows:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0 & \text{in } \Omega \times (0,T), \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \operatorname{div} \mathbf{S} + \nabla \mathfrak{p} = 0 & \text{in } \Omega \times (0,T), \\ \partial_t \rho_k + \operatorname{div}(\rho_k \mathbf{u}) + \operatorname{div} \mathbf{F}_k = 0 & \text{in } \Omega \times (0,T), \end{cases} \tag{1.1}$$

where ρ denotes the total density of the flow and is a sum of partial densities of the species $\rho = \rho_1 + \rho_2$, $\mathbf{u}$ denotes the velocity vector field, $\mathfrak{p}$ denotes the pressure, $\mathbf{F}_1$, $\mathbf{F}_2$

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denotes the diffusion fluxes for both species, and $\mathbf{S}$ denotes the stress tensor given by

$$\mathbf{S} = \mu \mathbf{D}(\mathbf{u}) + (\nu - \mu)\mathrm{div}\, \mathbf{u}\mathbf{I}, \tag{1.2}$$

where $\mathbf{D}(\mathbf{u}) = \nabla \mathbf{u} + \nabla^\top \mathbf{u}$ is the doubled deformation tensor. We assume the system (1.1) is supplied with the initial and boundary conditions

$$\begin{cases} \mathbf{u} = 0, \quad \mathbf{F}_1 \cdot \mathbf{n} = \mathbf{F}_2 \cdot \mathbf{n} = 0 & \text{on } \Gamma \times (0,T), \\ (\mathbf{u}, \rho_1, \rho_2)|_{t=0} = (\mathbf{u}_0, \rho_{10}, \rho_{20}) & \text{in } \Omega. \end{cases} \tag{1.3}$$

Note that assuming the constraint on the diffusion fluxes $\mathbf{F}_1 + \mathbf{F}_2 = 0$, the species equations, when summed, give the continuity equation. Therefore we have $\rho_1 = \rho - \rho_2$, and so, the unknowns of the system are ρ, $\mathbf{u}$, and one of the partial densities ρ_1 or ρ_2. For the derivation of system (1.1) from the kinetic theory of gases in the general multicomponent, heat-conducting, and reactive case we refer to the monograph of Giovangigli [18].

In this paper we consider the mixture of ideal gases, and therefore the internal pressure of the mixture is determined through the Boyle law

$$\mathfrak{p} = \frac{\rho_1}{m_1} + \frac{\rho_2}{m_2}. \tag{1.4}$$

Above, m_k denotes the molar mass of the species k and for simplicity, we set the gaseous constant equal to 1. We are interested in the case when the pressure essentially depends on the densities of different species, and therefore we assume

$$m_1 \neq m_2.$$

The simplest form of the diffusion fluxes widely used in particular applications is the Fick approximation $\mathbf{F}_k \approx -c\nabla(\frac{\rho_k}{\rho})$ (see [14]). The Fick law states that the flux of a species is proportional to the gradient of the concentration of this species and does not take into account the presence of all other components. However, in the real-word applications the cross-diffusion effects cannot be neglected (see, for example) [6, 45, 44, 1]). This issue can be solved by considering the so-called Maxwell–Stefan equations for multicomponent diffusion. These equations relate the diffusion velocities $\mathbf{V}_i$ defined as $\mathbf{F}_i = \rho_i \mathbf{V}_i$ and the molar and the mass fractions, respectively,

$$X_i = \frac{\mathfrak{p}_i}{\mathfrak{p}}, \quad Y_i = \frac{\rho_i}{\rho},$$

where $\mathfrak{p}_k = \frac{\rho_k}{m_k}$, in the implicit way:

$$\underbrace{\nabla X_i - (Y_i - X_i)\nabla \log \mathfrak{p}}_{:=\mathbf{d}_i} = \sum_{\substack{j=1 \\ j\neq i}}^{n} \left(\frac{X_i X_j}{\mathcal{D}_{ij}}\right)(\mathbf{V}_j - \mathbf{V}_i), \tag{1.5}$$

where $\mathcal{D}_{ij} > 0$ denotes the binary diffusion coefficient, $\mathcal{D}_{ij} = \mathcal{D}_{ji}$. The Maxwell–Stefan system (1.5) was first treated by Giovangigli [16, 17], who used iterative methods to solve these equations, i.e., to find the inverse matrix that allows one to characterize the fluxes as the functions of gradients of concentrations. It was proved that for positive concentrations Maxwell–Stefan relations lead to the following form of the fluxes:

$$\mathbf{F}_k = -\sum_{l=1}^{n} C_{kl}\mathbf{d}_l, \quad k = 1, \ldots, n, \tag{1.6}$$

where C_{kl} are multicomponent flux diffusion coefficients and $\mathbf{d}_l = (d_l^1, d_l^2, d_l^3)$ is the species l diffusion force

$$d_l^i = \nabla_{x_i}\left(\frac{\mathfrak{p}_l}{\mathfrak{p}}\right) + \left(\frac{\mathfrak{p}_l}{\mathfrak{p}} - \frac{\rho_k}{\rho}\right)\nabla_{x_i}\log\mathfrak{p} = \frac{1}{\mathfrak{p}}\left(\nabla_{x_i}\mathfrak{p}_l - \frac{\rho_l}{\rho}\nabla_{x_i}\mathfrak{p}\right), \tag{1.7}$$

appearing in the Maxwell–Stefan equations (1.5). The main properties of the flux diffusion matrix C discussed in [18, Chapter 7] are

$$C\mathcal{Y} = \mathcal{Y}C^T, \quad N(C) = \mathrm{lin}\{\vec{Y}\}, \quad R(C) = U^{\perp}, \tag{1.8}$$

where $\mathcal{Y} = \mathrm{diag}(Y_1, \dots, Y_N)$, $\vec{Y} = (Y_1, \dots, Y_n)^t$, $N(C)$ is the nullspace of C, $R(C)$ is the range of C, $\vec{U} = (1, \dots, 1)^T$, and $U^{\perp}$ is the orthogonal complement of $\mathrm{lin}\{\vec{U}\}$.

In this paper we will use the explicit form (1.6). In the case of two components it reduces to

$$F_1 = -\frac{1}{\mathfrak{p}}\left(\frac{\rho_2}{\rho}\nabla\left(\frac{\rho_1}{m_1}\right) - \frac{\rho_1}{\rho}\nabla\left(\frac{\rho_2}{m_2}\right)\right), \quad F_2 = -F_1. \tag{1.9}$$

Under the assumption (1.6), global in time strong (unique) solutions around the constant equilibrium for the Cauchy problem were proved by Giovangigli in [18]. He introduced the entropic and normal variables to symmetrize the system (1.1) and applied the Kawashima and Shizuta theory [23, 24] for symmetric hyperbolic-parabolic systems of conservation laws. For the local in time existence result to the species mass balances equations in the isobaric, isothermal case we refer to [2] (see also [20]). Later on, Jüngel and Stelzer generalized this result and combined it with the entropy dissipation method to prove the global in time existence of weak solutions [22], still in the case of constant pressure and temperature. For a detailed description of the method and its applicability for a range of models we refer to [21]. For the qualitative and quantitative analysis of the ternary gaseous system together with numerical simulations we refer to [6]. One should note that the constant pressure assumption in (1.5) not only significantly simplifies the cross-diffusion equations but basically decouples the fluid and the reaction-diffusion parts of the system (1.1). Stationary problems for compressible mixtures were considered in [48] under the assumption of Fick law and later in [19, 34, 35] with cross diffusion, however, for equal molar masses. Existence of weak solutions for the mixture of non-Newtonian fluids has been shown in [7]. Let us also mention some results on existence of weak solutions to equations of nonreactive multiphase systems [15, 25]. In these models each constituent has its own velocity vector field, and the part of momentum exchange due to difference of gradient of species densities is neglected. More recently, there have also been a couple of developments devoted to the incompressible model of mixtures, i.e., the model in which the barycentric velocity $\mathbf{u}$ is divergence free, but the partial densities/molar concentrations are not constant. For relevant literature on global in time existence of weak solutions we refer the reader to [26, 8] and to [4] for modelling and existence theory in an L_p-setting. We would also like to mention the theoretical results for the systems describing the compressible reacting electrolytes [10], where the authors prove the existence of global in time weak solutions to the Nernst–Planck–Poisson model originating from the modelling approach developed by Bothe and Dreyer in the previous paper [3]. The classical mixture models in the sense of [18] were studied in the series of papers [49, 50, 29, 30, 31], where the global in time existence of weak solutions was proved without any simplification of (1.7). This was possible thanks to the postulate of the so-called Bresch–Desjardins condition for the viscosity coefficients, which provides an extra estimate of the density gradient and a special form of the pressure. The last restriction was recently removed by Xi and Xie [51].

The global well-posedness in the framework of strong solutions for the compressible Navier–Stokes(–Fourier) system under smallness assumptions on the data is already well investigated; see among others [27] in L_2 framework, [47] in L_p setting with slip boundary condition, or [37, 12] for a free boudary problem. However, for the system coupled with reaction-diffusion equations admitting cross-diffusion the issue of global well posedness of initial-boundary value problems has remained open.

The purpose of this work is to prove the global in time existence of strong solutions to the system (1.1). Our basic observation is that this system enjoys some smoothing effect when written in terms of entropic variables [18]. Its symmetric structure enables us to apply an L_p-L_q maximal regularity estimate to show the local well posedness and exponential decay estimate to show the global well-posedness under additional smallness assumptions. The linear estimates are based on the theory of $\mathcal{R}$-bounded operators (see, for instance, [9], [32], [33], [36]). The symmetrized system is derived in the next section. Afterward we formulate our main results and discuss the structure of the remaining sections.

2. Symmetrization and main results. Since $\mathbf{F}_1$ and $\mathbf{F}_2$ are not independent, we reduce two diffusion equations to one diffusion equation introducing the normal form (see [18, Chapter 8]). Let

$$(h,\rho) = \left(\frac{1}{m_2}\log\rho_2 - \frac{1}{m_1}\log\rho_1\ ,\quad \rho_1+\rho_2\right) := \Psi(\rho_1,\rho_2). \tag{2.1}$$

Noting that $\Psi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R} \times \mathbb{R}_+$ is a bijection, let us denote its inverse by Φ. Computing $\nabla h, \nabla\rho$ from (2.1) and solving the resulting linear system for $\nabla\rho_1, \nabla\rho_2$ we get

$$\begin{aligned}\nabla\rho_1 &= \frac{m_1\rho_1}{m_1\rho_1+m_2\rho_2}\nabla\rho - \frac{m_1\rho_1 m_2\rho_2}{m_1\rho_1+m_2\rho_2}\nabla h,\\ \nabla\rho_2 &= \frac{m_2\rho_2}{m_1\rho_1+m_2\rho_2}\nabla\rho + \frac{m_1\rho_1 m_2\rho_2}{m_1\rho_1+m_2\rho_2}\nabla h.\end{aligned} \tag{2.2}$$

From (2.2) and the third equations in (1.1), we have

$$\begin{aligned}\partial_t h + \mathbf{u}\cdot\nabla h &= \frac{1}{m_2\rho_2}\partial_t\rho_2 - \frac{1}{m_1\rho_1}\partial_t\rho_1 + \frac{1}{m_2\rho_2}\mathbf{u}\cdot\nabla\rho_2 - \frac{1}{m_1\rho_1}\mathbf{u}\cdot\nabla\rho_1\\ &= \frac{1}{m_2\rho_2}\left(-\rho_2\operatorname{div}\mathbf{u} - \operatorname{div}\mathbf{F}_2\right) - \frac{1}{m_1\rho_1}\left(-\rho_1\operatorname{div}\mathbf{u} - \operatorname{div}\mathbf{F}_1\right)\\ &= -\left(\frac{1}{m_2} - \frac{1}{m_1}\right)\operatorname{div}\mathbf{u} - \frac{1}{m_2\rho_2}\operatorname{div}\mathbf{F}_2 + \frac{1}{m_1\rho_1}\operatorname{div}\mathbf{F}_1.\end{aligned}$$

Since $\mathbf{F}_1 = -\mathbf{F}_2$, we have

$$\partial_t h + \mathbf{u}\cdot\nabla h = -\left(\frac{1}{m_1\rho_1} + \frac{1}{m_2\rho_2}\right)\operatorname{div}\mathbf{F}_2 - \left(\frac{1}{m_2} - \frac{1}{m_1}\right)\operatorname{div}\mathbf{u},$$

which leads to

$$\frac{m_1 m_2\rho_1\rho_2}{m_1\rho_1+m_2\rho_2}(\partial_t h + \mathbf{u}\cdot\nabla h) + \frac{(m_1-m_2)\rho_1\rho_2}{m_1\rho_1+m_2\rho_2}\operatorname{div}\mathbf{u} = -\operatorname{div}\mathbf{F}_2. \tag{2.3}$$

Moreover, noting that m_1 and m_2 are positive constants, by (1.9) and (2.2) we have

(2.4)
$$\begin{aligned}-\mathbf{F}_2 = \mathbf{F}_1 &= \frac{1}{\mathfrak{p}}\left(\frac{\rho_1}{\rho m_2}\nabla\rho_2 - \frac{\rho_2}{\rho m_1}\nabla\rho_1\right)\\ &= \frac{1}{\mathfrak{p}}\left\{\left(\frac{\rho_1\rho_2}{\rho(m_1\rho_1+m_2\rho_2)} - \frac{\rho_1\rho_2}{\rho(m_1\rho_1+m_2\rho_2)}\right)\nabla\rho + \frac{m_1\rho_1^2\rho_2+m_2\rho_1\rho_2^2}{\rho(m_1\rho_1+m_2\rho_2)}\nabla h\right\}\\ &= \frac{\rho_1\rho_2}{\mathfrak{p}\rho}\nabla h.\end{aligned}$$

Combining (2.3) and (2.4) formulas gives

$$\frac{m_1m_2\rho_1\rho_2}{m_1\rho_1+m_2\rho_2}(\partial_t h + \mathbf{u}\cdot\nabla h) + \frac{(m_1-m_2)\rho_1\rho_2}{m_1\rho_1+m_2\rho_2}\operatorname{div}\mathbf{u} = \operatorname{div}\left(\frac{\rho_1\rho_2}{\mathfrak{p}\rho}\nabla h\right). \tag{2.5}$$

By (1.4) and (2.2), we have

$$\nabla\mathfrak{p} = \frac{1}{m_1}\nabla\rho_1 + \frac{1}{m_2}\nabla\rho_2 = \frac{\rho}{m_1\rho_1+m_2\rho_2}\nabla\rho + \frac{\rho_1\rho_2(m_1-m_2)}{m_1\rho_1+m_2\rho_2}\nabla h.$$

Inserting this formula into the second equation in (1.1), we obtain

$$\rho(\partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}) - \operatorname{div}\mathbf{S} + \frac{\rho}{m_1\rho_1+m_2\rho_2}\nabla\rho + \frac{\rho_1\rho_2(m_1-m_2)}{m_1\rho_1+m_2\rho_2}\nabla h = 0. \tag{2.6}$$

Concerning the boundary conditions, by (2.4) the condition $\mathbf{F}_1\cdot\mathbf{n}=0$ is transformed to $(\nabla h)\cdot\mathbf{n}=0$. Thus, setting

$$\Sigma_\rho = m_1\rho_1+m_2\rho_2, \quad \rho_0 = \rho_{10}+\rho_{20}, \quad h_0 = \frac{1}{m_2}\log\rho_{20} - \frac{1}{m_1}\log\rho_{10},$$

by (2.5) and (2.6) we have the following equations for ρ, $\mathbf{u}$, and h:

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho\mathbf{u}) = 0 & \text{in } \Omega\times(0,T),\\ \rho(\partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}) - \operatorname{div}\mathbf{S} + \dfrac{\rho}{\Sigma_\rho}\nabla\rho + \dfrac{(m_1-m_2)\rho_1\rho_2}{\Sigma_\rho}\nabla h = 0 & \text{in } \Omega\times(0,T),\\ \dfrac{m_1m_2\rho_1\rho_2}{\Sigma_\rho}(\partial_t h+\mathbf{u}\cdot\nabla h) + \dfrac{(m_1-m_2)\rho_1\rho_2}{\Sigma_\rho}\operatorname{div}\mathbf{u} = \operatorname{div}\left(\dfrac{\rho_1\rho_2}{p\rho}\nabla h\right) & \text{in } \Omega\times(0,T),\\ \mathbf{u}=0, \quad (\nabla h)\cdot\mathbf{n}=0 & \text{on } \Gamma\times(0,T),\\ (\rho,\mathbf{u},h)|_{t=0} = (\rho_0,\mathbf{u}_0,h_0) & \text{in } \Omega. \end{cases} \tag{2.7}$$

To solve (2.7) in the maximal L_p-L_q regularity class, we introduce Lagrange coordinates $\{y\}$. Let $\mathbf{v}(y,t)$ be the velocity field in the Lagrange coordinates and we consider the transformation:

$$x = y + \int_0^t \mathbf{v}(y,s)\,ds. \tag{2.8}$$

Then for any differentiable function f we have

$$\partial_t f(t,\phi(t,y)) = \partial_t f + \mathbf{v}\cdot\nabla_x f. \tag{2.9}$$

Moreover, since

$$\frac{\partial x_i}{\partial y_j} = \delta_{ij} + \int_0^t \frac{\partial v_i}{\partial y_j}(y,s)\,ds, \tag{2.10}$$

where δ_{ij} are Kronecker's delta symbols, assuming

$$\sup_{t\in(0,T)}\int_0^t \|\nabla\mathbf{v}(\cdot,s)\|_{L_\infty(\Omega)}\,ds \le \delta \tag{2.11}$$

with some small positive constant δ, the $N\times N$ matrix $\partial x/\partial y = (\partial x_i/\partial y_j)$ has the inverse

$$\left(\frac{\partial x_i}{\partial y_j}\right)^{-1} = \mathbf{I} + \mathbf{V}^0(\mathbf{k}_\mathbf{v}), \tag{2.12}$$

where $\mathbf{k}_\mathbf{v} = \int_0^t \nabla\mathbf{v}(y,s)\,ds$, $\mathbf{I}$ is the $N\times N$ identity matrix, and $\mathbf{V}^0(\mathbf{k})$ is the $N\times N$ matrix of smooth functions with respect to $\mathbf{k} = (k_{ij} \mid i,j = 1\ldots,N) \in \mathbb{R}^{N^2}$ defined on $|\mathbf{k}| < \delta$ with $\mathbf{V}^0(0) = 0$, where $\mathbf{k}$ are independent variables corresponding to $\mathbf{k}_\mathbf{v}$. We have

$$\nabla_x = (\mathbf{I} + \mathbf{V}^0(\mathbf{k}_\mathbf{v}))\nabla_y, \quad \frac{\partial}{\partial x_i} = \sum_{j=1}^N (\delta_{ij} + V^0_{ij}(\mathbf{k}_\mathbf{v}))\frac{\partial}{\partial y_j}. \tag{2.13}$$

Moreover, as was seen in Ströhmer [42], the map: $x = \Phi(y,t)$ is a bijection from Ω onto Ω, and so setting

$$\mathbf{v}(y,t) = \mathbf{u}(x,t), \quad \eta(y,t) = \rho(x,t), \quad \vartheta(y,t) = h(x,t) \tag{2.14}$$

we see that (2.7) is transformed into the following equations:

$$\begin{cases} \partial_t\eta + \eta\,\mathrm{div}\,\mathbf{v} = R_1(U) \quad \text{in } \Omega\times(0,T), \\ \eta\partial_t\mathbf{v} - \mu\Delta\mathbf{v} - \nu\nabla\mathrm{div}\,\mathbf{v} + \dfrac{\eta}{\Sigma_\rho}\nabla\eta + \dfrac{(m_1-m_2)\rho_1\rho_2}{\Sigma_\rho}\nabla\vartheta = R_2(U) \quad \text{in } \Omega\times(0,T), \\ \dfrac{m_1m_2\rho_1\rho_2}{\Sigma_\rho}\partial_t\vartheta + \dfrac{(m_1-m_2)\rho_1\rho_2}{\Sigma_\rho}\mathrm{div}\,\mathbf{v} - \mathrm{div}\left(\dfrac{\rho_1\rho_2}{\mathfrak{p}\rho}\nabla\vartheta\right) = R_3(U) \quad \text{in } \Omega\times(0,T), \\ \mathbf{v} = 0, \quad (\nabla\vartheta)\cdot\mathbf{n} = R_4(U) \quad \text{on } \Gamma\times(0,T), \\ (\eta,\mathbf{v},\vartheta)|_{t=0} = (\rho_0,\mathbf{u}_0,h_0) \quad \text{in } \Omega. \end{cases} \tag{2.15}$$

Here, $R_1(U)$, $R_2(U)$, $R_3(U)$, and $R_4(U)$ are nonlinear functions with respect to $U = (\eta,\mathbf{v},\vartheta)$, which are given in section 3 below.

Our main results are the following two theorems. The first one concerns the local well-posedness.

THEOREM 2.1. *Let $2 < p < \infty$, $3 < q < \infty$, and $L > 0$. Assume that $2/p + 3/q < 1$ and that Ω is a uniform C^3 domain in $\mathbb{R}^N$ $(N \geq 2)$. Let $\rho_{10}(x)$, $\rho_{20}(x)$, and $\mathbf{u}_0(x)$ be initial data for* (1.1). *Assume that there exist positive numbers a_1 and a_2 for which*

$$a_1 \leq \rho_{10}(x), \;\; \rho_{20}(x) \leq a_2 \quad \text{for any } x \in \overline{\Omega}. \tag{2.16}$$

Let $(h_0(x), \rho_0(x)) = \Psi(\rho_{10}(x), \rho_{20}(x))$. Then, there exists a time $T > 0$ depending on a_1, a_2, and L such that if ρ_{10}, ρ_{20}, $\mathbf{u}_0$, and h_0 satisfy the condition

$$\|\nabla(\rho_{10}, \rho_{20})\|_{L_q(\Omega)} + \|\mathbf{u}_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)} + \|h_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)} \leq L \tag{2.17}$$

and the compatibility condition

$$\mathbf{u}_0|_\Gamma = 0, \quad (\nabla h_0) \cdot \mathbf{n}|_\Gamma = 0, \tag{2.18}$$

then problem (2.15) *admits a unique solution $(\eta, \mathbf{v}, \vartheta)$ with*

$$\eta - \rho_0 \in H^1_p((0, T), H^1_q(\Omega)), \quad \mathbf{v} \in H^1_p((0, T), L_q(\Omega)^3) \cap L_p((0, T), H^2_q(\Omega)^3),$$
$$\vartheta \in H^1_p((0, T), L_q(\Omega)) \cap L_p((0, T), H^2_q(\Omega))$$

possessing the estimates

$$\|\eta - \rho_0\|_{H^1_p((0,T),H^1_q(\Omega))} + \|\partial_t(\mathbf{v}, \vartheta)\|_{L_p((0,T),L_q(\Omega))} + \|(\mathbf{v}, \vartheta)\|_{L_p((0,T),H^2_q(\Omega))} \leq CL,$$
$$a_1 \leq \rho(x, t) \leq 2a_2 + a_1 \quad \text{for } (x, t) \in \Omega \times (0, T), \qquad \int_0^T \|\nabla \mathbf{v}(\cdot, s)\|_{L_\infty(\Omega)}\, ds \leq \delta.$$

Here, C is some constant independent of L and δ is sufficiently small for (2.12) *to hold.*

The second main result gives the global well-posedness.

THEOREM 2.2. *Let $2 < p < \infty$, $3 < q < \infty$, and $L > 0$. Assume that $2/p + 3/q < 1$ and that Ω is a bounded domain whose boundary Γ is a compact C^3 hypersurface. Let ρ_{1*} and ρ_{2*} be any positive numbers and set $(h_*, \rho_*) = \Psi(\rho_{1*}, \rho_{2*}) \in \mathbb{R} \times \mathbb{R}_+$. Then, there exists a small number $\epsilon > 0$ depending on ρ_{1*}, ρ_{2*} such that if the initial data $(\rho_0, \mathbf{u}_0, h_0)$ satisfy the smallness condition*

$$\|(\rho_{10} - \rho_{1*}, \rho_{20} - \rho_{2*})\|_{H^1_q(\Omega)} + \|\mathbf{u}_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)} + \|h_0 - h_*\|_{B^{2(1-1/p)}_{q,p}(\Omega)} \leq \epsilon \tag{2.19}$$

and the compatibility condition (2.18), *then problem* (2.15) *with $T = \infty$ admits a unique solution $(\eta, \mathbf{v}, \vartheta)$ with*

$$\eta \in H^1_p((0, \infty), H^1_q(\Omega)), \quad \mathbf{v} \in H^1_p((0, T), L_q(\Omega)^N) \cap L_p((0, \infty), H^2_q(\Omega)^N),$$
$$\vartheta \in H^1_p((0, \infty), L_q(\Omega)) \cap L_p((0, T), H^2_q(\Omega))$$

possessing the estimates

$$\begin{aligned}
&\|e^{\gamma t}\nabla\eta\|_{L_p((0,\infty),L_q(\Omega))} + \|e^{\gamma t}\partial_t\eta\|_{L_p((0,\infty),H^1_q(\Omega))} \\
&\quad + \|e^{\gamma t}\partial_t(\mathbf{v}, \vartheta)\|_{L_p((0,\infty),L_q(\Omega))} + \|e^{\gamma t}\mathbf{v}\|_{L_p((0,\infty),H^2_q(\Omega))} \\
&\quad + \|e^{\gamma t}\nabla\vartheta\|_{L_p((0,\infty),H^1_q(\Omega))} + \|(\rho_1, \rho_2) - (\rho_{1*}, \rho_{2*})\|_{L_\infty((0,\infty),H^1_q(\Omega))} \leq C\epsilon,
\end{aligned}$$

$$\rho_{i*}/4 \leq \rho_i(x, t) \leq 4\rho_{i*} \quad \text{in } (x, t) \in \Omega \times (0, \infty) \text{ for } i = 1, 2, \qquad \int_0^T \|\nabla \mathbf{v}(\cdot, s)\|_{L_\infty(\Omega)} \leq \delta$$

for some constant $C > 0$ independent of ϵ.

The rest of the paper is organized as follows. In section 3, we derive the formulas $R_i(U)$ $(i=1,\dots,4)$ in the right side of (2.15). In section 4, assuming the maximal L_p-L_q theory for the linearized equations, we prove Theorem 2.1. In section 5, assuming the decay properties of solutions of the linearized equations, we prove Theorem 2.2. In section 6, we prove the maximal L_p-L_q regularity for the linearized equations, and in section 7 we prove the decay theorem for the linearized equations.

Notation. We conclude this section by summarizing the symbols used throughout the paper. For any domain G in $\mathbb{R}^N$, let $L_q(G)$, $H_q^m(G)$, and $B^s_{q,p}(G)$ be the standard Lebesgue, Sobolev, and Besov spaces on G, and let $\|\cdot\|_{L_q(G)}$, $\|\cdot\|_{H_q^m(G)}$, and $\|\cdot\|_{B^s_{q,p}(G)}$ denote their respective norms. Let $(\cdot,\cdot)_{\theta,p}$ and $(\cdot,\cdot)_{[\theta]}$ denote the real interpolation functor and complex interpolation functor, respectively. Note that $B^{m+\theta}_{q,p}(G) = (H^m_q(G), H^{m+1}_q(G))_{\theta,p}$. For a Banach space X with norm $\|\cdot\|_X$, let $X^d = \{(f_1,\dots,f_d) \mid f_i \in X \ (i=1,\dots,d)\}$, and write the norm of X^d as simply $\|\cdot\|_X$, which is defined by $\|f\|_X = \sum_{j=1}^d \|f_j\|_X$ for $f=(f_1,\dots,f_d)\in X^d$. Let

$$
\begin{aligned}
&\mathcal{H}_q(G) = \{F=(f_1,\mathbf{f}_2,f_3) \mid f_1\in H^1_q(G),\ \ \mathbf{f}_2\in L_q(G)^N,\ \ f_3\in L_q(G)\},\\
&\|F\|_{\mathcal{H}_q(G)} = \|f_1\|_{H^1_q(G)} + \|(\mathbf{f}_2,f_3)\|_{L_q(G)} \quad \text{for } F=(f_1,\mathbf{f}_2,f_3)\in\mathcal{H}_q(G),\\
&D_q(G) = \{U=(\zeta,\mathbf{v},\vartheta) \mid \zeta\in H^1_q(G),\ \ \mathbf{v}\in H^2_q(G)^N,\ \ \vartheta\in H^2_q(G)\},\\
&\|U\|_{D_q(\Omega)} = \|\zeta\|_{H^1_q(G)} + \|(\mathbf{v},\vartheta)\|_{H^2_q(G)} \quad \text{for } U=(\zeta,\mathbf{v},\vartheta)\in D_q(G),\\
&D_{p,q}(G) = \{U_0=(\zeta_0,\mathbf{v}_0,\vartheta_0) \mid \zeta_0\in H^1_q(G),\ \ \mathbf{v}_0\in B^{2(1-1/p)}_{q,p}(G)^N,\ \ \vartheta_0\in B^{2(1-1/p)}_{q,p}(G)\},\\
&\|U_0\|_{D_{p,q}(G)} = \|\zeta_0\|_{H^1_q(G)} + \|(\mathbf{v}_0,\vartheta_0)\|_{B^{2(1-1/p)}_{q,p}(G)} \quad \text{for } U_0=(\zeta_0,\mathbf{v}_0,\theta_0)\in D_{p,q}(G).
\end{aligned}
$$

Let $(\mathbf{u},\mathbf{v})_G = \int_G \mathbf{u}\cdot\mathbf{v}\,dx$ and let $(\mathbf{u},\mathbf{v})_{\partial G} = \int_{\partial G}\mathbf{u}\cdot\mathbf{v}\,d\omega$, where $d\omega$ denotes the surface element on ∂G. For $1\le p\le\infty$, $L_p((a,b),X)$ and $H^m_p((a,b),X)$ denote the standard Lebesgue and Sobolev spaces of X-valued functions defined on an interval (a,b), and $\|\cdot\|_{L_p((a,b),X)}$, $\|\cdot\|_{H^m_p((a,b),X)}$ denote their respective norms. Let $H^s_p(\mathbb{R},X)$ be the standard X-valued Bessel potential space and $\|\cdot\|_{H^s_p(\mathbb{R},X)}$ its norm. Let $C^\infty_0(G)$ be the set of all C^∞ functions whose supports are compact and contained in G. For a domain U in $\mathbb{C}$, $\mathrm{Hol}\,(U,\mathcal{L}(X,Y))$ denotes the set of all $\mathcal{L}(X,Y)$-valued holomorphic functions defined on U. Let $\Sigma_\epsilon = \{\lambda\in\mathbb{C}\setminus\{0\} \mid |\arg\lambda|\le\pi-\epsilon\}$ and $\Sigma_{\epsilon,\lambda_0} = \{\lambda\in\Sigma_\epsilon \mid |\lambda|\ge\lambda_0\}$. Moreover, the letter C denotes a generic constant and $C_{a,b,c,\dots}$ denotes that the constant $C_{a,b,c,\dots}$ depends on a, b, $c,\dots$. The value of C and $C_{a,b,c,\dots}$ may change from line to line.

