EQUIVARIANT THINNING OVER A FREE GROUP

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Abstract. We construct entropy increasing monotone factors in the context of a Bernoulli shift over the free group of rank at least two.

1. Introduction

Let κ be a probability measure on a finite set K. We will mainly be concerned with the simple case where $K = \{0, 1\}$, where we call $\kappa(1) := \kappa(\{1\}) \in (0, 1)$ the ***intensity*** of κ. Let G be a group. A ***Bernoulli shift over*** G ***with base*** (K, κ) is the measure-preserving system (G, K^G, κ^G), where G acts on K^G via $(gx)(f) = x(g^{-1}f)$ for $x \in K^G$ and $g, f \in G$. Let ι be a probability measure of lower intensity. We say that a measurable map $\phi : K^G \to K^G$ is an ***equivariant thinning from*** κ ***to*** ι if $\phi(x)(g) \leq x(g)$ for all $x \in K^G$ and $g \in G$, the push-forward of κ^G under ϕ is ι^G, and ϕ is equivariant κ^G-almost-surely; that is, on a set of full-measure, $\phi \circ g = g \circ \phi$ for all $g \in G$.

Theorem 1. *Let κ and ι be probability measures on $\{0, 1\}$ and ι be of lower intensity. For Bernoulli shifts over the free group of rank at least two, there exists an equivariant thinning from κ to ι.*

Theorem 1 does not hold with such generality in the case of a Bernoulli shift over an amenable group like the integers. Recall that the ***entropy*** of a probability measure κ on a finite set K is given by

$$H(\kappa) := -\sum_{i \in K} \kappa(i) \log \kappa(i).$$

Theorem 2 (Ball [3], Soo [16])**.** *Let κ and ι be probability measures on $\{0, 1\}$ and ι be of lower intensity. For Bernoulli shifts over the integers, there exists an equivariant thinning from κ to ι if and only if $H(\kappa) \geq H(\iota)$.*

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In Theorem 2, the necessity of $H(\kappa) \geq H(\iota)$ follows easily from the classical theory of Kolmogorov-Sinai entropy [8, 19], which we now recall. Let G be a group and let κ and ι be probability measures on a finite set K. An equivariant map ϕ is a ***factor*** from κ to ι if the push-forward of κ^G under ϕ is ι^G, and is an ***isomorphism*** if ϕ is a bijection and its inverse also serves as a factor from ι to κ. In the case $G = \mathbb{Z}$, Kolmogorov proved that entropy is non-increasing under factor maps; this implies the necessity of $H(\kappa) \geq H(\iota)$ in Theorem 2. Furthermore, Sinai [15] proved that there is a factor from κ to ι if $H(\kappa) \geq H(\iota)$, and Ornstein [12] proved there is an isomorphism from κ to ι if and only if $H(\kappa) = H(\iota)$. Thus entropy is a complete invariant for Bernoulli shifts over $\mathbb{Z}$. Ornstein and Weiss [13] generalized these results to the case where G is an amenable group. See also Keane and Smorodinsky for concrete constructions of factor maps and isomorphisms [9, 10].

The sufficiency of $H(\kappa) > H(\iota)$ in Theorem 2 was first proved by Ball [3]. The existence of an isomorphism that is also an equivariant thinning in the equal entropy case was proved by Soo [16]. Let us remark that the factor maps given in standard proofs of the Sinai and Ornstein theorems will not in general be monotone; that is, they may not satisfy $\phi(x)(i) \leq x(i)$ for all $x \in \{0,1\}^{\mathbb{Z}}$ and $i \in \mathbb{Z}$.

Towards the end of their 1987 paper, Ornstein and Weiss [13] give a simple but remarkable example of an entropy increasing factor in the case where G is the free group of rank at least two, which is further elaborated upon by Ball [2]. It was an open question until recently whether all Bernoulli shifts over a free group of rank at least two are isomorphic. This question was answered negatively by Lewis Bowen [5] in 2010, who proved that although entropy can increase under factor maps, in the context of a free group with rank at least two, it is still a complete isomorphism invariant. Recently, there has been much interest in studying factors in the non-amenable setting; see Russell Lyons [11] for more information.

Our proof of Theorem 1 will make use of a variation of the Ornstein and Weiss example in Ball [2] and a primitive version of a marker-filler type construction, in the sense of Keane and Smorodinsky [9, 10]. Our construction uses randomness already present in the process in a careful way as to mimic a construction that one would make if additional independent randomization were available. This approach was taken by Holroyd, Lyons, and Soo [7], Angel, Holroyd, and Soo [1], and Ball [4] for defining equivariant thinning in the context of Poisson point processes.

