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The Running Coupling BFKL Anomalous Dimensions and Splitting Functions

Robert S. Thorne[1]

Cavendish Laboratory, University of Cambridge
Madingley Road, Cambridge, CB3 0HE, U.K.

Abstract

I explicitly calculate the anomalous dimensions and splitting functions governing the Q^2 evolution of the parton densities and structure functions which result from the running coupling BFKL equation at LO, i.e. I perform a resummation in powers of $\ln(1/x)$ and in powers of β_0 simultaneously. This is extended as far as possible to NLO. These are expressed in an exact, perturbatively calculable analytic form, up to small power-suppressed contributions which may also be modelled to very good accuracy by analytic expressions. Infrared renormalons, while in principle present in a solution in terms of powers in $\alpha_s(Q^2)$, are ultimately avoided. The few higher twist contributions which are directly calculable are extremely small. The splitting functions are very different from those obtained from the fixed coupling equation, with weaker power-like growth $\sim x^{-0.25}$, which does not set in until extremely small x indeed. The NLO BFKL corrections to the splitting functions are moderate, both for the form of the asymptotic power-like behaviour and more importantly for the range of x relevant for collider physics. Hence, a stable perturbative expansion and predictive power at small x are obtained.

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[1] Royal Society University Research Fellow

1. Introduction.

Small x physics has been a particularly active area of particle physics research in the past few years, driven largely by the first data for $x < 0.005$ being obtained by the HERA experiments [1] [2]. However, as well as the need to describe this HERA data correctly, it will also be extremely important to understand the correct way of calculating physics at small x in order to interpret the results coming from the LHC in a truly quantitative manner. For example, for the production of a particle of mass $\sim$ 100GeV the typical value of x probed (at central rapidity) is 0.005, but values up to two orders of magnitude in either direction will also have an almost equally large influence.[2]

The potential complication at small x is that the splitting functions and coefficient functions governing the evolution of parton distributions and their conversion to physical quantities have terms in their perturbative expansions which behave like $\alpha_s^n \ln^m(1/x)$, where m can reach up to $n-1$. Therefore, as the power of the coupling increases the powers of $\xi = \ln(1/x)$ also increase, and rapid perturbative convergence is not really guaranteed if $\xi \gtrsim 1/\alpha_s$ i.e. ~ 5. This problem is not really diminished at the LHC, where the coupling is likely to be smaller than at HERA, since the parton distributions to be used will be those measured at HERA at much lower scales and evolved up to LHC scales. This question of large $\ln(1/x)$ terms is in principle addressed by the BFKL equation [4], which is an integral equation for the unintegrated 4-point gluon Green's function in the high energy limit. This sums the leading high-energy, or in the DIS case, small-x behaviour, which is dominated by the gluon, and thus allows the extraction of leading $\ln(1/x)$ terms for relevant quantities, such as splitting functions.

Hence, a major point of debate of the past decade has been whether the standard DGLAP approach based on renormalization group equations and conventionally ordered simply in powers of $\alpha_s(Q^2)$, or the BFKL equation, which sums leading logarithms in $(1/x)$, is most effective way of dealing with small x physics (most particularly structure functions), and/or whether the two approaches need to be combined in some way, and if so, how? While the conventional DGLAP approach has been relatively successful, it does have some significant problems (which are often overlooked): a valence-like, or even negative input gluon leading to a strange low-Q^2 $F_L(x, Q^2)$; undershooting of the data systematically for $x \sim 0.01$ at the highest Q^2 when a global fit is performed; and apparent instability at small x order-by-order in α_s up to NLLO [5].[3]

Nevertheless, the BFKL equation did not seem to help these problems. The original LO BFKL prediction of a behaviour of the form $x^{-\lambda}$ for structure functions and splitting functions at small x, with $\lambda \sim 0.5$, was clearly ruled out long ago. A combination of the two approaches, using the BFKL equation to supplement the Altarelli-Parisi splitting functions with higher terms

[2] For an illustration of the x and Q^2 of parton distributions sampled at the LHC see fig. 1 of [3].

[3] Of course the full NNLO splitting functions are not known, but good estimates are available [6] based on calculation of moments in [7].

of the form $\alpha_s^{n+1}\ln^n(1/x)$ was originally successful (so long as one avoided factorization scheme ambiguities by working in physical quantities) [8], but this success is not possible to sustain with the most recent data [9][10]. Moreover, the subject was thrown into confusion by the calculation of the NLO correction to the BFKL equation [11][12]. The results of this calculation were not very encouraging. Ignoring the running of the coupling at NLO, i.e. proceeding with the same sort of calculations as at LO but including the scale-independent NLO correction to the kernel, one obtains the "intercept" for the splitting function power-like behaviour, $x^{-\lambda}$, shifted from $\lambda = 4\ln 2\bar{\alpha}_s$ to $\lambda = 4\ln 2\bar{\alpha}_s(1 - 6.5\bar{\alpha}_s)$. This is clearly a huge correction, and implies the breakdown of the perturbative expansion for this quantity. More serious than this intercept is the power series for the splitting function, which may be calculated even taking into account the renormalization and scale dependence introduced at NLO. Expanding this out formally to NLO in $\ln(1/x)$ one finds that it is is dominated by the NLO corrections at all values of x below about $x = 0.01$. For example, using the formulae in [11] the first few terms in the power series for $P(x)$ go like

$$\begin{aligned} xP(x,Q^2) =& \bar{\alpha}_s + 2.4\bar{\alpha}_s^4\xi^3/6 + 2.1\bar{\alpha}_s^6\xi^5/120 + \cdots \\ & - \bar{\alpha}_s(0.43\bar{\alpha}_s + 1.6\bar{\alpha}_s^2\xi + 11.7\bar{\alpha}_s^3\xi^2/2 + 13.3\bar{\alpha}_s^4\xi^3/6 + 39.7\bar{\alpha}_s^5\xi^4/24 + 169.4\bar{\alpha}_s^6\xi^5/120 + \cdots), \end{aligned} \tag{1.1}$$

where $\xi = \ln(1/x)$ and $\alpha_s \equiv \alpha_s(Q^2)$. Clearly, the size of the coefficients more than compensates for the extra power of $\alpha_s(Q^2)$, particularly at low Q^2 where the perturbative analysis of structure function evolution often takes place.

Hence, this NLO correction left open the whole question of how to address the evolution of structure functions at small x. There has been considerable progress on the stability of the solutions to the BFKL equation in the intervening time. One major development was the observation that the resummation of double logarithmic terms in the transverse momentum k^2 is necessary in order to eliminate collinear divergences. This renders the intercept of the BFKL equation stable [13], even when ignoring the renormalization scale dependence. This initial idea has been further developed in [14][15][16], where the effect of running coupling is also considered in these later papers. This development is particularly important for the case of so-called "single scale" processes where both ends of the gluon Green's function are at high scales (not necessarily the same) where without this collinear resummation, all calculations are badly behaved over the full range of energy, not just in the asymptotic limit.

However, for the type of situation embodied by DIS, where one end of the gluon Green's function is at some low non-perturbative scale, the factorization theorem simplifies the problem. Although the growth of the coupling at low scales actually renders the solution of the BFKL equation formally divergent when the renormalization of the coupling is encountered, as realized as long ago as [17] and studied in detail in [19], all the uncertainty and indeed all the effects of the low Q^2 region are absorbed into the overall normalization of the gluon, leaving the evolution and

coefficient functions for hard scattering cross-sections calculable. However, these perturbatively calculable quantities are affected by the running of the coupling, and it was argued in [20] that the effective result was as if the usual LO BFKL splitting functions should be evaluated at an x-dependent scale, which grows with decreasing x, due to increasing diffusion into the ultraviolet, leading to a decrease in the coupling. Hence, the effect of running coupling totally transforms the more simplistic LO BFKL results, making overall normalization of quantities incalculable, but moderating the effect of those governing the evolution in Q^2. This moderation of the LO quantities also translated into a moderation of the effects of NLO corrections, leading to a much improved stability of the perturbative expansion, even without recourse to the type of resummation in [13]-[15]. Indeed, for this case of deep inelastic scattering further resummation of this type is redundant. These modified BFKL contributions to the splitting functions, when combined with the conventional LO-in-α_s contributions, also led to improved fits compared to the usual DGLAP approach [20] and a more sensible prediction for $F_L(x,Q^2)$. This concept was put on a firmer footing in [21] where an explicit calculation of the BFKL splitting functions in powers of $\beta_0\alpha_s(Q^2)$, i.e. a resummation of running coupling contributions, was outlined, and it was seen that over a wide range of the $x-Q^2$ range (including the HERA range) the previous hypothesis was largely correct, and precise results were also obtained outside this range.

The purpose of this paper is to explain in detail, and expand upon the results of this previous letter, i.e. to present in full the calculation of splitting functions and coefficient functions for deep inelastic scattering obtained from the BFKL equation (both LO and NLO) and incorporating running coupling contributions to all orders. Explicitly, while the usual BFKL equation presents an expression for these quantities which sums the leading power of ξ at each power in α_s, I will extend this by producing expressions which also include the leading power of β_0 at each power of $\alpha_s(Q^2)$ and ξ, e.g.

$$xP_{gg}(x,Q^2)=\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}a_{nm}\alpha_s^n(Q^2)\xi^{n-1-m}\beta_0^m, \tag{1.2}$$

though the formal divergence of the series will complicate this form a little. This presentation will begin, in section 2, a brief review of the standard solution to the BFKL equation at LO, and then a detailed presentation of the solution at LO with running coupling. This will result in a solution for the gluon splitting function in an analytic form up to a small, unambiguous, correction of the form Λ^2/Q^2 (which is *not* higher twist) which may be modelled by an analytic function to excellent accuracy. Despite the integration over the infrared region when solving the running coupling BFKL equation there is no ambiguity in this splitting function. Next, in section 3, will follow a discussion of some possible higher twist contributions at small x. It is argued that these are may be much smaller than generally supposed, though the possibility of some large power-suppressed corrections (not necessarily higher twist) is left open. In section 4 I discuss the solution of the BFKL equation at NLO, defining precisely what I mean by the "NLO BFKL splitting function", and showing that the

NLO corrections for the gluon splitting function are moderate. In section 5 I consider real physical quantities, i.e the structure functions. Firstly, I calculate the quark-gluon splitting function and coefficient functions, and then consider the rather more direct physical splitting functions [22]. I also consider how far one can calculate to NLO, defining a "nearly NLO" physical splitting function $P_{LL}(x,Q^2)$. The stability of the perturbative expansion is examined in detail, and seen to be very good. Finally, in section 6 phenomenology is briefly touched upon, and I present a summary and my conclusions.

2. BFKL Equation at LO.

The BFKL equation for zero momentum transfer is an integral equation for the 4-point, transverse momentum-dependent gluon Green's function for forward scattering in the high energy limit, $f(k_1,k_2,\alpha_s,N)$, where N is the Mellin conjugate variable to energy. In the case of DIS the second momentum k_2 is put equal to some non-perturbative scale Q_0, we let $k_1 = k$, and N becomes conjugate to x. In order to obtain a structure function we attach the non-perturbative bare gluon distribution $g_B(N,Q_0^2)$ to the non-perturbative end of the gluon Green's function and convolute a hard scattering cross-section $h(Q^2/k^2,\alpha_s,N)$ to the perturbative end.

In this section I will illustrate the effect that introducing the running coupling into the BFKL equation has. In order to do this I will first begin with a brief presentation of the fairly simple traditional case of fixed coupling before moving to the far more complicated case of running coupling. As will be seen, the introduction of renormalization, and hence running of the coupling, which is necessary except in the artificial model of no consideration beyond LO, completely changes not only the detail of the information one is able to extract from the BFKL equation, but also what type of information one is able to extract.

2.1. Fixed Coupling.

We simplify matters by working in moment space, i.e. defining the moment of a structure function by

$$\mathcal{F}(N,Q^2) = \int_0^1 x^{N-1} F(x,Q^2) dx, \tag{2.1}$$

and similarly for the parton distributions (scaled by x). Doing this the BFKL equation is

$$f(k^2,\bar{\alpha}_s/N) = f_I(k^2,Q_0^2) + \frac{\bar{\alpha}_s}{N}\int_0^\infty \frac{dq^2}{q^2} K_0(q^2,k^2) f(q^2), \tag{2.2}$$

where $f(k^2,\bar{\alpha}_s/N)$ is the unintegrated gluon four-point function, $f_I(k^2,Q_0^2)$ is the zeroth order input, $\bar{\alpha}_s = (3/\pi)\alpha_s$, and the LO kernel is defined by

$$K_0(q^2,k^2) f(q^2) = k^2 \left(\frac{f(q^2) - f(k^2)}{\mid k^2 - q^2 \mid} + \frac{f(k^2)}{(4q^4 + k^4)^{\frac{1}{2}}} \right). \tag{2.3}$$

It is convenient to define the input by $f_I(k^2, Q_0^2) = \delta(k^2 - Q_0^2)$. In fact in the leading twist factorization theorem this is the unique definition, and Q_0^2 is really just a regularization which we let $\to 0$ ultimately. Going beyond this approximation the dependence on Q_0^2 tells us about the higher twist due to the intrinsic transverse momentum of the gluon, and we will discuss this in section 3. The "gluon structure function" is now given by

$$\mathcal{G}(Q^2, N) = \int_0^{Q^2} \frac{dk^2}{k^2} f(N, k^2, Q_0^2) \times g_B(N, Q_0^2), \tag{2.4}$$

where $g_B(N, Q_0^2)$ is the bare gluon density in the proton which implicitly absorbs the collinear divergences in $f(k^2)$. The BFKL equation is most easily solved by taking the Mellin transformation to γ-space, i.e.

$$\tilde{f}(\gamma, N) = \int_0^{\infty} dk^2 (k^2)^{-1-\gamma} f(k^2, N), \tag{2.5}$$

where it reduces to

$$\tilde{f}(\gamma, N) = \tilde{f}_I(\gamma, Q_0^2) + (\bar{\alpha}_s/N)\chi_0(\gamma)\tilde{f}(\gamma, N), \tag{2.6}$$

where $\tilde{f}(\gamma, Q_0^2) = \exp(-\gamma \ln(Q_0^2))$ and $\chi(\gamma)$ is the characteristic function

$$\chi_0(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma). \tag{2.7}$$

A little simple manipulation leads to the expression

$$\mathcal{G}(Q^2, N) = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} d\gamma \exp(\gamma \ln(Q^2/Q_0^2)) \frac{g_B(N, Q_0^2)}{\gamma(1 - (\bar{\alpha}_s/N)\chi_0(\gamma))}. \tag{2.8}$$

This inverse transformation has a leading twist component given by the contribution of the leading pole at $1 - (\bar{\alpha}_s/N)\chi_0(\gamma) = 0$, and the solution is

$$\mathcal{G}(Q^2, N) = \frac{1}{-(\bar{\alpha}_s/N)\gamma_0 \chi_0'(\gamma_0)} \left(\frac{Q^2}{Q_0^2}\right)^{\gamma_0} g_B(N, Q_0^2). \tag{2.9}$$

The anomalous dimension $\gamma_0(\bar{\alpha}_s/N)$ may be transformed to x-space as a power series in $\bar{\alpha}_s \ln(1/x)$, and has a branch point at $N = \lambda = 4 \ln 2 \bar{\alpha}_s$ (at which $\gamma \to \frac{1}{2}$) leading to asymptotic small x behaviour for the splitting function

$$x P_{gg}^0(x, \bar{\alpha}_s) \to \frac{0.07 \bar{\alpha}_s x^{-\lambda.}}{(\bar{\alpha}_s \xi)^{3/2}}, \tag{2.10}$$

In a similar fashion, assuming that the leading small x behaviour is dominated by the perturbative physics rather than by $g_B(Q_0^2, N)$, one can transform to x-space the normalization $\frac{1}{-(\bar{\alpha}_s/N)\gamma_0 \chi_0'(\gamma_0)}$ finding that this leads to a gluon normalization $xg(x) \propto \frac{\bar{\alpha}_s x^{-\lambda.}}{(\bar{\alpha}_s \xi)^{1/2}}$.

2.2. Running Coupling.

Beyond strict leading order it is impossible to ignore the running of the coupling. At NLO ultraviolet regularization is required, resulting in a correction to the LO kernel of the form $-\beta_0\alpha_s(\mu_R^2)\ln(k^2/\mu_R^2)K_0(q^2,k^2)$, where μ_R is the renormalization scale which must now be introduced. Hence, it is unrealistic to simply use the LO kernel without considering the influence of such a correction. An obvious way in which to incorporate such a term is to simply use the running coupling constant evaluated at the scale k^2 in the previous LO BFKL equation. Since this, or something similar, is unavoidably forced upon us at NLO, it seems sensible to consider the fixed coupling LO BFKL equation as just a model which would apply in a conformally invariant world, and more realistically to work with the BFKL equation with running coupling [23] [24] [17][18] from the beginning. Doing this we obtain

$$f(k^2,Q_0^2,\bar{\alpha}_s(k^2)/N) = f_I(k^2,Q_0^2) + \frac{\bar{\alpha}_s(k^2)}{N}\int_0^\infty \frac{dq^2}{q^2}K_0(q^2,k^2)f(q^2), \tag{2.11}$$

where

$$\alpha_s = 1/(\beta_0\ln(k^2/\Lambda^2)), \tag{2.12}$$

$\beta_0 = (11-2N_f/3)/(4\pi)$, and N_f is the number of active flavours.

One can solve this equation in the same type of way as for the fixed coupling case, i.e. take the Mellin transformation, but now with respect to (k^2/Λ^2). It is most convenient first to multiply through by $\ln(k^2/\Lambda^2)$, in which case one obtains

$$\frac{d\tilde{f}(\gamma,N)}{d\gamma} = \frac{d\tilde{f}_I(\gamma,Q_0^2)}{d\gamma} - \frac{1}{\bar{\beta}_0 N}\chi(\gamma)\tilde{f}(\gamma,N), \tag{2.13}$$

where $\bar{\beta}_0 = (\pi\beta_0/3)$. Hence, the inclusion of the running coupling has completely changed the form of our double Mellin space equation, turning it into a first order differential equation. This has a profound effect on the form of the solutions. The equation may easily, if formally, be solved giving,

$$\tilde{f}(\gamma,N) = \exp(-X_0(\gamma)/(\bar{\beta}_0 N))\int_\gamma^\infty \frac{d\tilde{f}_I(\tilde{\gamma},N,Q_0^2)}{d\tilde{\gamma}}\exp(X_0(\tilde{\gamma})/(\bar{\beta}_0 N))d\tilde{\gamma}, \tag{2.14}$$

where

$$X_0(\gamma) = \int_{\frac{1}{2}}^{\gamma}\chi_0(\hat{\gamma})d\hat{\gamma} \equiv \left(2\psi(1)(\gamma-\tfrac{1}{2}) - \ln\left(\frac{\Gamma(\gamma)}{\Gamma(1-\gamma)}\right)\right). \tag{2.15}$$

$X_0(\gamma) \to \ln(\gamma)$ at $\gamma = 0$ and hence $\exp(-X_0(\gamma)/(\bar{\beta}_0 N))$ has a branch point at $\gamma = 0$ $(\exp(-X_0(\gamma)/(\bar{\beta}_0 N)) \to \gamma^{-1/\bar{\beta}_0 N})$ with similar branch points at all negative integers. It is easiest to choose each of the cuts along the negative real axis. $\exp(X_0(\gamma)/(\bar{\beta}_0 N))$ has similar branch points at every positive integer, and it is easiest to choose these cuts along the positive real axis. This means that the integral in (2.14) is ambiguous due to the available choice in avoiding the cuts. This ambiguity can only really be removed by regulating the Landau pole in the definition of

the coupling. However, this introduces model dependence, and also makes analytic progress rather more difficult, so I simply accept this ambiguity for this function.[4]

In order to simplify (2.14), and introduce factorization we trivially rewrite it as

$$\tilde{f}(\gamma,N)=\exp(-X_0(\gamma)/(\bar{\beta}_0 N))\left[\int_0^\infty-\int_0^\gamma\right]\frac{d\tilde{f}_I(\tilde{\gamma},N,Q_0^2)}{d\tilde{\gamma}}\exp(X_0(\tilde{\gamma})/(\bar{\beta}_0 N))d\tilde{\gamma}, \tag{2.16}$$

In the region of $\gamma=0$ the integrand in (2.16) is $\propto\gamma^{1/\bar{\beta}_0 N}$, so the integral of this from $0\to\gamma$ is $\propto\gamma^{1+1/\bar{\beta}_0 N}$. Hence, the leading singularity in the γ plane for $\exp(-X_0(\gamma)/(\bar{\beta}_0 N))$, is cancelled by the integral from $0\to\gamma$ of this integrand [18], and the new leading singularity is at $\gamma=-1$. Since $\mathcal{G}(Q^2,N)$ is obtained by an inverse Mellin transformation with respect to Q^2/Λ^2, the part of (2.16) coming from the integral from 0 to γ will behave like Λ^2/Q^2 (actually Q_0^2/λ^2 as we will see later). Hence, disguarding this power-suppressed correction, which will be considered in some detail in section 3, we keep only the first term in (2.16), obtaining for the gluon distribution

$$\begin{aligned}\mathcal{G}(Q^2,N)&=\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{1}{\gamma}\exp(\gamma\ln(Q^2/\Lambda^2)-X_0(\gamma)/(\bar{\beta}_0 N))d\gamma\\&\qquad\times\int_0^\infty\exp(-\tilde{\gamma}\ln(Q_0^2/\Lambda^2)+X_0(\tilde{\gamma})/(\bar{\beta}_0 N))d\tilde{\gamma}\,g_B(Q_0^2,N)\\&=\mathcal{G}_E(Q^2,N)\mathcal{G}_I(Q_0^2,N)g_B(Q_0^2,N).\end{aligned} \tag{2.17}$$

Therefore, we have factorization up to well-defined corrections of $\mathcal{O}(Q_0^2/Q^2)$, which genuinely do vanish as $Q_0^2\to 0$ (see section 3). As mentioned, $\exp(X_0(\gamma)/(\bar{\beta}_0 N))$, contains singularities at all positive integers, and $\mathcal{G}_I(Q_0^2,N)$ is not properly defined, since the integrand has singularities lying along the line of integration. However, since this factor is independent of Q^2, it does not contribute at all to the evolution of the structure function. It is also divergent as $Q_0^2\to 0$, and as usual in the factorization theorem these divergences are implicitly cancelled by $g_B(Q_0^2,N)$, and we can imagine the ambiguity to be cancelled in the same manner. So the overall normalization is incalculable, but there is a calculable function $\mathcal{G}_E(Q^2,N)$ whose form is determined by the singularities of $\exp(-X_0(\gamma)/(\bar{\beta}_0 N))$ in the γ plane. This also leads to a fundamental difference between the cases of the fixed and running couplings. Whereas previously the leading singularity was a pole at $(\bar{\alpha}_s/N)\chi(\gamma)=1$, i.e. at $\gamma\to\frac{1}{2}$ as $N\to 4\ln 2\bar{\alpha}_s$, now the leading singularity is an cut at $\gamma=0$: there is no power-like behaviour in Q^2. Similarly, the branch point in the N plane at $4\ln 2\bar{\alpha}_s$ has become an essential singularity at $N=0$: there is no power-like behaviour in x in the

[4] The problem due to the Landau pole is illustrated using an alternative method of solution in [19]. In this paper the solution of the equation where the NLO coupling effect is left simply as $-\beta_0\alpha_s(\mu_R^2)\ln(k^2/\mu_R^2)K_0(q^2,K^2)$ rather than resummed is also considered. This does not improve the situation, i.e. an ambiguity in the solution remains even in this case.

evolution factor for the gluon. The introduction of the running of the coupling has changed the character of the solution completely.

One can now proceed with the solution to the LO BFKL equation by acknowledging that the only real information contained in $\mathcal{G}_E(N,Q^2)$ is on the evolution of the structure function, i.e. defining

$$\frac{d\ln\mathcal{G}(N,Q^2)}{d\ln(Q^2)}=\frac{d\ln\mathcal{G}_E(N,Q^2)}{d\ln(Q^2)}\equiv\gamma_{gg}(N,Q^2). \tag{2.18}$$

$\mathcal{G}_E(N,Q^2)$ gives us an entirely perturbative effective anomalous dimension governing the evolution of the gluon structure function. The usual technique for solving for $\mathcal{G}_E(N,Q^2)$ is to expand the integrand in (2.17), about the saddle-point. This results in a contour of integration parallel to the imaginary axis, with real part $\to\frac{1}{2}$ for the small x solutions, see fig. 1. Using this results in an anomalous dimension

$$\gamma_{gg}(N,Q^2)=\gamma_0(\bar{\alpha}_s(Q^2)/N)+\sum_{n=1}^{\infty}(-\beta_0\alpha_s(Q^2))^n\tilde{\gamma}_n(\bar{\alpha}_s(Q^2)/N), \tag{2.19}$$

i.e., the effective anomalous dimension is the naive leading-order result with coupling at scale Q^2 plus an infinite series of corrections in increasing powers of $-\beta_0\alpha_s(Q^2)$ [20]. However, each of the $\tilde{\gamma}(\bar{\alpha}_s(Q^2)/N)$ is singular at $N=\lambda(Q^2)$, and the power of the singularity increases with increasing n. Hence, although the series for the resulting splitting function is in the small quantity $\alpha_s(Q^2)\beta_0$, the accompanying coefficients are progressively more singular as $x\to 0$. The saddle-point approximation is therefore not a reliable result as $x\to 0$ and explicit investigation reveals that it is only really quantitatively useful when $\bar{\alpha}_s(Q^2)\ln(1/x)$ is so small that the effective anomalous dimension is effectively the LO in α_s part, $xP_{gg}(x)=\bar{\alpha}_s(Q^2)$[20]. This translates into $x\gtrsim 0.01$ in the HERA range. Therefore the calculations of the anomalous dimension which rely on an expansion about the saddle-point, i.e. the conventional expansion in decreasing powers of $\ln(1/x)$ at fixed power of α_s, leads to very inaccurate and misleading results for small x. This instability is not surprising. If one examines the integrand along the saddle-point contour of integration one finds that it is very different from the Gaussian form the saddle-point method assumes [20]. Also this is an expansion obtained from approaching $\gamma=\frac{1}{2}$ and in terms of functions of N which are singular at $N=\lambda(Q^2)$, whereas we know that the full solution no longer sees these points as anything special. In fact, the known singularity structure of the integrand implies that $\gamma=0$ is the point on which to concentrate.

This suggests an alternative method of solution for the anomalous dimension. In order to concentrate on this leading singularity we may move the contour of integration to the left and simultaneously use the property that the integrand dies away very quickly at infinity (for $\Re e\gamma\leq 1/2$) to close the contour so that it simply encloses the real axis for $\gamma<0$ fig. 1. It is then useful to express $\chi_0(\gamma)$ in the form

$$\chi_0(\gamma)=1/\gamma+\sum_{n=1}^{\infty}2\zeta(2n+1)\gamma^{2n}, \tag{2.20}$$

which is, however, only strictly valid only for $|\gamma| < 1$. Doing this we may write

$$X_0(\gamma) = \ln(\gamma) + \gamma_E + \sum_{n=1}^{\infty} 2\frac{\zeta(2n+1)}{2n+1}\gamma^{2n+1}, \tag{2.21}$$

and the integrand for $\mathcal{G}_E(N, Q^2)$ becomes

$$\gamma^{-1/(\bar{\beta}_0 N)-1} \exp\left(\gamma t - \frac{1}{(\bar{\beta}_0 N)}(\gamma_E + \sum_{n=1}^{\infty} a_n \gamma^{2n+1})\right), \tag{2.22}$$

where $t = \ln(Q^2/\Lambda^2)$ and $a_n = 2\zeta(2n+1)/(2n+1)$. The contribution to the integral from $0 \to -\infty + i\epsilon$ is now the same as that from $-\infty - i\epsilon \to 0$ up to a phase factor, and we may write

$$\mathcal{G}_E(N,t) = -\sin\left(\frac{\pi}{(\bar{\beta}_0 N)}\right) \exp\left(-\frac{\gamma_E}{(\bar{\beta}_0 N)}\right) \int_{-\infty}^{0} \gamma^{-1/(\bar{\beta}_0 N)-1} \exp\left(\gamma t - \frac{1}{(\bar{\beta}_0 N)} \sum_{n=1}^{\infty} a_n \gamma^{2n+1}\right) d\gamma, \tag{2.23}$$

where the integral has to be understood as an analytic continuation, since there are singularities along the real axis, and strictly speaking the integrand is well defined only for $\gamma > -1$. Since the factor of $\exp(\gamma t)$ is present this latter point leads, in principle, to an error of order $\exp(-t)$, i.e. $\mathcal{O}(\Lambda^2/Q^2)$ into the value of $\mathcal{G}_E(N,t)$. This will be discussed in more detail below.

In order to evaluate the above integral it is convenient to let $y = \gamma t$, resulting in

$$\begin{aligned} \mathcal{G}_E(N,t) = &-\sin\left(\frac{\pi}{(\bar{\beta}_0 N)}\right) \exp\left(-\frac{\gamma_E}{(\bar{\beta}_0 N)}\right) \\ &t^{1/(\bar{\beta}_0 N)} \int_{-\infty}^{0} y^{-1/(\bar{\beta}_0 N)-1} \exp(y) \exp\left(-\frac{1}{(\bar{\beta}_0 N)} \sum_{n=1}^{\infty} a_n (y/t)^{2n+1}\right) dy. \end{aligned} \tag{2.24}$$

The latter exponential may be expanded as a power series in y/t and the each term in the integral then precisely evaluated using the standard result that

$$(-1)^n \Gamma(-1/(\bar{\beta}_0 N) + n) = \int_{-\infty}^{0} y^{-1/(\bar{\beta}_0 N)-1} \exp(y) y^n dy. \tag{2.25}$$

Hence, we may formally write

$$\begin{aligned} \mathcal{G}_E(N,t) = &-\sin\left(\frac{\pi}{(\bar{\beta}_0 N)}\right) \exp\left(-\frac{\gamma_E}{(\bar{\beta}_0 N)}\right) \Gamma(-1/(\bar{\beta}_0 N)) \\ &t^{1/(\bar{\beta}_0 N)} \left(1 + \sum_{n=3}^{\infty} A_n (1/(\bar{\beta}_0 N)) t^{-n} (-1)^n \frac{\Gamma(-1/(\bar{\beta}_0 N) + n)}{\Gamma(-1/(\bar{\beta}_0 N))}\right), \end{aligned} \tag{2.26}$$

plus an error of $\mathcal{O}(\Lambda^2/Q^2)$. We note that we could have reached this final expression (2.26) in a slightly more rigorous manner. After performing the expansion of $X_0(\gamma)$ in (2.21) we could have produced a well-defined integral in (2.23)by taking the lower limit of integration to be $-1 + \epsilon$ so that the expansion is valid over the region of integration. This would mean that there is region of

integration $\gamma \leq -1$ absent, which due to the factor of $\exp(\gamma t)$ would mean a missing contribution of $\mathcal{O}(\Lambda^2/Q^2)$. This new limit of integration would result in the lower limit of $-t$ in (2.24) and (2.25) and consequently we would obtain incomplete gamma functions $\gamma(-1/(\bar{\beta}_0 N) + n, t)$ rather than $\Gamma(-1/(\bar{\beta}_0 N) + n)$. However, $\gamma(-1/(\bar{\beta}_0 N) + n, t) = \Gamma(-1/(\bar{\beta}_0 N) + n) + \mathcal{O}(\Lambda^2/Q^2)$, so disguarding the contributions of $\mathcal{O}(\Lambda^2/Q^2)$ we regain (2.26), which is formally equivalent to (2.23), but we have seen explicitly the origin of the intuitively obvious $\mathcal{O}(\Lambda^2/Q^2)$ corrections to (2.26).

The result (2.26) was first noted in [25], and was simplified by using the relationship that as $N \to 0$, $(\Gamma(-1/(\bar{\beta}_0 N) + n)/\Gamma(-1/(\bar{\beta}_0 N))) \to (-1/(\bar{\beta}_0 N))^n$. However, it is important to notice the more general result that for all N

$$(-1)^n \frac{\Gamma(-1/(\bar{\beta}_0 N) + n)}{\Gamma(-1/(\bar{\beta}_0 N))} = \Delta_n(-1/(\bar{\beta}_0 N)), \tag{2.27}$$

where

$$\Delta_n(-1/(\bar{\beta}_0 N)) = \sum_{m=0}^{n-1} (-1)^m d_{mn} (\bar{\beta}_0 N)^{-n+m}, \tag{2.28}$$

and d_{mn} are positive coefficients and $d_{0n} = 1$. Explicitly the first few $\Delta_n(-1/(\bar{\beta}_0 N))$ are

$$\begin{aligned}
\Delta_1(-1/(\bar{\beta}_0 N)) &= \left(\frac{1}{(\bar{\beta}_0 N)}\right) \\
\Delta_2(-1/(\bar{\beta}_0 N)) &= \left(\frac{1}{(\bar{\beta}_0 N)}\right)^2 - \left(\frac{1}{(\bar{\beta}_0 N)}\right) \\
\Delta_3(-1/(\bar{\beta}_0 N)) &= \left(\frac{1}{(\bar{\beta}_0 N)}\right)^3 - 3\left(\frac{1}{(\bar{\beta}_0 N)}\right)^2 + 2\left(\frac{1}{(\bar{\beta}_0 N)}\right) \\
\Delta_4(-1/(\bar{\beta}_0 N)) &= \left(\frac{1}{(\bar{\beta}_0 N)}\right)^4 - 6\left(\frac{1}{(\bar{\beta}_0 N)}\right)^3 + 11\left(\frac{1}{(\bar{\beta}_0 N)}\right)^2 - 6\left(\frac{1}{(\bar{\beta}_0 N)}\right) \\
\Delta_5(-1/(\bar{\beta}_0 N)) &= \left(\frac{1}{(\bar{\beta}_0 N)}\right)^5 - 10\left(\frac{1}{(\bar{\beta}_0 N)}\right)^4 + 35\left(\frac{1}{(\bar{\beta}_0 N)}\right)^3 - 50\left(\frac{1}{(\bar{\beta}_0 N)}\right)^2 + 24\left(\frac{1}{(\bar{\beta}_0 N)}\right).
\end{aligned} \tag{2.29}$$

These functions oscillate a great deal and only approach the asymptotic values of $1/(\bar{\beta}_0 N)^n$ at low values of N which decrease with increasing n. The comparison of $\Delta_4(-1/(\bar{\beta}_0 N))$ with $1/(\bar{\beta}_0 N)^4$ is shown in fig. 2, and illustrates this feature clearly.