3. Lagrange transformation.

In this section we rewrite all necessary differential operators under the Lagrange transformation (2.8) under the assumption (2.11). This way we obtain the exact form of the right-hand side of (2.15). We have

$$
\operatorname{div}_x = \operatorname{div}_y + \sum_{i,j=1}^n V^0_{ij}(\mathbf{k_v})\frac{\partial v_i}{\partial y_j}, \tag{3.1}
$$

and therefore by (2.9), (2.13), and (2.14), we obtain $(2.15)_1$ with

$$
R_1(U) = -\eta\sum_{i,j=1}^N V^0_{ij}(\mathbf{k_v})\frac{\partial v_i}{\partial y_j}. \tag{3.2}
$$

Here and in the following, we set $U=(\eta,\mathbf{v},\vartheta)$. Now we have to transform second order operators. By (2.13), we have

$$\Delta \mathbf{u} = \sum_{k=1}^{3} \frac{\partial}{\partial x_k}\left(\frac{\partial \mathbf{u}}{\partial x_k}\right) = \sum_{k,\ell,m=1}^{3} \left(\delta_{k\ell} + V^0_{kl}(\mathbf{k_v})\right) \frac{\partial}{\partial y_\ell}\left((\delta_{km} + V^0_{km}(\mathbf{k_v}))\frac{\partial \mathbf{v}}{\partial y_m}\right),$$

and so setting

$$A_{2\Delta}(\mathbf{k})\nabla^2\mathbf{v} = 2\sum_{\ell,m=1} V^0_{k\ell}(\mathbf{k})\frac{\partial^2 \mathbf{v}}{\partial y_\ell \partial y_m} + \sum_{k,\ell,m=1}^{N} V^0_{k\ell}(\mathbf{k})V^0_{km}(\mathbf{k})\frac{\partial^2 \mathbf{v}}{\partial y_\ell \partial y_m},$$

$$A_{1\Delta}(\mathbf{k})\nabla\mathbf{v} = \sum_{\ell,m=1}^{3} (\nabla_{\mathbf{k}} V^0_{\ell m})(\mathbf{k}) \int_0^t (\partial_l \nabla \mathbf{v})\, ds \frac{\partial \mathbf{v}}{\partial y_m} + \sum_{k,\ell,m=1}^{3} V^0_{k\ell}(\mathbf{k})(\nabla_{\mathbf{k}} V^0_{km})(\mathbf{k}) \int_0^t \partial_\ell \nabla \mathbf{v}\, ds \frac{\partial \mathbf{v}}{\partial y_m}$$

we have

$$\Delta \mathbf{u} = \Delta \mathbf{v} + A_{2\Delta}(\mathbf{k_v})\nabla^2\mathbf{v} + A_{1\Delta}(\mathbf{k_v})\nabla\mathbf{v}.$$

And also, by (2.13), we have

$$\frac{\partial}{\partial x_j}\operatorname{div}\mathbf{u} = \sum_{k=1}^{3}(\delta_{jk} + V^0_{jk}(\mathbf{k_v}))\frac{\partial}{\partial y_k}\left(\operatorname{div}\mathbf{v} + \sum_{\ell,m=1}^{3} V^0_{\ell m}(\mathbf{k_v})\frac{\partial v_\ell}{\partial y_m}\right),$$

and so setting

$$A_{2\mathrm{div}\,,j}(\mathbf{k})\nabla^2\mathbf{v} = \sum_{\ell,m=1}^{3} V^0_{\ell m}(\mathbf{k})\frac{\partial^2 v_\ell}{\partial y_m \partial y_j} + \sum_{k=1}^{3} V^0_{jk}(\mathbf{k})\frac{\partial}{\partial y_k}\operatorname{div}\mathbf{v} + \sum_{k,\ell=1}^{3} V^0_{jk}(\mathbf{k})V^0_{\ell m}(\mathbf{k})\frac{\partial^2 v_\ell}{\partial y_k \partial y_m},$$

$$A_{1\mathrm{div}\,,j}(\mathbf{k})\nabla\mathbf{v} = \sum_{\ell,m=1}^{3} (\nabla_{\mathbf{k}} V^0_{\ell m})(\mathbf{k}) \int_0^t \partial_j \nabla \mathbf{v}\, ds \frac{\partial v_\ell}{\partial y_m} + \sum_{k,\ell,m=1}^{3} V^0_{jk}(\mathbf{k})(\nabla_{\mathbf{k}} V^0_{\ell m})(\mathbf{k}) \int_0^t \partial_k \nabla \mathbf{v}\, ds \frac{\partial v_\ell}{\partial y_m},$$

we have

$$\frac{\partial}{\partial x_j}\operatorname{div}\mathbf{u} = \frac{\partial}{\partial y_j}\operatorname{div}\mathbf{v} + A_{2\mathrm{div}\,,j}(\mathbf{k_v})\nabla^2\mathbf{v} + A_{1\mathrm{div}\,,j}(\mathbf{k_v})\nabla\mathbf{v}.$$

By (2.13), we have

$$\frac{\rho}{\Sigma_\rho}\nabla\rho + \frac{(m_1 - m_2)\rho_1\rho_2}{\Sigma_\rho}\nabla h = \frac{\eta}{\Sigma_\rho}(\nabla\eta + \mathbf{V}^0(\mathbf{k_v})\nabla\eta) + \frac{(m_1 - m_2)\rho_1\rho_2}{\Sigma_\rho}\left(\nabla\vartheta + \mathbf{V}^0(\mathbf{k_v})\nabla\vartheta\right).$$

Thus, noting that $\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = \partial_t\mathbf{v}$ and setting

$$\begin{aligned} R_2(U) = {} & \mu A_{2\Delta}(\mathbf{k_v})\nabla^2\mathbf{v} + \mu A_{1\Delta}(\mathbf{k_v})\nabla\mathbf{v} + \nu A_{2\mathrm{div}}(\mathbf{k_v})\nabla^2\mathbf{v} + \nu A_{1\mathrm{div}}(\mathbf{k_v})\nabla\mathbf{v} \\ & - \frac{\eta}{\Sigma_\rho}V^0(\mathbf{k_v})\nabla\eta - \frac{(m_1 - m_2)\rho_1\rho_2}{\Sigma_\rho}\mathbf{V}^0(\mathbf{k_v})\nabla\vartheta, \end{aligned} \tag{3.3}$$

where $A_{i\mathrm{div}}(\mathbf{k})\nabla^i\mathbf{v} = (A_{i\mathrm{div},1}(\mathbf{k})\nabla^i\mathbf{v}, \ldots, A_{i\mathrm{div},N}(\mathbf{k})\nabla^i\mathbf{v})^\top$ $(\nabla^1 = \nabla)$, we have

$$\eta\partial_t\mathbf{v} - \mu\Delta\mathbf{v} - \nu\nabla\mathbf{v} + \frac{\eta}{\Sigma_\rho}\nabla\eta + \frac{(m_1 - m_2)\rho_1\rho_2}{\Sigma_\rho}\nabla\vartheta = R_2(U) \quad \text{in } \Omega\times(0,T).$$

By (2.13), we have

$$\begin{aligned}\mathrm{div}_x\Big(\frac{\rho_1\rho_2}{\mathfrak{p}\rho}\nabla h\Big) &= \frac{\rho_1\rho_2}{\mathfrak{p}\rho}(\Delta\vartheta + A_{2\Delta}\nabla^2(\mathbf{k}_\mathbf{v})\vartheta + A_{1\Delta}(\mathbf{k}_\mathbf{v})\nabla\vartheta)\\ &\quad + \nabla_x\Big(\frac{\rho_1\rho_2}{\mathfrak{p}\rho}\Big)\cdot(\nabla\vartheta + \mathbf{V}^0(\mathbf{k}_\mathbf{v})\nabla\vartheta)\\ &= \mathrm{div}_y\Big(\frac{\rho_1\rho_2}{\mathfrak{p}\rho}\nabla\vartheta\Big) + \frac{\rho_1\rho_2}{\mathfrak{p}\rho}(A_{2\Delta}(\mathbf{k}_\mathbf{v})\nabla^2\vartheta + A_{1\Delta}(\mathbf{k}_\mathbf{v})\nabla\vartheta)\\ &\quad + \big(2\mathbf{V}^0(\mathbf{k}_\mathbf{v}) + (\mathbf{V}^0(\mathbf{k}_\mathbf{v}))^2\big)\,\nabla_y\Big(\frac{\rho_1\rho_2}{\mathfrak{p}\rho}\Big)\nabla\vartheta.\end{aligned}$$

Thus, noting that $\partial_t h + \mathbf{u}\cdot\nabla h = \partial_t\vartheta$ and setting

$$\begin{aligned}R_3(U) &= \frac{\rho_1\rho_2}{\mathfrak{p}\rho}(A_{2\Delta}(\mathbf{k}_\mathbf{v})\nabla^2\vartheta + A_{1\Delta}(\mathbf{k}_\mathbf{v})\nabla\vartheta) + \nabla\Big(\frac{\rho_1\rho_2}{\mathfrak{p}\rho}\Big)\mathbf{V}^0(\mathbf{k}_\mathbf{v})\nabla\vartheta\\ &\quad - \frac{(m_1 - m_2)\rho_1\rho_2}{\Sigma_\rho}\sum_{j,k=1}^{3} V^0_{jk}(\mathbf{k}_\mathbf{v})\frac{\partial v_j}{\partial y_k},\end{aligned} \tag{3.4}$$

we obtain $(2.15)_3$.

Finally, by the Taylor formula we have

$$\mathbf{n}(x) = \mathbf{n}\left(y + \int_0^t \mathbf{v}(y,s)\,ds\right) = \mathbf{n}(y) + \int_0^1(\nabla\mathbf{n})\left(y + \tau\int_0^t \mathbf{v}(y,s)\,ds\right)d\tau\int_0^t\mathbf{v}(y,s)\,ds,$$

and so setting

$$\begin{aligned}R_4(U) &= -\,\mathbf{n}\left(y + \int_0^t \mathbf{v}(y,s)\,ds)\cdot(\mathbf{V}^0(\mathbf{k}_\mathbf{v})\nabla\vartheta\right)\\ &\quad - \left(\int_0^1(\nabla\mathbf{n})(y + \tau\int_0^t\mathbf{v}(y,s)\,ds)\,d\tau\int_0^t\mathbf{v}(y,s)\,ds\right)\cdot\nabla\vartheta,\end{aligned} \tag{3.5}$$

we obtain (2.15).

4. Local well-posedness—Proof of Theorem 2.1.

Let $\rho_{10}(x)$, $\rho_{20}(x)$, and $\mathbf{u}_0(x)$ be initial data for (1.1). Let α_1 and α_2 be postive numbers for which we assume that

$$\alpha_1 \le \rho_{10}(x),\ \ \rho_{20}(x) \le \alpha_2 \ \text{ for any } x\in\overline{\Omega}, \quad \|\nabla(\rho_{10},\rho_{20})\|_{L_r(\Omega)} \le \alpha_2, \tag{4.1}$$

where α_1 and α_2 are some positive constants and $3 < r < \infty$. Let $(h_0(x), \rho_0(x)) = \Psi(\rho_{10}(x), \rho_{20}(x))$, where Ψ is defined in (2.1). Obviously, since $\rho_0(x) = \rho_{10}(x) + \rho_{20}(x)$, we have

$$2\alpha_1 \le \rho_0(x) \le 2\alpha_2, \quad |h_0(x)| \le \alpha_3, \tag{4.2}$$

where $\alpha_3 = (\frac{1}{m_1} + \frac{1}{m_2})(|\log\alpha_1| + |\log\alpha_2|)$. We linearize (2.15) at $(\rho_{10}(x), \rho_{20}(x), 0)$. Let

$$\rho = \rho_0(x) + \zeta, \quad \Sigma^0_\rho(x) = m_1\rho_{10}(x) + m_2\rho_{20}(x), \quad \gamma_1(x) = \frac{\rho_0(x)}{\Sigma^0_\rho(x)},$$
$$\gamma_2(x) = \frac{(m_1 - m_2)\rho_{10}(x)\rho_{20}(x)}{\Sigma^0_\rho(x)}, \quad \gamma_3(x) = \frac{m_1 m_2\rho_{10}(x)\rho_{20}(x)}{\Sigma^0_\rho(x)}, \tag{4.3}$$
$$\gamma_4(x) = \frac{\rho_{10}(x)\rho_{20}(x)}{\mathfrak{p}_0(x)\rho_0(x)}, \quad \mathfrak{p}_0(x) = \frac{\rho_{10}(x)}{m_1} + \frac{\rho_{20}(x)}{m_2}.$$

We then write (2.15) as

$$\begin{cases} \partial_t\zeta + \rho_0 \mathrm{div}\, \mathbf{v} = f_1(U) & \text{in } \Omega\times(0,T), \\ \rho_0\partial_t\mathbf{v} - \mu\Delta\mathbf{v} - \nu\nabla\mathrm{div}\,\mathbf{v} + \gamma_1\nabla\zeta + \gamma_2\nabla\vartheta = \mathbf{f}_2(U) & \text{in } \Omega\times(0,T), \\ \gamma_3\partial_t\vartheta + \gamma_2\mathrm{div}\,\mathbf{v} - \mathrm{div}\,(\gamma_4\nabla\vartheta) = f_3(U) & \text{in } \Omega\times(0,T), \\ \mathbf{v} = 0, \quad (\nabla\vartheta)\cdot\mathbf{n} = g(U) & \text{on } \Gamma\times(0,T), \\ (\zeta, \mathbf{v}, \vartheta)|_{t=0} = (0, \mathbf{u}_0, h_0) & \text{in } \Omega, \end{cases} \tag{4.4}$$

where we have set $U = (\rho, \mathbf{v}, \vartheta)$, $\rho = \rho_0(x) + \zeta$, and

$$\begin{aligned} f_1(U) &= R_1(U) - \zeta\mathrm{div}\,\mathbf{v}, \\ \mathbf{f}_2(U) &= R_2(U) - \zeta\partial_t\mathbf{v} - (\rho_0+\zeta)\left(\frac{1}{\Sigma_\rho} - \frac{1}{\Sigma^0_\rho}\right)\nabla(\rho_0+\zeta) - \frac{\rho_0+\zeta}{\Sigma^0_\rho}\nabla(\rho_0) \\ &\quad - \frac{\zeta}{\Sigma^0_\rho}\nabla\zeta - (m_1-m_2)\left(\frac{\rho_1\rho_2}{\Sigma_\rho} - \frac{\rho_{10}\rho_{20}}{\Sigma^0_\rho}\right)\nabla\vartheta, \\ f_3(U) &= R_3(U) - m_1m_2\left(\frac{\rho_1\rho_2}{\Sigma_\rho} - \frac{\rho_{10}\rho_{20}}{\Sigma^0_\rho}\right)\partial_t\vartheta - (m_1-m_2)\left(\frac{\rho_1\rho_2}{\Sigma_\rho} - \frac{\rho_{10}\rho_{20}}{\Sigma^0_\rho}\right)\mathrm{div}\,\mathbf{v} \\ &\quad + \mathrm{div}\,\left(\left(\frac{\rho_1\rho_2}{\mathfrak{p}\rho} - \frac{\rho_{10}\rho_{20}}{\mathfrak{p}_0\rho_0}\right)\nabla\vartheta\right), \\ g(U) &= R_4(U). \end{aligned} \tag{4.5}$$

To prove the local well-posedness, we use the Banach fixed point theorem and the maximal regularity result for the following equations:

$$\begin{cases} \partial_t\zeta + \rho_0(x)\mathrm{div}\,\mathbf{v} = f_1 & \text{in } \Omega\times(0,T), \\ \rho_0(x)\partial_t\mathbf{v} - \mu\Delta\mathbf{v} - \nu\nabla\mathrm{div}\,\mathbf{v} + \gamma_1(x)\nabla\zeta + \gamma_2(x)\nabla\vartheta = \mathbf{f}_2 & \text{in } \Omega\times(0,T), \\ \gamma_3(x)\partial_t\vartheta + \gamma_2(x)\mathrm{div}\,\mathbf{v} - \mathrm{div}\,(\gamma_4(x)\nabla\vartheta) = f_3 & \text{in } \Omega\times(0,T), \\ \mathbf{v}|_\Gamma = 0, \quad (\nabla\vartheta)\cdot\mathbf{n} = g & \text{on } \Gamma\times(0,T), \\ (\zeta, \mathbf{v}, \vartheta)|_{t=0} = (\zeta_0, \mathbf{v}_0, \vartheta_0) & \text{in } \Omega. \end{cases} \tag{4.6}$$

Here $\gamma_1(x)$, $\gamma_2(x)$, $\gamma_3(x)$, and $\gamma_4(x)$ have been given in (4.3). We assume that $\rho_{10}(x)$, $\rho_{20}(x)$ are uniformly continous functions defined on $\overline{\Omega}$ satisfying (4.1). Then we see immediately that there exist positive constants $\alpha_3 < \alpha_4$ depending on α_1 and α_2 for which

$$\begin{aligned} &\alpha_3 \le \rho_0(x), \gamma_1(x), \gamma_3(x), \gamma_4(x) \le \alpha_4 \ \text{ for } x \in \overline{\Omega}, \\ &\|\nabla(\rho_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4)\|_{L_r(\Omega)} \le \alpha_4. \end{aligned} \tag{4.7}$$

For a Banach space X with norm $\|\cdot\|_X$, let $H^s_p(\mathbb{R}, X)$ be an X valued Bessel potential space of order $s \in (0,1)$ defined by

$$H^s_p(\mathbb{R}, X) = \{f \in L_p(\mathbb{R}, X) \mid \|f\|_{H^s_p(\mathbb{R},X)} < \infty\},$$
$$\|f\|_{H^s_p(\mathbb{R},X)} = \|\mathcal{F}^{-1}[(1+\tau^2)^{s/2}\mathcal{F}[f](\tau)]\|_{L_p(\mathbb{R},X)},$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse formula. The following theorem gives a maximal L_p-L_q regularity estimate for the system (4.6).

Theorem 4.1. *Let $1 < p, q < \infty$, $2/p+1/q \neq 2$, and $2/p+1/q \neq 1$. Assume that Ω is a uniformly C^2 domain. Then, there exists a constant γ_0 for which the following assertion holds. Let*

$$\zeta_0 \in H^1_q(\Omega),\ \ \mathbf{v}_0 \in B^{2(1-1/p)}_{q,p}(\Omega)^3,\ \ \vartheta_0 \in B^{2(1-1/p)}_{q,p}(\Omega),$$
$$f_1 \in L_p((0,T), H^1_q(\Omega)),\ \ \mathbf{f}_2 \in L_p((0,T), L_q(\Omega)^3), \quad f_3 \in L_p((0,T), L_q(\Omega)),$$
$$e^{-\gamma t}g \in L_p(\mathbb{R}, H^1_q(\Omega)) \cap H^{1/2}_p(\mathbb{R}, L_q(\Omega))$$

for any $\gamma \geq \gamma_0$. Assume that $\mathbf{v}_0$ and ϑ_0 satisfy the compatibility conditions:

$$\mathbf{v}_0|_\Gamma = 0 \ \text{ on } \Gamma \text{ for } 2/p+1/q < 2, \quad (\nabla\vartheta_0)\cdot\mathbf{n} = g|_{t=0} \ \text{ on } \Gamma \text{ for } 2/p+1/q < 1.$$

Then, problem (4.6) admits unique solutions ζ, $\mathbf{v}$, and ϑ with

$$\zeta \in H^1_p((0,T), H^1_q(\Omega)), \quad \mathbf{v} \in H^1_p((0,T), L_q(\Omega)^3) \cap L_p((0,T), H^2_q(\Omega)^3),$$
$$\vartheta \in H^1_p((0,T), L_q(\Omega)) \cap L_p((0,T), H^2_q(\Omega))$$

possessing the estimate

$$\begin{aligned}
&\|\zeta\|_{H^1_p((0,T),H^1_q(\Omega))} + \|\partial_t(\mathbf{v},\vartheta)\|_{L_p((0,T),L_q(\Omega)^N)} + \|(\mathbf{v},\vartheta)\|_{L_p((0,T),H^2_q(\Omega))}\\
&\quad\leq C_\gamma e^{\gamma T}\{\|\rho_0\|_{H^1_q(\Omega)} + \|(\mathbf{v}_0,\vartheta_0)\|_{B^{2(1-1/p)}_{q,p}(\Omega)} + \|(f_1,\mathbf{f}_2,f_3)\|_{L_p((0,T),L_q(\Omega))}\\
&\qquad + \|e^{-\gamma t}g\|_{L_p(\mathbb{R},H^1_q(\Omega))} + \|e^{-\gamma t}g\|_{H^{1/2}_p(\mathbb{R},L_q(\Omega))}\}
\end{aligned}$$

for any $\gamma \geq \gamma_0$, where C is a constant depending on γ.

Remark 4.2. All the constants appearing in Theorem 4.1 depend on α_1 and α_2.

Postponing the proof of Theorem 4.1, we prove Theorem 2.1. Let $\mathcal{H}_{T,M}$ be the underlying space for our fixed point argument, which is defined by

$$\begin{aligned}
&\mathcal{H}_{T,M}\\
&= \{(\zeta,\mathbf{v},\vartheta) \mid \zeta \in H^1_p((0,T), H^1_q(\Omega)), \quad \mathbf{v} \in H^1_p((0,T), L_q(\Omega)^3) \cap L_p((0,T), H^2_q(\Omega)^3),\\
&\vartheta \in H^1_p((0,T), L_q(\Omega)) \cap L_p((0,T), H^2_q(\Omega)), \quad (\zeta,\mathbf{v},\vartheta)|_{t=0} = (0,\mathbf{u}_0,h_0) \quad \text{in } \Omega,\\
&[\zeta,\mathbf{v},\vartheta]_T = \|\zeta\|_{H^1_p((0,T),H^1_q(\Omega))} + \|\partial_t(\mathbf{v},\vartheta)\|_{L_p((0,T),L_q(\Omega))} + \|(\mathbf{v},\vartheta)\|_{L_p((0,T),H^2_q(\Omega))} \leq M\}.
\end{aligned} \tag{4.8}$$

Here, T and M are positive constants determined later. Since T will be chosen a positive small number eventually, we may assume that $0 < T \leq 1$.

Remark 4.3. In order to apply a fixed point argument combined with Theorem 4.1 we have to show that the nonlinearities $f_i(U), g(U)$ defined in (4.5) with $U \in \mathcal{H}_{T,M}$ satisfy the regularity assumptions required on the right-hand side in Theorem 4.1. Most of this section is devoted to the proof of these nonlinear estimates which are given by (4.29), (4.30), (4.31), and (4.37) below.

First, by Sobolev's inequality and Hölder's inequality, we have

$$\int_0^T \|\nabla \mathbf{v}(\cdot,t)\|_{L_\infty(\Omega)}\,dt \le C\int_0^T \|\mathbf{v}(\cdot,t)\|_{H^2_q(\Omega)}\,dt \le T^{1/p'}\bigg(\int_0^T \|\mathbf{v}(\cdot,t)\|^p_{H^2_q(\Omega)}\,dt\bigg)^{1/p} \le MT^{1/p'}.$$

Thus, choosing $T>0$ so small that $MT^{1/p'}\le\delta$, we may assume that the condition (2.11) holds for any $(\zeta,\mathbf{v},\vartheta)\in\mathcal{H}_{T,M}$. Let

$$\mathcal{I} = \|\nabla\rho_0\|_{H^1_q(\Omega)} + \|(\mathbf{v},h_0)\|_{B^{2(1-1/p)}_{q,p}(\Omega)},$$

and then by (2.17) we have

$$\mathcal{I}\le L \tag{4.9}$$

because $\rho_0(x)=\rho_{10}(x)+\rho_{20}(x)$. Let Ψ be the map defined in (2.1), which is a C^∞ diffeomorphism from $\mathbb{R}_+\times\mathbb{R}_+$ onto $\mathbb{R}\times\mathbb{R}_+$. Let Φ be its inverse map. Let $(\omega,\mathbf{w},\varphi)\in\mathcal{H}_{T,M}$, let $U=(\rho_0(x)+\omega,\mathbf{w},\varphi)$, and let $(\rho_1,\rho_2)=\Phi(\varphi,\rho_0+\omega)$. Since $(\omega,\mathbf{w},\varphi)|_{t=0}=(0,\mathbf{u}_0,h_0)$, we have

$$(\rho_{10}(x),\rho_{20}(x)) = \Phi(\varphi,\rho_0(x)+\omega)|_{t=0}. \tag{4.10}$$

Let $R_i(U)$ be functions given in (3.2), (3.3), (3.4), and (3.5), where η, $\mathbf{v}$ $(v_1,\dots,v_N)^\top$, and ϑ are replaced by $\rho_0+\omega$, $\mathbf{w}=(w_1,\dots,w_N)^\top$, and φ. Let $(\zeta,\mathbf{v},\vartheta)$ be a solution of (4.6) with $\zeta_0=0, \mathbf{v}_0=\mathbf{u}_0$, $\vartheta_0=h_0$, $f_1=f_1(U)$, $\mathbf{f}=\mathbf{f}_2(U)$, $f_3=f_3(U)$, and $g=g(U)$, where ζ, $\mathbf{v}$, and ϑ are replaced by ω, $\mathbf{w}$, and φ, respectively.

First, we estimate $f_1=f_1(U)$, $\mathbf{f}=\mathbf{f}_2(U)$, $f_3=f_3(U)$ and $g=g(U)$. Notice that

$$\begin{aligned} &\sup_{t\in(0,T)}\|\omega(\cdot,t)\|_{H^1_q(\Omega)}\le T^{1/p'}M\le M,\\ &\sup_{t\in(0,T)}\|\varphi(\cdot,t)-h_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)} + \sup_{t\in(0,T)}\|\mathbf{w}(\cdot,t)-\mathbf{u}_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)}\le C(M+L). \end{aligned} \tag{4.11}$$

In fact, since $\omega(\cdot,0)=0$, we have

$$\|\omega(\cdot,t)\|_{H^1_q(\Omega)}\le\int_0^t\|\partial_t\omega)(\cdot,s)\|_{H^1_q(\Omega)}\,ds\le T^{1/p'}\|\partial_t\omega\|_{L_p((0,T),H^1_q(\Omega))}\le T^{1/p'}M\le M,$$

where we have used the fact that $T\le1$ in the last step. To prove the bound for the second term in (4.11), we use the extension map e_T defined by

$$e_T[f](\cdot,t)=\begin{cases}0, & t<0,\\ f(\cdot,t), & 0<t<T,\\ f(\cdot,2T-t), & T<t<2T,\\ 0, & t>2T.\end{cases} \tag{4.12}$$

Obviously, $e_T[f](\cdot,t)=f(\cdot,t)$ for $t\in(0,T)$. If $f|_{t=0}=0$, then we have

$$\partial_t e_T[f](\cdot,t)=\begin{cases}0, & t<0,\\ (\partial_t f)(\cdot,t), & 0<t<T,\\ -(\partial_t f)(\cdot,2T-t), & T<t<2T,\\ 0, & t>2T.\end{cases} \tag{4.13}$$

Let X and Y be two Banach spaces such that X is a dense subset of Y and $X \subset Y$ is continuous, and then we know (cf. [43, p. 10]) that

$$H^1_p((0,\infty),Y) \cap L_p((0,\infty),X) \subset C([0,\infty),(X,Y)_{1/p,p}) \tag{4.14}$$

and

$$\sup_{t\in(0,\infty)} \|u(t)\|_{(X,Y)_{1/p,p}} \le (\|u\|^p_{L_p((0,\infty),X)} + \|u\|^p_{H^1_p((0,\infty),Y)})^{1/p} \tag{4.15}$$

for each $p \in (1,\infty)$. Applying this fact and using (4.12) and (4.13), we have

$$\begin{aligned}
&\sup_{t\in(0,T)} \|\varphi(\cdot,t) - h_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)} \le \sup_{t\in(0,\infty)} \|e_T[\varphi - h_0]\|_{B^{2(1-1/p)}_{q,p}(\Omega)}\\
&\quad = (\|e_T[\varphi-h_0]\|^p_{L_p((0,\infty),H^2_q(\Omega))} + \|e_T[\varphi-h_0]\|^p_{H^1_p((0,\infty),L_q(\Omega))})^{1/p}\\
&\quad \le C(\|\varphi - h_0\|_{L_p((0,\infty),H^2_q(\Omega))} + \|\partial_t\varphi\|_{L_p((0,T),L_q(\Omega))}) \le C(M + T^{1/p}L) \le C(M+L).
\end{aligned}$$

Here and in the following, C denotes a generic constant independent of M, L, and T. C depends at most on a_1 and a_2, for which (2.16) holds. Analogously, we have the third inequality in (4.11).

Since $2/p + 3/q < 1$, we have $1 + 3/q < 2(1-1/p)$, and so by Sobolev's imbedding theorem and (4.11) we have

$$\|(\varphi, \mathbf{w})\|_{L_\infty((0,T),H^1_\infty(\Omega))} \le CM. \tag{4.16}$$

Since $\rho_0(x) = \rho_{10}(x) + \rho_{20}(x)$, by (2.16) we have

$$2a_1 \le \rho_0(x) \le 2a_2 \quad \text{for } x \in \Omega. \tag{4.17}$$

If we choose $T > 0$ so small that $T^{1/p'}M \le a_1$, by (4.17) and (4.11), we have

$$a_1 \le \rho_0(x) + \omega \le 2a_2 + a_1 \tag{4.18}$$

for all $(x,t) \in \Omega \times (0,T)$. Since Φ is a C^∞ diffeomorphism from $\mathbb{R}\times\mathbb{R}_+$ onto $\mathbb{R}_+\times\mathbb{R}_+$, for any compact set $A \subset \mathbb{R}\times\mathbb{R}_+$ $\Phi(A)$ is a compact set in $\mathbb{R}_+\times\mathbb{R}_+$, and so by (4.18) and (4.16), there exist positive constants a_4 and a_5 depending on a_1, a_2, and M for which

$$a_4 \le \rho_1(x,t), \rho_2(x,t) \le a_5 \quad \text{for } (x,t) \in \Omega \times (0,T). \tag{4.19}$$

We now prove that

$$\|(\rho_1,\rho_2) - (\rho_{10},\rho_{20})\|_{L_\infty((0,T),H^1_q(\Omega))} \le C(L+M)T^{\theta/p'} \tag{4.20}$$

$\theta \in (0,1)$. By (4.10) we have

$$\begin{aligned}
&\sup_{t\in(0,T)} \|(\rho_1(\cdot,t),\rho_2(\cdot,t)) - (\rho_{10}(\cdot),\rho_{20}(\cdot))\|_{L_q(\Omega)}\\
&\quad \le \int_0^T \|\partial_t\Phi(\varphi(\cdot,t),\rho_0(\cdot)+\omega(\cdot,t))\|_{L_q(\Omega)}\,dt\\
&\quad \le \int_0^T \|\Phi'(\varphi(\cdot,t),\rho_0(\cdot)+\omega(\cdot,t))\|_{L_\infty(\Omega)}\|(\partial_t\varphi(\cdot,t),\partial_t\omega(\cdot,t))\|_{L_q(\Omega)}\,dt.
\end{aligned} \tag{4.21}$$

By (4.16) and (4.18), we have

$$\sup_{t\in(0,T)} \|\Phi'(\varphi(\cdot,t),\rho_0(\cdot)+\omega(\cdot,t))\|_{L_\infty(\Omega)} \leq a_6 \tag{4.22}$$

for some positive constant a_6 depending on a_1, a_2, M but independent of T. Thus, by (4.21) we have

$$\begin{aligned}
&\sup_{t\in(0,T)} \|(\rho_1(\cdot,t),\rho_2(\cdot,t))-(\rho_{10}(\cdot),\rho_{20}(\cdot))\|_{L_q(\Omega)} \\
&\quad\leq a_6\int_0^T \|(\partial_t\varphi(\cdot,t),\partial_t\omega(\cdot,t))\|_{L_q(\Omega)}\,dt \\
&\quad\leq a_6 T^{1/p'}\|\partial_t(\varphi,\omega)\|_{L_p((0,T),L_q(\Omega))} \leq a_6 M T^{1/p'}.
\end{aligned}\tag{4.23}$$

Moreover, by (2.17) and (4.9) we have

$$\begin{aligned}
&\|\nabla(\rho_1(\cdot,t),\rho_2(\cdot,t))-(\rho_{10}(\cdot),\rho_{20}(\cdot))\|_{L_q(\Omega)} \\
&\quad\leq \|\Phi'(\varphi(\cdot,t),\rho_0(\cdot)+\omega(\cdot,t)\|_{L_\infty(\Omega)}\|(\nabla\varphi(\cdot,t),\nabla\rho_0(\cdot)+\nabla\omega(\cdot,t))\|_{L_q(\Omega)} \\
&\qquad+\|\nabla(\rho_{10},\rho_{20})\|_{L_q(\Omega)} \\
&\quad\leq a_6(\|\nabla\varphi(\cdot,t)\|_{L_q(\Omega)}+\|\nabla\omega(\cdot,t)\|_{L_q(\Omega)})+a_6\|\nabla\rho_0\|_{L_q(\Omega)}+\|\nabla(\rho_{10},\rho_{20})\|_{L_q(\Omega)}.
\end{aligned}$$

Thus, by (4.11)

$$\sup_{t\in(0,T)} \|\nabla\{(\rho_1(\cdot,t),\rho_2(\cdot,t))-(\rho_{10}(\cdot),\rho_{20}(\cdot))\}\|_{L_q(\Omega)} \leq C(L+M). \tag{4.24}$$

Since $W_q^{3/q+\epsilon}(\Omega)\subset L_\infty(\Omega)$ with some small ϵ for which $3/q+\epsilon<1$ and this inclusion is continuous as follows from Sobolev's imbedding theorem, by real interpolation theorem

$$\begin{aligned}
&\sup_{t\in(0,T)} \|(\rho_1(\cdot,t),\rho_2(\cdot,t))-(\rho_{10}(\cdot),\rho_{20}(\cdot))\|_{L_\infty(\Omega)} \\
&\quad\leq \Big(\sup_{0\in(0,T)} \|(\rho_1(\cdot,t),\rho_2(\cdot,t))-(\rho_{10}(\cdot),\rho_{20}(\cdot))\|_{L_q(\Omega)}\Big)^\theta \\
&\qquad\times\Big(\sup_{0\in(0,T)} \|(\rho_1(\cdot,t),\rho_2(\cdot,t))-(\rho_{10}(\cdot),\rho_{20}(\cdot))\|_{H_q^1(\Omega)}\Big)^{1-\theta} \leq C(M+L)T^{\theta/p'}
\end{aligned}\tag{4.25}$$

with $\theta=1-(3/q+\epsilon)\in(0,1)$. By (4.25), (4.19), and (2.16), we have

$$\begin{aligned}
&\Big\|\frac{1}{\Sigma_\rho}-\frac{1}{\Sigma_\rho^0}\Big\|_{L_\infty((0,T),L_\infty(\Omega))}+\Big\|\frac{\rho_1\rho_2}{\Sigma_\rho}-\frac{\rho_{10}\rho_{20}}{\Sigma_\rho^0}\Big\|_{L_\infty((0,T),L_\infty(\Omega))} \\
&\quad+\Big\|\frac{\rho_1\rho_2}{\mathfrak{p}\rho}-\frac{\rho_{10}\rho_{20}}{\mathfrak{p}_0\rho_0}\Big\|_{L_\infty((0,T),L_\infty(\Omega))} \leq C(M+L)T^{\theta/p'}.
\end{aligned}\tag{4.26}$$

Moreover, by (4.24) we have

$$\sup_{t\in(0,T)} \|\nabla(\rho_1(\cdot,t),\rho_2(\cdot,t))\|_{L_q(\Omega)} \leq C(L+M),$$

and so by (4.19) and (2.16) we get

$$\Big\|\nabla\Big(\frac{\rho_1\rho_2}{\mathfrak{p}\rho}-\frac{\rho_{10}\rho_{20}}{\mathfrak{p}_0\rho_0}\Big)\Big\|_{L_\infty((0,T),L_q(\Omega))} \leq C(M+L). \tag{4.27}$$

Using (4.16), (4.11), (4.26), and (4.27), we conclude

$$
\begin{aligned}
&\Big\|(\rho_0+\omega)\Big(\frac{1}{\Sigma_\rho}-\frac{1}{\Sigma_\rho^0}\Big)\nabla(\rho_0+\omega)\Big\|_{L_p((0,T),L_q(\Omega))}\\
&\quad\le C\|\rho_0+\omega\|^2_{L_\infty((0,T),H^1_q(\Omega))}T^{1/p}(M+L)T^{\theta/p'}\le C(M+L)^3T^{(1/p+\theta/p')};\\
&\Big\|\frac{\rho_0+\omega}{\Sigma_\rho^0}\nabla\rho_0\Big\|_{L_p((0,\infty),L_q(\Omega))}\le C(M+L)LT^{1/p};\\
&\Big\|\frac{\omega}{\Sigma_\rho^0}\nabla\omega\Big\|_{L_p((0,T),L_q(\Omega))}\le C\|\omega\|_{L_\infty((0,T),H^1_q(\Omega))}T^{1/p}\le CL^2T^{1/p};\\
&\Big\|\Big(\frac{\rho_1\rho_2}{\Sigma_\rho}-\frac{\rho_{10}\rho_{20}}{\Sigma_\rho^0}\Big)\nabla\varphi\Big\|_{L_p((0,T),L_q(\Omega))}\le C(M+L)LT^{(\theta/p'+1/p)};\\
&\Big\|\Big(\frac{\rho_1\rho_2}{\Sigma_\rho}-\frac{\rho_{10}\rho_{20}}{\Sigma_\rho^0}\Big)\partial_t\omega\Big\|_{L_p((0,T),L_q(\Omega))}\le C(M+L)T^{\theta/p'}\|\partial_t\omega\|_{L_p((0,T),L_q(\Omega))}\\
&\quad\le CM(M+L)T^{\theta/p'};\\
&\Big\|\Big(\frac{\rho_1\rho_2}{\Sigma_\rho}-\frac{\rho_{10}\rho_{20}}{\Sigma_\rho^0}\Big)\mathrm{div}\,\mathbf{w}\Big\|_{L_p((0,T),L_q(\Omega))}\\
&\le C(M+L)T^{\theta/p'}T^{1/p}\|\mathbf{w}\|_{L_\infty((0,T),H^1_q(\Omega))}\le C(M+L)^2T^{(\theta/p'+1/p)};\\
&\Big\|\mathrm{div}\,\Big(\Big(\frac{\rho_1\rho_2}{\mathfrak{p}\rho}-\frac{\rho_{10}\rho_{20}}{\mathfrak{p}_0\rho_0}\Big)\nabla\varphi\Big)\Big\|_{L_p((0,T),L_q(\Omega))}\le C(M+L)T^{\theta/p'}\|\varphi\|_{L_p((0,T),H^2_q(\Omega))}\\
&+\Big\|\nabla\Big(\frac{\rho_1\rho_2}{\mathfrak{p}\rho}-\frac{\rho_{10}\rho_{20}}{\mathfrak{p}_0\rho_0}\Big)\Big\|_{L_\infty((0,T),L_q(\Omega))}\|\nabla\varphi\|_{L_\infty((0,T),L_\infty(\Omega))}T^{1/p}\\
&\quad\le C(M(M+L)T^{\theta/p'}+(M+L)^2T^{1/p}).
\end{aligned}
\tag{4.28}
$$