2. Tools

2.1. **Coupling.** Let (A, α) and (B, β) be probability spaces. A ***coupling*** of α and β is a probability measure on the product space $A \times B$ which has α and β as its marginals. For a random variable X, we will refer to the measure $\mathbb{P}(X \in \cdot)$ as the ***law*** or the ***distribution*** of X. If two random variables X and Y have the same law, we write $X \stackrel{d}{=} Y$. Similarly, a ***coupling*** of random variables X and Y is a pair of random variables (X', Y'), where X' and Y' are defined on the same probability space and have the same law as X and Y, respectively. Thus a coupling of random variables gives a coupling of the laws of the random variables. Often we will refer to the law of a pair of random variables as the ***joint distribution*** of the random variables. In the case that $A = B$ and A is a partially ordered by the relation $\preceq$, we say that a coupling γ is ***monotone*** if $\gamma \{(a, b) \subset A \times A : b \preceq a)\} = 1$. We will always endow the space of binary sequences $\{0, 1\}^I$ indexed by a set I with the partial order $x \preceq y$ if and only if $x_i \leq y_i$ for $i \in I$.

Example 3 (Independent thinning)**.** Let κ and ι be probability measures on $\{0, 1\}$, where $\kappa(1) := p \geq \iota(1) := q$. Let $r := \frac{p-q}{p}$. Then the measure ρ on $\{0, 1\}^2$ given by

$$\rho(0, 0) = 1 - p, \ \rho(0, 1) = 0, \ \rho(1, 0) = rp, \text{ and } \rho(1, 1) = (1 - r)p$$

is a monotone coupling of κ and ι. Thus under ρ, a 1 is thinned to a 0 with probability r and kept with probability $1 - r$. Clearly, the product measure ρ^n is a monotone coupling of κ^n and ι^n. We will refer to the coupling ρ^n as the ***independent thinning of*** κ^n ***to*** ι^n. $\diamondsuit$

The following simple lemma is one of the main ingredients in the proof of Theorem 1. In it we construct a coupling of κ^n and ι^n for n sufficiently large which will allow us to extract spare randomness from a related coupling of κ^G and ι^G. We will write $0^n 1^m$ to indicate the binary sequence of length $n + m$ of n zeros followed by m ones.

Lemma 4 (Key coupling)**.** *Let κ and ι be probability measures on $\{0, 1\}$, where κ is of greater intensity. For n sufficiently large, there exists a monotone coupling γ of κ^n and ι^n such that*

$$\gamma(100^{n-2}, 0^n) = \kappa^n(100^{n-2})$$

and

$$\gamma(010^{n-2}, 0^n) = \kappa^n(010^{n-2}).$$

Proof. Let $p = \kappa(1)$, $q = \iota(1)$, and ρ^n be the independent thinning of κ^n to ι^n as in Example 3. We will perturb ρ^n to give the required coupling.

We specify a probability measure ϱ on $\{0,1\}^n \times \{0,1\}^n$ by stating that it agrees with ρ^n except on the points $(100^{n-2}, 0^n)$, $(010^{n-2}, 0^n)$, $(100^{n-2}, 100^{n-2})$, and $(010^{n-2}, 010^{n-2})$, where we specify that

$$\varrho(100^{n-2}, 0^n) = \varrho(010^{n-2}, 0^n) = p(1-p)^{n-1}$$

and

$$\varrho(100^{n-2}, 100^{n-2}) = \varrho(010^{n-2}, 010^{n-2}) = 0.$$

Thus ϱ is almost a monotone coupling of κ^n and ι^n, except that from our changes to ρ^n we have

$$\begin{aligned}\sum_{x\in\{0,1\}^n} \varrho(x, 0^n) = \sum_{x\in\{0,1\}^n} &\rho^n(x, 0^n) - \rho^n(100^{n-2}, 0^n) - \rho^n(010^{n-2}, 0^n) \\ &+ \varrho(100^{n-2}, 0^n) + \varrho(010^{n-2}, 0^n) \\ = (1-q&)^n + 2p(1-p)^{n-1}(1-r),\end{aligned}$$

and

$$\begin{aligned}\sum_{x\in\{0,1\}^n} \varrho(x, 100^{n-2}) &= \sum_{x\in\{0,1\}^n} \rho^n(x, 100^{n-2}) - \rho^n(100^{n-2}, 100^{n-2}) \\ &\qquad + \varrho(100^{n-2}, 100^{n-2}) \\ &= q(1-q)^{n-1} - p(1-p)^{n-1}(1-r) + 0 \\ &= \sum_{x\in\{0,1\}^n} \varrho(x, 010^{n-2}),\end{aligned}$$

where $r = \frac{p-q}{p}$.

We perturb ϱ to obtain the desired coupling γ. Consider the set B_1 of all binary sequences of length n, where $x \in B_1$ if and only if $x_1 = 1$, $x_2 = 0$, and $\sum_{i=3}^n x_i = 1$. Similarly, let B_2 be the set of all binary sequences of length n, where $x \in B_2$ if and only if $x_1 = 0$, $x_2 = 1$, and $\sum_{i=3}^n x_i = 1$. The sets B_1 and B_2 are disjoint, and each have cardinality $n-2$.