Ignoring the common factor of $-\sin(\pi/(\bar{\beta}_0 N))\Gamma(-1/(\bar{\beta}_0 N))\exp(-\gamma_E/(\bar{\beta}_0 N))$, which has no t dependence, and is irrelevant for the calculation of the anomalous dimension,

$$\mathcal{G}_E(N, t) = t^{1/(\bar{\beta}_0 N)} \left(1 + \sum_{n=3}^{\infty} A_n(1/(\bar{\beta}_0 N)) t^{-n} \Delta_n(-1/(\bar{\beta}_0 N))\right) \tag{2.30}$$

where the A_n are simply calculable from the expansion of $\exp\left(-1/(\bar{\beta}_0 N)\sum_{n=1}^{\infty} a_n (y/t)^{2n+1}\right)$. The common factor of $t^{1/(\bar{\beta}_0 N)}$ is the well-known double-leading-log result coming from just the

LO $\alpha_s(Q^2)/N$ part of the anomalous dimension. Multiplying this we have an expansion as a power series in $1/t$ or equivalently in $\alpha_s(Q^2)$. In fact

$$t^{-n}\Delta_n(-1/(\bar{\beta}_0 N)) = (\bar{\alpha}_s(Q^2)/N)^n \sum_{m=0}^{n-1} d_{mn}(-\beta_0\alpha_s(Q^2))^m(\bar{\alpha}_s(Q^2)/N)^{-m}. \tag{2.31}$$

This explicitly demonstrates that we obtain a set of running coupling corrections to a LO result, i.e. in solving the BFKL equation we are now obtaining not only the leading power in $1/N$ (corresponding to the leading power of $\ln(1/x)$) at each order in $\alpha_s(Q^2)$, but we also obtain the leading power in β_0 at each power of $\alpha_s(Q^2)$ and $1/N$. Substituting this type of expansion into (2.30), putting the resulting expression for $\mathcal{G}_E(N,t)$ in (2.18) and expanding in inverse powers of t, one obtains an expression for the anomalous dimension as a power series in $\alpha_s(Q^2)$, where at each order we have the leading divergence in $1/N$ plus a sum of running coupling correction type terms. With a little work one may regain the whole leading $\gamma_0(\alpha_s(Q^2)/N)$ (though it is necessary to keep some subleading terms in the Δ_n to do this), along with a tower of terms which are subleading in powers of $\beta_0\alpha_s(Q^2)$ to this leading anomalous dimension – one obtains all the corrections to this naive LO anomalous dimension due to the running of the coupling, i.e. the whole of (2.19) is regained, but ordered in powers of $\alpha_s(Q^2)$ rather than in $\beta_0\alpha_s(Q^2)$.

The general features of this full, running coupling BFKL gluon Green's function and consequent anomalous dimension may be appreciated quite easily. The important fact to note is that although the $\Delta_n(-1/(\bar{\beta}_0 N))) \to (1/(\bar{\beta}_0 N))^n$ as $N \to 0$, the function oscillates a great deal with $1/(\bar{\beta}_0 N)$, and remains much smaller in magnitude than this asymptotic form until very small N, roughly until $1/N > n$. This coupled with the accompanying factor of t^{-n} means that for reasonable t, i.e $t \gtrsim 4-5$ ($Q^2 \gtrsim 1\mathrm{GeV}^2$), only the first 5 or so terms in (2.30) make a significant contribution for $N > 0.25$. Hence, to a very good approximation

$$\mathcal{G}_E(N,t) = t^{1/(\bar{\beta}_0 N)}\left(1 - \frac{2\zeta(3)}{3(\bar{\beta}_0 N)t^3}\Delta_3(-1/(\bar{\beta}_0 N)) - \frac{2\zeta(5)}{5(\bar{\beta}_0 N)t^5}\Delta_5(-1/(\bar{\beta}_0 N))\right), \tag{2.32}$$

and in fact the smallness of the coefficient makes even the t^{-5} term almost negligible in this case. $\mathcal{G}_E(N,t)$ initially grows as N falls due to the $t^{1/(\bar{\beta}_0 N)}$ term. However, for $N \sim 0.6$ the negative contribution from the t^{-3} term starts to become significant and ultimately drives the gluon structure function to negative values. The result is shown in fig. 3. $d\mathcal{G}_E(N,t)/dt$ may simply be evaluated also using (2.30), and shows the same general shape, but does not become negative until a slightly lower value of N as also seen in fig. 3. Hence the anomalous dimension develops a leading pole at a finite value of N, given by

$$t^3 = \frac{2\zeta(3)}{3(\bar{\beta}_0 N)}\left(\frac{1}{(\bar{\beta}_0 N)^3} - \frac{3}{(\bar{\beta}_0 N)^2} + \frac{2}{(\bar{\beta}_0 N)}\right). \tag{2.33}$$

This result is accurate to better than 10% even at $Q^2 \sim 1\text{GeV}^2$, and is much better at higher Q^2, the right-hand-side receiving corrections formally of $\mathcal{O}(1/(t^2\bar{\beta}_0^5 N^5))$, but which are numerically small. The value of N for this leading pole is shown as a function of t in fig. 4, and for the sort of values of t relevant at HERA is ~ 0.25. Going to $N < 0.25$ higher order terms in (2.30) become important, and the positive $1/((\bar{\beta}_0 N)^2 t^6)\Delta_6(-1/(\bar{\beta}_0 N))$ term absent in (2.30) pulls $\mathcal{G}_E(N,t)$ back to positive values, and another pole, with opposite sign residue, appears in $\gamma_{gg}(N,t)$. At even lower N the analytic expression eventually breaks down, as discussed below, but numerical results show a series of poles becoming closer together. Nevertheless, the position of the leading pole is essentially determined by the first handful of terms in the power series in $\alpha_s(Q^2)$ for $\mathcal{G}_E(N,t)$, and hence so is the asymptotic behaviour of the small x splitting function, i.e. $P_{gg}(x,t) \sim x^{-0.25}$. So we see that the introduction of the running coupling has a dramatic effect on the singularity structure of the LO BFKL anomalous dimension, turning the cut into a series of poles, and changing the position of the rightmost singularity by a factor of ~ 0.4. This result of the pole in the anomalous dimension was previously proved in detail in [15] using numerical techniques and in the context of the collinearly resummed NLO kernel, and also indicated here using an approximate analytical solution first suggested in [24]. However, in this paper I particularly stress the quantitative result of the huge modification of the naive LO BFKL anomalous dimension due to the running coupling contributions alone. This is apparent over a wide range of N, and in fig. 5.a I show the anomalous dimension as a function of N for all values right of the leading singularity. As one sees, it is much closer to the simple $\alpha_s(Q^2)/N$ expression than to the naive BFKL result.

Before going into more precise detail and more general situations there are two important points I should address. These are the choice of the scale of the running coupling in (2.11) as k^2 and the fact that the expansion of $\chi_0(\gamma)$ in powers of γ is not convergent over the whole range of the contour of integration. The former of these is the simpler, so firstly I shall address the choice of scale. It was known in [26] that the correct scale seemed as if it were really the symmetric choice $(k-q)^2$, but that k^2 could be used instead, leading to contribution to the NLO kernel which is proportional to β_0. In practice it is much easier to obtain analytic results using k^2, and this β_0-dependent NLO term leads to a contribution to the Mellin transformation of the NLO kernel, $\chi_1(\gamma)$, of the form $\frac{1}{2}\bar{\beta}_0(\chi_0^2(\gamma) + \chi_0'(\gamma))$. Including this in the integrand for the expression for $\mathcal{G}_E(N,t)$ at NLO (to be discussed in detail in section 4) leads to a multiplicative contribution of the form $\exp(\frac{1}{2}(\ln(\chi_0(\gamma)) + X_0(\gamma))) \equiv f^{\beta_0}(\gamma)$. This can be expanded as a power series which at low orders is

$$f^{\beta_0}(\gamma) = 1 + 1.60\gamma^3 + 1.24\gamma^5 - 0.163\gamma^6 + 1.15\gamma^7 + \cdots. \tag{2.34}$$

Including this additional factor in (2.23) modifies (2.32) to

$$\mathcal{G}_E(N,t) = t^{1/(\bar{\beta}_0 N)}\left(1 - \frac{(2/3\zeta(3) - 1.60(\bar{\beta}_0 N))}{(\bar{\beta}_0 N)t^3}\Delta_3(-1/(\bar{\beta}_0 N)) - \frac{(2/5\zeta(5) - 1.24(\bar{\beta}_0 N))}{(\bar{\beta}_0 N)t^5}\Delta_5(-1/(\bar{\beta}_0 N))\right). \tag{2.35}$$

For a given power of $\alpha_s(Q^2)$ these new contributions produce terms a power of $\bar{\beta}_0 N$ up on the other terms and hence, not surprisingly, result in additional running coupling corrections to the gluon and anomalous dimension. However, the new terms in the series in powers of γ do not start until third order and have rather small coefficients. The resulting change in the anomalous dimensions, both for general values of N and for the position of the leading pole is very minor. Therefore, the correction for my original "incorrect" choice of scale is very small. However, in principle it seems as though the factor just considered should really be taken as part of the LO result since it just gives running coupling corrections. I will adopt this convention and the LO anomalous dimensions and splitting functions presented in this paper will explicitly contain the corrections from this factor, and in fact the results already presented in fig. 3, fig. 4 and fig. 5 include these (very small) effects. In principle one could sum the corrections needed due to the simple choice of k^2 in the coupling, rather than $(k-q)^2$, by including contributions induced in the kernel at NNLO and beyond. In practice, beyond NLO the change seems too tiny for one to be concerned.

I should also comment on the limit of applicability of the analytic expression (2.30). As noted, it is obtained via a series expansion which is not valid over the whole contour of integration. This is reflected in the error of $\mathcal{O}(\Lambda^2/Q^2)$ we discovered for this expression but also in the fact that the overall magnitude of the $\Delta_n(-1/(\bar{\beta}_0 N))$ actually increases like $n!$ in general. This latter point means that the series in (2.30) is actually asymptotic. It turns out that it contains both infrared and ultraviolet renormalon contributions, and hence it must be truncated to obtain sensible results. The greatest accuracy may be obtained from (2.30) by truncating the series at order $n_0 \sim t$, the precise value depending on the size of the coefficients in the series expansion. For the LO gluon these are small and one could use $n_0 \sim 10$, but from experience with other variables (see later) and the desire to go down to $Q^2 \sim 1\text{GeV}^2$, i.e. $t \sim 4-5$, in practice I always use $n_0 = 5$. (For the LO gluon the contribution from $n = 6 \to 10$ is practically negligible.) Using the truncated expression for $\mathcal{G}_E(N,t)$ in the manner already discussed, then results in an infinite series in $\alpha_s(Q^2)$ for $\gamma_{gg}(N,t)$ which is convergent for any N right of the leading pole, but different to the real, divergent series beyond 6th order in $\alpha_S(Q^2)$.

It is vital to note that although the formal expression for the gluon, and hence anomalous dimension, as a power series in $\alpha_s(Q^2)$ (2.30) contains infrared renormalons[5], and hence has an ambiguity of $\mathcal{O}(\Lambda^2/Q^2)$, the integral in (2.17), which properly defines the leading twist gluon and anomalous dimension, does exist and produces well-defined results. The ambiguity of $\mathcal{O}(\Lambda^2/Q^2)$ in (2.30) cancels with an ambiguity in the $\mathcal{O}(\Lambda^2/Q^2)$ correction to this power-series expansion which

[5] In unphysical regularization schemes, such as $\overline{\text{MS}}$, the anomalous dimensions are not expected to contain renormalons (see section 3.4 of [27] for a discussion), these being confined to the coefficient functions relating the parton distributions to physical quantities. However, by regularizing via a finite Q_0, and defining the gluon density as the bare density convoluted with the gluon Green's function we have implicitly chosen a more physically motivated factorization scheme which allows the presence of renormalons.

we discovered in the derivation of (2.30). The accuracy of the (truncated) analytic expression can be found by comparing with results obtained from evaluating (2.17) using numerical integration along the contour shown in fig. 1. For the gluon structure function for N to the right of the leading pole the analytic approximation to the anomalous dimension is found to be a fraction of a percent for $t = 6$, and falls like $\exp(-t)$. Strictly speaking there is an $\exp(-t)$ contribution from the correction to (2.30) (with the renormalon ambiguity removed) plus a $1/t^7$ correction due to the truncation. However, $1/t^7$ is similar to $\exp(-t)$ in the range considered. Hence, we have a power-like correction to the power series in $\alpha_s(Q^2)$ obtained from the truncated expression which is completely well-defined. This illustrates that the presence of infrared renormalons in a physical quantity is not necessarily due to an inherent ambiguity in the quantity itself (due, for example, to the Landau pole in the coupling) as is commonly thought, but rather due to the impossibility of completely expressing the physical quantity as a power series in $\alpha_s(Q^2)$ [28]. In truncating the power-series expansion in (2.30) I simply choose to split the expression for the gluon as some general function of N and Q^2 into a perturbatively calculable part as a power-series in $\alpha_s(Q^2)$ and a remainder which is approximately of order $\mathcal{O}(\Lambda^2/Q^2)$. The point of truncation is then chosen empirically so as to make this remainder term as small as possible. This seems to be the way to obtain the most accurate analytic results. It is important to note that the remainder term, although power-suppressed, is not in any way higher twist, since it is obtained from the leading twist part of the solution to the BFKL equation.

Having got these two points out of the way we can now begin to discuss the quantitative results of the running coupling BFKL equation. In order to investigate the real effect of the BFKL anomalous dimension on structure function evolution it is necessary to calculate the BFKL splitting function as a function of x. This is where an analytic expression for the anomalous dimension is particularly useful. A series of numerically obtained values of $\gamma_{gg}(N,t)$ allows an approximate determination of $P(x,t)$, but it is extremely difficult to be accurate, especially for the wildly oscillating functions of $1/N$ which do in fact make up $\mathcal{G}_E(N,t)$. However, I now have an explicit series for $\gamma_{gg}(N,t)$ in powers of $\alpha_s(Q^2)$, obtained from the truncated expression for $\mathcal{G}_E(N,t)$. The N-dependent functions at each power of $\alpha_s(Q^2)$ become larger at small N as the series progresses, of course, and to reach small enough x more and more terms are needed. However, at a fixed value of N there is no such growth, and the same is therefore true for fixed x. Hence, one only needs to work to a finite order. Limiting oneself to $x > 10^{-5}$ and $t > 4.5$, i.e. $Q^2 \gtrsim 1\mathrm{GeV}^2$, the suppression of the $\Delta_n(-1/(\bar{\beta}_0 N))$ is quite significant and seventh order in $\alpha_s(Q^2)$ is easily sufficient. This results in a power-series contribution to the splitting function

$$\begin{aligned} xP_{gg}^{LO}(\xi,\alpha_s(Q^2)) &= \bar{\alpha}_s(Q^2) + \bar{\alpha}_s^4(Q^2)\left(2.4\frac{\xi^3}{3!} - 12.01\bar{\beta}_0\frac{\xi^2}{2} + 9.206\bar{\beta}_0^2\xi - 9.60\bar{\beta}_0^3\right) \\ &+ \bar{\alpha}_s^6(Q^2)\left(2.08\frac{\xi^5}{5!} - 26.95\bar{\beta}_0\frac{\xi^4}{4!} + 134.6\bar{\beta}_0^2\frac{\xi^3}{3!} - 320.7\bar{\beta}_0^3\frac{\xi^2}{2} + 359.8\bar{\beta}_0^4\xi - 148.8\bar{\beta}_0^5\right) \\ &+ \bar{\alpha}_s^7(Q^2)\left(\frac{1.92}{\bar{\beta}_0}\frac{\xi^7}{7!} - 19.23\frac{\xi^6}{6!} + 78.94\bar{\beta}_0\frac{\xi^5}{5!} - 169.2\bar{\beta}_0^2\frac{\xi^4}{4!} + 199.8\bar{\beta}_0^3\frac{\xi^3}{3!} - 122.9\bar{\beta}_0^4\frac{\xi^2}{2} + 30.72\bar{\beta}_0^5\xi\right). \end{aligned} \tag{2.36}$$

This contribution to the splitting function for $t = 6$ and is shown in fig. 6.a. Note that because of the truncation of $\mathcal{G}_E(N,t)$, beyond 6th order the expression for $P_{gg}^{LO}(\xi, \alpha_s(Q^2))$ is not what one would really get from the true power-series. In particular there are higher powers of ξ than strictly allowed. Nevertheless, it represents a very accurate approximation to the full result whereas the correct series would simply diverge.

We also have to consider the power-suppressed contribution. Although this is only calculated numerically in N-space it is only a small correction of order 0.05% for $\gamma_{gg}^{LO}(N,t)$ a t $t = 6$, and can also be calculated for a wide variety of values of N and t without too much work. It can then be modelled by an analytic function which may easily be converted to x-space. Hence, I choose to calculate it for $t = 4.5$ ($Q^2 \sim 1\text{GeV}^2$) and $t = 6$ ($Q^2 \sim 6\text{GeV}^2$) and N-values $0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.5, 2, 3, 5, \infty$. The lower t value is the lower limit at which we will trust this LO perturbative result, and for t above 6 the power-suppressed effect is very small. The N values go low enough to correspond safely to $x > 0.00001$ and are sufficient that very accurate modeling can be done. The values are fit to a function of the form

$$a_0 \exp(-b_0 t) + \exp(-t)\left(\sum_{n=1}^{7} a_n \left(\frac{\alpha_s(t)}{\alpha_s(t=4.5)}\right)^{b_n} \frac{1}{N^n}\right). \tag{2.37}$$

Introducing further degrees of freedom beyond this does not seem to change the results. This expression can then be trivially converted to x-space. Performing this procedure in the case of the power-suppressed contributions to the LO gluon anomalous dimension I obtain the explicit result

$$\begin{aligned} 4.92\exp(-1.62t)\delta(1-x) + \exp(-t)\Bigg(&1.068\left(\frac{\alpha_s(t)}{\alpha_s(4.5)}\right)^{1.98} + 5.257\left(\frac{\alpha_s(t)}{\alpha_s(4.5)}\right)^{3.06}\xi \\ &- 18.73\left(\frac{\alpha_s(t)}{\alpha_s(4.5)}\right)^{2.90}\frac{\xi^2}{2!} + 21.56\left(\frac{\alpha_s(t)}{\alpha_s(4.5)}\right)^{2.90}\frac{\xi^3}{3!} - 11.60\left(\frac{\alpha_s(t)}{\alpha_s(4.5)}\right)^{2.79}\frac{\xi^4}{4!} \\ &+ 3.00\left(\frac{\alpha_s(t)}{\alpha_s(4.5)}\right)^{2.55}\frac{\xi^5}{5!} - 0.301\left(\frac{\alpha_s(t)}{\alpha_s(4.5)}\right)^{2.17}\frac{\xi^6}{6}\Bigg). \end{aligned} \tag{2.38}$$

This power-suppressed correction is shown along with the power-series part and the full LO splitting function in fig. 6.a. Although the power-suppressed contribution in x space turns out to be a larger fraction of the total than in N-space, it still only makes a very small correction to the evolution. However, one notices than the logarithmic terms in (2.38) are such that it falls more quickly than (Λ^2/Q^2), or alternatively, grows more quickly than this as Q^2 falls. This may be due to the presence of a significant (Λ^4/Q^4) term in practice.

The full LO splitting function is shown in fig. 6.b along with the purely order $\alpha_s(Q^2)$ contribution and the naive BFKL splitting function. One sees that it is hugely suppressed compared with the naive LO BFKL splitting function, and is even lower than the $\mathcal{O}(\alpha_s(Q^2))$ contribution for x between about 0.1 and 0.001. Finally I note that the LO running coupling BFKL equation has also

been calculated in [29], but numerically, with coupling scale equal to $(k-q)^2$, and with the coupling frozen below a particular scale and Q_0 taken to be a finite value. The results are displayed for high t (where my power-series is essentially exact) and despite the above differences seem to be in very good agreement with the results in [21] and this paper. The freezing of the coupling and the finite Q_0 introduce choice-dependent non-perturbative effects which become important at extremely low values of x, which in general become lower as Q_0 and the scale of freezing decrease. This seems to support the results obtained by my method of formally factorizing the non-perturbative effects into $\mathcal{G}_I(Q_0^2, N)$ and extracting as much information as possible in an analytic model-independent manner.

3. Higher Twist at Small x.

In this section I will show that as far as the information from the BFKL equation is concerned calculable higher twist contributions are small. I will also suggest that some other powerlike corrections at small x may perhaps be less significant than often claimed. As a first point I note that it has been claimed that there are likely to be large infrared renormalon contributions to structure functions at small x [30]. As shown in the previous section for the case of the gluon both infrared and ultraviolet renormalons do show up in the solution to the BFKL equation if one insists upon trying to express results entirely in terms as a power-series in $\alpha_s(Q^2)$ and uses the whole of (2.30) rather than truncating. Presumably these are an extension of the small x divergent contribution to the renormalons in [30]. However, these renormalons are circumvented if one considers the full solution to the Q^2-dependent part of the BFKL equation. Precisely the same argument works for the case of real structure functions, as will be shown explicitly in section 5. This is not to say that there are not relatively large power-suppressed corrections to the (truncated) perturbative-series. We have already seen a non-negligible contribution to $P_{gg}^{LO}(x, Q^2)$, and the power-suppressed contributions turn out to be larger for physical quantities. However, these contributions are calculable and unambiguous. Hence, solution of the BFKL equation, which provides results more general than a power series in $\alpha_s(Q^2)$ avoids the renormalon ambiguity. This means that renormalons obtained from unresummed (in $\ln(1/x)$) calculations require not only a $\ln(1/x)$ resummation but also the consideration of results beyond the power-series expansion. This implies they do not really tell us anything truely quantitative about power corrections in practice.

Now let us consider genuine higher twist effects. Some of these are contained within the BFKL equation, since if Q_0^2 is allowed to be non-zero a series in powers of (Q_0^2/Q^2) is obtained which tells us about the higher twist contributions due to the intrinsic transverse momentum in the two-gluon operator. This is the only information, however, and we learn nothing about the other three contributions to next-to-leading twist (discussed, for example in [31]), in particular those due to the

four-gluon operator and hence possible saturation effects. However, it is possible to obtain some useful and interesting results.

Let us first consider the fixed coupling BFKL equation. When solving (2.8) it is straightforward to also calculate the higher twist contributions by picking up the non-leading poles in γ. The easiest way to proceed is to obtain $G(Q^2,x)$ by first taking the exact inverse Mellin transformation back to x-space by picking up the simple pole at $N=\bar{\alpha}_s\chi_0(\gamma)$ resulting in

$$xG(Q^2,x) \propto \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} d\gamma \exp(\gamma \ln(Q^2/Q_0^2))\exp(\xi\bar{\alpha}_s\chi_0(\gamma)). \tag{3.1}$$

This can now accurately be evaluated in the asymptotic small x limit using the saddle-point technique, i.e. integrating along the contour determined by the condition $(d\chi_0(\gamma)/d\gamma)=0$ which defines γ_0. At leading twist, $0\geq \Re e\gamma \geq 1$, $\gamma_0=1/2$ and $\chi_0(\gamma_0)=4\ln(2)$, leading to the usual power-like growth at small x. However, looking for the solutions to $(d\chi_0(\gamma)/d\gamma)=0$ for $-1\geq \Re e\gamma \geq 0$, i.e. examining the higher twist operator and its anomalous dimension, one finds

$$\gamma_0^{HT}=-0.425\pm 0.474i \qquad\qquad \chi_0(\gamma_0^{HT})=-2.64\pm 2.393i. \tag{3.2}$$

Hence, the features of the saddle-point are completely different at next-to-leading twist. Not only are there complex conjugate saddle-points leading to an oscillatory behaviour, but the real part of $\chi_0(\gamma_0^{HT})$ is negative rather than positive. Inserting (3.2) into (3.1) one obtains

$$xG^{HT}(Q^2,x) \propto x^{2.64\bar{\alpha}_s}\cos(2.393\bar{\alpha}_s\xi), \tag{3.3}$$

i.e. a valence-like gluon rather than one growing at small x. The corresponding higher twist splitting function has the same general behaviour as the gluon as $x\to 0$.

One can also find the splitting function by solving $1-(\bar{\alpha}_s/N)\chi_0(\gamma)$ as a power-series in $(\bar{\alpha}_s/N)$ for the next-to-leading twist solution. This results in the explicit series

$$\begin{aligned}\gamma_0^{HT}(\bar{\alpha}_s/N)+1 = &\left(\frac{\bar{\alpha}_s}{N}\right)-2\left(\frac{\bar{\alpha}_s}{N}\right)^2+2\left(\frac{\bar{\alpha}_s}{N}\right)^3+4.4\left(\frac{\bar{\alpha}_s}{N}\right)^4-29.2\left(\frac{\bar{\alpha}_s}{N}\right)^5+80.2\left(\frac{\bar{\alpha}_s}{N}\right)^6\\
&-90.6\left(\frac{\bar{\alpha}_s}{N}\right)^7-298\left(\frac{\bar{\alpha}_s}{N}\right)^8+2084\left(\frac{\bar{\alpha}_s}{N}\right)^9-6446\left(\frac{\bar{\alpha}_s}{N}\right)^{10}\\
&+9157\left(\frac{\bar{\alpha}_s}{N}\right)^{11}+20919\left(\frac{\bar{\alpha}_s}{N}\right)^{12}-187924\left(\frac{\bar{\alpha}_s}{N}\right)^{13}+666008\left(\frac{\bar{\alpha}_s}{N}\right)^{14}\\
&-1.2\times 10^6\left(\frac{\bar{\alpha}_s}{N}\right)^{15}+1.3\times 10^6\left(\frac{\bar{\alpha}_s}{N}\right)^{16}+1.9\times 10^7\left(\frac{\bar{\alpha}_s}{N}\right)^{17}-7.7\times 10^7\left(\frac{\bar{\alpha}_s}{N}\right)^{18}\\
&-1.7\times 10^8\left(\frac{\bar{\alpha}_s}{N}\right)^{19}-2.1\times 10^7\left(\frac{\bar{\alpha}_s}{N}\right)^{20}-2.0\times 10^9\left(\frac{\bar{\alpha}_s}{N}\right)^{21}+\cdots,\end{aligned} \tag{3.4}$$

which can be easily converted to x-space. The corresponding splitting function is plotted for $\bar{\alpha}_s = 0.2$ in fig. 7, and it clearly fits the expectation that $xP_{gg}^{HT}(x,\bar{\alpha}_s) \sim x^{0.5}\cos(0.5\xi)$ as $x \to 0$.[6] Hence, although the first term in the series is the same as at leading twist, and implies a growth at small x, the summation of the series is extremely different, and the next-to-leading twist contributions from the BFKL equation are not only suppressed by (Q_0^2/Q^2), but also become negligible at small x. This can also be shown to be true for the even higher twist contributions using the same techniques. This highlights the danger of using low order terms in the series for the splitting functions to estimate higher twist corrections, as in [31] - the summation of leading $\ln(1/x)$ terms may be very important, in this case of the two-gluon operator leads to a complete change of conclusion on the import of higher twist. Unfortunately, there is no knowledge at all of the corresponding series for the four-gluon operators.

Given that the results from the fixed coupling BFKL equation were altered so dramatically at leading twist by the inclusion of the running coupling, we should see what happens at higher twist. As already mentioned, the higher twist contribution to the running coupling BFKL equation is given by

$$\begin{aligned}\mathcal{G}^{HT}(Q^2,N) = \frac{1}{2\pi i}\int_{-\epsilon-i\infty}^{-\epsilon+i\infty}\frac{1}{\gamma}\exp(\gamma\ln(Q^2/\Lambda^2) - X_0(\gamma)/(\bar{\beta}_0 N))d\gamma \\ \times\int_{\gamma}^{0}\exp(-\tilde{\gamma}\ln(Q_0^2/\Lambda^2) + X_0(\tilde{\gamma})/(\bar{\beta}_0 N))d\tilde{\gamma}\, g_B(Q_0^2,N),\end{aligned} \tag{3.5}$$

where the contour in the first integral has been moved to the left since the leading singularity at $\gamma = 0$ is eliminated by the second integral.

Let us consider first the case where $t = \ln(Q^2/\Lambda^2) \gg t_0 = \ln(Q_0^2/\Lambda^2)$, which would be the case for deep inelastic scattering. Let us also, without justification for the moment, let the lower limit on the second integral be a constant, $k \sim -1$, so that we have factorization imposed. In this case we can evaluate the two integrals separately. Both the integrals can be calculated accurately using the saddle-point method. Thus, using the type of steps outlined in (4.1)–(4.5) of [20] one obtains

$$\frac{\exp\left(\int^{Q^2}\gamma_0^{HT}(\bar{\alpha}_s(q^2)/N)d\,\ln q^2\right)}{\gamma_0^{HT}(\bar{\alpha}_s(Q^2)/N)[-\chi_0'(\gamma_0^{HT}(\bar{\alpha}_s(Q^2)/N))]^{\frac{1}{2}}}, \tag{3.6}$$

for the first integral and

$$\frac{\exp\left(-\int^{Q_0^2}\gamma_0^{HT}(\bar{\alpha}_s(q^2)/N)d\,\ln q^2\right)}{[-\chi_0'(\gamma_0^{HT}(\bar{\alpha}_s(Q_0^2)/N))]^{\frac{1}{2}}}, \tag{3.7}$$

[6] Unfortunately, because of large cancellations, the first 21 terms in the series for $xP_{gg}^{HT}(x,\bar{\alpha}_s)$ are needed for $x \geq 0.00001$.

for the second. It can be verified numerically that these expressions are indeed good approximations to the precise results. Combining these we get the full next-to-leading twist gluon Green's function.

$$\frac{\exp\left(\int_{Q_0^2}^{Q^2}\gamma_0^{HT}(\bar{\alpha}_s(q^2)/N)d\,\ln q^2\right)}{\gamma_0^{HT}(\bar{\alpha}_s(Q^2)/N)[-\chi_0'(\gamma_0^{HT}(\bar{\alpha}_s(Q^2)/N))]^{\frac{1}{2}}[-\chi_0'(\gamma_0^{HT}(\bar{\alpha}_s(Q_0^2)/N))]^{\frac{1}{2}}}. \tag{3.8}$$

Hence, the anomalous dimension for the higher twist operator is simply that obtained for the fixed coupling, but with the coupling constant allowed to run with the scale, while the normalization is (roughly) the root of the fixed coupling normalization evaluated for $\alpha_s(Q^2)$ multiplied by the same for $\alpha_s(Q_0^2)$. Hence, the result is much the same as for the fixed coupling case, with both the splitting function and the normalization decreasing and oscillating as $x \to 0$.

It order to justify this conclusion it is only necessary to explain why we could assume the factorization. To do this we note that the saddle-point for the first integrand is at $t = (1/\bar{\beta}_0 N)\chi_0(\gamma_0^{HT}(t))$ and similarly for the second integrand with $t \to t_0$. However, since $t \gg t_0$ $\gamma_0^{HT}(t_0)$ is significantly to the right of $\gamma_0^{HT}(t)$. The value of $\exp(-\tilde{\gamma}t_0 + X_0(\tilde{\gamma})/(\bar{\beta}_0 N))$ along the real axis along with $\tilde{\gamma} = \gamma_0^{HT}(t_0), \gamma_0^{HT}(t)$ is shown in fig. 8. It is simple to rewrite (3.5) in the equivalent form

$$\begin{aligned}\mathcal{G}^{HT}(Q^2,N) = \frac{1}{2\pi i}\int_{\gamma_0^{HT}(t)-i\infty}^{\gamma_0^{HT}(t)+i\infty}\frac{1}{\gamma}\exp(\gamma\ln(Q^2/\Lambda^2) - X_0(\gamma)/(\bar{\beta}_0 N))d\gamma \\ \times\left[\int_{\gamma_0^{HT}(t)}^{0}\exp(-\tilde{\gamma}\ln(Q_0^2/\Lambda^2) + X_0(\tilde{\gamma})/(\bar{\beta}_0 N))d\tilde{\gamma}\right. \\ \left. + \int_{\gamma}^{\gamma_0^{HT}(t)}\exp(-\tilde{\gamma}\ln(Q_0^2/\Lambda^2) + X_0(\tilde{\gamma})/(\bar{\beta}_0 N))\right]d\tilde{\gamma}\, g_B(Q_0^2,N).\end{aligned} \tag{3.9}$$

Using fig. 8, and remembering that the saddle-point integral for the first integral is parallel to the imaginary axis, and that the integrand very quickly decreases away from $\gamma_0^{HT}(t)$, we conclude that the value of the second integral in the second line of (3.9) is negligible compared with the first. Also noting from fig. 8 that there is little change if we alter the lower limit of the first integral in the second line to $k \sim -1$, we obtain the factorization assumed above. Hence, in this $t \gg t_0$ limit we find that we obtain factorization of the next-to-leading twist solution and that as for the fixed coupling case this is negligible as $x \to 0$.

Even if t_0 approaches t the results can be shown to be similar by numerical calculation. For example, in the extreme limit of $t = t_0$ the first integral in the second line of (3.9) gives only half the saddle-point contribution, but one can check that the previously negligible second integral now gives a roughly equal contribution for all N. However, factorization is now clearly broken. Detailed numerical investigation shows that for t_0 not much smaller than t we can write the higher twist contribution in the form $(Q_0^2/Q^2)f(Q^2,Q_0^2,N)$ where the total is a function of N which grows slowly with N, approaching a constant as $N \to 0$. This is consistent with the form $x^a\cos(b\ln(1/x))$

which we get for the factorized next-to-leading twist solution (the Mellin transformation of which is $(N+a)/((N+a)^2+b^2)$), and certainly confirms that the gluon Green's function is falling as $x \to 0$.