Next, we estimate nonlinear terms from the Lagrange transformation. In (4.4), we set $U=(\omega,\mathbf{w},\varphi)$. Recall that $3<q<\infty$. By Sobolev's inequality and (4.11), we have

$$\|\omega\mathrm{div}\,\mathbf{w}\|_{H^1_q(\Omega)}\le C\|\omega\|_{H^1_q(\Omega)}\|\mathbf{w}\|_{H^2_q(\Omega)}\le CT^{1/p'}M\|\mathbf{w}\|_{H^2_q(\Omega)},$$

and so we have

$$\|\omega\mathrm{div}\,\mathbf{w}\|_{L_p((0,T),H^1_q(\Omega))}\le CT^{1/p'}M\|\mathbf{w}\|_{L_p((0,T),H^2_q(\Omega))}\le CT^{1/p'}M^2.$$

Replacing $\mathbf{v}$ by $\mathbf{w}$ in (3.2), by Sobolev's inequality and (4.11), we have

$$
\begin{aligned}
\|R_1\|_{H^1_q(\Omega)}&\le C(\|\rho_0\|_{H^1_q(\Omega)}+\|\omega\|_{H^1_q(\Omega)})\int_0^t\|\mathbf{w}(\cdot,s)\|_{H^2_q(\Omega)}\,ds\|\mathbf{w}(\cdot,t)\|_{H^2_q(\Omega)}\\
&\le C(L+M)T^{1/p'}\|\mathbf{w}\|_{L_p((0,T),H^2_q(\Omega))}\|\mathbf{w}(\cdot,t)\|_{H^2_q(\Omega)},
\end{aligned}
$$

and so we have

$$\|R_1\|_{L_p((0,T),H^1_q(\Omega))}\le C(L+M)M^2T^{1/p'}.$$

Thus, we obtain

$$\|f_1(U)\|_{L_p((0,T),H^1_q(\Omega))}\le C(M^2+(L+M)M^2)T^{1/p'}.\tag{4.29}$$

Next, we consider $\mathbf{f}_2(U)$. By (4.16), we have

$$\Big\| \int_0^t \nabla \mathbf{w}(\cdot, s)\, ds \nabla^2 \mathbf{w}(\cdot, t) \Big\|_{L_q(\Omega)} \le T \|\nabla \mathbf{w}\|_{L_\infty(0,T), L_\infty(\Omega)} \|\nabla^2 \mathbf{w}(\cdot, t)\|_{L_q(\Omega)} \le CMT \|\nabla^2 \mathbf{w}(\cdot, t)\|_{L_q(\Omega)},$$

and therefore

$$\Big\| \int_0^t \nabla \mathbf{w}(\cdot, s)\, ds \nabla^2 \mathbf{w}(\cdot, t) \Big\|_{L_p((0,T), L_q(\Omega))} \le CTML.$$

By Hölder's inequality and (4.16), we also get

$$\Big\| \int_0^t \nabla^2 \mathbf{w}(\cdot, s)\, ds \nabla \mathbf{w}(\cdot, t) \Big\|_{L_q(\Omega)} \le T^{1/p'} \left(\int_0^T \|\nabla^2 \mathbf{w}(\cdot, t)\|_{L_q(\Omega)}^p \right)^{1/p} \|\nabla \mathbf{w}(\cdot, t)\|_{L_\infty(\Omega)} \le CMT^{1/p'} \|\mathbf{w}\|_{L_p((0,T), H_q^2(\Omega))} \le CTML.$$

In this way, setting $\mathbf{k_w} = \int_0^t \nabla \mathbf{w}\, ds$, we have

$$\|(A_{2\Delta}(\mathbf{k_w})\nabla^2 \mathbf{w}, A_{1\Delta}(\mathbf{k_w})\nabla \mathbf{w}, A_{2\mathrm{div}}(\mathbf{k_w})\nabla^2 \mathbf{w}, A_{1\mathrm{div}}(\mathbf{k_w})\nabla \mathbf{w})\|_{L_p((0,T), L_q(\Omega))} \le CTLM.$$

By (4.11), (4.18), (4.19), and Sobolev's inequality we obtain

$$\begin{aligned} &\Big\| \frac{\rho_0 + \omega}{\Sigma_\rho} \mathbf{V}^0(\mathbf{k_w}) \nabla(\rho_0 + \omega) \Big\|_{L_q(\Omega)} \\ &\le C \int_0^T \|\nabla \mathbf{w}(\cdot, s)\|_{H_q^1(\Omega)}\, ds (\|\nabla \rho_0\|_{L_q(\Omega)} + \|\nabla \omega(\cdot, t)\|_{L_q(\Omega)}) \\ &\le CT^{1/p'} \|\mathbf{w}\|_{L_p((0,T), H_q^2(\Omega))} (L + \|\nabla \omega(\cdot, t)\|_{L_q(\Omega)}) \le CT(L+M)M. \end{aligned}$$

Analogously, (4.18), (4.19), and Sobolev's inequality give

$$\Big\| \frac{(m_1 - m_2)\rho_1\rho_2}{\Sigma_\rho} \mathbf{V}^0(\mathbf{k_w}) \nabla \varphi \Big\|_{L_q(\Omega)} \le CT^{1/p'} \|\mathbf{w}\|_{L_p((0,T), H_q^2(\Omega))} \|\nabla \varphi(\cdot, t)\|_{L_q(\Omega)} \le CTM^2.$$

Putting the estimates above and the estimates obtained in (4.28) together gives

$$\tag{4.30} \begin{aligned} \|\mathbf{f}_2(U)\|_{L_p((0,T), L_q(\Omega))} \le C\Big\{ &(LM + M^2 + L^2)T + (M+L)^3 T^{(\theta/p' + 1/p)} \\ &+ (M+L)T^{1/p} + L^2 T^{1/p} + (M+L)LT^{(\theta/p'+1/p)} \Big\}. \end{aligned}$$

Next, we consider R_3 defined in (3.4) replacing ϑ and $\mathbf{v}$ by φ and $\mathbf{w}$. By (4.27), (4.16), Sobolev's inequality, and Hölder's inequality

$$\begin{aligned} &\|\nabla \Big(\frac{\rho_1\rho_2}{\mathfrak{p}\rho} \Big) \left(2\mathbf{V}^0(\mathbf{k_w}) + (\mathbf{V}^0(\mathbf{k_w}))^2 \right) \nabla \varphi \|_{L_q(\Omega)} \\ &\le C(M+L) \int_0^T \|\mathbf{w}(\cdot, s)\|_{H_q^2(\Omega)}\, ds \|\nabla \varphi(\cdot, t)\|_{L_q(\Omega)} \\ &\le C(M+L)M^2 T. \end{aligned}$$

Other terms in R_3 can be estimated in a similar manner to the estimate of R_2, and hence we obtain

$$\|R_3\|_{L_p((0,T),L_q(\Omega))} \le C(M+L)LT,$$

which, combined with the estimates obtained in (4.28), leads to

$$\begin{aligned}\|f_3(U)\|_{L_p((0,T),L_q(\Omega))} \le{}& C((LM+M^2+L^2)T + M(M+L)T^{\theta/p'} \\ &+ (M+L)^2T^{(\theta/p'+1/p)} + M(M+L)T^{\theta/p'} + (M+L)^2T^{1/p}).\end{aligned} \tag{4.31}$$

Finally, we estimate R_4 defined in (3.5) replacing $\mathbf{v}$ and ϑ by $\mathbf{w}$ and φ. For this purpose, we have to extend R_4 to the whole time interval $\mathbb{R}$. Let e_T be the extension operator defined in (4.12). Let $\tilde h_0$ be a function in $B^{2(1-1/p)}_{q,p}(\mathbb{R}^N)$ such that $\tilde h_0 = h_0$ in Ω and

$$\|\tilde h_0\|_{B^{2(1-1/p)}_{q,p}(\mathbb{R}^N)} \le C\|h_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)}.$$

Let

$$T(t)h_0 = e^{(\Delta-2)t}\tilde h_0 = \mathcal{F}^{-1}[e^{-(|\xi|^2+2)t}\mathcal{F}[\tilde h_0](\xi)],$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform on $\mathbb{R}^N$ and its inverse transform. We know that

$$\|e^tT(\cdot)h\|_{L_p((0,\infty),H^2_q(\mathbb{R}^N))} + \|e^t\partial_tT(\cdot)h\|_{L_p((0,\infty),L_q(\mathbb{R}^N))} \le C\|h\|_{B^{2(1-1/p)}_{q,p}(\Omega)}. \tag{4.32}$$

Let $\psi(t) \in C^\infty(\mathbb{R})$ be one for $t > -1$ and zero for $t < -2$. Since $\omega|_{t=0} - T(t)h|_{t=0} = h - h = 0$ in Ω, we set

$$\tilde e_T[\omega] = e_T[\omega - T(\cdot)h] + \psi(t)T(|t|)h.$$

Then, by (4.12), (4.13), and (4.32), we have

$$\|e^{-\gamma t}\tilde e_T[\omega]\|_{L_p(\mathbb{R},H^2_q(\Omega))} + \|e^{-\gamma t}\partial_t\tilde e_T[\omega]\|_{L_p(\mathbb{R},L_q(\Omega))} \le C(e^{2\gamma}L+M) \tag{4.33}$$

for any $\gamma \ge 0$, where C is a constant independent of γ, T, L, and M. To treat R_4, setting

$$\begin{aligned}\mathcal{R}_{\mathbf{w}} = -\Big\{&\mathbf{n}\left(y + \int_0^t \mathbf{w}(y,s)\,ds\right)\mathbf{V}^0(\mathbf{k}_{\mathbf{w}}) \\ &+ \int_0^1 (\nabla\mathbf{n})\left(y + \tau\int_0^t \mathbf{w}(y,s)\,ds\right)\,d\tau\int_0^t \mathbf{w}(y,s)\,ds\Big\},\end{aligned}$$

we write it as $R_4 = \mathcal{R}_{\mathbf{w}}\nabla\varphi$. Here, we may assume that $\mathbf{n}$ is defined in $\mathbb{R}^N$ and $\|\mathbf{n}\|_{H^2_\infty(\mathbb{R}^N)} \le C$. Notice that $\mathcal{R}_{\mathbf{w}}|_{t=0} = 0$. We then define $\tilde R_4$ by

$$\tilde R_4 = e_T[\mathcal{R}_{\mathbf{w}}]\nabla(\tilde e_T[\varphi]).$$

$\tilde R_4$ is an extension of R_4 to the whole time interval $\mathbb{R}$. Obviously, $\tilde R_4 = R_4$ in $(0,T)$.

To estimate $\tilde R_4$, we use the following lemma due to Shibata and Shimizu [39].

Lemma 4.4. *Let $1 < p < \infty$, $3 < q < \infty$, and $0 < T \le 1$. Assume that Ω is a uniformly C^2 domain. Let*

$$f \in H^1_\infty(\mathbb{R}, L_q(\Omega)) \cap L_\infty(\mathbb{R}, H^1_q(\Omega)), \quad g \in L_p(\mathbb{R}, H^1_q(\Omega)) \cap H^{1/2}_p(\mathbb{R}, L_q(\Omega)).$$

If we assume that $f \in L_p(\mathbb{R}, H^1_q(\Omega))$ and that f vanishes for $t \in [0, 2T]$ in addition, then we have

$$\|fg\|_{L_p(\mathbb{R},H^1_q(\Omega))} + \|fg\|_{H^{1/2}_p(\mathbb{R},L_q(\Omega))} \le C(\|f\|_{L_\infty(\mathbb{R},H^1_q(\Omega))} + T^{(q-3)/(pq)}\|\partial_t f\|^{(1-3/(2q))}_{L_\infty(\mathbb{R},L_q(\Omega))}\|\partial_t f\|^{3/(2q)}_{L_p((\mathbb{R},H^1_q(\Omega))})(\|g\|_{p(\mathbb{R},H^1_q(\Omega))} + \|g\|_{H^{1/2}_p(\mathbb{R},L_q(\Omega))}).$$

Remark 4.5. (1) The boundary of Ω was assumed to be bounded in Shibata and Shimizu [39]. But, Lemma 4.4 can be proved with the help of Sobolev's inequality and complex interpolation theorem, and so employing the same argument as that in the proof of [39, Lemma 2.7], we can prove Lemma 4.4.

(2) By Sobolev's inequality, $\|fg\|_{H^1_q(\Omega)} \le C\|f\|_{H^1_q(\Omega)}\|g\|_{L_q(\Omega)}$, and so the essential part of Lemma 4.4 is the estimate of $\|fg\|_{H^{1/2}_p(\mathbb{R},L_q(\Omega))}$.

Since Ω is a uniformly C^3 domain, we may assume that $\mathbf{n}$ is defined on the whole $\mathbb{R}^N$ and $\|\mathbf{n}\|_{H^2_\infty(\mathbb{R}^N)} < \infty$. We then have

$$\begin{aligned}\|e_T[\mathcal{R}_{\mathbf{w}}](\cdot,t)\|_{H^1_q(\Omega)} &\le C\left\{\int_0^T \|\mathbf{w}(\cdot,s)\|_{H^2_q(\Omega)}\,ds + \left(\int_0^T \|\mathbf{w}(\cdot,s)\|_{H^1_q(\Omega)}\,ds\right)^2\right\}\\ &\le C(T^{1/p'}M + T^{2/p'}M^2),\end{aligned}$$

and so

$$\|e_T[\mathcal{R}_{\mathbf{w}}]\|_{L_\infty(\mathbb{R},H^1_q(\Omega))} \le C(T^{1/p'}M + T^{2/p'}M^2). \tag{4.34}$$

Choosing $T >$ so small that $T^{1/p'}M \le 1$, by (4.13) we have

$$\|\partial_t e_T[\mathcal{R}_{\mathbf{w}}](\cdot,t)\|_{H^1_q(\Omega)} \le C\begin{cases}0 & \text{for } t < 0,\\ \|\mathbf{w}(\cdot,t)\|_{H^2_q(\Omega)} & \text{for } 0 < t < T,\\ \|\mathbf{w}(\cdot,2T-t)\|_{H^2_q(\Omega)} & \text{for } T < t < 2T,\\ 0 & \text{for } t > 2T,\end{cases}$$

and therefore

$$\|\partial_t e_T[\mathcal{R}_{\mathbf{w}}]\|_{L_p(\mathbb{R},H^1_q(\Omega))} \le C\|\mathbf{w}\|_{L_p((0,T),H^2_q(\Omega))} \le CM. \tag{4.35}$$

To estimate $\nabla(\tilde{e}_T[\varphi])$, we use the following lemma.

LEMMA 4.6. *Let $1 < p, q < \infty$. Assume that Ω is a uniform C^2 domain. Then*

$$H^1_p(\mathbb{R}, L_q(\Omega)) \cap L_p(\mathbb{R}, H^2_q(\Omega)) \subset H^{1/2}_p(\mathbb{R}, H^1_q(\Omega))$$

and

$$\|\nabla u\|_{H^{1/2}_p(\mathbb{R},L_q(\Omega))} \le C(\|u\|_{L_p(\mathbb{R},H^2_q(\Omega))} + \|\partial_t u\|_{L_p(\mathbb{R},L_q(\Omega))}).$$

Remark 4.7. As was mentioned in Shibata and Shimizu [40], in the case that $\Omega = \mathbb{R}^N$, Lemma 4.6 can be proved by Weis's operator valued Fourier multiplier theorem. In the uniformly C^2 domain case, localizing the estimate and using the uniformity of the domain and the partition of unity, we can prove Lemma 4.6. The detailed proof was given in Shibata [38]. In the case that $p = q$ and Ω is bounded, Lemma 4.6 was proved by Meyries and Schnaubelt [28].

Applying Lemma 4.6 and using (4.32), we have

$$\begin{aligned} &\|e^{-\gamma t}\nabla\tilde e_T[\varphi]\|_{H^{1/2}_p(\mathbb{R},L_q(\Omega))} + \|e^{-\gamma t}\nabla\tilde e_T[\varphi]\|_{L_p(\mathbb{R},H^1_q(\Omega))} \\ &\quad\le C(\|e^{-\gamma t}\tilde e_T[\varphi]\|_{H^1_p(\mathbb{R},L_q(\Omega))} + \|e^{-\gamma t}\tilde e_T[\varphi]\|_{L_p(\mathbb{R},H^2_q(\Omega))} \\ &\quad\le C(\|\varphi\|_{L_p((0,T),H^2_q(\Omega))} + \|\varphi\|_{H^1_p((0,T),L_q(\Omega))} + e^{2\gamma}L + M) \le C(e^{2\gamma}L+M) \end{aligned} \tag{4.36}$$

for any $\gamma > 0$. Since $e_T[\mathcal{R}_{\mathbf{w}}] = 0$ for $t \notin (0, 2T)$, applying Lemma 4.4 to $\tilde R_4$ and using estimates (4.34), (4.35), and (4.36), we have

$$\|e^{-\gamma t}\tilde R_4\|_{L_p(\mathbb{R},H^1_q(\Omega))} + \|e^{-\gamma t}\tilde R_4\|_{H^{1/2}_p(\mathbb{R},L_q(\Omega))} \le C(T^{1/p'}M + T^{(q-3)/(pq)}M)(e^{2\gamma}L+M) \tag{4.37}$$

for any $\gamma > 0$.

Applying Theorem 4.1 to (4.4), using (4.29), (4.30), (4.31), and (4.37), noting that $0 < T \le 1$, and fixing $\gamma > 0$ a large positive number, we see that there exists three positive constants C and $C_{M,L,\gamma}$ and τ for which

$$[\zeta, \mathbf{v}, \vartheta]_T \le Ce^{2\gamma T}(L + T^\tau C_{M,L,\gamma}). \tag{4.38}$$

Here, $C_{M,L,\gamma}$ is a constant depending on L, M, and γ. Letting $M = 2Ce^{2\gamma}L$ and choosing $T > 0$ so small that $T^\tau C_{M,L,\gamma} \le L$, we have

$$[\zeta, \mathbf{v}, \vartheta]_T \le M. \tag{4.39}$$

Let $\mathbf{S}$ be a map acting on $U = (\omega, \mathbf{w}, \varphi) \in \mathcal{H}_{T,M}$ defined by $\mathbf{S}U = V$, where $V = (\zeta, \mathbf{v}, \vartheta)$ is a unique solution of (4.6), and then by (4.39) we see that $\mathbf{S}$ maps $\mathcal{H}_{T,M}$ into itself. Let U_1, $U_2 \in \mathcal{H}_{T,M}$, and then applying the same argument as that in the proof of (4.38) to $V_1 - V_2$ with $V_i = \mathbf{S}U_i$, we see that there exists a constant K depending on M and L for which

$$[\mathbf{S}U_1 - \mathbf{S}U_2]_T \le KT^\tau[U_1 - U_2]_T. \tag{4.40}$$

Here, $(U_1 - U_2)|_{t=0} = 0$, and so constructing the extension of the term corresponding to R_4 in the previous argument we can use $e_T[\varphi_1 - \varphi_2]$ instead of $\tilde e_T[\varphi_1 - \varphi_2]$. Namely, we de not need to use the operator $T(\cdot)$, and so γ does not appear in the estimate, because $e_T[\varphi_1 - \varphi_2]$ vanishes for $t \notin (0, 2T)$.

From (4.40) we see that $\mathbf{S}$ is a contraction map from $\mathcal{H}_{T,M}$ into itself, and so by the Banach fixed point theorem there exists a unique $V = (\zeta, \mathbf{v}, \vartheta) \in \mathcal{H}_{T,M}$ with $M = 2CL$ such that $V = \mathbf{S}V$. This V is a unique solution of (4.4), which completes the proof of Theorem 2.1.

Employing the same argument as that in the proof of Theorem 2.1 we can prove the following theorem, which is the so-called almost global existence theorem and is used to prove the global well-posedness.

THEOREM 4.8. *Let* $2 < p < \infty$, $3 < q < \infty$, *and* $T > 0$. *Assume that* $2/p + 3/q < 1$ *and that* Ω *is a uniform* C^3 *domain in* $\mathbb{R}^N$ *(*$N \ge 2$*). Let* $\rho_{10}(x)$, $\rho_{20}(x)$, *and* $\mathbf{u}_0(x)$ *be initial data for* (1.1). *Assume that there exist positive numbers* a_1 *and* a_2 *for which*

$$a_1 \le \rho_{10}(x), \ \ \rho_{20}(x) \le a_2 \quad \text{for any } x \in \overline{\Omega}. \tag{4.41}$$

Let $(h_0(x), \rho_0(x)) = \Psi(\rho_{10}(x), \rho_{20}(x))$. *Then, there exists a small constant* $\epsilon_0 > 0$ *depending on* a_1, a_2, *and* T *such that if* ρ_{10}, ρ_{20}, $\mathbf{u}_0$, *and* h_0 *satisfy the condition*

$$\|\nabla(\rho_{10}, \rho_{20})\|_{L_q(\Omega)} + \|\mathbf{u}_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)} + \|h_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)} \le \epsilon_0 \tag{4.42}$$

and the compatibility condition

$$\mathbf{u}_0|_\Gamma = 0, \quad (\nabla h_0)\cdot \mathbf{n}|_\Gamma = 0, \tag{4.43}$$

then problem (2.15) *admits a unique solution* $(\eta, \mathbf{v}, \vartheta)$ *with*

$$\eta - \rho_0 \in H^1_p((0,T), H^1_q(\Omega)), \quad \mathbf{v} \in H^1_p((0,T), L_q(\Omega)^3) \cap L_p((0,T), H^2_q(\Omega)^3),$$
$$\vartheta \in H^1_p((0,T), L_q(\Omega)) \cap L_p((0,T), H^2_q(\Omega))$$

possessing the estimates

$$\|\eta - \rho_0\|_{H^1_p((0,T),H^1_q(\Omega))} + \|\partial_t(\mathbf{v},\vartheta)\|_{L_p((0,T),L_q(\Omega))} + \|(\mathbf{v},\vartheta)\|_{L_p((0,T),H^2_q(\Omega))} \le C\epsilon_0,$$
$$a_1 \le \rho(x,t) \le 2a_2 + a_1 \quad \textit{for } (x,t) \in \Omega\times(0,T), \quad \int_0^T \|\nabla \mathbf{v}(\cdot,s)\|_{L_\infty(\Omega)} \le \delta.$$

Here, C is some constant independent of ϵ_0.

5. Global well-posedness—Proof of Theorem 2.2. In this section, Ω is a bounded domain whose boundary Γ is a compact hypersurface of C^3 class. Let ρ_{1*} and ρ_{2*} be any positive numbers and set $(h_*, \rho_*) = \Psi(\rho_{1*}, \rho_{2*}) \in \mathbb{R}\times\mathbb{R}_+$. Let $T > 0$ and let $(\eta, \mathbf{v}, \vartheta)$ be a solution of (2.15) such that

$$\eta \in H^1_p((0,T), H^1_q(\Omega)), \quad \mathbf{v} \in H^1_p((0,T), L_q(\Omega)^3) \cap L_p((0,T), H^2_q(\Omega)^3),$$
$$\vartheta \in H^1_p((0,T), L_q(\Omega)) \cap L_p((0,T), H^2_q(\Omega)), \quad \int_0^T \|\nabla \mathbf{v}(\cdot,s)\|_{L_\infty(\Omega)}\,ds \le \delta,$$
$$\rho_*/4 \le \eta(x,t) \le 4\rho_*, \quad |\vartheta(x,t)| \le 4|h_*| \quad \text{for } (x,t)\in\Omega\times(0,T). \tag{5.1}$$

To prove the global well-posedness, we prolong $(\eta, \mathbf{v}, \vartheta)$ to any time interval beyond T. Let $\zeta = \eta - \rho_*$ and $h = \vartheta - h_*$, and let

$$\mathcal{I} = \|\rho_0 - \rho_*\|_{H^1_q(\Omega)} + \|(\mathbf{u}_0, h_0 - h_*)\|_{B^{2(1-1/p)}_{q,p}},$$
$$\langle e^{\gamma t} V\rangle_T = \|e^{\gamma t}\nabla\zeta\|_{L_p((0,T),L_q(\Omega))} + \|e^{\gamma t}\partial_t\zeta\|_{L_p((0,T),H^1_q(\Omega))} + \|e^{\gamma t}\mathbf{v}\|_{L_p((0,T),H^2_q(\Omega)}$$
$$+ \|e^{\gamma t}\nabla h\|_{L_p((0,T),H^1_q(\Omega))} + \|e^{\gamma t}\partial_t(\mathbf{v},h)\|_{L_p((0,T),L_q(\Omega))}.$$

Here, γ is a positive constant appearing in Theorem 5.1 below. The key step is to prove the estimate

$$\langle e^{\gamma t} V\rangle_T \le C(\mathcal{I} + \langle e^{\gamma t} V\rangle_T^2) \tag{5.2}$$

for some constant $C > 0$.

To prove (5.2), we linearize (2.15) at $(\rho_1, \rho_2) = (\rho_{1*}, \rho_{2*})$, $\eta = \rho_*$, $\mathbf{v} = 0$, and $\vartheta = h_*$. Namely, $\eta = \rho_* + \zeta$, $\mathbf{v}$, and $\vartheta = h_* + h$ satisfy the following equations:

$$\begin{cases} \partial_t\zeta + a_{0*}\operatorname{div}\mathbf{v} = \tilde f_1(U) & \text{in } \Omega\times(0,T), \\ a_{0*}\partial_t\mathbf{v} - \mu\Delta\mathbf{v} - \nu\nabla\operatorname{div}\mathbf{v} + a_{1*}\nabla\zeta + a_{2*}\nabla h = \tilde{\mathbf{f}}_2(U) & \text{in } \Omega\times(0,T), \\ a_{3*}\partial_t h + a_{2*}\operatorname{div}\mathbf{v} - a_{4*}\Delta h = \tilde f_3(U) & \text{in } \Omega\times(0,T), \\ \mathbf{v} = 0, \quad (\nabla h)\cdot\mathbf{n} = g(U) & \text{on } \Gamma\times(0,T), \\ (\zeta, \mathbf{v}, h)|_{t=0} = (\rho_0 - \rho_*, \mathbf{u}_0, h_0 - h_*) & \text{in } \Omega. \end{cases} \tag{5.3}$$

Here, we have set

$$
\begin{aligned}
&a_{0*} = \rho_*, \quad a_{1*} = \frac{a_0}{\Sigma_{\rho_*}}, \quad a_{2*} = \frac{(m_1 - m_2)\rho_{1*}\rho_{2*}}{\Sigma_{\rho_*}},\\
&a_{3*} = \frac{m_1 m_2 \rho_{1*}\rho_{2*}}{\Sigma_{\rho_*}}, \quad a_{4*} = \frac{\rho_{1*}\rho_{2*}}{\mathfrak{p}_*\rho_*},\\
&\Sigma_{\rho_*} = m_1\rho_{1*} + m_2\rho_{2*}, \quad \mathfrak{p}_* = \frac{\rho_{1*}}{m_1} + \frac{\rho_{2*}}{m_2}, \quad U = (\eta, \mathbf{v}, \vartheta) = (\rho_* + \zeta, \mathbf{v}, h_* + h),\\
&\tilde f_1(U) = R_1(U) - \zeta \mathrm{div}\, \mathbf{v},\\
&\tilde{\mathbf{f}}_2(U) = R_2(U) - \zeta\partial_t \mathbf{v} - \Big(\frac{\eta}{\Sigma_\rho} - \frac{\rho_*}{\Sigma_{\rho_*}}\Big)\nabla\zeta - (m_1 - m_2)\Big(\frac{\rho_1\rho_2}{\Sigma_\rho} - \frac{\rho_{1*}\rho_{2*}}{\Sigma_{\rho_*}}\Big)\nabla h,\\
&\tilde f_3(U) = R_3(U) - m_1 m_2\Big(\frac{\rho_1\rho_2}{\Sigma_\rho} - \frac{\rho_{1*}\rho_{2*}}{\Sigma_{\rho_*}}\Big)\nabla h - (m_1 - m_2)\Big(\frac{\rho_1\rho_2}{\Sigma_\rho} - \frac{\rho_{1*}\rho_{2*}}{\Sigma_{\rho_*}}\Big)\mathrm{div}\, \mathbf{v}\\
&\qquad + \mathrm{div}\, \Big(\Big(\frac{\rho_1\rho_2}{\mathfrak{p}\rho} - \frac{\rho_{1*}\rho_{2*}}{\mathfrak{p}_*\rho_*}\Big)\nabla h\Big), \qquad (5.4)
\end{aligned}
$$

$$g(U) = R_4(U). \qquad (5.5)$$

Notice that a_{0*}, a_{1*}, a_{3*}, and a_{4*} are positive constants, while a_{2*} is a real number. We consider the system of linear equations:

$$
(5.6) \qquad \begin{cases}
\partial_t\zeta + a_{0*}\mathrm{div}\, \mathbf{v} = g_1 & \text{in } \Omega\times(0,T),\\
a_{0*}\partial_t\mathbf{v} - \mu\Delta\mathbf{v} - \nu\nabla\mathrm{div}\, \mathbf{v} + a_{1*}\nabla\zeta + a_{2*}\nabla\vartheta = \mathbf{g}_2 & \text{in } \Omega\times(0,T),\\
a_{3*}\partial_t\vartheta + a_{2*}\mathrm{div}\, \mathbf{v} - a_{4*}\Delta\vartheta = g_3 & \text{in } \Omega\times(0,T),\\
\mathbf{v} = 0, \quad (\nabla\vartheta)\cdot\mathbf{n} = g_4 & \text{on } \Gamma\times(0,T),\\
(\zeta, \mathbf{v}, \vartheta)|_{t=0} = (\zeta_0, \mathbf{v}_0, \vartheta_0) & \text{in } \Omega.
\end{cases}
$$

For (5.6), we have the following decay theorem.

THEOREM 5.1. *Let $1 < p, q < \infty$, $2/p + 1/q \neq 1$, and $2/p + 1/q \neq 2$. Assume that Ω is a bounded domain whose boundary Γ is a compact hypersurface of C^3 class. Let*

$$
\begin{aligned}
&\rho_0 \in H^1_q(\Omega), \quad \mathbf{v}_0 \in B^{2(1-1/p)}_{q,p}(\Omega)^3, \quad \vartheta_0 \in B^{2(1-1/p)}_{q,p}(\Omega),\\
&g_1 \in L_p((0,T), H^1_q(\Omega)), \quad \mathbf{g}_2 \in L_p((0,T), L_q(\Omega)^3) \cap H^1_p((0,T), L_q(\Omega)^3),\\
&g_3 \in L_p((0,T), L_q(\Omega)), \quad E[e^{\gamma_1 t} g_4] \in H^{1/2}_p(\mathbb{R}, L_q(\Omega)) \cap L_p(\mathbb{R}, H^1_q(\Omega))
\end{aligned}
$$

for some $\gamma_1 > 0$. Here, $E[e^{\gamma_1 t} g_4]$ denotes some extension of $e^{\gamma_1 t} g_4$ to the whole time interval $\mathbb{R}$. Assume that $\mathbf{v}_0$, ϑ_0, and g_4 satisfy the compatibility conditions:

$$\mathbf{v}_0 = 0 \ \text{ on } \Gamma \text{ for } 2/p + 1/q < 2, \quad (\nabla\vartheta_0)\cdot\mathbf{n} = g_4|_{t=0} \ \text{ on } \Gamma \text{ for } 2/p + 1/q < 1.$$

Then, problem (5.6) admits unique solutions η, $\mathbf{v}$, and ϑ with

$$
\begin{aligned}
&\eta \in H^1_p((0,T), H^1_q(\Omega)), \quad \mathbf{v} \in L_p((0,T), H^2_q(\Omega)^3) \cap H^1_p((0,T), L_q(\Omega)^3),\\
&\vartheta \in L_p((0,T), H^2_q(\Omega)) \cap H^1_p((0,T), L_q(\Omega))
\end{aligned}
$$

possessing the estimate

$$\begin{aligned}&\|e^{\gamma t}\nabla\eta\|_{L_p((0,T),L_q(\Omega))}+\|e^{\gamma t}\partial_t\eta\|_{L_p((0,T),H^1_q(\Omega))}+\|e^{\gamma t}\mathbf{v}\|_{L_p((0,T),H^2_q(\Omega)}\\&\quad+\|e^{\gamma t}\nabla\vartheta\|_{L_p((0,T),H^1_q(\Omega))}+\|e^{\gamma t}\partial_t(\mathbf{v},\vartheta)\|_{L_p((0,T),L_q(\Omega))}\\&\le C(\|\zeta_0\|_{H^1_q(\Omega)}+\|(\mathbf{v}_0,\vartheta_0)\|_{B^{2(1-1/p)}_{q,p}(\Omega)}+\|e^{\gamma t}(g_1,\mathbf{g}_2,g_3)\|_{L_p((0,T),L_q(\Omega))}\\&\quad+\|E[e^{\gamma_1 t}g_4]\|_{H^{1/2}_p(\mathbb{R},L_q(\Omega))}+\|E[e^{\gamma_1 t}g_4]\|_{L_p(\mathbb{R},H^1_q(\Omega))})\end{aligned}$$

for some constants $\gamma\in(0,\gamma_1]$ and $C>0$.

