A local version of the Projection Theorem and other results in Geometric Measure Theory.

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Abstract

In this thesis we investigate how knowledge of the local behaviour of a Borel measure on $\mathbf{R}^n$ enables us to deduce information about its global behaviour. The main concept we use for this is that of tangent measures as introduced by Preiss.

In order to illustrate the limitations of tangent measures we first construct a Borel measure μ on $\mathbf{R}^n$ such that for μ-a.e. x, all non-zero, locally finite Borel measures on $\mathbf{R}^n$ are tangent measures of μ at x. Furthermore we show that the set of measures for which this fails to be true is of first category in the space of Borel measures on $\mathbf{R}^n$.

The main result of the thesis is the following:

Suppose that $1 \leq m \leq n$ are integers and μ is a Borel measure on $\mathbf{R}^n$ such that for μ-a.e. x,

1. *The upper and lower m-densities of μ at x are positive and finite.*
2. *If ν is a tangent measure of μ at x then for all $V \in \mathrm{G}(n,m)$ the orthogonal projection of the support of ν onto V is a convex set.*

Then μ is m-rectifiable.

By considering a measure derived from a variation of an example given by Dickinson, we are able to illustrate the necessity of a condition such as (*2*) in our main theorem. Moreover this measure has its average density equal to its upper density and possesses a unique tangent measure distribution almost everywhere.

Our final example is based upon one given by Besicovitch. We show that there is a Borel measure μ with positive and finite upper and lower 1-density almost everywhere and with average 1-density existing almost everywhere but with non-unique tangent measure distributions.

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Chapter 1

Preliminaries

1.1 Introduction

The tools introduced by Besicovitch in his seminal papers of the late twenties and early thirties [Bes28, Bes38, Bes39] have had an enormous influence on the development of modern geometric measure theory. In these papers he introduced, amongst others, the notions of approximate tangents and densities. Many mathematicians working in geometric measure theory today make constant use of these fundamental ideas.

Besicovitch was particularly interested in the dichotomy between regular and irregular sets and one of his most striking results in this direction was his Projection theorem for 1-sets in the plane: it stated that a 1-set, E, in the plane is irregular if and only if for almost every line, L, the projection of E onto L has zero length. This result was eventually extended to general measures in Euclidean space by Federer [Fed47a, Fed47b] and a detailed account of this result may be found in [Fed69, 3.3].

More recently Preiss [Pre87] introduced new tools for studying the local structure of measures — tangent measures. Using these new notions he has succeeded in answering many of the remaining problems concerning rectifiable measures. (See, for example, Theorem 1.5.2.)

In this thesis we investigate how information about the tangent measures of a measure determines its global structure.

In this Chapter we briefly summarise the background material required for the rest of the thesis and provide a basic introduction to the theory of tangent measures. The final section of this chapter illustrates the limitations of tangent measures and shows that they are not a universal panacea.

1.2 Notation

We use $\mathbf{R}^n$ to denote n-dimensional Euclidean space with $\|.\|$ denoting the usual Euclidean norm and $\langle .,. \rangle$ the associated inner product. For $E \subset \mathbf{R}^n$ and $x \in \mathbf{R}^n$ we define

$$\mathrm{d}(x, E) := \inf\{\|y - x\| : y \in E\}$$

and for $r \geq 0$ we set

$$\mathrm{B}(E, r) := \{y \in \mathbf{R}^n : \mathrm{d}(y, E) \leq r\},$$

$$\mathrm{U}(E, r) := \{y \in \mathbf{R}^n : \mathrm{d}(y, E) < r\}.$$

Observe that $\mathrm{U}(E, 0) = \emptyset$ and $\mathrm{B}(E, 0)$ is just the usual topological closure of E in $\mathbf{R}^n$ (usually denoted by $\mathrm{clos}\,(E)$). We abbreviate $\mathrm{B}(\{x\}, r)$ by $\mathrm{B}(x, r)$

and similarly for $\mathrm{U}(x,r)$. If $r > 0$ then $\mathrm{B}(x,r)$ is a non-degenerate ball. If $V \subset \mathbf{R}^n$ let $\operatorname{int}_V(E)$ denote the interior of E with respect to the induced topology on V and let $\partial_V E$ denote the boundary of $E \cap V$ (when considered as embedded in V). Define $\operatorname{int}(E) := \operatorname{int}_{\mathbf{R}^n}(E)$ and $\partial E := \partial_{\mathbf{R}^n}(E)$. $\operatorname{conv}(E)$ will denote the closed convex hull of E. For a set E, $\operatorname{card}(E)$ will denote the cardinality of E.

$\mathbf{N}$ will denote the natural numbers, $\mathbf{Z}$ the integers, $\mathbf{Q}$ the rationals and $\mathbf{Q}^+$ will be the positive rationals. There will also be occasion to refer to $\mathbf{Q}^n$ — n-tuples of rational numbers. For $x \in \mathbf{R}$, $\lceil x \rceil$ will denote the least integer greater than or equal to x and $\lfloor x \rfloor$ will denote the largest integer less than or equal to x.

Let

$$\mathrm{G}(n,m) := \{V \subset \mathbf{R}^n : V \text{ is an } m\text{-dimensional linear subspace of } \mathbf{R}^n\}.$$

For $V \in \mathrm{G}(n,m)$ let P_V denote orthogonal projection onto V thus $\mathrm{P}_V \colon \mathbf{R}^n \to \mathbf{R}^n$ and has range V. Define $V_i \to V$ to mean that $\|\mathrm{P}_{V_i} - \mathrm{P}_V\| \to 0$ in the usual operator norm. Let $\mathrm{P}_V^{\perp}$ be the orthogonal projection onto the $(n-m)$–dimensional subspace of $\mathbf{R}^n$ which is orthogonal to V.

For $x \in \mathbf{R}^n$, $h > 0$, $k \geq 1$ and $V \in \mathrm{G}(n,m)$ define $\mathrm{X}(x,h,k,V)$ by

$$\mathrm{X}(x,h,k,V) := \{y \in \mathbf{R}^n : \|x - y\| \leq k\,[h + \|\mathrm{P}_V(x-y)\|]\}.$$

This is an expanded cone around x with central axis V.

1.3 Some results from Measure Theory

Throughout this thesis by saying that μ is a Borel measure over $\mathbf{R}^n$ we shall understand that μ is a Borel regular outer measure over $\mathbf{R}^n$ such that all Borel sets are μ-measurable. A measure μ is locally finite if for all $x \in \mathbf{R}^n$ there is an $r > 0$ such that $\mu\mathrm{U}(x,r) < \infty$. Recall that locally finite, Borel measures on $\mathbf{R}^n$ are Radon measures (see [Fed69, 2.2.5]). Observe that this implies that for all compact sets $K \subset \mathbf{R}^n$ $\mu(K) < \infty$. All measures we shall consider in this thesis are Borel measures and consequently we shall often just write 'measure' for 'Borel measure'. A measure μ is almost finite if

$$\mu\{x \in \mathbf{R}^n : \text{For all } r > 0,\ \mu\mathrm{U}(x,r) = \infty\} = 0.$$

We define the support of a measure μ by

$$\operatorname{Spt}\mu := \mathbf{R}^n \setminus \{x : \text{There is an } r > 0 \text{ with } \mu\mathrm{U}(x,r) = 0\}.$$

Notice that $\operatorname{Spt}\mu$ is a closed set and $\mu(\mathbf{R}^n \setminus \operatorname{Spt}\mu) = 0$. For a set $E \subset \mathbf{R}^n$ we define the restriction of μ to E, $\mu\lfloor_E$, by

$$\mu\lfloor_E(A) := \mu(A \cap E) \text{ for } A \subset \mathbf{R}^n.$$

Observe that if E is a Borel set and μ is a Borel measure then $\mu\lfloor_E$ is also a Borel measure.

A function $f\colon \mathbf{R}^n \to X$, where X is a topological space, is Borel-measurable if for all open sets $U \subset X$ we find that $f^{-1}(U)$ is a Borel set in $\mathbf{R}^n$. Observe that if μ is a Borel measure on $\mathbf{R}^n$ then f is μ-measurable.

For $x \in \mathbf{R}^n$, $A \subset \mathbf{R}^n$ and $r > 0$ define

$$A_{x,r} := \{x + ra : a \in A\}$$

and for a measure μ on $\mathbf{R}^n$ define a new measure $\mu_{x,r}$ by, for $E \subset \mathbf{R}^n$,

$$\mu_{x,r}(E) := \mu\left(\{x + re : e \in E\}\right).$$

Thus $\mu_{x,r}(E) = \mu(E_{x,r})$. Observe that if $y \in \mathbf{R}^n$ and $s > 0$ then

$$\left(\mu_{x,r}\right)_{y,s} = \mu_{x+ry,rs}$$

and consequently

$$\left(\mu_{x,r}\right)_{-x/r,1/r} = \mu_{0,1} = \mu.$$

One measure which will appear on numerous occasions in this work is m-dimensional Hausdorff measure (where $0 \leq m \leq n$ and we are working in $\mathbf{R}^n$.) If $\alpha(m)$ denotes the Lebesgue measure of a unit ball in $\mathbf{R}^m$ then we define the m-dimensional Hausdorff measure of a set $E \subset \mathbf{R}^n$ by

$$\mathcal{H}^m(E) := \sup_{\delta>0} \inf \left\{ \sum_{i=1}^{\infty} \alpha(m) \left(\frac{\operatorname{diam}(U_i)}{2} \right)^m : E \subset \bigcup_i U_i,\ U_i \text{ are open and} \right.$$
$$\left. \text{for all } i, \operatorname{diam}(U_i) \leq \delta \right\}.$$

For further discussion of Hausdorff measures and proofs that they are indeed Borel measures see either [Rog70] or [Fed69, 2.10].

It will also be helpful to define for integer m between 0 and n

$$\mathcal{G}(n,m) := \{c\mathcal{H}^m \lfloor_V : c > 0,\ V \in \mathrm{G}(n,m)\}$$

which may be thought of as the set of flat m-dimensional measures.

One classical result on approximation of measurable functions which we shall use is Lusin's Theorem which states:

Theorem 1.3.1 *If μ is a Radon measure over a locally compact Hausdorff space X, f is a μ-measurable function with values in a separable metric space Y, A is a μ-measurable set for which $\mu(A) < \infty$ and $\epsilon > 0$ then A contains a compact set K such that $\mu(A \setminus K) < \epsilon$ and $f\lfloor_K$ is continuous. (Where $f\lfloor_K$ denotes the restriction of f to K.)*

Proof: See [Fed69, 2.3.5]. ∎

For a measure μ on $\mathbf{R}^n$, $x \in \mathbf{R}^n$ and $0 \leq m \leq n$ define the lower m-density of μ at x, $\underline{\mathrm{D}}_m(\mu, x)$, by

$$\underline{\mathrm{D}}_m(\mu, x) := \liminf_{r \to 0} \frac{\mu\mathrm{B}(x, r)}{\alpha(m) r^m}$$

and define the upper m-density of μ at x, $\overline{\mathrm{D}}_m(\mu, x)$, by

$$\overline{\mathrm{D}}_m(\mu, x) := \limsup_{r \to 0} \frac{\mu\mathrm{B}(x, r)}{\alpha(m) r^m}.$$

If these two limits are the same at a point x then we call their common value the m-density of μ at x and denote it by $\mathrm{D}_m(\mu, x)$.

The following lemma allows us to compare a measure μ with m-dimensional Hausdorff measure.

Lemma 1.3.2 *Suppose that μ is a locally finite, Borel regular measure on $\mathbf{R}^n$, $0 < \chi < \infty$ and $0 \leq m \leq n$.*

1. *If $E \subset \mathbf{R}^n$ is a Borel set such that for μ-a.e. x in E, $\overline{\mathrm{D}}_m(\mu, x) \leq \chi$ then $\mu(E) \leq 2^m \chi \mathcal{H}^m(\mathrm{Spt}\, \mu \cap E)$.*

2. *If $E \subset \mathbf{R}^n$ is a Borel set such that for μ-a.e. x in E, $\overline{\mathrm{D}}_m(\mu, x) \geq \chi$ then $\mu(E) \geq \chi \mathcal{H}^m(\mathrm{Spt}\, \mu \cap E)$.*

Proof: The first statement follows immediately from [Fed69, 2.1.19(1)]. The second statement follows from [Fed69, 2.10.19(3)] which states that if all closed subsets of $\mathbf{R}^n$ are μ-measurable, G is open, $F \subset G$ and $\overline{\mathrm{D}}_m(\mu, x) > t$ whenever $x \in F$ then

$$\mu(G) \geq t\mathcal{H}^m(F).$$

For, as E is Borel and μ is a locally finite, Borel measure, we can find an open set $G \supset E$ such that $\mu(G \setminus E)$ is arbitrarily small. Approximation now gives the result. ■

For a sequence of measures (μ_i) on $\mathbf{R}^n$ we say that μ_i converges to a (locally finite) measure μ (denoted by $\mu_i \to \mu$) if for all continuous functions $f : \mathbf{R}^n \to \mathbf{R}$ with compact support (that is, the set $\{x : f(x) \neq 0\}$ is compact) we have

$$\int f\, d\mu_i \to \int f\, d\mu.$$

If $D > 0$ and two Borel measures μ and ν are such that $(\mu + \nu)\mathrm{U}(0, D) < \infty$ then we define their distance apart on $\mathrm{U}(0, D)$ by

$$\mathrm{F}_D(\mu, \nu) := \sup\left\{\left|\int f\, d\mu - \int f\, d\nu\right| : f \geq 0,\ \mathrm{Spt}\,(f) \subset \mathrm{B}(0, D) \text{ and } \mathrm{lip}\,(f) \leq 1\right\}.$$

It can be shown, [Pre87, 1.11], that

$$\mu_i \to \mu$$

if and only if for all $D > 0$

$$\mathrm{F}_D(\mu_i, \mu) \to 0.$$

Four elementary observations to make about $\mathrm{F}_D(\mu, \nu)$ are

1. $\mathrm{F}_D(\mu,\nu) = D\mathrm{F}_1(\mu_{0,D},\nu_{0,D})$,

2. if $x \in \mathbf{R}^n$ then $D\mathrm{F}_1(\mu_{x,D},\nu_{x,D}) \leq \mathrm{F}_{|x|+D}(\mu,\nu)$,

3. if $D \leq E$ then $\mathrm{F}_D(\mu,\nu) \leq \mathrm{F}_E(\mu,\nu)$,

4. if ω is also a measure then

$$\mathrm{F}_D(\mu,\nu) \leq \mathrm{F}_D(\mu,\omega) + \mathrm{F}_D(\omega,\nu).$$

Let $\mathcal{M}(\mathbf{R}^n)$ denote the set of all locally finite, Borel measures on $\mathbf{R}^n$ (we shall usually write $\mathcal{M}$ for $\mathcal{M}(\mathbf{R}^n)$).

If for $\mu,\nu \in \mathcal{M}$ we define

$$\mathrm{dist}(\mu,\nu) := \sum_{i=1}^{\infty} 2^{-i} \min\{F_i(\mu,\nu),\, 1\}$$

then this is a metric on $\mathcal{M}$ and with this notion of distance $\mathcal{M}$ is both complete and separable (see [Pre87, 1.12(2)].) Notice that if $D > 1$ and $\mathrm{dist}(\mu,\nu) < \epsilon < 2^{1-D}$ then for $0 < m \leq D$, $\mathrm{F}_m(\mu,\nu) < 2^{\lceil m \rceil - 1}\epsilon$. Also observe that for $\mu \in \mathcal{M}$, $\mathrm{F}_D(\mu,.)$ is an upper semicontinuous function with respect to the topology induced by dist on $\mathcal{M}$.

We shall have frequent recourse to the following Lemma which is a consequence of Prohorov's Theorem.

Lemma 1.3.3 *If (μ_i) is a sequence of Borel measures on $\mathbf{R}^n$ such that for all $T > 0$*

$$\limsup_{i\to\infty} \mu_i \mathrm{B}(0,T) < \infty$$

then (μ_i) possesses a convergent subsequence.

Proof: See, for example, [Pre87, Lemma 1.12]. A version of this result for probability measures can be found in [Par67]. ∎

The following Lemmas provide us with some basic techniques to compare two measures.

Lemma 1.3.4 *Suppose that μ and ν are in $\mathcal{M}(\mathbf{R}^n)$ and $D > 0$. If $\tau > 0$ and $E \subset \mathbf{R}^n$ are such that* $\mathrm{B}(E, \tau) \subset \mathrm{B}(0, D)$ *then*

$$\mu(E) \leq \nu \mathrm{B}(E, \tau) + \mathrm{F}_D(\mu,\, \nu)/\tau.$$

Proof: This is [Pre87, Proposition 1.10(3)]. ∎

Lemma 1.3.5 *Suppose $R \geq 1$ and both $\mu, \nu \in \mathcal{M}(\mathbf{R}^n)$. Suppose that $\mathcal{A}$ is a finite family of Borel sets such that:*

1. *If $A, B \in \mathcal{A}$ are distinct then $\mu(A \cap B) = \nu(A \cap B) = 0$,*
2. $[\mathrm{Spt}\,(\mu) \cup \mathrm{Spt}\,(\nu)] \cap \mathrm{B}(0, R) \subset \cup \mathcal{A}$.