Therefore, the higher twist operators and their anomalous dimensions derived from either the fixed coupling or running coupling BFKL equation are negligible at small x, and for these higher twist contributions the use of the running coupling equation does not qualitatively change anything. However, we are currently not able to say anything about the contributions from the four-gluon operators, and hence about shadowing corrections etc., beyond relatively simple results, e.g. anomalous dimensions in the small x limit at LO in α_s. There have been various suggestions that such shadowing corrections are large, but I feel that these estimates may well be severely exaggerated by the use of the approximation of this LO in α_s anomalous dimension, and also by the fact that the even more restrictive double-leading logarithmic approximation is often used. This often seriously overestimates the size of the anomalous dimensions, coefficient functions, and also the gluon distribution. I hope I have demonstrated that for the evolution of the higher twist two-gluon operator the LO-in-α_s double-leading-log approximations is indeed totally misleading. It is also interesting to note that a more complete calculation of the higher twist coefficient functions for the evolution of $F_2(x,Q^2)$ due to the four-gluon operators [32] implies that the double leading log approximation is a vast overestimate. Even using very small values of the screening length ($R = 2\text{GeV}^{-2}$ rather than the more usual $R \sim 10\text{GeV}^{-2}$) and the very large LO GRV gluon distribution [33], it seems that the shadowing correction is almost negligible in the perturbative HERA range. Saturation effects will no doubt eventually set in for low enough x and Q^2, but presently I feel the technology is not such as to predict where with any real accuracy. Certainly, resummations in $\ln(1/x)$ tend to decrease the size of the gluon extracted from data, and this combined with the above considerations suggests a much smaller saturation effect, and total higher twist effect, than often supposed. Certainly the model-independent "rule of thumb" for strong saturation contributions that $dF_2(x,Q^2)/d\ln Q^2 \approx Q^2\sigma(x)$ and hence $d\ln(F_2(x,Q^2))/d\ln Q^2 \approx 1$ is not even closely approached for any HERA data with $Q^2 \geq 1\text{GeV}^2$.

However, I note that in my examination of higher twist I have not examined the mixing between leading twist and higher twist operators or included any nonperturbative contributions due to, for example, the behaviour of the coupling constant at low scales. These two effects are related to each other. Such questions have been considered for toy models in [15] and [29], and numerically for the full LO running coupling BFKL equation [29]. These papers have considered the full anomalous dimension defined by $d\ln(\mathcal{G}(Q^2,N))/dt$, and the way this is affected by the higher twist corrections, rather than just $d\ln(\mathcal{G}^{HT}(Q^2,N))/dt$ considered above. They demonstrate that there are potentially serious modifications to the leading twist anomalous dimension due to the higher twist corrections introducing sensitivity to the form of the normalization factor $\mathcal{G}_I(Q_0^2,N)$ which depends on the regularization of the coupling at low scales and on the Q_0^2 dependence. Depending

on the assumptions about the nonperturbative physics, these contributions can be be important at extremely small x, generally changing the precise form of the power-like behaviour, and for more severe imposition of nonperturbative effects, i.e. letting them set in at higher scales, introducing a completely different asymptotic behaviour. Unfortunately, within the framework of my paper the formal divergence of $\mathcal{G}_I(Q_0^2, N)$ makes a similar study impossible and, as mentioned at the end of the previous section, I simply have to appeal to these alternative results, in particular the smallness of x at which the power-suppressed modifications set in, in order to support the reliability of my more formal calculations. However, I also note that the smallness of the higher twist operators and their anomalous dimensions calculated in this section suggest that whilst these contributions from non-perturbative sources only set in at low Q^2 or very small x indeed it seems perfectly possible that they will give a comparable, or even larger contribution at low x and low Q^2 than the genuine higher twist contributions.

4. NLO Corrections.

In section 2 I demonstrated that using $\alpha_s(k^2)$ in the BFKL equation, as in (2.11), has a profound effect on the form of the solution both for the normalization and for the anomalous dimension. However, given the first conclusions regarding NLO corrections in the essentially fixed coupling case, it is particularly necessary to check that the results presented are not severely modified by the inclusion of the NLO kernel, i.e. that the perturbative calculations are stable. The NLO kernel was presented in [11] and the way in which to solve at NLO with a running coupling was presented in [14]. Writing the NLO equation as

$$f(k^2, Q_0^2) = f_I(k^2, Q_0^2) + \left(\frac{\bar{\alpha}_s(k^2)}{N}\right) \int_0^\infty \frac{dq^2}{q^2} (K_0(q^2, k^2) - \alpha_s(k^2) K_1(q^2, k^2)) f(q^2), \tag{4.1}$$

and using just the one-loop expression for the coupling[7] leads to a 2nd order differential equation in γ-space

$$\frac{d^2 \tilde{f}(\gamma, N)}{d\gamma^2} = \frac{d^2 \tilde{f}_I(\gamma, Q_0^2)}{d\gamma^2} - \frac{1}{\bar{\beta}_0 N} \frac{d(\chi_0(\gamma) \tilde{f}(\gamma, N))}{d\gamma} - \frac{\pi}{3\bar{\beta}_0^2 N} \chi_1(\gamma) \tilde{f}(\gamma, N). \tag{4.2}$$

This can be solved in a very similar way to LO, i.e. it factorizes into the same form as (2.17) with Q^2-dependent part given by

$$\mathcal{G}_E^1(N, t) = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{1}{\gamma} \exp(\gamma t - X_1(\gamma, N)/(\bar{\beta}_0 N)) d\gamma. \tag{4.3}$$

[7] Using the full NLO expression for the running coupling would lead to a huge degree of complication, and this has never been attempted. Since, so long as Λ is chosen appropriately, the one- and two-loop couplings are very similar, I do not imagine any major errors in the results below.

However, $X_1(\gamma, N)$ is rather more complicated than the previous $X_0(\gamma)$. It can still be expressed in the form

$$X_1(\gamma, N) = \int_{\frac{1}{2}}^{\gamma} \chi_{NLO}(\hat{\gamma}, N) d\hat{\gamma}, \tag{4.4}$$

but now $\chi_{NLO}(\gamma, N)$ can be written as a power series in N beginning at zeroth order with $\chi_0(\gamma)$. As seen in [14], though here ignoring resummations in N, the explicit form is

$$\chi_{NLO}(\gamma, N) = \chi_0(\gamma) - N\frac{\chi_1(\gamma)}{\chi_0(\gamma)} + \frac{N^2}{\chi_0}\left(-\left(\frac{\chi_1(\gamma)}{\chi_0(\gamma)}\right)^2 - \beta_0\left(\frac{\chi_1(\gamma)}{\chi_0(\gamma)}\right)'\right) + \cdots, \tag{4.5}$$

where the currently unknown NNLO contribution to the kernel, $\chi_2(\gamma)$, would also appear at order N^2 in principle.

As already discussed in section 2 there is a contribution to $\chi_1(\gamma)$ from the β_0-dependent terms induced by an "incorrect" choice of the scale for the coupling – k^2 rather than $(k-q)^2$. Taking this contribution to the term in (4.5) which is linear in N, and combining with the LO expression we find the previously discussed result of only a minor change in the anomalous dimension and splitting function extracted. Hence, the choice of $\alpha_s(k^2)$ is reliable, and is easily corrected for. In this section I consider the rest of the NLO correction to the kernel, which is much larger, and henceforth I denote $\chi_1(\gamma)$ as the NLO kernel with the β_0-dependent part $\frac{1}{2}\bar{\beta}_0(\chi_0^2(\gamma)+\chi_0'(\gamma))$ already extracted, and include the multiplicative factor $f^{\beta_0}(\gamma)$ in the integrand in (4.3). This still leaves a decision as to precisely what I take "the NLO calculation" to mean. There are various possibilities. I could work at the level of the NLO correction to the kernel, and hence the BFKL equation, and solve (4.1), producing the infinite series in (4.5). Alternatively, I could truncate $\chi_{NLO}(\gamma, N)$ in (4.5) after the second term. However, doing this still leaves the question of whether to use the whole of $\exp(1/\bar{\beta}_0 \int_{\frac{1}{2}}^{\gamma}(\chi_1(\hat{\gamma})/\chi_0(\hat{\gamma}))d\hat{\gamma})$ or just expand it out to first order in $\bar{\beta}_0^{-1}$.

There are particular problems associated with all choices. If one solves using the full NLO corrected kernel then there is an infinite series in powers of N to consider in (4.5), which turns out to be important in practice (see below). Also, the gluon Green's function and anomalous dimensions obtained from this solution contain many subleading terms beyond just LO and NLO in $\ln(1/x)$ (and running coupling type corrections to these), as is essentially obvious from looking at (4.1) - iteration of f leads the last term producing NNLO then NNNLO and so on. Hence, this choice is disguarded. If one instead truncates (4.5) at order N, one still generates a subset of higher order terms beyond those one wishes, though it is possible to proceed in this case at least. One can see the explicit form of the solution by substituting the truncated (4.5) into (4.3) and proceeding as in section 2. The contribution to $X_1(\gamma, N)$ coming from the second term, $-N(\chi_1(\gamma)/\chi_0(\gamma))$, leads to an expression of the same form as in (2.21), i.e.

$$X_1(\gamma, N) = X_0(\gamma) - c_l N \ln(\gamma) - N c_0 - N\sum_{n=1}^{\infty} c_n \gamma^n, \tag{4.6}$$

where the c_n may be calculated easily by performing a power-series expansion of the known functions of γ, i.e.

$$\sum_{n=1}^{\infty} c_n \gamma^n = 0.424\gamma + 0.805\gamma^2 + 0.521\gamma^3 + 2.290\gamma^4 + 1.287\gamma^5 + 2.980\gamma^6 + \cdots . \tag{4.7}$$

Hence, the integrand for $\mathcal{G}_E^1(N, Q^2)$ becomes

$$\gamma^{-(1-c_l N)/(\bar{\beta}_0 N)-1} f^{\beta_0}(\gamma) \exp\biggl(\gamma t - \frac{1}{(\bar{\beta}_0 N)}(\gamma_E - c_0 N + \sum_{n=1}^{\infty}(a_n \gamma^{2n+1} - N c_n \gamma^n))\biggr). \tag{4.8}$$

Performing precisely the same type of manipulations as in section 2 results in the expression

$$\begin{aligned}\mathcal{G}_E^1(N,t) = &-\sin\biggl(\frac{\pi(1-c_l N)}{(\bar{\beta}_0 N)}\biggr)\Gamma(-(1-c_l N)/(\bar{\beta}_0 N)) \exp\biggl(-\frac{\gamma_E - c_0 N}{(\bar{\beta}_0 N)}\biggr) t^{(1-c_l N)/(\bar{\beta}_0 N)} \\ &\times\biggl(1 + \sum_{n=1}\sum_{m=1}\biggl[\biggl(1 + \tilde{A}_n(1/(\bar{\beta}_0 N))\biggr)\biggl(1 + C_m(1/\bar{\beta}_0)\biggr) - 1\biggr] t^{-n-m} \Delta_{n+m}\biggl(\frac{-(1-c_l N)}{(\bar{\beta}_0 N)}\biggr)\biggr),\end{aligned} \tag{4.9}$$

where

$$1 + \sum_{m=1}^{\infty} C_m(1/\bar{\beta}_0)\gamma^m = \exp\biggl(\frac{1}{\bar{\beta}_0}\sum_{n=1}^{\infty} c_n \gamma^n\biggr), \tag{4.10}$$

and the $\tilde{A}(1/(\bar{\beta}_0 N))$ include the contributions from $f^{\beta_0}(\gamma)$, i.e. are of the form in (2.35). The factoring of the terms independent of t then results in the expression

$$\begin{aligned}\mathcal{G}_E^1(N,t) = t^{(1-c_l N)/(\bar{\beta}_0 N)}&\biggl(1 + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\biggl[\biggl(1 + \tilde{A}_n(1/(\bar{\beta}_0 N))\biggr) \\ &\times\biggl(1 + C_m(1/\bar{\beta}_0)\biggr) - 1\biggr] t^{-n-m}\Delta_{n+m}\biggl(\frac{-(1-c_l N)}{(\bar{\beta}_0 N)}\biggr)\biggr).\end{aligned} \tag{4.11}$$

There are two sources of corrections beyond NLO in $\ln(1/x)$, other than running coupling corrections, in (4.11). Firstly, $C_n(1/\bar{\beta}_0)$ can be expanded as a power series in $1/(\bar{\beta}_0)$. Only the first term in this series is genuinely a NLO correction to the LO result. Terms of higher order lead to contributions to the anomalous dimensions which are beyond NLO in $\ln(1/x)$ without compensating factors of β_0 which would enable them to be interpreted as running coupling corrections. Secondly, when one expands terms of the form $((1-c_l N)/(\bar{\beta}_0 N))^n$ which appear in the Δ_n in (4.11), one obtains a power-series of the form,

$$\biggl(\frac{(1-c_l N)}{(\bar{\beta}_0 N)}\biggr)^n = \biggl(\frac{1}{(\bar{\beta}_0 N)}\biggr)^n\biggl[1 - n c_l N + \frac{n(n-1)}{2}(c_l N)^2 + \cdots\biggr]. \tag{4.12}$$

The second term in this series gives the NLO in $\ln(1/x)$ correction while the remainder give higher corrections without compensating powers of β_0. Therefore, both these power-series expansions, i.e of the C_n in powers of $1/(\bar{\beta}_0)$, and the Δ_n in powers of N should be stopped at first order in $\bar{\beta}_0^{-1}$

or N, and the cross-terms coming from first-order in both expansions, which are of overall second order, should be eliminated to obtain truly NLO results.[8]

Ultimately I define NLO by appealing to the perturbative form of the gluon Green's function and anomalous dimension produced and hence by choosing the NLO definition such that the Green's function does receive only corrections which are no more than one power of $\alpha_s(Q^2)$ (without compensating factors of β_0) down on the leading order one. This means using an expression for the gluon Green's function of the form

$$\begin{aligned}\mathcal{G}_E^1(N,t) = t^{(1-c_l N)/(\bar{\beta}_0 N)}\Bigg(1+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Bigg[\Big(1+\tilde{A}_n(1/(\bar{\beta}_0 N))\Big)\Big(1+c_n/\bar{\beta}_0\Big)-1\Bigg] \\ t^{-n-m}\Delta_{n+m}(-1/(\bar{\beta}_0 N)) - \frac{c_l}{\bar{\beta}_0}\sum_{n=1}^{\infty}\tilde{A}_n(1/(\bar{\beta}_0 N))t^{-n}\frac{d\,\Delta_n(-1/(\bar{\beta}_0 N))}{d(-1/(\bar{\beta}_0 N))}\Bigg),\end{aligned} \tag{4.13}$$

where the $c_n/\bar{\beta}_0$ are obtained by expanding the exponential expression $\exp(1/\bar{\beta}_0\big(\int_{\frac{1}{2}}^{\gamma}(\chi_1(\hat{\gamma})/\chi_0(\hat{\gamma})+ c_l + c_0\big))d\hat{\gamma})$, out to just first order in $1/\bar{\beta}_0$. Implicitly there is also a factor of $-\sin\Big(\frac{\pi(1-c_l N)}{(\bar{\beta}_0 N)}\Big)\Gamma\Big(\frac{-(1-c_l N)}{(\bar{\beta}_0 N)}\Big)\exp(-\gamma_E/(\bar{\beta}_0 N) + c_0/\bar{\beta}_0)$ which contributes to the normalization in (4.13).

Now that we have this NLO expression for the gluon Green's function it is necessary to make one more decision regarding the definition of the anomalous dimension. This is obtained from $\gamma^{LO+NLO}(N,t) = (d\ln(\mathcal{G}_E^1(N,t))/dt)$. However, strictly speaking, in order to obtain only NLO contributions to the anomalous dimension $(\mathcal{G}_E^1(N,t))^{-1}$ in this expression should be expanded only to NLO. This leads to a formal problem already pointed out in section 6 of [20]. Using the whole of $(\mathcal{G}_E^1(N,t))^{-1}$ in the expression for the anomalous dimension we notice that the position of the first zero is changed from that at LO, leading to a shift, in fact a decrease, in the leading pole for the anomalous dimension, and hence in the power of leading behaviour of the splitting function as $x \to 0$. So the $x \to 0$ behaviour of the splitting function becomes $P_{gg}(x) = \exp(\lambda_0\xi - \Delta\lambda\xi)$. However, since $\Delta\lambda$ is due to NLO corrections, the strict NLO expansion is just $P_{gg}(x) = \exp(\lambda_0\xi) - \Delta\lambda\xi\exp(\lambda_0\xi)$. This definition does not explicitly retain the shift in the power-like behaviour, and also leads to the NLO correction ultimately becoming larger than the LO result. Hence, I choose to retain the whole of $(\mathcal{G}_E^1(N,t))^{-1}$ in the definition of the NLO anomalous dimension, thus obtaining

[8] Ignoring this requirement and using the whole of (4.11), it turns out that the resultant expression is very badly behaved - blowing up at large N. This is almost entirely due to the higher-order terms in the expansion of the Δ_n. Using the full $C_n(1/\bar{\beta}_0)$ does not change things much in practice. This large N instability translates into huge corrections in the splitting function at large x. Presumably this instability at large N and x is cured if one resums the whole series in (4.5). Including just the $\mathcal{O}(N^2)$ term does seem to improve matters.

the full $P_{gg}(x) = \exp(\lambda_0\xi - \Delta\lambda\xi)$ as $x \to 0$, even though in practice the choice makes little difference at the values of x relevant to HERA.

So now I can use (4.13) to determine analytic expressions for the NLO gluon Green's function and anomalous dimension. However, the formal definition again results in a divergent power series, and as at LO I really truncate the series in (4.13) at $n_0 = 5$. This leaves the problem of calculating the power-suppressed corrections. In order to do this it is necessary to have an exact definition for $\mathcal{G}_E^1(N,t)$ in the form of an inverse Mellin transformation, as in (4.3). This requires finding the integral expression which would lead to (4.13) if a power-series expansion of the integrand is performed. Unfortunately this is not that simple. The problem comes with the manner of treating the $-c_l N \ln(\gamma)$ term in (4.6). In order to have the leading $t^{(1-c_l N)/(\bar{\beta}_0 N)}$ factor in (4.13), and hence obtain the correct expression for the $\mathcal{O}(\alpha_s(Q^2))$ part of the anomalous dimension, it is necessary to keep $-c_l N \ln(\gamma)$ in the exponential in the integrand, giving a factor $\gamma^{-c_l/\bar{\beta}_0}$. Expanding out $\exp(-c_l \ln(\gamma)/\bar{\beta}_0)$ to first order would lead to $\ln(t)$ contributions to the anomalous dimension. However, keeping the full $\gamma^{-c_l/\bar{\beta}_0}$ factor results in the argument of the Δ_n being $-(1-c_l N)/(\bar{\beta}_0 N)$ as in (4.11). Hence, there is no simple way to generate only NLO corrections from this term. In order to obtain an expression equivalent to (4.13) I choose to effectively put the known factor of $t^{(-c_l N/(\bar{\beta}_0 N))}$ in by hand and to generate the derivatives of the Δ_n within the integral with respect to γ.

In order to see how to do this I consider the LO expressions (2.17) and (2.26). It is quite simple to generate the first part of (4.13). All one needs do is insert the series expansion $1 + 1/(\bar{\beta}_0)\sum_{n=1}^{\infty} c_n \gamma^n$ expanded to first order in $1/\bar{\beta}_0$ into the integral representation, i.e.

$$\mathcal{G}_E^{1,I}(N,t) = \int_C \gamma^{-1/(\bar{\beta}_0 N)-1} f^{\beta_0}(\gamma) \exp\left(\gamma t - \frac{1}{(\bar{\beta}_0 N)} \sum_{n=1}^{\infty} a_n \gamma^{2n+1}\right)\left(1 + \sum_{m=0} (1/\bar{\beta}_0) c_m \gamma^m\right) d\gamma, \tag{4.14}$$

where the integral is over the full, unspecified contour, and generates the t-independent factor $\sin\left(\frac{-\pi}{(\beta_0 N)}\right)\Gamma\left(\frac{-1}{(\beta_0 N)}\right)$, as well as the t-dependent parts explicitly in (4.13). On top of this one must also insert the $t^{-c_l N/(\bar{\beta}_0 N)}$ factor by hand. If one is also concerned with the N-dependent normalization it is probably most consistent to also multiply by the factor

$$\frac{\sin(\pi(1-c_l N)/(\bar{\beta}_0 N))\Gamma(-(1-c_l N)/(\bar{\beta}_0 N))\exp(-(\gamma_E - c_0 N)/(\bar{\beta}_0 N))}{\sin(\pi/(\bar{\beta}_0 N))\Gamma(-1/(\bar{\beta}_0 N))}, \tag{4.15}$$

in order to obtain the overall factor of

$$-\sin((\pi(1-c_l N)/(\bar{\beta}_0 N))\Gamma(-(1-c_l N)/(\bar{\beta}_0 N))\exp\left(\frac{-\gamma_E + c_0 N}{\bar{\beta}_0 N}\right). \tag{4.16}$$

Generating the second part of (4.13) is rather more complicated. One has to somehow modify the integral representation so that the derivatives of the $\Delta_n(-1/(\bar{\beta}_0 N))$ are obtained. To see how

to do this we let $1/(\bar{\beta}_0 N) = z$, in which case the equivalence of (2.23) and (2.30) (ignoring the divergence of the series) is

$$\int_C \gamma^{-z-1} \exp\Big(\gamma t - z\sum_{n=1}^{\infty} a_n \gamma^{2n+1}\Big) d\gamma = -\sin(\pi z)\Gamma(-z) t^z \Big(1 + \sum_{n=3}^{\infty} A_n(z) t^{-n} \Delta(-z+n)\Big), \quad (4.17)$$

where I have removed the trivial factor of $\exp(-\gamma_E/(\bar{\beta}_0 N))$ from each side. Differentiating both sides with respect to z we obtain

$$\begin{aligned}
-\int_C \ln(\gamma)\gamma^{-z-1} \exp\Big(\gamma t - z\sum_{n=1}^{\infty} a_n \gamma^{2n+1}\Big) d\gamma \\
&- \int_C \gamma^{-z-1} \sum_{m=1}^{\infty} a_m \gamma^{2m+1} \exp\Big(\gamma t - z\sum_{n=1}^{\infty} a_n \gamma^{2n+1}\Big) d\gamma \\
&= \Psi(-z)\sin(\pi z)\Gamma(-z) t^z \Big(1 + \sum_{n=3}^{\infty} A_n(z) t^{-n} \Delta(-z+n)\Big) \\
&- \pi\cot(\pi z)\sin(\pi z)\Gamma(-z) t^z \Big(1 + \sum_{n=3}^{\infty} A_n(z) t^{-n} \Delta(-z+n)\Big) \\
&- \ln(t)\sin(\pi z)\Gamma(-z) t^z \Big(1 + \sum_{n=3}^{\infty} A_n(z) t^{-n} \Delta(-z+n)\Big) \\
&- \sin(\pi z)\Gamma(-z) t^z \Big(1 + \sum_{n=3}^{\infty} A_n(z) t^{-n} \frac{d\Delta(-z+n)}{dz}\Big) \\
&- \sin(\pi z)\Gamma(-z) t^z \Big(\sum_{n=3}^{\infty} \frac{dA_n(z)}{dz} t^{-n} \Delta(-z+n)\Big).
\end{aligned} \quad (4.18)$$

The last term on each side are equivalent, and rearranging the rest we obtain an expression for a series containing the derivatives of the $\Delta_n(z)$ -

$$\begin{aligned}
\sin(\pi z)\Gamma(-z) t^z \Big(\sum_{n=3}^{\infty} A_n(z) t^{-n} \frac{d\,\Delta(-z+n)}{dz}\Big) &= \int_C \ln(\gamma)\gamma^{-z-1} \exp\Big(\gamma t - z\sum_{n=1}^{\infty} a_n \gamma^{2n+1}\Big) d\gamma \\
&+ (\Psi(-z) - \pi\cot(\pi z) - \ln t)\sin(\pi z)\Gamma(-z) t^z \Big(1 + \sum_{n=3}^{\infty} A_n(z) t^{-n} \Delta(-z+n)\Big),
\end{aligned} \quad (4.19)$$

which using (4.17) becomes

$$\begin{aligned}
\sin(\pi z)\Gamma(-z) t^z \Big(\sum_{n=3}^{\infty} A_n(z) t^{-n} \frac{d\Delta(-z+n)}{dz}\Big) \\
&= \int_C \Big[\ln(\gamma) - (\Psi(-z) - \pi\cot(\pi z) - \ln t)\Big]\gamma^{-z-1} \exp\Big(\gamma t - z\sum_{n=1}^{\infty} a_n \gamma^{2n+1}\Big) d\gamma.
\end{aligned} \quad (4.20)$$

Therefore, the right-hand-side of (4.20), multiplied by $-c_l/(\bar{\beta}_0) t^{(-c_l N/(\bar{\beta}_0 N))}$, gives the second term in (4.13) with some t-independent normalization which should be multiplied by (4.16) to be consistent with the first term in the preceding paragraph. Thus, we have a prescription for the full

calculation at NLO which is equivalent to the series expansion in (4.13), i.e

$$\mathcal{G}_E^1(N,t) \propto t^{-c_l/\bar{\beta}_0} \int_C \Bigg[\gamma^{-1/(\bar{\beta}_0 N)-1} f^{\beta_0}(\gamma) \exp\Bigg(\gamma t - \frac{1}{(\bar{\beta}_0 N)} \sum_{n=1}^{\infty} a_n \gamma^{2n+1}\Bigg)\Bigg(1 + \sum_{m=0} (1/\bar{\beta}_0) c_m \gamma^m\Bigg)$$
$$- \frac{c_l}{\bar{\beta}_0}\Bigg[\ln(\gamma t) - \Psi\Bigg(-\frac{1}{\bar{\beta}_0 N}\Bigg) + \pi \cot\Bigg(\frac{\pi}{\bar{\beta}_0 N}\Bigg)\Bigg]\gamma^{-(1/(\bar{\beta}_0 N))-1} f^{\beta_0}(\gamma) \exp\Bigg(\gamma t - \frac{1}{\bar{\beta}_0 N} \sum_{n=1}^{\infty} a_n \gamma^{2n+1}\Bigg)\Bigg] d\gamma, \tag{4.21}$$

and once again one should multiply by (4.16) to get the most suitable normalization. We can now insert the above expression into $\gamma^{LO+NLO}(N,t) = (d \ln(\mathcal{G}_E^1(N,t))/dt)$ and evaluate numerically in order to get the NLO anomalous dimension without recourse to the truncated series expansion.

We are now in a position to solve for the anomalous dimension and splitting function at NLO. Unlike the case of fixed coupling, or the simplistic results of the saddle-point evaluation, the NLO corrections to the LO anomalous dimension are under control. This is simply illustrated by the positions of the leading pole in the anomalous dimensions which are shown in fig. 4, and one can see that they change from about 0.25 for $\gamma_{gg}(N,t)$ at LO to 0.17 at NLO, and that the Q^2-dependence reduces a little. However, as already noted at LO, the value of the intercepts has little to do with physics at HERA – the power-like behaviour only really settling down for lower x, and this is even more true at NLO. Being more particular one notices that the anomalous dimension $\gamma_{gg}(N,t)$ over a wide range of N shows only a relatively small change going from LO to NLO. This is shown in fig. 5.b where the part of the NLO anomalous dimension at first order in $\alpha_s(Q^2)$, i.e $-0.935\alpha_s(Q^2)$, is not included, since this should properly be included at LO in a combined leading order in $\alpha_s(Q^2)$ and $\alpha_s(Q^2)\ln(1/x)$ expansion scheme. Alternative definitions of NLO lead to very similar results except at very high values of N, where less sophisticated definitions lead to blowing up at large N, as already mentioned. For this case of the gluon structure function the NLO correction is negative except for very large N. I should also note that the power-like correction to the purely analytic result is a larger proportion of the NLO correction than of the LO contribution, but would still be almost impossible to spot if shown on fig. 5.b. The correction to the analytic value for the intercept is about 7% at $t = 6$ however.

One can also make the transformation to x-space and calculate the NLO corrected splitting function. Unfortunately, due to the increase in size of the c_n coefficients compared to the a_n (particularly the absence of zeros) and also to the factors of n invoked by differentiating the Δ_n in (4.11) the power-series in $\alpha_s(Q^2)$ is much less convergent than at LO. In order to obtain an expression which is reliable down to $x = 0.00001$ at $Q^2 = 1\text{GeV}^2$ it is necessary to go to 20th order in $\alpha_s(Q^2)$. Hence we can write the NLO correction to the splitting function as

$$xP_{gg}^{NLO}(\xi, \alpha_s(Q^2)) = \bar{\alpha}_s(Q^2) \sum_{n=1}^{19} \sum_{m=0}^{m_{max}} \bar{\alpha}_s^n(Q^2)\Bigg(K_{nm} \frac{\xi^m \bar{\beta}_0^{n-m-1}}{m!} + K_{n\delta} \bar{\beta}_0^n \delta(1-x)\Bigg), \tag{4.22}$$

where because we have truncated the series for the gluon structure function m_{max} can be greater than the naive expectation of $m_{max} = n - 1$. The coefficients for the series are shown in table 1. If one is only concerned with $x > 0.0001$ or $Q^2 > 4\text{GeV}^2$ then the series can be truncated at about $12th$ order.

As at LO we also have to model the N dependence of the power-suppressed correction by an analytic function. Fortunately, exactly the same type of function is sufficient and we obtain the power-suppressed NLO correction to the splitting function of the form

$$\begin{aligned}-2.86\exp(-1.02t)\delta(1-x) + \exp(-t)\Bigg(13.59\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{0.88} - 29.61\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{1.21}\xi \\ + 39.76\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{1.315}\frac{\xi^2}{2!} - 33.765\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{1.48}\frac{\xi^3}{3!} + 16.89\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{1.77}\frac{\xi^4}{4!} \\ - 4.479\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{2.16}\frac{\xi^5}{5!} + 0.4839\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{2.63}\frac{\xi^6}{6}\Bigg).\end{aligned} \tag{4.23}$$

The full NLO correction $xP_{gg}^{NLO}(x)$ and its power-series and power-suppressed contributions are shown in fig. 9.a, where the relatively unimportant terms $\propto \delta(1-x)$ are absent. As at LO the power-suppressed correction is proportionally much larger in x-space than in moment space and certainly needs to be considered at $t = 6$ and below. Also as at LO it tends to oppose the form of the power-series expression, hence reducing the total NLO correction. The powers of α_s in (4.23) are slightly smaller than for LO, and hence the power-suppressed correction does not fall quite so quickly with Q^2.

The total NLO splitting function, i.e LO plus the NLO correction, is shown for $t = 6$ in fig. 9.b, where the contributions $\propto \delta(1-x)$ both from the $\mathcal{O}(\alpha_s(Q^2))$ part and the running coupling corrections to this are absent. The latter of these is a very small contribution. The NLO corrected splitting function is clearly not qualitatively different from that at LO, though it is quite a lot smaller at small x. Hence it seems as though by including the infinite series of running coupling corrections the perturbative expansion of the BFKL splitting function has been stabilized. However, the real importance of the NLO corrections as far as physics is concerned is the effect they have on the evolution of the gluon structure function. This is demonstrated in fig. 10 where the evolution of a suitable model for the structure function $G(x, Q^2)$, i.e. $(1-x)^6 x^{-0.2}$, is shown both for the LO running coupling splitting function, and for the NLO corrected one (all $\delta(1-x)$ contributions other that at first order in $\alpha_s(Q^2)$ one are included). Also shown is the evolution due just to the double-leading-log term $P(x) = \bar{\alpha}_s(Q^2)/x$. As one sees, at this (fairly low) value of t, i.e. $Q^2 \sim 6\text{GeV}^2$, the evolution driven by the LO splitting function is very similar to that from the double-leading-log contribution, and is even slightly smaller for x from 0.007 to 0.00001, corresponding to the dip in the splitting function seen in fig. 6. Below this the growth of the splitting function increases the evolution above the double-leading-log result. One also sees that the effect of the NLO corrections is

certainly significant, and increases relatively with falling x, but it is clearly a correction rather than the complete change in qualitative behaviour induced by the NLO corrections without resummation of running coupling effects.

A further way often used to investigate the perturbative stability of a fixed order perturbative calculation is to investigate the renormalization-scale dependence. This is often used fallaciously, e.g. if one calculates $P_{gg}(\alpha_s, x)$ to NLO in the standard perturbative expansion and then investigates variation of renormalization scales one will never notice the influence of the terms at higher orders in α_s which are also of higher order in $\ln(1/x)$. This is symptomatic of the fact that the expansion purely in powers of α_s is not really a correct expansion scheme for splitting functions (for a full discussion see [8]). However, once we have performed a resummation of large logarithms, as here, renormalization-scale variation should be more reliable. The renormalization scheme dependence may be investigated by letting

$$\alpha_s(Q^2) \to \alpha_s(kQ^2) + \beta_0 \ln(k) \alpha_s^2(kQ^2) \tag{4.24}$$

and in the LO part of the splitting function expanding out to first order in $\ln(k)$, whilst in the NLO part using only the zeroth order, i.e. just letting $\alpha_s(Q^2) \to \alpha_s(kQ^2)$. In this case we must also use a similar procedure for the power-suppressed corrections, i.e. these are really of the form (Λ^2/μ_R^2) rather than (Λ^2/Q^2). The results for $k = 0.5$ and $k = 2$ are shown in fig. 11 for $Q^2 \sim 6\mathrm{GeV}^2$. As with the NLO corrections to LO the variation is significant but leads only to a correction rather than a qualitative change. This implies that the series expansion is stable, if not as rapidly converging as one might ideally hope for.