Postponing the proof of Theorem 5.1, we prove (5.2). For this purpose we have to find the estimates for the nonlinearities $\tilde f_i(U), g(U)$ defined in (5.4). Let $(\rho_1(x),\rho_2(x))=\Phi(\vartheta,\eta)$. Following the ideas from [36], we first prove that

$$\begin{aligned}\|\eta-\rho_0\|_{L_\infty((0,T),H^1_q(\Omega))}&\le C\langle e^{\gamma t}V\rangle_T,\\\|\vartheta-h_0\|_{L_\infty((0,T),B^{2(1-1/p)}_{q,p}(\Omega))}&\le C(\mathcal{I}+\langle e^{\gamma t}V\rangle_T),\end{aligned}\tag{5.7}$$

where $(h_0,\rho_0)=(\vartheta,\eta)|_{t=0}$ (cf. (2.15) in the introduction). In fact, by Hölder's inequality we have

$$\begin{aligned}\|\eta(\cdot,t)-\eta(\cdot,0)\|_{H^1_q(\Omega)}&\le\int_0^T\|\partial_t\eta(\cdot,t)\|_{H^1_q(\Omega)}\,dt\\&\le\Big(\int_0^T e^{-p'\gamma t}\,dt\Big)^{1/p'}\Big(\int_0^T(e^{\gamma t}\|\partial_t\eta(\cdot,t)\|_{H^1_q(\Omega)})^p\,dt\Big)^{1/p}\\&\le C\langle e^{\gamma t}V\rangle_T.\end{aligned}$$

Recalling that $\vartheta-h_*=h$ and $\vartheta_0-h_*=h_0-h_*$, we have

$$\|\vartheta(\cdot,t)-\vartheta_0\|_{L_q(\Omega)}\le\int_0^T\|\partial_s h(\cdot,s)\|_{L_q(\Omega)}\,ds+\|h_0-h_*\|_{L_q(\Omega)}\le C(\langle e^{\gamma t}V\rangle_T+\mathcal{I}).$$

Let $H(x,t)=h(x,t)-|\Omega|^{-1}\int_\Omega h(x,t)\,dx$. Since $\int_\Omega H(x,t)\,dx=0$, by Poincaré's inequality we have

$$\|H(\cdot,t)\|_{H^2_q(\Omega)}\le C\|\nabla H(\cdot,t)\|_{H^1_q(\Omega)}=C\|\nabla h(\cdot,t)\|_{H^1_q(\Omega)}.$$

Moreover, noting that $2(1-1/p)>1$, we have

$$\begin{aligned}\|H|_{t=0}\|_{B^{2(1-1/p)}_{q,p}(\Omega)}&\le\|H|_{t=0}\|_{L_q(\Omega)}+\|\nabla H|_{t=0}\|_{B^{1-2/p}_{q,p}(\Omega)}\\&\le C(\|h_0-h_*\|_{L_q(\Omega)}+\|\nabla h_0\|_{B^{1-2/p}_{q,p}(\Omega)})\\&=C\|h_0-h_*\|_{B^{2(1-1/p)}_{q,p}(\Omega)}.\end{aligned}$$

On the other hand, employing the same argument as that in the proof of (4.11), by real interpolation theory, we have

$$\begin{aligned}&\sup_{t\in(0,T)}\|H(\cdot,t)\|_{B^{2(1-1/p)}_{q,p}(\Omega)}\\&\quad\le\sup_{t\in(0,T)}\|\tilde e_T[H](\cdot,t)\|_{B^{2(1-1/p)}_{q,p}(\Omega)}\\&\quad\le C(\|H\|_{L_p((0,T),H^2_q(\Omega))}+\|\partial_t H\|_{L_p((0,T),L_q(\Omega))}+\|H|_{t=0}\|_{B^{2(1-1/p)}_{q,p}(\Omega)}).\end{aligned}$$

Therefore, since $\|\partial_t H\|_{L_q(\Omega)} \le C\|\partial_t h\|_{L_q(\Omega)}$, we obtain

$$\sup_{t\in(0,T)} \|H(\cdot,t)\|_{B^{2(1-1/p)}_{q,p}(\Omega)} \le C(\|\nabla h\|_{L_p((0,T),H^1_q(\Omega))} + \|\partial_t h\|_{L_p((0,T),L_q(\Omega))} + \|h_0 - h_*\|_{B^{2(1-1/p)}_{q,p}(\Omega)}).$$

Since

$$\sup_{t\in(0,T)} \|h(\cdot,t)\|_{L_q(\Omega)} \le \|h_0 - h_*\|_{L_q(\Omega)} + \int_0^T \|\partial_t h(\cdot,t)\|_{L_q(\Omega)}\,dt \le C(\|h_0 - h_*\|_{L_q(\Omega)} + C\langle e^{\gamma t}V\rangle_T),$$

we have

$$\sup_{t\in(0,T)} \|h\|_{B^{2(1-1/p)}_{q,p}(\Omega)} \le \sup_{t\in(0,T)} \|H\|_{B^{2(1-1/p)}_{q,p}(\Omega)} + \sup_{t\in(0,T)} \|h(\cdot,t)\|_{L_q(\Omega)} \le C(\|h_0 - h_*\|_{B^{2(1-1/p)}_{q,p}(\Omega)} + \langle e^{\gamma t}V\rangle_T) \le C(\mathcal{I} + \langle e^{\gamma t}V\rangle_T),$$

which shows the second inequality in (5.7). Next we show that

$$\|(\rho_1,\rho_2) - (\rho_{1*},\rho_{2*})\|_{L_\infty((0,T),H^1_q(\Omega))} \le C(\mathcal{I} + \langle e^{\gamma t}V\rangle_T). \tag{5.8}$$

In fact, by the Taylor formula, we have

$$(\rho_1,\rho_2) - (\rho_{10},\rho_{20}) = \Phi(\vartheta,\eta) - \Phi(h_0,\rho_0) \le \int_0^1 \Phi'((h_0,\rho_0) + \theta(\vartheta-\vartheta_0,\eta-\eta_0))\,d\theta(\vartheta - h_0, \eta - \rho_0),$$

where $(h_0,\rho_0) = (\vartheta,\eta)|_{t=0}$. Set

$$D = \{(\zeta,\eta)\in\mathbb{R}^2 \mid |\zeta| \le |h_*|/4, \quad \rho_*/4 \le \eta \le 4\rho_*\},$$

and then by (5.1), $(\vartheta,\eta)\in D$ for any $(x,t)\in\Omega\times(0,T)$. Let C_0 be a positive constant for which

$$\sup_{(\vartheta,\eta)\in D} |\Phi'(\vartheta,\eta)| \le C_0, \qquad \sup_{(\vartheta,\eta)\in D} |\Phi''(\vartheta,\eta)(\vartheta,\eta)| \le C_0.$$

We then have

$$\|(\rho_1,\rho_2) - (\rho_{10},\rho_{20})\|_{L_\infty((0,T),H^1_q(\Omega))} \le 3C_0\|(\vartheta - h_0, \eta - \rho_0)\|_{L_\infty((0,T),H^1_q(\Omega))},$$

which, combined with (5.7), leads to (5.8), because $\|(\rho_{10},\rho_{20}) - (\rho_{1*},\rho_{2*})\|_{H^1_q(\Omega)} \le \mathcal{I}$.

By (5.1) we may assume that there exist two positive constants a_1 and a_2 depending on ρ_* and h_* for which

$$a_1 \le \rho_1(x,t), \rho_2(x,t) \le a_2 \quad \text{for any } (x,t)\in\Omega\times(0,T). \tag{5.9}$$

By (5.8) and (5.9), we have the following estimates:

$$\begin{aligned}&\Big\|e^{\gamma t}\Big(\frac{\eta}{\Sigma_\rho}-\frac{\rho_*}{\Sigma_{\rho_*}}\Big)\nabla\zeta\Big\|_{L_\infty((0,T),L_q(\Omega))}+\Big\|e^{\gamma t}\Big(\frac{\rho_1\rho_2}{\Sigma_\rho}-\frac{\rho_{1*}\rho_{2*}}{\Sigma_{\rho_*}}\Big)\nabla h\Big\|_{L_\infty((0,T),L_q(\Omega))}\\&+\Big\|e^{\gamma t}\mathrm{div}\,\Big(\Big(\frac{\rho_1\rho_2}{\mathfrak{p}\rho}-\frac{\rho_{1*}\rho_{2*}}{\mathfrak{p}_*\rho_*}\Big)\nabla h\Big)\Big\|_{L_\infty((0,T),L_q(\Omega))}\leq C(\mathcal{I}+\langle e^{\gamma t}V\rangle_T)\langle e^{\gamma t}V\rangle_T.\end{aligned} \tag{5.10}$$

By Sobolev's inequality and (5.7), we have

$$\|\zeta\|_{H^1_q(\Omega)}\leq C\|\eta-\rho_0\|_{H^1_q(\Omega)}+\|\rho_0-\rho_*\|_{H^1_q(\Omega)}\leq C(\mathcal{I}+\langle e^{\gamma t}V\rangle_T),$$

and so

$$\begin{aligned}\|e^{\gamma t}\zeta\mathrm{div}\,\mathbf{v}\|_{L_p((0,T),H^1_q(\Omega))}&\leq C\|\zeta\|_{L_\infty((0,T),H^1_q(\Omega))}\|e^{\gamma t}\nabla\mathbf{v}\|_{L_p((0,T),H^1_q(\Omega))}\\&\leq C(\mathcal{I}+\langle e^{\gamma t}V\rangle_T)\langle e^{\gamma t}V\rangle_T.\end{aligned}$$

Similarly, by (5.8),

$$\Big\|e^{\gamma t}\Big(\frac{\rho_1\rho_2}{\Sigma_\rho}-\frac{\rho_{1*}\rho_{2*}}{\Sigma_{\rho_*}}\Big)\nabla h\Big\|_{L_\infty((0,T),L_q(\Omega))}\leq C(\mathcal{I}+\langle e^{\gamma t}V\rangle_T)\langle e^{\gamma t}V\rangle_T. \tag{5.11}$$

By (4.32) and real interpolation theory, we have

$$\|\mathbf{v}\|_{L_\infty((0,T),B^{2(1-1/p)}_{q,p}(\Omega))}+\|\mathbf{v}\|_{L_\infty((0,T),H^1_\infty(\Omega))}\leq C(\mathcal{I}+\langle e^{\gamma t}V\rangle_T). \tag{5.12}$$

In fact,

$$\begin{aligned}&\sup_{t\in(0,T)}\|\mathbf{v}(\cdot,t)\|_{B^{2(1-1/p)}_{q,p}(\Omega)}\\&\quad\leq\sup_{t\in(0,T)}\|\tilde e_T[\mathbf{v}](\cdot,t)\|_{B^{2(1-1/p)}_{q,p}(\Omega)}\\&\quad\leq C(\|\mathbf{v}\|_{L_p((0,T),H^2_q(\Omega))}+\|\partial_t\mathbf{v}\|_{L_p((0,T),L_q(\Omega))}+\|T(\cdot)\tilde{\mathbf{v}}_0\|_{L_p((0,\infty),H^2_q(\Omega))}\\&\qquad\quad+\|\partial_tT(\cdot)\tilde{\mathbf{v}}_0\|_{L_p((0,\infty),L_q(\Omega))}),\end{aligned}$$

where $\tilde{\mathbf{v}}_0\in B^{2(1-1/p)}_{q,p}(\mathbb{R}^3)$ equals to $\mathbf{v}_0$ in Ω and $\|\tilde{\mathbf{v}}_0\|_{B^{2(1-1/p)}_{q,p}(\mathbb{R}^3)}\leq C\|\mathbf{v}_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)}$. Thus, by (4.32), we have the estimate of the first term in (5.12). Since $2/p+3/q<1$, we have $\|\mathbf{v}\|_{H^1_\infty(\Omega)}\leq C\|\mathbf{v}\|_{B^{2(1-1/p)}_{q,p}(\Omega)}$, which completes the proof of (5.12).

Now we shall estimate $R_i(U)$. By Sobolev's inequality and Hölder's inequality, we have

$$\begin{aligned}&\Big\|\int_0^t\nabla\mathbf{v}(\cdot,s)\,ds\nabla^2f\Big\|_{L_q(\Omega)}\\&\quad\leq\Big(\int_0^Te^{-\gamma p's}\,ds\Big)^{1/p'}\Big(\int_0^T(e^{\gamma s}\|\nabla\mathbf{v}(\cdot,s)\|_{L_\infty(\Omega)})^p\,ds\Big)^{1/p}\|\nabla^2f(\cdot,t)\|_{L_q(\Omega)}\\&\quad\leq C\langle^{\gamma t}V\rangle_T\|f(\cdot,t)\|_{H^2_q(\Omega)},\end{aligned}$$

and therefore

$$\Big\|e^{\gamma t}\int_0^t\nabla\mathbf{v}(\cdot,s)\,ds\nabla^2f\Big\|_{L_p((0,T),L_q(\Omega))}\leq C\langle e^{\gamma t}V\rangle_T\|e^{\gamma t}f\|_{L_p((0,T),H^2_q(\Omega))}. \tag{5.13}$$

A similar estimate of $l\|\int_0^t \nabla^2\mathbf{v}(\cdot,s)\,ds\nabla f\|_{L_q(\Omega)}$ yields

$$\Big\|e^{\gamma t}\int_0^t \nabla^2\mathbf{v}(\cdot,s)\,ds\nabla f\Big\|_{L_p((0,T),L_q(\Omega))} \le C\langle e^{\gamma t}V\rangle_T\|e^{\gamma t}f\|_{L_p((0,T),H^2_q(\Omega))}. \tag{5.14}$$

By (5.1) and (5.13), we have

$$\begin{aligned}\|e^{\gamma t}R_1(U)\|_{L_p((0,T),L_q(\Omega))} &\le C\Big\|\int_0^t \nabla\mathbf{v}(\cdot,s)\,ds\nabla\mathbf{v}\Big\|_{L_p((0,T),L_q(\Omega))}\\ &\le C\langle^{\gamma t}V\rangle_T\|\nabla\mathbf{v}(\cdot,t)\|_{L_p((0,T),L_q(\Omega))} \le C\langle e^{\gamma t}V\rangle_T^2.\end{aligned}$$

Noting that $\nabla\eta = \nabla\zeta$, we have

$$\begin{aligned}&\nabla R_1(U)\\ &= -\sum_{i,j=1}^3\Big(\nabla\zeta V^0_{ij}(\mathbf{k_v})\frac{\partial v_i}{\partial x_j} + \eta(\nabla_{\mathbf{k}}V^0_{ij})(\mathbf{k_v})\int_0^t \nabla^2\mathbf{v}(\cdot,s)\,ds\,\frac{\partial v_i}{\partial x_j} + \eta V^0_{ij}(\mathbf{k_v})\nabla\frac{\partial v_i}{\partial x_j}\Big).\end{aligned}$$

Noting that $|\mathbf{k_v}|\le\delta$, we have

$$\begin{aligned}&\|e^{\gamma t}\nabla R_1(U)\|_{L_p((0,T),L_q(\Omega))}\\ &\le C(\|\nabla\zeta\|_{L_\infty((0,T),L_q(\Omega))}\|e^{\gamma t}\nabla\mathbf{v}\|_{L_p((0,T),L_q(\Omega))} + \langle e^{\gamma t}V\rangle_T\|e^{\gamma t}\nabla\mathbf{v}\|_{L_p((0,T),H^1_q(\Omega))}).\end{aligned}$$

Since

$$\|\nabla\zeta\|_{L_q(\Omega)} \le \|\nabla(\eta-\rho_0)\|_{L_q(\Omega)} + \|\nabla\rho_0\|_{L_q(\Omega)},$$

by (5.7) we have

$$\|\nabla\zeta\|_{L_\infty((0,T),L_q(\Omega))} \le C(\mathcal{I} + \langle e^{\gamma t}V\rangle_T),$$

and so (5.13) and (5.14) imply

$$\|e^{\gamma t}\nabla R_1(U)\|_{L_p((0,T),L_q(\Omega))} \le C(\mathcal{I} + \langle e^{\gamma t}V\rangle_T)\langle e^{\gamma t}V\rangle_T.$$

Summing up, we have obtained

$$\|e^{\gamma t}R_1(U)\|_{L_p((0,T),H^1_q(\Omega))} \le C(\mathcal{I} + \langle e^{\gamma t}V\rangle_T)\langle e^{\gamma t}V\rangle_T. \tag{5.15}$$

We next consider $R_2(U)$ given in (3.3). By (5.13) and (5.14), we have

$$\begin{aligned}&\|e^{\gamma t}(A_{2\Delta}(\mathbf{k_v})\nabla^2\mathbf{v}, A_{1D}(\mathbf{k_v})\nabla\mathbf{v}, A_{2\mathrm{div}}(\mathbf{k_v})\nabla^2\mathbf{v}, A_{1\mathrm{div}}(\mathbf{k_v})\nabla\mathbf{v})\|_{L_p((0,T),L_q(\Omega))}\\ &\quad\le C\langle e^{\gamma t}V\rangle_T^2.\end{aligned}$$

By (5.1) and (5.13) we have

$$\Big\|e^{\gamma t}\Big(\frac{\eta}{\Sigma_\rho}\mathbf{V}^0(\mathbf{k_v})\nabla\zeta, \frac{\rho_1\rho_2}{\Sigma_\rho}\mathbf{V}^0(\mathbf{k_v})\nabla h\Big)\Big\|_{L_p((0,T),L_q(\Omega))} \le C\langle e^{\gamma t}V\rangle_T^2.$$

Summing up, we have obtained

$$\|e^{\gamma t}R_2(U)\|_{L_p((0,T),H^1_q(\Omega))} \le C(\mathcal{I} + \langle e^{\gamma t}V\rangle_T)\langle e^{\gamma t}V\rangle_T. \tag{5.16}$$

We next consider $R_3(U)$ given in (3.4). By (5.1) and (5.8), we have

$$a_1 \le \rho_1(x,t), \rho_2(x,t) \le a_2 \quad \text{for } (x,t) \in \Omega \times (0,T),$$
$$\|\nabla \rho_i\|_{L_\infty((0,T),L_q(\Omega))} \le C(\mathcal{I} + \langle e^{\gamma t}V\rangle_T) \quad (i=1,2),$$

where a_1 and a_2 are some positive constants depending on ρ_{1*} and ρ_{2*}, and therefore

$$\Big\|\nabla\Big(\frac{\rho_1\rho_2}{\mathfrak{p}\rho}\Big)\Big\|_{L_\infty((0,T),L_q(\Omega))} \le C(\mathcal{I} + \langle e^{\gamma t}V\rangle_T).$$

Thus, by (5.1), (5.13), and (5.14), we have

$$\|e^{\gamma t}R_3(U)\|_{L_p((0,T),H^1_q(\Omega))} \le C(\mathcal{I} + \langle e^{\gamma t}V\rangle_T)\langle e^{\gamma t}V\rangle_T. \tag{5.17}$$

Finally, we estimate $R_4(U)$ given in (3.5). Similarly to section 5, we set

$$\mathcal{R}_{\mathbf{v}} = -\Big\{\mathbf{n}\Big(y+\int_0^t \mathbf{v}(y,s)\,ds\Big)\mathbf{V}^0(\mathbf{k}_{\mathbf{v}})+\int_0^1 (\nabla\mathbf{n})\Big(y+\tau\int_0^t \mathbf{v}(y,s)\,ds\Big)\,d\tau \int_0^t \mathbf{v}(y,s)\,ds\Big\}.$$

Let $H(x,t) = h(x,t) - |\Omega|^{-1}\int_\Omega h(x,t)\,dx$. Obviously, $\nabla H = \nabla h$. Moreover, by Poincaré's inequality, we have

$$\begin{aligned}\|e^{\gamma t}H\|_{L_p((0,T),H^2_q(\Omega))} &+ \|e^{\gamma t}\partial_t H\|_{L_p((0,T),L_q(\Omega))} \\ &\le C(\|e^{\gamma t}\nabla h\|_{L_p((0,T),H^1_q(\Omega))} + \|e^{\gamma t}\partial_t h\|_{L_p((0,T),L_q(\Omega))}).\end{aligned} \tag{5.18}$$

In particular, we can write $R_4(U)$ as $R_4(U) = \mathcal{R}_{\mathbf{v}}\nabla H$. We define the extension of $e^{\gamma t}R_4(U)$ by

$$E[e^{\gamma t}R_4(U)] = e_T[\mathcal{R}_{\mathbf{v}}](\nabla \tilde{e}_T[e^{\gamma t}H]).$$

To estimate $E[e^{\gamma t}R_4(U)]$, we use the following lemma.

Lemma 5.2. *Let $1 < p < \infty$ and $3 < q < \infty$. Then, the following two assertions hold.*

(1) *If $f \in H^1_\infty(\mathbb{R}, L_\infty(\Omega))$ and $g \in H^{1/2}_p(\mathbb{R}, L_q(\Omega))$, then*

$$\|fg\|_{H^{1/2}_p(\mathbb{R},L_q(\Omega))} \le C\|f\|_{H^1_\infty(\mathbb{R},L_\infty(\Omega))}\|g\|_{H^{1/2}_p(\mathbb{R},L_q(\Omega))}.$$

(2) *If $f \in L_\infty(\mathbb{R}, H^1_q(\Omega))$ and $g \in L_p(\mathbb{R}, H^1_q(\Omega))$, then*

$$\|fg\|_{L_p(\mathbb{R},H^1_q(\Omega))} \le C\|f\|_{L_\infty(\mathbb{R},H^1_q(\Omega))}\|g\|_{L_p(\mathbb{R},H^1_q(\Omega))}.$$

Proof. To prove the first assertion, we use the fact that

$$H^{1/2}_p(\mathbb{R}, L_q(\Omega)) = (L_p(\mathbb{R}, L_q(\Omega)), H^1_p(\mathbb{R}, L_q(\Omega)))_{[1/2]}, \tag{5.19}$$

where $(\cdot,\cdot)_{[\theta]}$ denotes a complex interpolation functor for $\theta \in (0,1)$. Since

$$\|\partial_t(fg)\|_{L_p(\mathbb{R},L_q(\Omega))} \le \|f\|_{H^1_\infty(\mathbb{R},L_\infty(\Omega))}\|g\|_{H^1_p(\mathbb{R},L_q(\Omega))},$$
$$\|fg\|_{L_p(\mathbb{R},L_q(\Omega))} \le \|f\|_{L_\infty(\mathbb{R},L_\infty(\Omega))}\|g\|_{L_p(\mathbb{R},L_q(\Omega))},$$

by (5.19) we have the first assertion. The second assertion follows immediately from the Banach algebra property of $H^1_q(\Omega)$ for $3 < q < \infty$. ∎

Recalling that $\mathbf{n}$ is defined on $\mathbb{R}^3$ and $\|\mathbf{n}\|_{H^2_\infty(\mathbb{R}^3)} < \infty$, by (5.12) we have

$$\|\partial_t e_T[\mathcal{R}_{\mathbf{v}}]\|_{L_\infty(\mathbb{R},L_\infty(\Omega))} \le C\|\mathbf{v}\|_{L_\infty((0,T),H^1_\infty(\Omega))} \le C(\mathcal{I} + \langle e^{\gamma t}V\rangle_T).$$

By Sobolev's inequality and Hölder's inequality, we have

$$\begin{aligned}\|e_T[\mathcal{R}_{\mathbf{v}}]\|_{L_\infty(\mathbb{R},L_\infty(\Omega))} &\le C\int_0^T \|\mathbf{v}(\cdot,s)\|_{H^2_q(\Omega)}\,ds \le C\Big(\int_0^T (e^{\gamma t}\|\mathbf{v}(\cdot,s)\|_{H^2_q(\Omega)})^p\,ds\Big)^{1/p}\\ &\le C\langle e^{\gamma t}V\rangle_T.\end{aligned}$$

Noting that $|\mathbf{k}_{\mathbf{v}}| \le \delta$, we also have

$$\|e_T[\mathcal{R}_{\mathbf{v}}]\|_{L_\infty(\mathbb{R},H^1_q(\Omega))} \le C\langle^{\gamma t}V\rangle_T.$$

Thus, applying Lemmas 5.2 and 4.6, we obtain

$$\begin{aligned}&\|E[e^{\gamma t}R_4(U)]\|_{L_p(\mathbb{R},H^1_q(\Omega))} + \|E[e^{\gamma t}R_4(U)]\|_{H^{1/2}_p(\mathbb{R},L_q(\Omega))}\\ &\quad\le C(\mathcal{I} + \langle e^{\gamma t}V\rangle_T)(\|\tilde e_T[e^{\gamma t}H]\|_{L_p(\mathbb{R},H^2_q(\Omega))} + \|\partial_t\tilde e_T[e^{\gamma t}H]\|_{L_p(\mathbb{R},L_q(\Omega))}).\end{aligned}$$

Since $e^{\gamma t}H|_{t=0} = H|_{t=0}$, we have

$$\tilde e_T[e^{\gamma t}H] = e_T[e^{\gamma t}H - T(t)\tilde H_0] + \psi(t)T(|t|)\tilde H_0,$$

where $\tilde H_0$ is a function in $B^{2(1-1/p)}_{q,p}(\mathbb{R}^N)$ such that

$$\tilde H_0 = H|_{t=0} \quad \text{in } \Omega, \quad \|\tilde H_0\|_{B^{2(1-1/p)}_{q,p}(\mathbb{R}^N)} \le C\|H|_{t=0}\|_{B^{2(1-1/p)}_{q,p}(\Omega)}.$$

Thus, using (4.32) and (5.18), we get

$$\begin{aligned}&\|\tilde e_T[e^{\gamma t}H]\|_{L_p(\mathbb{R},H^2_q(\Omega))} + \|\partial_t\tilde e_T[e^{\gamma t}H]\|_{L_p(\mathbb{R},L_q(\Omega))}\\ &\quad\le C(\|e^{\gamma t}\nabla h\|_{L_p((0,T),H^1_q(\Omega))} + \|e^{\gamma t}\partial_t h\|_{L_p((0,T),L_q(\Omega))} + \|H|_{t=0}\|_{B^{2(1-1/p)}_{q,p}(\Omega)}).\end{aligned}$$

Finally, by Poincaré's inequality, we have

$$\|H|_{t=0}\|_{B^{2(1-1/p)}_{q,p}(\Omega)} = \|H|_{t=0}\|_{L_q(\Omega)} + \|\nabla(H|_{t=0})\|_{B^{1-2/p}_{q,p}(\Omega)} \le C\|\nabla h_0\|_{B^{1-2/p}_{q,p}(\Omega)}.$$

Summing up, we have obtained

$$\|E[e^{\gamma t}R_4(U)]\|_{H^{1/2}_p(\mathbb{R},L_q(\Omega))} + \|E[e^{\gamma t}R_4(U)]\|_{L_p(\mathbb{R},H^1_q(\Omega))} \le C(\mathcal{I} + \langle e^{\gamma t}V\rangle_T)^2. \tag{5.20}$$

Applying Theorem 5.1 to (5.3) and using the estimates (5.10), (5.11), (5.15), (5.16), (5.17), and (5.20), we have

$$\langle e^{\gamma t}V\rangle_T \le C(\mathcal{I} + (\mathcal{I} + \langle e^{\gamma t}V\rangle_T)^2).$$

We assume that $\mathcal{I} \le \epsilon < 1$, and so $(\mathcal{I} + \langle e^{\gamma t}V\rangle_T)^2 \le 2(\mathcal{I} + \langle e^{\gamma t}V\rangle_T^2)$, which completes the proof of (5.2).

We now prolong a local solution to $(0,\infty)$. Let $T > 0$ and η, $\mathbf{v}$, and ϑ be solutions of (2.15) satisfying (5.1). Then, by (5.2) we have

$$\langle e^{\gamma s}V\rangle_t \le C(\mathcal{I} + \langle e^{\gamma s}V\rangle_t^2) \tag{5.21}$$

for any $t \in (0, T)$, where C is independent of $t \in (0, T)$ and $T > 0$. Let $r_\pm(\epsilon)$ be two roots of the quadratic equation $C^{-1}x = \epsilon + x^2$, that is, $r_\pm(\epsilon) = (2C)^{-1} \pm \sqrt{(2C)^{-2} - \epsilon}$. We find a small positive number $\epsilon_1 > 0$ such that

$$0 < r_-(\epsilon) \le 2C\epsilon < 2C^{-1} < r_+(\epsilon)$$

for $0 < \epsilon < \epsilon_1$. Since $\langle e^{\gamma s} V\rangle_t$ satisfies the inequality (5.21), we have $\langle e^{\gamma s} V\rangle_t \le r_-(\epsilon)$ or $\langle e^{\gamma s} V\rangle_t \ge r_+(\epsilon)$. Since

$$\langle e^{\gamma s} V\rangle_t \to 0 \quad \text{as } t \to 0,$$

for small $t \in (0, T)$, we have $\langle e^{\gamma s} V\rangle_t \le r_-(\epsilon)$. But, $\langle e^{\gamma s} V\rangle_t$ is continuous with respect to $t \in (0, T)$, and so $\langle e^{\gamma s} V\rangle_t \le r_-(\epsilon)$ for any $t \in (0, T)$. Thus, we have

$$\langle e^{\gamma s} V\rangle_T \le 2C\epsilon. \tag{5.22}$$

By (5.7), (5.8), and (5.12), we see that there exists a constant $M > 0$ for which

$$\|\eta - \rho_0\|_{L_\infty((0,T),H^1_q(\Omega))} \le M\epsilon, \quad \|(\mathbf{v}, \vartheta - h_0)\|_{L_\infty((0,T),B^{2(1-1/p)}_{q,p}(\Omega))} \le M\epsilon,$$
$$\|(\rho_1, \rho_2) - (\rho_{1*}, \rho_{2*})\|_{L_\infty((0,T),H^1_q(\Omega))} \le M\epsilon. \tag{5.23}$$

Let η', $\mathbf{v}'$, and ϑ' be solutions of the following equations:

(5.24)
$$\begin{cases} \partial_t\eta' + \eta \operatorname{div}\mathbf{v}' = R'_1(U) \\ \qquad \text{in } \Omega \times (T, T+T_1), \\ \eta\partial_t\mathbf{v}' - \mu\Delta\mathbf{v}' - \nu\nabla\operatorname{div}\mathbf{v}' + \dfrac{\eta}{\Sigma_{\rho'}}\nabla\eta' \dfrac{(m_1 - m_2)\rho'_1\rho'_2}{\Sigma_{\rho'}}\nabla\vartheta' = R'_2(U) \\ \qquad \text{in } \Omega \times (T, T+T_1), \\ \dfrac{m_1m_2\rho'_1\rho'_2}{\Sigma_{\rho'}}\partial_t\vartheta' + \dfrac{(m_1 - m_2)\rho'_1\rho'_2}{\Sigma_{\rho'}}\operatorname{div}\mathbf{v}' - \operatorname{div}\Big(\dfrac{\rho'_1\rho'_2}{\mathfrak{p}'\rho'}\nabla\vartheta'\Big) = R'_3(U) \\ \qquad \text{in } \Omega \times (T, T+T_1), \\ \mathbf{v}' = 0, \quad (\nabla\vartheta)\cdot\mathbf{n} = R'_4(U) \\ \qquad \text{on } \Gamma \times (T, T+T_1), \\ (\eta', \mathbf{v}', \vartheta')|_{t=T} = (\eta(\cdot, T), \mathbf{v}(\cdot, T), \vartheta(\cdot, T)) \\ \qquad \text{in } \Omega. \end{cases}$$

Here, $\Sigma_{\rho'} = m_1\rho'_1 + m_2\rho'_2$, $\mathfrak{p}' = \rho'_1/m_1 + \rho'_2/m_2$, and $R_i(U)$ are defined by replacing $\int_0^t \nabla\mathbf{v}(\cdot, s)\,ds$, η, ρ_1, ρ_2, ρ $\mathbf{v}$, and ϑ by $\int_0^T \nabla\mathbf{v}(\cdot, s)\,ds + \int_T^t \nabla\mathbf{v}'(\cdot, s)\,ds$, η', ρ'_1, ρ'_2, ρ' $\mathbf{v}'$, and ϑ'. Employing the same argument as that in the proof of Theorem 2.1, we can show that there exists a T_1 depending on $\epsilon > 0$ such that problem (5.24) admits unique solutions η', $\mathbf{v}'$, and ϑ' with

$$\eta' \in H^1_p((T, T+T_1), H^1_q(\Omega)), \ \mathbf{v}' \in H^1_p((T, T+T_1), L_q(\Omega)^3) \cap L_p((T, T+T_1), H^2_q(\Omega)^3),$$
$$\vartheta' \in H^1_p((T, T+T_1), L_q(\Omega)) \cap L_p((T, T+T_1), H^2_q(\Omega)), \int_T^{T+T_1} \|\nabla\mathbf{v}'(\cdot, s)\|_{L_\infty(\Omega)}\,ds \le \delta,$$
$$\rho_*/4 \le \eta'(x, t) \le 4\rho_*, \quad |\vartheta'(x, t)| \le 4|h_*| \quad \text{for } (x, t) \in \Omega \times (T, T+T_1). \tag{5.25}$$

Choosing $\epsilon > 0$ small enough, in view of (5.23) we may assume that

$$\rho_{i*}/2 \le \rho_i(x, T) \le 2\rho_{i*} \quad \text{in } x \in \Omega \text{ for } i = 1, 2.$$

Thus, setting

$$f'' = \begin{cases} f & \text{for } t \in (0, T), \\ f' & \text{for } t \in (T, T+T_1), \end{cases}$$

for $f \in \{\eta, \mathbf{v}, \vartheta\}$, η'', $\mathbf{v}''$, and ϑ'' are solutions of (2.15) satisfying (5.1), where T is replaced by $T + T_1$. The repeated use of this argument implies the existence of solutions η, $\mathbf{v}$, ϑ of (2.15) with $T = \infty$, which satisfies the estimate $\langle e^{\gamma t} V\rangle_\infty \leq C\epsilon$. This completes the proof of Theorem 2.2.

6. Maximal L_p-L_q regularity—Proof of Theorem 4.1. In this section, we consider the linear problem (4.6) in a uniformly C^2 domain in the N-dimensional Euclidean space $\mathbb{R}^N$ ($N \geq 2$). To prove Theorem 4.1, we use the $\mathcal{R}$-bounded solution operators for the generalized resolvent problem corresponding to (4.6). We first make a definition.

Definition 6.1. *Let X and Y be two Banach spaces, and $\|\cdot\|_X$ and $\|\cdot\|_Y$ their norms. A family of operators $\mathcal{T} \subset \mathcal{L}(X, Y)$ is called $\mathcal{R}$-bounded on $\mathcal{L}(X, Y)$ if there exist constants $C > 0$ and $p \in [1, \infty)$ such that for any $n \in \mathbb{N}$, $\{T_j\}_{j=1}^n \subset \mathcal{T}$, and $\{f_j\}_{j=1}^n \subset X$, the inequality*

$$\int_0^1 \left\| \sum_{j=1}^n r_j(u) T_j f_j \right\|_Y^p du \leq C \int_0^1 \left\| \sum_{j=1}^n r_j(u) f_j \right\|_X^p du.$$

Here, the Rademacher functions $r_j : [0,1] \to \{-1, 1\}$, $j \in \mathbb{N}$, are given by $r_j(t) = \mathrm{sign}(\sin(2^j \pi t))$. The smallest such C is called $\mathcal{R}$-bound of $\mathcal{T}$ on $\mathcal{L}(X, Y)$ which is written by $\mathcal{R}_{\mathcal{L}(X,Y)}\mathcal{T}$.

The generalized resolvent problem corresponding to (4.6) is the following system:

$$\begin{cases} \lambda\zeta + \rho_0(x)\mathrm{div}\,\mathbf{v} = f_1 & \text{in } \Omega, \\ \rho_0(x)\lambda\mathbf{v} - \mu\Delta\mathbf{v} - \nu\nabla\mathrm{div}\,\mathbf{v} + \gamma_1(x)\nabla\zeta + \gamma_2(x)\nabla\vartheta = \mathbf{f}_2 & \text{in } \Omega, \\ \gamma_3(x)\lambda\vartheta + \gamma_2(x)\mathrm{div}\,\mathbf{v} - \mathrm{div}\,(\gamma_4(x)\nabla\vartheta) = f_3 & \text{in } \Omega, \\ \mathbf{v} = 0, \quad (\nabla\vartheta)\cdot\mathbf{n} = f_4 & \text{on } \Gamma. \end{cases} \tag{6.1}$$

We assume that the coefficients $\rho_0(x)$, $\gamma_i(x)$, $\quad i = 1, \dots, 4$, are uniformly continous on $\overline{\Omega}$ and satisfy the conditions (4.7). The main part of this section is to prove the following theorem concerning the existence of $\mathcal{R}$-bounded solution operators for (6.1).