Then

$$\mathrm{F}_R(\mu,\, \nu) \leq \sum_{A \in \mathcal{A}} \mathrm{diam}\,(A)(\mu + \nu)(A \cap \mathrm{B}(0, R)).$$

If for all $A \in \mathcal{A}$, $\mathrm{diam}\,(A) \leq d$ *then we have*

$$\mathrm{F}_R(\mu,\, \nu) \leq d(\mu + \nu)(\mathrm{B}(0, R)).$$

Proof: Observe that the second statement follows immediately from the first since (1) and (2) imply that

$$\sum_{A \in \mathcal{A}} (\mu + \nu)(A \cap \mathrm{B}(0, R)) = (\mu + \nu)(\mathrm{B}(0, R)).$$

In order to verify the first statement suppose that $f : \mathbf{R}^n \to [0, \infty)$ is such that $\mathrm{Spt}\,(f) \subset \mathrm{B}(0,R)$ and $\mathrm{lip}\,(f) \leq 1$. We need to estimate

$$\left|\int f\,d(\mu-\nu)\right| = \left|\int_{B(0,R)} f\,d(\mu-\nu)\right|$$

but on using (1) and (2) of the hypotheses we find that

$$\begin{aligned}\left|\int_{B(0,R)} f\,d(\mu-\nu)\right| &= \left|\sum_{A\in\mathcal{A}}\int_{A\cap B(0,R)} f\,d(\mu-\nu)\right| \\ &\leq \sum_{A\in\mathcal{A}}\left|\int_{A\cap B(0,R)} f\,d(\mu-\nu)\right|.\end{aligned}$$

Therefore let us investigate $\left|\int_{A\cap B(0,R)} f\,d(\mu-\nu)\right|$ for some $A \in \mathcal{A}$. For any measure ω we have

$$\omega(A\cap \mathrm{B}(0,R))\inf_{x\in A} f(x) \leq \int_{A\cap B(0,R)} f\,d\omega \leq \omega(A\cap\mathrm{B}(0,R))\sup_{x\in A} f(x)$$

but $\mathrm{lip}\,(f) \leq 1$ and so

$$\inf_A f \geq \sup_A f - \mathrm{diam}\,(A).$$

Hence

$$\omega(A\cap\mathrm{B}(0,R))\left[\sup_A f - \mathrm{diam}\,(A)\right] \leq \int_{A\cap B(0,R)} f\,d\omega \leq \omega(A\cap\mathrm{B}(0,R))\sup_A f.$$

Thus on considering $\int_{A\cap B(0,R)} f\,d(\mu-\nu)$ we find

$$\begin{aligned}\int_{A\cap B(0,R)} f\,d(\mu-\nu) &\leq \mu(A\cap\mathrm{B}(0,R))[\sup_A f] \\ &\qquad -\nu(A\cap\mathrm{B}(0,R))[\sup_A f - \mathrm{diam}\,(A)] \\ &= (\mu-\nu)(A\cap\mathrm{B}(0,R))\sup_A f - \mathrm{diam}\,(A)\nu(A\cap\mathrm{B}(0,R)) \\ &\leq \mathrm{diam}\,(A)\nu(A\cap\mathrm{B}(0,R).\end{aligned}$$

Similarly we find that

$$
\begin{aligned}
\int_{A\cap B(0,R)} f\,d(\mu-\nu) &\geq \mu(A\cap \mathrm{B}(0,R))[\sup_A f - \mathrm{diam}\,(A)] \\
&\qquad - \nu(A\cap \mathrm{B}(0,R))\sup_A f \\
&= (\mu-\nu)(A\cap \mathrm{B}(0,R))\sup_A f - \mathrm{diam}\,(A)\mu(A\cap \mathrm{B}(0,R)) \\
&\geq -\mathrm{diam}\,(A)\mu(A\cap \mathrm{B}(0,R).
\end{aligned}
$$

Thus

$$
\left|\int_{A\cap B(0,R)} f\,d(\mu-\nu)\right| \leq \mathrm{diam}\,(A)(\mu+\nu)(A\cap \mathrm{B}(0,R))
$$

and so returning to our original estimates we find that

$$
\left|\int f\,d(\mu-\nu)\right| \leq \sum_{A\in\mathcal{A}} \mathrm{diam}\,(A)(\mu+\nu)(A\cap \mathrm{B}(0,R))
$$

which implies the Lemma. ∎

1.4 Tangent Measures

Tangent measures were introduced in [Pre87] and are an extension of ideas in [Mar61] and [Mat75]. They provide a natural framework within which to describe and investigate the local behaviour of measures.

Suppose that μ is an almost finite measure on $\mathbf{R}^n$ and that $x \in \mathbf{R}^n$. We say that a non-zero, measure $\nu \in \mathcal{M}$ on $\mathbf{R}^n$ is a tangent measure of μ at x if there are sequences $r_k \searrow 0$ and $c_k > 0$ such that

$$
\nu = \lim_{k\to\infty} c_k \mu_{x,r_k}.
$$

By $\mathrm{Tan}(\mu, x)$ we denote the set of all tangent measures of μ at x.

It is clear that $\mathrm{Tan}(\mu, x)$ has the following property: If $\nu \in \mathrm{Tan}(\mu, x)$ then for $c, r > 0$, $c\nu_{0,r} \in \mathrm{Tan}(\mu, x)$. Also $\mathrm{Tan}(\mu, x) \cup \{\mathbf{0}\}$ is a closed set.

We should first verify that tangent measures exist:

Lemma 1.4.1 *If μ is a almost finite Borel measure on $\mathbf{R}^n$ then for μ-a.e. x*

$$\mathrm{Tan}(\mu, x) \neq \emptyset.$$

Proof: See [Pre87, 2.5] ∎

One of the important properties of tangent measures is that of shift invariance:

Lemma 1.4.2 *Suppose that μ is an almost finite measure on $\mathbf{R}^n$. Then μ-a.e. x has the following property: Whenever $\nu \in \mathrm{Tan}(\mu, x)$ and $\zeta \in \mathrm{Spt}\,\nu$ then*

$$\nu_{\zeta,1} \in \mathrm{Tan}(\mu, x).$$

Proof: This is [Pre87, 2.12]. ∎

If μ is a measure and x is a point in $\mathbf{R}^n$ such that for some $0 \leq m \leq n$

$$0 < \underline{\mathrm{D}}_m(\mu, x) \leq \overline{\mathrm{D}}_m(\mu, x) < \infty$$

then we may define the standardised tangent measures of μ at x, $\mathrm{Tan}_S(\mu, x)$, as follows:

$$\mathrm{Tan}_S(\mu, x) := \left\{ \nu \in \mathcal{M} : \nu = \lim_{k \to \infty} r_k^{-m} \mu_{x,r_k} \text{ for some sequence } r_k \searrow 0 \right\}.$$

It is easy to verify that if ν is a standardised tangent measure of μ at x then for $\rho > 0$

$$0 < \alpha(m)\underline{\mathrm{D}}_m(\mu, x)\rho^m \leq \nu\mathrm{B}(0, \rho) \leq \alpha(m)\overline{\mathrm{D}}_m(\mu, x)\rho^m < \infty \tag{1.1}$$

and hence $0 \in \operatorname{Spt} \nu$. It is interesting and useful to observe that shift invariance holds within the (smaller) set of standardised tangent measures:

Lemma 1.4.3 *Suppose that μ is a measure on $\mathbf{R}^n$, $0 \leq m \leq n$ and for μ-a.e. x*

$$0 < \underline{\mathrm{D}}_m(\mu, x) \leq \overline{\mathrm{D}}_m(\mu, x) < \infty$$

then for μ-a.e. x if $\nu \in \operatorname{Tan}_S(\mu, x)$ and $\zeta \in \operatorname{Spt} \nu$ then $\nu_{\zeta,1} \in \operatorname{Tan}_S(\mu, x)$.

Proof: P. Mörters observed that it suffices to replace every occurrence of the letter 'c' by 'r^{-m}' in the proof of shift invariance for normal tangent measures [Pre87, 2.12]. ■

As an immediate corollary of this Lemma and equation 1.1 preceding it we deduce that:

Corollary 1.4.4 *Suppose that μ is a measure on $\mathbf{R}^n$, $0 \leq m \leq n$ and for μ-a.e. x*

$$0 < \underline{\mathrm{D}}_m(\mu, x) \leq \overline{\mathrm{D}}_m(\mu, x) < \infty$$

then for μ-a.e. x if $\nu \in \operatorname{Tan}_S(\mu, x)$, $\zeta \in \operatorname{Spt} \nu$ and $\rho > 0$ then

$$0 < \alpha(m)\underline{\mathrm{D}}_m(\mu, x)\rho^m \leq \nu \mathrm{B}(\zeta, \rho) \leq \alpha(m)\overline{\mathrm{D}}_m(\mu, x)\rho^m < \infty.$$

Moreover there are tangent measures $\nu, \omega \in \operatorname{Tan}_S(\mu, x)$ such that

$$\underline{\mathrm{D}}_m(\nu, 0) = \underline{\mathrm{D}}_m(\mu, x) \text{ and } \overline{\mathrm{D}}_m(\omega, 0) = \overline{\mathrm{D}}_m(\nu, x).$$

If $0 < \underline{\mathrm{D}}_m(\mu, x) \leq \overline{\mathrm{D}}_m(\mu, x) < \infty$ then $\limsup_{r \to 0} \frac{\mu \mathrm{B}(x,2r)}{\mu \mathrm{B}(x,r)} < \infty$ and so we may use Lemma 1.3.3 to deduce that for every sequence $r_i \searrow 0$, $(r_i^{-m} \mu_{x,r_i})$

possesses a convergent subsequence and hence we deduce that $\mathrm{Tan}_S(\mu, x)$ will be a compact set.

For non-empty compact subsets M, $N \subset \mathcal{M}$ define their Hausdorff distance to be

$$\mathrm{H}(M, N) := \max\{d(M, N),\ d(N, M)\}$$

where

$$d(M, N) := \sup_{\mu \in M} \inf_{\nu \in N} \mathrm{dist}(\mu, \nu).$$

If $\mathcal{K}$ is defined to be the collection of non-empty compact subsets of $\mathcal{M}$ then $(\mathcal{K}, \mathrm{H})$ is a complete separable metric space (see [Mic51, Propositions 4.5(1), 4.1(3)]).

Lemma 1.4.5 *Suppose that $\mu \in \mathcal{M}$, $0 < a \leq b < \infty$, $0 \leq m \leq n$ and $E \subset \mathbf{R}^n$ is a Borel set such that for all $x \in E$*

$$a \leq \underline{\mathrm{D}}_m(\mu, x) \leq \overline{\mathrm{D}}_m(\mu, x) \leq b$$

then the function $t\colon E \to \mathcal{K}$ defined by $t(x) := \mathrm{Tan}_S(\mu, x)$ is Borel-measurable.

Proof: Since $\mathcal{K}$ is a complete separable metric space it suffices to show that for all $0 < \delta < 1$ and $x \in E$

$$F := \{y \in E : \ \mathrm{H}(t(x), t(y)) < \delta\}$$

is a Borel set. However

$$F = \{y \in E : \ d(t(x), t(y)) < \delta\} \cap \{y \in E : \ d(t(y), t(x)) < \delta\}.$$

Thus as, for $u, v \in E$, we have

$$d(t(u), t(v)) = \sup_{\omega \in t(u)} \inf_{\nu \in t(v)} \mathrm{dist}(\omega, \nu)$$

it suffices to verify that for all $0 < \sigma < 1$ and all $R \in \mathbf{N}$ both

$$F_A := \left\{ y \in E : \sup_{\omega \in t(y)} \inf_{\nu \in t(x)} \mathrm{F}_R(\omega, \nu) < \sigma \right\}$$

and

$$F_B := \left\{ y \in E : \sup_{\nu \in t(x)} \inf_{\omega \in t(y)} \mathrm{F}_R(\omega, \nu) < \sigma \right\}$$

are Borel sets. However as, for any $R > 0$ and $\kappa \in \mathcal{M}$, $\mathrm{F}_R(\kappa, .)$ is upper semicontinuous, we find that

$$F_A = \bigcup_{j,k \in \mathbf{N}} \bigcap_{r \in (0,1/k] \cap \mathbf{Q}} \bigcap_{l,T \in \mathbf{N}} \bigcup_{s \in (0,1/T] \cap \mathbf{Q}} F_A(j,k,l,r,s,T)$$

where

$$F_A(j,k,l,r,s,T) := \left\{ y \in E : \mathrm{F}_R\left(r^{-m}\mu_{y,r}, s^{-m}\mu_{x,s}\right) < \sigma\left(1 - j^{-1}\right)\left(1 - k^{-1}\right)\right\}.$$

Also we find that

$$F_B = \bigcup_{j,k \in \mathbf{N}} \bigcap_{r \in (0,1/k] \cap \mathbf{Q}} \bigcup_{l \in \mathbf{N}} \bigcap_{T \in \mathbf{N}} \bigcup_{s \in (0,1/T] \cap \mathbf{Q}} F_B(j,k,l,r,s,T)$$

where

$$F_B(j,k,l,r,s,T) := \left\{ y \in E : \mathrm{F}_R\left(s^{-m}\mu_{y,s}, r^{-m}\mu_{x,r}\right) < \sigma\left(1 - j^{-1}\right)\left(1 - l^{-1}\right)\right\}.$$

Hence, as these sets are clearly Borel, the result follows. ∎

Lemma 1.4.6 *Suppose that $\mathcal{N} \subset \mathcal{M}(\mathbf{R}^n)$, μ is an almost finite measure on $\mathbf{R}^n$, $x \in \mathbf{R}^n$ and for all $R \geq 1$ and $\epsilon > 0$ there is an $s > 0$ such that for all $0 < r \leq s$ we can find a $\nu \in \mathcal{N}$ with*

$$\mathrm{F}_R\left(r^{-m}\mu_{x,r},\, \nu\right) \leq \epsilon$$

then

$$\mathrm{Tan}_S(\mu, x) \subset \mathrm{clos}\,(\mathcal{N}).$$

Proof: Suppose there is an $\omega \in \mathcal{M}$ such that for some sequence $r_i \searrow 0$, $r_i^{-m}\mu_{x,r_i} \to \omega$ and $\omega \notin \operatorname{clos}(\mathcal{N})$. Then we can find an $R \geq 1$ and an $\epsilon > 0$ so that for all $\nu \in \operatorname{clos}(\mathcal{N})$

$$\mathrm{F}_R(\omega,\, \nu) > 2\epsilon.$$

However there is an $s > 0$ such that for all $r_i \leq s$

$$\mathrm{F}_R(\omega, r_i^{-m}\mu_{x,r_i}) \leq \epsilon$$

and so for such an r_i and any $\nu \in \operatorname{clos}(\mathcal{N})$ we have

$$\mathrm{F}_R\left(r_i^{-m}\mu_{x,r_i},\, \nu\right) > \epsilon.$$

In particular, for all $\nu \in \mathcal{N}$

$$\mathrm{F}_R\left(r_i^{-m}\mu_{x,r_i},\, \nu\right) > \epsilon$$

which, if i is sufficiently large, contradicts the hypotheses of the Lemma. ∎

This suggests a strategy for finding the tangent measures of a measure μ. Suppose that we wish to show that for μ-a.e. x, $\operatorname{Tan}_S(\mu, x) = \mathcal{N}$ (assuming that $\mathcal{N}$ is closed). First we should find a small (ideally finite) set $\mathcal{N}' \subset \mathcal{N}$ such that

$$\operatorname{clos}\{r^{-m}\nu_{\zeta,r} : r > 0,\, \zeta \in \operatorname{Spt}\nu,\, \nu \in \mathcal{N}'\} = \mathcal{N}.$$

We should then show that for μ-a.e. x

$$\mathcal{N}' \subset \operatorname{Tan}_S(\mu, x).$$

We may then use shift invariance to conclude that for μ-a.e. x, $\mathcal{N} \subset \operatorname{Tan}_S(\mu, x)$. Finally show that the hypotheses of Lemma 1.4.6 are satisfied by $\mathcal{N}$ for

μ-a.e. x. We may then conclude that for μ-a.e. x

$$\mathcal{N} \subset \mathrm{Tan}_S(\mu, x) \subset \mathrm{clos}\,(\mathcal{N}) = \mathcal{N}.$$

This technique will be used in Chapter 3.