Hence, the NLO corrections to the running coupling BFKL derived splitting function are well under control, both in terms of the asymptotic power-like behaviour of the splitting functions and in terms of the evolution in the range currently accessible to experiments. For deep-inelastic scattering, or indeed any process where there is factorization of the infrared physics into the input parton distributions, e.g. Drell-Yan scattering in proton-proton collisions, no further resummation is necessary, or even useful, beyond the running coupling corrections. This in distinct contrast to the case where both ends of the gluon ladder are associated with a hard scale. In this case the conventional BFKL expansion is fundamentally flawed due to progressively higher order poles at $\gamma = 0$ and $\gamma = 1$ (corresponding to large logs in the ratios of the two scales k_1^2 and k_2^2) as shown in [13]. These large order poles need to be resummed, and without this resummation calculations are badly behaved over the whole range of N (in fact explicit calculation shows that this is particularly the case at large N). In the case of deep inelastic scattering the collinear factorization procedure automatically orders the poles at $\gamma = 0$ correctly, and the above problem shows up in high order poles at $\gamma = 1$ only. The anomalous dimension is totally dominated by the region very close to $\gamma = 0$, as this paper shows, and is very insensitive to effects at $\gamma = 1$. Including the type of resummation in [13][14] alters results from the NLO corrected case by only a very small amount, and is likely to be

no more influential than the remaining NNLO effects for which it does not account. Resummation of poles near $\gamma = 1$ would be essential if one attempted to obtain information about the input form of the gluon, i.e. $\mathcal{G}_I(Q_0^2, N)$. However, as well as the fact that Q_0^2 is an essentially nonperturbative scale, this type of calculation, along with the whole subject of single-scale processes, is also plagued by the infrared ambiguity problem caused by behaviour of the coupling at low scales. A discussion of such issues can be found in [15] and [29].

I close this section by noting that although the above results all look promising it is important to realize that they are all in a sense ambiguous because they deal with a particular way of defining the gluon parton distribution, which is a factorization scheme-dependent quantity. In this paper it is defined in a manner which is natural from the point of view of the solution of the BFKL equation, and which one may think of as perhaps a good "physical" definition of the gluon. However, it is very different from, for example, the gluon defined in the $\overline{\rm MS}$ scheme. In order to investigate the real success of the approach in this paper it is necessary to look at the results for the real physical quantities – the structure functions.

5. Small x Structure Functions.

One may define a real structure function by a simple extension of the above methods, i.e. by including a hard scattering cross section at the top of the gluon ladder. This modifies (2.4) to

$$\mathcal{F}_i(Q^2, N) = \alpha_s \int_0^\infty \frac{dk^2}{k^2} \sigma_{i,g}(k^2/Q^2) f(N, k^2, Q_0^2) g_B(N, Q_0^2), \tag{5.1}$$

where $\sigma_{i,g}(k^2/Q^2)$ is the cross-section for scattering of a virtual photon from a gluon with transverse momentum k^2. For the case of the longitudinal structure function this cross-section is well defined even in the limit $k^2 \to 0$, but for $\mathcal{F}_2(N, Q^2)$ the cross-section diverges like $\ln(Q^2/k^2)$ as $k^2 \to 0$ (for details see [34]). This demonstrates that for $\mathcal{F}_L(x, Q^2)$ the solution in the leading $1/N$ limit factorizes neatly into the gluon distribution and a multiplicative coefficient function, while for $\mathcal{F}_2(N, Q^2)$ there is interference at this order between the coefficient function and the result of solving the evolution equation including the anomalous dimension $\alpha_s \gamma_{qg}^0(\alpha_s, N)$. In this latter case it is simplest instead to differentiate with respect to $\ln(Q^2)$ obtaining

$$\frac{d\mathcal{F}_2(Q^2, N)}{d\ln Q^2} = \alpha_s \int_0^\infty \frac{dk^2}{k^2} \frac{d\sigma_{i,g}(k^2/Q^2)}{d\ln Q^2} f(N, k^2, Q_0^2) g_B(N, Q_0^2), \tag{5.2}$$

where $\frac{d\sigma_{i,g}(k^2/Q^2)}{d\ln Q^2}$ is finite as $k^2 \to 0$. In this case, if we work in an DIS-type scheme, i.e. one in which the quark-gluon coefficient function vanishes beyond zeroth order, there is a simple factorization between the anomalous dimension $\alpha_s \gamma_{qg}^0(\alpha_s, N)$ and the gluon distribution.[9]

[9] Note that in this article I ignore the mixing with the quark input distribution in general for simplicity. However, it does implicitly appear in the NLO correction to the kernel, i.e. it is the NLO correction to the anomalous dimension eigenvalue rather than to γ_{gg} which I use since this is the quantity directly calculated in [11][12]. The contribution to this due to the quark mixing is very small in practice.

In order to progress it is first necessary to consider the overall factor of α_s in the above expressions, and particularly its scale. One might think that it should be $\alpha_s(k^2)$, and thus appear within the integrals with respect to k^2. However, this could only come about due to double counting of diagrams, since the resummation of bubble diagrams required to make this equal to $\alpha_s(k^2)$ has already been performed in defining the coupling in the BFKL equation as $\alpha_s(k^2)$. Q^2 is the only remaining scale, so it must be the scale of this coupling. One can also justify this by considering the fact that there is a NLO correction to the input of the BFKL equation of the form $-\beta_0\alpha_s\ln(Q_0^2/\mu_R^2)\delta(k^2-Q_0^2)$ (coming from bubbles in a gluon propagator). Introducing this into calculations leads to multiplying each result by a factor $(1-\beta_0\alpha_s\ln(Q_0^2/\mu_R^2))$. This splits into $-\beta_0\alpha_s\ln(Q^2/\mu_R^2)+\beta_0\alpha_s\ln(Q^2/Q_0^2)$, and the latter term is an infrared divergence which contributes to the one-loop gluon-gluon splitting function while the second goes into making the overall factor of α_s have renormalization scale Q^2.

Now removing the overall factor of $\alpha_s(Q^2)$ (or in fact the normalization factor $\alpha_s(Q^2)N_f/(3\pi)$) from (5.1), and taking the Mellin transformation with respect to (Q^2/Λ^2) leads to the simple expression

$$\tilde{\mathcal{F}}_i(\gamma,N)=h_{i,g}(\gamma)\tilde{\mathcal{G}}(\gamma,N). \tag{5.3}$$

Thus we may solve for $\mathcal{F}_i(N,t)$ in exactly the same way as for $\mathcal{G}(N,t)$, obtaining exactly the same divergent Q^2-independent part and a Q^2-dependent part given by solving

$$\mathcal{F}_{E,i}(N,t)=\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{h_{i,g}(\gamma)}{\gamma}f^{\beta_0}(\gamma)\exp(\gamma t-X_0(\gamma)/(\bar{\beta}_0N))d\gamma. \tag{5.4}$$

This may be evaluated numerically, using the same contour as for the gluon, or in order to find the power-series solution we may proceed as with the gluon structure function by expanding the $h_{i,g}(\gamma)$ (which were calculated in [34]) as a power series about $\gamma=0$. For the two cases we discussed above we have

$$h_{L,g}(\gamma)f^{\beta_0}(\gamma)=1-0.33\gamma+2.13\gamma^2+0.67\gamma^3+2.58\gamma^4+2.99\gamma^5+1.92\gamma^6+\cdots, \tag{5.5}$$

and

$$h_{2,g}(\gamma)f^{\beta_0}(\gamma)=1+2.17\gamma+2.30\gamma^2+6.67\gamma^3+7.05\gamma^4+12.92\gamma^5+15.47\gamma^6+\cdots. \tag{5.6}$$

It seems natural to absorb the (in some sense) NLO corrections from $f^{\beta_0}(\gamma)$ into the contributions from the $h_{i,g}(\gamma)$ since they are of exactly the same form, whereas the other NLO corrections have inverse powers of β_0. Following the same steps as in section 2.2 then results in an expression

$$\mathcal{F}_{E,i}(N,t)=t^{1/(\bar{\beta}_0N)}\left(1+\sum_{n=1}^{n_0}B_{i,n}(1/(\bar{\beta}_0N))t^{-n}\Delta_n(-1/(\bar{\beta}_0N))\right), \tag{5.7}$$

where the $B_{i,n}(1/(\bar{\beta}_0 N))$ are now determined not only by the power series in γ obtained from the expansion of $X_0(\gamma)$, but also from the expansion of $h_{i,g}(\gamma)$. In particular they now contain parts at zeroth order in $1/(\bar{\beta}_0 N)$.

Using these results it is now a simple matter to derive the longitudinal gluon coefficient function at leading powers of $\ln(1/x)$ plus running coupling corrections and similarly for the quark-gluon anomalous dimension, i.e.

$$C_{L,g}(\alpha_s(Q^2), N) = \frac{\alpha_s(Q^2) N_f}{3\pi} \frac{\mathcal{F}_{E,L}(N,t)}{\mathcal{G}_E(N,t)}, \tag{5.8}$$

with obvious generalization to $\gamma_{qg}(\alpha_s(Q^2), N)$. These moment space expressions may easily be converted to x-space. Truncating the series for the structure functions and the gluon at $n_0 = 5$ results in the perturbative series for $xC_{L,g}(\alpha_s(Q^2), x)$,

$$\begin{aligned}
xC_{L,g}(\alpha_s(Q^2),x) = \frac{\alpha_s(Q^2)N_f}{3\pi}\Bigg[&\delta(1-x) - 0.33\alpha_s(Q^2) + 2.13\alpha_s^2(Q^2)\big(\xi - \bar{\beta}_0\big) \\
&+ \alpha_s^3(Q^2)\big(-0.933\frac{\xi^2}{2!} + 2.79\bar{\beta}_0\xi - 1.86\bar{\beta}_0^2\big) + \alpha_s^4(Q^2)\big(2.32\frac{\xi^3}{3!} - 14.69\bar{\beta}_0\frac{\xi^2}{2!} + 27.85\bar{\beta}_0^2\xi - 15.48\bar{\beta}_0^3\big) \\
&+ \alpha_s^5(Q^2)\big(8.41\frac{\xi^4}{4!} - 54.45\bar{\beta}_0\frac{\xi^3}{3!} + 125.2\bar{\beta}_0^2\frac{\xi^2}{2!} - 121.2\bar{\beta}_0^3\xi + 42.0\bar{\beta}_0^4\big) \\
&+ \alpha_s^6(Q^2)\big(\frac{-0.89}{\bar{\beta}_0}\frac{\xi^6}{6!} + 7.76\frac{\xi^5}{5!} - 27.53\bar{\beta}_0\frac{\xi^4}{4!} + 49.48\bar{\beta}_0^2\frac{\xi^3}{3!} - 44.59\bar{\beta}_0^3\frac{\xi^2}{2!} + 15.77\bar{\beta}_0^4\xi\big) \\
&+ \alpha_s^7(Q^2)\big(\frac{2.74}{\bar{\beta}_0}\frac{\xi^7}{7!} - 33.41\frac{\xi^6}{6!} + 164.8\bar{\beta}_0\frac{\xi^5}{5!} - 419.3\bar{\beta}_0^2\frac{\xi^4}{4!} + 577.2\bar{\beta}_0^3\frac{\xi^3}{3!} - 404.9\bar{\beta}_0^4\frac{\xi^2}{2!} + 112.9\bar{\beta}_0^5\xi\big) \\
&+ \alpha_s^8(Q^2)\big(\frac{6.48}{\bar{\beta}_0}\frac{\xi^8}{8!} - 72.27\frac{\xi^7}{7!} + 335.7\beta_0\frac{\xi^6}{6!} - 838.2\bar{\beta}_0^2\frac{\xi^5}{5!} + 1210\bar{\beta}_0^3\frac{\xi^4}{4!} - 1004\bar{\beta}_0^4\frac{\xi^3}{3!} \\
&+ 441.7\bar{\beta}_0^5\frac{\xi^2}{2!} - 79.05\bar{\beta}_0^6\xi\big)\Bigg].
\end{aligned} \tag{5.9}$$

However, as for the gluon splitting function we have to calculate the power-suppressed correction by evaluating the inverse Mellin transformations numerically. This is done in precisely the same way as for the gluon, and results in the correction to $xC_{L,g}(\alpha_s(Q^2), x)$ of the form

$$\begin{aligned}
\frac{\alpha_s(Q^2)N_f}{3\pi}\Bigg[&(-1.168 - 0.482t + 0.1106)\exp(-t)\delta(1-x) + \exp(-t)\Bigg(-4.685\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{-3.026} \\
&+ 34.25\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{-0.875}\xi - 59.47\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{0.074}\frac{\xi^2}{2!} + 45.81\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{0.78}\frac{\xi^3}{3!} \\
&- 17.94\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{1.37}\frac{\xi^4}{4!} + 3.365\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{1.77}\frac{\xi^5}{5!} - 0.2942\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{1.78}\frac{\xi^6}{6}\Bigg)\Bigg],
\end{aligned} \tag{5.10}$$

where in this case it was necessary to model the $N \to \infty$, i.e. the $\delta(1-x)$ part with a slightly more complicated form than previously. Both expressions have been shown in a form which is sufficient

for $Q^2 > 1\text{GeV}^2$ and $x > 0.00001$. The full $xC_{L,g}(x,t)$ is shown in fig. 12 a. along with the two contributions above. Note that the $\delta(1-x)$ term at $\mathcal{O}(\alpha_s(Q^2))$ in the power-series is obtained from the inverse Mellin transformation of the limit as $N \to 0$ of the full $\mathcal{O}(\alpha_s(Q^2))$ coefficient function and in the figure we replace it by the full $\mathcal{O}(\alpha_s(Q^2))$ contribution, $6x^2(1-x)$, for ease of presentation (it not being easy to represent the normalization of the $\delta(1-x)$ term). The $\delta(1-x)$ term is simply missing from the power-suppressed part, though this is insignificant. We see that the power-suppressed contribution is now a much larger fraction of the total than for the gluon, though it does not increase as quickly with falling Q^2. In fig. 12 b. we show $xC_{L,g}(x,t)$ along with the $\mathcal{O}(\alpha_s(Q^2))$ contribution and with the naive LO BFKL result in this factorization scheme, which grows far more quickly than the resummed result.

Similarly we can calculate the perturbative series $xP_{qg}(\alpha_s(Q^2),x)$,

$$
\begin{aligned}
xP_{qg}(\alpha_s(Q^2),x) = \frac{\alpha_s(Q^2)N_f}{3\pi}\Big[&\delta(1-x) + 2.17\alpha_s(Q^2) + 2.30\alpha_s^2(Q^2)\big(\xi - \bar\beta_0\big)\\
&+ \alpha_s^3(Q^2)\big(5.07\frac{\xi^2}{2!} - 15.21\bar\beta_0\xi + 10.14\bar\beta_0^2\big) + \alpha_s^4(Q^2)\big(8.80\frac{\xi^3}{3!} - 47.50\bar\beta_0\frac{\xi^2}{2!} + 81.02\bar\beta_0^2\xi - 42.30\bar\beta_0^3\big)\\
&+ \alpha_s^5(Q^2)\big(18.88\frac{\xi^4}{4!} - 156.7\bar\beta_0\frac{\xi^3}{3!} + 478.0\bar\beta_0^2\frac{\xi^2}{2!} - 620.4\bar\beta_0^3\xi + 280.3\bar\beta_0^4\big)\\
&+ \alpha_s^6(Q^2)\big(\frac{4.95}{\bar\beta_0}\frac{\xi^6}{6!} - 44.15\frac{\xi^5}{5!} + 159.9\bar\beta_0^1\frac{\xi^4}{4!} - 293.4\bar\beta_0^2\frac{\xi^3}{3!} + 269.7\bar\beta_0^3\frac{\xi^2}{2!} - 97.03\bar\beta_0^4\xi\big)\\
&+ \alpha_s^7(Q^2)\big(\frac{7.98}{\bar\beta_0}\frac{\xi^7}{7!} - 86.53\frac{\xi^6}{6!} + 385.6\bar\beta_0\frac{\xi^5}{5!} - 899.9\bar\beta_0^2\frac{\xi^4}{4!} + 1153\bar\beta_0^3\frac{\xi^3}{3!} - 764.0\bar\beta_0^4\frac{\xi^2}{2!} + 203.8\bar\beta_0^5\xi\big)\\
&+ \alpha_s^8(Q^2)\big(\frac{17.15}{\bar\beta_0}\frac{\xi^8}{8!} - 234.6\frac{\xi^7}{7!} + 1354\beta_0\frac{\xi^6}{6!} - 4263\bar\beta_0^2\frac{\xi^5}{5!} + 7882.9\bar\beta_0^3\frac{\xi^4}{4!} - 8519\bar\beta_0^4\frac{\xi^3}{3!}\\
&+ 4962\bar\beta_0^5\frac{\xi^2}{2!} - 1199\bar\beta_0^6\xi\big)\\
&+ \alpha_s^9(Q^2)\big(\frac{3.97}{\bar\beta_0^2}\frac{\xi^{10}}{10!} - \frac{51.57}{\bar\beta_0}\frac{\xi^9}{9!} + 269.5\frac{\xi^8}{8!} - 647.5\bar\beta_0\frac{\xi^7}{7!} + 258.8\beta_0^2\frac{\xi^6}{6!} + 2451\bar\beta_0^3\frac{\xi^5}{5!}\\
&- 6962\bar\beta_0^4\frac{\xi^4}{4!} + 8473\bar\beta_0^5\frac{\xi^3}{3!} - 5145\bar\beta_0^6\frac{\xi^2}{2!} + 1259\bar\beta_0^7\xi\big)\Big].
\end{aligned}
\tag{5.11}
$$

and we have a power-suppressed contribution to $xP_{qg}(\alpha_s(Q^2),x)$ of the form

$$
\begin{aligned}
\frac{\alpha_s(Q^2)N_f}{3\pi}\Big[&12.86\exp(-1.521t)\delta(1-x) + \exp(-t)\bigg(-14.31\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{2.695} + 36.297\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{2.93}\xi\\
&- 41.14\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{3.03}\frac{\xi^2}{2!} + 25.34\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{3.20}\frac{\xi^3}{3!} - 9.096\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{3.44}\frac{\xi^4}{4!}\\
&+ 1.85\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{3.695}\frac{\xi^5}{5!} - 0.1693\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{3.80}\frac{\xi^6}{6}\bigg)\Big].
\end{aligned}
\tag{5.12}
$$

The full $xP_{qg}(\alpha_s(Q^2), x)$ is shown in fig. 13 a. along with the two contributions above. As with $xC_{L,g}(x,t)$ the $\delta(1-x)$ term at $\mathcal{O}(\alpha_s(Q^2))$ in the power-series is replaced by the full $\mathcal{O}(\alpha_s(Q^2))$ contribution which is $1.5x(x^2+(1-x)^2)$. Again the $\delta(1-x)$ term is missing from the power-suppressed part, and again this is insignificant. In this case the power-suppressed part is tiny at $t=6$, though from the large powers of $\alpha_s(Q^2)$ in (5.12) we see that it grows very quickly at lower Q^2. In fig. 13 b. we show $xP_{qg}(x,t)$ along with the $\mathcal{O}(\alpha_s(Q^2))$ contribution and with the naive LO BFKL result in this factorization scheme, which again grows far more quickly than the resummed result.

These above results, along with the LO gluon splitting function, allow for a LO in $\ln(1/x)$ (with running coupling corrections) calculation and analysis of structure functions. In previous papers [8] I have strongly warned against the use of factorization-scheme dependent splitting functions and coefficient functions within the $\ln(1/x)$ expansion. It is still true that it is always possible to make huge redefinitions of the unphysical parton distributions by factorization scheme changes at a given order (or even at all orders) but the changes invoked by transfer between the commonly used schemes are diminished somewhat by the reduction of the size of the splitting functions and coefficient functions by the inclusion of the running coupling effects. It is also true that many of the changes invoked by factorization scheme changes are themselves due to running coupling effects, and the resummation of these stabilizes the whole procedure a great deal. Hence, it is now possible to work in terms of these unphysical quantities if one wishes, without potential disasters, as long as the ordering of the expressions is done with particular care. Nevertheless, it is still very convenient in some ways to eliminate the partons completely and work directly in terms of the structure functions $F_L(x,Q^2)$ and $F_2(x,Q^2)$ and the physical anomalous dimensions [22]. In fact we can easily argue a case for improved stability. At LO the longitudinal coefficient function is positive and quite large at small x, and hence $F_L(x,Q^2)$ will be enhanced compared to the gluon at small x. At NLO the gluon evolution is smaller than at LO. Hence, evolving down from a given gluon at very high Q^2 (where everything is simpler and more reliable) the NLO gluon will be larger at small Q^2 than the LO gluon. However, we expect the NLO corrections to $C_{L,g}(x,Q^2)$ to be negative, and thus counteract this increase in the NLO gluon in the calculation of $F_L(x,Q^2)$. Hence $F_L(x,Q^2)$ is (probably) a more stable perturbative quantity at small x than $G(x,Q^2)$.

The physical anomalous dimension which is most closely related to the gluon anomalous dimension is

$$\Gamma_{LL}(N,t) = \frac{d\ln(\mathcal{F}_L(N,t))}{dt}. \tag{5.13}$$

Ignoring the mixing with the quark sector this is given in terms of the parton related quantities by

$$\Gamma_{LL}(N,t) = \gamma_{gg}(N,t) + \frac{d\ln(C_{L,g}(N,t))}{dt}, \tag{5.14}$$

where I will use the convention of ignoring the overall power of $\alpha_s(Q^2)$ in the coefficient function which would just result in a single contribution of $-\beta_0\alpha_s^2(Q^2)$ to (5.14). Using the LO $\gamma_{gg}(N,t)$ plus running coupling corrections, and similarly for $C_{L,g}(N,t)$ we see that the latter gives entirely running coupling corrections, and the total is the LO $\gamma_{gg}(N,t)$ with an extended set of running coupling corrections. This total expression could be calculated from the $\gamma_{gg}(N,t)$ and $C_{L,g}(N,t)$ already calculated, but part of the advantage in using physical anomalous dimensions is that it reduces the number of perturbative quantities governing the structure function evolution, i.e. the four splitting functions and four coefficient functions used to define $F_2(x,Q^2)$ and $F_L(x,Q^2)$ are reduced to four truly independent physical splitting functions. Hence, we notice that using (5.4) for the longitudinal structure function we can calculate $\Gamma_{LL}(N,t)$ and $P_{LL}(x,t)$ directly, rather than from (5.14). Of course, the two definitions are equivalent, but the latter allows a single power-suppressed correction to be calculated rather than having to combine those for $\gamma_{gg}(N,t)$ and $C_{L,g}(N,t)$ and thus the potential error is minimized. The asymptotic power-like behaviour for $P_{LL}^{LO}(x,t)$ is not identical to that of $P_{gg}^{LO}(x,t)$ and is shown in fig. 4. The difference is only relatively minor, but one sees that the power-like growth for $F_L(x,Q^2)$ is slightly smaller than for the gluon, and is also slightly less Q^2-dependent. The result for the LO in $\ln(1/x)$ power-series solution $xP_{LL}^{LO}(\alpha_s(Q^2),x)$ is unfortunately a little less convergent than the previous LO quantities, due to large coefficients generated in taking the derivative with respect to t of the expression for $\mathcal{F}_L(N,t)$ (or of $C_{L,g}(N,t)$). Hence, in order to obtain an expression which is sufficiently accurate for $Q^2 > 1\text{GeV}^2$ and $x > 0.00001$ we need to go to about 12th order. This results in the explicit

expression

$$
\begin{aligned}
xP_{LL}^{LO}(\alpha_s(Q^2),x) &= \bar{\alpha}_s(Q^2) + 0.333\alpha_s^2(Q^2)\bar{\beta}_0 + \alpha_s^3(Q^2)\big(-4.157\bar{\beta}_0\xi + 4.266\bar{\beta}_0^2\big) \\
&+ \alpha_s^4(Q^2)\big(2.4\frac{\xi^3}{3!} - 11.29\bar{\beta}_0\frac{\xi^2}{2!} + 12.94\bar{\beta}_0^2\xi - 4.02\bar{\beta}_0^3\big) \\
&+ \alpha_s^5(Q^2)\big(0.121\bar{\beta}_0\frac{\xi^3}{3!} + 37.85\bar{\beta}_0^2\frac{\xi^2}{2!} - 99.88\bar{\beta}_0^3\xi + 61.92\bar{\beta}_0^4\big) \\
&+ \alpha_s^6(Q^2)\big(2\frac{\xi^5}{5!} - 75.14\bar{\beta}_0\frac{\xi^4}{4!} + 454.7\bar{\beta}_0^2\frac{\xi^3}{3!} - 1034\bar{\beta}_0^3\frac{\xi^2}{2!} + 1011\bar{\beta}_0^4\xi - 358.8\bar{\beta}_0^5\big) \\
&+ \alpha_s^7(Q^2)\big(\frac{1.92}{\bar{\beta}_0}\frac{\xi^7}{7!} - 13.94\frac{\xi^6}{6!} + 23.68\bar{\beta}_0\frac{\xi^5}{5!} - 39.48\bar{\beta}_0^2\frac{\xi^4}{4!} + 121.9\bar{\beta}_0^3\frac{\xi^3}{3!} - 155.2\bar{\beta}_0^4\frac{\xi^2}{2!} + 61.14\bar{\beta}_0^5\xi\big) \\
&+ \alpha_s^8(Q^2)\big(-16.91\frac{\xi^7}{7!} + 348.8\beta_0\frac{\xi^6}{6!} - 2087\bar{\beta}_0^2\frac{\xi^5}{5!} + 5522\bar{\beta}_0^3\frac{\xi^4}{4!} - 7305\bar{\beta}_0^4\frac{\xi^3}{3!} + 4754\bar{\beta}_0^5\frac{\xi^2}{2!} - 1215\bar{\beta}_0^6\xi\big) \\
&+ \alpha_s^9(Q^2)\big(\frac{2.56}{\bar{\beta}_0}\frac{\xi^9}{9!} - 119.5\frac{\xi^8}{8!} + 1173\bar{\beta}_0\frac{\xi^7}{7!} - 5052\beta_0^2\frac{\xi^6}{6!} + 12044\bar{\beta}_0^3\frac{\xi^5}{5!} - 17444\bar{\beta}_0^4\frac{\xi^4}{4!} \\
&+ 15528\bar{\beta}_0^5\frac{\xi^3}{3!} - 7859\bar{\beta}_0^6\frac{\xi^2}{2!} + 1728\bar{\beta}_0^7\xi\big) \\
&+ \alpha_s^{10}(Q^2)\big(\frac{1.536}{\bar{\beta}_0^2}\frac{\xi^{11}}{11!} - \frac{16.73}{\bar{\beta}_0}\frac{\xi^{10}}{10!} + 83.37\frac{\xi^9}{9!} - 492.2\bar{\beta}_0\frac{\xi^8}{8!} + 1559\bar{\beta}_0^2\frac{\xi^7}{7!} + 2043\beta_0^3\frac{\xi^6}{6!} - 24427\bar{\beta}_0^4\frac{\xi^5}{5!} \\
&+ 61280\bar{\beta}_0^5\frac{\xi^4}{4!} - 72753\bar{\beta}_0^6\frac{\xi^3}{3!} + 42720\bar{\beta}_0^7\frac{\xi^2}{2!} - 9998\bar{\beta}_0^8\xi\big) \\
&+ \alpha_s^{11}(Q^2)\big(-\frac{18.53}{\bar{\beta}_0}\frac{\xi^{11}}{11!} + 444.0\frac{\xi^{10}}{10!} - 2988\bar{\beta}_0\frac{\xi^9}{9!} + 5290\bar{\beta}_0^2\frac{\xi^8}{8!} - 22253\bar{\beta}_0^3\frac{\xi^7}{7!} - 135896\beta_0^4\frac{\xi^6}{6!} \\
&+ 321404\bar{\beta}_0^5\frac{\xi^5}{5!} - 425485\bar{\beta}_0^6\frac{\xi^4}{4!} + 330620\bar{\beta}_0^7\frac{\xi^3}{3!} - 141370\bar{\beta}_0^8\frac{\xi^2}{2!} + 25747\bar{\beta}_0^9\xi\big) \\
&+ \alpha_s^{12}(Q^2)\big(\frac{2.82}{\bar{\beta}_0^2}\frac{\xi^{13}}{13!} - \frac{141.7}{\bar{\beta}_0}\frac{\xi^{12}}{12!} + 1757\frac{\xi^{11}}{11!} - 10347\bar{\beta}_0\frac{\xi^{10}}{10!} + 39345\bar{\beta}_0^2\frac{\xi^9}{9!} - 119096\bar{\beta}_0^3\frac{\xi^8}{8!} \\
&+ 295058\bar{\beta}_0^4\frac{\xi^7}{7!} - 538834\beta_0^5\frac{\xi^6}{6!} + 658339\bar{\beta}_0^6\frac{\xi^5}{5!} - 499685\bar{\beta}_0^7\frac{\xi^4}{4!} + 211914\bar{\beta}_0^8\frac{\xi^3}{3!} - 38311\bar{\beta}_0^9\frac{\xi^2}{2!}\big).
\end{aligned}
\tag{5.15}
$$

The power-suppressed correction is calculated in the usual manner and is of the form

$$
\begin{aligned}
&36.57\exp(-1.75t)\delta(1-x) + \exp(-t)\bigg(4.626\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{-2.78} - 37.84\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{-0.58}\xi \\
&\quad + 67.22\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{0}\frac{\xi^2}{2!} - 51.30\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{0.17}\frac{\xi^3}{3!} + 18.82\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{-0.01}\frac{\xi^4}{4!} \\
&\quad - 3.136\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{-0.69}\frac{\xi^5}{5!} + 0.1706\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{-2.27}\frac{\xi^6}{6}\bigg).
\end{aligned}
\tag{5.16}
$$

The anomalous dimension $\Gamma_{LL}^{LO}(N,t)$ is plotted in fig. 14.a. Until N is very small it is similar to $\gamma_{gg}^{LO}(N,t)$ and both are close to the common $\alpha_s(Q^2)/N$ contribution, though $\Gamma_{LL}^{LO}(N,t)$ is a little larger at large N. However, at lower N $\Gamma_{LL}^{LO}(N,t)$ dips below the others before eventually

rising above $\alpha_s(Q^2)/N$ but staying below $\gamma_{gg}^{LO}(N,t)$. Clearly the effect of the additional coefficient function, and hence additional running coupling corrections, is to make $\Gamma_{LL}(N,t)$ dip significantly below the $\mathcal{O}(\alpha_s(Q^2))$ contribution $\bar{\alpha}_s(Q^2)/N$ for a region and to reduce the value of the intercept compared to the gluon structure function. The effective splitting function $xP_{LL}^{LO}(x,t)$ is shown in fig. 15. In fig. 15.a we see that the power-suppressed contribution is larger for $xP_{LL}^{LO}(x,t)$ than it was for $xP_{gg}^{LO}(x,t)$. In fig. 15.b we see the outcome of the comparison of the anomalous dimensions for F_L and the gluon. $xP_{LL}^{LO}(x,t)$ starts a little higher at $x=0$ and the dip below the $\mathcal{O}(\alpha_s(Q^2))$ part is considerably more pronounced than for $xP_{gg}^{LO}(x,t)$. Also, going to $x \sim 10^{-5}$, we see that the splitting function dips again, showing that the subleading poles in the anomalous dimension may have large residues compared to the leading pole, and that the increase in $xP_{LL}^{LO}(x)$ with decreasing x is not monotonic. This corresponds to the significant fall of $\Gamma_{LL}(N,t)$ below $\bar{\alpha}_s(Q^2)/N$ at $N \sim 0.6$. The eventual rise of $\Gamma_{LL}(N,t)$ guarantees that the splitting function will eventually rise again with the calculated intercept, i.e. like $x^{-0.23}$, at even smaller x. However, for $t=6$ this asymptotic power behaviour does not set in until $x < 10^{-10}$ and in the region of $x \sim 10^{-7}$ $xP_{LL}^{LO}(x)$ even becomes slightly negative. For higher t even smaller x is required, e.g. $t=8$ ($Q^2 \approx 30\text{GeV}^2$) needs x to become as low as 10^{-13} before the power-like behaviour sets in, though the size of the dip before this is smaller than for $t=6$. This illustrates very clearly that as far as phenomenology at HERA, or any foreseeable collider, is concerned the value of the intercept for the anomalous dimension is simply not relevant to the evolution of structure functions. Indeed, it is very possible that before the power-like behaviour has set it unitarization effects have already become important. For collider phenomenology it is the splitting functions over the relevant x and Q^2 range which one needs, and this requires the sort of detailed calculation in this paper.