Theorem 6.2. *Let $1 < q < \infty$ and $0 < \epsilon < \pi/2$. Assume that Ω is a uniform C^2 domain. Let*

$$X_q(\Omega) = \{(f_1, \mathbf{f}_2, f_3, f_4) \mid f_1, f_4 \in H_q^1(\Omega),\ \ \mathbf{f}_2 \in L_q(\Omega)^N,\ \ f_3 \in L_q(\Omega)\},$$
$$\mathcal{X}_q(\Omega) = \{(F_1, F_2, F_3, F_4, F_5) \mid F_1, F_5 \in H_q^1(\Omega),\ \ F_3, F_4 \in L_q(\Omega),\ \ F_2 \in L_q(\Omega)^N\}.$$

Then, there exist a positive constant λ_0 and operator families $\mathcal{A}(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_q(\Omega), H_q^1(\Omega)))$, $\mathcal{B}_1(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_q(\Omega), H_q^2(\Omega)^N))$, and $\mathcal{B}_2(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_q(\Omega), H_q^2(\Omega)))$ such that for any $(f_1, \mathbf{f}_2, f_3, f_4) \in X_q(\Omega)$ and $\lambda \in \Sigma_{\epsilon,\lambda_0}$, $\zeta = \mathcal{A}(\lambda)F_\lambda$, $\mathbf{v} = \mathcal{B}_1(\lambda)F_\lambda$, and $\vartheta = \mathcal{B}_2(\lambda)F_\lambda$ are unique solutions of (6.1), *where*

$F_\lambda = (f_1, \mathbf{f}_2, f_3, \lambda^{1/2} f_4, f_4)$, *and*

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), H^1_q(\Omega))}(\{(\tau\partial_\tau)^\ell \mathcal{A}(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le r_b,$$
$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), H^{2-j}_q(\Omega)^N)}(\{(\tau\partial_\tau)^\ell (\lambda^{j/2} \mathcal{B}_1(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le r_b,$$
$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), H^{2-j}_q(\Omega))}(\{(\tau\partial_\tau)^\ell (\lambda^{j/2} \mathcal{B}_2(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le r_b$$

for $\ell = 0, 1$, $j = 0, 1, 2$, *and for some constant* r_b.

Remark 6.3. F_1, F_2, F_3, F_4, and F_5 are variables corresponding to f_1, f_2, f_3, $\lambda^{1/2} f_4$, and f_4. The norm of $\mathcal{X}_q(\Omega)$ is defined by

$$\|(F_1, F_2, F_3, F_4, F_5)\|_{\mathcal{X}_q(\Omega)} = \|(F_1, F_5)\|_{H^1_q(\Omega)} + \|(F_2, F_3, F_4)\|_{L_q(\Omega)}.$$

Since we consider the case that $\lambda \in \Sigma_{\epsilon,\lambda_0}$ with $\lambda_0 > 0$, setting $\zeta = \lambda^{-1}(f_1 - \rho_0(x)\mathrm{div}\, \mathbf{v})$, and inserting this formula into the second equation in (6.1), we rewrite it as

(6.2)
$$\rho_0(x)\lambda \mathbf{v} - \mu\Delta \mathbf{v} - \nu\nabla \mathrm{div}\, \mathbf{v} - \gamma_1(x)\lambda^{-1}\nabla(\rho_0(x)\mathrm{div}\, \mathbf{v}) + \gamma_2(x)\nabla\vartheta = \mathbf{f}_2 - \lambda^{-1}\gamma_1(x)\nabla f_1.$$

Since $\gamma_2(x)\nabla\vartheta$ and $\gamma_2(x)\mathrm{div}\, \mathbf{v}$ are lower order terms, our main concern is to prove the existence of $\mathcal{R}$-bounded solution operators for the following two equations:

$$\rho_0(x)\lambda \mathbf{v} - \mu\Delta \mathbf{v} - \nu\nabla \mathrm{div}\, \mathbf{v} - \gamma_1(x)\lambda^{-1}\nabla(\rho_0(x)\mathrm{div}\, \mathbf{v}) = \mathbf{g} \quad \text{in } \Omega, \quad \mathbf{v}|_\Gamma = 0; \tag{6.3}$$
$$\gamma_3(x)\lambda\vartheta - \mathrm{div}\,(\gamma_4(x)\nabla\vartheta) = h_1 \quad \text{in } \Omega, \quad (\nabla\vartheta)\cdot \mathbf{n}|_\Gamma = h_2. \tag{6.4}$$

Let us denote

$$Y_q(G) = \{(h_1, h_2) \mid h_1 \in L_q(G), \quad h_2 \in H^1_q(G)\}, \quad \|(h_1, h_2)\|_{Y_q(G)} = \|h_1\|_{L_q(G)}$$
$$\mathcal{Y}_q(G) = \{(F_1, F_2, F_3) \mid F_1, F_2 \in L_q(\Omega), \quad F_3 \in H^1_q(\Omega)\},$$
$$\|(F_1, F_2, F_3)\|_{\mathcal{Y}_q(G)} = \|(F_1, F_2)\|_{L_q(G)} + \|F_3\|_{H^1_q(G)}.$$

Then, we shall prove the following theorem.

THEOREM 6.4. *Let* $1 < q < \infty$ *and* $0 < \epsilon < \pi/2$. *Assume that* Ω *is a uniform* C^2 *domain in* $\mathbb{R}^N$. *Then, there exists a positive constant* λ_0 *such that the following assertions hold:*

(1) *There exists an operator family* $\mathcal{C}(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(L_q(\Omega)^N, H^2_q(\Omega)^N))$ *such that for any* $\lambda \in \Sigma_{\epsilon,\lambda_0}$ *and* $\mathbf{g} \in L_q(\Omega)^N$, $\mathbf{v} = \mathcal{C}(\lambda)\mathbf{g}$ *is a unique solution of* (6.3), *and*

$$\mathcal{R}_{\mathcal{L}(L_q(\Omega)^N, H^{2-j}_q(\Omega)^N)}(\{(\tau\partial_\tau)^\ell \mathcal{C}(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le r_b$$

for $\ell = 0, 1$ *and* $j = 0, 1, 2$.

(2) *Let* $Y_q(\Omega)$ *and* $\mathcal{Y}_q(\Omega)$ *be the spaces defined above with* $G = \Omega$. *Then, there exists an operator family* $\mathcal{D}(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{Y}_q(\Omega), H^2_q(\Omega))$ *such that for any* $\lambda \in \Sigma_{\epsilon,\lambda_0}$ *and* $(h_1, h_2) \in Y_q(\Omega)$, $\vartheta = \mathcal{D}(\lambda)(h_1, \lambda^{1/2} h_2, h_2)$ *is a unique solution of* (6.4), *and*

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\Omega), H^{2-j}_q(\Omega))}(\{(\tau\partial_\tau)^\ell \mathcal{D}(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le r_b$$

for $\ell = 0, 1$ *and* $j = 0, 1, 2$.

6.1. The model problems in $\mathbb{R}^N$ and $\mathbb{R}^N_+$. First, we consider the model problem in $\mathbb{R}^N$. In what follows, let ρ_{0*}, γ_{1*}, γ_{3*}, and γ_{4*} be positive constants. Assume that there exist two positive constants b_1 and b_2 for which

$$b_1 \le \rho_{0*}, \gamma_{*1}, \gamma_{3*}, \gamma_{4*} \le b_2. \tag{6.5}$$

Let us consider the following problems:

$$\rho_{0*}\lambda\mathbf{v} - \mu\Delta\mathbf{v} - \nu\nabla\mathrm{div}\,\mathbf{v} - \gamma_{1*}\rho_{0*}\lambda^{-1}\nabla\mathrm{div}\,\mathbf{v} = \mathbf{g} \quad \text{in } \mathbb{R}^N; \tag{6.6}$$

$$\gamma_{3*}\lambda\vartheta - \gamma_{4*}\Delta\vartheta = h \quad \text{in } \mathbb{R}^N. \tag{6.7}$$

THEOREM 6.5. *Let $1 < q < \infty$ and $0 < \epsilon < \pi/2$. Then, we have the following assertions:*

(1) *There exist a large constant $\lambda_0 > 0$ and an operator family $\mathcal{C}_1(\lambda)$ with*

$$\mathcal{C}_1(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(L_q(\mathbb{R}^N)^N, H^2_q(\mathbb{R}^N)^N))$$

such that for any $\mathbf{g} \in L_q(\mathbb{R}^N)^N$ and $\lambda \in \Sigma_{\epsilon,\lambda_0}$, $\mathbf{v} = \mathcal{C}_1(\lambda)\mathbf{g}$ is a unique solution of (6.6)*, and*

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N)^N, H^{2-j}_q(\mathbb{R}^N)^N)}(\{(\tau\partial_\tau)^\ell \mathcal{C}_1(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le r_{b1}$$

for $\ell = 0, 1$ and $j = 0, 1, 2$. Here, λ_0 and r_{1b} depend solely on N, q, μ, ν, b_1, and b_2.

(2) *Let $\lambda_0 \ge 1$. Then, there exists an operator family*

$$\mathcal{D}_1(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(L_q(\mathbb{R}^N), H^2_q(\mathbb{R}^N)))$$

such that for any $h \in Y_q(\mathbb{R}^N)$ and $\lambda \in \Sigma_{\epsilon,\lambda_0}$, $\vartheta = \mathcal{D}_1(\lambda)h$ is a unique solution of (6.7)*, and*

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N), H^{2-j}_q(\mathbb{R}^N)))}(\{(\tau\partial_\tau)^\ell \mathcal{D}_1(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le r_{b2}$$

for $\ell = 0, 1$ and $j = 0, 1, 2$. Here, r_{b2} depends solely on N, q, λ_0, b_1, and b_2.

Proof. The assertion (1) was proved in Enomoto and Shibata [11, Theorem 3.2], and so we may omit the proof. To prove (2), using the Fourier tranform $\mathcal{F}$ and its inversion formula $\mathcal{F}^{-1}$, we define ϑ by

$$\vartheta = \mathcal{F}^{-1}\left[\frac{\mathcal{F}[h](\xi)}{\gamma_{3*}\lambda + \gamma_{4*}|\xi|^2}\right](x).$$

Thus, by Lemma 3.1 and Theorem 3.3 in [11], we can show the assertion (2). Thus, we also may omit the detailed proof. ∎

Next we consider the half space problem. Let

$$\mathbb{R}^N_+ = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N > 0\}, \quad \mathbb{R}^N_0 = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N = 0\},$$

and $\mathbf{n}_0 = (0, \ldots, 0, -1)^\top$. We consider the following problems in $\mathbb{R}^N_+$:

$$\rho_{0*}\lambda\mathbf{v} - \mu\Delta\mathbf{v} - \nu\nabla\mathrm{div}\,\mathbf{v} - \gamma_{1*}\rho_{0*}\lambda^{-1}\nabla\mathrm{div}\,\mathbf{v} = \mathbf{g} \quad \text{in } \mathbb{R}^N_+, \quad \mathbf{v}|_{\mathbb{R}^N_0} = 0; \tag{6.8}$$

$$\gamma_{3*}\lambda\vartheta - \gamma_{4*}\Delta\vartheta = h_1 \quad \text{in } \mathbb{R}^N_+, \quad (\nabla\vartheta)\cdot\mathbf{n}_0 = h_2 \quad \text{on } \mathbb{R}^N_0. \tag{6.9}$$

THEOREM 6.6. *Let $1 < q < \infty$, $0 < \epsilon < \pi/2$, and $\lambda_0 \geq 1$.*

(1) *There exist a large constant $\lambda_0 > 0$ and an operator family $\mathcal{C}_2(\lambda)$ with*

$$\mathcal{C}_2(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(L_q(\mathbb{R}^N_+)^N, H^2_q(\mathbb{R}^N_+)^N)$$

such that for any $\mathbf{g} \in L_q(\mathbb{R}^N_+)^N$ and $\lambda \in \Sigma_{\epsilon,\lambda_0}$, $\mathbf{v} = \mathcal{C}_2(\lambda)\mathbf{g}$ is a unique solution of (6.8), *and*

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+)^N, H^{2-j}_q(\mathbb{R}^N_+)^N)}(\{(\tau\partial_\tau)^\ell \mathcal{C}_2(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \leq r_b$$

for $\ell = 0, 1$ and $j = 0, 1, 2$.

(2) *Let $\lambda_0 \geq 1$. Then, there exists an operator family*

$$\mathcal{D}_2(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{Y}_q(\mathbb{R}^N_+), H^2_q(\mathbb{R}^N_+))$$

such that for any $(h_1, h_2) \in Y_q(\mathbb{R}^N_+)$ and $\lambda \in \Sigma_{\epsilon,\lambda_0}$, $\vartheta = \mathcal{D}_2(\lambda)(h_1, \lambda^{1/2}h_2, h_2)$ is a unique solution of (6.9), *and*

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\mathbb{R}^N_+), H^{2-j}_q(\mathbb{R}^N_+))}(\{(\tau\partial_\tau)^\ell \mathcal{D}_2(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \leq r_b$$

for $\ell = 0, 1$ and $j = 0, 1, 2$.

Here, $Y_q(\mathbb{R}^N_+)$ and $\mathcal{Y}_q(\mathbb{R}^N_+)$ are spaces defined in section 1 *with $G = \mathbb{R}^N_+$, and r_b is a constant depending solely on N, q, λ_0, b_1, and b_2.*

Proof. The first assertion has been proved in [11, Theorem 4.1]. To prove the second one we divide a solution of (6.9) into two parts: $\vartheta = \vartheta_1 + \vartheta_2$, where ϑ_1 and ϑ_2 are solutions of the problems

$$\gamma_{3*}\lambda\vartheta_1 - \gamma_{4*}\Delta\vartheta_1 = h_1 \quad \text{in } \mathbb{R}^N_+, \quad (\nabla\vartheta_1)\cdot\mathbf{n}_0 = 0 \quad \text{on } \mathbb{R}^N_0; \tag{6.10}$$

$$\gamma_{3*}\lambda\vartheta_2 - \gamma_{4*}\Delta\vartheta_2 = 0 \quad \text{in } \mathbb{R}^N_+, \quad (\nabla\vartheta_2)\cdot\mathbf{n}_0 = h_2 \quad \text{on } \mathbb{R}^N_0. \tag{6.11}$$

Given function F_1 defined on $\mathbb{R}^N_+$, let F^e_1 be the even extension of F_1 to $x_N < 0$, that is, $F^e_1(x) = F_1(x)$ for $x_N > 0$ and $F^e_1(x) = F_1(x', -x_N)$ for $x_N < 0$, where $x' = (x_1, \dots, x_{N-1})$. We then define an $\mathcal{R}$ bounded solution operator $\mathcal{D}_{21}(\lambda)$ acting on $F_1 \in L_q(\mathbb{R}^N_+)$ by

$$\mathcal{D}_{21}(\lambda)[F_1] = \mathcal{F}^{-1}\left[\frac{\mathcal{F}[F^e_1](\xi)}{\gamma_{3*}\lambda + \gamma_{4*}|\xi|^2}\right].$$

Obviously, $\vartheta_1 = \mathcal{D}_1(\lambda)[h_1]$ is a unique solution of (6.10).

To construct an $\mathcal{R}$ bounded solution operator for (6.11), we introduce the partial Fourier transform $\mathcal{F}'$ and its inversion formula $\mathcal{F}^{-1}_{\xi'}$, which are defined by

$$\hat{f}(\xi', x_N) = \mathcal{F}'[f](\xi', x_N) = \int_{\mathbb{R}^{N-1}} e^{-ix'\cdot\xi'} f(x', x_N)\,dx',$$
$$\mathcal{F}^{-1}_{\xi'}[g(\xi', x_N)](x') = \frac{1}{(2\pi)^{N-1}}\int_{\mathbb{R}^{N-1}} e^{ix'\cdot\xi'} g(\xi', x_N)\,d\xi',$$

where $\xi' = (\xi_1, \dots, \xi_{N-1}) \in \mathbb{R}^{N-1}$ and $x'\cdot\xi' = \sum_{j=1}^{N-1} x_j\xi_j$. Applying the partial Fourier transform to (6.11), we have

$$(\gamma_{3*}\lambda + \gamma_{4*}|\xi'|^2)\hat{\vartheta} - \gamma_{*4}\partial^2_N\hat{\vartheta} = 0 \quad \text{for } x_N > 0, \quad \partial_N\vartheta|_{x_N=0} = -\hat{h}_2(\xi', 0),$$

where $|\xi'|^2 = \sum_{j=1}^{N-1} \xi_j^2$ and $\partial_N = \partial/\partial_N$. Thus, ϑ_2 is given by

$$\begin{aligned}\vartheta_2 &= \mathcal{F}^{-1}\left[\frac{e^{-\sqrt{\gamma_{3*}\gamma_{4*}^{-1}\lambda+|\xi'|^2}\,x_N}}{\sqrt{\gamma_{3*}\gamma_{4*}^{-1}\lambda+|\xi'|^2}}\hat{h}_2(\xi',0)\right](x')\\
&= \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\frac{e^{-\sqrt{\gamma_{3*}\gamma_{4*}^{-1}\lambda+|\xi'|^2}\,(x_N+y_N)}}{\sqrt{\gamma_{3*}\gamma_{4*}^{-1}\lambda+|\xi'|^2}}\sqrt{\gamma_{3*}\gamma_{4*}^{-1}\lambda+|\xi'|^2}\hat{h}_2(\xi',y_N)\right](x')\\
&\quad - \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\frac{e^{-\sqrt{\gamma_{3*}\gamma_{4*}^{-1}\lambda+|\xi'|^2}\,(x_N+y_N)}}{\sqrt{\gamma_{3*}\gamma_{4*}^{-1}\lambda+|\xi'|^2}}\partial_N\hat{h}_2(\xi',y_N)\right](x').\end{aligned}$$

Writing

$$\sqrt{\gamma_{3*}\gamma_{4*}^{-1}\lambda+|\xi'|^2} = \frac{\gamma_{3*}\gamma_{4*}^{-1}\lambda}{\sqrt{\gamma_{3*}\gamma_{4*}^{-1}\lambda+|\xi'|^2}} - \sum_{j=1}^{N-1}\frac{i\xi_j i\xi_j}{\sqrt{\gamma_{3*}\gamma_{4*}^{-1}\lambda+|\xi'|^2}}$$

we have

$$\begin{aligned}&\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\frac{e^{-\sqrt{\gamma_{3*}\gamma_{4*}^{-1}\lambda+|\xi'|^2}\,(x_N+y_N)}}{\sqrt{\gamma_{3*}\gamma_{4*}^{-1}\lambda+|\xi'|^2}}\sqrt{\gamma_{3*}\gamma_{4*}^{-1}\lambda+|\xi'|^2}\hat{h}_2(\xi',y_N)\right](x')\\
&= \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\lambda^{1/2}e^{-\sqrt{\gamma_{3*}\gamma_{4*}^{-1}\lambda+|\xi'|^2}\,(x_N+y_N)}\frac{\gamma_{3*}\gamma_{4*}^{-1}\lambda^{1/2}}{\sqrt{\gamma_{3*}\gamma_{4*}^{-1}\lambda+|\xi'|^2}}\hat{h}_2(\xi',y_N)\right](x')\\
&- \sum_{j=1}^{N-1}\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[|\xi'|e^{-\sqrt{\gamma_{3*}\gamma_{4*}^{-1}\lambda+|\xi'|^2}\,(x_N+y_N)}\frac{i\xi_j}{|\xi'|\sqrt{\gamma_{3*}\gamma_{4*}^{-1}\lambda+|\xi'|^2}}\mathcal{F}'[\partial_j h_2](\xi',y_N)\right](x').\end{aligned}$$

We then define an operator $\mathcal{D}_{22}(\lambda)$ acting on $(F_2, F_3) \in L_q(\mathbb{R}^N_+ \times H^1_q(\mathbb{R}^N_+)$ by

$$\begin{aligned}&\mathcal{D}_{22}(\lambda)(F_2,F_3) = \int_0^\infty \mathcal{F}_{\xi'}^{-1}\\
&\left[\lambda^{1/2}e^{-\sqrt{\gamma_{3*}\gamma_{4*}^{-1}\lambda+|\xi'|^2}\,(x_N+y_N)}\frac{\gamma_{3*}\gamma_{4*}^{-1}}{\sqrt{\gamma_{3*}\gamma_{4*}^{-1}\lambda+|\xi'|^2}}\mathcal{F}'[F_2](\xi',y_N)\right](x')\\
&- \sum_{j=1}^{N-1}\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[|\xi'|e^{-\sqrt{\gamma_{3*}\gamma_{4*}^{-1}\lambda+|\xi'|^2}\,(x_N+y_N)}\frac{i\xi_j}{|\xi'|\sqrt{\gamma_{3*}\gamma_{4*}^{-1}\lambda+|\xi'|^2}}\mathcal{F}'[\partial_j F_3](\xi',y_N)\right](x')\\
&- \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\frac{e^{-\sqrt{\gamma_{3*}\gamma_{4*}^{-1}\lambda+|\xi'|^2}\,(x_N+y_N)}}{\sqrt{\gamma_{3*}\gamma_{4*}^{-1}\lambda+|\xi'|^2}}\mathcal{F}'[\partial_N F_3](\xi',y_N)\right](x').\end{aligned}$$

Obviously, $\vartheta_2 = \mathcal{D}_2(\lambda)(\lambda^{1/2}h_2, h_2)$. Moreover, the $\mathcal{R}$ boundedness of the operator $\mathcal{D}_{22}(\lambda)$ follows from Lemma 4.2 in [11]. This completes the proof of the assertion (2). ∎

6.2. Problem in a bent half space. Let $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ be a bijection of C^2 class and let Φ^{-1} be its inverse map. Writing $\nabla\Phi = \mathcal{A} + B(x)$ and $\nabla\Phi^{-1} = \mathcal{A}_- +$

$B_-(x)$, we assume that $\mathcal{A}$ and $\mathcal{A}_-$ are orthogonal matrices with constant coefficients and $B(x)$ and $B_-(x)$ are matrices of functions in $C^1(\mathbb{R}^N)$ with $N < r < \infty$ such that

$$\|(B, B_-)\|_{L_\infty(\mathbb{R}^N)} \leq M_1, \quad \|\nabla(B, B_-)\|_{L_\infty(\mathbb{R}^N)} \leq M_2. \tag{6.12}$$

We will choose M_1 small enough eventually, and so we may assume that $0 < M_1 \leq 1 \leq M_2$ in the following. Set $\Omega_+ = \Phi(\mathbb{R}^N_+)$ and $\Gamma_+ = \Phi(\mathbb{R}^N_0)$. Let $\mathbf{n}_+$ be the unit outer normal to Γ_+. Since Γ_+ is represented by $\Phi_{-1,N}(y) = 0$, where $\Phi^{-1} = (\Phi_{-1,1}, \ldots, \Phi_{-1,N})^\top$, $\mathbf{n}_+$ is given by

$$\mathbf{n}_+ = -\frac{\nabla\Phi_{-1,N}}{|\nabla\Phi_{-1,N}|} = -\frac{(\mathcal{A}_{N1} + B_{N1}, \ldots, \mathcal{A}_{NN} + B_{NN})^\top}{\sqrt{\sum_{j=1}^N (A_{Nj} + B_{Nj})^2}}. \tag{6.13}$$

Choosing $M_1 > 0$ small enough, by (6.12) we have

$$\mathbf{n}_+ = -(\mathcal{A}_{N1}, \ldots, \mathcal{A}_{NN})^\top + \tilde{\mathbf{n}}_+, \tag{6.14}$$

where $\tilde{\mathbf{n}}_+$ has the estimates

$$\|\tilde{\mathbf{n}}_+\|_{L_\infty(\mathbb{R}^N)} \leq C_N M_1, \quad \|\nabla\tilde{\mathbf{n}}_+\|_{L_\infty(\mathbb{R}^N)} \leq C_{M_2}. \tag{6.15}$$

We consider the following two equations:

$$\rho_{0*}\lambda\mathbf{v} - \mu\Delta\mathbf{v} - \nu\nabla\mathrm{div}\,\mathbf{v} - \gamma_{1*}\rho_{0*}\lambda^{-1}\nabla\mathrm{div}\,\mathbf{v} = \mathbf{g} \quad \text{in } \Omega_+, \quad \mathbf{v}|_{\Gamma_+} = 0; \tag{6.16}$$

$$\gamma_{3*}\lambda\vartheta - \gamma_{4*}\Delta\vartheta = h_1 \quad \text{in } \Omega_+, \quad (\nabla\vartheta)\cdot\mathbf{n}_0 = h_2 \quad \text{on } \Gamma_+. \tag{6.17}$$

THEOREM 6.7. *Let $1 < q < \infty$ and $0 < \epsilon < \pi/2$. Then, we have the following assertions:*

(1) *There exist a large constant $\lambda_0 > 0$ and an operator family $\mathcal{C}_3(\lambda)$ with*

$$\mathcal{C}_3(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(L_q(\Omega_+)^N, H^2_q(\Omega_+)^N)$$

such that for any $\mathbf{g} \in L_q(\Omega_+)^N$ and $\lambda \in \Sigma_{\epsilon,\lambda_0}$, $\mathbf{v} = \mathcal{C}_3(\lambda)\mathbf{g}$ is a unique solution of (6.16)*, and*

$$\mathcal{R}_{\mathcal{L}(L_q(\Omega_+)^N, H^{2-j}_q(\Omega_+)^N)}(\{(\tau\partial_\tau)^\ell\mathcal{C}_3(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \leq r_{b1}$$

for $\ell = 0, 1$ and $j = 0, 1, 2$. Here, r_{b1} is a constant depending solely on N, q, μ, ν, b_1, and b_2.

(2) *Let $Y_q(\Omega_+)$ and $\mathcal{Y}_q(\Omega_+)$ be spaces defined by replacing Ω by Ω_+ in Theorem* 6.4*. Then, there exist a positive constant λ_0 and an operator family $\mathcal{D}_3(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{Y}_q(\Omega_+), H^2_q(\Omega_+))$ such that for any $(h_1, h_2) \in Y_q(\Omega_+)$ and $\lambda \in \Sigma_{\epsilon,\lambda_0}$, $\vartheta = \mathcal{D}_3(\lambda)(h_1, \lambda^{1/2}h_2, h_2)$ is a unique solution of* (6.17)*, and*

$$\mathcal{R}_{\mathcal{L}(L_q(\Omega_+), H^{2-j}_q(\Omega_+))}(\{(\tau\partial_\tau)^\ell\mathcal{D}_2(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \leq r_{b2}$$

for $\ell = 0, 1$ and $j = 0, 1, 2$. Here, r_{b2} is a constant depending solely on N, q, b_1, and b_2.

Proof. The first assertion was proved in Enomoto and Shibata [11, Theorem 5.1], and so we may omit the proof. Thus, we prove the assertion (2) below. For this purpose, we shall transform (6.17) into the equations in $\mathbb{R}^N_+$ by the change of variables: $x = \Phi^{-1}(y)$ with $x \in \mathbb{R}^N_+$ and $y \in \Omega_+$. We have

$$\frac{\partial}{\partial y_j} = \sum_{k=1}^N (\mathcal{A}_{kj} + B_{kj}(x))\frac{\partial}{\partial x_k}, \tag{6.18}$$

where $\mathcal{A}_{kj}$ is the $(k,j)^{\rm th}$ component of $\mathcal{A}_-$ and $B_{kj}(x)$ is the $(k,j)^{\rm th}$ component of $B_-(\Phi(x))$. Let $\varphi(x) = \vartheta(\Phi(x))$ in (6.17), and then by (6.14) and (6.18) we have

$$\gamma_{3*}\lambda\varphi - \gamma_{4*}[\Delta\varphi + A_1\nabla^2\varphi + A_2\nabla\varphi] = H_1 \quad \text{in } \mathbb{R}^N_+, \quad (\nabla\varphi)\cdot\mathbf{n}_0 + (\nabla\varphi)\cdot\mathbf{n}_1 = H_2 \quad \text{on } \mathbb{R}^N_0. \tag{6.19}$$

Here, we have set

$$\begin{aligned} A_1\nabla^2\varphi &= \sum_{j,k,\ell=1}^N (\mathcal{A}_{kj}B_{\ell j}(x) + \mathcal{A}_{\ell j}B_{kj}(x) + B_{kj}(x)B_{\ell j}(x))\frac{\partial^2\varphi}{\partial x_k \partial_\ell}, \\ A_2\nabla\varphi &= \sum_{j,k,\ell=1}^N (\mathcal{A}_{kj} + B_{kj}(x))\Big(\frac{\partial}{\partial x_k}B_{\ell j}(x)\Big)\frac{\partial\varphi}{\partial x_\ell}, \\ (\nabla\varphi)\cdot\mathbf{n}_1 &= \sum_{j,k=1}^N (\mathcal{A}_{Nj}B_{kj}(x) + \tilde{n}_j(x)\mathcal{A}_{kj} + \tilde{n}_j(x)B_{kj}(x))\frac{\partial\varphi}{\partial x_k}. \end{aligned}$$

Notice that

$$\begin{aligned} \|A_1\nabla^2\varphi\|_{L_q(\mathbb{R}^N_+)} &\le CM_1\|\nabla^2\varphi\|_{L_q(\mathbb{R}^N_+)}, \\ \|A_2\nabla\varphi\|_{L_q(\mathbb{R}^N_+)} &\le C_{M_2}\|\nabla\varphi\|_{L_q(\mathbb{R}^N_+)}, \\ \|(\nabla\varphi)\cdot\mathbf{n}_1\|_{L_q(\mathbb{R}^N_+)} &\le CM_1\|\nabla\varphi\|_{L_q(\mathbb{R}^N_+)}, \\ \|(\nabla\varphi)\cdot\mathbf{n}_1\|_{H^1_q(\mathbb{R}^N_+)} &\le C(M_1\|\nabla^2\varphi\|_{L_q(\mathbb{R}^N_+)} + C_{M_2}\|\nabla\varphi\|_{L_q(\mathbb{R}^N_+)}). \end{aligned} \tag{6.20}$$

Let $\mathcal{C}_2(\lambda)$ be an $\mathcal{R}$-bounded solution operator given in Theorem 6.6 and set $\psi = \mathcal{C}_2(\lambda)F_\lambda(H_1,H_2)$. Here and in the following, F_λ is an operator acting on $(H_1,H_2) \in Y_q(\mathbb{R}^N_+)$ defined by $F_\lambda(H_1,H_2) = (H_1, \lambda^{1/2}H_2, H_2) \in \mathcal{Y}_q(\mathbb{R}^N_+)$. We then have

$$\begin{aligned} \gamma_{3*}\lambda\psi - \gamma_{4*}(\Delta\psi + A_1\nabla^2\psi + A_2\nabla\psi) &= H_1 + R_1(\lambda)(H_1,H_2) \quad \text{in } \mathbb{R}^N_+, \\ (\nabla\psi)\cdot\mathbf{n}_0 + (\nabla\psi)\cdot\mathbf{n}_1 &= H_2 + R_2(\lambda)(H_1,H_2) \quad \text{on } \mathbb{R}^N_0, \end{aligned} \tag{6.21}$$

where

$$\begin{aligned} &R_1(\lambda)(H_1,H_2) = \gamma_{4*}(A_1\nabla^2\mathcal{C}_2(\lambda)F_\lambda(H_1,H_2) + A_2\nabla\mathcal{C}_2(\lambda)F_\lambda(H_1,H_2)), \\ &R_2(\lambda)(H_1,H_2) = (\nabla\mathcal{C}_2(\lambda)F_\lambda(H_1,H_2))\cdot\mathbf{n}_1. \end{aligned} \tag{6.22}$$

For $F = (F_1,F_2,F_3) \in \mathcal{Y}_q(\mathbb{R}^N_+)$, let

$$\mathcal{R}_1(\lambda)F = \gamma_{4*}(A_1\nabla^2\mathcal{C}_2(\lambda)F) + A_2\nabla\mathcal{C}_2(\lambda)F, \quad \mathcal{R}_2(\lambda)F = [\nabla\mathcal{C}_2(\lambda)F]\cdot\mathbf{n}_1,$$

and let $\mathcal{R}(\lambda)F = (\mathcal{R}_1(\lambda)F, \mathcal{R}_2(\lambda)F) \in Y_q(\mathbb{R}^N_+)$ and

$$R(\lambda)(H_1,H_2) = (R_1(\lambda)(H_1,H_2), R_2(\lambda)(H_1,H_2)).$$

We then have

$$\mathcal{R}(\lambda)F_\lambda(H_1,H_2) = R(\lambda)(H_1,H_2). \tag{6.23}$$

We now use the following two lemmas to calculate the $\mathcal{R}$-norm.

LEMMA 6.8. (1) *Let X and Y be Banach spaces, and let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families in $\mathcal{L}(X,Y)$. Then, $\mathcal{T}+\mathcal{S}=\{T+S \mid T\in\mathcal{T},\ S\in\mathcal{S}\}$ is also an $\mathcal{R}$-bounded family in $\mathcal{L}(X,Y)$ and*

$$\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T}+\mathcal{S}) \le \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T}) + \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{S}).$$

(2) *Let X, Y, and Z be Banach spaces, and let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families in $\mathcal{L}(X,Y)$ and $\mathcal{L}(Y,Z)$, respectively. Then, $\mathcal{S}\mathcal{T}=\{ST \mid T\in\mathcal{T},\ S\in\mathcal{S}\}$ is also an $\mathcal{R}$-bounded family in $\mathcal{L}(X,Z)$ and*

$$\mathcal{R}_{\mathcal{L}(X,Z)}(\mathcal{S}\mathcal{T}) \le \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})\mathcal{R}_{\mathcal{L}(Y,Z)}(\mathcal{S}).$$

LEMMA 6.9. *Let $1<p,\, q<\infty$ and let D be a domain in $\mathbb{R}^N$.*

(1) *Let $m(\lambda)$ be a bounded function defined on a subset Λ in a complex plane $\mathbb{C}$ and let $M_m(\lambda)$ be a multiplication operator with $m(\lambda)$ defined by $M_m(\lambda)f=m(\lambda)f$ for any $f\in L_q(D)$. Then,*

$$\mathcal{R}_{\mathcal{L}(L_q(D))}(\{M_m(\lambda) \mid \lambda\in\Lambda\}) \le C_{N,q,D}\|m\|_{L_\infty(\Lambda)}.$$

(2) *Let $n(\tau)$ be a C^1 function defined on $\mathbb{R}\setminus\{0\}$ that satisfies the conditions $|n(\tau)|\le\gamma$ and $|\tau n'(\tau)|\le\gamma$ with some constant $\gamma>0$ for any $\tau\in\mathbb{R}\setminus\{0\}$. Let T_n be an operator valued Fourier multiplier defined by $T_n f=\mathcal{F}^{-1}[n\mathcal{F}[f]]$ for any $f\in\mathcal{S}(\mathbb{R},X)$ with $\mathcal{F}[f]\in\mathcal{D}(\mathbb{R},X)$. Then, T_n is extended to a bounded linear operator from $L_p(\mathbb{R},L_q(D))$ into itself. Moreover, denoting this extension also by T_n, we have*

$$\|T_n\|_{\mathcal{L}(L_p(\mathbb{R},L_q(D)))} \le C_{p,q,D}\gamma.$$

Remark 6.10. For proofs of Lemmas 6.8 and 6.9, we refer to [9, Proposition 3.4, p. 28 and Remarks (4), p. 27] (cf. also Bourgain [5]), respectively.