For $x \in B_1 \cup B_2$,

$$\varrho(x, 0^n) = \rho^n(x, 0^n) = p^2(1-p)^{n-2}r^2,$$

for $x \in B_1$,

$$\varrho(x, 100^{n-2}) = \rho^n(x, 100^{n-2}) = p^2(1-p)^{n-2}r(1-r),$$

and for $x \in B_2$,

$$\varrho(x, 010^{n-2}) = \rho^n(x, 010^{n-2}) = p^2(1-p)^{n-2}r(1-r).$$

Note that for n sufficiently large

$$\sum_{x\in B_1\cup B_2} \varrho(x, 0^n) = 2(n-2)p^2(1-p)^{n-2}r^2 > 2p(1-p)^{n-1}(1-r).$$

Let γ be equal to ϱ except on the set of points

$\{(x,0^n) : x \in B_1 \cup B_2\} \cup \{(x,100^{n-2}) : x \in B_1\} \cup \{(x,010^{n-2}) : x \in B_2\}$,

where we make the following adjustments. For $x \in B_1 \cup B_2$, set

$$\gamma(x,0^n) = p^2(1-p)^{n-2}r^2 - \frac{p(1-p)^{n-1}(1-r)}{n-2} > 0,$$

for $x \in B_1$, set

$$\gamma(x,100^{n-2}) = p^2(1-p)^{n-2}(1-r)r + \frac{p(1-p)^{n-1}(1-r)}{n-2},$$

and for $x \in B_2$, set

$$\gamma(x,010^{n-2}) = p^2(1-p)^{n-2}(1-r)r + \frac{p(1-p)^{n-1}(1-r)}{n-2}.$$

That γ has the required properties follows from its construction. □

To illustrate the utility of Lemma 4, we will give a different proof of the following result of Peled and Gurel-Gurevich [6]. Let $\mathbb{N} = \{0,1,2,\ldots\}$.

Theorem 5 (Peled and Gurel-Gurevich [6])**.** *Let κ and ι be probability measures on $\{0,1\}$, where κ is of greater intensity. There exists a measurable map $\phi : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ such that the push-forward of $\kappa^{\mathbb{N}}$ under ϕ is $\iota^{\mathbb{N}}$ and $\phi(x)(i) \le x(i)$ for all $x \in \{0,1\}^{\mathbb{N}}$ and all $i \in \mathbb{N}$.*

We note that in [6, Theorem 1.3], they use the dual terminology of *thickenings*; their equivalent theorem states that for probability measures ι and κ on $\{0,1\}$, where ι is of lesser intensity, there is a measurable map $\phi : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ such that the push-forward of $\iota^{\mathbb{N}}$ under ϕ is $\kappa^{\mathbb{N}}$ and $\phi(x)(i) \ge x(i)$ for all $x \in \{0,1\}^{\mathbb{N}}$ and all $i \in \mathbb{N}$.

In the proof of Theorem 5, we will make use of the following two lemmas. We say that a random variable U is ***uniformly distributed*** in $[0,1]$ if the probability that U lies in a Borel subset of the unit interval is given by the Lebesgue measure of the set.

Lemma 6. *Let (X,Y) be a pair of discrete random variables taking values on the finite set $A \times B$ with joint distribution γ. There exists a measurable function $\Gamma : A \times [0,1] \to B$ such that if U is uniformly distributed in $[0,1]$ and independent of X, then $(X, \Gamma(X,U))$ has joint distribution γ.*

Proof. Assume that $\mathbb{P}(X = a) > 0$, for all $a \in A$. Let $B = \{b_1,\ldots,b_n\}$. For each $a \in A$, let

$$q_a(j) := \mathbb{P}(Y \in \{b_1,\ldots,b_j\} \,|X = a) = \frac{\mathbb{P}(Y \in \{b_1,\ldots,b_j\}, X = a)}{\mathbb{P}(X = a)}$$

for all $1 \leq j \leq n$. Set $q_a(0) = 0$ and note that $q_a(n) = 1$, so that

$$\mathbb{P}(q_a(j-1) \leq U < q_a(j)) = \frac{\mathbb{P}(Y = b_j, X = a)}{\mathbb{P}(X = a)}.$$

For each $1 \leq j \leq n$, let

$$\Gamma(a, u) := b_j \text{ if } q_a(j-1) \leq u < q_a(j).$$

□

We call a $\{0,1\}$-valued random variable a ***Bernoulli random variable***. The following lemma allows us to code sequences of independent coin-flips into sequences of uniformly distributed random variables.

Lemma 7. *There exists a measurable function* $c : \{0,1\}^{\mathbb{N}} \to [0,1]^{\mathbb{N}}$ *such that if* $B = (B_i)_{i\in\mathbb{N}}$ *is a sequence of i.i.d. Bernoulli random variables with mean* $\frac{1}{2}$*, then* $(c(B)_i)_{i\in\mathbb{N}}$ *is a sequence of i.i.d. random variables that are uniformly distributed in* $[0,1]$.