Finally, it is interesting to note that if a normalisation other than r^m is used in the definition of standardised tangent measures then all these results remain true for a measure μ provided that the new normalisation, $h(r)$ say, is such that for μ-a.e. x

1. $0 < \liminf_{r\to 0} \frac{\mu \mathrm{B}(x,r)}{h(r)} \leq \limsup_{r\to 0} \frac{\mu \mathrm{B}(x,r)}{h(r)} < \infty$,

2. the set $\{h(r)^{-1}\mu_{x,r} : 0 < r \leq 1\}$ is such that every sequence of measures in this set possesses a convergent subsequence (not necessarily converging to an element of the set.)

1.5 Rectifiability

A set $E \subset \mathbf{R}^n$ is m-rectifiable for some m, an integer between 0 and n, if there is a countable set of lipschitz maps $f_i : \mathbf{R}^m \to \mathbf{R}^n$ such that

$$\mathcal{H}^m \left(E \setminus \bigcup_i f_i\left(\mathbf{R}^m\right) \right) = 0.$$

A set E is purely m-unrectifiable if for all Lipschitz maps $f : \mathbf{R}^m \to \mathbf{R}^n$

$$\mathcal{H}^m \left(E \cap f\left(\mathbf{R}^m\right)\right) = 0.$$

A measure μ is m-rectifiable if there is an m-rectifiable Borel set E such that

$$\mu\left(\mathbf{R}^n \setminus E\right) = 0.$$

A measure μ is purely m-unrectifiable if for all m-rectifiable Borel sets E

$$\mu(E) = 0.$$

One deep result about rectifiability which we shall use is the Besicovitch-Federer Projection Theorem:

Theorem 1.5.1 *Suppose that E is a purely m-unrectifiable, $\mathcal{H}^m$-measurable set with $\mathcal{H}^m(E) < \infty$ then for almost every $V \in \mathrm{G}(n,m)$*

$$\mathcal{H}^m \mathrm{P}_V(E) = 0.$$

Proof: See [Mat95, Theorem 18.1]. ■

More recently Preiss has proved the following deep theorem about rectifiability:

Theorem 1.5.2 *Whenever μ is an almost finite Borel measure on $\mathbf{R}^n$, the following conditions are equivalent:*

1. *μ is m-rectifiable and is absolutely continuous with respect to $\mathcal{H}^m$.*
2. *For μ-a.e. x, $0 < \mathrm{D}_m(\mu, x) < \infty$.*
3. *For μ-a.e. x, $0 < \underline{\mathrm{D}}_m(\mu, x) < \infty$ and $\mathrm{Tan}(\mu, x) \subset \mathcal{G}(n,m)$.*

Proof: See [Pre87, Theorem 5.6]. ■

1.6 A measure with a large set of tangent measures

Throughout this section we shall be working in $\mathbf{R}^n$. Let $\mathcal{M}$ be the space of all locally finite Borel measures on $\mathbf{R}^n$.

The main advantage of tangent measures is that they often possess more regularity than the original measure and thus a wider range of analytical techniques may be used upon them. In this section we show that there exist non-zero, finite measures, $\mu \in \mathcal{M}$ such that for μ-a.e. x the set of tangent measures of μ at x is equal to $\mathcal{M} \setminus \{\mathbf{0}\}$.(Where $\mathbf{0}$ denotes the zero measure.) We shall then show that for most (in the sense of category) measures $\mu \in \mathcal{M}$ we have that, for μ-a.e. x, $\mathrm{Tan}(\mu, x)$ equals $\mathcal{M} \setminus \{\mathbf{0}\}$. Hence for most measures consideration of their tangent measures does not aid in their analysis.

Before we construct an example of a measure with a large set of tangent measures observe that the definition of tangent measures implies that for $\mu \in \mathcal{M}$ and $x \in \mathbf{R}^n$ the following two simple lemmas hold:

Lemma 1.6.1 *If $\mathcal{N} \subset \mathrm{Tan}(\mu, x)$ then $\bigcup_{r,s>0} r\mathcal{N}_{0,s} \subset \mathrm{Tan}(\mu, x)$ where $r\mathcal{N}_{0,s} := \{r\nu_{0,s} : \nu \in \mathcal{N}\}$.*

Lemma 1.6.2 *If $\mathcal{N} \subset \mathrm{Tan}(\mu, x)$ and $\mathcal{N}$ is dense in $\mathcal{M}$ then $\mathrm{Tan}(\mu, x) = \mathcal{M} \setminus \{\mathbf{0}\}$.*

We can now construct our example.

Theorem 1.6.3 *There exists a non-zero measure $\mu \in \mathcal{M}$ such that for μ-a.e. x, $\mathrm{Tan}(\mu, x) = \mathcal{M} \setminus \{\mathbf{0}\}$.*

Proof: First let us define for $x \in \mathbf{R}^n$ the Dirac measure at x as follows: For $E \subset \mathbf{R}^n$

$$\delta_x(E) := \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{otherwise.} \end{cases}$$

We have that

$$\mathcal{S} = \left\{ \alpha_0\delta_0 + \sum_{i=1}^{m-1} \alpha_i\delta_{x_i} : m \in \{2,3,\ldots\},\ \alpha_i \in \mathbf{Q}^+,\ x_i \in \mathbf{Q}^n,\ \|x_i\| \le 1 \text{ for } \right.$$

$$\left. i \in \{0,\ldots,m-1\} \text{ and } \sum_{i=0}^{m-1} \alpha_i = 1 \text{ and if } i \neq j \text{ then } x_i \neq x_j \right\}$$

is a countable set and if $\nu \in \mathcal{S}$ then it is a probability measure with support in $\mathrm{B}(0,1)$. Moreover

$$\bigcup_{p,q\in\mathbf{Q}^+} p\mathcal{S}_{0,q}$$

is a countable set which is dense in $\mathcal{M}$. Thus by Lemmas 1.6.1 and 1.6.2 it suffices to construct a measure μ such that $\operatorname{Tan}(\mu,x) \supset \mathcal{S}$ for μ-a.e. x.

Let $(\mu_k)_{k=1}^{\infty}$ be a sequence of elements of $\mathcal{S}$ such that every element of $\mathcal{S}$ occurs infinitely many times in this sequence. Thus each μ_k is of the form

$$\mu_k = \alpha(k,0)\delta_0 + \sum_{i=1}^{m_k-1} \alpha(k,i)\delta_{x(k,i)}$$

where the $\alpha(k,i)$, $x(k,i)$ fulfill the appropriate conditions of $\mathcal{S}$ (in particular $x(k,0)=0$). For each μ_k define

$$\sigma_k = \min_{0\le i,j\le m_k-1} \{\|x(k,i)-x(k,j)\| : i \neq j\}.$$

From this define an increasing sequence of real numbers (r_k) by setting $r_1 = 8$ and choosing $r_{k+1} > 8^{k+2} r_k/\sigma_k$.

Let $\Sigma := \prod_{k=1}^{\infty}\{0,\ldots,m_k-1\}$ and let P be the probability measure on Σ obtained by setting for $j \ge 1$

$$P(\eta|_j) := \prod_{k=1}^{j} \alpha_{k,\eta_k}$$

where $\eta|_j := (\eta_1, \ldots, \eta_j) \times \prod_{k=j+1}^{\infty}\{0, \ldots, m_k - 1\}$. Define $\pi : \Sigma \to \mathrm{B}(0,1)$ by

$$\pi(\eta) := \sum_{k=1}^{\infty}(r_k)^{-1}x(k, \eta_k).$$

Notice that π is a well defined $1-1$ map. Set $\mu := \pi_{\#}P$, that is, for $E \subset \mathbf{R}^n$ define

$$\mu(E) := P\left[\pi^{-1}(E)\right].$$

I claim that μ is our required measure. The Borel regularity of μ follows from the continuity of the mapping π with respect to the product topology on Σ.

Lemma 1.6.4 *For a given $\nu \in \mathcal{S}$, let $(v_i)_{i=1}^{\infty}$ be a strictly increasing sequence such that $\mu_{v_i} = \nu$ for all i. Let*

$$V_\nu = \{\eta \in \Sigma \,:\, \eta_{v(i)} = 0 \text{ i.o.}\}.$$

Then $P(V_\nu) = 1$ and so $\mu[\pi(V_\nu)] = 1$.

Proof: We have that for all i

$$P(\eta_{v(i)} = 0) = \alpha(v(i), 0) = \alpha(v(1), 0) > 0$$

therefore $\sum P(\eta_{v(i)} = 0) = \infty$ and so, by the Borel-Cantelli lemma and independence, the lemma follows. ∎

Let $V = \bigcap_{\nu \in \mathcal{S}} V_\nu$ then, as S is countable, $P(V) = 1$ and so $\mu[\pi(V)] = 1$. For $x \in \pi(\Sigma)$ define $x_i := x(i, [\pi^{-1}(x)]_i)$ and so $x = \sum_{i=1}^{\infty} x_i/r_i$. Let $\overline{x} \in \pi(V)$ and let $\overline{\eta}$ be the associated element of V. Fix $\nu \in \mathcal{S}$ and define $(v_i)_{i=1}^{\infty}$ as in the lemma (so $\mu_{v(i)} = \nu$ for all i). Then, as $\overline{\eta} \in V$, there is an infinite set $N \subset \bigcup_{i=1}^{\infty}\{v(i)\}$ such that for all $k \in N$, $\overline{x}_k = 0$ and $\mu_k = \nu$.

We wish to show that $\nu \in \operatorname{Tan}(\mu, \overline{x})$. So we need to find sequences $c_j > 0$ and $s_j \searrow 0$ such that $c_j \mu_{\overline{x}, s_j} \to \nu$ as $j \to \infty$.

Let $s_j = 1/r_{k(j)}$ where $k(j)$ is the j^{th} element of N and so $s_j \searrow 0$.

Define

$$c_j = [\mu\{x \in \pi(\Sigma) : x_i = \overline{x}_i \text{ for } i = 1, \dots, k(j) - 1\}]^{-1}.$$

From Chapter 1 we know that $\phi_k \to \phi$ if and only if for all $R \geq 1$, $\mathrm{F}_R(\phi_k, \phi) \to 0$. So fix $R \geq 1$ and choose $g : \mathbf{R}^n \to [0, \infty)$ such that $\operatorname{Spt}(g) \subset \mathrm{B}(0, R)$ and $\operatorname{lip}(g) \leq 1$. We need to verify that

$$\left| \int g \, d(c_j \mu_{\overline{x}, s_j}) - \int g \, d\nu \right| \to 0.$$

Choose $J \in \mathbf{N}$ such that $\frac{27}{28} 8^{k(J)} > R$.

For $j \geq J$ we have (letting $k := k(j)$)

$$\begin{aligned} \int_{\mathbf{R}^n} g \, d(c_j \mu_{\overline{x}, s_j}) &= c_j \int_{\mathbf{R}^n} g(r_{k(j)}(x - \overline{x})) \, d\mu(x) \\ &= c_j \int_{\pi(\Sigma)} g \left(r_k \sum_{i=1}^{\infty} \frac{x_i - \overline{x}_i}{r_i} \right) d\mu(x) \, . \end{aligned}$$

Let us consider $r_k \sum_{i=1}^{\infty} \frac{x_i - \overline{x}_i}{r_i}$ in more detail. There are two possible cases:

Case 1 : $x_i = \overline{x}_i$ for $i = 1, \dots, k - 1$.

Then since

$$r_k \sum_{i=1}^{\infty} \frac{x_i - \overline{x}_i}{r_i} = x_k - \overline{x}_k + r_k \sum_{i=k+1}^{\infty} \frac{x_i - \overline{x}_i}{r_i}$$

and as

$$\left\| r_k \sum_{i=k+1}^{\infty} \frac{x_i - \overline{x}_i}{r_i} \right\| \leq \frac{2}{7} 8^{-k}$$

we have (as $\overline{x}_k = 0$)

$$\left\| x_k - r_k \sum_{i=1}^{\infty} \frac{x_i - \overline{x}_i}{r_i} \right\| \leq \frac{2}{7} 8^{-k} \, .$$

Case 2 : There exists $u \in \{1, \ldots, k-1\}$ such that $x_i = \overline{x}_i$ for $i = 1, \ldots, u-1$ but $x_u \neq \overline{x}_u$.

Thus

$$\sum_{i=1}^{\infty} \frac{x_i - \overline{x}_i}{r_i} = \frac{x_u - \overline{x}_u}{r_u} + \sum_{i=u+1}^{\infty} \frac{x_i - \overline{x}_i}{r_i}$$

and both

$$\left\| \sum_{i=u+1}^{\infty} \frac{x_i - \overline{x}_i}{r_i} \right\| \leq \frac{\sigma_u}{7r_u} 8^{-k} \text{ and } \left\| \frac{x_u - \overline{x}_u}{r_u} \right\| \geq \frac{\sigma_u}{r_u}$$

therefore

$$\begin{aligned} \left\| r_k \sum_{i=1}^{\infty} \frac{x_i - \overline{x}_i}{r_i} \right\| &\geq \frac{r_k}{r_u} \sigma_u \frac{27}{28} \\ &> \frac{27}{28} 8^k > R. \end{aligned}$$

Thus in Case 2, $g[r_k(x - \overline{x})] = 0$ and so

$$c_j \int_{\pi(\Sigma)} g[r_k(x - \overline{x})] d\mu(x) = c_j \int_X g[r_k(x - \overline{x})] d\mu(x)$$

where $X = \{x \in \pi(\Sigma) : x_i = \overline{x}_i \text{ for } i = 1, \ldots, k-1\}$. Notice that $c_j = [\mu(X)]^{-1}$.

As $\operatorname{lip}(g) \leq 1$, we have, by case 1, that for $x \in X$

$$|g[r_k(x - \overline{x})] - g(x_k)| \leq \frac{2}{7} 8^{-k}.$$

Thus integrating over X and multiplying by c_j gives

$$\left| c_j \int_{\pi(\Sigma)} g[r_k(x - \overline{x})]\, d\mu(x) - \frac{1}{\mu(X)} \int_X g(x_k) d\mu(x) \right| \leq \frac{2}{7} 8^{-k}$$

but, by independence,

$$\begin{aligned} \int_X g(x_k)\, d\mu(x) &= \mu(X) \int_{\pi(\Sigma)} g(x_k) d\mu(x) \\ &= \mu(X) \int_{\mathbf{R}^n} g(x)\, d\nu(x) \end{aligned}$$

and so

$$\mathrm{F}_R\left(c_j \mu_{\overline{x},s_j},\, \nu\right) \leq \frac{2}{7} 8^{-k(j)}$$

and the theorem follows. ∎

In order to reduce unnecessary effort later we make the following observations concerning the measure μ constructed in the proof of this theorem.

Lemma 1.6.5 *The measure μ which was constructed in the proof of Theorem 1.6.3 has the following properties:*

1. $\mu(\mathbf{R}^n) = 1$ *and* $\operatorname{Spt}\mu \subset \mathrm{B}(0,1)$,
2. *for all $\nu \in \mathcal{S}$, $R \geq 1$ and $\theta, \gamma \in (0,1)$ there are a $\rho \in (0, 1/R)$ and $C > 0$ such that*

$$\mu\left\{x : \textit{There is an } r \in (\rho, 1/R) \textit{ and } c \in (0,C) \textit{ with } \mathrm{F}_R(c\mu_{x,r}, \nu) < \gamma\right\}$$
$$> 1 - \theta.$$

Proof: The first item, (1), follows immediately from the definition of μ.