One can follow exactly the same procedure for the other important physical anomalous dimension defined by

$$\frac{\partial \mathcal{F}_2(N,Q^2)}{\partial \ln Q^2} = \Gamma_{2L}(Q^2,N)\mathcal{F}_L(N,Q^2), \tag{5.17}$$

simply by using the LO expressions for $\frac{d\mathcal{F}_2(N,Q^2)}{d\ln Q^2}$ and $\mathcal{F}_L(N,t)$. The power-like behaviour as $x \to 0$ is governed by the poles in $\mathcal{F}(N,t)$ as in the previous case, so the position of the intercepts is identical. The power-series expression requires the first 10 powers in order to be valid over the

required range of x and Q^2, so I write it as

$$
\begin{aligned}
xP_{2L}^{LO}(\alpha_s(Q^2),x) = \Big[&\delta(1-x) + 2.5\alpha_s(Q^2) + \alpha_s^2(Q^2)\big(\xi - 0.167\bar{\beta}_0\big) \\
&+ \alpha_s^3(Q^2)\big(\frac{\xi^2}{2!} - 12.72\bar{\beta}_0\xi + 12.0\bar{\beta}_0^2\big) + \alpha_s^4(Q^2)\big(7.007\frac{\xi^3}{3!} - 41.41\bar{\beta}_0\frac{\xi^2}{2!} + 61.42\bar{\beta}_0^2\xi - 26.82\bar{\beta}_0^3\big) \\
&+ \alpha_s^5(Q^2)\big(5.78\frac{\xi^4}{4!} - 52.95\bar{\beta}_0\frac{\xi^3}{3!} + 253.0\bar{\beta}_0^2\frac{\xi^2}{2!} - 444.1\bar{\beta}_0^3\xi + 238.32\bar{\beta}_0^4\big) \\
&+ \alpha_s^6(Q^2)\big(\frac{5.80}{\bar{\beta}_0}\frac{\xi^6}{6!} - 87.30\frac{\xi^5}{5!} + 409.7\bar{\beta}_0\frac{\xi^4}{4!} - 773.3\bar{\beta}_0^2\frac{\xi^3}{3!} + 621.7\bar{\beta}_0^3\frac{\xi^2}{2!} - 176.6\bar{\beta}_0^4\xi\big) \\
&+ \alpha_s^7(Q^2)\big(\frac{9.348}{\bar{\beta}_0}\frac{\xi^7}{7!} - 117.8\frac{\xi^6}{6!} + 591.4\bar{\beta}_0\frac{\xi^5}{5!} - 1701\bar{\beta}_0^2\frac{\xi^4}{4!} + 2792\bar{\beta}_0^3\frac{\xi^3}{3!} - 2315\bar{\beta}_0^4\frac{\xi^2}{2!} + 741.5\bar{\beta}_0^5\xi\big) \\
&+ \alpha_s^8(Q^2)\big(\frac{-4.380}{\bar{\beta}_0}\frac{\xi^8}{8!} + 52.41\frac{\xi^7}{7!} + 70.68\beta_0\frac{\xi^6}{6!} - 1954\bar{\beta}_0^2\frac{\xi^5}{5!} + 6623\bar{\beta}_0^3\frac{\xi^4}{4!} - 9500\bar{\beta}_0^4\frac{\xi^3}{3!} \\
&+ 6307\bar{\beta}_0^5\frac{\xi^2}{2!} - 1596\bar{\beta}_0^6\xi\big) \\
&+ \alpha_s^9(Q^2)\big(\frac{4.64}{\bar{\beta}_0^2}\frac{\xi^{10}}{10!} - \frac{95.79}{\bar{\beta}_0}\frac{\xi^9}{9!} + 657.3\frac{\xi^8}{8!} - 1775.7\bar{\beta}_0\frac{\xi^7}{7!} + 450.0\beta_0^2\frac{\xi^6}{6!} + 9410\bar{\beta}_0^3\frac{\xi^5}{5!} \\
&- 26327\bar{\beta}_0^4\frac{\xi^4}{4!} + 33805\bar{\beta}_0^5\frac{\xi^3}{3!} - 21743\bar{\beta}_0^6\frac{\xi^2}{2!} + 5614\bar{\beta}_0^8\xi\big) \\
&+ \alpha_s^{10}(Q^2)\big(\frac{7.478}{\bar{\beta}_0^2}\frac{\xi^{11}}{11!} - \frac{115.7}{\bar{\beta}_0}\frac{\xi^{10}}{10!} + 765.7\frac{\xi^9}{9!} - 3293\bar{\beta}_0\frac{\xi^8}{8!} + 8687\beta_0^2\frac{\xi^7}{7!} - 5511\bar{\beta}_0^3\frac{\xi^6}{6!} \\
&- 35038\bar{\beta}_0^4\frac{\xi^5}{5!} + 104536\bar{\beta}_0^5\frac{\xi^4}{4!} - 127198\bar{\beta}_0^6\frac{\xi^3}{3!} + 74260\bar{\beta}_0^8\frac{\xi^2}{2!} - 17101\bar{\beta}_0^9\xi\big)\Big].
\end{aligned}
\tag{5.18}
$$

The power-suppressed correction is

$$
\begin{aligned}
&(3.558 + 0.4216t - 0.1542t^2)\exp(-t)\delta(1-x) + \exp(-t)\bigg(72.17\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{0.93}x \\
&\quad - 78.03\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{1.66} + 56.85\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{2.66}\xi - 24.16\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{0.58}\frac{\xi^2}{2!} \\
&\quad + 13.50\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{2.50}\frac{\xi^3}{3!} - 10.32\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{-2.27}\frac{\xi^4}{4!} + 3.918\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{-1.584}\frac{\xi^5}{5!} \\
&\quad - 0.5141\bigg(\frac{\alpha_s(t)}{\alpha_s(4.5)}\bigg)^{-1.05}\frac{\xi^6}{6!}\bigg),
\end{aligned}
\tag{5.19}
$$

where it is necessary to introduce a term $\propto x$ in order to get a good description at high N. The full $xP_{2L}(x,t)$ is shown in fig. 16. a along with the two contributions above. The $\delta(1-x)$ term is replaced in the power series by the x-dependence in the $\mathcal{O}(\alpha_s(Q^2))$ quark-gluon splitting function, i.e. $x(x^2+(1-x)^2)$, normalized by 1.5 to give the correct $N \to 0$ limit. This corresponds to a slight modification of the usual physical anomalous dimension in terms of the $\mathcal{O}(\alpha_s(Q^2))$ longitudinal gluon coefficient function, but may be viewed as an analytic function with the correct $N \to 0$ limit

which aids presentation here.[10] The $\delta(1-x)$ terms in the power-suppressed contribution are very small, and are simply left out. In fig. 16.b we see $xP_{2L}(x,t)$ plotted as a function of x along with the naive LO BFKL calculation with coupling $\alpha_S(Q^2)$, and in order to illustrate the contribution of the higher-order terms, also the zeroth order contribution $1.5x(x^2+(1-x)^2)$. As with $P_{LL}(x,t)$ one can see that $P_{2L}(x,t)$ has a dip at small x before the eventual power-like growth sets in, again only for $x < 10^{10}$, and as with all calculated quantities the running coupling corrections severely diminish the strength of the small x growth.

We can also try to investigate the effect of NLO corrections on physical quantities. In terms of partons the only known NLO correction is that to the gluon splitting function – there is simply no information on the NLO corrections to coefficient functions or the quark splitting functions. In terms of the physical anomalous dimensions, similarly there is no real information for $\Gamma_{2L}(N,t)$, but the situation is better for $\Gamma_{LL}(N,t)$. Let us look at the expression in terms of the partonic quantities (5.14), for the moment in the leading $\ln(1/x)$ expansion without resummed running coupling corrections. At LO in $1/N$, $\Gamma_{LL}^{LO}(N,t)$ is equal to $\gamma_{gg}^{LO}(N,t)$ since the differentiation of the log of the coefficient function with respect to t automatically introduces an extra factor of $\beta_0\alpha_s(Q^2)$. At NLO in $1/N$ $\Gamma_{LL}^{NLO}(N,t)$ picks up a contribution from $\gamma_{gg}^{NLO}(N,t)$ which is (largely) independent of the running coupling, and the contribution from the derivative of the LO coefficient function, which is entirely running coupling dependent. Hence, by knowing $\gamma_{gg}^{NLO}(N,t)$ we know the whole of $\Gamma_{LL}^{NLO}(N,t)$ before resuming running coupling corrections. Hence, we might hope that using an expression of the form (5.4), but corrected in the way described in the previous section for the NLO corrections to the kernel, we might calculate the full NLO, running coupling corrected BFKL expression for $\Gamma_{LL}(N,t)$. Unfortunately, this is not quite the case. This can be appreciated by again using (5.14). When solving this NLO corrected expression for $\mathcal{F}_{E,L}(N,t)$ one includes all the running coupling corrections to $\gamma_{gg}^{NLO}(N,t)$ just by the manner of solving the equation. But without knowing the NLO correction to the coefficient function one misses a whole series of terms of the form $\alpha_s(Q^2)(\bar{\beta}_0\alpha_s(Q^2))^n f(\bar{\alpha}_s(Q^2)/N)$ which would come form the $\frac{d\ln(C_{L,g}(N,t))}{dt}$ term.[11] Thus, we do not yet know the full running coupling corrections to the NLO contribution to $\Gamma_{LL}(N,t)$.

I will proceed to calculate the "NLO" corrected $\Gamma_{LL}(N,t)$ on the assumption that since the resummation of the running coupling corrections stabilizes the perturbative expansion the missing

[10] This modification to the physical splitting function will be discussed in a future paper.

[11] Some of these are automatically generated by using the NLO kernel in our solution, but the full set requires also the NLO correction to the hard scattering cross-section which will lead to NLO corrections to $h_{L,g}(\gamma)$.

running coupling corrections will not lead to anything other than minor corrections. It is straightforward to generalize the results of section 4 to the case of the physical quantity. Essentially we just replace (4.21) by

$$\mathcal{F}^1_{E,L}(N,t) \propto t^{-c_l/\bar{\beta}_0} \int_C \Bigg[\gamma^{-1/(\bar{\beta}_0 N)-1} h_{L,g}(\gamma) f^{\beta_0}(\gamma) \exp\left(\gamma t - \frac{1}{(\bar{\beta}_0 N)} \sum_{n=1}^{\infty} a_n \gamma^{2n+1}\right)\left(1 + \sum_{m=0} (1/\bar{\beta}_0) c_m \gamma^m\right)$$
$$- \frac{c_l}{\bar{\beta}_0}\left[\ln(\gamma t) - \Psi\left(-\frac{1}{\bar{\beta}_0 N}\right) + \pi \cot\left(\frac{\pi}{\bar{\beta}_0 N}\right)\right] \gamma^{-(1/(\bar{\beta}_0 N))-1} h_{L,g}(\gamma) f^{\beta_0}(\gamma) \exp\left(\gamma t - \frac{1}{\bar{\beta}_0 N} \sum_{n=1}^{\infty} a_n \gamma^{2n+1}\right)\Bigg] d\gamma, \tag{5.20}$$

where we are currently missing a further term of the form

$$-N t^{-c_l/\bar{\beta}_0} \int_C \gamma^{-1/(\bar{\beta}_0 N)-1} \delta h_{L,g}(\gamma, \bar{\beta}_0 N) f^{\beta_0}(\gamma) \exp\left(\gamma t - \frac{1}{(\bar{\beta}_0 N)} \sum_{n=1}^{\infty} a_n \gamma^{2n+1}\right). \tag{5.21}$$

Using (5.20) we can calculate both the power-series and power-suppressed NLO contributions to $\Gamma_{LL}(N,t)$ and hence $P_{LL}(x,t)$. The LO +"NLO" values of the intercept for the asymptotic power-like behaviour are shown in fig. 4. These lie very slightly below the LO+NLO intercepts for the gluon, and hint at perhaps a more rapid convergence for the physical F_L than for the gluon. However, we would expect the missing contributions to lower the intercept a little more. The "NLO" corrected anomalous dimension $\Gamma^{LO+NLO}_{LL}(N,t)$ is shown as a function of N for $t = 6$ in fig. 14.b. It is very similar to that at LO until very low N where the difference in the leading intercept starts to become apparent.

As for the NLO correction to $xP_{gg}(x,t)$ the power-series is not very convergent an to work all the way down to $Q^2 = 1\text{GeV}^2$ and $x = 0.00001$ we again need the first 20 or so terms. Hence the power-series contribution is

$$xP^{NLO}_{LL}(\alpha_s(Q^2), x) = \bar{\alpha}_s(Q^2) \sum_{n=1}^{19} \sum_{m=0}^{m_{max}} \bar{\alpha}_s^n(Q^2) \left(K_{nm} \frac{\xi^m \bar{\beta}_0^{n-m-1}}{m!} + K_{n\delta} \bar{\beta}_0^n \delta(1-x)\right), \tag{5.22}$$

where the coefficients are listed in table 2. The power-suppressed contribution is

$$\begin{aligned}(-0.183\exp(-0.51t)\delta(1-x) + \exp(-t)\Bigg(&31.90\left(\frac{\alpha_s(t)}{\alpha_s(4.5)}\right)^{-0.274} - 80.22\left(\frac{\alpha_s(t)}{\alpha_s(4.5)}\right)^{0.346}\xi \\ &+ 56.67\left(\frac{\alpha_s(t)}{\alpha_s(4.5)}\right)^{0.60}\frac{\xi^2}{2!} + 9.017\left(\frac{\alpha_s(t)}{\alpha_s(4.5)}\right)^{3.15}\frac{\xi^3}{3!} - 25.925\left(\frac{\alpha_s(t)}{\alpha_s(4.5)}\right)^{1.715}\frac{\xi^4}{4!} \\ &+ 10.28\left(\frac{\alpha_s(t)}{\alpha_s(4.5)}\right)^{1.875}\frac{\xi^5}{5!} - 1.298\left(\frac{\alpha_s(t)}{\alpha_s(4.5)}\right)^{2.09}\frac{\xi^6}{6}\Bigg).\end{aligned} \tag{5.23}$$

The NLO correction to the splitting function $xP^{NLO}_{LL}(x,t)$ is shown, minus the contributions $\propto \delta(1-x)$, in fig. 17.a. Clearly there is a very large cancellation between the power-series and power-suppressed contributions resulting in a relatively small total NLO correction. We can see that

unlike for the the gluon this NLO correction is actually positive in some regions of x, rather than everywhere negative. We also see from fig. 17.b that the NLO splitting function is quite similar to the LO splitting function over the whole x range.

However, as with the gluon, the real test of perturbative stability is the evolution of the structure function itself. This is shown in fig. 18 where the evolution of a model for the structure function $F_L(x,Q^2)$, i.e. $(1-x)^6 x^{-0.2}$, is shown both for the LO running coupling splitting function, and for the "NLO" corrected one (all $\delta(1-x)$ contributions other that at first order in $\alpha_s(Q^2)$ are included). Also shown is the evolution due just to the double-leading-log term $P(x) = \bar{\alpha}_s(Q^2)/x$. Compared to the evolution of the gluon shown in the previous section we see that the additional running coupling contributions due to the t-derivative of the coefficient function have slowed the LO evolution below that of the double-leading-log result over the whole range of x (except very high x), and this will only cease to be true at very small x indeed, when the power-like growth of the physical splitting function finally sets in. In this case, however, the difference between LO and LO+"NLO" is much smaller than for the gluon, and the perturbative expansion seems very stable indeed. As with the NLO corrections to the intercepts this might be a sign that the expansion converges more quickly for the physical structure functions than for the unphysical gluon structure function. However, as a note of caution, the missing contributions at NLO are likely to be negative in general, and this difference between LO and NLO evolution will probably be increased a little. In fact it is desirable for these missing contributions to be non-negligible. Whilst if we decrease t to 4.5, i.e. $Q^2 \sim 1\text{GeV}^2$, at NLO everything remains relatively stable for the gluon, the physical splitting function starts to develop extreme behaviour at this low scale – the minimum at $x \sim 0.01$ becomes much lower and the peak at $x \sim 0.0001$ becomes very much higher. This trend is illustrated in fig. 11 b. which shows the renormalization scale dependence of $P_{LL}^{LO+NLO}(x,t)$ for $t = 6$. Clearly there is very good stability for an increase in scale, but it is not so good for a decrease in scale (though since the splitting function oscillates, the variation washes out to a large extent when evolution is calculated). There is very good stability in both directions if one examines the variation for a slightly higher t, say $t = 8$ ($Q^2 \sim 30\text{GeV}^2$). This instability in the physical splitting function results in instabilities in the evolution at $t = 4.5$, even though it appeared to be very stable at $t = 6$. Hopefully, the inclusion of the missing terms will help stabilize this evolution, though it may simply be a sign that at this low Q^2 some non-perturbative contribution is becoming essential.

6. Conclusions.

In this paper I have shown that it is possible to obtain analytic solutions to the LO running coupling BFKL equation for the Q^2-dependent parts of the gluon structure function and for the real physical structure functions $F_2(x,Q^2)$ and $F_L(x,Q^2)$. This results in a resummation of the leading

$\ln(1/x)$ terms at each power in $\alpha_s(Q^2)$ and also of the leading powers in β_0 at each power of $\alpha_s(Q^2)$ and $\ln(1/x)$. However, the Q_0^2-dependent gluon input is plagued by contamination from infrared nonperturbative physics, and has an inherent ambiguity of $\mathcal{O}(\Lambda^2/Q_0^2)$. The analytic expressions may be expressed in form of a power-series in $\alpha_s(Q^2)$. In practice the main features of the solution are almost completely determined by only the first handful (~ 5) of terms in the expansion, in complete contrast with the case of fixed coupling, where an all orders summation is needed. In fact the perturbative series for the structure functions is not convergent, and the analytic expression is most accurately obtained by this truncation. The small remainder, which roughly speaking is suppressed by powers of (Λ^2/Q^2), may be calculated from the difference between a numerical solution with the analytic solution, and then modelled by an analytic expression of Q^2 and N, which may easily be transformed to x-space. There are two points to note here. Firstly, this power-suppressed condition is both well-defined and is nothing to do with higher twist operators. Even though there are infrared (and ultraviolet) renormalons in the untruncated perturbative expansion, they only appear due to the impossibility of expressing the Q^2-dependent part of the the structure functions as a power series in $\alpha_s(Q^2)$, not because of some inherent ambiguity at leading twist as is often the case with renormalons. Hence, they are circumvented completely by this manner of calculation. Secondly, this procedure of an analytic calculation as a truncated power series plus a numerical calculation of the power-suppressed part, which is then modelled, seems to allow for the most accurate determination of x-space quantities. Transformation of numerical moment space expressions to x-space are subject to errors, and the magnification of the power-suppressed contributions in x-space, compared to moment space, seen in this paper highlights the potential effect of small errors in moment space when ultimately working in x-space. Hence, obtaining as accurate an analytic moment space expression as possible is vital in ultimately obtaining good accuracy for splitting functions and the evolution of structure functions.

It is also demonstrated that there are well-defined, calculable higher-twist contributions due to the transverse degrees of freedom of the two-gluon operator. However, both the normalization and splitting functions of these genuinely higher twist operators decrease quickly as $x \to 0$ (roughly like $x^{0.5}\cos(0.5\ln(1/x))$) when the small x resummation is performed. Unlike leading twist, this is largely insensitive to the running coupling corrections. This result is only apparent from resummation, and a fixed (small) order in $\alpha_s(Q^2)$, particularly first order only, gives very misleading results. Hence, this one form of higher twist does not lead to any sizable correction at all at small x and Q^2. It is possible that this unambiguous, small-x vanishing higher-twist contribution to the two-gluon operator is responsible for the absence of a genuine ambiguity in the leading twist anomalous dimensions. However, I note that this paper has nothing to say about the size of shadowing corrections coming from four gluon operators, except to point out that the double-leading-log type calculations often performed are likely to lead to huge overestimations. Neither does it consider the

power-suppressed corrections due to nonperturbative effects which mix with higher twist, leading to mixing with leading twist, and may well be important at extremely small x [15][29].

The calculated expressions for leading twist structure functions may be used to produce LO expressions for the splitting functions and coefficient functions for physical processes, and also the physical splitting functions which allow one to work directly in terms of physical quantities. My results prove that the effect of the running of the coupling is to weaken the asymptotic power-like growth of the splitting functions severely compared to the naive BFKL results, and even to lower the splitting function below the $\alpha_s(Q^2)/x$ contribution for $0.001 \gtrsim x \gtrsim 0.2$. It is also noted that the asymptotic behaviour of the form $x^{-\lambda}$ is often not approached even approximately until $x << 0.00001$, with the required x decreasing with increasing Q^2, and is therefore by no means a good indicator of physics at present or future colliders. In fact it is very likely that unitarization will stop this true power-like behaviour ever being seen. Rather than the intercept, the detailed expressions for the splitting functions and coefficient functions are needed in order to really calculate the evolution at realistic values of x.

The procedure can also be extended to NLO without any real modification, though there is some ambiguity in precisely what the best definition of NLO is.[12] The choice is made so that the expressions for the structure functions are genuinely only a single power of $\alpha_s(Q^2)$ down on LO, up to $\beta_0\alpha_s(Q^2)$ corrections, but in $\gamma(N,t) = d\ln(G(N,t))/dt$ the full NLO expression for $(G(N,t))^{-1}$ is used, rather than truncating its expansion at NLO, and hence the full NLO correction to the intercept is obtained. This has little effect until extremely small x. Unlike leading $\ln(1/x)$ calculations without resummation of running coupling effects the NLO correction to the gluon splitting function here is moderate, both for the value of the intercept and for the exact size of the splitting function and the evolution of gluon structure function for $x > 10^{-5}$. Hence, this running coupling resummation does a great deal to stabilize the perturbative series. Unfortunately it is not yet possible to calculate the complete NLO correction to any real physical quantity, though one may come close for $P_{LL}(x,t)$, the splitting function governing the evolution of the longitudinal structure function in terms of itself, which is very similar to $P_{gg}(x,t)$. In this case only a subset of the running coupling corrections to the NLO in $\ln(1/x)$ part are still unknown. For F_L the stability of the perturbative series looks even better than for the gluon as long as $Q^2 \gtrsim 4\mathrm{GeV}^2$, but begins to deteriorate below this, perhaps due to the missing corrections.

Let me also comment briefly on other methods which attempt to incorporate the NLO corrections (and beyond) to the BFKL equation. Firstly I note that my previous conjecture that the effect of the running coupling in the BFKL equation could be accounted for using an x-dependent scale for the coupling [20], resulting in falling coupling for decreasing x, turns out to be essentially correct

[12] The power-series expressions also become very complicated at NLO. It will probably ultimately be more convenient to model them accurately with some simpler function of x and t similar to the manner in which the power-suppressed contributions are treated at present.

so long as the change in the scale of the coupling is moderate compared to the scale itself, though it fails if this condition is not satisfied. In practice this condition is identical to that specifying that diffusion in the fixed coupling BFKL equation is not too large, and therefore that the virtualities sampled in the running coupling equation are not too far away from Q^2. This results in the requirement that $t^3 \gtrsim 20 \ln(1/x)$ [35]. This is true for all but the lowest x and Q^2 at HERA. I also note that my approach is completely consistent with that in [14][15], with both being built upon the running coupling BFKL equation essentially introduced long ago [23][24][17][18] and generalized beyond LO in [14]. The differences to this approach are that I ignore the collinear resummation which is a central theme in this work, since as I stress it is an unnecessary complication in the calculation of splitting functions - the running coupling effects being the most important and stabilizing the calculation themselves; that I concentrate on solving very accurately and precisely for the Q^2-dependent part of the gluon and structure functions, obtaining splitting functions over the range of x and Q^2 relevant for a phenomenological treatment; and that I also ignore the complication of a real regularization of the coupling in the infrared region (this latter point is also considered in [36]). Hence, I obtain detailed accurate results for all splitting functions and coefficient functions in closed form, but ignore contributions considered in these papers which are necessary if investigating single-scale processes and/or potential nonperturbative effects (which may be important for splitting functions at low Q^2 and very small x [29]). There is less similarity with other approaches. Even though that in [37] claims to in some sense be dealing with the scale appropriate for the coupling in this problem, it has no overlap with the approach in this paper, and comments on this approach can be found in [20]. Also there is no connection with the approach in [38] which adopts a phenomenological approach to resummation beyond fixed orders in $\ln(1/x)$ in terms of the asymptotic power-like behaviour, which is a free parameter, and which consequently loses true predictive power for the evolution at small x. Finally, there also seems to be no overlap with the approach in the first part of [39] which incorporates subleading effects via a kinematic constraint while solving the BFKL equation - resulting in an anomalous dimension which includes a resummation of some subset of higher order contributions, none of which is concerned with the running of the coupling, but which stabilizes the calculation. (The latter part of [39] also includes a running coupling and infrared regularization, but concentrates on the normalization rather than the evolution.) In this sense it has some similarities to the resummation of collinear logs in [13], which also stabilizes results even with fixed coupling (and which is essential in single scale processes). Hence, there appear to be a number of ways in which the apparent poor convergence of the perturbative series at small x can be improved. However, since one must ultimately deal with the contribution of the running coupling in all perturbative QCD calculations I prefer to concentrate on this feature and consider just the resulting β_0 resummation combined with the $\ln(1/x)$ resummation, which results in explicit results in terms of an ordered power-series in the well-defined quantities $\alpha_s(Q^2)$, $\ln(1/x)$ and β_0. This stabilizes the small x expansion without consideration of these other effects, indeed

it leads to the most divergent terms as $x \to 0$ [20] and alters the complete singularity structure, and moreover is easy to directly incorporate into the usual calculation of partons and structure functions in terms of the coefficient functions and splitting functions.

It will, of course, be interesting to examine the effect of incorporating my resummed corrections to splitting function in a global fit to structure function and related data. Such an analysis will also need to include a precise explanation of how the small x relevant expansions derived in this paper must be combined with the normal order-by-order in $\alpha_s(Q^2)$ expansion, and potentially large $\ln(1-x)$ expansions. Full details of such a fit, and the complete procedure used, will appear in a future paper which awaits the release of new data from a number of experimental collaborations. From the analysis of presently published data it is clear that the quality of such a fit is improved by inclusion of these small x resummed corrections[13], and that the predicted $F_L(x,Q^2)$ is smaller than that form a NLO-in-$\alpha_s(Q^2)$ fit, but much more regular in shape at low Q^2 than that seen in [5].[14] This can be seen as a solution to the lack of convergence of $F_L(x,Q^2)$ apparent as one goes from LO to NLO to NLO in the conventional expansion scheme which is seen in [5].

Hence, I conclude by claiming that this paper outlines a method for including the most complete resummation of splitting functions (and coefficient functions) which is needed at small x, and satisfies the theoretical requirements of stability of the perturbative expansion and the minimum of model dependence as well as the more practical considerations of being in a closed form which is easy to implement. It will prove useful in an analysis of structure function data, and in a prediction of related quantities relevant for the Tevatron and the LHC. However, at present it only really exists at LO (and not even that for many quantities), and for full implementation the calculation of the NLO impact factors within the BFKL framework is urgently needed. Once this is done, a truly full NLO analysis of structure functions, which will be equally valid over the full perturbative range will be possible.

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[13] A prediction for $dF_2(x,Q^2)/d\ln Q^2$ for preliminary ZEUS data is shown in fig. 35 of [40] which shows the clear improvement compared to a convention NLO-in-$\alpha_s(Q^2)$ treatment.

[14] Figures showing this can be seen in [41].

Table 1.

The coefficients K_{nm} in

$$xP_{gg}^{NLO}(\xi,\alpha_s(Q^2)) = \bar{\alpha}_s(Q^2)\sum_{n=1}^{19}\sum_{m=1}^{m_{max}}\bar{\alpha}_s^n(Q^2)\left(K_{nm}\frac{\xi^m\bar{\beta}_0^{n-m}}{m!} + K_{n\delta}\bar{\beta}_0^n\delta(1-x)\right). \qquad (6.1)$$

n	$m=5$ $m=11$ $m=17$ $m=23$	$m=4$ $m=10$ $m=16$ $m=22$	$m=3$ $m=9$ $m=15$ $m=21$	$m=2$ $m=8$ $m=14$ $m=20$	$m=1$ $m=7$ $m=13$ $m=19$	$m=0$ $m=6$ $m=12$ $m=18$
1						-0.4236
2					-1.354	1.611
3				-7.000	30.22	-34.63
4			-5.686	46.92	-103.2	63.85
5		-16.14	193.5	-797.2	1373	-918.0
6	186.0	-971.0	2518	-3323	2045	-458.9 -14.35
7	-1386	5051	-9865	10113	-4281 -10.60	709.2 192.0
8	21431	-59800	99225	-95325 -24.48	49058 511.5	-11483 -4497
9	70532	-46099 -17.21	-25896 349.4	59631 -3100	-29684 15284	2798 -44034
10	-126387 -12.01	-261087 326.1	735693 -3758	-761882 23801	373984 -88010	-77690 179647
11	8688676 1117.6	-9665206 -15044	6981022 119789	-3087487 -620744	771318 2179220	-102010 -5256680 -37.57
12	-1.621×10^7 46536	1.864×10^7 -211318	-1.288×10^7 563130	5044618 -547416 -18.36	-962638 -1766356 506.4	64963 8225690 -6353
13	1.139×10^8 -400076	-1.236×10^8 1467162	8.576×10^7 -2855626 -12.58	-3.677×10^7 -645255 453.6	9106782 2.136×10^7 -7298	-1186015 -6.623×10^7 68149
14	4.425×10^8 1.425×10^7	-2.089×10^8 -5.541×10^7 -46.85	6.225×10^7 $+1.589\times10^8$ 180.1	-1.129×10^7 -3.345×10^8 -32842	2061380 5.135×10^8 359398	-386008 -5.625×10^8 -2678626
15	-7.723×10^8 -1.146×10^7 639.2	1.340×10^8 1.108×10^8 -103283	1.391×10^8 -4.028×10^8 101509	-1.075×10^8 9.069×10^8 -640342	3.170×10^7 -1.361×10^9 2523533	-4331143 1.339×10^9 -4527424 -18.33

n	$m=5$ $m=11$ $m=17$ $m=23$	$m=4$ $m=10$ $m=16$ $m=22$	$m=3$ $m=9$ $m=15$ $m=21$	$m=2$ $m=8$ $m=14$ $m=20$	$m=1$ $m=7$ $m=13$ $m=19$	$m=0$ $m=6$ $m=12$ $m=18$
16	1.403×10^{10} 2.987×10^{9} -11408	-8.066×10^{9} -1.490×10^{9} 137101	3.172×10^{9} 4.598×10^{9} -1076349	-8.188×10^{8} -9.865×10^{9} 5431540	1.324×10^{8} 1.529×10^{10} -1.481×10^{7} -12.37	-1.227×10^{7} -1.724×10^{10} -9014694 552.7
17	-1.108×10^{10} 9.306×10^{9} 803905	7.809×10^{9} -1.676×10^{10} -7783462	-3.096×10^{9} 2.172×10^{10} 5.544×10^{7}	6.807×10^{8} -1.824×10^{10} -2.982×10^{8} -57.64	-7.184×10^{7} 6.057×10^{9} 1.227×10^{9} 2645	1672556 6.571×10^{9} -3.869×10^{9} -58085
18	7.129×10^{10} -3.690×10^{10} 7633219	-4.890×10^{10} 6.574×10^{10} -2.006×10^{7} -17.59	2.107×10^{10} -8.024×10^{11} -4.467×10^{7} 741.8	-5.733×10^{9} 5.604×10^{10} 7.555×10^{8} -14788	9.531×10^{8} 3.195×10^{9} -4.185×10^{9} 180754	-8.884×10^{7} -5.773×10^{10} 1.480×10^{10} -1460081
19	4.467×10^{11} 4.962×10^{11} -4.778×10^{7} -11.75	-1.573×10^{11} -9.477×10^{11} -6.248×10^{7} 620.5	3.986×10^{10} 1.402×10^{12} 1.948×10^{9} -15217	-7.340×10^{9} -1.602×10^{12} -1.372×10^{10} 225792	9.737×10^{8} 1.399×10^{12} 6.153×10^{10} -2200274	-8.398×10^{7} -9.198×10^{11} -2.005×10^{11} 1.398×10^{7}

The series for the part proportional to $\delta(1-x)$ is more convergent in $\alpha_s(Q^2)$ and for all $Q^2 \gtrsim 1\text{GeV}^2$ is given accurately by

$$\bar{\alpha}_s(Q^2)\delta(1-x)\big[9.0(\bar{\beta}_0\bar{\alpha}_s(Q^2))^3 + 139.5(\bar{\beta}_0\bar{\alpha}_s(Q^2))^5 + 38.88(\bar{\beta}_0\bar{\alpha}_s(Q^2))^6 + 964.2(\bar{\beta}_0\bar{\alpha}_s(Q^2))^8 + 167.0(\bar{\beta}_0\bar{\alpha}_s(Q^2))^9 + 5605(\bar{\beta}_0\bar{\alpha}_s(Q^2))^{10}\big]. \tag{6.2}$$

Table 2.