Proof. The result follows from the Borel isomorphism theorem. See [17, Theorem 3.4.23] for more details. □

Proof of Theorem 5. Let γ be the monotone coupling of κ^n and ι^n given by Lemma 4, so that γ is a measure on $\{0,1\}^n \times \{0,1\}^n \equiv (\{0,1\} \times \{0,1\})^n$. Thus the product measure γ^2 is a monotone coupling of κ^{2n} and ι^{2n} and $\gamma^{\mathbb{N}}$ gives a monotone coupling of $\kappa^{\mathbb{N}}$ and $\iota^{\mathbb{N}}$. We will modify the coupling $\gamma^{\mathbb{N}}$ to become the required map ϕ. In order to do this, it will be easier to think in terms of random variables rather than measures.

Let $X = (X_i)_{i\in\mathbb{N}}$ be an i.i.d. sequence of Bernoulli random variables with mean $\kappa(1)$. For each $j \geq 0$, let

$$X^j := (X_{jn}, \ldots, X_{(j+1)n-1}),$$

so that the random variables are partitioned into blocks of size n. Let $U = (U_i)_{i\in\mathbb{N}}$ be an i.i.d. sequence of random variables that are uniformly distributed in $[0,1]$. Also assume that U is independent of X, and let $Y = (Y_i)_{i\in\mathbb{N}}$ be an i.i.d. sequence of Bernoulli random variables with mean $\iota(1)$.

By Lemmas 4 and 6, let $\Gamma : \{0,1\}^n \times [0,1] \to \{0,1\}^n$ be a measurable map such that $(X^1, \Gamma(X^1, U_1))$ has joint law γ and $\Gamma(w, v) = 0^n$ for all $v \in [0,1]$ if $w \in \{100^{n-2}, 010^{n-2}\}$. We have that

$$\big(X, (\Gamma(X^i, U_i))_{i\in\mathbb{N}}\big)$$

gives a monotone coupling of X and Y with law $\gamma^{\mathbb{N}}$.

For each $j \in \mathbb{N}$, call X^j ***special*** if $X^j \in \{100^{n-2}, 010^{n-2}\}$ and let $S \subset \mathbb{N}$ be the random set of $j \in \mathbb{N}$ for which X^j are special. Note that almost surely, S is an infinite set. Let $\bar{X} = (\bar{X}_i)_{i\in\mathbb{N}}$ be the sequence of binary digits such that $\bar{X}^j = X^j$ if $j \notin S$ and $\bar{X}^j = 0^n$ if $j \in S$. We have that

$$(\Gamma(X^i, U_i))_{i\in\mathbb{N}} = (\Gamma(\bar{X}^i, U_i))_{i\in\mathbb{N}}.$$

Let $(s_i)_{i\in\mathbb{N}}$ be the enumeration of S, where $s_0 < s_1 < s_2 < s_3 \cdots$. Consider the sequence of random variables given by

$$b(X) := (\mathbf{1}[X^{s_i} = 100^{n-2}])_{i\in\mathbb{N}} = (X_{s_i n})_{i\in\mathbb{N}}$$

Since 100^{n-2} and 010^{n-2} occur with equal probability, we have that $b(X)$ is an i.i.d. sequence of Bernoulli random variables with mean $\frac{1}{2}$. Furthermore, we have that $b(X)$ is independent of $\bar{X}$, since $b(X)$ only depends on the values of X on the special blocks. Let c be the function from Lemma 7, so that $c(b(X)) \stackrel{d}{=} U$. Since $b(X)$ is independent of $\bar{X}$,

$$\begin{aligned}[]
[\Gamma(X^i, U_i)]_{i\in\mathbb{N}} &= [\Gamma(\bar{X}^i, U_i)]_{i\in\mathbb{N}} \\
&\stackrel{d}{=} \left[\Gamma(\bar{X}^i, c(b(X))_i)\right]_{i\in\mathbb{N}} \\
&= \left[\Gamma(X^i, c(b(X))_i)\right]_{i\in\mathbb{N}}.
\end{aligned}$$

Thus $\Big(X, \left[\Gamma(X^i, c(b(X))_i)\right]_{i\in\mathbb{N}}\Big)$ is another monotone coupling of X and Y. Hence, we define

$$\phi(x) := \left[\Gamma(x^i, c(b(x))_i)\right]_{i\in\mathbb{N}}$$

for all $x \in \{0,1\}^{\mathbb{N}}$ when the set S is infinite, and set $\phi(x) = 0^{\mathbb{N}}$ when S is finite–an event that occurs with probability zero. ☐

2.2. **Joinings.** Let $T : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ be the left-shift given by $(Tx)_i = x_{i+1}$ for all $x \in \{0,1\}^{\mathbb{Z}}$ and all $i \in \mathbb{Z}$. Let κ and ι be probability measures on $\{0,1\}$. A ***joining*** of $\kappa^{\mathbb{Z}}$ and $\iota^{\mathbb{Z}}$ is a coupling ϱ of the two measures with the additional property that $\varrho \circ (T \times T) = \varrho$. We will make use of the following joining in the proof of Theorem 1.