For (2): Suppose (in the notation of the proof of Theorem 1.6.3) that $k(j) \nearrow \infty$ is such that for all j, $\mu_{k(j)} = \nu$. (That such a sequence exists is clear from the definition of μ_k.) Choose N_1 such that

$$R < \frac{27}{28} 8^{k(N_1)} \text{ and } \frac{2}{7} 8^{-k(N_1)} < \gamma.$$

Now choose N_2 such that if

$$\nu = \alpha_0 \delta_0 + \sum_{i=1}^{m-1} \alpha_i \delta_{x_i}$$

(where the $\alpha_i > 0$ for all i) then

$$(1-\alpha_0)^{N_2} < \theta.$$

Finally let ρ be chosen so that

$$0 < \rho < r_{k(N_1+N_2)}^{-1} \ (< 1/R \text{ by the definition of } N_1.)$$

Then we find that

$$\mu\left\{\pi(\eta) : \eta \in \Sigma \text{ and for some } j \in \{N_1, \ldots, N_1+N_2-1\},\ \eta_{k(j)} = 0\right\}$$
$$\geq 1-(1-\alpha_0)^{N_2} > 1-\theta$$

and for any one of these $\pi(\eta)$ it is clear from the calculations in Theorem 1.6.3 that if $j \in \{N_1, \ldots, N_1+N_2-1\}$ is such that $\eta_{k(j)} = 0$ then on setting $c_j := [P\{\sigma \in \Sigma : \sigma_i = \eta_i \text{ for } i \in \{1, \ldots, k(j)-1\}\}]^{-1}$ we find that

$$\begin{aligned} c_j &\leq [P\{\sigma \in \Sigma : \sigma_i = \eta_i \text{ for } i \in \{1, \ldots, k(N_1+N_2-1)-1\}\}]^{-1} \\ &= \prod_{i=1}^{N_1+N_2-1} \alpha_i^{-1} =: C, \text{ say} \end{aligned}$$

and

$$\mathrm{F}_R\left(c_j \mu_{\pi(\eta), r_{k(j)}^{-1}}, \nu\right) \leq \frac{2}{7} 8^{-k(j)} < \gamma.$$

Hence the Lemma follows. ∎

Having shown that there exists a measure with a large set of tangent measures we can now show that, in fact, most measures possess this property. In order to show this we need the notion of sets of first category: A set A contained in a topological space X is of first category if it may be written as a countable union of nowhere dense sets. (A set B in X is nowhere dense if for all non-empty open sets $U \subset X$ there is a non-empty open set $V \subset U$ such that $B \cap V = \emptyset$.) A discussion of these notions may be found in either Kelley [Kel55] or Oxtoby [Oxt71].

Let $\mathcal{N}$ denote the set of locally finite, Borel measures λ on $\mathbf{R}^n$ which possess the following property:

(*) There is a set $A \subset \mathbf{R}^n$ of positive λ-measure such that for all $x \in A$ there is a non-zero measure $\omega \in \mathcal{M}$ with $\omega \notin \mathrm{Tan}(\lambda, x)$.

Thus $\mathcal{M} \setminus \mathcal{N}$ consists of those measures $\kappa \in \mathcal{M}$ such that for κ a.e.-x

$$\mathrm{Tan}(\kappa, x) = \mathcal{M} \setminus \{\mathbf{0}\}.$$

Theorem 1.6.6 *$\mathcal{N}$ is of first category in $\mathcal{M}$.*

Proof: This reduces to showing that we can find a countable union of nowhere dense sets in $\mathcal{M}$ which contains $\mathcal{N}$. This leads us to the following:

Lemma 1.6.7

$$\mathcal{N} \subset \bigcup_{\nu \in \mathcal{S}} \bigcup_{i,R \in \mathbf{N}} E(i, R, \nu)$$

where $E(i, R, \nu)$ is defined to be the set of $\lambda \in \mathcal{M}$ such that $\lambda \mathrm{U}(0, R) > 0$ and

$$\lambda \left\{ x \in \mathrm{U}(0, R+2) : \textit{For } r \in (0, R^{-1}),\ c > 0,\ F_R(c\lambda_{x,r}, \nu) > R^{-1} \right\} > \frac{\lambda \mathrm{U}(0, R)}{2i}.$$

Proof: Suppose $\lambda \in \mathcal{N}$ then there is a set $A \subset \mathbf{R}^n$ of positive λ-measure such that for all $x \in A$ there is a non-zero measure $\omega \notin \mathrm{Tan}(\lambda, x)$. On recalling that $\bigcup_{p,q \in \mathbf{Q}^+} p\mathcal{S}_{0,q}$ is dense in $\mathcal{M}$ and that if $\omega \in \mathrm{Tan}(\lambda, x)$ then $r\omega_{0,s} \in \mathrm{Tan}(\lambda, x)$ for any $r, s > 0$ we deduce that there is a $\nu \in \mathcal{S}$ and a set

$B \subset \mathbf{R}^n$ of positive λ-measure such that for all $x \in B$, $\nu \notin \mathrm{Tan}(\lambda, x)$. Hence we can find an $R \geq 1$ and a set $C \subset B$ of positive λ-measure such that for all $x \in C$, all $0 < r < 1/R$ and all $c > 0$, $\mathrm{F}_R(c\lambda_{x,r}, \nu) > 1/R$. Hence if R is chosen so large that $\lambda(C \cap \mathrm{U}(0, R)) > 0$ then we can find an $i \in \mathbf{N}$ so that

$$\lambda\left\{x \in \mathrm{U}(0, R+2) : \text{For } r \in (0, R^{-1}),\ c > 0,\ F_R(c\lambda_{x,r}, \nu) > R^{-1}\right\} > \frac{\lambda \mathrm{U}(0, R)}{2i}.$$

and so $\lambda \in E(i, R, \nu)$ as required. ∎

It now only remains to show that:

Lemma 1.6.8 *For all $i, R \in \mathbf{N}$ and $\nu \in \mathcal{S}$, $E(i, R, \nu)$ is nowhere dense in $\mathcal{M}$.*

Proof: We may suppose that $E(i, R, \nu)$ is not empty. Suppose that U is an open set with $U \cap E(i, R, \nu) \neq \emptyset$ then we need to find a non-empty open set $V \subset U$ such that $V \cap E(i, R, \nu) = \emptyset$. Suppose that $\lambda \in U \cap E(i, R, \nu)$ and choose $\epsilon > 0$ such that:

(i) $2^{1-(R+3)} > \epsilon > 0$,

(ii) if $\mathrm{dist}(\omega, \lambda) < \epsilon$ then $\omega \mathrm{U}(0, R) > 0$,

(iii) the open set $\{\omega : \mathrm{dist}(\omega, \lambda) < \epsilon\} \subset U$.

Observe that if $\mathrm{dist}(\omega, \lambda) < \epsilon$ then for $0 < m \leq R + 3$, $\mathrm{F}_m(\omega, \lambda) < 2^{\lceil m \rceil - 1}\epsilon$. Since

$$D := \left\{\sum_{j=1}^{N} \beta_j \delta_{y_j} : \beta_j \in \mathbf{Q}^+,\ y_j \in \mathbf{Q}^n\right\}$$

is a countable dense subset of $\mathcal{M}$ we can find an $\omega \in D$, $\omega = \sum_{j=1}^{N} \beta_j \delta_{y_j}$ say, such that

$$\mathrm{dist}(\omega, \lambda) < \epsilon/4.$$

Notice that **(ii)** ensures that $\omega \mathrm{U}(0,R) > 0$. Suppose that T is chosen so that $T \geq R+3$ and $\operatorname{Spt} \omega \subset \mathrm{B}(0,T)$. We now wish to perturb ω slightly to form a new measure, ω', which is a positive distance from $E(i,R,\nu)$. By Lemma 1.6.5 we can find a $\rho \in (0,1/R)$ and $C > 0$ such that

$$\mu\left\{x : \text{There is an } r \in (\rho, R^{-1}) \text{ and } c \in (0,C) \text{ with } \mathrm{F}_R(c\mu_{x,r}, \nu) < 1/(2R)\right\}$$
$$> 1 - (7i)^{-1}.$$

Recall that $\operatorname{Spt} \mu \subset \mathrm{B}(0,1)$ and $\mu(\mathbf{R}^n) = 1$. Define for any measure κ, any $\zeta \in \mathbf{R}^n$ and $r > 0$ a new measure $\kappa^{\zeta,r}$ by, for $G \subset \mathbf{R}^n$

$$\kappa^{\zeta,r}(G) := \kappa\left(\{(y-\zeta)/r : y \in G\}\right).$$

Choose $s \in (0,1/3)$ such that if $i \neq j$ then

$$\mathrm{B}(y_i,2s) \cap \mathrm{B}(y_j,2s) = \emptyset$$

and

$$s < \frac{\epsilon}{4\omega \mathrm{B}(0,T)}.$$

Now observe that for $j \in \{1,\ldots,N\}$ we have

$$\operatorname{Spt} \mu^{y_j,s} \subset \mathrm{B}(y_j,s) \subset \mathrm{B}(0,T+1),$$

$$\mu^{y_j,s}\mathrm{B}(y_j,s) = 1$$

and for all $m \geq T+1$

$$\mathrm{F}_m\left(\mu^{y_j,s}, \delta_{y_j}\right) \leq s.$$

Thus if we define

$$\omega' := \sum_{i=1}^{N} \beta_j \mu^{y_j,s}$$

then we find that for $m \geq T + 1$

$$\mathrm{F}_m(\omega, \omega') \leq s\omega\mathrm{B}(0, T)$$

and so

$$\mathrm{dist}(\omega, \omega') \leq s\omega\mathrm{B}(0, T) < \epsilon/4.$$

Hence the open set

$$\{\kappa : \mathrm{dist}(\kappa, \omega') < \epsilon/2\}$$

is a subset of U. Also, as ω' is made up of identically scaled copies of μ, we find that if

$$W := \{x \in \mathrm{U}(0, R+1+s) : \text{There is an } r \in (\rho s, s/R) \text{ and } c \in (0, C) \text{ with} \\ \mathrm{F}_R(c\omega'_{x,r}, \nu) < 1/(2R)\Big\}$$

then

$$\begin{aligned} \omega'(W) &> \left[1 - (7i)^{-1}\right] \omega\mathrm{U}(0, R+1) \\ &\geq \left[1 - (7i)^{-1}\right] \omega'\mathrm{U}(0, R+1-s). \end{aligned}$$

We shall now find an open ball around ω' which is disjoint from $E(i, R, \nu)$. Suppose that $0 < \sigma < s$ is such that

$$\sigma < \frac{\rho s}{8RC\omega'\mathrm{B}(0, T+1)}.$$

Fix $x \in W$, $r \in (\rho s, s/R)$ and $c \in (0, C)$ such that

$$\mathrm{F}_R\left(c\omega'_{x,r}, \nu\right) < 1/(2R)$$

and suppose that $y \in \mathrm{B}(x, \sigma)$ and $\mathrm{dist}(\kappa, \omega') < 2^{-(R+3)}\sigma(7i)^{-1}\omega'\mathrm{U}(0, R)$. A straightforward application of Lemma 1.3.4 verifies that

$$\omega'\mathrm{U}(0, R+1-s) \geq \frac{1-3(7i)^{-1}}{1-2(7i)^{-1}}\kappa\mathrm{U}(0, R+1-2s).$$

Also we find that

$$\begin{aligned}
\mathrm{F}_R\left(\omega'_{x,r}, \omega'_{y,r}\right) &\leq r^{-1}\mathrm{F}_{Rr}\left(\omega'_{x,1}, \left(\omega'_{y-x,1}\right)_{x,1}\right) \\
&\leq r^{-1}\mathrm{F}_{\|x\|+Rr}\left(\omega', \omega'_{y-x,1}\right) \\
&\leq 2\sigma r^{-1}\omega'\mathrm{B}(0, \|x\| + Rr + \sigma) \\
&\leq 2\sigma r^{-1}\omega'\mathrm{B}(0, T+1).
\end{aligned}$$

This enables us to deduce that

$$\begin{aligned}
\mathrm{F}_R(c\kappa_{y,r}, \nu) &\leq \mathrm{F}_R\left(c\kappa_{y,r}, c\omega'_{y,r}\right) + \mathrm{F}_R\left(c\omega'_{y,r}, c\omega'_{x,r}\right) \\
&\qquad + \mathrm{F}_R\left(c\omega'_{x,r}, \nu\right) \\
&< C\left[r^{-1}\mathrm{F}_{\|y\|+Rr}(\kappa, \omega') + \mathrm{F}_R\left(\omega'_{y,r}, \omega'_{x,r}\right)\right] + 1/(2R) \\
&< 4C\sigma[s\rho]^{-1}\omega'\mathrm{B}(0, T+1) + 1/(2R) \\
&< 1/R.
\end{aligned}$$

Thus if

$$K := \{y \in \mathrm{U}(0, R+1+s+\sigma) : \text{There is an } r \in (\rho s, s/R) \text{ and } c \in (0, C) \text{ with } \mathrm{F}_R(c\kappa_{y,r}, \nu) < 1/R\}$$

then

$$K \supset \mathrm{B}(W, \sigma)$$

and so use of Lemma 1.3.4 gives that

$$\begin{aligned}\kappa(K) \;\geq\; \kappa\mathrm{B}(W,\sigma) \;\geq\;& \omega'(W) - \frac{\mathrm{F}_{R+1+s+\sigma}(\omega',\kappa)}{\sigma} \\ >\;& \left[1-(7i)^{-1}\right]\omega'\mathrm{U}(0,R+1-s) - \frac{\mathrm{F}_{R+2}(\omega',\kappa)}{\sigma}\end{aligned}$$

and as $\mathrm{F}_{R+2}(\omega',\kappa) < \sigma(7i)^{-1}\omega'\mathrm{U}(0,R+1-s)$

$$> \;\left[1-2(7i)^{-1}\right]\omega'\mathrm{U}(0,R+1-s)$$

but, from our earlier estimate, this is

$$\begin{aligned}\geq\;& \left[1-3(7i)^{-1}\right]\kappa\mathrm{U}(0,R+1-2s) \\ >\;& \left[1-(2i)^{-1}\right]\kappa\mathrm{U}(0,R) \;>\; 0.\end{aligned}$$

Hence $\kappa \notin E(i,R,\nu)$ which implies that the open set

$$U := \left\{\kappa : \mathrm{dist}(\kappa,\omega') < \min\{2^{-(R+2)}\sigma\omega'\mathrm{U}(0,R),\epsilon/2\}\right\} \subset V$$

is disjoint from $E(i,R,\nu)$ as required. ∎

Hence the theorem follows immediately from these two Lemmas. ∎

Chapter 2

A local version of the Projection Theorem

2.1 Introduction

The Besicovitch-Federer Projection Theorem (Theorem 1.5.1) has helped to extend the understanding of the structure of sets in $\mathbf{R}^n$. It is however a qualitative result whose hypotheses require global information about the behaviour of a set. It has been observed by G. David and S. Semmes in [DS91] that one of the difficulties in trying to find quantitative characterisations of rectifiability is the lack of a local version of the Projection Theorem. The result in this chapter is a first step towards this goal:

Theorem 2.1.1 *Suppose μ is a non-zero, almost finite, Borel measure on $\mathbf{R}^n$ such that for μ-a.e. x*

1. $0 < \underline{\mathrm{D}}_m(\mu, x) \leq \overline{\mathrm{D}}_m(\mu, x) < \infty.$

2. If ν is a tangent measure of μ at x then for all $V \in \mathrm{G}(n,m)$, $\mathrm{P}_V(\mathrm{Spt}\,\nu)$ is convex.

Then μ is m-rectifiable.

We shall split the proof of the Theorem into two sections; the first section contains many of the preliminary Lemmas which are required for proving the Theorem and the second section contains the actual proof.

2.2 Lemmas

Unless otherwise stated we shall always be working in $\mathbf{R}^n$ and $\mathcal{M} = \mathcal{M}(\mathbf{R}^n)$. Define for $0 < a \leq b < \infty$ and $0 \leq m \leq n$

$$\mathcal{M}^m(a,b) := \{0 \neq \nu \in \mathcal{M} : \text{ For all } \zeta \in \mathrm{Spt}\,\nu, \text{ for all } \rho > 0,$$
$$\alpha(m) a \rho^m \leq \nu \mathrm{B}(\zeta,\rho) \leq \alpha(m) b \rho^m\}$$

and let

$$\mathcal{M}_C^m := \{\nu \in \mathcal{M} : \text{For all } V \in \mathrm{G}(n,m), \mathrm{P}_V(\mathrm{Spt}\,\nu) \text{ is convex }\}.$$

Finally define

$$\mathcal{M}_C^m(a,b) := \mathcal{M}_C^m \cap \mathcal{M}^m(a,b).$$

First let us observe that:

Lemma 2.2.1 *If $0 < a \leq b < \infty$ and if $\nu \in \mathcal{M}_C^m(a,b)$ then for all $\zeta \in \mathbf{R}^n$ and all $\rho > 0$, $\rho^{-m}\nu_{\zeta,\rho} \in \mathcal{M}_C^m(a,b)$.*

Proof: This is immediate. ∎

Lemma 2.2.2 *Suppose W is an $(\mathcal{H}^m, m)$-rectifiable set in $\mathbf{R}^n$. Define*

$$\Lambda(n, m) := \{\lambda : \lambda\colon \{1, \ldots, m\} \to \{1, \ldots, n\} \text{ and if } i < j \text{ then } \lambda(i) < \lambda(j)\}$$

and let $\{e_1, \ldots, e_n\}$ be the standard orthonormal basis in $\mathbf{R}^n$ and for $x = \sum_{i=1}^n x_i e_i$ in $\mathbf{R}^n$ and $\lambda \in \Lambda(n, m)$ define

$$P_\lambda(x) := \sum_{i=1}^m x_{\lambda(i)} e_{\lambda(i)})$$

and

$$V_\lambda := \{P_\lambda(x) : x \in \mathbf{R}^n\}.$$

Then if

$$a_\lambda := \int \operatorname{card}\left(P_\lambda^{-1}(y) \cap W\right) \, d\mathcal{H}^m \lfloor_{V_\lambda}(y)$$

we have that

$$\left[\sum_{\lambda \in \Lambda(n,m)} (a_\lambda)^2\right]^{1/2} \leq \mathcal{H}^m(W) \leq \sum_{\lambda \in \Lambda(n,m)} a_\lambda.$$

Proof: This is [Fed69, 3.2.27] . ∎

Using the same notation as the last Lemma we have:

Lemma 2.2.3 *Suppose that E is an m-dimensional linear subspace of $\mathbf{R}^n$ and there is a $\sigma \in \Lambda(n, m)$ such that*

$$P_\sigma(E) = V_\sigma$$

and for all $\lambda \neq \sigma$ we have

$$\mathcal{H}^m(P_\lambda(E)) = 0.$$

Then $E = V_\sigma$.