By Lemmas 6.8 and 6.9, (6.20), and Theorem 6.6(2) we have

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\mathbb{R}^N_+))}(\{(\tau\partial_\tau)^\ell F_\lambda\mathcal{R}(\lambda) \mid \lambda\in\Sigma_{\epsilon,\lambda_1}\}) \le r_b(CM_1+C_{M_2}\tilde\lambda_0^{-1/2}) \tag{6.24}$$

for any $\tilde\lambda_0\ge\lambda_0$. In fact, by (6.20), we have

$$\int_0^1\Big\|\sum_{j-1}^n r_j(u)A_2\nabla\mathcal{C}_2(\lambda_j)F_j\Big\|^q_{L_q(\mathbb{R}^N_+)}du \le C^q_{M_2}\int_0^1\Big\|\sum_{j=1}^n r_j(u)\nabla\mathcal{C}_2(\lambda_j)F_j\Big\|^q_{L_q(\mathbb{R}^N_+)}du.$$

By Lemma 6.9, we have

$$\begin{aligned}\int_0^1\Big\|\sum_{j=1}^n r_j(u)\nabla\mathcal{C}_2(\lambda_j)F_j\|^q_{L_q(\mathbb{R}^N_+)}du &= \int_0^1\Big\|\sum_{j=1}^n r_j(u)\lambda_j^{-1/2}\lambda_j^{1/2}\nabla\mathcal{C}_2(\lambda_j)F_j\Big\|^q_{L_q(\mathbb{R}^N_+)}du\\ &\le \tilde\lambda_0^{-q/2}\int_0^1\Big\|\sum_{j=1}^n r_j(u)\lambda_j^{1/2}\nabla\mathcal{C}_2(\lambda_j)F_j\Big\|^q_{L_q(\mathbb{R}^N_+)}du\end{aligned}$$

for any $\lambda_j\in\Sigma_{\epsilon,\tilde\lambda_0}$ and $\tilde\lambda_0\ge\lambda_0$. Thus, by Theorem 6.4(2), we have

$$\int_0^1\Big\|\sum_{j-1}^n r_j(u)A_2\nabla\mathcal{C}_2(\lambda_j)F_j\Big\|^q_{L_q(\mathbb{R}^N_+)}du \le \tilde\lambda_0^{-q/2}r_b^q\int_0^1\Big\|\sum_{j=1}^n r_j(u)F_j\Big\|^q_{L_q(\mathbb{R}^N_+)}du.$$

Analogously, we can estimate $\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\mathbb{R}^N_+),L_q(\mathbb{R}^N_+))}$ norm of B_1 and B_2 and $\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\mathbb{R}^N_+),H^1_q(\mathbb{R}^N_+))}$ norm of B_3, where $B_1 = A_1\nabla^2\mathcal{C}_2(\lambda)F$, $B_2 = \lambda^{1/2}[\nabla\mathcal{C}_2(\lambda)F]\cdot\mathbf{n}$, $B_3 = [\nabla\mathcal{C}_2(\lambda)F]\cdot\mathbf{n}$, and so we have (6.24).

Choosing M_1 so small that $r_bCM_1 \le 1/4$ and choosing $\tilde\lambda_0$ so large that $r_bC_{M_2}\tilde\lambda_0^{-1/2} \le 1/4$ in (6.24), we have

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\mathbb{R}^N_+))}(\{(\tau\partial_\tau)^\ell F_\lambda\mathcal{R}(\lambda) \mid \lambda\in\Sigma_{\epsilon,\tilde\lambda_0}\}) \le 1/2. \tag{6.25}$$

Since $\mathcal{R}$-boundedness implies the usual boundedness, we have

$$\|F_\lambda\mathcal{R}(\lambda)F_\lambda(H_1,H_2)\|_{\mathcal{Y}_q(\mathbb{R}^N_+)} \le (1/2)\|F_\lambda(H_1,H_2)\|_{L_q(\mathcal{Y}_q(\mathbb{R}^N_+)}.$$

Here and in the following, the norm of $\mathcal{Y}_q(\mathbb{R}^N_+)$ is given by

$$\|(F_1,F_2,F_3)\|_{\mathcal{Y}_q(\mathbb{R}^N_+)} = \|(F_1,F_2)\|_{L_q(\mathbb{R}^N_+)} + \|F_3\|_{H^1_q(\mathbb{R}^N_+)}.$$

Thus, $\|F_\lambda(H_1,H_2)\|_{\mathcal{Y}_q(\mathbb{R}^N_+)}$ gives the equivalent norm of $Y_q(\mathbb{R}^N_+)$. By (6.23) and (6.25) we see that $(\mathbf{I}+R(\lambda))^{-1} = \sum_{j=0}^\infty(-R(\lambda))^j$ exists as an operator from $Y_q(\mathbb{R}^N_+)$ into itself and its operator norm does not exceed 2. Thus, in view of (6.21), $\varphi = \mathcal{C}_2(\lambda)F_\lambda(\mathbf{I}+R(\lambda))^{-1}(H_1,H_2)$ is a solution of (6.19).

On the other hand, by (6.25) and Lemma 6.8, we see that $(\mathbf{I}+F_\lambda\mathcal{R}(\lambda))^{-1} = \sum_{j=1}^\infty(F_\lambda\mathcal{R}(\lambda))^j$ exists and

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\mathbb{R}^N_+))}(\{(\tau\partial_\tau)^\ell(\mathbf{I}+F_\lambda\mathcal{R}(\lambda))^{-1} \mid \lambda\in\Sigma_{\epsilon,\tilde\lambda_0}\}) \le 4. \tag{6.26}$$

Set $\tilde{\mathcal{C}}_3(\lambda) = \mathcal{C}_2(\lambda)(\mathbf{I}+F_\lambda\mathcal{R}(\lambda))^{-1}$. Since $\mathcal{R}(\lambda)F_\lambda = R(\lambda)$ as follows from (6.23), we have

$$\begin{aligned}(\mathbf{I}+F_\lambda\mathcal{R}(\lambda))^{-1}F_\lambda &= \sum_{j=1}^\infty(-1)^j(F_\lambda\mathcal{R}(\lambda))^jF_\lambda = \sum_{j=0}^\infty(-1)^jF_\lambda(\mathcal{R}(\lambda)F_\lambda)^j\\ &= F_\lambda\sum_{j=0}^\infty(-R(\lambda))^j = F_\lambda(\mathbf{I}+R(\lambda))^{-1},\end{aligned}$$

which leads to $\varphi = \tilde{\mathcal{C}}_3(\lambda)F_\lambda(H_1,H_2)$. Thus, $\tilde{\mathcal{C}}_3(\lambda)$ is an $\mathcal{R}$-bounded solution operator for (6.19). Set

$$\mathcal{C}_3(\lambda)F = [\tilde{\mathcal{C}}_3(\lambda)(F\circ\Phi^{-1})]\circ\Phi,$$

and then $\mathcal{C}_3(\lambda)$ is an $\mathcal{R}$-bounded solution operator of (6.17). This completes the proof of the assertion (2). ∎

6.3. Proof of Theorem 6.4. To prove Theorem 6.4, we need to use several properties of uniform C^2 domain, which are stated in the following proposition.

PROPOSITION 6.11. *Let Ω be a uniform C^2-domain in $\mathbb{R}^N$ with boundary Γ. Then, for any positive constant M_1, there exist constants $M_2 > 0$, $d\in(0,1)$, at most countably many functions $\Phi_j\in C^2(\mathbb{R}^N)$, and points $x^1_j\in\Omega$ and $x^2_j\in\Gamma$ ($j\in\mathbb{N}$) such that the following assertions hold:*

(1) *For every $j\in\mathbb{N}$, the map $\mathbb{R}^N\ni x\to\Phi_j(x)\in\mathbb{R}^N$ is bijective.*
(2) $\Omega = (\bigcup_{j=1}^\infty B_d(x^1_j))\cup(\bigcup_{j=1}^\infty(\Phi_j(\mathbb{R}^N_+)\cap B_d(x^2_j)))$, $B_d(x^1_j)\subset\Omega$, $\Phi_j(\mathbb{R}^N_+)\cap B_d(x^2_j) = \Omega\cap B_d(x^2_j)$, *and* $\Phi_j(\mathbb{R}^N_0)\cap B_d(x^2_j) = \Gamma\cap B_d(x^2_j)$.

(3) *There exist C^∞ functions $\zeta^i_j (i = 1, 2, j \in \mathbb{N})$ such that* $\operatorname{supp}\zeta^i_j$, $\operatorname{supp}\tilde\zeta^i_j \subset B_d(x^i_j)$, $\|\zeta^i_j\|_{H^2_\infty(\mathbb{R}^N)} \le c_0$, $\|\tilde\zeta^i_j\|_{H^2_\infty(\mathbb{R}^N)} \le c_0$, $\tilde\zeta^i_j = 1$ *on* $\operatorname{supp}\zeta^i_j$, $\sum_{i=1,2}\sum_{j=1}^\infty \zeta^i_j = 1$ *on* $\overline{\Omega}$, $\sum_{j=1}^\infty \zeta^1_j = 1$ *on* Γ. *Here, c_0 is a constant which depends on M_2, N, q, q', and r, but is independent of $j \in \mathbb{N}$.*

(4) $\nabla\Phi_j = \mathcal{R}_j + R_i, \nabla(\Phi_j)^- = \mathcal{R}^-_j + R^-_j$, *where $\mathcal{R}_j$ and $\mathcal{R}^-_j$ are $N \times N$ constant orthogonal matrices, and R_j and R^-_j are $N \times N$ matrices of H^1_∞ functions defined on $\mathbb{R}^N$ which satisfies the conditions* $\|R_j\|_{L_\infty(\mathbb{R}^N)} \le M_1$, $\|R^-_j\|_{L_\infty(\mathbb{R}^N)} \le M_1$, $\|\nabla R_j\|_{L_\infty(\mathbb{R}^N)} \le M_2$, *and* $\|\nabla R^-_j\|_{L_\infty(\mathbb{R}^N)} \le M_2$ *for any* $j\mathbb{N}$.

(5) *There exist a natural number $L > 2$ such that any $L+1$ distinct sets of* $\{B_d(x^i_j) \mid i = 1, 2,\ \ j \in \mathbb{N}\}$ *have an empty intersection.*

In what follows, we write $\Omega_\ell = \Phi_\ell(\mathbb{R}^N_+)$ and $\Gamma_\ell = \Phi_\ell(\mathbb{R}^N_0)$ for $\ell \in \mathbb{N}$. Moreover, we write $B_d(x^i_j)$ simply by B^i_j. Since $\rho_0(x)$ and $\gamma_k(x)$ $(k = 1, 3, 4)$ are uniformly continuous functions on $\overline{\Omega}$, choosing d smaller if necessary, we may assume that

$$|\rho_0(x) - \rho_0(x^i_j)| \le M_1, \quad |\gamma_k(x) - \gamma_k(x^i_j)| \le M_1 \quad \text{for } x \in B^i_J \cap \overline{\Omega}, \quad k = 1, 3, 4. \tag{6.27}$$

By the finite intersection property stated in Proposition 6.11(5), we have

$$\Big(\sum_{i=1,2}\sum_{j=1}^\infty \|f\|^q_{L_q(B^i_j\cap\Omega)}\Big)^{1/q} \le C_q\|f\|_{L_q(\Omega)} \tag{6.28}$$

for any $f \in L_q(\Omega)$ and $1 \le q < \infty$. In particular, by (6.28) we have the following.

LEMMA 6.12. *Let $i = 1, 2$ and $1 < q < \infty$. Let $\{f_j\}_{j=0}^\infty$ be a sequence of functions in $L_q(\Omega)$ such that $\sum_{j=0}^\infty \|f_j\|^q_{L_q(\Omega)} < \infty$, and $\operatorname{supp} f_j \subset B^i_j$ $(j \in \mathbb{N})$. Then, $\sum_{j=0}^\infty f_j \in L_q(\Omega)$ and $\|\sum_{j=1}^\infty f_j\|_{L_q(\Omega)} \le (\sum_{j=1}^\infty \|f_j\|^q_{L_q(\Omega)})^{1/q}$.*

We first prove the assertion (1) in Theorem 6.4. We construct a parametrix. Let $\mathbf{v}^1_j \in H^2_q(\mathbb{R}^N)^N$ be solutions of the equations

$$\rho_0(x^1_j)\lambda\mathbf{v}^1_j - \mu\Delta\mathbf{v}^1_j - \nu\nabla\operatorname{div}\mathbf{v}^1_j - \gamma_1(x^1_j)\rho_0(x^1_j)\lambda^{-1}\nabla\operatorname{div}\mathbf{v}^1_j = \zeta^1_j\mathbf{g} \quad \text{in } \mathbb{R}^N, \tag{6.29}$$

and $\mathbf{v}^2_j \in H^2_q(\Omega_j)^N$ solutions of the equations

$$\rho_0(x^2_j)\lambda\mathbf{v}^2_j - \mu\Delta\mathbf{v}^2_j - \nu\nabla\operatorname{div}\mathbf{v}^2_j - \gamma_1(x^2_j)\rho_0(x^2_j)\lambda^{-1}\nabla\operatorname{div}\mathbf{v}^2_j = \zeta^2_j\mathbf{g} \quad \text{in } \Omega_j, \quad \mathbf{v}^2_j|_{\Gamma_j} = 0. \tag{6.30}$$

By Theorems 6.5(1) and 6.7(1), there are $\mathcal{R}$-bounded solution operators $\mathcal{C}^i_j(\lambda)$ with

$$\mathcal{C}^i_j(\lambda) \in \operatorname{Hol}(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(L_q(\Omega^i_j)^N, H^2_q(\Omega^i_j)^N))$$

such that for any $\mathbf{g} \in L_q(\Omega)$ and $\lambda \in \Sigma_{\epsilon,\lambda_0}$, $\mathbf{v}^1_j = \mathcal{C}^1_j(\lambda)\zeta^1_j\mathbf{g}$ are solutions of (6.29) and $\mathbf{v}^2_j = \mathcal{C}^2_j(\lambda)\zeta^2_j\mathbf{g}$ solutions of (6.30), where we have set $\Omega^1_j = \mathbb{R}^N$ and $\Omega^2_j = \Omega_j$. Moreover, we have

$$\mathcal{R}_{\mathcal{L}(L_q(\Omega^i_j)^N, H^{2-k}_q(\Omega^i_j)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{k/2}\mathcal{C}^i_j(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le r_b \tag{6.31}$$

for $\ell = 0, 1$ and $k = 0, 1, 2$. Notice that λ_0 and r_b are independent of $i = 1, 2$ and $j \in \mathbb{N}$. Let

$$\mathcal{U}_1(\lambda)\mathbf{g} = \sum_{i=1,2}\sum_{j=1}^\infty \tilde\zeta^i_j\mathcal{C}^i_j(\lambda)\zeta^i_j\mathbf{g}$$

for $\mathbf{g} \in L_q(\Omega)^N$. By Lemma 6.12, we have

$$\mathcal{R}_{\mathcal{L}(L_q(\Omega)^N, H_q^{2-k}(\Omega)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{k/2}\mathcal{U}(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le C_{N,q} r_b. \tag{6.32}$$

In fact, by (6.31) and (6.28) we have

$$\begin{aligned}
&\sum_{i=1,2}\sum_{j=1}^\infty \int_0^1 \int_\Omega \Big|\sum_{k=1}^n r_k(u)\tilde\zeta_j^i \mathcal{C}_j^i(\lambda_k)\zeta_j^i \mathbf{g}_k\Big|^q\,dx\,du \\
&\quad\le c_0^q \sum_{i=1,2}\sum_{j=1}^\infty \int_0^1 \int_{\Omega_j^i} \Big|\sum_{k=1}^n r_k(u)\mathcal{C}_j^i(\lambda_k)\zeta_j^i \mathbf{g}_k\Big|^q\,dx\,du \\
&\quad\le (c_0 r_b)^q \sum_{i=1,2}\sum_{j=1}^\infty \int_0^1 \int_{\Omega_j^i} \Big|\sum_{k=1}^n r_k(u)\zeta_j^i \mathbf{g}_k\Big|^q\,dx\,du \\
&\quad\le (c_0^2 r_b)^q \sum_{i=0,1}\sum_{j=1}^\infty \int_0^1 \int_{\Omega\cap B_j^i} \Big|\sum_{k=1}^n r_k(u) \mathbf{g}_k\Big|^q\,dx\,du \\
&\quad= (c_0^2 r_b)^q \int_0^1 \left(\sum_{i=0,1}\sum_{j-1}^\infty \int_{\Omega\cap B_j^i} \Big|\sum_{k=1}^n r_k(u) \mathbf{g}_k\Big|^q\,dx\right)du \\
&\quad\le (C_q c_0^2 r_b)^q \int_0^1 \Big\|\sum_{k=1}^n r_k(u)\mathbf{g}_k\Big\|_{L_q(\Omega)}^q\,du,
\end{aligned}$$

and so by Lemma 6.12 we have

$$\Big\|\sum_{k=1}^n r_k(u)\mathcal{U}_1(\lambda_k)\mathbf{g}_k\Big\|_{L_q(\Omega\times(0,1))} \le C_q c_0^2 r_b \Big\|\sum_{k=1}^n r_k(u)\mathbf{g}_k\Big\|_{L_q(\Omega\times(0,1))}.$$

In this way, we can show (6.32). Next, since

$$\begin{aligned}
\Delta(\tilde\zeta_j^i \mathbf{v}_j^i) &= \tilde\zeta_j^i \Delta \mathbf{v}_j^i + 2(\nabla\tilde\zeta_j^i)\nabla \mathbf{v}_j^i + (\Delta\tilde\zeta_j^i)\mathbf{v}_j^i, \\
\nabla \mathrm{div}\,(\tilde\zeta_j^i \mathbf{v}_j^i) &= \tilde\zeta_j^i \nabla\mathrm{div}\,\mathbf{v}_j^i + (\nabla\zeta_j^i)\mathrm{div}\,\mathbf{v}_j^i + \nabla((\nabla\zeta_j^i)\cdot \mathbf{v}_j^i), \\
\nabla(\rho_0(x)\mathrm{div}\,(\tilde\zeta_j^i \mathbf{v}_j^i)) &= \tilde\zeta_j^i \rho_0(x)\nabla\mathrm{div}\,\mathbf{v}_j^i + \nabla(\rho_0(x)(\nabla\zeta_j^i)\cdot \mathbf{v}_j^i) + (\nabla\zeta_j^i)\rho_0(x)\mathrm{div}\,\mathbf{v}_j^i
\end{aligned}$$

with $\mathbf{v}_j^i = \mathcal{C}_j^i(\lambda)\zeta_j^i\mathbf{g}$, setting

(6.33)

$$\begin{aligned}
\mathcal{V}_1(\lambda)\mathbf{g} = &-\sum_{i=1,2}\sum_{j=1}^\infty \{\mu(2(\nabla\tilde\zeta_j^i)\nabla\mathcal{C}_j^i(\lambda)\zeta_j^i\mathbf{g} + (\Delta\tilde\zeta_j^i)\mathcal{C}_j^i(\lambda)\zeta_j^i\mathbf{g} \\
&\qquad + \nu((\nabla\zeta_j^i)\mathrm{div}\,\mathcal{C}_j^i(\lambda)\zeta_j^i\mathbf{g} + \nabla((\nabla\zeta_j^i)\mathcal{C}_j^i(\lambda)\zeta_j^i\mathbf{g}) \\
&\qquad + \lambda^{-1}\gamma_1(x)(\nabla(\rho_0(x)(\nabla\zeta_j^i)\cdot\mathcal{C}_j^i(\lambda)\zeta_j^i\mathbf{g}) + (\nabla\zeta_j^i)\rho_0(x)\mathrm{div}\,\mathcal{C}_j^i(\lambda)\zeta_j^i\mathbf{g})\} \\
&+ \sum_{i=1,2}\sum_{j=1}^\infty \tilde\zeta_j^i((\rho_0(x) - \rho_0(x_j^i))\lambda\mathcal{C}_j^i(\lambda)\zeta_j^i\mathbf{g} \\
&- (\gamma_1(x)\rho_0(x) - \gamma_1(x_j^i)\rho_0(x_j^i))\lambda^{-1}\nabla\mathcal{C}_j^i(\lambda)\zeta_j^i\mathbf{g})
\end{aligned}$$

and setting $\mathbf{v} = \mathcal{U}_1(\lambda)\mathbf{g}$, we have

(6.34)

$$\rho_0(x)\lambda\mathbf{v} - \mu\Delta\mathbf{v} - \nu\nabla\mathrm{div}\,\mathbf{v} - \gamma_1(x)\lambda^{-1}\nabla(\rho_0(x)\mathrm{div}\,\mathbf{v}) = \mathbf{g} + \mathcal{V}_1(\lambda)\mathbf{g} \quad \text{in } \Omega, \quad \mathbf{v}|_\Gamma = 0,$$

because $\Gamma \cap B^1_j = \Gamma_j$ and $\tilde\zeta^i_j\zeta^i_j = \zeta^i_j$. Using (4.7), (6.27), Lemma 6.12, (6.31), and (6.32), we obtain

$$\mathcal{R}_{\mathcal{L}(L_q(\Omega)^N)}(\{(\tau\partial_\tau)^\ell\mathcal{R}(\lambda) \mid \lambda \in \Sigma_{\epsilon,\tilde\lambda_0}\}) \leq c_0^2C_q\{(1+2\alpha_4)M_1 + \alpha_4 r_b\tilde\lambda_0^{-1/2}\} \tag{6.35}$$

for any $\tilde\lambda_0 \geq \lambda_0$, where we have assumed that $\tilde\lambda_0 \geq 1$. To prove (6.35), we have to estimate $(\nabla\rho_0)(\nabla\zeta^i_j)\cdot\mathcal{C}^i_j(\lambda)\zeta^i_j\mathbf{g}$. For this purpose, we use the following lemma, which can be proved easily with the help of Sobolev's imbedding theorem.

LEMMA 6.13. *Let $1 < q \leq r < \infty$ and $N < r < \infty$. Then, the following two inequalities hold:*

(1) *There exists a constant C depending only on N, q, and r for which*

$$\|ab\|_{L_q(\Omega)} \leq C\|a\|_{L_r(\Omega)}\|b\|_{H^1_q(\Omega)}.$$

(2) *For any $\sigma > 0$, there exists a constant $C = C_{\sigma,\|a\|_{L_r(\Omega)}}$ for which*

$$\|ab\|_{L_q(\Omega)} \leq \sigma\|b\|_{H^1_q(\Omega)} + C\|b\|_{L_q(\Omega)}.$$

For any $\lambda_k \in \Sigma_{\epsilon,\tilde\lambda_0}$, and $\mathbf{g}_k \in L_q(\Omega)^N$ $(k = 1,\dots,n)$, by Lemma 6.13, (6.31), and (4.7),

$$\begin{aligned}
&\sum_{i=1,2}\sum_{j=1}^\infty\int_0^1\int_\Omega\Big|\sum_{k=1}^n r_k(u)(\nabla\rho_0)(\nabla\zeta^i_j)\cdot\mathcal{C}^i_j(\lambda_k)\zeta^i_j\mathbf{g}_k\Big|^q\,dx\,du\\
&\quad\leq (c_0\alpha_4)^q\sum_{i=1,2}\sum_{j=1}^\infty\int_0^1\Big\|\sum_{k=1}^n r_k(u)\mathcal{C}^i_j(\lambda_k)\zeta^i_j\mathbf{g}_k\Big\|^q_{H^1_q(\Omega^i_j)}\,du\\
&\quad\leq (c_0\alpha_4)^q\sum_{i=1,2}\sum_{j=1}^\infty\int_0^1\Big\|\sum_{k=1}^n r_k(u)\lambda_k^{-1/2}\lambda_k^{1/2}\mathcal{C}^i_j(\lambda_k)\zeta^i_j\mathbf{g}_k\Big\|^q_{H^1_q(\Omega^i_j)}\,du\\
&\quad\leq (c_0\alpha_4\tilde\lambda_0^{-1/2})^q\sum_{i=1,2}\sum_{j=1}^\infty\int_0^1\Big\|\sum_{k=1}^n r_k(u)\lambda_k^{1/2}\mathcal{C}^i_j(\lambda_k)\zeta^i_j\mathbf{g}_k\Big\|^q_{H^1_q(\Omega^i_j)}\,du\\
&\quad\leq (c_0\alpha_4\tilde\lambda_0^{-1/2}r_b)^q\sum_{i=1,2}\sum_{j=1}^\infty\int_0^1\Big\|\sum_{k=1}^n r_k(u)\zeta^i_j\mathbf{g}_k\Big\|^q_{L_q(\Omega^i_j)}\,du\\
&\quad\leq (c_0^2\alpha_4\tilde\lambda_0^{-1/2}r_b)^q\sum_{i=1,2}\sum_{j=1}^\infty\int_0^1\Big\|\sum_{k=1}^n r_k(u)\mathbf{g}_k\Big\|^q_{L_q(\Omega\cap B^i_j)}\,du\\
&\quad= (c_0^2\alpha_4\tilde\lambda_0^{-1/2}r_b)^q\int_0^1\Big(\sum_{i=1,2}\sum_{j=1}^\infty\Big\|\sum_{k=1}^n r_k(u)\mathbf{g}_k\Big\|^q_{L_q(\Omega\cap B^i_j)}\Big)\,du\\
&\quad= (C_qc_0^2\alpha_4\tilde\lambda_0^{-1/2}r_b)^q\int_0^1\Big\|\sum_{k=1}^n r_k(u)\mathbf{g}_k\Big\|^q\,du.
\end{aligned}$$

Other terms can be estimated similarly, and so by Lemma 6.12 and (6.28) we have (6.35). Choosing $M_1 > 0$ so small that $c_0^2C_q(1+2\alpha_4)M_1 \leq 1/4$ and choosing $\tilde\lambda_0 \geq$

$\max(\lambda_0, 1)$ so large that $c_0^2 C_q \alpha_4 r_b \tilde\lambda_0^{-1/2} \le 1/4$, by (6.35) we have

$$\mathcal{R}_{\mathcal{L}(L_q(\Omega)^N)}(\{(\tau\partial_\tau)^\ell \mathcal{V}_1(\lambda) \mid \lambda \in \Sigma_{\epsilon,\tilde\lambda_0}\}) \le 1/2. \tag{6.36}$$

Thus, $(\mathbf{I} + \mathcal{V}_1(\lambda))^{-1} = \sum_{j=1}^\infty (-\mathcal{V}_1(\lambda))^j$ exists and satisfies the estimate

$$\mathcal{R}_{\mathcal{L}(L_q(\Omega)^N)}(\{(\tau\partial_\tau)^\ell (\mathbf{I} + \mathcal{V}_1(\lambda))^{-1} \mid \lambda \in \Sigma_{\epsilon,\tilde\lambda_0}\}) \le 4.$$

Let $\mathcal{C}(\lambda) = \mathcal{U}_1(\lambda)(\mathbf{I} + \mathcal{V}_1(\lambda))^{-1}$, and then in view of (6.34) we see that $\mathbf{u} = \mathcal{C}(\lambda)\mathbf{g}$ is a solution of (6.3). The uniqueness of solutions follows from the existence of solutions of the dual problem. Moreover, by (6.32) and (6.36) we see that $\mathcal{C}(\lambda)$ satisfies the estimate

$$\mathcal{R}_{\mathcal{L}(L_q(\Omega)^N, H_q^{2-k}(\Omega)^N)}(\{(\tau\partial_\tau)^\ell (\lambda^{k/2}\mathcal{C}(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le 4C_{N,q} r_b.$$

This completes the proof of assertion (1) of Theorem 6.4.

We next prove the assertion (2). By Theorems 6.5(2) and 6.7(2), there are $\mathcal{R}$-bounded solution operators $\mathcal{D}_j^i(\lambda)$ with

$$\mathcal{D}_j^1(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(L_q(\mathbb{R}^N), H_q^2(\mathbb{R}^N))), \quad \mathcal{D}_j^2(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{Y}_q(\Omega_1), H_q^2(\mathbb{R}^N))) \tag{6.37}$$

such that for any $(h_1, h_2) \in Y_q(\Omega)$ and $\lambda \in \Sigma_{\epsilon,\lambda_0}$, $\vartheta_j^1 = \mathcal{D}_j^1(\lambda)\zeta_j^1 h_1$ are solutions of the equations

$$\gamma_3(x_j^1)\lambda\vartheta_j^1 - \gamma_4(x_j^1)\Delta\vartheta_j^1 = \zeta_j^1 h_1 \quad \text{in } \mathbb{R}^N, \tag{6.38}$$

and $\vartheta_j^2 = \mathcal{D}_j^2(\lambda)\zeta_j^2(h_1, \lambda^{1/2}h_2, h_2)$ are solutions of the equations

$$\gamma_3(x_j^2)\lambda\vartheta_j^2 - \gamma_4(x_j^2)\Delta\vartheta_j^2 = \zeta_j^2 h_1 \quad \text{in } \Omega_j, \quad (\nabla\vartheta_j^2)\cdot\mathbf{n}_j|_{\Gamma_j} = 0, \tag{6.39}$$

where $\mathbf{n}_j$ is the unit outer normal to Γ_j. Notice that $\mathbf{n}_j = \mathbf{n}$ on $\Gamma_j \cap B_j^2 = \Gamma \cap B_j^2$. In particular, by (6.37) we have

$$\sum_{k=0}^2 |\lambda|^{k/2} \|\vartheta_j^i\|_{H_q^{2-k}(\Omega_j^i} \le r_b\{\|\zeta_j^i h_1\|_{L_q(\Omega_j^i)} + \sigma^i(\|\lambda^{1/2}h_2\|_{L_q(\Omega_j^2)} + \|h_2\|_{H_q^1(\Omega_j^2)})\} \ (i = 1, 2), \tag{6.40}$$

where $\sigma^1 = 0$ and $\sigma^2 = 1$. Let

$$\mathbf{U}_2(\lambda)(h_1, h_2) = \sum_{i=1,2}\sum_{j=1}^\infty \tilde\zeta_j^i \vartheta_j^i, \quad \mathcal{U}_2(\lambda)F = \sum_{j=1}^\infty \tilde\zeta_j^1 \mathcal{D}_j^1(\lambda)\zeta_j^1 F_1 + \sum_{j=1}^\infty \tilde\zeta_j^2 \mathcal{D}_j^2(\lambda)\zeta_j^2 F$$

for $(h_1, h_2) \in Y_q(\Omega)$ and $F = (F_1, F_2, F_3) \in \mathcal{Y}_q(\Omega)$. By Lemma 6.12 and (6.28), we have

$$\sum_{k=0}^2 |\lambda|^{k/2} \|\mathbf{U}_2(\lambda)(h_1, h_2)\|_{H_q^{2-k}(\Omega)} \le C_{N,q} r_b(\|h_1\|_{L_q(\Omega)} + |\lambda|^{1/2}\|h_2\|_{L_q(\Omega)} + \|h_2\|_{H_q^1(\Omega)}) \tag{6.41}$$

for any $\lambda \in \Sigma_{\epsilon,\lambda_0}$ and $(h_1, h_2) \in Y_q(\Omega)$, and

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\Omega), H_q^{2-k}(\Omega))}(\{(\tau\partial_\tau)^\ell (\lambda^{k/2}\mathcal{U}_2(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le C_{N,q} r_b \tag{6.42}$$

for $k=0,1,2$ and $\ell=0,1$. For $F=(F_1,F_2,F_3)\in\mathcal{Y}_q(\Omega)$, let

$$\begin{aligned}
&\mathcal{V}_{21}(\lambda)F\\
&\quad=-\sum_{i=1,2}\sum_{j=1}^\infty\{\mathrm{div}\,(\gamma_4(x)(\nabla\zeta^i_j)\mathcal{D}^i_j(\lambda)\zeta^i_jF+\nabla(\gamma_4\zeta^i_j)\cdot\nabla(\mathcal{D}^i_j(\lambda)\zeta^i_jF)\}\\
&\qquad+\sum_{i=1,2}\sum_{j=1}^\infty\zeta^i_j\{(\gamma_3(x)-\gamma_3(x^i_j))\lambda\mathcal{D}^i_j(\lambda)\zeta^i_jF-\mathrm{div}\,((\gamma_4(x)-\rho(x^i_j))\nabla\mathcal{D}^i_j(\lambda)\zeta^i_jF)\},\\
&\mathcal{V}_{22}(\lambda)F=-\sum_{j=1}^\infty(\nabla\zeta^2_j)\cdot\mathbf{n}_j\mathcal{D}^2_j(\lambda)\zeta^i_jF,
\end{aligned}$$

where we have set $\mathcal{D}^1_j(\lambda)\zeta^i_jF=\mathcal{D}^1_j(\lambda)\zeta^i_jF_1$. We then have

$$\begin{aligned}
&\gamma_3(x)\lambda\mathbf{U}_2(\lambda)(h_1,h_2)-\mathrm{div}\,(\gamma_4(x)\nabla\mathbf{U}_2(\lambda)(h_1,h_2))=h_1+\mathcal{V}_{21}(\lambda)F_\lambda(h_1,h_2)\quad\text{in }\Omega,\\
&(\nabla\mathbf{U}_2(\lambda)(h_1,h_2))\cdot\mathbf{n}=h_2+\mathcal{V}_{22}(\lambda)F_\lambda(h_1,h_2)\quad\text{on }\Gamma
\end{aligned}\tag{6.43}$$

for any $(h_1,h_2)\in Y_q(\Omega)$, where we have set $F_\lambda(h_1,h_2)=(h_1,\lambda^{1/2}h_2,h_2)\in\mathcal{Y}_q(\Omega)$. Since

$$\|(\nabla\gamma_4)\cdot\nabla\mathcal{D}^i_j(\lambda)\zeta^i_jF\|_{L_q(\Omega)}\le\sigma\|\nabla\mathcal{D}^i_j(\lambda)\zeta^i_jF\|_{H^1_q(\Omega)}+C_{\sigma,\alpha_4}\|\nabla\mathcal{D}^i_j(\lambda)\zeta^i_jF\|_{L_q(\Omega)}$$

as follows from Lemma 6.13(2), by (6.37), Lemma 6.12, (6.27), (6.28), and Lemma 6.13, we have

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\Omega))}(\{(\tau\partial_\tau)^\ell(F_\lambda(\mathcal{V}_{21}(\lambda),\mathcal{V}_{22}(\lambda)))\mid\lambda\in\Sigma_{\epsilon,\tilde\lambda_0}\})\le\{2M_1+\sigma+c_0^2C_qC_{\sigma,\alpha_4}\tilde\lambda_0^{-1/2}\}r_b$$

for any $\tilde\lambda_0\ge\max(\lambda_0,1)$. Choosing M_1 and $\sigma>0$ so small that $2M_1r_b<1/8$, $\sigma r_b<1/8$ and choosing $\tilde\lambda_0$ so large that $c_0^2C_qC_{\sigma,\alpha_4}r_b\tilde\lambda_0^{-1/2}\le1/4$, we have

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\Omega))}(\{(\tau\partial_\tau)^\ell(F_\lambda(\mathcal{V}_{21}(\lambda),\mathcal{V}_{22}(\lambda)))\mid\lambda\in\Sigma_{\epsilon,\tilde\lambda_0}\})\le1/2,\tag{6.44}$$

and so $(\mathbf{I}+F_\lambda(\mathcal{V}_{21}(\lambda),\mathcal{V}_{22}(\lambda)))^{-1}=\sum_{j=0}^\infty(-F_\lambda(\mathcal{V}_{21}(\lambda),\mathcal{V}_{22}(\lambda)))^j$ exists and

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\mathbf{I}+F_\lambda(\mathcal{V}_{21}(\lambda),\mathcal{V}_{22}(\lambda)))^{-1}\mid\lambda\in\Sigma_{\epsilon,\tilde\lambda_0}\})\le4.\tag{6.45}$$

On the other hand, by (6.44) we have

$$\|F_\lambda(\mathcal{V}_{21}(\lambda)F_\lambda(h_1,h_2),\mathcal{V}_{22}(\lambda)F_\lambda(h_1,h_2))\|_{\mathcal{Y}_q(\Omega)}\le(1/2)\|F_\lambda(h_1,h_2)\|_{L_q(\Omega)}$$

for any $\lambda\in\Sigma_{\epsilon,\tilde\lambda_0}$. Since $\|F_\lambda(h_1,h_2)\|_{\mathcal{Y}_q(\Omega)}=\|h_1\|_{L_q(\Omega)}+|\lambda|^{1/2}\|h_2\|_{L_q(\Omega)}+\|h_2\|_{H^1_q(\Omega)}$ gives equivalent norms in $Y_q(\Omega)$, we see that for each $\lambda\in\Sigma_{\epsilon,\tilde\lambda_0}$, $\mathbf{I}+(\mathcal{V}_{21}(\lambda),\mathcal{V}_{22}(\lambda))F_\lambda)^{-1}$ $=\sum_{j=0}^\infty(-(\mathcal{V}_{21}(\lambda),\mathcal{V}_{22}(\lambda))F_\lambda)^j$ exists as an operator in $\mathcal{L}(Y_q(\Omega))$ whose operator norm does not exceed 2. Thus, in view of (6.43), $\vartheta=\mathbf{U}_2(\lambda)(\mathbf{I}+(\mathcal{V}_{21}(\lambda),\mathcal{V}_{22}(\lambda)))^{-1}(h_1,h_2)$ is a solution of (6.4). The uniqueness of the solution follows from the existence of solutions for the dual problem. Notice that $\mathcal{U}_2(\lambda)F_\lambda(h_1,h_2)=\mathbf{U}_2(\lambda)(h_1,h_2)$. We then define an operator $\mathcal{D}(\lambda)$ by

$$\mathcal{D}(\lambda)F=\mathcal{U}_2(\lambda)(\mathbf{I}+F_\lambda(\mathcal{V}_{21}(\lambda),\mathcal{V}_{22}(\lambda)))^{-1}$$

for $F = (F_1, F_2, F_3) \in \mathcal{Y}_q(\Omega)$. Since

$$
\begin{aligned}
(\mathbf{I} + F_\lambda(\mathcal{V}_{21}(\lambda), \mathcal{V}_{22}(\lambda)))^{-1} F_\lambda &= \sum_{j=0}^{\infty} (-F_\lambda(\mathcal{V}_{21}(\lambda), \mathcal{V}_{22}(\lambda)))^j F_\lambda \\
&= F_\lambda \sum_{j=0}^{\infty} (-(\mathcal{V}_{21}(\lambda), \mathcal{V}_{22}(\lambda)) F_\lambda)^j \\
&= F_\lambda (\mathbf{I} + (\mathcal{V}_{21}(\lambda), \mathcal{V}_{22}(\lambda)) F_\lambda)^{-1},
\end{aligned}
$$

we have

$$
\begin{aligned}
\vartheta &= \mathbf{U}_2(\lambda)(\mathbf{I} + (\mathcal{V}_{21}(\lambda), \mathcal{V}_{22}(\lambda))F_\lambda)^{-1}(h_1, h_2) \\
&= \mathcal{U}_2(\lambda) F_\lambda (\mathbf{I} + (\mathcal{V}_{21}(\lambda), \mathcal{V}_{22}(\lambda))F_\lambda)^{-1}(h_1, h_2) \\
&= \mathcal{U}_2(\lambda)(\mathbf{I} + F_\lambda(\mathcal{V}_{21}(\lambda), \mathcal{V}_{22}(\lambda)))^{-1} F_\lambda (h_1, h_2) \\
&= \mathcal{D}(\lambda) F_\lambda (h_1, h_2) = \mathcal{D}(\lambda)(h_1, \lambda^{1/2} h_2, h_2).
\end{aligned}
$$

By (6.42) and (6.45), we have

$$
\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\Omega), H_q^{2-k}(\Omega))}(\{(\tau\partial_\tau)^\ell (\lambda^{k/2} \mathcal{D}(\lambda)) \mid \lambda \in \Sigma_{\epsilon, \tilde{\lambda}_0}\}) \le 4 C_{N,q} r_b.
$$

This completes the proof of the assertion (2) of Theorem 6.4.