The coefficients K_{nm} in

$$xP_{LL}^{NLO}(\xi,\alpha_s(Q^2)) = \bar{\alpha}_s(Q^2)\sum_{n=1}^{19}\sum_{m=1}^{m_{max}}\bar{\alpha}_s^n(Q^2)\left(K_{nm}\frac{\xi^m\bar{\beta}_0^{n-m}}{m!} + K_{n\delta}\bar{\beta}_0^n\delta(1-x)\right). \qquad (6.3)$$

n	$m=5$ $m=11$ $m=17$ $m=23$	$m=4$ $m=10$ $m=16$ $m=22$	$m=3$ $m=9$ $m=15$ $m=21$	$m=2$ $m=8$ $m=14$ $m=20$	$m=1$ $m=7$ $m=13$ $m=19$	$m=0$ $m=6$ $m=12$ $m=18$
1						-0.4236
2					-1.354	9.494
3				-7.040	25.89	-29.49
4			-5.672	63.20	-222.22	251.90
5		-15.84	310.8	-1504	2766	-1964
6	243.4	-1444	4540	-7293	5206	-1100 -17.45
7	-5265	24975	-61945	82368	-55633 -19.57	16210 521.2
8	27358	-84630	162654	-187932 -6.545	116668 448.6	-31108 -5158
9	215634	-122925 -20.94	-266550 468.9	550451 -5027	-383797 31574	100196 -114142
10	-1552522 -22.36	1019004 814.7	567195 -12094	-1582395 91607	1103037 -396924	-286332 1031187
11	3.965×10^7 1291	-5.343×10^7 -23513	4.627×10^7 232273	-2.425×10^7 -1492972	7013331 6537249	-1016798 -1.956×10^7 -20.00
12	-2.424×10^8 60402	2.970×10^8 -153923	-1.702×10^8 -952951	1.214×10^8 1.057×10^7 -22.33	-3.553×10^7 -4.805×10^7 641	5390954 1.328×10^8 -8519
13	-1.514×10^8 -2256260	-7.7974×10^7 1.199×10^7	1.993×10^8 -4.586×10^7 -23.25	-1.426×10^8 1.252×10^8 1171	4.814×10^7 -2.356×10^8 -25002	-7741384 2.796×10^8 296133
14	4.934×10^9 5.595×10^7	-2.902×10^9 -2.675×10^8 -33.40	9.784×10^8 9.236×10^8 2462	-1.227×10^8 -2.317×10^9 -61331	-1.883×10^7 4.213×10^9 876031	6764209 -5.475×10^9 -8361477
15	-2.576×10^{10} -3.263×10^8 781.7	1.702×10^{10} 1.473×10^9 -12818	-7.397×10^9 -4.775×10^9 114646	1.968×10^9 1.147×10^{10} -395405	-2.961×10^8 -2.052×10^{10} -2857628	2.201×10^7 2.708×10^{10} 4.706×10^7 -22.34

n	$m=5$ $m=11$ $m=17$ $m=23$	$m=4$ $m=10$ $m=16$ $m=22$	$m=3$ $m=9$ $m=15$ $m=21$	$m=2$ $m=8$ $m=14$ $m=20$	$m=1$ $m=7$ $m=13$ $m=19$	$m=0$ $m=6$ $m=12$ $m=18$
16	1.725×10^{11} -1.550×10^{9} -40001	-1.209×10^{11} -7.835×10^{8} 623658	5.738×10^{10} 1.583×10^{10} -6411756	-1.764×10^{10} -5.601×10^{9} 4.638×10^{7}	3.314×10^{9} 1.187×10^{11} -2.376×10^{8} -2.90	-3.434×10^{8} -1.710×10^{11} 8.205×10^{8} 1464
17	-3.968×10^{11} 5.446×10^{10} 1959861	3.104×10^{11} -1.036×10^{11} -2.351×10^{7}	-1.572×10^{11} 1.304×10^{11} 2.013×10^{8}	5.059×10^{10} -6.795×10^{10} -1.264×10^{9} -45.09	-9.869×10^{9} 1.099×10^{11} 5.914×10^{9} 3674	1.057×10^{9} 3.145×10^{11} -2.077×10^{10} -110727
18	-2.261×10^{12} -8.652×10^{11} -1.287×10^{7}	8.240×10^{11} 2.018×10^{12} 2.343×10^{8} -21.44	-1.747×10^{11} -3.648×10^{12} -1.611×10^{9} 885.6	1.319×10^{10} 5.066×10^{12} 1.497×10^{10} -171708	2.293×10^{9} -5.319×10^{12} -7.492×10^{10} 177961	-5.601×10^{8} 4.119×10^{12} 2.888×10^{11} -514627
19	1.562×10^{13} 5.886×10^{12} -6.036×10^{8} -21.75	-6.662×10^{12} -1.334×10^{13} 1.710×10^{9} 1680	1.980×10^{12} 2.338×10^{13} 4.335×10^{9} -54959	-3.942×10^{11} -3.179×10^{13} -7.920×10^{10} 1031287	4.967×10^{10} 3.328×10^{13} 4.895×10^{11} -1.271×10^{7}	-3.565×10^{9} -2.645×10^{13} -1.982×10^{12} 1.077×10^{8}

The series for the part proportional to $\delta(1-x)$ is more convergent in $\alpha_s(Q^2)$ and for all $Q^2 \gtrsim 1\text{GeV}^2$ is given accurately by

$$\bar{\alpha}_s(Q^2)\delta(1-x)\big[-0.3094(\bar{\beta}_0\bar{\alpha}_s(Q^2)) - 3.856(\bar{\beta}_0\bar{\alpha}_s(Q^2))^2 + 6.376(\bar{\beta}_0\bar{\alpha}_s(Q^2))^3 - 50.36(\bar{\beta}_0\bar{\alpha}_s(Q^2))^4 + 340.0(\bar{\beta}_0\bar{\alpha}_s(Q^2))^5 + 55.51(\bar{\beta}_0\bar{\alpha}_s(Q^2))^6 - 1600(\bar{\beta}_0\bar{\alpha}_s(Q^2))^7 + 2838(\bar{\beta}_0\bar{\alpha}_s(Q^2))^8 - 8457(\bar{\beta}_0\bar{\alpha}_s(Q^2))^9 + 24526(\bar{\beta}_0\bar{\alpha}_s(Q^2))^{10} + 57602(\bar{\beta}_0\bar{\alpha}_s(Q^2))^{11} - 325984(\bar{\beta}_0\bar{\alpha}_s(Q^2))^{12} + 477536(\bar{\beta}_0\bar{\alpha}_s(Q^2))^{13}\big]. \tag{6.4}$$

Figure Captions

Fig. 1. The branch points and cuts associated with $\exp(-X_0(\gamma)/(\bar{\beta}_0 N))$ and the saddle-point contour, the Gamma-function contour and the numerical integration contour.

Fig. 2. The expression $\Delta_4(-1/(\bar{\beta}_0 N))$ as a function of N compared to $1/(\bar{\beta}_0 N)^4$.

Fig. 3. The Q^2-dependent part of the gluon structure function, $\mathcal{G}_E(N,t)$, and of $d\mathcal{G}_E(N,t)/dt$ as a function of N for $t = 6$ ($Q^2 \sim 6\text{GeV}^2$). The Q^2-independent factor of $-\sin(\pi/(\bar{\beta}_0 N))\Gamma(-1/(\bar{\beta}_0 N))\exp(-\gamma_E/(\bar{\beta}_0 N))$ is included in both in order to produce a smoother N-dependent normalization of the functions.

Fig. 4. The positions of the leading poles in the anomalous dimensions for the gluon structure function at LO and NLO, and for F_L at LO and NLO.

Fig. 5. a. The anomalous dimension for the gluon structure function at LO plotted as a function of N for $t = 6$ ($Q^2 \sim 6\text{GeV}^2$). Also shown is the $\mathcal{O}(\alpha_s(Q^2))$ contribution $\bar{\alpha}_s(Q^2)/N$, and the full naive LO BFKL anomalous dimension. b. The anomalous dimensions for the gluon at LO and at NLO plotted as functions of N for $t = 6$.

Fig. 6. a. The splitting function $xP_{gg}^{LO}(x)$ and its power series and power-suppressed contributions plotted as functions of x for $t = 6$. b. The splitting function $xP_{gg}^{LO}(x)$ plotted as a function of x for $t = 6$ ($Q^2 \sim 6\text{GeV}^2$). Also shown is the $\mathcal{O}(\alpha_s(Q^2))$ contribution $\bar{\alpha}_s(Q^2)$, and the naive LO BFKL splitting function with coupling $\alpha_s(Q^2)$.

Fig. 7. The next-to-leading twist splitting function for $\bar{\alpha}_s = 0.2$.

Fig. 8. The value of $\exp(-\tilde{\gamma}t_0 + X_0(\tilde{\gamma})/(\bar{\beta}_0 N))$, along the real axis for $N = 0.4$ and $t_0 = 2$, along with $\tilde{\gamma} = \gamma_0^{HT}(t_0), \gamma_0^{HT}(t)$ for $t \gg t_0$.

Fig. 9. a. The splitting function $xP_{gg}^{NLO}(x)$ and its power-series and power-suppressed contributions plotted as functions of x for $t = 6$. b. The splitting functions $xP_{gg}^{LO+NLO}(x)$ plotted as a function of x for $t = 6$ ($Q^2 \sim 6\text{GeV}^2$). Also shown is the $\mathcal{O}(\alpha_s(Q^2))$ contribution $\bar{\alpha}_s(Q^2)$, and the LO contribution $xP_{gg}^{LO}(x)$.

Fig. 10. The values of $dG(x,Q^2)/d\ln Q^2$, for $G(x,Q^2) = x^{-0.2}(1-x)^6$, due to the LO splitting function $P_{gg}^{LO}(x)$ and the LO+NLO splitting function $P_{gg}^{LO+NLO}(x)$, plotted as functions of x for $t = 6$ ($Q^2 \sim 6\text{GeV}^2$). Also shown is the evolution due to the $\mathcal{O}(\alpha_s(Q^2))$ contribution $P(x) = \bar{\alpha}_s(Q^2)/x$.

Fig. 11. a. The renormalization scale variation of the LO+NLO splitting function P_{gg}^{LO+NLO}. Shown are the three choices of scale Q^2, $0.5Q^2$ and $2Q^2$ for $t = 6$, i.e. $Q^2 \sim 6\text{GeV}^2$. b. The same for the LO+NLO physical splitting function P_{LL}^{LO+NLO}.

Fig. 12. a. The full leading $\ln(1/x)$ plus running coupling corrections coefficient function $xC_{L,g}(x,t)$ plotted as a function of x for $t = 6$ and $N_f = 4$. Also shown are the contributions from the power-series and the power-suppressed part. Note that the term $\propto \delta(1-x)$ in the power-series is replaced by the full $\mathcal{O}(\alpha_s(Q^2))$ contribution $6x^2(1-x)$, and the terms $\propto \delta(1-x)$ in the power-suppressed part are absent. b. $xC_{L,g}^{LO}(x,t)$ plotted

as a function of x for $t = 6$ and $N_f = 4$. Also shown is the coefficient function obtained from the naive LO BFKL calculation, and the contribution at $\mathcal{O}(\alpha_s(Q^2))$ alone.

Fig. 13. a. The full leading $\ln(1/x)$ plus running coupling corrections coefficient function $xP_{qg}(x,t)$ plotted as a function of x for $t = 6$ and $N_f = 4$. Also shown are the contributions from the power-series and the power-suppressed part. Note that the term $\propto \delta(1-x)$ in the power-series is replaced by the full $\mathcal{O}(\alpha_s(Q^2))$ contribution $1.5x(x^2 + (1-x)^2)$, and the terms $\propto \delta(1-x)$ in the power-suppressed part are absent. b. $xP_{qg}^{LO}(x,t)$ plotted as a function of x for $t = 6$ and $N_f = 4$. Also shown is the coefficient function obtained from the naive LO BFKL calculation, and the contribution at $\mathcal{O}(\alpha_s(Q^2))$ alone.

Fig. 14. a. The anomalous dimensions for the gluon structure function at LO and for $\mathcal{F}_L(N,t)$ at LO plotted as functions of N for $t = 6$. Also shown is the $\mathcal{O}(\alpha_s(Q^2))$ contribution common to each. b. The anomalous dimensions for $\mathcal{F}_L(N,t)$ at LO and "NLO" plotted as functions of N for $t = 6$.

Fig. 15. a. The splitting functions $xP_{LL}^{LO}(x)$ and its power series and power-suppressed contributions plotted as a function of x for $t = 6$. b. The splitting function $xP_{LL}^{LO}(x)$ plotted as a function of x for $t = 6$ ($Q^2 \sim 6\text{GeV}^2$). Also shown is the $\mathcal{O}(\alpha_s(Q^2))$ contribution $\bar{\alpha}_s(Q^2)$, the gluon splitting function $P_{gg}^{LO}(x)$ and the naive LO BFKL splitting function with coupling $\alpha_s(Q^2)$.

Fig. 16. a. The full leading $\ln(1/x)$ plus running coupling corrections physical splitting function $xP_{2L}(x,t)$ plotted as a function of x for $t = 6$. Also shown are the contributions from the power-series and the power-suppressed part. In the power-series the part $\propto \delta(1-x)$ is replaced by $1.5x(x^2 + (1-x)^2)$ while in the power-suppressed part this contribution is simply absent. b. The physical splitting function $xP_{2L}(x,t)$ plotted as a function of x for $t = 6$ along with the physical splitting function obtained from the naive LO BFKL calculation with coupling $\alpha_s(Q^2)$ and the zeroth order contribution.

Fig. 17. a. The splitting functions $xP_{LL}^{NLO}(x)$ and its power series and power-suppressed contributions plotted as functions of x for $t = 6$. b. The splitting function $xP_{LL}^{LO+NLO}(x)$ plotted as a function of x for $t = 6$ ($Q^2 \sim 6\text{GeV}^2$). Also shown is the $\mathcal{O}(\alpha_s(Q^2))$ contribution $\bar{\alpha}_s(Q^2)$, and the LO contribution $xP_{LL}^{LO}(x)$.

Fig. 18. The values of $dF_L(x,Q^2)/d\ln Q^2$, for $F_L(x,Q^2) = x^{-0.2}(1-x)^6$, due to the LO splitting functions $P_{LL}^{LO}(x)$ and the LO+NLO splitting function $P_{LL}^{LO+NLO}(x)$, plotted as a function of x for $t = 6$ ($Q^2 \sim 6\text{GeV}^2$). Also shown is the evolution due to the $\mathcal{O}(\alpha_s(Q^2))$ contribution $P(x) = \bar{\alpha}_s(Q^2)/x$.

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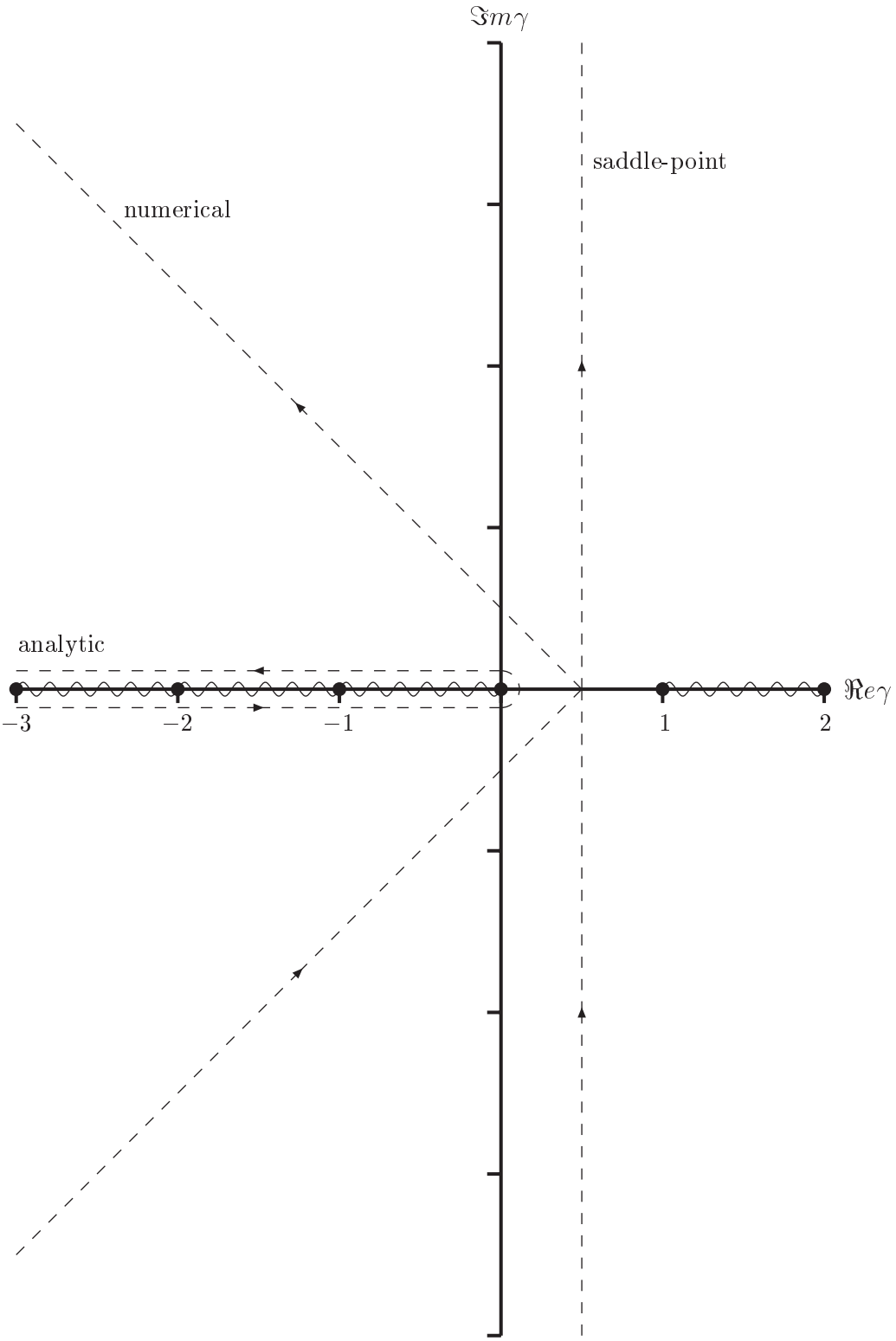