Proof: As $E \in \mathrm{G}(n,m)$ there is a simple m-vector ξ representing E. The m-vector ξ may be written (uniquely) as

$$\xi = \sum_{\lambda \in \Lambda(n,m)} \xi_\lambda \boldsymbol{e}_\lambda$$

where $\boldsymbol{e}_\lambda = e_{\lambda(1)} \wedge \ldots \wedge e_{\lambda(m)}$. Since projection onto a fixed coordinate m-plane is linear we deduce that

$$P_\lambda(\xi) = \xi_\lambda \boldsymbol{e}_\lambda.$$

Thus if $\lambda \neq \sigma$ we conclude that as $\mathcal{H}^m(P_\lambda E) = 0$

$$\xi_\lambda = 0$$

and hence the result follows. ∎

Lemma 2.2.4 (Covering) *Suppose that $A \subset \mathbf{R}^m$ is a bounded set and that $\{\mathrm{B}(x, r(x)) : x \in A\}$ is a collection of non-degenerate balls in $\mathbf{R}^m$ such that $\sup_{x \in A} r(x) < \infty$. Then we may find a countable (possibly finite) set $D \subset A$ and an associated disjoint collection of Borel sets $\mathcal{C} := \{C_x : x \in D\}$ such that*

1. *for all $x \in D$, $\mathrm{B}(x, r(x)) \subset C_x \subset \mathrm{B}(x, 4r(x))$,*

2. *$A \subset \cup \mathcal{C}$,*

3. *for all $0 < \epsilon < [2/(3m)]^{m+1}$ and for all $x \in D$*

$$\mathcal{H}^m \left[\mathrm{B}(\partial C_x, \epsilon r(x))\right] \leq c(m) \epsilon^{1/(m+1)} [r(x)]^m.$$

(The constant $c(m)$ depends only on m.)

Proof: Since $\sup_{x\in A} r(x) < \infty$ we may use [Fed69, 2.8.4] to find a countable set $D \subset A$ such that $\{\mathrm{B}(x, r(x)) : x \in D\}$ is a disjoint collection and yet $\{\mathrm{B}(x, 4r(x)) : x \in D\}$ covers A. Moreover as A is a bounded set and $\{\mathrm{B}(x, r(x)) : x \in D\}$ is a disjoint collection we conclude that for all $x \in D$ the set

$$\{y \in D : \mathrm{B}(x, 4r(x)) \subset \mathrm{B}(y, 4r(y))\}$$

is finite. Therefore we may assume that if x and y are distinct elements of D then both $\mathrm{B}(x, 4r(x)) \setminus \mathrm{B}(y, 4r(y))$ and $\mathrm{B}(y, 4r(y)) \setminus \mathrm{B}(x, 4r(x))$ are non-empty. As A is bounded we can find an enumeration $x_1, x_2, \ldots$ of D such that the sequence $(r(x_k))$ is decreasing. Define the collection $\mathcal{C}$ inductively as follows: For $k \geq 1$,

$$C_k := \mathrm{B}(x_k, 4r(x_k)) \setminus \left[\bigcup_{1\leq i \leq k-1} C_i \cup \bigcup_{i \geq k+1} \mathrm{B}(x_i, r(x_i)) \right].$$

Clearly the family $\mathcal{C} := \{C_k : k \geq 1\}$ is disjoint and each member of it is a Borel set. It remains only to verify the other claims.

From the definition of the C_k we have that $C_k \subset \mathrm{B}(x_k, 4r(x_k))$ for all k. In order to verify that $C_k \supset \mathrm{B}(x_k, r(x_k))$ observe that as $\{\mathrm{B}(x_j, r(x_j)) : j \geq 1\}$ is a disjoint family and for all $i < k$, $C_i \cap \mathrm{B}(x_k, r(x_k)) = \emptyset$ then

$$\mathrm{B}(x_k, r(x_k)) \cap \left[\bigcup_{i<k} C_i \cup \bigcup_{i>k} \mathrm{B}(x_i, r(x_i)) \right] = \emptyset.$$

Hence $\mathrm{B}(x_k, r(x_k)) \subset C_k$ and so the first claim holds.

In order to verify the second claim it suffices to show that

$$\cup\mathcal{C} \supset \bigcup_{k\geq 1} \mathrm{B}(x_k, 4r(x_k)).$$

Suppose that $y \in \cup_{k\geq 1}\mathrm{B}(x_k, 4r(x_k))$ and let k be such that $\mathrm{B}(x_k, 4r(x_k))$ is the first ball of which y is a member. Then as for all i, $C_i \subset \mathrm{B}(x_i, 4r(x_i))$ we know that if $i < k$ then $y \notin C_i$. Thus, from the definition of C_k, we conclude that either $y \in C_k$ or there is a $j > k$ such that $y \in \mathrm{B}(x_j, r(x_j))$. But $\mathrm{B}(x_j, r(x_j)) \subset C_j$ and so the second claim holds.

For the third claim fix $j \geq 1$ and for all $i \geq 1$ let $r_i = r(x_i)$. Observe that if

$$D_1 := \{x_i : i < j \text{ and } \mathrm{B}(x_i, 4r_i) \cap \mathrm{B}(x_j, 4r_j) \neq \emptyset\}$$

and

$$D_2 := \{x_i : i > j \text{ and } \mathrm{B}(x_i, r_i) \cap \mathrm{B}(x_j, 4r_j) \neq \emptyset\}$$

then

$$\partial C_j \subset \operatorname{clos}\left[\partial\mathrm{B}(x_j, r_j) \cup \partial\mathrm{B}(x_j, 4r_j) \cup \bigcup_{x\in D_1} \partial\mathrm{B}(x, 4r(x)) \cup \bigcup_{x\in D_2} \partial\mathrm{B}(x, r(x))\right].$$

Thus in order to estimate $\mathcal{H}^m[\mathrm{B}(\partial C_j, \xi r_j)]$ it suffices to investigate the behaviour of the right hand side of the above expression. Let $\mathcal{D}_1 := \{\mathrm{B}(x, 4r(x)) : x \in D_1\}$ and recall that for distinct x and y in D_1, $\mathrm{B}(x, r(x)) \cap \mathrm{B}(y, r(y)) = \emptyset$. Hence (by [Fed69, 2.8.12]) there is a constant $a(m)$ such that $\mathcal{D}_1$ may be written as the union of at most $a(m)$ subfamilies of disjoint balls. Suppose $\mathcal{D}'$ is such a subfamily then each $B \in \mathcal{D}'$ has a radius which is at least r_j and also has a non-empty intersection with $\mathrm{B}(x_j, 4r_j)$. Thus we may obtain from $\mathcal{D}'$ a new family of disjoint sets, $\mathcal{D}''$, by replacing each $B \in \mathcal{D}'$ by $\mathrm{B}(x, r_j)$ where x is chosen such that $\mathrm{B}(x, r_j) \subset \mathrm{B}(x_j, 6r_j) \cap B$. As the family $\mathcal{D}'$ was disjoint it follows that $\operatorname{card}(\mathcal{D}') = \operatorname{card}(\mathcal{D}'')$ and, as the family $\mathcal{D}''$ is disjoint, it follows that

$$\operatorname{card}(\mathcal{D}'') \leq \frac{\alpha(m)6^m r_j^m}{\alpha(m) r_j^m} = 6^m.$$

Thus

$$\operatorname{card}(\mathcal{D}_1) \leq a(m)6^m.$$

In addition, there is a constant $b(m)$ such that for each $B \in \mathcal{D}_1$

$$\mathcal{H}^m\left[\mathrm{B}(\partial(B \cap \mathrm{B}(x_j, 4r_j)), \epsilon r_j)\right] \leq b(m)\epsilon r_j^m.$$

Hence combining the above gives us that there is a constant $a'(m)$ with

$$\mathcal{H}^m\left[\mathrm{B}(\cup_{B\in\mathcal{D}_1}\partial(B \cap \mathrm{B}(x_j, 4r_j)), \epsilon r_j)\right] \leq a'(m)\epsilon r_j^m.$$

Now let us consider $\mathcal{D}_2 := \{\mathrm{B}(x, r(x)) : x \in D_2\}$. If $\mathrm{B}(x,\rho) \in \mathcal{D}_2$ then, since for all $x \in D$ there is no $y \in D$ different from x such that $\mathrm{B}(x, 4r(x)) \subset \mathrm{B}(y, 4r(y))$, we know that

$$\mathrm{B}(x, 4\rho) \setminus \mathrm{B}(x_j, 4r_j) \neq \emptyset$$

and so

$$4r_j + \rho > |x| > 4(r_j - \rho).$$

Thus

$$\mathrm{B}(x,\rho) \subset \mathrm{B}(x_j, 4r_j + 2\rho) \setminus \mathrm{B}(x_j, 4r_j - 5\rho).$$

Hence if $\rho \leq \epsilon^{1/(m+1)} r_j$ then

$$\mathrm{B}\left(\partial\mathrm{B}(x,\rho), \epsilon r_j\right) \subset \mathrm{B}\left(x_j, (4 + 3\epsilon^{1/(m+1)})r_j\right) \setminus \mathrm{B}\left(x_j, (4 - 6\epsilon^{1/(m+1)})r_j\right)$$

and so if $D_2' := \{x \in D_2 : r(x) \leq \epsilon^{1/(m+1)} r_j\}$ then

$$\mathrm{B}\left(\bigcup_{x\in D_2'} \partial\mathrm{B}(x,\rho), \epsilon r_j\right) \subset \mathrm{B}\left(x_j, (4 + 3\epsilon^{1/(m+1)})r_j\right) \setminus \mathrm{B}\left(x_j, (4 - 6\epsilon^{1/(m+1)})r_j\right).$$

Hence

$$\begin{aligned}\mathcal{H}^m\left[\mathrm{B}\left(\bigcup_{x\in D_2'}\partial\mathrm{B}(x,\rho),\epsilon r_j\right)\right] &\leq \alpha(m)4^m r_j^m\Big[(1+3\epsilon^{1/(m+1)}/2)^m \\ &\qquad -(1-3\epsilon^{1/(m+1)}/2)^m\Big]\end{aligned}$$

which, as $\epsilon < [2/(3m)]^{m+1}$, is

$$\begin{aligned}&\leq 3\times 4^m m^2\alpha(m)\epsilon^{1/(m+1)}r_j^m \\ &= c'(m)\epsilon^{1/(m+1)}r_j^m, \text{ say.}\end{aligned}$$

Finally let us estimate the contribution due to balls in $\mathcal{D}_2 \setminus \mathcal{D}_2'$ — balls in this set have radius between $\epsilon^{1/(m+1)}r_j$ and r_j. Since they are disjoint and are all contained in $\mathrm{B}(x_j, 6r_j)$ we deduce that

$$\begin{aligned}\operatorname{card}(\mathcal{D}_2\setminus\mathcal{D}_2') &\leq \frac{\alpha(m)6^m r_j^m}{\alpha(m)\epsilon^{m/(m+1)}r_j^m} \\ &= 6^m\epsilon^{-m/(m+1)}.\end{aligned}$$

Moreover there is a constant $c''(m)$ such that each ball in this collection, B, has

$$\mathcal{H}^m\left[\mathrm{B}(\partial B,\epsilon r_j)\right] \leq c''(m)\epsilon r_j^m.$$

Thus combining these two inequalities we find that

$$\mathcal{H}^m\left[\mathrm{B}\left(\bigcup_{B\in\mathcal{D}_2\setminus\mathcal{D}_2'}\partial B,\epsilon r_j\right)\right] \leq c''(m)6^m\epsilon^{1/(m+1)}r_j^m.$$

Finally putting all these estimates together we deduce that there is a constant $c(m)$ so that

$$\mathcal{H}^m\left[\mathrm{B}\left(\partial C_j,\epsilon r_j\right)\right] \leq c(m)\epsilon^{1/(m+1)}r_j^m$$

as required. ∎

Lemma 2.2.5 *Suppose $0 < a \leq b < \infty$, $0 < \epsilon < (m+1)^{-1/2}/3$ and both R and $D > 0$. If $\mu \in \mathcal{M}$ and $y \in \operatorname{Spt}\mu \cap \mathrm{B}(0, D)$ are such that for $0 \leq r \leq R$*

$$\mu \mathrm{B}(y, r) \geq \alpha(m) a r^m$$

and if $\nu \in \mathcal{M}^m(a, b)$ is such that

$$\mathrm{F}_{D+R}(\mu_{0,r}/r^m, \nu) < \alpha(m) a \epsilon^{m+3}$$

then there is a $\zeta \in \operatorname{Spt}\nu$ such that

$$\|\zeta - y/r\| \leq \epsilon.$$

Proof: This is a consequence of Lemma 1.3.4 with $E = \mathrm{B}(y, m\epsilon/(m+1))$ and $\tau = \epsilon/(m+1)$. ∎

Lemma 2.2.6 *If $\mu \in \mathcal{M}$ is such that for μ-a.e. x*

$$0 < \underline{\mathrm{D}}_m(\mu, x) \leq \overline{\mathrm{D}}_m(\mu, x) < \infty$$

then there is a Borel set B of positive μ-measure such that if $x_i \in B$ for all i and $x_i \to x \in B$ and if $\nu_i \in \operatorname{Tan}_S(\mu, x_i)$ is a sequence of measures converging to a measure ν then ν is in $\operatorname{Tan}_S(\mu, x)$.