Example 8. Let κ and ι be probability measures on $\{0,1\}$. Assume that the intensity of κ is greater than the intensity of ι. Let $x \in \{0,1\}^{\mathbb{Z}}$, and let n be sufficiently large as in Lemma 4. Call the subset $[j, j + 2n+1] \subset \mathbb{Z}$ a ***marker*** if $x_i = 0$ for all $i \in [j, j+2n]$ and $x_{j+2n+1} = 1$. Notice that two distinct markers have an empty intersection. Call an interval a ***filler*** if it is nonempty and lies between two markers. Thus each $x \in \{0,1\}^{\mathbb{Z}}$ partitions $\mathbb{Z}$ into intervals of markers and fillers. Call a filler ***fitted*** if it is of size n, and call a filler ***special*** if it is both fitted and of the form 100^{n-2} or 010^{n-2}.

Let X have law $\kappa^{\mathbb{Z}}$ and Y have law $\iota^{\mathbb{Z}}$. In what follows we describe explicitly how to obtain a monotone joining of X and Y, where the independent thinning is used everywhere, except at the fitted fillers, where the coupling from Lemma 4 is used. Let $U = (U_i)_{i\in\mathbb{Z}}$ be an i.i.d. sequence of random variables that are uniformly distributed in $[0, 1]$ and independent of X. By Example 3 and Lemma 6, let $R : \{0, 1\} \times [0, 1] \to \{0, 1\}$ be a measurable function such that $R(X_1, U_1) \le X_1$ is a Bernoulli random variable with mean $\iota(1)$. Let Γ and γ be as in the proof of Theorem 5, so that

$$\big((X_1, \ldots, X_n), \Gamma(X_1, \ldots, X_n, U_1)\big)$$

has law γ. Consider the function $\Phi : \{0, 1\}^{\mathbb{Z}} \times [0, 1]^{\mathbb{Z}} \to \{0, 1\}^{\mathbb{Z}}$ defined by $\Phi(x, u)_i = R(x_i, u_i)$ if i is not in a fitted filler. For $(j, j+1, \ldots, j+n)$ in a fitted filler, we set

$$(\Phi(x, u)_j, \ldots, \Phi(x, u)_{j+n}) = \Gamma(x_j, \ldots, x_{j+n}, u_j).$$

The law of X restricted to a filler interval is the law of a finite sequence of i.i.d. Bernoulli random variables with mean $\kappa(1)$, conditioned not to contain a marker. Note that since a fitted interval is of size n, and a marker is of size $2n + 1$, the law of X restricted to a fitted interval is just the law of a finite sequence of i.i.d. Bernoulli random variables with mean $\kappa(1)$. Furthermore, conditioned on the locations of the markers, the restrictions of X to each filler interval are independent (see for example Keane and Smorodinsky [9, Lemma 4] for a detailed proof). Hence, $\Phi(X, U) \stackrel{d}{=} Y$. In addition, since all the couplings involved are monotone, we easily have that $\Phi(X, U)_i \le X_i$ for all $i \in \mathbb{Z}$. $\Diamond$

Remark 9. To emphasize the strong form of independence in Example 8, we note that if $A = (A_i)_{i\in\mathbb{Z}}$ are independent Bernoulli random variables with mean $\frac{1}{2}$ that are independent of X, then $(A_{jn})_{j\in S}$ has the same law as $(X_{jn})_{j\in S}$. Recall if $j \in S$ then $X^j = (X_{jn}, \ldots, X_{(j+1)n-1})$ is special. In addition, if X' is such that $X'_i = X_i$ for every i not in a special filler of X and on each special filler of X we set $X'_{jn} = A_{jn}$, $X'_{jn+1} = 1 - A_{jn}$, and

$$X'_{jn+2} = X'_{jn+3} = \cdots = X'_{(j+1)n-1} = 0,$$

then $X' \stackrel{d}{=} X$. Thus we can independently resample on the special fillers without affecting the distribution of X. $\Diamond$

2.3. **The example of Ornstein and Weiss.** Let $\mathbb{F}_r$ be the free group of rank $r \ge 2$. Let a and b be two of its generators. The Ornstein and Weiss [13] entropy increasing factor map is given by

$$\phi(x)(g) = (x(g) \oplus x(ga), x(g) \oplus x(gb))$$

for all $x \in \{0, 1\}^{\mathbb{F}_r}$ and all $g \in \mathbb{F}_2$, where

$$\phi : \{0, 1\}^{\mathbb{F}_r} \to (\{0, 1\} \times \{0, 1\})^{\mathbb{F}_r} \equiv \{00, 01, 10, 11\}^{\mathbb{F}_r}$$

pushes the uniform product measure $(\frac{1}{2}, \frac{1}{2})^{\mathbb{F}_r}$ forward to the uniform product measure $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})^{\mathbb{F}_r}$; the required independence follows from the observation that if $m \oplus n := m + n \bmod 2$, if X, X', and Y are independent Bernoulli random variables with mean $\frac{1}{2}$, and if $Z := X \oplus Y$ and $Z' := X' \oplus Y$, then Z and Z' are independent, even though they both depend on Y.