Fig. 1

LO Longitudinal Coefficient Function

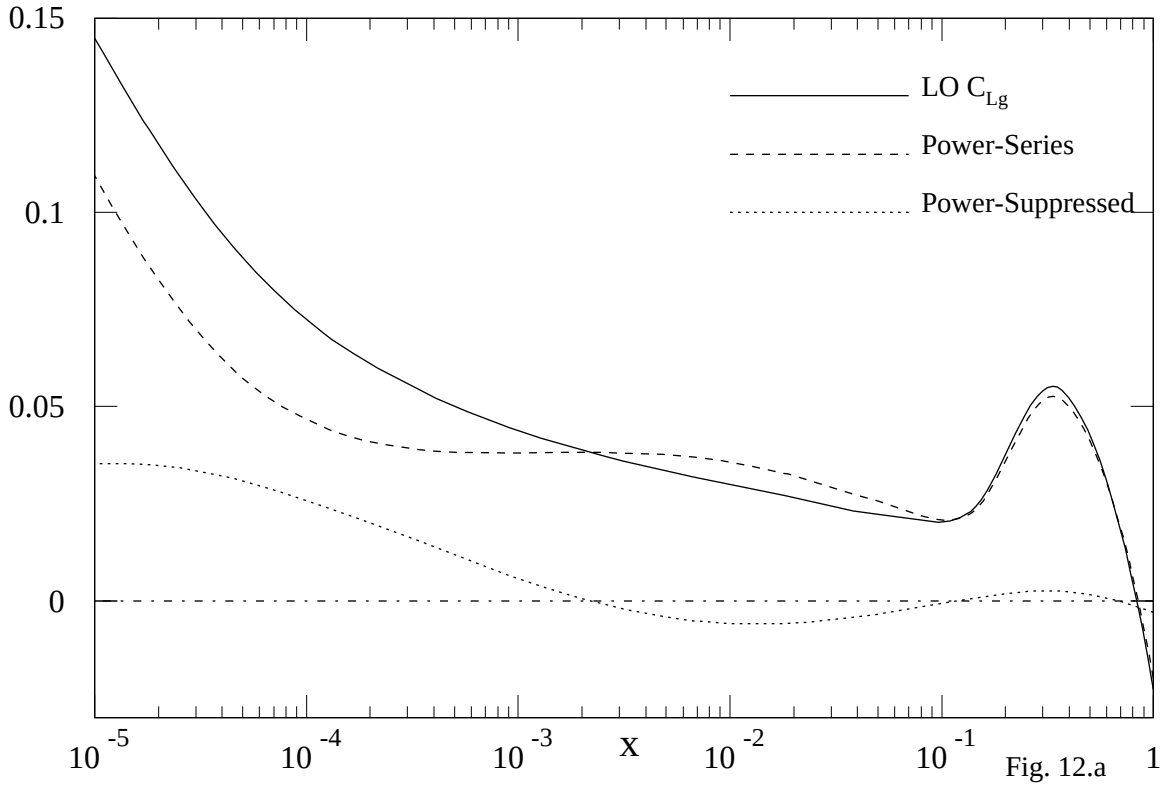

Fig. 12.a

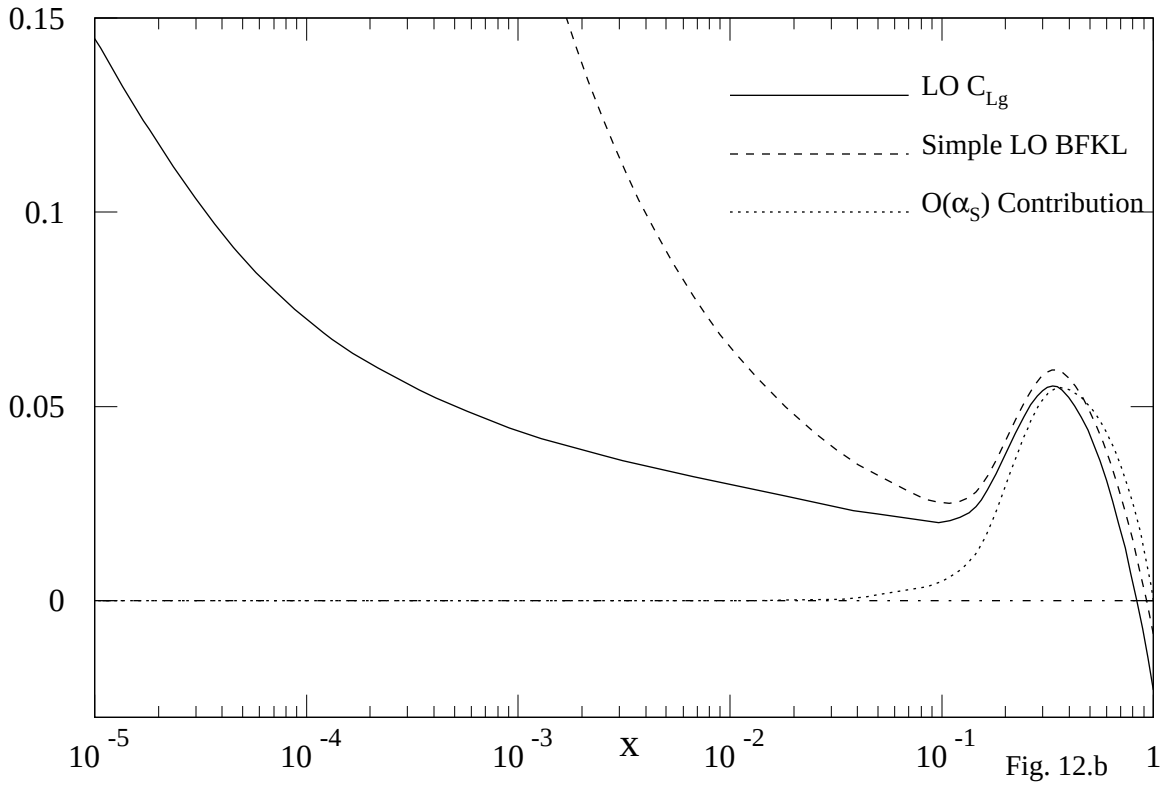

Fig. 12.b

Comparison of $\Delta_4(1/\bar{\beta}_0 N)$ and $1/(\bar{\beta}_0 N)^4$

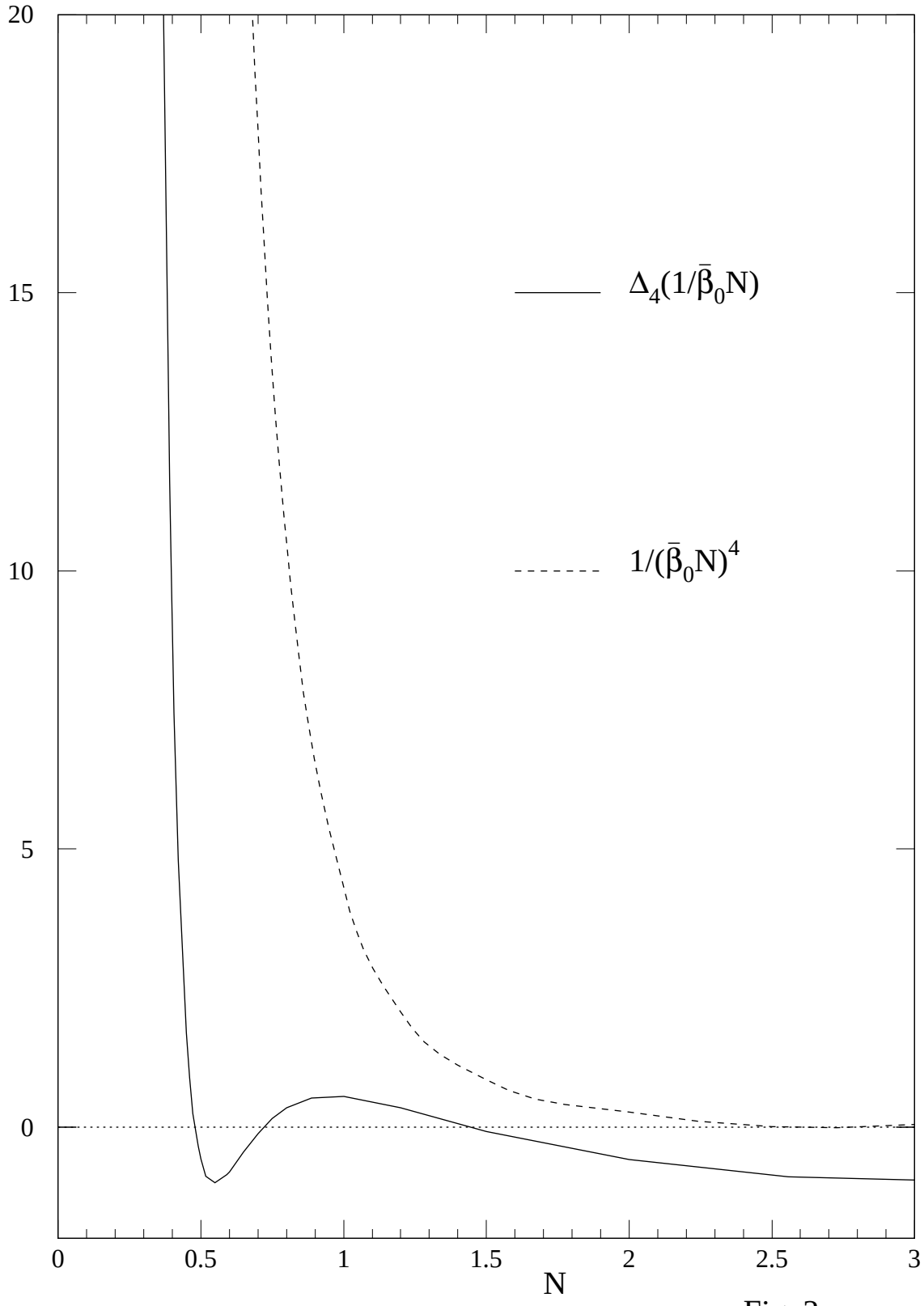

Fig. 2

Evolution of F_L, $\ln(Q^2/\Lambda^2)=6$, $F_L=(1-x)^6 x^{-0.2}$

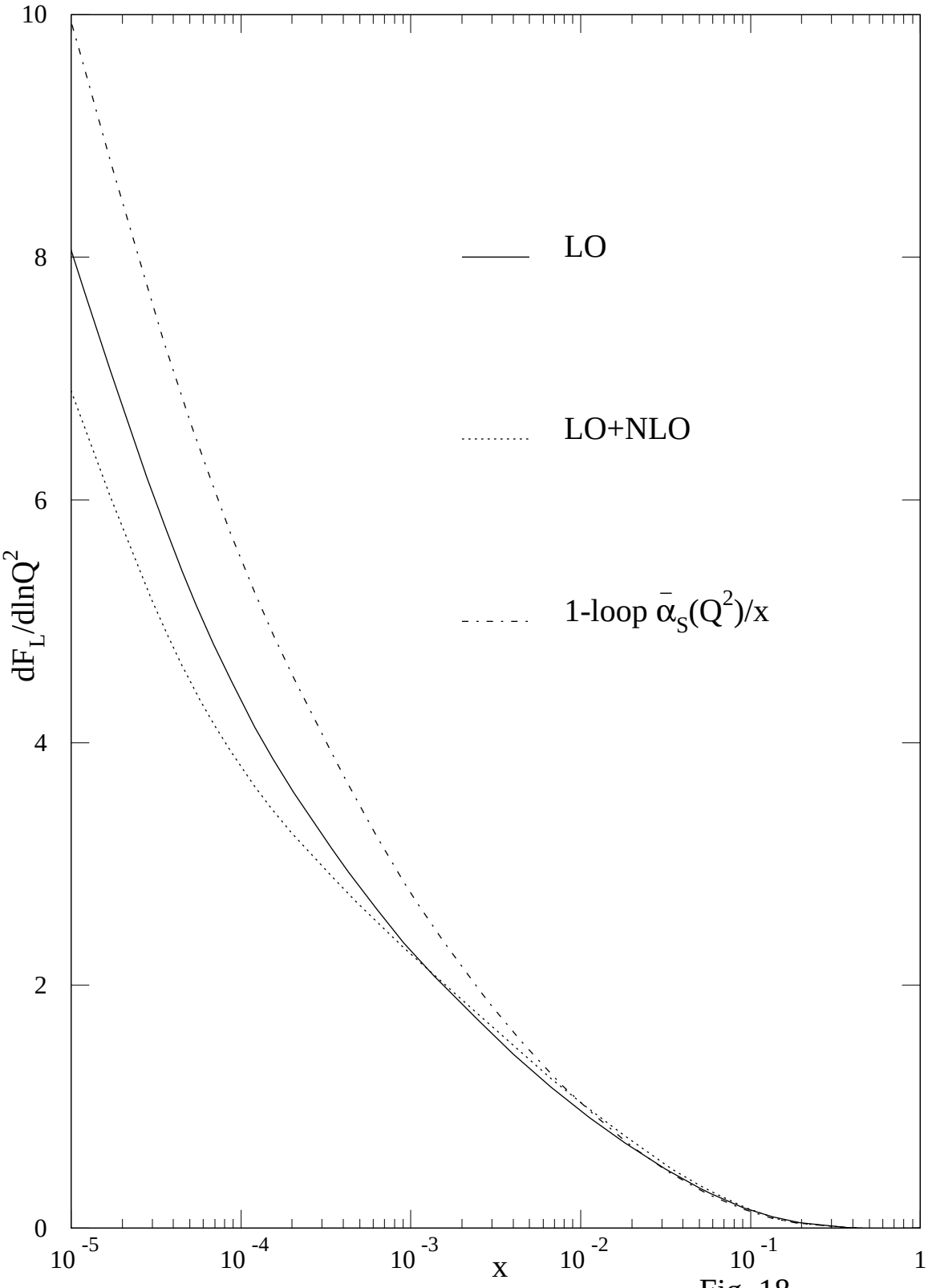

Fig. 18

NLO Physical Splitting Function

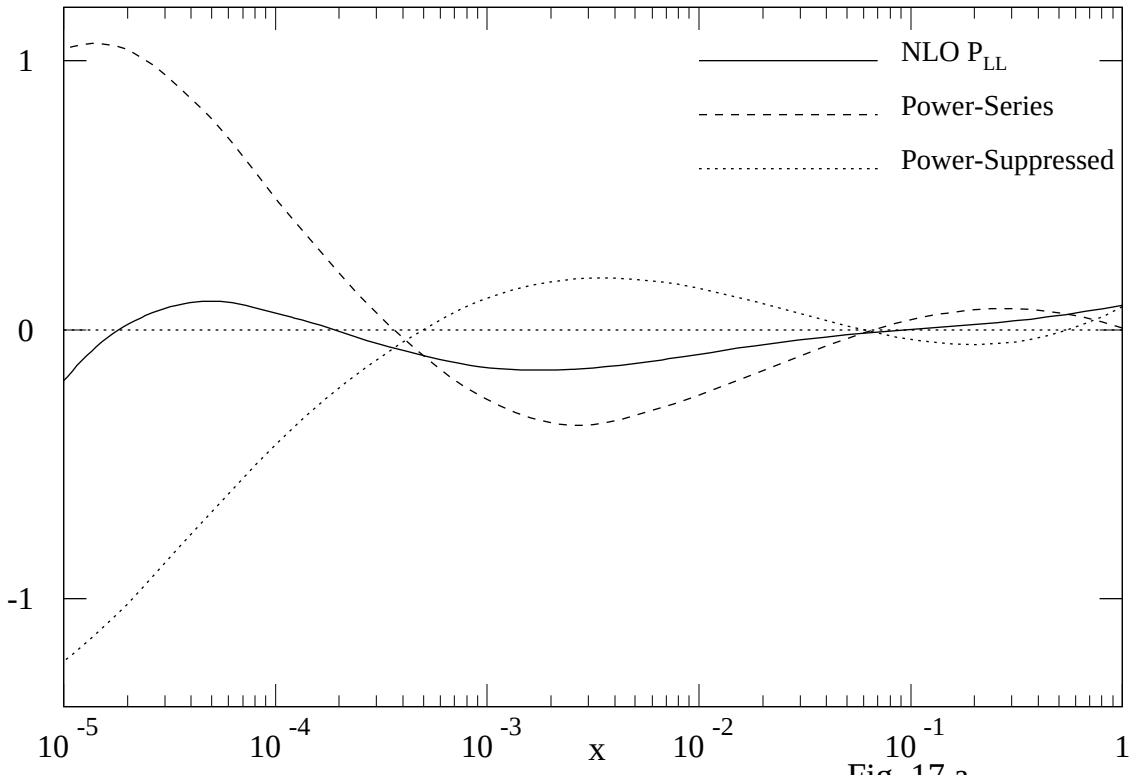

Fig. 17.a

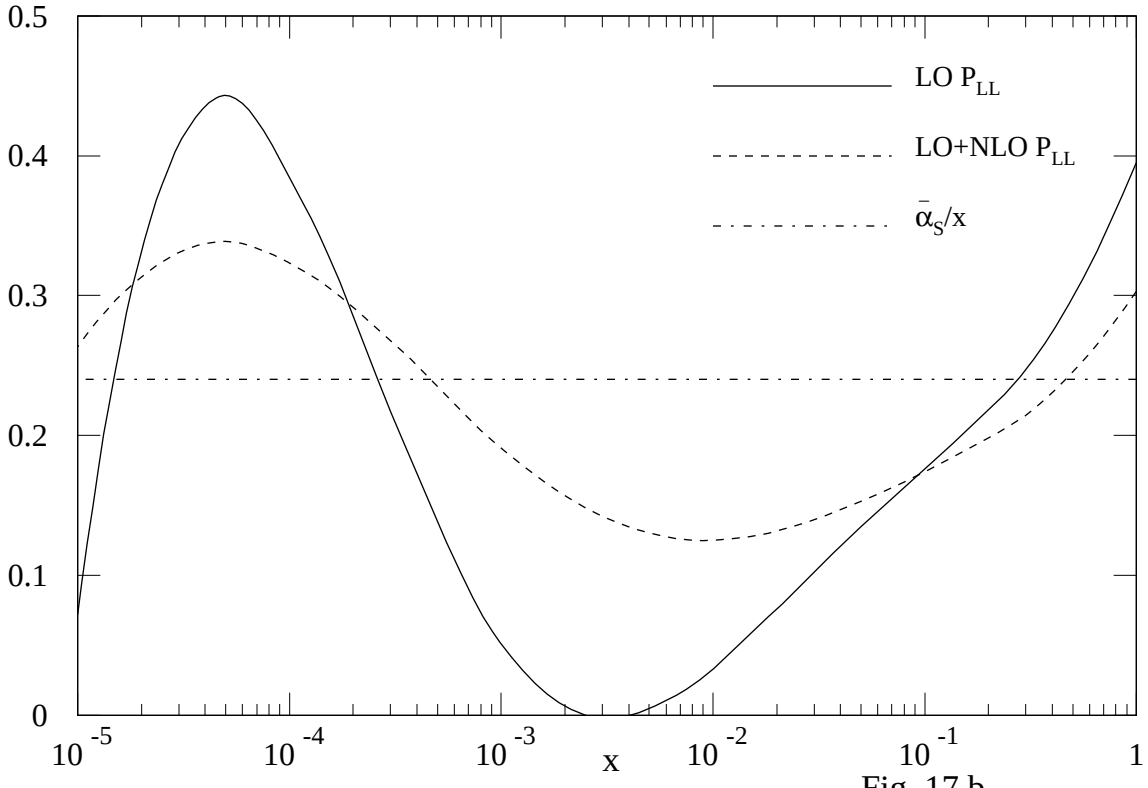

Fig. 17.b

Comparison of Anomalous Dimensions

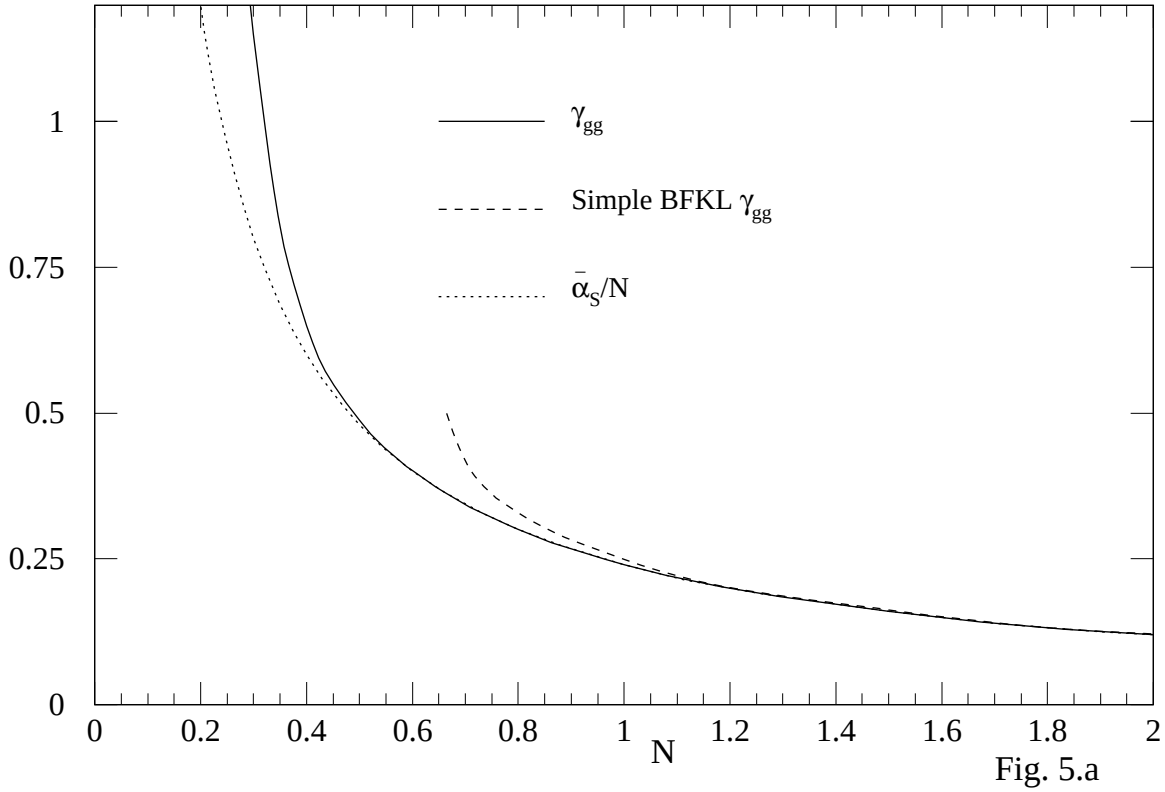

Fig. 5.a

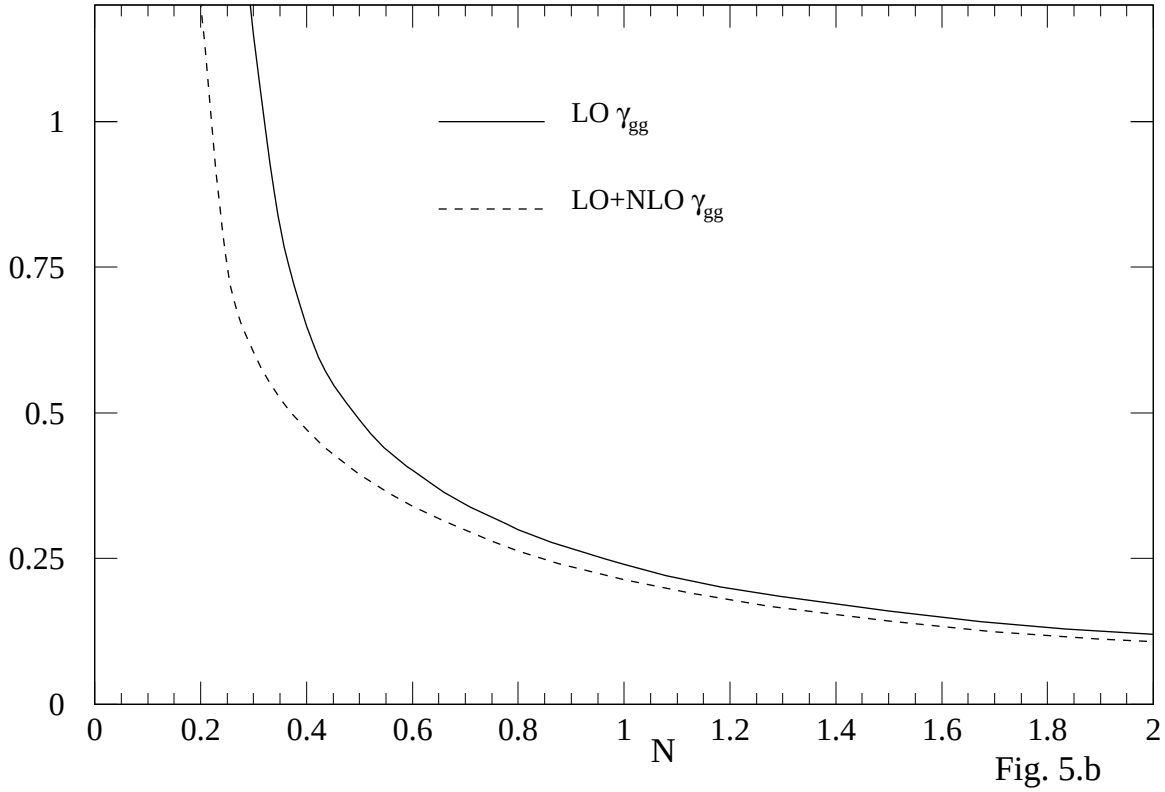

Fig. 5.b

Comparison of Physical Anomalous Dimensions

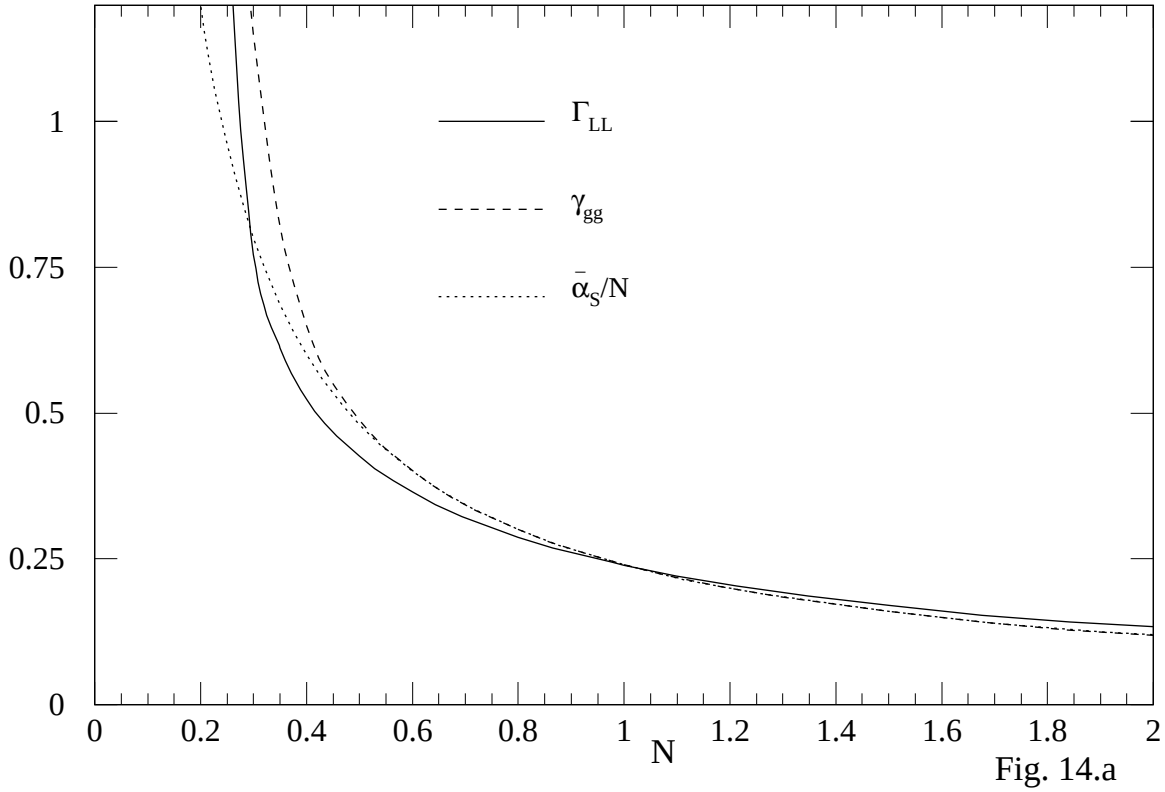

Fig. 14.a

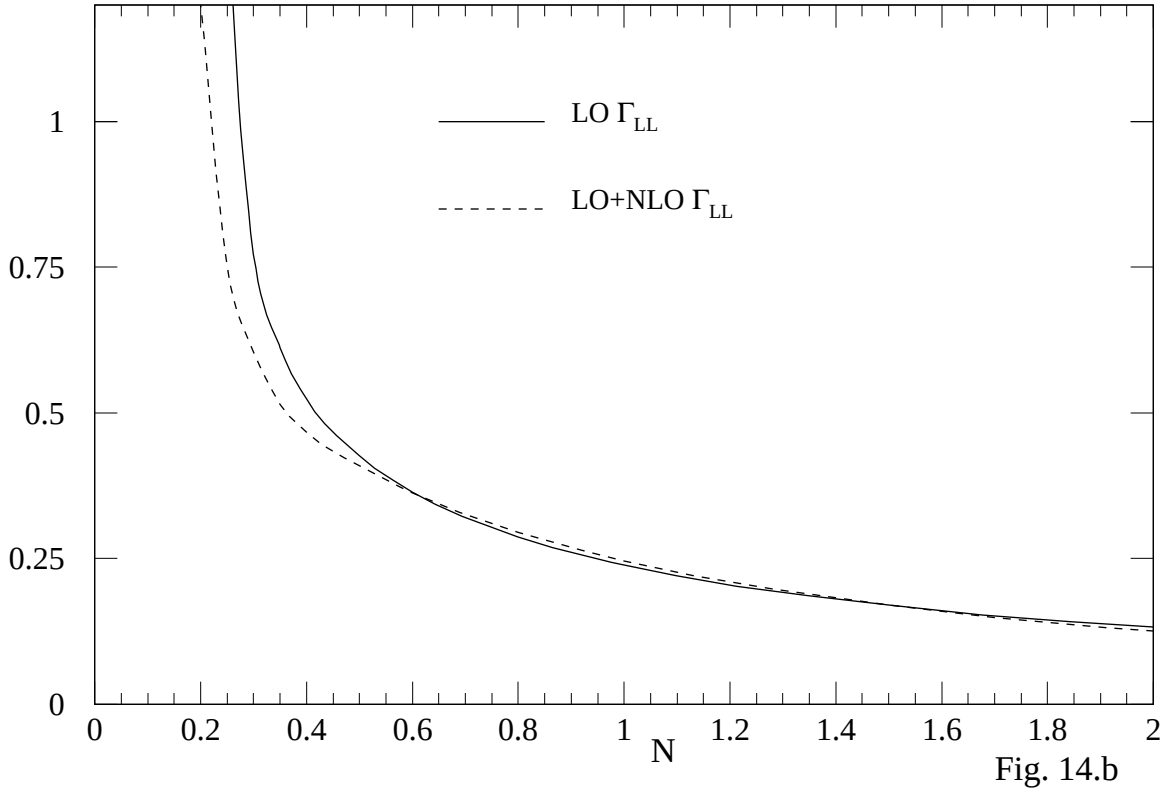

Fig. 14.b

Evolution of $G(x,t)$, $\ln(Q^2/\Lambda^2)=6$, $G(x)=(1-x)^6 x^{-0.2}$

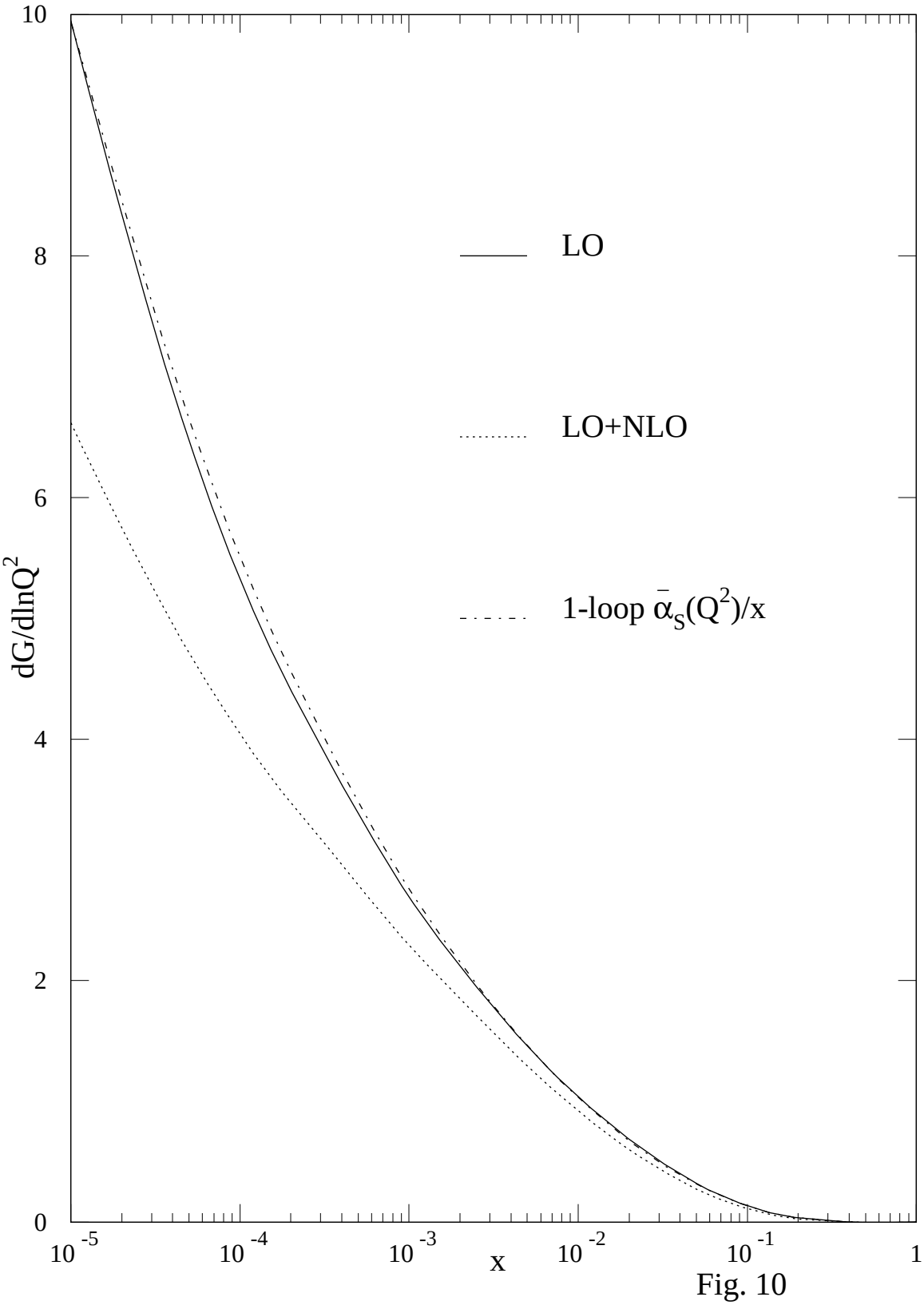

Fig. 10

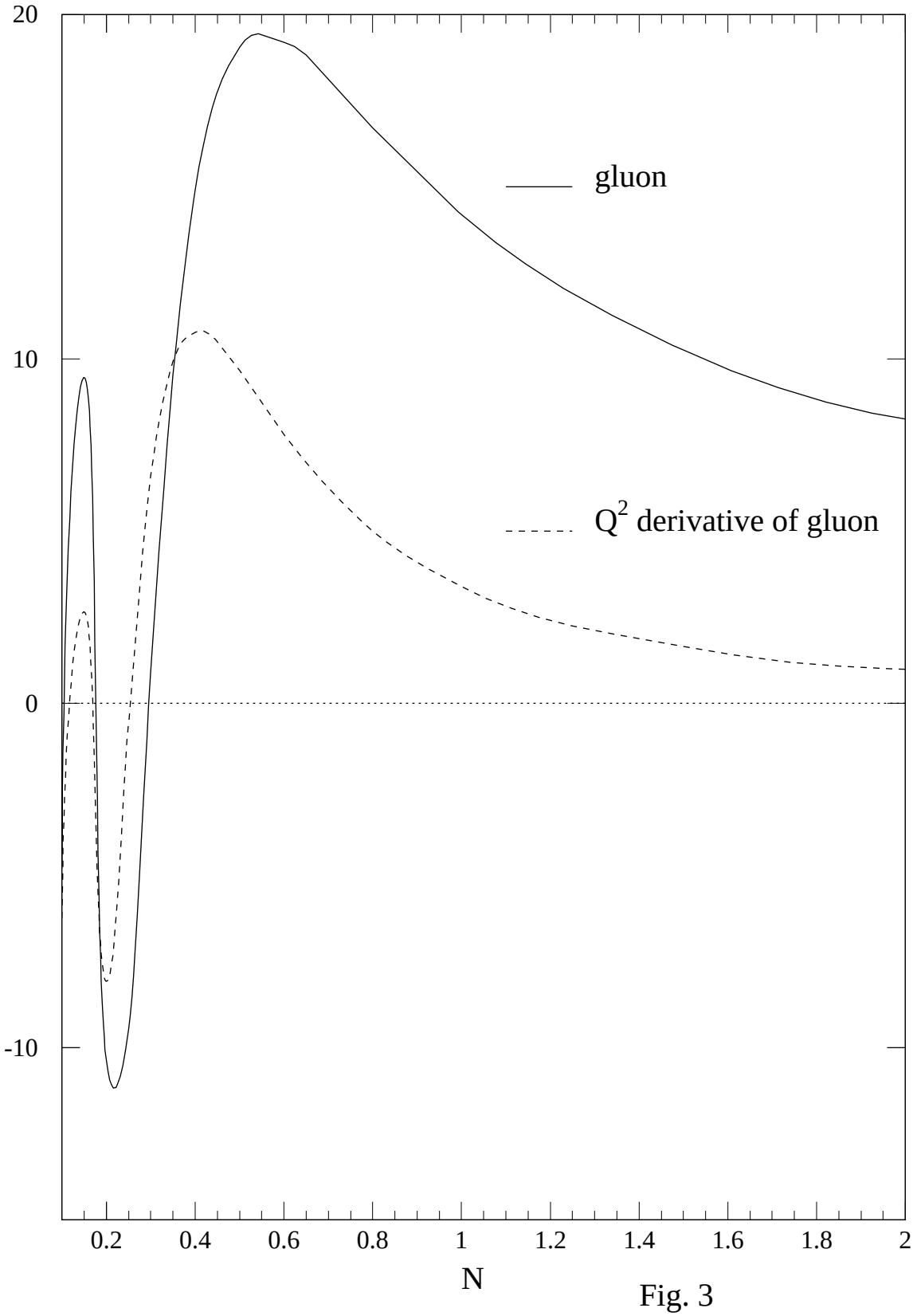
Q^2-dependent gluon and its $\ln Q^2$-derivative
20
10
0
-10
gluon
Q^2 derivative of gluon
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
N