6.4. Proof of Theorem 6.2. Let $\mathcal{C}(\lambda)$ and $\mathcal{D}(\lambda)$ be the operators given in Theorem 6.4. Let $\vartheta_0 = \mathcal{D}(\lambda)(0, \lambda^{1/2} h_2, h_2)$, and then the third equation of (6.1) and the boundary condition for ϑ are reduced to the equations

$$
\gamma_3(x)\lambda\varphi + \gamma_2(x)\mathrm{div}\,\mathbf{v} - \mathrm{div}\,(\gamma_4(x)\nabla\varphi) = f_3 \quad \text{in } \Omega, \quad (\nabla\varphi)\cdot\mathbf{n}|_\Gamma = 0. \tag{6.46}
$$

Thus, in view of (6.2) and (6.46), instead of (6.1) we consider the equations

$$
\begin{cases}
\rho_0(x)\lambda\mathbf{v} - \mu\Delta\mathbf{v} - \nu\nabla\mathrm{div}\,\mathbf{v} - \gamma_1(x)\lambda^{-1}\nabla(\rho_0(x)\mathrm{div}\,\mathbf{v}) + \gamma_2(x)\nabla\varphi = \mathbf{f} & \text{in } \Omega, \\
\gamma_3(x)\lambda\varphi + \gamma_2(x)\mathrm{div}\,\mathbf{v} - \mathrm{div}\,(\gamma_4(x)\nabla\varphi) = g & \text{in } \Omega, \\
\mathbf{v} = 0, \quad (\nabla\varphi)\cdot\mathbf{n} = 0 & \text{on } \Gamma.
\end{cases} \tag{6.47}
$$

In the following, we write $\mathcal{D}(\lambda)(g, 0, 0)$ simply by $\mathcal{D}(\lambda)g$. Let

$$
\mathbf{v} = \mathcal{C}(\lambda)\mathbf{f}, \quad \varphi = \mathcal{D}(\lambda)g
$$

in (6.47), and then we have

$$
\begin{cases}
\rho_0(x)\lambda\mathbf{v} - \mu\Delta\mathbf{v} - \nu\nabla\mathrm{div}\,\mathbf{v} - \gamma_1(x)\lambda^{-1}\nabla(\rho_0(x)\mathrm{div}\,\mathbf{v}) + \gamma_2(x)\nabla\varphi = \mathbf{f} + \mathcal{E}_1(\lambda)(\mathbf{f}, g) \\
\hfill \text{in } \Omega, \\
\gamma_3(x)\lambda\varphi + \gamma_2(x)\mathrm{div}\,\mathbf{v} - \mathrm{div}\,(\gamma_4(x)\nabla\varphi) = g + \mathcal{E}_2(\lambda)(\mathbf{f}, g) \\
\hfill \text{in } \Omega, \\
\mathbf{v} = 0, \quad (\nabla\varphi)\cdot\mathbf{n} = 0 \\
\hfill \text{on } \Gamma,
\end{cases} \tag{6.48}
$$

where we have set

$$
\mathcal{E}_1(\lambda)(\mathbf{f}, g) = \gamma_3(x)\nabla\mathcal{D}(\lambda)g, \quad \mathcal{E}_2(\lambda)(\mathbf{f}, g) = \gamma_2(x)\mathrm{div}\,\mathcal{C}(\lambda)\mathbf{f}.
$$

Let $\mathcal{E}(\lambda)(\mathbf{f}, g) = (\mathcal{E}_1(\lambda)(\mathbf{f}, g), \mathcal{E}_2(\lambda)(\mathbf{f}, g))$, and then by Theorem 6.4 we have

$$\mathcal{R}_{\mathcal{L}(L_q(\Omega)^{N+1})}(\{(\tau\partial_\tau)^\ell \mathcal{E}(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_1}\}) \leq r_b \lambda_1^{-1/2} \tag{6.49}$$

for any $\lambda_1 \geq \lambda_0$. Thus, choosing $\lambda_1 > 0$ so large that $r_b\lambda_1^{-1/2} \leq 1/2$, by (6.49) and Lemma 6.8 we see that $(\mathbf{I}+\mathcal{E}(\lambda))^{-1} = \sum_{j=0}^{\infty}(-\mathcal{E}(\lambda))^j$ exists and satisfies the estimate

$$\mathcal{R}_{\mathcal{L}(L_q(\Omega)^{NL+1})}(\{(\tau\partial_\tau)^\ell (\mathbf{I} + \mathcal{E}(\lambda))^{-1} \mid \lambda \in \Sigma_{\epsilon,\lambda_1}\}) \leq 4. \tag{6.50}$$

Let $\tilde{\mathcal{B}}_1(\lambda) = \mathcal{C}(\lambda)(\mathbf{I} + \mathcal{E}(\lambda))^{-1}$ and $\tilde{\mathcal{B}}_2(\lambda) = \mathcal{D}(\lambda)(\mathbf{I} + \mathcal{E}(\lambda))^{-1}$, and then by Theorem 6.4, (6.50), and Lemma 6.8, we see that for any $\lambda \in \Sigma_{\epsilon,\lambda_1}$ and $(\mathbf{f}, g) \in L_q(\Omega)^{N+1}$, $\mathbf{v} = \tilde{\mathcal{B}}_1(\lambda)(\mathbf{f}, g)$ and $\varphi = \tilde{\mathcal{B}}_2(\lambda)(\mathbf{f}, g)$ are solutions of (6.48) and

$$\mathcal{R}_{\mathcal{L}(L_q(\Omega)^{N+1}, H_q^{2-k}(\Omega)^{N+1})}(\{(\tau\partial_\tau)^\ell (\lambda^{k/2}(\tilde{\mathcal{B}}_1(\lambda), \tilde{\mathcal{B}}_2(\lambda))) \mid \lambda \in \Sigma_{\epsilon,\lambda_1}\}) \leq 4r_b. \tag{6.51}$$

Finally, setting

$$\begin{aligned}
&\mathbf{v} = \tilde{\mathcal{B}}_1(\lambda)(\mathbf{f}_2 - \lambda^{-1}\gamma_1(x)\nabla f_1, f_3),\\
&\zeta = \lambda^{-1}(f_1 - \rho_0(x)\mathrm{div}\, \mathbf{v}),\\
&\vartheta = \tilde{\mathcal{B}}_2(\lambda)(\mathbf{f}_2 - \lambda^{-1}\gamma_1(x)\nabla f_1, f_3) + \mathcal{D}(\lambda)(0, \lambda^{1/2} f_4, f_4)
\end{aligned}$$

we see that ζ, $\mathbf{v}$, and ϑ are solutions of (6.1). The uniqueness of solutions follows from the existence of solutions for the dual problem. For $F = (F_1, F_2, F_3, F_4, F_5) \in \mathcal{X}_q(\Omega)$, we set

$$\begin{aligned}
\mathcal{B}_1(\lambda)F &= \tilde{\mathcal{B}}_1(\lambda)(F_2, F_3) - \lambda^{-1}\tilde{\mathcal{B}}_1(\lambda)(\gamma_1(x)\nabla F_1, 0),\\
\mathcal{A}(\lambda)F &= \lambda^{-1}F_1 - \lambda^{-1}\rho_0(x)\mathrm{div}\, \mathcal{B}_1(\lambda)F,\\
\mathcal{B}_2(\lambda)F &= \tilde{\mathcal{B}}_2(\lambda)(F_2, F_3) - \lambda^{-1}\tilde{\mathcal{B}}_2(\lambda)(\gamma_1(x)\nabla F_1, 0) + \mathcal{D}(\lambda)(0, F_4, F_5),
\end{aligned}$$

and then we have $\zeta = \mathcal{A}(\lambda)\mathbf{F}_\lambda$, $\mathbf{v} = \mathcal{B}_1(\lambda)\mathbf{F}_\lambda$, and $\vartheta = \mathcal{B}_2(\lambda)\mathbf{F}_\lambda$, where $\mathbf{F}_\lambda = (f_1, \mathbf{f}_2, f_3, \lambda^{1/2}f_4, f_4)$. Moreover, by Lemma 6.9, (4.7), and (6.51) we have

$$\begin{aligned}
\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), H_q^{2-k}(\Omega)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{k/2}\mathcal{B}_1(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_1}\}) &\leq (4 + \lambda_1^{-1}\alpha_4)r_b,\\
\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), H_q^{2-k}(\Omega))}(\{(\tau\partial_\tau)^\ell(\lambda^{k/2}\mathcal{B}_2(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_1}\}) &\leq (4 + \lambda_1^{-1}\alpha_4)r_b,\\
\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), H_q^1(\Omega))}(\{(\tau\partial_\tau)^\ell(\lambda\mathcal{A}(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_1}\}) &\leq 1 + (4 + \lambda_1^{-1}\alpha_4)r_b.
\end{aligned}$$

This completes the proof of Theorem 6.2.

6.5. Proof of Theorem 4.1. We first prove the generation of a C_0 analytic semigroup associated with (4.6). Let

$$\begin{aligned}
&\mathcal{D}_q(\Omega) = \{(\zeta, \mathbf{v}, \vartheta) \in D_q(\Omega) \mid \mathbf{v}|_\Gamma = 0, \quad (\nabla\vartheta)\cdot\mathbf{n}|_\Gamma = 0\},\\
&A(\zeta, \mathbf{v}, \vartheta) = \begin{pmatrix} -\rho_0(x)\mathrm{div}\, \mathbf{v} \\ \rho_0(x)^{-1}(\mu\Delta\mathbf{v} + \nu\nabla\mathrm{div}\, \mathbf{v} - \gamma_1(x)\nabla\zeta - \gamma_2(x)\nabla\vartheta) \\ \gamma_3(x)^{-1}(-\gamma_2(x)\mathrm{div}\, \mathbf{v} + \mathrm{div}\,(\gamma_4(x)\nabla\vartheta)) \end{pmatrix}, \qquad (6.52)\\
&\mathcal{A}_q(\zeta, \mathbf{v}, \vartheta) = A(\zeta, \mathbf{v}, \vartheta) \quad \text{for } (\zeta, \mathbf{v}, \vartheta) \in \mathcal{D}_q(\Omega).
\end{aligned}$$

And then, (4.6) with$f_1 = \mathbf{f}_2 = f_3 = g = 0$ is formally written as

$$\partial_t U - \mathcal{A}_q U = 0 \quad \text{for } t > 0, \quad U|_{t=0} = U_0, \tag{6.53}$$

where $U_0 = (\zeta_0, \mathbf{v}_0, \vartheta_0) \in \mathcal{H}_q(\Omega)$ and U with

$$U \in C^0[0, \infty, \mathcal{H}_q(\Omega)) \cap C^0((0,\infty), \mathcal{D}_q(\Omega) \cap C^1((0,\infty), \mathcal{H}_q(\Omega)).$$

The resolvent equation corresponding to (6.53) is

$$\lambda V - \mathcal{A}_q V = F \quad \text{in } \Omega, \tag{6.54}$$

where $F = (f_1, \mathbf{f}_2, f_3) \in \mathcal{H}_q(\Omega)$ and $V \in \mathcal{D}_q(\Omega)$. By Theorem 6.2, we see that the resolvent set $\rho(\mathcal{A}_q)$ of $\mathcal{A}_q$ contains $\Sigma_{\epsilon,\lambda_0}$ and for any $F \in \mathcal{H}_q(\Omega)$ and $\lambda \in \Sigma_{\epsilon,\lambda_0}$, $V = (\lambda \mathbf{I} - \mathcal{A}_q)^{-1} F \in \mathcal{D}_q(\Omega)$ satisfies the estimate

$$|\lambda| \|V\|_{\mathcal{H}_q(\Omega)} + \|V\|_{\mathcal{D}_q(\Omega)} \le r_b \|F\|_{\mathcal{H}_q(\Omega)}, \tag{6.55}$$

where

$$\|F\|_{\mathcal{H}_q(\Omega)} = \|f_1\|_{H^1_q(\Omega)} + \|(\mathbf{f}_2, f_3)\|_{L_q(\Omega)}, \quad \|V\|_{\mathcal{D}_q(\Omega)} = \|\zeta\|_{H^1_q(\Omega)} + \|(\mathbf{v}, \vartheta)\|_{H^2_q(\Omega)}$$

for $F = (f_1, \mathbf{f}_2, f_3) \in \mathcal{H}_q(\Omega)$ and $V = (\zeta, \mathbf{v}, \vartheta) \in \mathcal{D}_q(\Omega)$. Since $0 < \epsilon < \pi/2$, the operator $\mathcal{A}_q$ generates a C_0 analytic semigroup $\{T(t)\}_{t \ge 0}$ on $\mathcal{H}_q(\Omega)$ possessing the estimate

$$\|T(t)F\|_{\mathcal{H}_q(\Omega)} \le C e^{\gamma t} \|F\|_{\mathcal{H}_q(\Omega)} \quad (t > 0)$$

for some constants C and γ.

We now consider the maximal L_p-L_q regularity for (4.6) in the case that $f_1 = \mathbf{f}_2 = f_3 = g = 0$. Let

$$\mathcal{E}_{p,q}(\Omega) = (\mathcal{H}_q(\Omega), \mathcal{D}_q(\Omega))_{1-1/p,p}. \tag{6.56}$$

Notice that $\mathcal{E}_{p,q}(\Omega) \subset D_{p,q}(\Omega)$ and that for $(\zeta, \mathbf{v}, \vartheta) \in \mathcal{E}_{p,q}(\Omega)$ we have

$$\mathbf{v}|_\Gamma = 0 \quad \text{for } 2/p + 1/q < 2, \quad (\nabla\vartheta) \cdot \mathbf{n}|_\Gamma = 0 \quad \text{for } 2/p + 1/q < 1. \tag{6.57}$$

By real interpolation theory, we have the following.

THEOREM 6.14. *Let $1 < p, q < \infty$. Assume that Ω is a uniformly C^2 domain. Then, for $(\zeta_0, \mathbf{v}_0, \vartheta_0) \in \mathcal{E}_{p,q}(\Omega)$, $(\zeta, \mathbf{v}, \vartheta) = T(t)(\zeta_0, \mathbf{v}_0, \vartheta_0)$ satisfies* (4.6) *with $f_1 = \mathbf{f}_2 = f_3 = g = 0$ and possesses the estimate*

$$\|e^{-\gamma t}(\zeta, \mathbf{v}, \vartheta)\|_{H^1_p((0,\infty), \mathcal{H}_q(\Omega))} + \|e^{-\gamma t}(\mathbf{v}, \vartheta)\|_{L_p((0,\infty), H^2_q(\Omega))} \le C \|(\zeta_0, \mathbf{v}_0, \vartheta_0)\|_{D_{p,q}(\Omega)}.$$

Remark 6.15. Theorem 6.14 can be shown employing the same argument as that in the proof of Theorem 3.9 in Shibata and Shimizu [40], so we may omit the proof.

We next consider (4.6) in the case that $(\zeta_0, \mathbf{v}_0, \vartheta_0) = 0$. Notice that g is defined on $\mathbb{R}$ with respect to t. We extend f_1, $\mathbf{f}_2$, and f_3 to functions f_{10}, $\mathbf{f}_{20}$, and f_{30} defined on $\mathbb{R}$ setting zero for negative times. We then consider the following equations:

$$\partial_t V_1 - A V_1 = (f_{10}, \mathbf{f}_{20}, f_{30}) \quad \text{in } \Omega \times \mathbb{R}, \quad \mathbf{v}_1 = 0, \quad (\nabla \vartheta_1) \cdot \mathbf{n} = g \quad \text{on } \Gamma \times \mathbb{R}, \tag{6.58}$$

where $V_1 = (\zeta_1, \mathbf{v}_1, \vartheta_1)$. We use the Laplace transform $\mathcal{L}$ with respect to t and its inversion formula $\mathcal{L}^{-1}$, which are defined by

$$\mathcal{L}[f](\cdot,\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} f(\cdot,t)\,dt = \mathcal{F}[e^{-\gamma t}f](\tau) \quad (\lambda = \gamma + i\tau \in \mathbb{C}),$$
$$\mathcal{L}^{-1}[g](\cdot,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\lambda t} g(\cdot,\tau)\,d\tau = e^{\gamma t}\mathcal{F}^{-1}[g](\cdot,t),$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform with respect to t and its inverse. Applying the Laplace transform to (6.58), we have

$$\lambda\hat{V}_1 - A\hat{V}_1 = (\mathcal{L}[f_{10}], \mathcal{L}[\mathbf{f}_{20}], \mathcal{L}[f_{30}]) \quad \text{in } \Omega, \quad \hat{\mathbf{v}}_1 = 0, \quad (\nabla\hat{\vartheta}_1)\cdot\mathbf{n} = \mathcal{L}[g] \quad \text{on } \Gamma. \tag{6.59}$$

Let $\mathcal{S}(\lambda) = (\mathcal{A}(\lambda), \mathcal{B}_1(\lambda), \mathcal{B}_2(\lambda))$ be $\mathcal{R}$ bounded solution operators given in Theorem 6.2. We then have $\hat{V}_1(\lambda) = \mathcal{S}(\lambda)\mathbf{F}_\lambda$, where we have set

$$\mathbf{F}_\lambda = (\mathcal{L}[f_{10}](\lambda), \mathcal{L}[\mathbf{f}_{20}](\lambda), \mathcal{L}[f_{30}](\lambda), \lambda^{1/2}\mathcal{L}[g](\lambda), \mathcal{L}[g](\lambda)).$$

We now introduce an operator $\Lambda_\gamma^{1/2}$ by

$$\Lambda_\gamma^{1/2} g = \mathcal{L}^{-1}[\lambda^{1/2}\mathcal{L}[g](\lambda)].$$

Since

$$|(\tau\partial_\tau)^\ell(\lambda^{1/2}/(1+\tau^2)^{1/4})| \le C_\gamma$$

for any $\lambda = \gamma + i\tau \in \mathbb{C}$ with some constant C_γ depending solely on $\gamma \in \mathbb{R}$, by the Bourgain theorem (cf. Lemma 6.9), we have

$$\|e^{-\gamma t}\Lambda_\gamma^{1/2} g\|_{L_p(\mathbb{R},L_q(\Omega))} \le C_\gamma \|e^{-\gamma t} g\|_{H_p^{1/2}(\mathbb{R},L_q(\Omega))} \tag{6.60}$$

for any $\gamma > 0$. Since $\lambda^{1/2}\mathcal{L}[g](\lambda) = \mathcal{L}[\Lambda_\gamma^{1/2} g](\lambda)$, using Theorem 6.2 and Weis's operator valued Fourier multiplier theorem [46], we have

$$\begin{aligned}
&\|e^{-\gamma t}(\zeta_1, \mathbf{v}_1, \vartheta_1)\|_{H_p^1(\mathbb{R},\mathcal{H}_q(\Omega))} + \|e^{-\gamma t}(\mathbf{v}_1\vartheta_1)\|_{L_p(\mathbb{R},H_q^2(\Omega))} \\
&\le r_b(\|e^{-\gamma t}(f_{10}, \mathbf{f}_{20}, f_{30})\|_{L_p(\mathbb{R},\mathcal{H}_q(\Omega))} + \|e^{-\gamma t}\Lambda_\gamma^{1/2} g\|_{L_p(\mathbb{R},L_q(\Omega))} + \|e^{-\gamma t} g\|_{L_p(\mathbb{R},H_q^1(\Omega))}),
\end{aligned}$$

which, combined with (6.60), leads to

$$\begin{aligned}
&\|(\zeta_1, \mathbf{v}_1, \vartheta_1)\|_{H_p^1((0,T),\mathcal{H}_q(\Omega))} + \|(\mathbf{v}_1, \vartheta_1)\|_{L_p((0,T),H_q^2(\Omega))} \\
&\quad \le r_b e^{\gamma T}(\|(f_1, \mathbf{f}_2, f_3)\|_{L_p((0,T),\mathcal{H}_q(\Omega))} + C_\gamma\|e^{-\gamma t} g\|_{H_p^{1/2}(\mathbb{R},L_q(\Omega))} + \|e^{-\gamma t} g\|_{L_p(\mathbb{R},H_q^1(\Omega))}).
\end{aligned} \tag{6.61}$$

Finally, let V_2 be a solution of the system

$$\begin{cases} \partial_t V_2 - AV_2 = 0 & \text{in } \Omega \times (0,\infty), \\ \mathbf{v}_2 = 0, \quad (\nabla\vartheta_2)\cdot\mathbf{n} = 0 & \text{on } \Gamma \times (0,\infty), \\ V_2 = V_0 - V_1|_{t=0} & \text{in } \Omega. \end{cases}$$

By the compatibility condition, $V_0 - V_1|_{t=0} \in \mathcal{D}_{p,q}(\Omega)$ provided that $2/p + 1/q \ne 1$ and $2/p + 1/q \ne 1$. Thus, by Theorem 6.14 we see that $V_2 = (\zeta_2, \mathbf{v}_2, \vartheta_2)$ exists and satisfies the following estimate:

$$\begin{aligned}&\|e^{-\gamma t}(\zeta_2, \mathbf{v}_2, \vartheta_2)\|_{H^1_p((0,\infty),\mathcal{H}_q(\Omega))} + \|e^{-\gamma t}(\mathbf{v}_2, \vartheta_2)\|_{L_p((0,\infty),H^2_q(\Omega))}\\ &\quad \leq C(\|(\zeta_0 - \zeta_1|_{t=0}, \mathbf{v}_0 - \mathbf{v}_1|_{t=0}, \vartheta_0 - \vartheta_1|_{t=0})\|_{D_{p,q}(\Omega)}).\end{aligned}$$

By real interpolation theorem, we have

$$\begin{aligned}&\|(\mathbf{v}_1|_{t=0}, \vartheta_1|_{t=0})\|_{B^{2(1-1/p)}_{q,p}(\Omega)}\\ &\quad \leq C(\|e^{-\gamma t}\partial_t(\mathbf{v}_1, \vartheta_1)\|_{L_p((0,\infty),L_q(\Omega))} + \|e^{-\gamma t}(\mathbf{v}_1, \vartheta_1)\|_{L_p((0,\infty),H^2_q(\Omega))})\end{aligned}$$

because $e^{-\gamma t}(\mathbf{v}_1, \vartheta_1)|_{t=0} = (\mathbf{v}_1|_{t=0}, \vartheta_1|_{t=0})$. Putting $\zeta = \zeta_1 + \zeta_2$, $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, and $\vartheta = \vartheta_1 + \vartheta_2$, we see that ζ, $\mathbf{v}$, and ϑ are required solutions of (4.6). The uniqueness follows from the existence of solutions for the dual problem (cf. Shibata and Shimizu [40, Proof of Theorem 4.3]). This completes the proof of Theorem 4.1.

7. Decay estimate—Proof of Theorem 5.1. To prove Theorem 5.1, we first prove the existence of a C^0 analytic semigroup associated with (5.6) that is exponentially stable. For this purpose, we consider the resolvent problem:

$$(7.1)\qquad \begin{cases} \lambda\zeta + a_{0*}\mathrm{div}\,\mathbf{v} = f_1 & \text{in } \Omega,\\ \lambda\mathbf{v} - a_{0*}^{-1}(\mu\Delta\mathbf{v} + \nu\nabla\mathrm{div}\,\mathbf{v} - a_{1*}\nabla\zeta - a_{2*}\nabla\vartheta) = \mathbf{f}_2 & \text{in } \Omega,\\ \lambda\vartheta + a_{3*}^{-1}(a_{2*}\mathrm{div}\,\mathbf{v} - a_{4*}\Delta\vartheta) = f_3 & \text{in } \Omega,\\ \mathbf{v}|_\Gamma = 0, \quad (\nabla\vartheta)\cdot\mathbf{n}|_\Gamma = 0. \end{cases}$$

We shall prove the following.

THEOREM 7.1. *Let $1 < q < \infty$ and $0 < \epsilon < \pi/2$. Assume that Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 2$) whose boundary Γ is a compact hypersurface of C^2 class. Assume that a_{0*}, a_{1*}, μ, ν, a_{3*}, and a_{4*} are positive constans and that a_{2*} is a nonzero constant. Let*

$$(7.2)\qquad \hat{\mathcal{H}}_q(\Omega) = \{(f_1, \mathbf{f}_2, f_3) \in \mathcal{H}_q(\Omega) \mid \int_\Omega f_1\,dx = \int_\Omega f_3\,dx = 0\}.$$

Set $\mathbf{C}_+ = \{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda \geq 0\}$. Then, for any $\lambda \in \mathbf{C}_+$ and $(f_1, \mathbf{f}_2, f_3) \in \hat{H}_q(\Omega)$, problem (7.1) admits a unique solution $U = (\zeta, \mathbf{v}, \vartheta) \in D_q(\Omega) \cap \hat{\mathcal{H}}_q(\Omega)$ possessing the esitmate

$$(7.3)\qquad (|\lambda| + 1)\|(\zeta, \mathbf{v}, \vartheta)\|_{\mathcal{H}_q(\Omega)} + \|(\mathbf{v}, \vartheta)\|_{H^2_q(\Omega)} \leq C\|(f_1, \mathbf{f}_2, f_3)\|_{\mathcal{H}_q(\Omega)}.$$

Proof. Employing the same argument as that in the proof of Theorem 6.2, we can prove the existence of $\mathcal{R}$-bounded solution operators corresponding to (7.1), and so there exists $\lambda_0 \geq 1$ such that for any $\lambda \in \Sigma_{\epsilon,\lambda_0}$ and $(f_1, \mathbf{f}_2, f_3) \in \mathcal{H}_q(\Omega)$, problem (7.1) admits a unique solution $(\zeta, \mathbf{v}, \vartheta) \in D_q(\Omega)$ possessing the estimate (7.3). Moreover, if f_1 and f_3 satisfy zero average condition, then ζ and ϑ also satisfy this condition in the case that $\lambda \neq 0$, which can be easily observed intergating $(7.1)_1$ and $(7.1)_3$ and applying the boundary conditions. Thus, for $\lambda \in \Sigma_{\epsilon,\lambda_0}$ the solutions obtained above belong to $\hat{\mathcal{H}}_q(\Omega)$.