Ornstein and Weiss's example can be iterated to produce an infinite number of bits at each vertex in the following way. As in Ball [2, Proposition 2.1], we will define $\phi_k : \{0,1\}^{\mathbb{F}_r} \to (\{0,1\}^k)^{\mathbb{F}_r}$ inductively for $k \geq 2$. Let $\tilde{\phi_k} : \{0,1\}^{\mathbb{F}_r} \to \{0,1\}^{\mathbb{F}_r}$ be the last coordinate of ϕ_k so that $\tilde{\phi_k}(x)(g) = [\phi_k(x)(g)]_k$ for all $x \in \{0,1\}^{\mathbb{F}_r}$ and all $g \in \mathbb{F}_2$. Set $\phi_2 = \phi$. For $k \geq 3$, let ϕ_k be given by

$$\phi_k(x)(g) = \Big([\phi_{k-1}(x)(g)]_1, \ldots, [\phi_{k-1}(x)(g)]_{k-2}, (\phi \circ \tilde{\phi}_{k-1})(x)(g)\Big)$$

for all $x \in \{0,1\}^{\mathbb{F}_r}$ and all $g \in \mathbb{F}_2$. At each step we are saving one bit to generate two new bits using the original map ϕ. The map ϕ_k pushes the uniform product measure $(\frac{1}{2}, \frac{1}{2})^{\mathbb{F}_r}$ forward to the uniform product measure on $(\{0,1\}^k)^{\mathbb{F}_r}$. By taking the limit, we obtain the mapping

$$\phi_\infty : \{0,1\}^{\mathbb{F}_r} \to (\{0,1\}^{\mathbb{Z}^+})^{\mathbb{F}_r}$$

which yields a sequence of i.i.d. fair bits at each coordinate $g \in \mathbb{F}_2$, independently. Note that $\phi_\infty(x)(g)_k = \phi_n(x)(g)_k$ for all $n > k$. In our proof of Theorem 1 we will use this iteration, which Ball attributes to Timár.

3. Proof of the main theorem

Proof of Theorem 1. Let $r \geq 2$. We begin by extending the same monotone joining defined in Example 8 to a monotone joining of $\kappa^{\mathbb{F}_r}$ and $\iota^{\mathbb{F}_r}$. Let X have law $\kappa^{\mathbb{F}_r}$ and Y have law $\iota^{\mathbb{F}_r}$; then $X = (X_g)_{g \in \mathbb{F}_r} = (X(g))_{g \in \mathbb{F}_r}$ are i.i.d. Bernoulli random variables with mean $\kappa(1)$. As in the Ornstein and Weiss example, it will be sufficient to use only two generators a and b in the expression of our equivariant thinning. We refer to the string of generators and their inverses that make up the representation of an element in $\mathbb{F}_r$ as a ***word***, and the individual generators and inverses as ***letters***. We call a word ***reduced*** if its string of letters has no possible cancellations.

Consider $\mathbb{F}_r$ as being partitioned into infinitely many $\mathbb{Z}$ copies $Z(w)$ in the following way. Let $\mathbb{F}'_r$ be the set of reduced words in $\mathbb{F}_r$ that do not end in either b or b^{-1}. For each $w \in \mathbb{F}'_r$, set $Z(w) := \{wb^i\}_{i \in \mathbb{Z}}$. Indeed, any element in $\mathbb{F}_r$ may be written as wb^i for unique reduced $w \in \mathbb{F}'_r$ and $i \in \mathbb{Z}$.

Let n be sufficiently large for the purposes of Lemma 4. We define markers, fillers, fitted fillers, and special fillers on each of the $\mathbb{Z}$ copies in the obvious way. For example, if $x \in \{0,1\}^{\mathbb{F}_r}$ and $w \in \mathbb{F}_r'$, then the set $\{wb^j, \ldots, wb^{j+2n+1}\}$ is a marker if $x(wb^i) = 0$ for all $i \in [j, 2n]$ and $x(wb^{2n+1}) = 1$.

Let $U' = (U'_g)_{g \in \mathbb{F}_r}$ be i.i.d. uniform random variables independent of X. Let Φ be as in Example 8. Define $\hat{\Phi} : \{0,1\}^{\mathbb{F}_r} \times [0,1]^{\mathbb{F}_r} \to \{0,1\}^{\mathbb{F}_r}$ by

$$\hat{\Phi}(x, u')_{wb^i} = \Phi\big(x(Z(w)), u'(Z(w))\big)_i$$

for all $w \in \mathbb{F}_r'$ and all $i \in \mathbb{Z}$, where $x(Z(w)) := (x(wb^j))_{j \in \mathbb{Z}}$ and $u'(Z(w)) := (u'(wb^j))_{j \in \mathbb{Z}}$. Thus we have the monotone joining Φ on each $\mathbb{Z}$ copy $Z(w)$ in $\mathbb{F}_r$, so that

$$\hat{\Phi}(X, U') \stackrel{d}{=} Y \tag{1}$$

and $\hat{\Phi}(X, U')_g \leq X_g$ for all $g \in \mathbb{F}_r$. Additionally, since Φ is a joining, the joint law of $(X, \hat{\Phi}(X, U'))$ is invariant under $\mathbb{F}_r$-actions.