Proof: We may find $0 < a \leq b < \infty$ and a Borel set E of finite and positive μ-measure which is contained in the support of μ such that for all $x \in E$

$$a \leq \underline{\mathrm{D}}_m(\mu, x) \leq \overline{\mathrm{D}}_m(\mu, x) \leq b.$$

This implies that for all $x \in E$, $\operatorname{Tan}_S(\mu, x)$ is a non-empty, compact set and moreover, $\operatorname{Tan}_S(\mu, x) \subset \mathcal{M}(a, b)$. Recall from Section 1.4 the definition of

$(\mathcal{K}, \mathrm{H})$ and that it is a complete, separable metric space. From Lemma 1.4.5 recall that $t: E \to \mathcal{K}$ defined by $t(x) := \mathrm{Tan}_S(\mu, x)$ is Borel-measurable. Hence we may use Lusin's Theorem (Theorem 1.3.1) to find a compact subset B contained in E which is of positive μ-measure and upon which t is continuous. Thus if $x_i \to x$ in B and if $\nu_i \in \mathrm{Tan}_S(\mu, x_i)$ converge to a measure ν then since $t(x_i) \to t(x)$ we conclude that $\nu \in \mathrm{Tan}_S(\mu, x)$ — this implies the Lemma. ∎

Lemma 2.2.7 *Suppose* $\mu \in \mathcal{M}(\mathbf{R}^n)$*,* B *is a compact set of positive* μ*-measure and* $0 < a \leq b < \infty$ *are such that*

1. *for all* $x \in B$*,* $a \leq \underline{\mathrm{D}}_m(\mu, x) \leq \overline{\mathrm{D}}_m(\mu, x) \leq b$*,*
2. *if* $(x_i) \subset B$ *and* $x_i \to x \in B$ *and* $\nu_i \in \mathrm{Tan}_S(\mu, x_i)$ *converge to a measure* ν *then* $\nu \in \mathrm{Tan}_S(\mu, x)$*,*
3. *for all* $x \in B$*,* $\mathrm{Tan}_S(\mu, x) \subset \mathcal{M}^m_C(a, b)$*.*
4. *for all* $x \in B$*, if* $\nu \in \mathrm{Tan}_S(\mu, x)$ *and* $\zeta \in \mathrm{Spt}\,\nu$ *then* $\nu_{\zeta,1} \in \mathrm{Tan}_S(\mu, x)$*.*

Then for all $\xi, \gamma \in (0,1)$ *and integer* $M \geq 2$*, there is an* $R \geq 1$ *so that for all* $x \in B$*, all* $\nu \in \mathrm{Tan}_S(\mu, x)$*, all* $V \in \mathrm{G}(n, m)$ *and all distinct points* $\{\zeta^1, \ldots, \zeta^M\} \subset \mathrm{Spt}\,\nu$ *which satisfy*

$$\min_{i \neq j}\{\|\mathrm{P}_V(\zeta^i - \zeta^j)\|\} \geq \gamma \max_{i,j}\{\|\zeta^i - \zeta^j\|\}$$

we have that if $u \in \mathrm{conv}\,\{\mathrm{P}_V\zeta^1, \ldots, \mathrm{P}_V\zeta^M\}$ *then there is a* $Y \in \mathrm{Spt}\,\nu \cap \mathrm{B}(\zeta^1, R\max_{i,j}\{\|\zeta^i - \zeta^j\|\})$ *with*

$$\mathrm{P}_V Y \in \mathrm{B}(u, \xi \min_{i \neq j}\{\|\mathrm{P}_V(\zeta^i - \zeta^j)\|\}).$$

Proof: Suppose that the Lemma is false. Then there are $\xi, \gamma \in (0,1)$ and an integer $M \geq 2$ such that for all $R \geq 1$ there are an $x_R \in B$, $\nu_R \in \mathrm{Tan}_S(\mu, x_R)$, $V_R \in \mathrm{G}(n,m)$ and $\{\zeta_R^1, \ldots, \zeta_R^M\} \subset \mathrm{Spt}\, \nu_R$ with

$$\min_{i \neq j}\{\|\mathrm{P}_{V_R}(\zeta_R^i - \zeta_R^j)\|\} \geq \gamma \max_{i,j}\{\|\zeta_R^i - \zeta_R^j\|\}$$

and yet there is a $u \in \mathrm{conv}\,\{\mathrm{P}_{V_R}\zeta_R^1, \ldots, \mathrm{P}_{V_R}\zeta_R^M\}$ with

$$\mathrm{P}_{V_R}^{-1}[\mathrm{P}_{V_R}\mathrm{B}(u, \xi \min_{i \neq j}\{\|\mathrm{P}_V(\zeta_R^i - \zeta_R^j)\|\})] \cap \mathrm{Spt}\, \nu_R \cap \mathrm{B}(\zeta_R^1, R \max_{i,j}\{\|\zeta_R^i - \zeta_R^j\|\}) = \emptyset.$$

We may suppose, without loss of generality, that

$$\|\zeta_R^1 - \zeta_R^M\| = \max_{i,j}\{\|\zeta_R^i - \zeta_R^j\|\}.$$

There is a $c > 0$ such that $\omega_R := c\nu_{\zeta_R^1, \|\zeta_R^1 - \zeta_R^M\|} \in \mathrm{Tan}_S(\mu, x)$. Let $y_R^i := (\zeta_R^i - \zeta_R^1)/\|\zeta_R^1 - \zeta_R^M\|$. Then we have that

(i) for all i, $y_R^i \in \mathrm{Spt}\, \omega_R$,

(ii) $\min_{i \neq j}\{\|\mathrm{P}_{V_R}(y_R^i - y_R^j)\| \geq \gamma$,

(iii) there is a $u_R \in \mathrm{conv}\,\{\mathrm{P}_{V_R}y_R^1, \ldots, \mathrm{P}_{V_R}y_R^M\}$ such that

$$\mathrm{P}_{V_R}^{-1}[\mathrm{P}_{V_R}\mathrm{B}(u_R, \xi\gamma)] \cap \mathrm{Spt}\, \omega_R \cap \mathrm{B}(0, R) = \emptyset.$$

Upon recalling that for all $x \in B$, $\overline{\mathrm{D}}_m(\mu, x) \leq b < \infty$ we may make appropriate use of compactness and Lemma 1.3.3 to find a sequence $R(k) \to \infty$ such that

(iv) $x_{R(k)} \to x \in B$,

(v) $\omega_{R(k)} \to \omega \in \mathrm{Tan}_S(\mu, x)$ (this follows from the definition of B),

(vi) $V_{R(k)} \to V \in \mathrm{G}(n,m)$,

(vii) for all i, $y^i_{R(k)} \to y^i \in \partial \mathrm{B}(0,1)$,

(viii) $u_{R(k)} \to u \in V \cap \mathrm{conv}\,\{\mathrm{P}_V y^1, \ldots \mathrm{P}_V y^M\}$.

As a consequence of the lower density estimate on μ we find that $\{y^1, \ldots, y^M\} \subset \mathrm{Spt}\,\omega$ and since for all k and all $i \neq j$

$$\|\mathrm{P}_{V_{R(k)}}(y^i_{R(k)} - y^j_{R(k)})\| \geq \gamma$$

it follows that for $i \neq j$

$$\|\mathrm{P}_V(y^i - y^j)\| \geq \gamma.$$

As $\omega \in \mathcal{M}^m_C$, there is an $R \geq 1$ such that

$$\omega[\mathrm{B}(0,R) \cap \mathrm{P}_V^{-1}(V \cap \mathrm{B}(u, \xi\gamma/2))] > 0.$$

However if k is sufficiently large then

$$\mathrm{B}(0,R) \cap \mathrm{P}_V^{-1}(V \cap \mathrm{B}(u, \xi\gamma/2))$$

is a subset of

$$\mathrm{B}(0,R(k)) \cap \mathrm{P}^{-1}_{V_{R(k)}}\left[V_{R(k)} \cap \mathrm{B}(u_{R(k)}, \xi\gamma)\right]$$

and so

$$\begin{aligned}
\omega[\mathrm{B}(0,R) \cap \mathrm{P}_V^{-1}(V \cap \mathrm{B}(u, \xi\gamma/2))] &\leq \limsup_{k\to\infty} \omega_k[\mathrm{B}(0,R) \cap \\
&\qquad \mathrm{P}_V^{-1}(V \cap \mathrm{B}(u, \xi\gamma/2))] \\
&\leq \limsup_{k\to\infty} \omega_k \Big[\mathrm{B}(0,R(k)) \cap \\
&\qquad \mathrm{P}^{-1}_{V_{R(k)}}(V_{R(k)} \cap \mathrm{B}(u_{R(k)}, \xi\gamma))\Big] \\
&= 0, \text{ from the definition of } u_{R(k)}.
\end{aligned}$$

But this is impossible and so the Lemma holds. ∎

Lemma 2.2.8 *If $0 < a \le b < \infty$, μ is a measure, $x \in \operatorname{Spt}\mu$, $s > 0$ and $\nu \in \mathcal{M}^m(a,b)$ are such that for some $R \ge 1$ and $0 < \epsilon < 1/m$*

$$\mathrm{F}_{R+3}\left(\nu, \frac{\mu_{x,s}}{s^m}\right) < \alpha(m)a\epsilon^{m+3}$$

then for all $z \in (x + s\operatorname{Spt}\nu) \cap \mathrm{B}(x, Rs)$ and all $t \in [\epsilon s, s]$ we have that

$$\alpha(m)a(1-3m\epsilon)t \le \mu\mathrm{B}(z,t) \le \alpha(m)b(1+3m\epsilon)t.$$

Proof: This is an application of Lemma 1.3.4 with $E = \mathrm{B}((z-x)/s, t/s)$ and $\tau = \epsilon t/s$. ∎

Lemma 2.2.9 *Suppose $0 \in E \subset \mathbf{R}^n$ is such that for all $V \in \mathrm{G}(n,m)$, $\mathrm{P}_V(E)$ is a convex set and for almost every $V \in \mathrm{G}(n,m)$, $\mathcal{H}^m[\mathrm{P}_V(E)] = 0$ then there is an $(m-1)$-dimensional subspace of $\mathbf{R}^n$ which contains E.*

Proof: If there were $m+1$ points , $\{0, e_1, \ldots, e_m\}$, such that the linear span, V, of $\{0, e_1, \ldots, e_m\}$ was m-dimensional then $\mathcal{H}^m(\mathrm{P}_V\mathrm{conv}\,\{0, e_1, \ldots, e_m\}) > 0$ and moreover for all $W \in \mathrm{G}(n,m)$ sufficiently close to V we would have

$$\mathcal{H}^m(\mathrm{P}_W(\mathrm{conv}\,\{0, e_1, \ldots, e_m\})) > 0$$

which is impossible by the hypotheses of the Lemma. Thus we conclude that for any m points, $\{e_1, \ldots, e_m\}$ in E the linear span of $\{0, e_1, \ldots, e_m\}$ has dimension strictly less than m. This implies the Lemma. ∎

As a consequence of this we have:

Lemma 2.2.10 *If $0 < a \le b < \infty$ and ν is a purely m-unrectifiable, locally finite measure on $\mathbf{R}^n$ with*

1. *for all* $\zeta \in \operatorname{Spt}\nu$, $a \leq \underline{\mathrm{D}}_m(\nu,x) \leq \overline{\mathrm{D}}_m(\nu,x) \leq b$,

2. *for all* $V \in \mathrm{G}(n,m)$, $\mathrm{P}_V(\operatorname{Spt}\nu)$ *is a convex set.*

Then $\nu \equiv 0$.

Proof: From Lemma 1.3.2 we know that for all Borel sets E we have

$$2^m b\mathcal{H}^m(\operatorname{Spt}\nu \cap E) \geq \nu(E) \geq a\mathcal{H}^m(\operatorname{Spt}\nu \cap E)$$

and so we deduce from the unrectifiability of ν that if E is m-rectifiable then $\nu(E) = 0$. Thus $\operatorname{Spt}\nu$ is a purely $(\mathcal{H}^m, m)$-unrectifiable set. Hence the Besicovitch-Federer Projection Theorem enables us to deduce that for almost every $V \in \mathrm{G}(n,m)$ and all $R \geq 0$

$$\mathcal{H}^m\left[\mathrm{P}_V(\operatorname{Spt}\nu \cap \mathrm{B}(0,R))\right] = 0.$$

Thus for almost every $V \in \mathrm{G}(n,m)$

$$\mathcal{H}^m\left[\mathrm{P}_V(\operatorname{Spt}\nu)\right] = 0$$

and so we can use Lemma 2.2.9 to deduce that there is an $(m-1)$-dimensional subspace, W say, which contains $\operatorname{Spt}\nu$. But then for any $\zeta \in \operatorname{Spt}\nu$ and $r > 0$

$$\begin{aligned}
\nu[\mathrm{B}(\zeta,r)] &\leq 2^m b\mathcal{H}^m\left[\operatorname{Spt}\nu \cap \mathrm{B}(\zeta,r)\right] \\
&= 2^m b\mathcal{H}^m\left[W \cap \operatorname{Spt}\nu \cap \mathrm{B}(\zeta,r)\right] \\
&= 0
\end{aligned}$$

which implies that $\operatorname{Spt}\nu = \emptyset$ and so the Lemma follows. ∎

Lemma 2.2.11 *Suppose that* $0 < a \leq \chi \leq b < \infty$, $x \in \mathbf{R}^n$ *and* $\mu \in \mathcal{M}$ *are such that*

1. $\mathrm{Tan}_S(\mu, x) \subset \mathcal{M}_C^m(a, b)$,

2. if $\nu \in \mathrm{Tan}_S(\mu, x)$ *and* $\zeta \in \mathrm{Spt}\,\nu$ *then* $\nu_{\zeta,1} \in \mathrm{Tan}_S(\mu, x)$.

3. $\chi \leq \inf\{\mathrm{D}_m(\omega, 0) : \omega \in \mathrm{Tan}_S(\mu, x) \cap \mathcal{G}(n, m)\} \leq b$.

Then for all $\nu \in \mathrm{Tan}_S(\mu, x)$ *we have that for* ν*-a.e.* ζ

$$\overline{\mathrm{D}}_m(\nu, \zeta) \geq \chi.$$

Proof: Suppose that the Lemma is false: Then there is a $\nu \in \mathrm{Tan}_S(\mu, x)$, a Borel set C of positive ν-measure and a $\theta \in (0, 1)$ such that for all $\zeta \in C$

(i) $\overline{\mathrm{D}}_m(\nu, \zeta) \leq \chi(1 - \theta)$,

(ii) if $\omega \in \mathrm{Tan}_S(\nu, \zeta)$ and $\xi \in \mathrm{Spt}\,\omega$ then $\omega_{\xi,1} \in \mathrm{Tan}_S(\nu, \zeta)$.

Fix $\zeta \in C$ and consider $\omega \in \mathrm{Tan}_S(\mu, \zeta)$ then (ii) enables us to conclude that for all $\xi \in \mathrm{Spt}\,\omega$

$$\overline{\mathrm{D}}_m(\omega, \xi) \leq \chi(1 - \theta).$$

However for any $\xi \in \mathrm{Spt}\,\omega$

$$\omega_{\xi,1} \in \mathrm{Tan}_S(\nu, \zeta) = \mathrm{Tan}_S(\nu_{\zeta,1}, 0) \subset \mathrm{Tan}_S(\mu, x)$$

and so

$$\mathrm{Tan}_S(\omega, \xi) \subset \mathrm{Tan}_S(\mu, x)\,(\subset \mathcal{M}_C^m(a, b)).$$

Thus (2) and (3) of the hypotheses of the Lemma force us to conclude that ω is purely m-unrectifiable. But then Lemma 2.2.10 implies that $\omega = 0$. This is impossible and so our claim holds. ■

It is easy to see that conditions (1),(2),(3) and (5) of the following Lemma are not sufficient to guarantee its conclusion. For example we may consider the measure $\mathcal{H}^1\lfloor_{\partial B(0,1)}+\mathcal{H}^1\lfloor_C$ where $C \subset \mathrm{B}(0,1)$ is any purely 1-unrectifiable 1-set with positive and finite upper and lower 1-densities.

Lemma 2.2.12 *Suppose that $V \in \mathrm{G}(n,m)$ and ν is a measure satisfy the following:*

1. *For all $W \in \mathrm{G}(n,m)$, $\mathrm{P}_W(\mathrm{Spt}\,\nu)$ is a convex set,*
2. *there is a $\chi > 0$ such that for ν-a.e. ζ, $\overline{\mathrm{D}}_m(\nu,\zeta) \geq \chi$,*
3. *there are $0 < a \leq b < \infty$ such that for ν-a.e. ζ*

$$a \leq \underline{\mathrm{D}}_m(\nu,\zeta) \leq \overline{\mathrm{D}}_m(\nu,\zeta) \leq b,$$

4. *for all $I \subset V$, $\nu(\mathrm{P}_V^{-1}(I)) = \chi\mathcal{H}^m(I)$,*
5. *there is an $h \geq 0$ and $k \geq 0$ so that $\mathrm{Spt}\,\nu \subset \mathrm{X}(0,h,k,V)$.*

Then ν is an m-rectifiable measure.

Proof: From the density estimates of (3) we may use Lemma 1.3.2 to deduce that for all Borel sets $E \subset \mathbf{R}^n$

$$\chi\mathcal{H}^m(E \cap \mathrm{Spt}\,\nu) \leq \nu(E) \leq 2^m b\mathcal{H}^m(E \cap \mathrm{Spt}\,\nu). \tag{2.1}$$

These density estimates enable us to conclude that a set A is (ν,m)-rectifiable if and only if $A \cap \mathrm{Spt}\,\nu$ is an $(\mathcal{H}^m,m)$-rectifiable set. Hence we may split $\mathrm{Spt}\,\nu$ into an $(\mathcal{H}^m,m)$-rectifiable Borel set, R, and a purely $(\mathcal{H}^m,m)$-unrectifiable Borel set, U, such that $\mathcal{H}^m(R \cap U) = \nu(R \cap U) = 0$ and $\mathcal{H}^m[(R \cup U) \setminus \mathrm{Spt}\,\nu] = 0$.

If $\mathcal{H}^m[\mathrm{P}_V U] = 0$ then, from (4), we conclude that $\nu(U) = 0$ and hence equation 2.1 implies that $\mathcal{H}^m(U) = 0$ and we are done.