Let $\mathcal{B}_{\lambda_0} = \{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda \geq 0,\ |\lambda| \leq \lambda_0\}$. Our task now is to prove the unique existence theorem for $\lambda \in \mathcal{B}_{\lambda_0}$. We first consider the case where $\lambda \neq 0$. Inserting the formula $\zeta = \lambda^{-1}(f_1 - a_{*0}\mathrm{div}\,\mathbf{v})$ into the second equation in (7.1), it becomes

$$\lambda\mathbf{v} - a_{0*}^{-1}\{\mu\Delta\mathbf{v} + (\nu + \lambda^{-1}a_{1*}a_{0*})\nabla\mathrm{div}\,\mathbf{v} - a_{2*}\nabla\vartheta\} = \mathbf{f}_2 - a_{0*}^{-1}a_{1*}\lambda^{-1}\nabla f_1.$$

Thus, we consider the following equations:

$$(7.4)\qquad \begin{cases} \lambda\mathbf{v} - a_{0*}^{-1}\{\mu\Delta\mathbf{v} + (\nu + \lambda^{-1}a_{1*}a_{0*})\nabla\mathrm{div}\,\mathbf{v}\} + a_{0*}^{-1}a_{2*}\nabla\vartheta = \mathbf{f}_2 & \text{in } \Omega, \\ \lambda\vartheta + a_{2*}a_{3*}^{-1}\mathrm{div}\,\mathbf{v} - a_{3*}^{-1}a_{4*}\Delta\vartheta = f_3 & \text{in } \Omega, \\ \mathbf{v}|_\Gamma = 0, \quad (\nabla\vartheta)\cdot\mathbf{n}|_\Gamma = 0. \end{cases}$$

To solve (7.4), we introduce a new resolvent parameter $\tau > 0$ and we consider the auxiliary problem

$$(7.5)\qquad \tau(\mathbf{v}, \vartheta) - \mathcal{A}_\lambda(\mathbf{v}, \vartheta) = (\mathbf{g}_1, g_2) \quad \text{in } \Omega,$$

where we have set

$$\begin{aligned} \mathcal{A}_\lambda(\mathbf{v}, \vartheta) &= (A_{1\lambda}\mathbf{v} - a_{0*}^{-1}a_{2*}\nabla\vartheta,\, a_{3*}^{-1}a_{4*}\Delta\vartheta - a_{2*}a_{3*}^{-1}\mathrm{div}\,\mathbf{v}) \\ &\qquad \text{for } (\mathbf{v}, \vartheta) \in \mathcal{D}_q^1(\Omega)\times\mathcal{D}_q^2(\Omega), \\ \mathcal{D}_q^1(\Omega) &= \{\mathbf{v} \in H_q^2(\Omega)^N \mid \mathbf{v}|_\Gamma = 0\}, \quad \mathcal{D}_q^2(\Omega) = \{\vartheta \in H_q^2(\Omega) \mid (\nabla\vartheta)\cdot\mathbf{n}|_\Gamma = 0\}, \\ A_{1\lambda}\mathbf{v} &= a_{0*}^{-1}(\mu\Delta\mathbf{v} + (\nu + \lambda^{-1}a_{1*}a_{0*})\nabla\mathrm{div}\,\mathbf{v}) \quad \text{for } \mathbf{v} \in H_q^2(\Omega). \end{aligned}$$

Let

$$\begin{aligned} \mathcal{A}_{1\lambda}\mathbf{v} &= a_{0*}^{-1}(\mu\Delta\mathbf{v} + (\nu + \lambda^{-1}a_{1*}a_{0*})\nabla\mathrm{div}\,\mathbf{v}) \quad \text{for } \mathbf{v} \in \mathcal{D}_q^1(\Omega), \\ \mathcal{A}_2\vartheta &= a_{3*}^{-1}a_{4*}\Delta\vartheta \quad \text{for } \vartheta \in \mathcal{D}_q^2(\Omega). \end{aligned}$$

Shibata and Tanaka [41] proved that there exists a $\tau_0 > 0$ such that $(\tau\mathbf{I} - \mathcal{A}_{1\lambda})^{-1}$ exists as a bounded linear operator from $L_q(\Omega)^N$ into $\mathcal{D}_q^1(\Omega)$ for $\tau \geq \tau_0$ possessing the estimate

$$(7.6)\qquad \tau\|\mathbf{w}\|_{L_q(\Omega)} + \tau^{1/2}\|\mathbf{w}\|_{H_q^1(\Omega)} + \|\mathbf{w}\|_{H_q^2(\Omega)} \leq C\|\mathbf{g}_1\|_{L_q(\Omega)}$$

for any $\tau \geq \tau_0$ and $\mathbf{g}_1 \in L_q(\Omega)^N$, where we have set $\mathbf{w} = (\tau\mathbf{I} - \mathcal{A}_{1\lambda})^{-1}\mathbf{g}_1$. And, by Theorem 6.4, we see that there exists a $\tau_0 > 0$ such that $(\tau\mathbf{I} - \mathcal{A}_2)^{-1}$ exists as a bounded linear operator from $L_q(\Omega)$ into $\mathcal{D}_q^2(\Omega)$ for $\tau \geq \tau_0$ possessing the estimate

$$(7.7)\qquad \tau\|\varphi\|_{L_q(\Omega)} + \tau^{1/2}\|\varphi\|_{H_q^1(\Omega)} + \|\varphi\|_{H_q^2(\Omega)} \leq C\|g_2\|_{L_q(\Omega)}$$

for any $\tau \geq \tau_0$ and $g_2 \in L_q(\Omega)$, where we have set $\varphi = (\tau\mathbf{I} - A_2)^{-1}\mathbf{g}_1$. To solve (7.5), we set $(\mathbf{v}, \vartheta) = ((\tau\mathbf{I} - \mathcal{A}_{1\lambda})^{-1}\mathbf{g}_1, (\tau\mathbf{I} - \mathcal{A}_2)^{-1}g_2)$. We then have

$$(7.8)\qquad \tau(\mathbf{v}, \vartheta) - \mathcal{A}_\lambda(\mathbf{v}, \vartheta) = (\mathbf{g}_1, g_2), + \mathcal{R}_\tau(\mathbf{g}_1, g_2),$$

where we have set

$$\mathcal{R}_\tau(\mathbf{g}_1, g_2) = (a_{0*}^{-1}a_{2*}\nabla(\tau\mathbf{I} - \mathcal{A}_2)^{-1}g_2,\, a_{2*}a_{3*}^{-1}\mathrm{div}\,(\tau\mathbf{I} - \mathcal{A}_1)^{-1}\mathbf{g}_1).$$

By (7.6) and (7.7), we have

$$\|\mathcal{R}_\tau(\mathbf{g}_1, g_2)\|_{L_q(\Omega)} \leq C\tau^{-1/2}\|(\mathbf{g}_1, g_2)\|_{L_q(\Omega)},$$

and so for large $\tau > 0$, $(\mathbf{I} - \mathcal{R}_\tau)^{-1}$ exists as an element in $\mathcal{L}(L_q(\Omega)^{N+1})$ and $\|(\mathbf{I} + \mathcal{R}_\tau)^{-1}\|_{\mathcal{L}(L_q(\Omega)^{N+1})} \leq 2$. Let $(\mathbf{I} + \mathcal{R}_\tau)^{-1}(\mathbf{g}_1, g_2) = (\mathbf{h}_{1\tau}, h_{2\tau})$, and then $\mathbf{v}_\tau =$

$(\tau\mathbf{I} - \mathcal{A}_1)\mathbf{h}_{\tau 1} \in \mathcal{D}^1_q(\Omega)$ and $\vartheta_\tau = (\tau - \mathcal{A}_2)^{-1}h_{2\tau} \in \mathcal{D}^2_q(\Omega)$ are unique solutions of (7.5) possessing the estimate

$$\tau\|(\mathbf{v}_\tau, \vartheta_\tau)\|_{L_q(\Omega)} + \|\mathbf{v}_\tau, \vartheta_\tau)\|_{H^2_q(\Omega)} \le C\|(\mathbf{g}_1, g_2)\|_{L_q(\Omega)} \tag{7.9}$$

for any large $\tau > 0$. Namely, the resolvent set $\rho(\mathcal{A}_\lambda)$ of $\mathcal{A}_\lambda$ contains (τ_1, ∞) for some $\tau_1 > 0$. We then write the resolvent operator by $(\tau\mathbf{I} - \mathcal{A}_\lambda)^{-1}$ as usual. If we set $(\mathbf{v}_\tau, \vartheta_\tau) = (\mathbf{I} - \mathcal{A}_\lambda)^{-1}(\mathbf{g}_1, g_2)$, then $(\mathbf{v}_\tau, \vartheta_\tau)$ satisfies the estimate (7.9). Using $(\tau\mathbf{I} - \mathcal{A}_\tau)^{-1}$, we write (7.4) as

$$(\mathbf{v}, \vartheta) + (\lambda - \tau)(\tau\mathbf{I} - \mathcal{A}_\lambda)^{-1}(\mathbf{v}, \vartheta) = (\tau\mathbf{I} - \mathcal{A}_\lambda)^{-1}(\mathbf{g}_1, g_2). \tag{7.10}$$

Since $(\lambda - \tau)(\tau\mathbf{I} - \mathcal{A}_\lambda)^{-1}$ is a compact operator on $L_q(\Omega)^{N+1}$, in view of Riesz–Schauder theory, in particular the Fredholm alternative principle, it is sufficient to prove that the kernel of $\mathbf{I} + (\lambda - \tau)(\tau\mathbf{I} - \mathcal{A}_\lambda)^{-1}$ is trivial in order to prove the existence of $(\mathbf{I} + (\lambda - \tau)(\tau\mathbf{I} - \mathcal{A}_\lambda)^{-1})^{-1} \in \mathcal{L}(L_q(\Omega)^{N+1})$. Thus, let $(\mathbf{g}_1, g_2)$ be an element in $L_q(\Omega)^{N+1}$ for which

$$(\mathbf{I} + (\lambda - \tau)(\tau\mathbf{I} - \mathcal{A}_\lambda)^{-1})(\mathbf{g}_1, g_2) = (0, 0).$$

Since $(\mathbf{g}_1, g_2) = (\tau - \lambda)(\tau\mathbf{I} - \mathcal{A}_\lambda)^{-1}(\mathbf{g}_1, g_2) \in \mathcal{D}^1_q(\Omega) \times \mathcal{D}^2_q(\Omega)$, setting $(\mathbf{v}, \vartheta) = (\tau\mathbf{I} - \mathcal{A}_\lambda)(\mathbf{g}_1, g_2)$, we have

$$(0, 0) = (\tau - \mathcal{A}_\lambda)(\mathbf{v}, \vartheta) + (\lambda - \tau)(\mathbf{v}, \vartheta) = (\lambda\mathbf{I} - \mathcal{A}_\lambda)(\mathbf{v}, \vartheta),$$

that is, $(\mathbf{v}, \vartheta) \in H^2_q(\Omega)^{N+1}$ satisfies the homogeneous equations

$$\begin{cases} a_{0*}\lambda\mathbf{v} - \mu\Delta\mathbf{v} - (\nu + \lambda^{-1}a_{1*}a_{0*})\nabla\mathrm{div}\,\mathbf{v} + a_{2*}\nabla\vartheta = 0 & \text{in } \Omega, \\ a_{3*}\lambda\vartheta + a_{2*}\mathrm{div}\,\mathbf{v} - a_{4*}\Delta\vartheta = 0 & \text{in } \Omega, \\ \mathbf{v}|_\Gamma = 0, \quad (\nabla\vartheta)\cdot\mathbf{n}|_\Gamma = 0. \end{cases} \tag{7.11}$$

To prove $(\mathbf{v}, \vartheta) = (0, 0)$, we first consider the case where $2 \le q < \infty$. Since $(\mathbf{v}, \vartheta) \in H^2_q(\Omega)^{N+1} \subset H^2_2(\Omega)^{N+1}$, by (7.11) and the divergence theorem of Gauss we have

$$\begin{aligned} 0 &= a_{0*}\lambda\|\mathbf{v}\|^2_{L_2(\Omega)} + \mu\|\nabla\mathbf{v}\|^2_{L_2(\Omega)} + (\nu + \lambda^{-1}a_{1*}a_{0*})\|\mathrm{div}\,\mathbf{v}\|^2_{L_2(\Omega)} \\ &\quad + a_{3*}\lambda\|\vartheta\|^2_{L_2(\Omega)} + a_{4*}\|\nabla\vartheta\|^2_{L_2(\Omega)} + a_{2*}\{(\nabla\vartheta, \mathbf{v})_\Omega - (\mathbf{v}, \nabla\vartheta)_\Omega\}. \end{aligned}$$

Taking the real part, we have

$$\begin{aligned} 0 &= a_{0*}\mathrm{Re}\,\lambda\|\mathbf{v}\|^2_{L_2(\Omega)} + \mu\|\nabla\mathbf{v}\|^2_{L_2(\Omega)} + (\nu + a_{1*}a_{0*}\mathrm{Re}\,\lambda^{-1})\|\mathrm{div}\,\mathbf{v}\|^2_{L_2(\Omega)} \\ &\quad + a_{3*}\mathrm{Re}\,\lambda\|\vartheta\|^2_{L_2(\Omega)} + a_{4*}\|\nabla\vartheta\|^2_{L_2(\Omega)}. \end{aligned}$$

Since $\mathrm{Re}\,\lambda \ge 0$, we have $\nabla(\mathbf{v}, \vartheta) = (0, 0)$ in Ω, that is, $\mathbf{v}$ and ϑ are constants. But, $\mathbf{v}|_\Gamma = 0$, and so $\mathbf{v} = 0$. Thus, by the second equation and boundary condition $(\nabla\vartheta)\cdot\mathbf{n}|_\Gamma = 0$ in (7.11), we have

$$0 = a_{3*}\lambda\int_\Omega \vartheta\,dx - a_{4*}\int_\Omega \Delta\vartheta\,dx = a_{3*}\lambda\int_\Omega \vartheta\,dx,$$

and so $\vartheta = 0$. Thus, in the case that $2 \le q < \infty$, we see that (7.4) admits a unique solution $(\mathbf{v}, \vartheta) \in \mathcal{D}^1_q(\Omega) \times \mathcal{D}^2_q(\Omega)$ possessing the estimate

$$\|(\mathbf{v}, \vartheta)\|_{H^2_q(\Omega)} \leq C_\lambda \|(\mathbf{f}_2, f_3)\|_{L_q(\Omega)} \tag{7.12}$$

for some constant C_λ depending on λ.

We next consider the case $1 < q < 2$. Let $q^* = q/(q-1) \in (2, \infty)$. For any $(\mathbf{g}_1, g_2) \in L_q(\Omega)^{N+1}$, let $(\mathbf{w}, \varphi) \in \mathcal{D}^1_{q^*}(\Omega) \times \mathcal{D}^2_{q^*}(\Omega)$ be a solution of the equation

$$\begin{cases} \bar\lambda \mathbf{w} - a_{0*}^{-1}\{\mu\Delta\mathbf{w} + (\nu + \bar\lambda^{-1}a_{1*}a_{0*})\nabla \mathrm{div}\,\mathbf{w}\} - a_{0*}^{-1}a_{2*}\nabla\varphi = \mathbf{g}_1 & \text{in } \Omega, \\ \lambda\varphi - a_{2*}a_{3*}^{-1}\mathrm{div}\,\mathbf{w} - a_{3*}^{-1}a_{4*}\Delta\varphi = g_2 & \text{in } \Omega, \\ \mathbf{v}|_\Gamma = 0, \quad (\nabla\vartheta)\cdot\mathbf{n}|_\Gamma = 0. \end{cases} \tag{7.13}$$

Replacing λ and a_{2*} by $\bar\lambda$ and $-a_{2*}$ in (7.4), we can prove the unique existence of solutions $(\mathbf{w}, \varphi) \in \mathcal{D}^1_q(\Omega) \times \mathcal{D}^2_q(\Omega)$ of (7.13). By the divergence theorem

$$\begin{aligned} 0 &= (a_{0*}\lambda\mathbf{v} - \mu\Delta\mathbf{v} - (\nu + \lambda^{-1}a_{1*}a_{0*})\nabla\mathrm{div}\,\mathbf{v} + a_{2*}\nabla\vartheta, \mathbf{w})_\Omega \\ &\quad + (a_{3*}\lambda\vartheta + a_{2*}\mathrm{div}\,\mathbf{v} - a_{4*}\Delta\vartheta, \varphi)_\Omega \\ &= a_{0*}(\mathbf{v}, \mathbf{g}_1)_\Omega + a_{3*}(\vartheta, g_2)_\Omega. \end{aligned}$$

Thus, the arbitrariness of $(\mathbf{g}_1, g_2) \in L_{q^*}(\Omega)^{N+1}$ yields $(\mathbf{v}, \vartheta) = (0, 0)$, which leads to the unique existence of solutions $(\mathbf{v}, \vartheta) \in \mathcal{D}^1_q(\Omega) \times \mathcal{D}^2_q(\Omega)$ of (7.4) possessing the estimate (7.12). Thus, we have proved that for any $\lambda \in \mathcal{B}_{\lambda_0} \setminus \{0\}$ and $(f_1, \mathbf{f}_2, f_3) \in \mathcal{H}_q(\Omega)$, (7.1) admits a unique solution $(\zeta, \mathbf{v}, \vartheta) \in \mathcal{D}_q(\Omega)$ possessing the estimate

$$\|\zeta, \mathbf{v}, \vartheta)\|_{D_q(\Omega)} \leq C_\lambda \|(f_1, \mathbf{f}_2, f_3)\|_{\mathcal{H}_q(\Omega)}. \tag{7.14}$$

We now consider the case that $\lambda = 0$. Inserting the relation $\mathrm{div}\,\mathbf{v} = a_{0*}^{-1}f_1$, we rewrite (7.1) as

$$\begin{cases} \mathrm{div}\,\mathbf{v} = a_{0*}^{-1}f_1 & \text{in } \Omega, \\ -\mu\Delta\mathbf{v} + a_{1*}\nabla\zeta = \mathbf{f}_2 + \nu a_{0*}^{-1}\nabla f_1 - a_{2*}\nabla\vartheta & \text{in } \Omega, \\ -a_{4*}\Delta\vartheta = f_3 - a_{0*}^{-1}a_{2*}f_1 & \text{in } \Omega, \\ \mathbf{v}|_\Gamma = 0, \quad (\nabla\vartheta)\cdot\mathbf{n}|_\Gamma = 0. \end{cases} \tag{7.15}$$

We first consider the Laplace equation

$$-a_{4*}\Delta\vartheta = g_2 \quad \text{in } \Omega, \quad (\nabla\vartheta)\cdot\mathbf{n}|_\Gamma = 0, \tag{7.16}$$

and then, for any $g_2 \in L_q(\Omega)$ with $\int_\Omega g_2\,dx = 0$, problem (7.16) admits a unique solution $\vartheta \in H^2_q(\Omega)$ with $\int_\Omega \vartheta\,dx = 0$ possessing the estimate $\|\vartheta\|_{H^2_q(\Omega)} \leq C\|g_2\|_{L_q(\Omega)}$. Therefore the third equation of (7.15) admits a unique solution $\vartheta \in H^2_q(\Omega)$ satisfying the estimate $\|\vartheta\|_{H^2_q(\Omega)} \leq C\|(f_1, f_3)\|_{L_q(\Omega)}$ and $\int_\Omega \vartheta\,dx = 0$.

Finally, setting $g_1 = a_{0*}^{-1}f_1$ and $\mathbf{g}_2 = \mathbf{f}_2 - \nu a_{0*}^{-1}\nabla f_1 - a_{2*}\nabla\vartheta$, we consider the Cattabriga problem

$$-\mu\Delta\mathbf{v} + a_{1*}\nabla\zeta = \mathbf{g}_2, \quad \mathrm{div}\,\mathbf{v} = g_1 \quad \text{in } \Omega, \quad \mathbf{v}|_\Gamma = 0. \tag{7.17}$$

By Farwig and Sohr [13], there exists a $\lambda_0 > 0$ for which the equation

$$\lambda_0\mathbf{v} - \mu\Delta\mathbf{v} + a_{1*}\nabla\zeta = \mathbf{g}_2, \quad \mathrm{div}\,\mathbf{v} = g_1 \quad \text{in } \Omega, \quad \mathbf{v}|_\Gamma = 0,$$

admits a unique solution $(\zeta, \mathbf{v}) \in H^1_q(\Omega) \times H^2_q(\Omega)^N$ with $\int_\Omega \zeta\, dx = 0$ for any $(g_1, \mathbf{g}_2) \in H^1_q(\Omega) \times L_q(\Omega)^N$ with $\int_\Omega g_2\, dx = 0$. Thus, by the Fredholm alternative principle, the uniqueness of solutions of (7.17) yields the unique existence theorem, that is, for any $(g_1, \mathbf{g}_2) \in H^1_q(\Omega) \times L_q(\Omega)^N$ with $\int_\Omega g_2\, dx = 0$, problem (7.17) admits a unique solution $(\zeta, \mathbf{v}) \in H^1_q(\Omega) \times H^2_q(\Omega)^N$ with $\int_\Omega \zeta\, dx = 0$ possessing the estimate

$$\|\zeta\|_{H^1_q(\Omega)} + \|\mathbf{v}\|_{H^2_q(\Omega)} \le C(\|g_1\|_{H^1_q(\Omega)} + \|\mathbf{g}_2\|_{L_q(\Omega)}).$$

Therefore the problem of existence for (7.17) is reduced to showing uniqueness for the homogeneous problem which is an immediate consequence of the divergence theorem.

Summing up, we have proved that for any $(f_1, \mathbf{f}_2, f_3) \in \hat{\mathcal{H}}_q(\Omega)$, problem (7.15) admits a unique solution $(\zeta, \mathbf{v}, \vartheta) \in \mathcal{D}_q(\Omega) \cap \hat{\mathcal{H}}_q(\Omega)$ possessing the estimate

$$\|\zeta\|_{H^1_q(\Omega)} + \|(\mathbf{v}, \vartheta)\|_{H^2_q(\Omega)} \le C(\|f_1\|_{H^1_q(\Omega)} + \|(\mathbf{f}_2, f_3)\|_{L_q(\Omega)}).$$

Since the resolvent operator is continuous and the set $\mathcal{B}_{\lambda_0}$ is compact, we can take the constants C_λ in the estimate (7.14) independent of $\lambda \in \mathcal{B}_{\lambda_0}$. This completes the proof of Theorem 7.1. ∎

We now give a proof.

Proof of Theorem 5.1. Let

$$PU = \begin{pmatrix} -\rho_{0*}\operatorname{div}\mathbf{v} \\ a_{0*}^{-1}(\mu\Delta\mathbf{v} + \nu\nabla\operatorname{div}\mathbf{v} - a_{1*}\nabla\zeta - a_2\nabla\vartheta) \\ -a_{3*}^{-1}(a_{2*}\operatorname{div}\mathbf{v} - a_{4*}\Delta\vartheta) \end{pmatrix} \quad \text{for } U = (\zeta, \mathbf{v}, \vartheta) \in D_q(\Omega),$$
$$\mathcal{P}U = PU \quad \text{for } U = (\zeta, \mathbf{v}, \vartheta) \in \mathcal{D}_q(\Omega) \cap \hat{\mathcal{H}}_q(\Omega).$$

Here, $\hat{\mathcal{H}}_q(\Omega)$ and $\mathcal{D}_q(\Omega)$ are the spaces given in (7.2) and (6.52), respectively. Let us consider the Cauchy problem

$$\partial_t U - \mathcal{P}U = 0 \quad \text{for } t > 0, \quad U|_{t=0} = U_0 = (\zeta_0, \mathbf{v}_0, \vartheta_0) \in \hat{\mathcal{H}}_q(\Omega). \tag{7.18}$$

The resolvent problem corresponding to (7.18) is (7.1). Thus, by Theorem 7.1, we see that $\mathcal{P}$ generates a C_0 analytic semigroup $\{\dot{T}(t)\}_{t\ge 0}$ that is exponentially stable on $\hat{\mathcal{H}}_q(\Omega)$, that is,

$$\|\dot{T}(t)U_0\|_{\mathcal{H}_q(\Omega)} \le Ce^{-\gamma_1 t}\|U_0\|_{\mathcal{H}_q(\Omega)} \tag{7.19}$$

for any $U_0 \in \hat{\mathcal{H}}_q(\Omega)$ and $t > 0$ with some positive constants C and γ_1.

Let $\lambda_1 > 0$ be a sufficiently large number and let $0 < \gamma < \gamma_1$ be a small positive number determined later. We consider the time-shifted equations

$$\begin{cases} \partial_t U_1 + \lambda_1 U_1 - PU_1 = G & \text{in } \Omega \times (0, T), \\ BU_1 = (0, g_4) & \text{on } \Gamma \times (0, T), \\ U_1|_{t=0} = U_0 & \text{in } \Omega, \end{cases} \tag{7.20}$$

where $G = (g_1, \mathbf{g}_2, g_3)$ and $BU = (\mathbf{v}, (\nabla\vartheta) \cdot \mathbf{n})$. Multiplying (7.20) by $e^{\gamma t}$, we have

$$\begin{cases} \partial_t(e^{\gamma t}U_1) + (\lambda_1 - \gamma)e^{\gamma t}U_1 - P(e^{\gamma t}U_1) = e^{\gamma t}G & \text{in } \Omega \times (0, T), \\ B(e^{\gamma t}U_1) = (0, e^{\gamma t}g_4) & \text{on } \Gamma \times (0, T), \\ e^{\gamma t}U_1|_{t=0} = U_0 & \text{in } \Omega. \end{cases} \tag{7.21}$$

Let G_0 be the zero extension of G to $\mathbb{R}$ with respect to t, that is, $G_0(\cdot,t) = G(\cdot,t)$ for $t \in (0,T)$ and $G_0(\cdot,t) = 0$ for $t \notin (0,T)$. To estimate $e^{\gamma t}U_1$, we consider the equations

$$\begin{cases} \partial_t U_2 + (\lambda_1 - \gamma)U_2 - PU_2 = e^{\gamma t}G_0 & \text{in } \Omega \times \mathbb{R}, \\ BU_2 = (0, e^{\gamma t}g_4) & \text{on } \Gamma \times \mathbb{R}. \end{cases} \tag{7.22}$$

Applying the Fourier transform with respect to t to (7.22), we have

$$\begin{cases} (\lambda_1 - \gamma + i\tau)\mathcal{F}[U_2](\cdot,\tau) - P\mathcal{F}[U_2](\cdot,\tau) = \mathcal{F}[e^{\gamma t}G_0](\cdot,\tau) & \text{in } \Omega, \\ B\mathcal{F}[U_2](\cdot,\tau) = (0, \mathcal{F}[e^{\gamma t}g_4](\cdot,\tau)) & \text{on } \Gamma. \end{cases} \tag{7.23}$$

Let $\mathcal{S}(\lambda) = (\mathcal{A}(\lambda), \mathcal{B}_1(\lambda), \mathcal{B}_2(\lambda))$ be the $\mathcal{R}$-bounded solution operators given in Theorem 6.2. If we choose $\lambda_1 > 0$ so large that $\lambda_1 - \gamma \geq \lambda_0$, then we have $\mathcal{F}[\hat{U}_2](\cdot,\tau) = \mathcal{S}(\lambda_1 - \gamma + i\tau)\mathbf{F}_{\lambda_1-\gamma+i\tau}$, where

$$\mathbf{F}_{\lambda_1-\gamma+i\tau} = (\mathcal{F}[e^{\gamma t}G_0](\cdot,\tau), (\lambda_1 - \gamma + i\tau)^{1/2}\mathcal{F}[e^{\gamma t}g_4](\cdot,\tau), \mathcal{F}[e^{\gamma t}g_4](\cdot,\tau)).$$

Since

$$(\tau\partial_\tau)^\ell(i\tau/\lambda_1 - \gamma + i\tau)| \leq C_{\lambda_1}, \quad |(\tau\partial_\tau)^\ell((\lambda_1 - \gamma + i\tau)^{1/2}/(1+\tau^2)^{1/4})| \leq C_{\lambda_1}$$

for $\ell = 0, 1$ and $\tau \in \mathbb{R} \setminus \{0\}$, applying Weis's operator valued Fourier multiplier theorem and Bourgain's theorem (cf. Lemma 6.9) to

$$U_1 = \mathcal{F}^{-1}[\mathcal{F}[U_1](\cdot,\tau)] = \mathcal{F}^{-1}[\mathcal{S}(\lambda_1 - \gamma + i\tau)\mathbf{F}_{\lambda_1-\gamma+i\tau}],$$

we have

$$\begin{aligned} &\|\partial_t U_2\|_{L_p(\mathbb{R},\mathcal{H}_q(\Omega))} + \|U_2\|_{L_p(\mathbb{R},D_q(\Omega))} \\ &\quad \leq C(\|e^{\gamma t}G_0\|_{L_p(\mathbb{R},\mathcal{H}_q(\Omega))} + \|e^{\gamma t}g_4\|_{H_p^{1/2}(\mathbb{R},L_q(\Omega))} + \|e^{\gamma t}g_4\|_{L_p(\mathbb{R},H_q^1(\Omega))}) \\ &\quad \leq C(\|e^{\gamma t}G\|_{L_p((0,T),\mathcal{H}_q(\Omega))} + \|e^{\gamma t}g_4\|_{H_p^{1/2}(\mathbb{R},L_q(\Omega))} + \|e^{\gamma t}g_4\|_{L_p(\mathbb{R},H_q^1(\Omega))}). \end{aligned} \tag{7.24}$$

We next consider the Cauchy problem

$$\begin{cases} \partial_t U_3 + (\lambda_1 - \gamma)U_3 - PU_3 = 0 & \text{in } \Omega \times (0,\infty), \\ BU_3 = (0,0) & \text{on } \Gamma \times (0,\infty), \\ U_3|_{t=0} = U_0 - U_2|_{t=0} & \text{in } \Omega. \end{cases} \tag{7.25}$$

If we choose $\lambda_1 > 0$ sufficiently large, by Theorem 6.2 we see that there exists a C^0 analytic semigroup $\{T_1(t)\}_{t\geq 0}$ associated with (7.21), which is exponentially stable. Setting $U_3 = T_1(t)(U_0 - U_2|_{t=0})$, we then see that U_2 satisfies (7.21) and the estimate

$$\|e^{\gamma t}\partial_t U_3\|_{L_p((0,\infty),\mathcal{H}_q(\Omega))} + \|e^{\gamma t}U_3\|_{L_p((0,\infty),D_q(\Omega))} \leq C\|U_0 - U_2|_{t=0}\|_{D_{p,q}(\Omega)}. \tag{7.26}$$

By the uniqueness of solutions, we have $e^{\gamma t}U_1 = U_2 + U_3$, and so by (7.24), (7.26), and real interpolation theorem (4.14) and (4.15), we have

$$
\begin{aligned}
&\|e^{\gamma t}\partial_t U_1\|_{L_p((0,T),\mathcal{H}_q(\Omega))} + \|e^{\gamma t}U_1\|_{L_p((0,T),D_q(\Omega))} \\
&\quad \le C(\|(\zeta_0,\mathbf{v}_0,\vartheta_0)\|_{D_{p,q}(\Omega)} + \|e^{\gamma t}G\|_{L_p((0,T),\mathcal{H}_q(\Omega))} \\
&\qquad + \|e^{\gamma t}g_4\|_{H_p^{1/2}(\mathbb{R},L_q(\Omega))} + \|e^{\gamma t}g_4\|_{L_p(\mathbb{R},H_q^1(\Omega))}).
\end{aligned} \tag{7.27}
$$

We next consider the equations

$$
\partial_t V - PV = -\lambda_0 U_1 \quad \text{in } \Omega\times(0,T), \quad BV|_\Gamma = 0, \quad V|_{t=0} = 0 \quad \text{in } \Omega. \tag{7.28}
$$

Let $U_1 = (\zeta_1, \mathbf{v}_1, \vartheta_1)$ and set

$$
\tilde U_1(x,t) = (\zeta_1(x,t) - \frac{1}{|\Omega|}\int_\Omega \zeta_1(y,t)\,dy, \mathbf{v}_1(x,t), \vartheta_1(x,t) - \frac{1}{|\Omega|}\int_\Omega \vartheta_1(y,t)\,dy). \tag{7.29}
$$

Then $\tilde U(\cdot,t) \in \hat{\mathcal{H}}_q(\Omega)$ for any $t\in(0,T)$. We consider the equations

$$
\partial_t \tilde V - P\tilde V = -\lambda_0 \tilde U_1 \quad \text{in } \Omega\times(0,T), \quad B\tilde V|_\Gamma = 0, \quad \tilde V|_{t=0} = 0 \quad \text{in } \Omega. \tag{7.30}
$$

In view of (7.18), by the Duhamel principle we have $\tilde V = \int_0^t \dot T(t-s)\tilde U_1(\cdot,s)\,ds$. Moreover, by (7.19) we have

$$
\|e^{\gamma t}\tilde V\|_{L_p((0,T),\mathcal{H}_q(\Omega))} \le C(\gamma_1-\gamma)^{-1/p}\|e^{\gamma t}\tilde U_1\|_{L_p((0,T),\mathcal{H}_q(\Omega))}. \tag{7.31}
$$

In fact, by (7.19) and Hölder's inequality with exponent $p' = p/(p-1)$ we have

$$
\begin{aligned}
&e^{\gamma t}\|\tilde V(\cdot,t)\|_{L_q(\Omega)} \\
&\quad \le C\int_0^t e^{\gamma t}e^{-\gamma_1(t-s)}\|\tilde U_1(\cdot,s)\|_{\mathcal{H}_q(\Omega)}\,ds = C\int_0^t e^{-\gamma_1(t-s)}e^{\gamma s}\|\tilde U_1(\cdot,s)\|_{\mathcal{H}_q(\Omega)}\,ds \\
&\quad \le \left(\int_0^t e^{-(\gamma_1-\gamma)(t-s)}\,ds\right)^{1/p'}\left(\int_0^t e^{-(\gamma_1-\gamma)(t-s)}(e^{\gamma s}\|\tilde U(\cdot,s)\|_{\mathcal{H}_q(\Omega)})^p\,ds\right)^{1/p},
\end{aligned}
$$

and so by the change of integration order we have

$$
\begin{aligned}
&\int_0^T (e^{\gamma t}\|\tilde V(\cdot,t)\|_{L_q(\Omega)})^p\,dt \\
&\quad \le C^p(\gamma_1-\gamma)^{-p/p'}\int_0^T (e^{\gamma s}\|\tilde U(\cdot,s)\|_{\mathcal{H}_q(\Omega)})^p\,ds\int_s^T e^{-(\gamma_1-\gamma)(t-s)}\,dt \\
&\quad = C^p(\gamma_1-\gamma)^{-p}\int_0^T (e^{\gamma s}\|\tilde U(\cdot,s)\|_{\mathcal{H}_q(\Omega)})^p\,ds.
\end{aligned}
$$

Thus, we have (7.31). Since $\tilde V$ satisfies the shifted equations

$$
\partial_t\tilde V + \lambda_0\tilde V - P\tilde V = -\lambda\tilde U_1 + \lambda\tilde V \quad \text{in } \Omega\times(0,T), \quad B\tilde V|_\Gamma = 0, \quad \tilde V|_{t=0} = 0,
$$

we have

$$
\begin{aligned}
&\|e^{\gamma t}\partial_t\tilde V\|_{L_p((0,T),\mathcal{H}_q(\Omega))} + \|e^{\gamma t}\tilde V\|_{L_p((0,T),D_q(\Omega))} \\
&\quad \le C(\|e^{\gamma t}\tilde U_1\|_{L_p((0,T),\mathcal{H}_q(\Omega))} + \|e^{\gamma t}\tilde V\|_{L_p((0,T),\mathcal{H}_q(\Omega))}),
\end{aligned}
$$

which, combined with (7.27) and (7.31), leads to

$$
\begin{aligned}
&\|e^{\gamma t}\partial_t\tilde{V}\|_{L_p((0,T),\mathcal{H}_q(\Omega))} + \|e^{\gamma t}\tilde{V}\|_{L_p((0,T),D_q(\Omega))} \\
&\quad \le C(\|(\zeta_0,\mathbf{v}_0,\vartheta_0)\|_{D_{p,q}(\Omega)} + \|e^{\gamma t}G\|_{L_p((0,T),\mathcal{H}_q(\Omega))} \\
&\qquad + \|e^{\gamma t}g_4\|_{H_p^{1/2}(\mathbb{R},L_q(\Omega))} + \|e^{\gamma t}g_4\|_{L_p(\mathbb{R},H_q^1(\Omega))}).
\end{aligned} \tag{7.32}
$$

In view of (7.29), we define V by

$$
V = \tilde{V} - \left(\frac{1}{|\Omega|}\int_0^t\int_\Omega \zeta_1(x,s)\,dx, 0, \frac{1}{|\Omega|}\int_0^t\int_\Omega \vartheta_1(x,s)\,dxds\right),
$$

and then V satisfies (7.28). Moreover, setting $V = (\zeta_2,\mathbf{v}_2,\vartheta_2)$, by (7.32) and (7.27) we have

(7.33)

$$
\begin{aligned}
&\|e^{\gamma t}\partial_t(\zeta_2,\mathbf{v}_2,\vartheta_2)\|_{L_p((0,T),\mathcal{H}_q(\Omega))} + \|e^{\gamma t}\nabla\zeta_2\|_{L_p((0,T),H_q^1(\Omega))} + \|e^{\gamma t}\mathbf{v}_2\|_{L_p((0,T),H_q^2(\Omega))} \\
&\quad + \|e^{\gamma t}\nabla\vartheta_2\|_{L_p((0,T),H_q^1(\Omega))} \le C(\|(\zeta_0,\mathbf{v}_0,\vartheta_0)\|_{D_{p,q}(\Omega)} + \|e^{\gamma t}G\|_{L_p((0,T),\mathcal{H}_q(\Omega))} \\
&\qquad + \|e^{\gamma t}g_4\|_{H_p^{1/2}(\mathbb{R},L_q(\Omega))} + \|e^{\gamma t}g_4\|_{L_p(\mathbb{R},H_q^1(\Omega))}).
\end{aligned}
$$

Let $(\zeta,\mathbf{v},\vartheta) = U_1 + V$, and then $(\zeta,\mathbf{v},\vartheta)$ is a unique solution of (5.6). Moreover, by (7.33) and (7.27) $(\zeta,\mathbf{v},\vartheta)$ satisfies the decay estimate

$$
\begin{aligned}
&\|e^{\gamma t}\partial_t(\zeta,\mathbf{v},\vartheta)\|_{L_p((0,T),\mathcal{H}_q(\Omega))} + \|e^{\gamma t}\nabla\zeta\|_{L_p((0,T),H_q^1(\Omega))} \\
&\quad + \|e^{\gamma t}\mathbf{v}\|_{L_p((0,T),H_q^2(\Omega))} + \|e^{\gamma t}\nabla\vartheta\|_{L_p((0,T),H_q^1(\Omega))} \\
&\le C(\|(\zeta_0,\mathbf{v}_0,\vartheta_0)\|_{D_{p,q}(\Omega)} + \|e^{\gamma t}(g_1,\mathbf{g}_2,g_3)\|_{L_p((0,T),\mathcal{H}_q(\Omega))} \\
&\qquad + \|e^{\gamma t}g_4\|_{H_p^{1/2}(\mathbb{R},L_q(\Omega))} + \|e^{\gamma t}g_4\|_{L_p(\mathbb{R},H_q^1(\Omega))}).
\end{aligned}
$$

This completes the proof of Theorem 5.1. ∎

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