Recall that a special filler has length exactly n, and the filler has two choices of values 010^{n-2} or 100^{n-2}, which occur with equal probability. We define an ***initial vertex*** of a special filler in $Z(w)$ to be an element $wb^{n_0} \in Z(w)$ where the entire special filler takes values sequentially at vertices on the minimal path from wb^{n_0} to wb^{n_0+n}. For each $x \in \{0,1\}^{\mathbb{F}_r}$, let $V = V(x)$ be the set of initial vertices in $\mathbb{F}_r$. Note that as in Example 8, the law of X restricted to a fitted interval is just the law of a finite sequence of i.i.d. Bernoulli random variables with mean $\kappa(1)$. Furthermore, conditioned on the locations of the markers, the restrictions of X to each filler interval are independent. Thus for all $v \in V(X)$, $X(v)$ is a Bernoulli random variable with mean $\frac{1}{2}$, and conditioned on $V(X)$, the random variables $(X(v))_{v \in V}$ are independent.

We have the same strong form of independence here as emphasized in Remark 9 for Example 8, again by Keane and Smorodinsky [9, Lemma 4]. This is key in our construction: we will use the Bernoulli random variables $(X(v))_{v \in V}$ to build deterministic substitutes for U'.

Now we adapt the iteration of the Ornstein and Weiss example to assign a sequence of i.i.d. Bernoulli random variables to each $v \in V$. For each $v \in V$, let k be the smallest positive integer such that $va^k \in V$; set $\alpha(v) = va^k$. Similarly, let k' be the smallest positive integer such that $vb^{k'} \in V$ and set $\beta(v) = vb^{k'}$. For each $v \in V$, define

$$\psi(x)(v) = \big(x(v) \oplus x(\alpha(v)), x(v) \oplus x(\beta(v))\big).$$

Conditioned on V, we have that $(\psi(X))_{v \in V}$ is a family of independent random variables uniformly distributed on $\{00, 01, 10, 11\}$. We iterate the map ψ as we did with the Ornstein and Weiss map ϕ. Set $\psi_2 = \psi$. For $k \geq 3$, let

$$\psi_k(x)(v) = \Big([\psi_{k-1}(x)(v)]_1, \ldots, [\psi_{k-1}(x)(v)]_{k-2}, (\psi \circ \tilde{\psi}_{k-1})(x)(v)\Big),$$

where $\tilde{\psi}_{k-1}(x)(v) = [\psi_{k-1}(x)(v)]_{k-1}$ is the last coordinate of ψ_k. Let ψ_∞ be the limit, and let $B_v = \psi_\infty(X)(v)$, so that conditioned on V, the random variables $(B_v)_{v\in V}$ are independent, and each B_v is an i.i.d. sequence of Bernoulli random variables with mean $\frac{1}{2}$.

For all $x \in \{0,1\}^{\mathbb{F}_r}$, let $\bar{x}(g) = x(g)$ for all g not in a special filler, and let $\bar{x}(g) = 0$ if g belongs to a special filler. It follows from Remark 9 that if $B' = (B'_g)_{g\in\mathbb{F}_r}$ are independent Bernoulli random variables with mean $\frac{1}{2}$ independent of X, then $(B'_v)_{v\in V(X)}$ has the same law as $(B_v)_{v\in V(X)}$. Moreover,

$$\big(\bar{X}, (B_v)_{v\in V(X)}\big) \overset{d}{=} \big(\bar{X}, (B'_v)_{v\in V(X)}\big). \tag{2}$$

We assign, in an equivariant way, one uniform random variable to each element in $\mathbb{F}_r$ using the randomness provided by $(B_v)_{v\in V}$. Let $c : \{0,1\}^{\mathbb{N}} \to [0,1]^{\mathbb{N}}$ be the function from Lemma 7, and let $g \in \mathbb{F}_r$. Then almost surely there exist $v \in V$ and a minimal $j > 0$ such that $gb^j = v$; set $U_g = c(B_v)_j$. Define $\mathbf{u} : \{0,1\}^{\mathbb{F}_r} \to [0,1]^{\mathbb{F}_r}$ by setting $\mathbf{u}(X) := (U_g)_{g\in\mathbb{F}_r}$. Recall that $U' = (U'_g)_{g\in\mathbb{F}_r}$ are independent random variables uniformly distributed in $[0,1]$ independent of X. From (2),

$$\big(\bar{X}, \mathbf{u}(X)\big) \overset{d}{=} \big(\bar{X}, U'\big). \tag{3}$$

Let $R : \{0,1\}\times[0,1] \to \{0,1\}$ and $\Gamma : \{0,1\}^n\times[0,1] \to \{0,1\}^n$ be the functions that appear in the definition of Φ in Example 8. Recall that R facilitated independent thinning and Γ the key monotone coupling of Lemma 4. Also recall $\Gamma(100^{n-2}, t) = 0 = \Gamma(010^{n-2}, t)$ for all $t \in [0,1]$.