So suppose instead that $\mathcal{H}^m[\mathrm{P}_V U] > 0$. We may suppose (by a suitable translation and relabeling of h and k) that 0 is a density point of $\mathrm{P}_V(U)$. Fix $0 < \xi < 1$ and recall that $\mathrm{Spt}\,\nu \subset \mathrm{X}(0, h, k, V)$. We can find an $r > 0$ such that for $0 < s \le r$

$$\mathcal{H}^m(\mathrm{P}_V(U) \cap \mathrm{B}(0,s)) > (1-\xi)\mathcal{H}^m(\mathrm{B}(0,s) \cap V).$$

So for such an s we find that

$$\nu[\mathrm{P}_V^{-1}(\mathrm{B}(0,s)) \cap R] \le \chi\xi\mathcal{H}^m(\mathrm{B}(0,s) \cap V)$$

and so

$$\mathcal{H}^m[\mathrm{P}_V^{-1}(\mathrm{B}(0,s)) \cap R] \le \xi\mathcal{H}^m(\mathrm{B}(0,s) \cap V).$$

Fix $[1-\xi^{1/m}]r < s < r$. Since $\mathrm{P}_V^{-1}(\mathrm{B}(0,s) \cap \mathrm{X}(a,h,k,V)$ is compact we may find a $\delta > 0$ such that if $W \in \mathrm{G}(n,m)$ has $\|\mathrm{P}_V - \mathrm{P}_W\| < \delta$ then

$$\mathrm{P}_W^{-1}(\mathrm{B}(0,s)) \cap \mathrm{X}(0,h,k,V) \subset \mathrm{P}_V^{-1}(\mathrm{B}(0,r)) \cap \mathrm{X}(0,h,k,V).$$

On observing that (1) and (4) imply that $\mathrm{P}_V \mathrm{Spt}\,\nu = V$ we deduce that we can find $0 < \delta' \le \delta$ such that for all $W \in \mathrm{G}(n,m)$ satisfying $\|\mathrm{P}_V - \mathrm{P}_W\| < \delta'$ we have

$$\mathrm{P}_W[\mathrm{Spt}\,\nu \cap \mathrm{P}_V^{-1}(\mathrm{B}(0,r))] \cap \mathrm{B}(0,s) \supset \mathrm{B}(0,s) \cap W.$$

Hence

$$\begin{aligned}\mathcal{H}^m&[\mathrm{P}_W[U \cap \mathrm{P}_V^{-1}(\mathrm{B}(0,r))] \cap \mathrm{B}(0,s)] \\ &\ge \mathcal{H}^m[\mathrm{B}(0,s) \cap W) - \mathcal{H}^m(\mathrm{P}_W(R \cap \mathrm{P}_V^{-1}\mathrm{B}(0,r))]\end{aligned}$$

$$\begin{aligned} &\geq \alpha(m)s^m - \alpha(m)(\xi\chi/a)r^m \\ &> 0. \end{aligned}$$

However, from the Projection Theorem, we know that for almost every $W \in \mathrm{G}(n,m)$, $\mathcal{H}^m[\mathrm{P}_W U] = 0$. But this contradicts the above and thus $\mathcal{H}^m(U) = 0$ as required. ∎

The following lemma is a key part of the proof of the main theorem: The proof of this theorem (in the next section) is directed towards showing that if there is a purely m-unrectifiable measure μ satisfying the hypotheses of the theorem then we can construct a tangent measure, ν satisfying the hypotheses of the following lemma. However the construction of ν is such that the following condition must also hold: For all $A \subset V$ (defined below) we have

$$\nu \mathrm{P}_V^{-1}(A) = \chi \mathcal{H}^m(A).$$

This is in direct contradiction with the conclusion of the lemma.

Lemma 2.2.13 *Suppose* $0 < a \leq \chi \leq b < \infty$ *and* $\nu \in \mathcal{M}_C^m(a,b)$ *is an* m*-rectifiable measure such that*

1. *for* ν*-a.e.* x, $\overline{\mathrm{D}}_m(\nu, x) \geq \chi$
2. *there is a* $V \in \mathrm{G}(n,m)$ *with* $\operatorname{diam}(\mathrm{P}_V^{\perp}\operatorname{Spt}\nu) > 0$ *and* $V = \mathrm{P}_V(\operatorname{Spt}\nu)$.

Then there is a Borel set $B \subset V$ *such that*

$$\nu(\mathrm{P}_V^{-1}(B)) > \chi \mathcal{H}^m(B).$$

Proof: This follows from Lemma 2.2.3 and Lemma 2.2.2: For suppose we choose an orthonormal basis of $\mathbf{R}^n$, $\{e_1, \ldots, e_n\}$, such that V is the linear subspace spanned by $\{e_1, \ldots, e_m\}$ and σ is its associated map in $\Lambda(n,m)$. If for all $\lambda \neq \sigma$ we have that

$$\mathcal{H}^m[P_\lambda(\mathrm{Spt}\,\nu)] = 0$$

then from Lemma 2.2.3 we conclude that $\mathrm{Spt}\,\nu \subset V$ which contradicts the fact that $\mathrm{diam}\,[\mathrm{P}_V^\perp \mathrm{Spt}\,\nu] > 0$. Thus there is a $\lambda \in \Lambda(n,m)$ which is different from σ and with

$$\mathcal{H}^m[P_\lambda(\mathrm{Spt}\,\nu)] > 0.$$

Hence we can find a closed ball $B \subset V$ such that for some positive ξ

$$\mathcal{H}^m[P_\lambda((\mathrm{P}_V^{-1}B) \cap \mathrm{Spt}\,\nu)] > \xi.$$

By Lemma 2.2.2 we conclude that

$$\mathcal{H}^m(\mathrm{P}_V^{-1}(B) \cap \mathrm{Spt}\,\nu) \geq \left[(\mathcal{H}^m(B))^2 + \xi^2\right]^{1/2}$$

and so as for ν-a.e. x, $\overline{\mathrm{D}}_m(\nu, x) \geq \chi$ we deduce from Lemma 1.3.2 that

$$\begin{aligned} \nu(\mathrm{P}_V^{-1}(B)) &\geq \chi\left[(\mathcal{H}^m(B))^2 + \xi^2\right]^{1/2} \\ &> \chi\mathcal{H}^m(B) \end{aligned}$$

as required. ∎

Our final Lemma in this section is a technical result introduced to avoid unnecessary repetition later.

Lemma 2.2.14 *Fix L (possibly ∞),$a, b, q > 0$ and suppose that S_i, Θ_i and Ξ_i are sequences of positive real numbers with* $\limsup S_i \geq L$, $S_i\Theta_i \to 0$ *and*

$\Xi_i \to 0$. *If* $V_i \in \mathrm{G}(n,m)$, $\nu_i \in \mathcal{M}_C^m(l,u)$ *and* $Y_i \in \operatorname{Spt}\nu_i \cap \mathrm{P}_{V_i}^{-1}[\mathrm{B}(0,1+\Xi_i)] \cap \mathrm{X}(0,a+\Xi_i,b,V_i)$ *are such that*

$$V_i \to V \in \mathrm{G}(n,m),\ Y_i \to Y \ \textit{and}\ \nu_i \to \nu \in \mathcal{M}_C^m(l,u),$$

$$\textit{for all } i,\ \|\mathrm{P}_{V_i}^{\perp} Y_i\| \geq q - \Xi_i$$

and for all $w \in \mathrm{B}(0,S_i) \cap V_i$

$$\mathrm{P}_{V_i}^{-1}[\mathrm{B}(w,\Theta_i S_i)] \cap \operatorname{Spt}\nu_i \cap \mathrm{X}(0,a+\Xi_i,b,V_i) \neq \emptyset.$$

Then (on interpreting $\mathrm{B}(0,\infty)$ *as* $\mathbf{R}^n$*)*

1. $\mathrm{P}_V[\operatorname{Spt}\nu \cap \mathrm{X}(0,a,b,V)] \supset V \cap \mathrm{B}(0,L)$,
2. $\|\mathrm{P}_V^{\perp} Y\| \geq q$,
3. $Y \in \operatorname{Spt}\nu \cap \mathrm{P}_V^{-1}\mathrm{B}(0,1) \cap \mathrm{X}(0,a,b,V)$.

Proof: First observe that an immediate consequence of the density estimates on the ν_i is that if $y_i \in \operatorname{Spt}\nu_i \in \mathcal{M}_C^m(l,u)$ for all i and $y_i \to y$ then $y \in \operatorname{Spt}\nu$. Hence we may immediately conclude that $Y \in \operatorname{Spt}\nu$. Moreover it is clear that $\|\mathrm{P}_V^{\perp} Y\| \geq q$. If there was a $\theta > 0$ such that

$$\mathrm{B}(Y,\theta) \cap \mathrm{P}_V^{-1}(\mathrm{B}(0,1)) \cap \mathrm{X}(0,a,b,V) = \emptyset$$

then we would find that for all i sufficiently large

$$\mathrm{B}(Y_i,\theta/2) \cap \mathrm{P}_{V_i}^{-1}(\mathrm{B}(0,1+\Xi_i)) \cap \mathrm{X}(0,a+\Xi_i,b,V_i) = \emptyset$$

which is impossible and so claim (3) holds.

In order to verify claim (1) fix $\theta > 0$ and (interpreting $\mathrm{B}(0,\infty) = \mathbf{R}^n$) suppose that there is a $v \in \operatorname{int}[\mathrm{B}(0,L)] \cap L$ and an $r > 0$ such that

$$\mathrm{P}_V^{-1}[\mathrm{B}(v,r)] \cap \operatorname{Spt}\nu \cap \mathrm{X}(0, a+\theta, b, V) = \emptyset.$$

Let $v_i \in V_i$ be such that $v_i \to v$. We may find a $0 < \rho < r$ such that there are arbitrarily large i so that

$$\mathrm{P}_{V_i}^{-1}[\mathrm{B}(v_i,\rho)] \cap \mathrm{X}(0, a+\Xi_i, b, V_i)$$

is a subset of

$$\mathrm{P}_V^{-1}[\mathrm{B}(v,r)] \cap \mathrm{X}(0, a+\theta, b, V)$$

and

$$\mathrm{P}_{V_i}^{-1}[\mathrm{B}(v_i,\rho)] \cap \operatorname{Spt}\nu_i \cap \mathrm{X}(0, a+\Xi_i, b, V) \neq \emptyset$$

which in view of our earlier note enables us to deduce that

$$\mathrm{P}_V^{-1}[\mathrm{B}(v,r)] \cap \operatorname{Spt}\nu \cap \mathrm{X}(0, a+\theta, b, V) \neq \emptyset,$$

contradicting the choice of v and r. Hence, as $\operatorname{Spt}\nu$ is a closed set

$$V \cap \mathrm{B}(0,L) \subset \mathrm{P}_V[\operatorname{Spt}\nu \cap \mathrm{X}(0, a+\theta, b, V)]$$

and thus as θ was arbitrary

$$V \cap \mathrm{B}(0,L) \subset \mathrm{P}_V[\operatorname{Spt}\nu \cap \mathrm{X}(0, a, b, V)]$$

as required. ∎

2.3 Proof of Theorem

It suffices to prove the Theorem in the case that μ is a locally finite measure. We shall prove the Theorem by contradiction: Suppose that there is a purely m-unrectifiable, non-zero, locally finite, Borel measure μ which satisfies the hypotheses of Theorem 2.1.1.

Our first task is to find a set in $\mathbf{R}^n$ within which we shall work.

(1) From Section 1.3 we know that for μ-a.e. x both

$$\operatorname{Tan}_S(\mu, x) \neq \emptyset$$

and if $\nu \in \operatorname{Tan}_S(\mu, x)$ and $\zeta \in \operatorname{Spt} \nu$ then $\nu_{\zeta,1} \in \operatorname{Tan}_S(\mu, x)$. From the hypotheses of Theorem 2.1.1 we know that for μ-a.e. x

$$\operatorname{Tan}_S(\mu, x) \subset \mathcal{M}_C^m.$$

Thus we can find a Borel set $B \subset \operatorname{Spt} \mu$ of positive and finite μ-measure such that for all $x \in B$

(i) $\emptyset \neq \operatorname{Tan}_S(\mu, x) \subset \mathcal{M}_C^m$ and

(ii) if $\nu \in \operatorname{Tan}_S(\mu, x)$ and $\zeta \in \operatorname{Spt} \nu$ then $\nu_{\zeta,1} \in \operatorname{Tan}_S(\mu, x)$.

By decomposing B into a set of measure zero and a countable number of (Borel) sets of the form $\{x \in B : p \leq \underline{\mathrm{D}}_m(\mu, x) \leq \overline{\mathrm{D}}_m(\mu, x) \leq q\}$ (where p and q are positive rationals) we may find a Borel set $B^{(0)} \subset B$ of positive μ-measure and $0 < l \leq u < \infty$ such that for all $x \in B^{(0)}$

(iii) $2l \leq \underline{\mathrm{D}}_m(\mu, x) \leq \overline{\mathrm{D}}_m(\mu, x) \leq u/2$.

By Lemma 1.3.2, for all Borel sets E we have

$$2l\mathcal{H}^m(E \cap B^{(0)}) \leq \mu(E \cap B^{(0)}) \leq 2^{m+1}u\mathcal{H}^m(E \cap B^{(0)})$$

and so if E is $\mathcal{H}^m$-rectifiable then $\mu(E \cap B^{(0)}) = 0$ which implies that $\mathcal{H}^m(E \cap B^{(0)}) = 0$. Thus $B^{(0)}$ is purely $(\mathcal{H}^m, m)$-unrectifiable and of positive and finite $\mathcal{H}^m$-measure. By the Projection Theorem we may conclude that for almost every $V \in \mathrm{G}(n, m)$

$$\mathcal{H}^m\left[\mathrm{P}_V B^{(0)}\right] = 0.$$

(2) By applying Lemma 2.2.6 we can find a compact subset $B^{(1)}$ of $B^{(0)}$ of positive μ measure such that if $x_i \in B^{(1)}$ for all i and $x_i \to x \in B^{(1)}$ and $\nu_i \in \mathrm{Tan}_S(\mu, x_i)$ are such that they converge to a measure ν then $\nu \in \mathrm{Tan}_S(\mu, x)$.
(3) Let $K := (5m)^{2m}u/l$, and let $\xi := 3/82$ and $\gamma := 1/(100K)$ and define M to be the maximum number of balls of radius $5/4$ and with centres in the boundary of $\mathrm{B}(0, 4)$ in $\mathbf{R}^m$ which may be packed disjointly. Then, by Lemma 2.2.7, there is an $R \geq 1$ so that for all $x \in B^{(1)}$, all $\nu \in \mathrm{Tan}_S(\mu, x)$,

2.3.1 Properties of $B^{(1)}$ dependent upon ϵ

(4) If ν is a standardised tangent measure of μ at $x \in B^{(1)}$ then ν is not the zero measure and so Lemma 2.2.10 implies that ν is not purely m-unrectifiable. Thus for all $x \in B^{(1)}$ we conclude that $\mathrm{Tan}_S(\mu,x) \cap \mathcal{G}(n,m) \neq \emptyset$ and so we may define

$$\chi := \inf \Big\{ \lambda \; : \; \text{There is a Borel set } C \subset B^{(1)} \text{ of positive } \mu\text{-measure so}$$
$$\text{that } \forall x \in C,\ \exists \nu \in \mathrm{Tan}_S(\mu,x) \cap \mathcal{G}(n,m) \text{ with } \mathrm{D}_m(\nu,0) \leq \lambda \Big\}.$$

Thus

$$\mu \Big\{ x \in B^{(1)} \; : \text{There is a } \nu \in \mathrm{Tan}_S(\mu,x) \cap \mathcal{G}(n,m) \text{ with } \mathrm{D}_m(\nu,0) < \chi \Big\} = 0$$

and so we may find a compact subset $B^{(2)}$ of $B^{(1)}$ which is of positive μ-measure such that for all $x \in B^{(2)}$, there is an $\omega \in \mathrm{Tan}_S(\mu,x) \cap \mathcal{G}(n,m)$ with

$$\chi \leq \mathrm{D}_m(\omega,0) \leq \chi(1+\epsilon)$$

and if $\nu \in \mathrm{Tan}_S(\mu,x) \cap \mathcal{G}(n,m)$ then $\mathrm{D}_m(\nu,0) \geq \chi$. Observe also that from Lemma 2.2.11 we have that if $x \in B^{(2)}$ and $\nu \in \mathrm{Tan}_S(\mu,x)$ then for ν-a.e. ζ

$$\overline{\mathrm{D}}_m(\nu,\zeta) \geq \chi.$$

(5) We can find a Borel subset $B^{(3)}$ of $B^{(2)}$ of positive μ-measure and $0 < r' \leq 1$ such that for all $x \in B^{(3)}$ and all $0 \leq r \leq r'$

(iv) $\alpha(m) l r^m \leq \mu \mathrm{B}(x,r) \leq \alpha(m) u r^m$.

(6) We can find a Borel subset $B^{(4)}$ of $B^{(3)}$ of positive μ-measure and $0 < r'' \leq r'$ so that for all $x \in B^{(4)}$ and all $0 < r \leq r''$ there is a $\nu \in \mathrm{Tan}_S(\mu,\, x)\,(\subset \mathcal{M}_C^m(l, u))$ so that

$$\mathrm{F}_{R+3+\epsilon^{-1}}\left(\frac{\mu_{x,r}}{r^m}, \nu\right) < \alpha(m) l \epsilon^{m(m+3)}.$$

(7) Let $B^{(5)}$ be a compact subset of $B^{(4)}$ of positive μ-measure and recall that as $B^{(5)}$ is a subset of $B^{(0)}$ we have that for almost every $V \in \mathrm{G}(n,m)$ $\mathcal{H}^m[\mathrm{P}_V B^{(5)}] = 0$.