Fig. 3

Next-to-Leading Twist Gluon Splitting Function

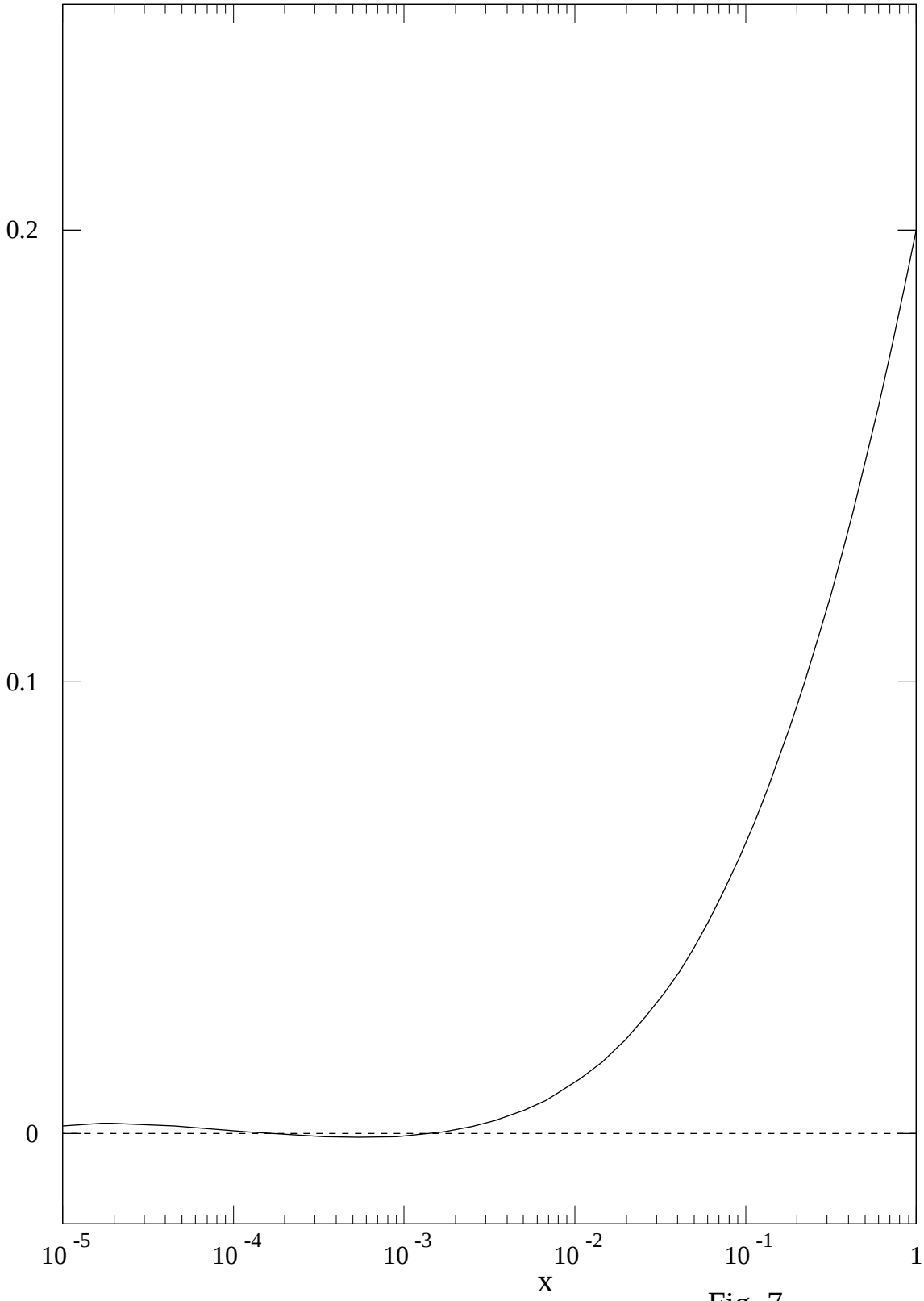

Fig. 7

$\exp(-\gamma t_0 + X_0/(\bar{\beta}_0 N))$ along negative real axis

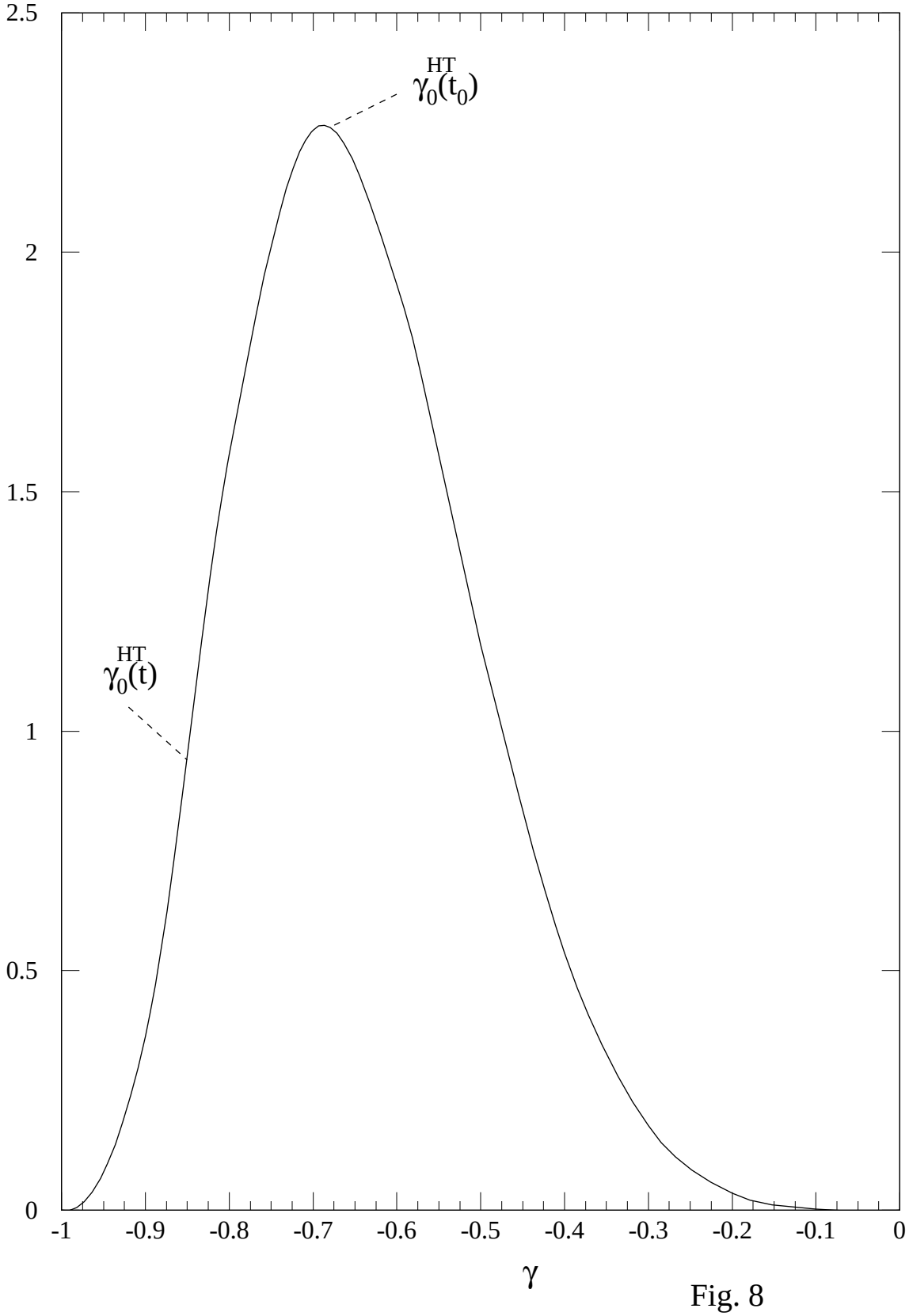

Fig. 8

Comparison of Intercepts

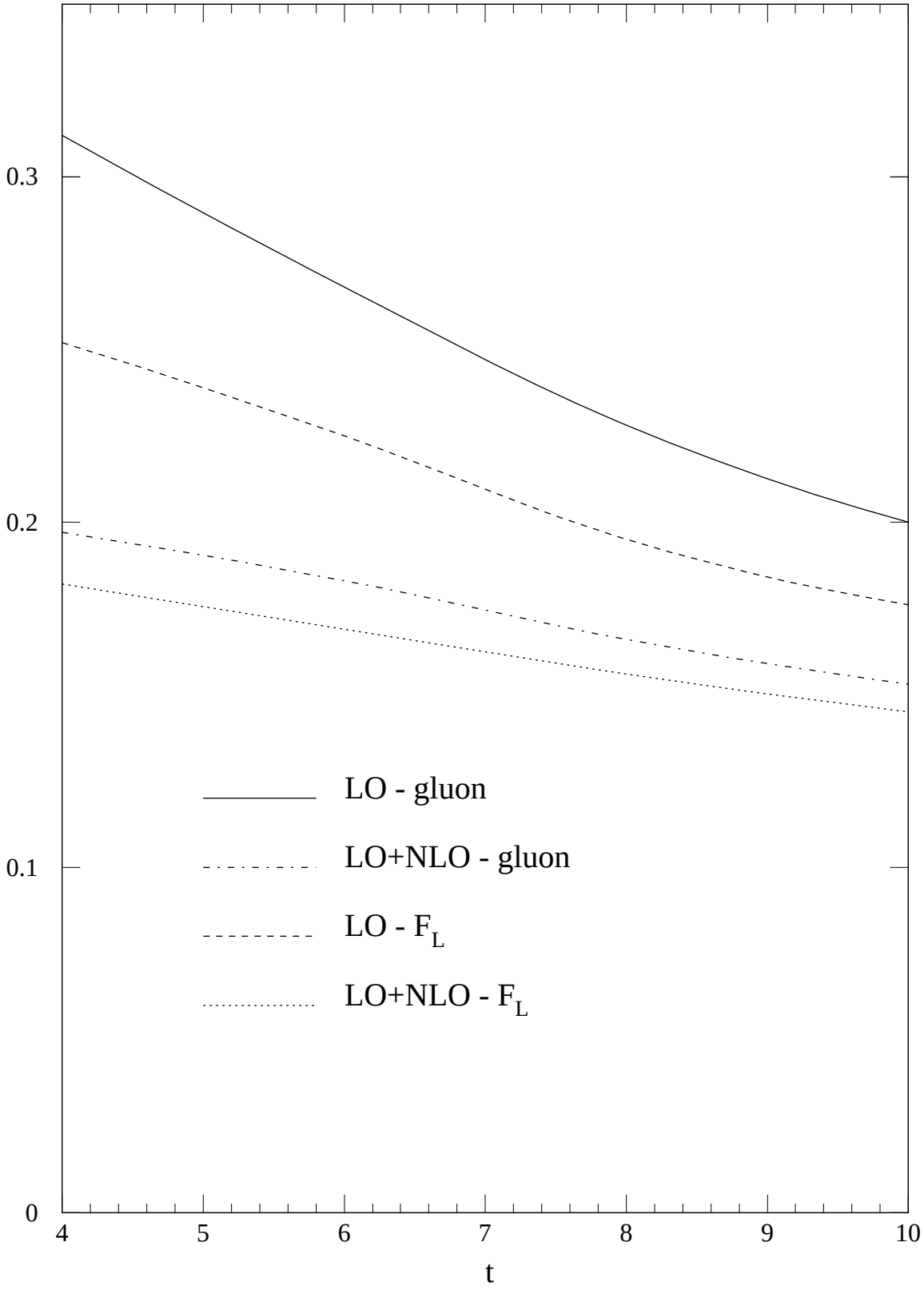

Fig. 4

LO Splitting Function P_{2L}

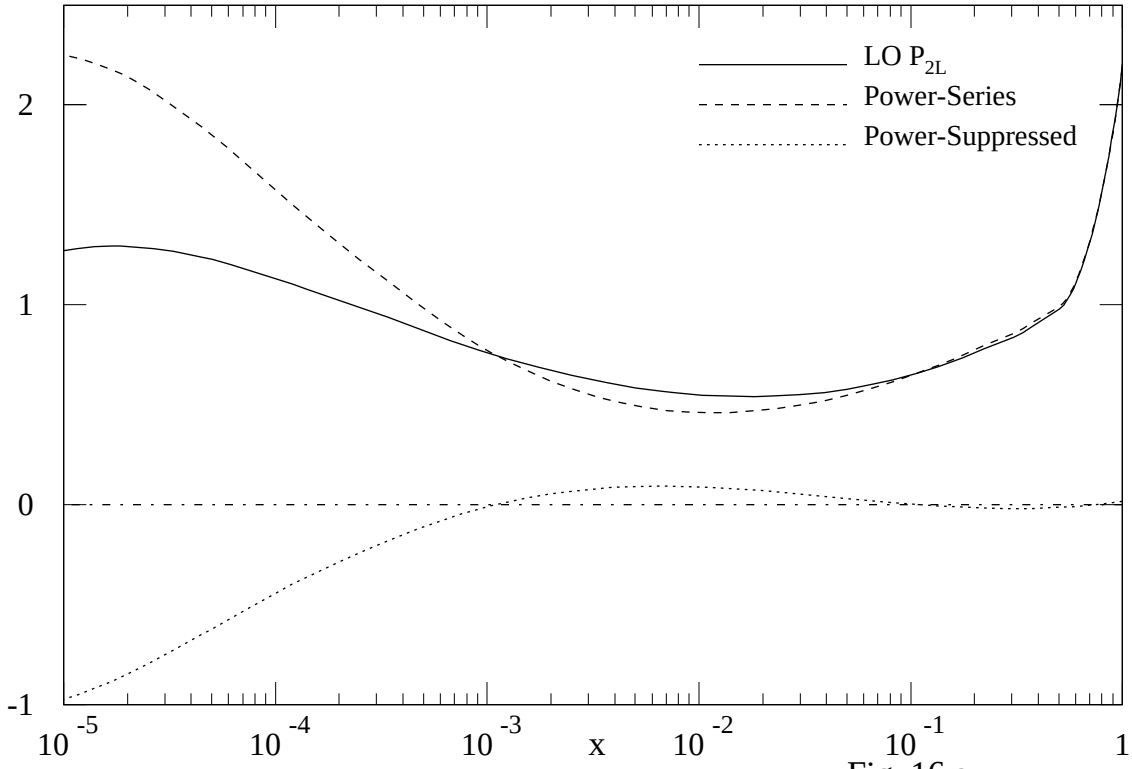

Fig. 16.a

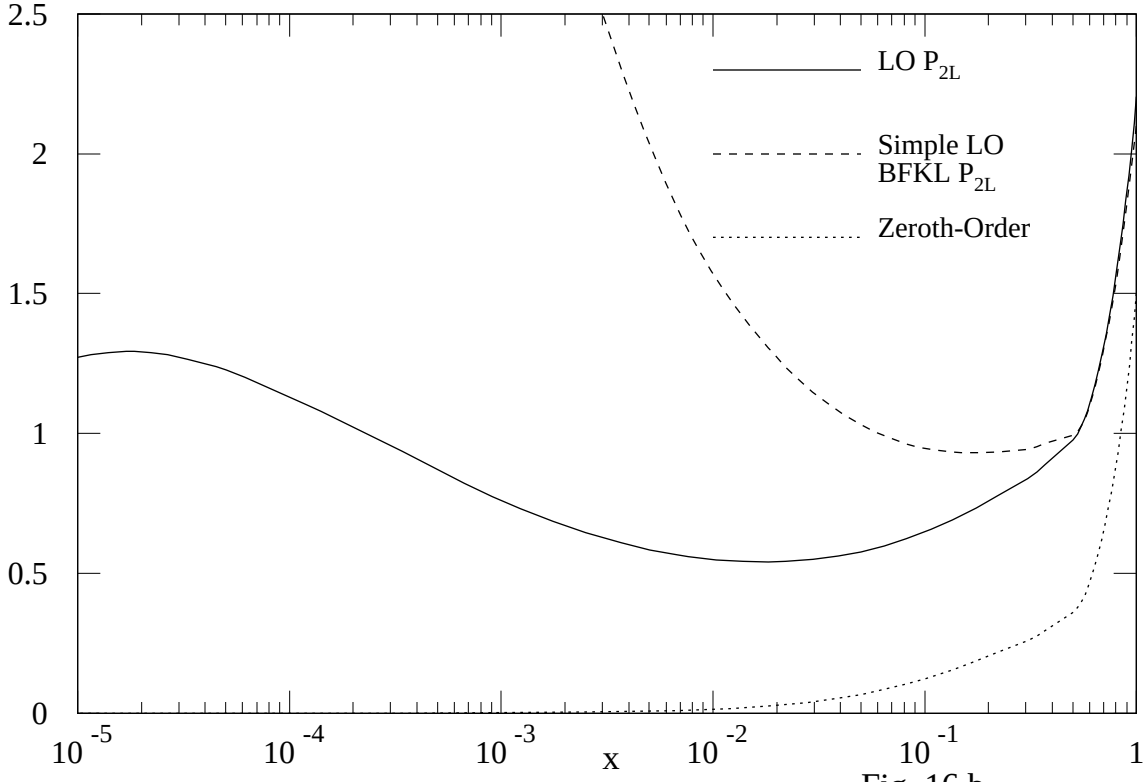

Fig. 16.b

LO Longitudinal Splitting Function

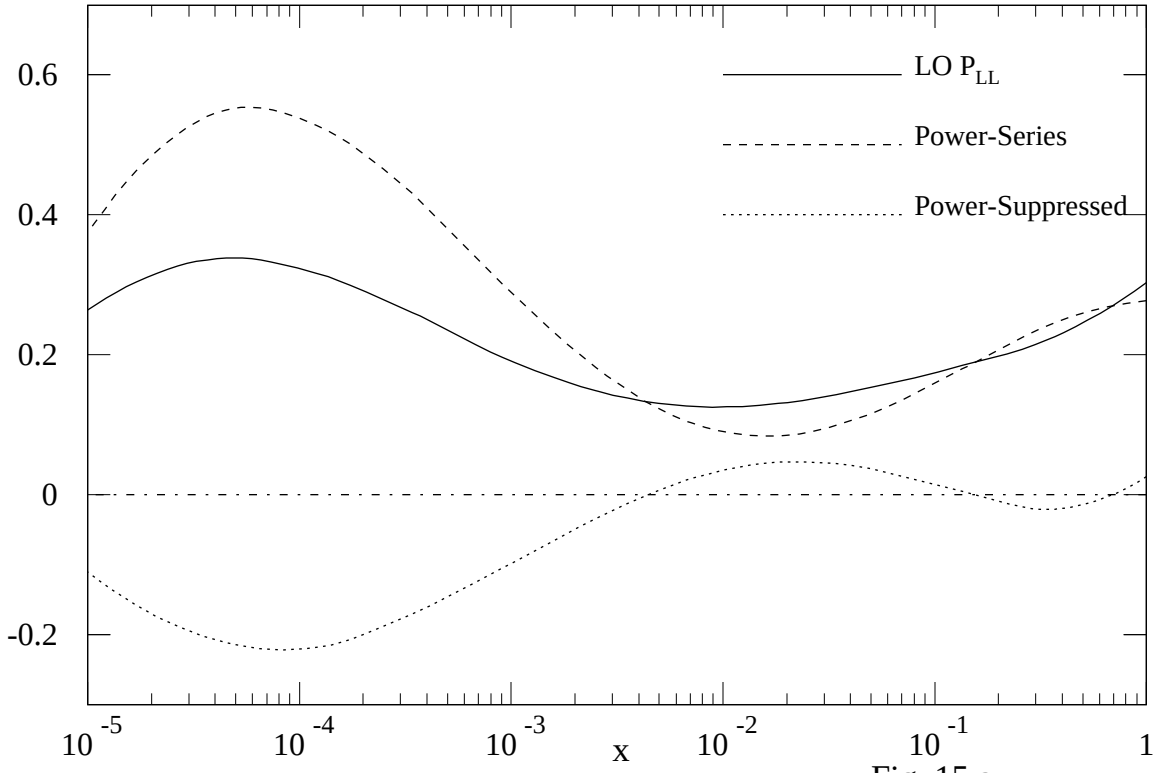

Fig. 15.a

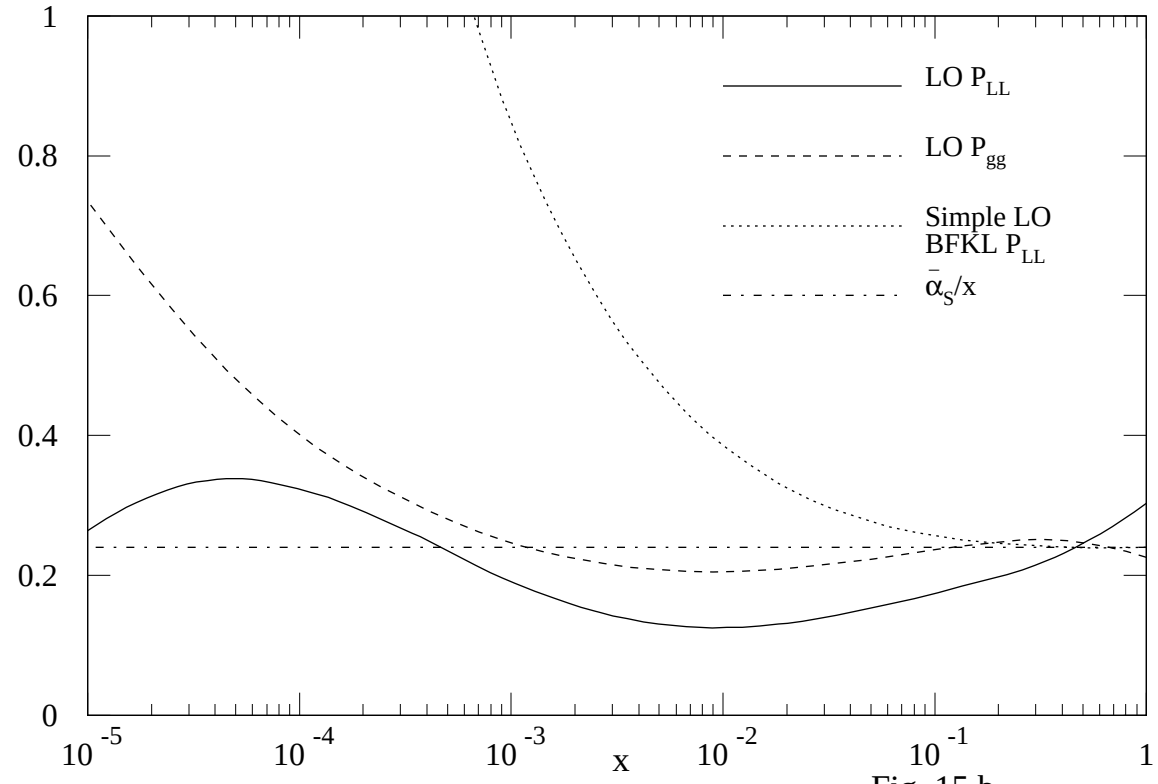

Fig. 15.b

LO Gluon Splitting Function

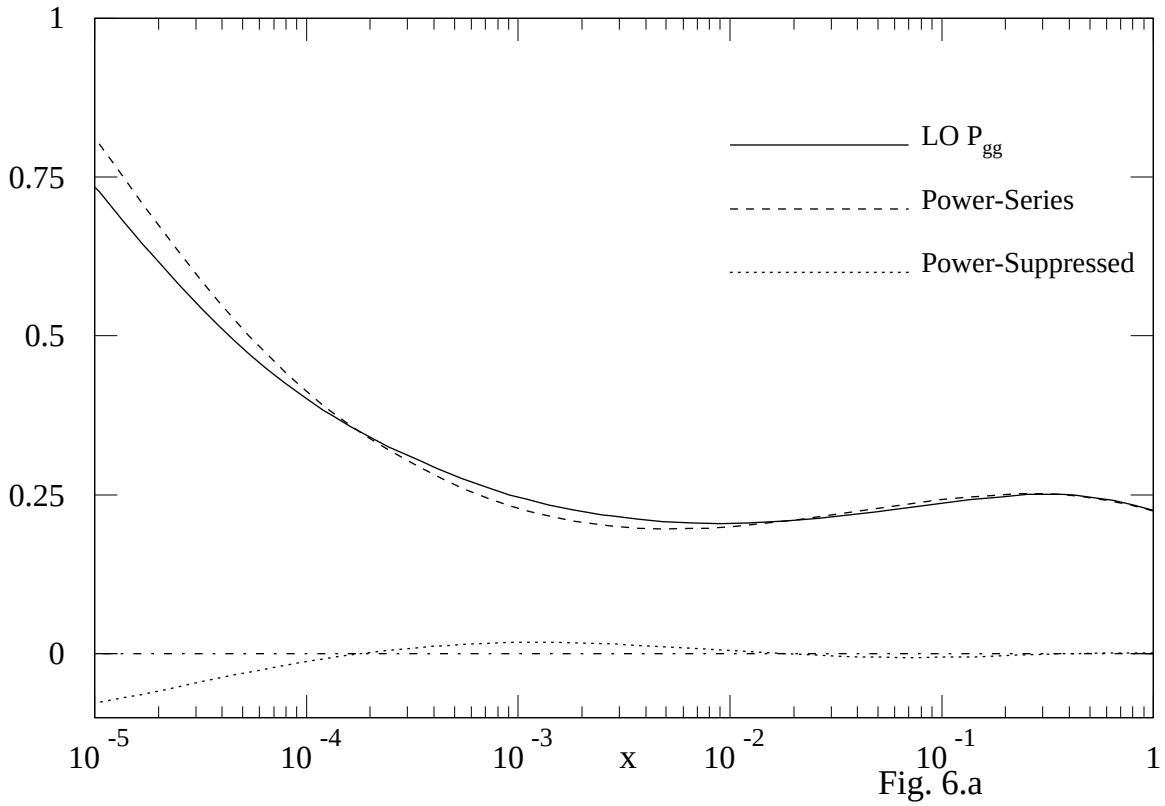

Fig. 6.a

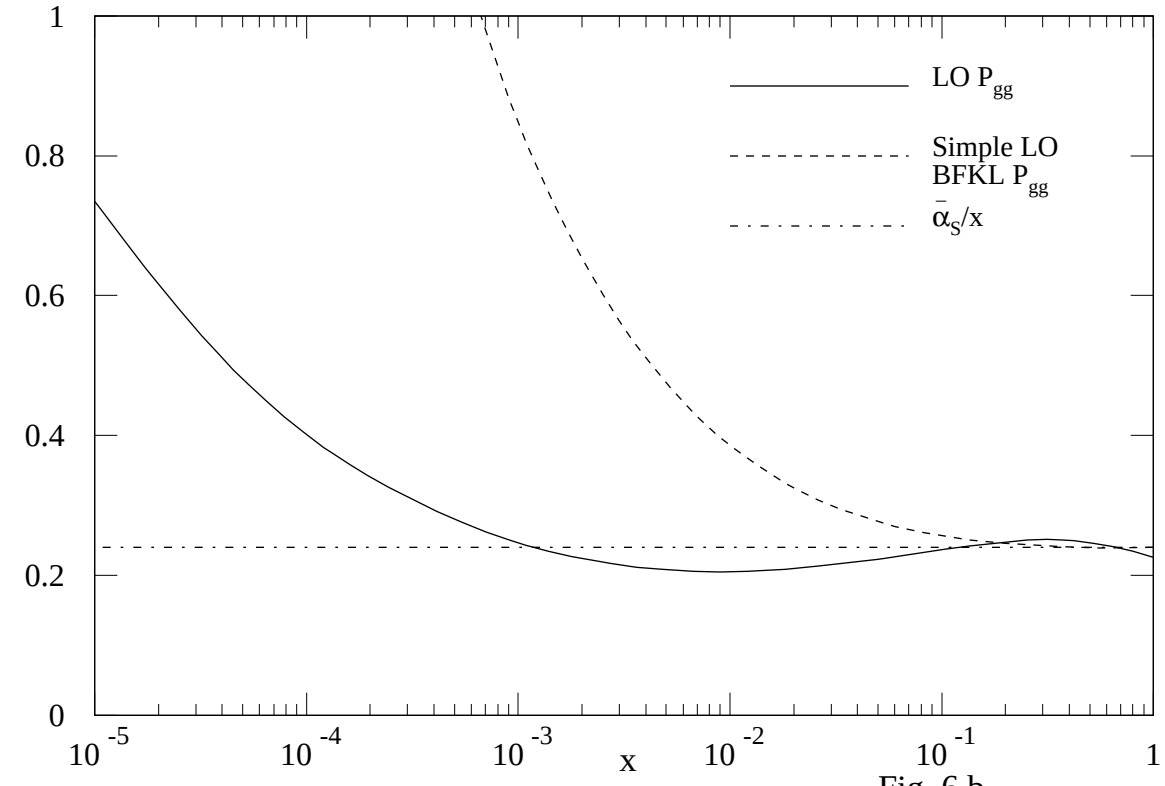

Fig. 6.b

NLO Gluon Splitting Function

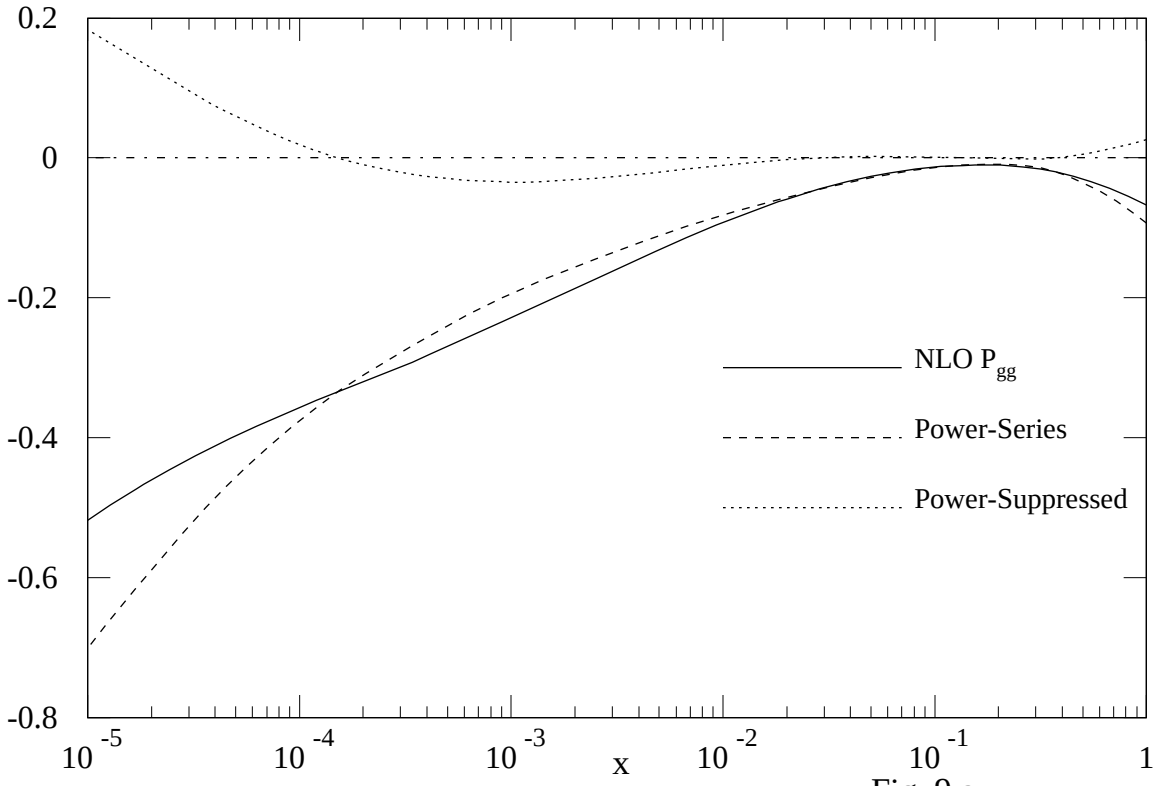

Fig. 9.a

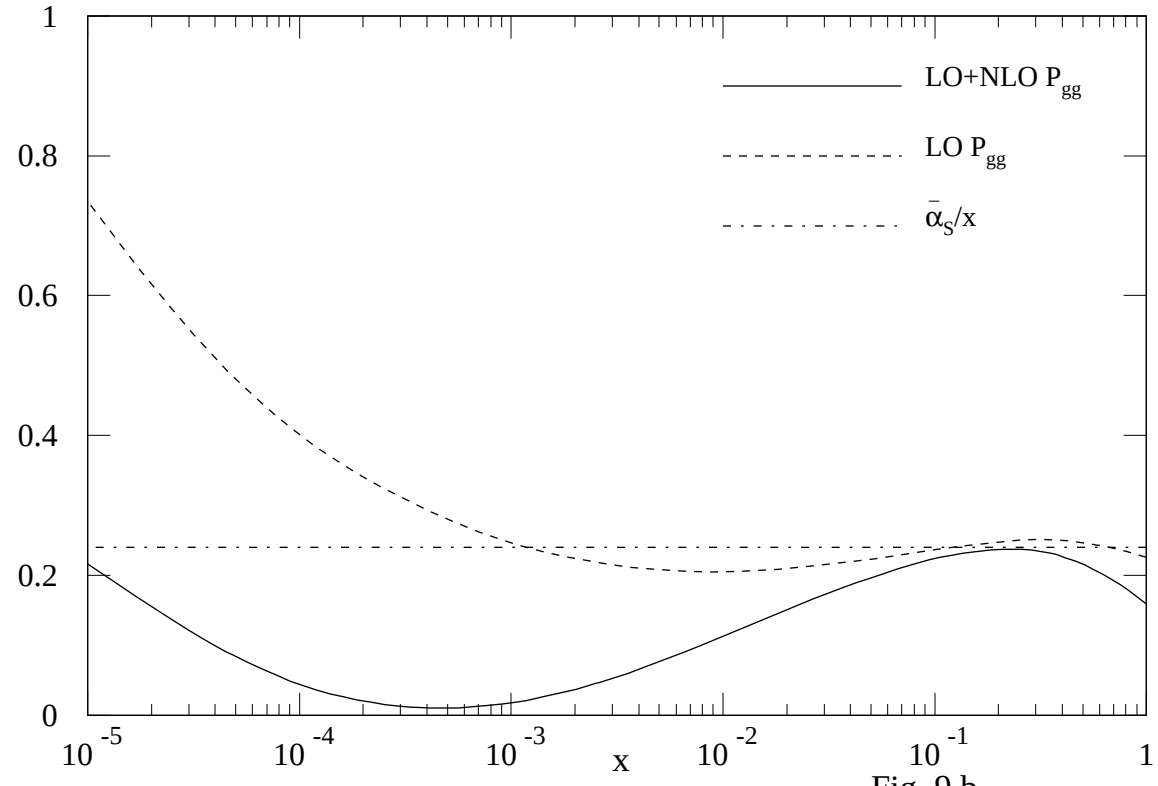

Fig. 9.b

LO Quark-Gluon Splitting Function

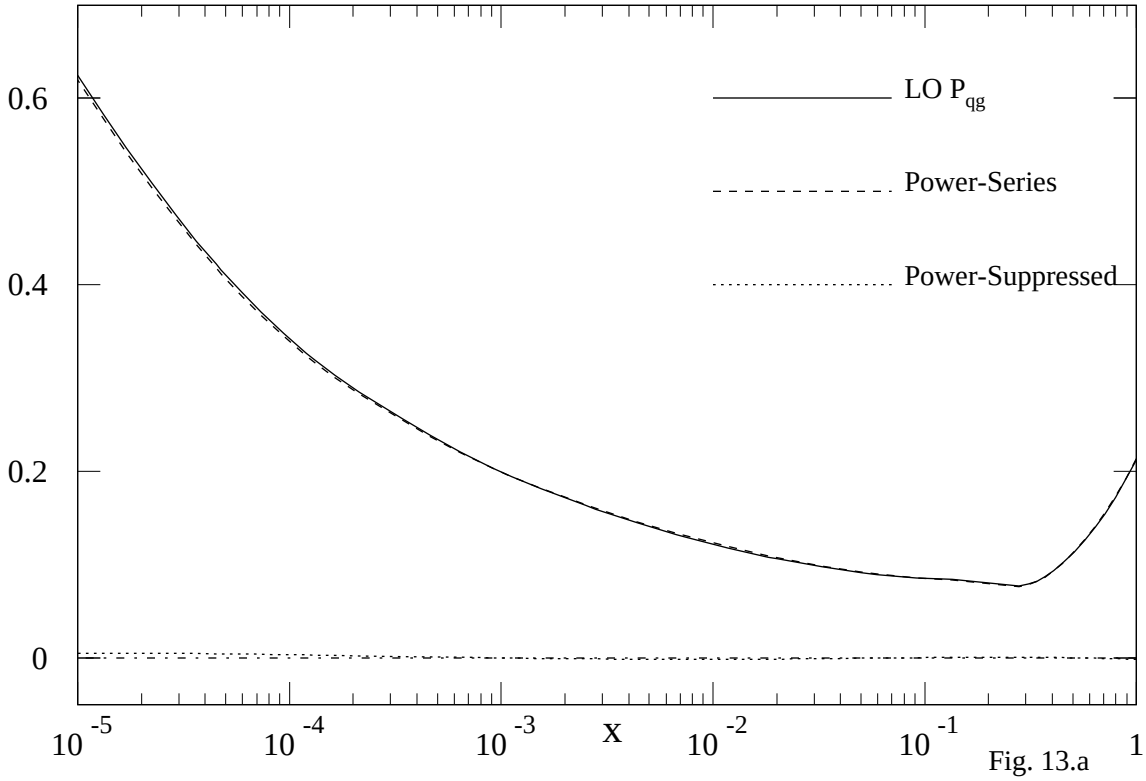

Fig. 13.a

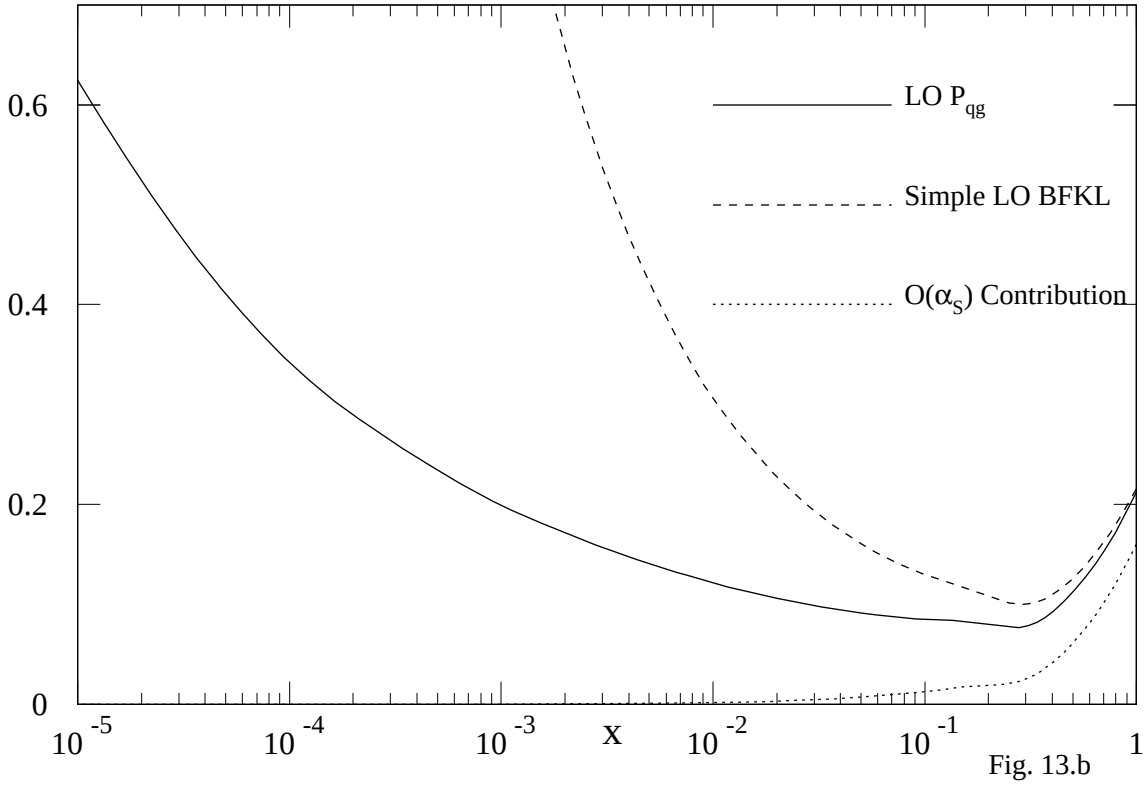

Fig. 13.b

Renormalization Scale Variation

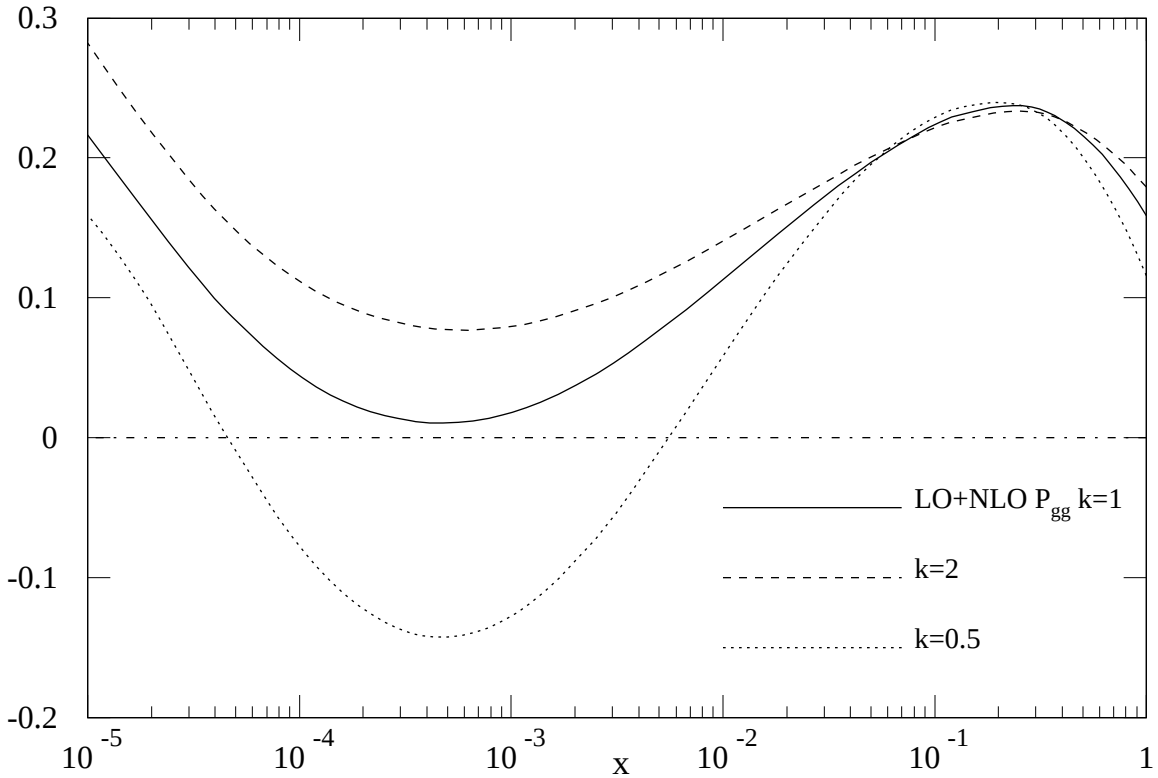

Fig. 11.a

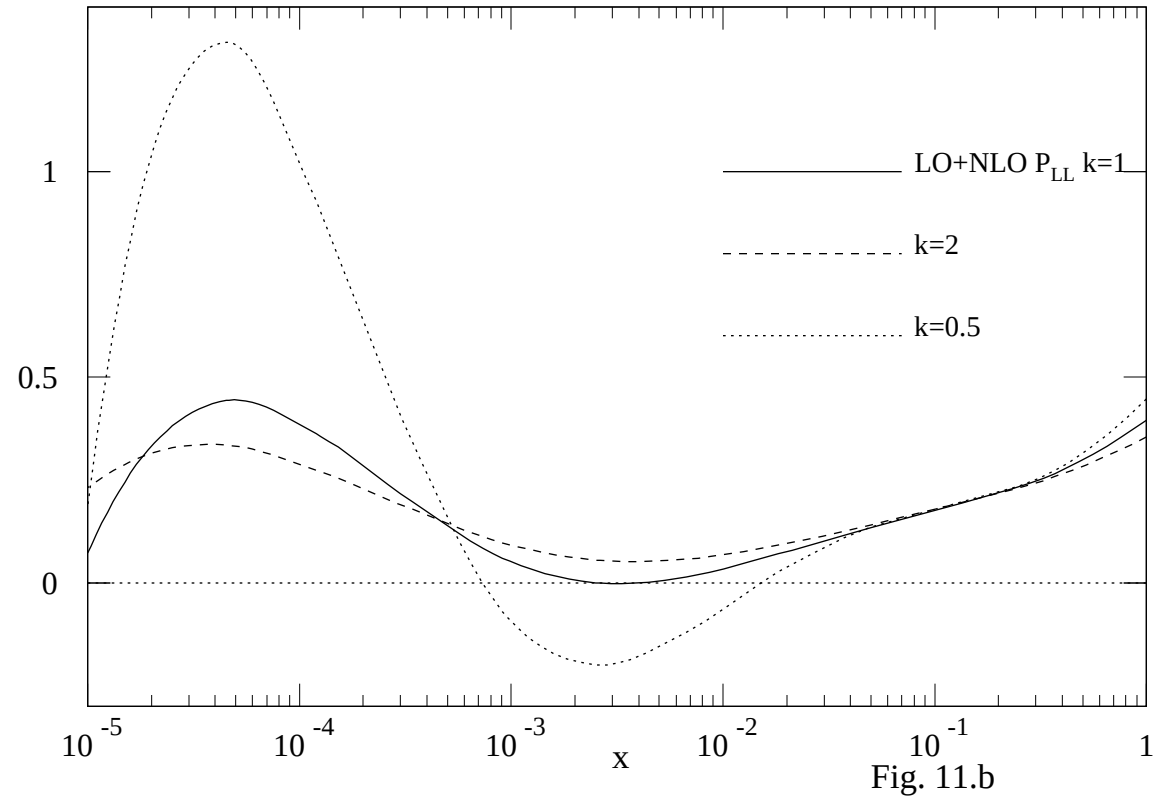

Fig. 11.b