Now define $\phi : \{0,1\}^{\mathbb{F}_r} \to \{0,1\}^{\mathbb{F}_r}$ by

$$\phi(x)(g) = R\big(x(g), \mathbf{u}(x)(g)\big)$$

for g not in a fitted filler; if $\{wb^i, \ldots, wb^{i+n-1}\}$ is a fitted filler, then set

$$(\phi(x)(wb^i), \ldots, \phi(x)(wb^{i+n-1})) = \Gamma\big(x(wb^i), \ldots, x(wb^{i+n-1}), \mathbf{u}(x)(wb^i)\big).$$

Note ϕ is defined so that $\phi(x) = \hat{\Phi}(x, \mathbf{u}(x))$. The map ϕ is equivariant and satisfies $\phi(x)(g) \leq x(g)$ by construction. It remains to verify that $\phi(X) \overset{d}{=} Y$.

By the definition of Γ, we have $\phi(X) = \phi(\bar{X})$; that is, all special fillers are sent to 0^n. A similar remark applies to the map $\hat{\Phi}$. From (1) and (3),

$$\phi(X) = \hat{\Phi}(X, \mathbf{u}(X)) = \hat{\Phi}(\bar{X}, \mathbf{u}(X)) \stackrel{d}{=} \hat{\Phi}(\bar{X}, U') = \hat{\Phi}(X, U') \stackrel{d}{=} Y. \quad \square$$

4. Generalizations and questions

4.1. **Stochastic domination.** Let $[N] = \{0, 1, \ldots, N-1\}$ be endowed with the usual total ordering. Let κ and ι be probability measures on $[N]$. We say that κ ***stochastically dominates*** ι if $\sum_{i=0}^{j} \kappa_i \leq \sum_{i=0}^{j} \iota_i$ for all $j \in [N]$. An elementary version of Strassen's theorem [18, Theorem 11] gives that κ stochastically dominates ι if and only if there exists a monotone coupling of κ and ι. Notice that in the case $N = 2$, we have that κ stochastically dominates ι if and only if ι is not of higher intensity than κ. Thus Theorem 1 gives a positive answer to a special case of the following question.

Question 1. *Let κ and ι be probability measures on $[N]$, where κ stochastically dominates ι, and κ gives positive measure to at least two elements of $[N]$. Let G be the free group of rank at least two. Does there exist a measurable equivariant map $\phi : [N]^G \to [N]^G$ such that the push-forward of κ^G is ι^G and $\phi(x)(g) \leq x(g)$ for all $x \in [N]^G$ and $g \in G$?*

In Question 1, we call the map ϕ a ***monotone factor from κ to ι***. A necessary condition for the existence of a monotone factor from κ to ι is that κ stochastically dominates ι. In the case $G = \mathbb{Z}$, Ball [3] proved that there exists a monotone factor from κ to ι provided that κ stochastically dominates ι, $H(\kappa) > H(\iota)$, and ι is supported on two symbols; Quas and Soo [14] removed the two symbol condition on ι.

In the non-amenable case, where G is a free group of rank at least two, one can hope that Question 1 can be answered positively, without any entropy restriction. However, the analogue of Lemma 4 that was key to the proof of Theorem 1 does not apply in the simple case where $\kappa = (0, \frac{1}{2}, \frac{1}{2})$ and $\iota = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. In particular, for all $n \geq 1$, there is no coupling ρ of κ^n and ι^n for which there exists $x \in \{1, 2\}^n$ and $y \in \{0, 1, 2\}^n$ such that $\rho(x, y) = \kappa^n(x) = (\frac{1}{2})^n$, since $\rho(x, y) \leq \iota^n(y) = (\frac{1}{3})^n$.

4.2. **Automorphism-equivariant factors.** The Cayley graph of $\mathbb{F}_n$ is the regular tree $\mathbb{T}_{2n}$ of degree $2n$. We note that $\mathbb{F}_n$ is a strict subset of the group of graph automorphisms of $\mathbb{T}_{2n}$. The map that we constructed in Theorem 1 is not equivariant with respect to the full automorphism group of $\mathbb{T}_{2n}$. In particular, our definition of a marker is not equivariant with respect to the automorphism which exchanges

a-edges and b-edges in $\mathbb{T}_{2n}$. However, Ball generalizes the Ornstein and Weiss example to the full automorphism group in [2, Theorem 3.3] by proving that for any $d \geq 3$, there exists a measurable mapping $\phi : \{0,1\}^{\mathbb{T}_d} \to [0,1]^{\mathbb{T}_d}$ which pushes the uniform product measure on two symbols forward to the product measure of Lebesgue measure on the unit interval, equivariant with respect to the group of automorphisms of $\mathbb{T}_d$. Moreover, she proved the analogous result for any tree with bounded degree, no leaves, and at least three ends.

Question 2. *Let T be a tree with bounded degree, no leaves, and at least three ends. Let κ and ι be probability measures on $\{0,1\}$ and ι be of lower intensity. Does there exists a thinning from κ to ι that is equivariant with respect to the full automorphism group of T?*

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