(8) By a suitable translation of μ (and hence the corresponding $B^{(i)}$) we may suppose without loss of generality that $0 \in B^{(5)}$ and it is a density point of $B^{(5)}$. Thus we can find a $0 < r''' \leq r''$ so that for all $0 < r < r'''$

$$\mu[\mathrm{B}(0,r)] > (1 - (\epsilon^m/4))\mu \mathrm{B}(0,r).$$

By (4) we can find an $\omega_0 \in \mathrm{Tan}_S(\mu, 0) \cap \mathcal{G}(n,m)$ such that

$$\chi \leq \mathrm{D}_m(\omega_0, 0) \leq \chi(1+\epsilon).$$

Thus as for almost every $V \in \mathrm{G}(n,m)$, $\mathcal{H}^m[\mathrm{P}_V B^{(5)}] = 0$ and for infinitely many $0 < \rho \leq r'''$

$$\mathrm{F}_{R+3+\epsilon^{-1}}\left(\frac{\mu_{0,\rho}}{\rho^m}, \omega_0\right) < \alpha(m) l \epsilon^{m(m+3)}/2$$

we can find a $\nu_0 \in \mathcal{G}(n,m)$ with $\mathrm{Spt}\,\nu_0 = V_0$, say, such that

$$\mathcal{H}^m[\mathrm{P}_{V_0} B^{(5)}] = 0,$$

$$\chi \leq \mathrm{D}_m(\nu_0, 0) \leq \chi(1+\epsilon)$$

and

$$\mathrm{F}_{R+3+\epsilon^{-1}}(\omega_0,\, \nu_0) < \alpha(m) l \epsilon^{m(m+3)}/2.$$

Thus, in view of the density estimates for μ at 0, there is an $r_0 \leq r'''$ such that

$$\mu \partial \mathrm{B}(0, r_0) = 0 \text{ and } \mathrm{F}_{R+3+\epsilon^{-1}}\left(\frac{\mu_{0,r_0}}{r_0^m},\, \nu_0\right) < \alpha(m) l \epsilon^{m(m+3)}$$

Hence we have, in summary,

(v) $\mathcal{H}^m[\mathrm{P}_{V_0} B^{(5)}] = 0$.

(vi) $\mu \partial \mathrm{B}(0, r_0) = 0$.

(vii) $\chi \leq \mathrm{D}_m(\nu_0, 0) \leq \chi(1+\epsilon)$.

(viii) $\mathrm{F}_{R+3+\epsilon^{-1}}\left(\frac{\mu_{0,r_0}}{r_0^m},\, \nu_0\right) < \alpha(m) l \epsilon^{m(m+3)}$.

(ix) If we define

$$F := B^{(5)} \cap \mathrm{B}(0, r_0)$$

then F is compact and (by (7)) for $0 \leq r \leq r_0$

$$\mu[F \cap \mathrm{B}(0, r)] > (1 - (\epsilon^m/4)) \mu \mathrm{B}(0, r).$$

Henceforth let P denote orthogonal projection onto V_0. Let

$$r_1 := (1-\alpha) r_0, \qquad L := l r_0^m / \mu \mathrm{B}(0, r_0),$$

$$\Lambda_1 := \alpha(m) 2^{-1} 5^{-m} m^{-m/2} L K \qquad \text{and} \qquad \Lambda_2 := \alpha(m) 3^m 2^{-3m-1} L \delta^m.$$

For $u \in V_0$ and $s \geq 0$ define

$$\mathrm{S}(u, s) := \{y \in \mathbf{R}^n : \|P(y) - u\| \leq s\} \cap B(0, r_0)$$

and

$$S^o(u,s) := \{y \in \mathbf{R}^n : \|P(y) - u\| < s\} \cap B(0, r_0).$$

We now define a real-valued function on points of $V_0 \cap \mathrm{B}(0, r_1)$:

For $u \in V_0 \cap \mathrm{B}(0, r_1)$ define $s(u)$ to be the least (non-negative) number such that if

$$s(u) < s \leq r_0 - \|u\|$$

then

[I] for all $v \in V_0 \cap \mathrm{B}(u, (1-\delta)s)$,

$$F \cap \mathrm{S}(v, \delta s) \neq \emptyset$$

and

[II] there is a $W \in \mathrm{G}(n,m)$ and $t \in \mathbf{R}^n$ such that

$$F \cap \mathrm{S}(u, s) \subset \mathrm{B}(t + W, \delta s)$$

and if $x, y \in W$ then

$$K\|\mathrm{P}x - \mathrm{P}y\| \geq \|x - y\|.$$

Let

$$A := \{u \in V_0 \cap \mathrm{B}(0, r_1) : s(u) = 0\},$$

$$A_1 := \left\{u \in [V_0 \cap \mathrm{B}(0, r_1)] \setminus A : \mu S^o(u, s(u)) \geq \Lambda_1 \left[\frac{s(u)}{r_0}\right]^m \mu \mathrm{B}(0, r_0)\right\},$$

$$A_2 := \left\{u \in [V_0 \cap \mathrm{B}(0, r_1)] \setminus A : \mu[\mathrm{S}(u, s(u)) \setminus F] \geq \Lambda_2 \left[\frac{s(u)}{r_0}\right]^m \mu \mathrm{B}(0, r_0)\right\}$$

and let

$$A_3 := \Big\{ u \in [V_0 \cap \mathrm{B}(0,r_1)] \setminus [A \cup A_1 \cup A_2] : \text{[I] holds for } s(u) \text{ at } u \text{ and}$$
$$\delta s(u) \le \operatorname{diam}[\mathrm{P}^{\perp}[F \cap \mathrm{S}(u,s(u))]] \le 2K(1+\delta)s(u) \Big\}.$$

As F is compact, if $u \in A$ then $\mathrm{P}^{-1}(u) \cap F \neq \emptyset$ and hence (from (8v)) we conclude that $\mathcal{H}^m A = 0$.

The function $s(u)$ provides us with a tool to investigate the properties of the set F. Our next task is to establish some of the elementary properties of $s(u)$ and the sets associated with it. We shall say that a positive real number s is good for a point $u \in V_0 \cap \mathrm{B}(0,r_1)$ if it satisfies both [I] and [II]. It is bad if it doesn't!

Our first task is to ensure that $s(u)$ is well defined:

Lemma 2.3.1 *For all $u \in V_0 \cap \mathrm{B}(0,r_1)$, $s(u) \le \epsilon r_0/\delta$.*

Proof: Fix $u \in V_0 \cap \mathrm{B}(0,r_1)$ and $\epsilon r_0/\delta \le s \le r_0 - \|u\|$. As $r_0 - \|u\| \ge \alpha r_0$ and $\epsilon/\delta < \alpha$ this is a non-trivial interval. As $\nu_0 \in \mathcal{M}^m_C(\chi, \chi(1+\epsilon)) \cap \mathcal{G}(n,m)$ has $\operatorname{Spt}\nu_0 = V_0$ and $\mathrm{F}_{R+3}(\mu_{0,r_0}/r_0, \nu_0) < \alpha(m) l \epsilon^{m+3}$ then for $v \in V_0 \cap \mathrm{B}(u,(1-\delta)s)$ we have by Lemma 2.2.8, since $\delta s \in [\epsilon r_0, r_0]$, that

$$\begin{aligned}
\mu\mathrm{B}(v,\delta s) \ge \alpha(m)\chi(1-3m\epsilon^m)(\delta s)^m &\ge \alpha(m)\chi(1-3m\epsilon^m)(\epsilon r_0)^m \\
&\ge \left[\frac{1-3m\epsilon^m}{1+3m\epsilon^m}\right](1+\epsilon)^{-1}\epsilon^m \mu\mathrm{B}(0,r_0) \\
&\ge (\epsilon^m/4)\mu\mathrm{B}(0,r_0) \\
&> \mu[\mathrm{B}(0,r_0) \setminus F].
\end{aligned}$$

Thus $\mathrm{S}(v,\delta s) \cap F \neq \emptyset$ and so [I] is satisfied. I claim that [II] holds with $W = V_0$ and $t = 0$. For suppose there is an $x \in [F \setminus \mathrm{B}(V_0,\delta s)] \cap \mathrm{S}(u,s)$ and consider $\mathrm{B}(x, \delta s)$ (which is disjoint from V_0). Let $\zeta = x/r_0$ and $\rho = \delta s/r_0$. By the definition of ϵ^m we know that $\delta s \in [\epsilon^m r_0, r_0]$. Moreover as $\|x\| \leq r_0$ and $\delta s \leq \delta r_0$ then $\mathrm{B}(\zeta, \rho) \subset \mathrm{B}(0, R+3+\epsilon^{-1})$ and so by Lemma 1.3.4 (with $E = \mathrm{B}(\zeta, \rho(1-\epsilon^m))$ and $\tau = \epsilon^m \rho$)

$$\begin{aligned}
\frac{1}{\epsilon^m \rho} \mathrm{F}_{R+3+\epsilon^{-1}} \left(\frac{\mu_{0,r_0}}{r_0^m}, \nu_0 \right) &\geq \frac{1}{r_0^m} \mu_{0,r_0} \mathrm{B}(\zeta, (1-\epsilon^m)\rho) - \nu_0 \mathrm{B}(\zeta, \rho) \\
&= \frac{1}{r_0^m} \mu \mathrm{B}(x, (1-\epsilon^m)\delta s) \\
&\geq \alpha(m) l (1 - 2m\epsilon^m)(\delta s/r_0)^m \\
&\geq \alpha(m) l (1 - 2m\epsilon^m)\epsilon^{m^2}
\end{aligned}$$

and so

$$\begin{aligned}
\mathrm{F}_{R+3+\epsilon^{-1}} \left(\frac{\mu_{0,r_0}}{r_0^m}, \nu_0 \right) &\geq \alpha(m) l \epsilon^{m(m+1)} (1 - 2m\epsilon^m)\rho \\
&\geq \alpha(m) l \epsilon^{m(m+2)} (1 - 2m\epsilon^m) \\
&> \alpha(m) l \epsilon^{m(m+3)} \text{ — a contradiction.}
\end{aligned}$$

Hence [II] holds. ∎

Lemma 2.3.2 *For all $u \in V_0 \cap \mathrm{B}(0, r_1)$, if $s(u) > 0$ then it is good for u.*

Proof: This is just an exercise in using the compactness of F. If $s \in (s(u), 2s(u))$ it is good for u. Hence for all v in $V_0 \cap \mathrm{B}(u, (1-\delta)s(u))$, $F \cap \mathrm{S}(v, \delta s)$ is a compact non-empty set . Intersecting these compact sets over s gives that $F \cap \mathrm{S}(v, \delta s(u))$ is not empty and so [I] holds.

[II] follows in a similar manner using the compactness of $\mathrm{G}(n, m)$. ∎

Lemma 2.3.3 *For all* $u \in V_0 \cap \mathrm{B}(0, r_1)$ *if* $s(u) > 0$ *then there is an* $x_u \in \mathrm{S}(u, 0)$ *such that*

$$F \cap \mathrm{S}(u, r_0 - \|u\|) \subset \mathrm{X}(x_u, s(u)/2, 2K(1+2\delta)s(u), V_0).$$

Also

$$\operatorname{diam}\left(\mathrm{P}^{\perp}[F \cap \mathrm{S}(u, s(u))]\right) \leq 2K(1+\delta)s(u).$$

Proof: The second part of the lemma follows from noticing that, as $s(u)$ is good for u (Lemma 2.3.2), there is an $x_u \in \mathrm{S}(u, 0)$ and $W \in \mathrm{G}(n, m)$ such that

$$F \cap \mathrm{S}(u, s(u)) \subset \mathrm{B}(x_u + W, \delta s(u))$$

and if $x, y \in W$ then

$$K\|\mathrm{P}(x-y)\| \geq \|x - y\|.$$

Thus if $\zeta \in F \cap \mathrm{S}(u, s(u))$ then

$$\|\mathrm{P}^{\perp}(\zeta - x_u)\| \leq K(1+\delta)s(u).$$

For the main statement suppose that x_u is as defined above and fix $\zeta \in F \cap \mathrm{S}(u, r_0 - \|u\|)$. If $\zeta \in F \cap \mathrm{S}(u, s(u))$ then by the above it follows that it is in $\mathrm{X}(x_u, s(u)/2, 2K(1+2\delta), V_0)$ as required. So suppose $\zeta \in F \cap [\mathrm{S}(u, r_0 - \|u\|) \setminus \mathrm{S}(u, s(u))]$. As $s(u)$ is good for u we can find $X \in F \cap \mathrm{S}(u, \delta s(u))$ and

$$\|x_u - X\| \leq K\delta s(u) + K\delta s(u).$$

Now consider $\|\mathrm{P}(\zeta) - u\|$ which is good for u and so we can find $W \in \mathrm{G}(n, m)$ and $y_\zeta \in \mathrm{S}(u, 0)$ such that

$$F \cap \mathrm{S}(u, \|\mathrm{P}(\zeta) - u\|) \subset \mathrm{B}(y_\zeta + W, \delta\|\mathrm{P}(\zeta) - u\|)$$

and for all $x, y \in W$

$$K\|\mathrm{P}(x-y)\| \geq \|x-y\|.$$

In particular both X and ζ are in $\mathrm{B}(y_\zeta + W, \delta\|\mathrm{P}(\zeta)-u\|)$. Hence

$$\begin{aligned} \|X-\zeta\| &\leq K[\|\mathrm{P}(X-y_\zeta)\| + \|\mathrm{P}(\zeta-y_\zeta)\|] + 2K\delta\|\mathrm{P}(\zeta)-u\| \\ &\leq K\delta s(u) + K(1+2\delta)\|\mathrm{P}(\zeta)-u\| \end{aligned}$$

and so as $\delta \leq 1/4$

$$\begin{aligned} \|x_u-\zeta\| &\leq K(\delta+2\delta)s(u) + K(1+2\delta)\|\mathrm{P}(\zeta-x_u)\| \\ &\leq K(1+2\delta)s(u)/2 + K(1+2\delta)\|\mathrm{P}(\zeta-x_u)\| \end{aligned}$$

as required. ∎

Lemma 2.3.4 $\mathcal{H}^m(A_2) \leq 4^{m-1}\alpha(m)\epsilon^m\Lambda_2^{-1}r_0^m$.

Proof: Let $D_2 \subset A_2$ be a countable set such that

$$\{\mathrm{B}(u, s(u)) \,:\, u \in D_2\}$$

is disjoint and

$$\{\mathrm{B}(u, 4s(u)) \,:\, u \in D_2\}$$

covers A_2 [Fed69, 2.8.4]. Then as $D_2 \subset A_2$

$$\begin{aligned} \mu[\mathrm{B}(0,r_0)\setminus F] &\geq \sum_{u\in D_2} \mu[\mathrm{S}(u,s(u))\setminus F] \\ &\geq \Lambda_2\mu[\mathrm{B}(0,r_0)]r_0^{-m}\sum_{D_2}[s(u)]^m. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{u\in D_2}[s(u)]^m &\leq \Lambda_2^{-1}r_0^m\frac{\mu[\mathrm{B}(0,r_0)\setminus F]}{\mu\mathrm{B}(0,r_0)} \\ &\leq (\epsilon^m/4)\Lambda_2^{-1}r_0^m. \end{aligned}$$

Thus

$$\mathcal{H}^m(A_2) \leq \alpha(m) \sum_{D_2} [4s(u)]^m \leq \alpha(m) 4^{m-1} \epsilon^m \Lambda_2^{-1} r_0^m$$

as required. ∎

Lemma 2.3.5

$$V_0 \cap \mathrm{B}(0, r_1) \subset A \cup A_1 \cup A_2 \cup A_3.$$

Proof: Suppose that $u \in [V_0 \cap \mathrm{B}(0, r_1)] \setminus [A \cup A_1 \cup A_2]$ and so $s(u) > 0$.

By the definition of $s(u)$ it is possible to find an $s \in ((3/4)s(u), s(u))$ such that it is bad for u— that is either [I] or [II] fails for s at u. Thus there are two cases to consider for s:

1. Either [I] fails for s at u or

2. [I] holds for s at u but [II] fails to hold.

Case 1: [I] fails.

In this situation there is a $v \in V_0 \cap \mathrm{B}(u, (1-\delta)s)$ such that $\mathrm{S}(v, \delta s) \cap F = \emptyset$. In this case let $t = (4/3)s$ and so $s(u) \leq t < \alpha r_0$ and thus t is good for u.

Consider

$$\mathcal{C} := \{\mathrm{B}(w, 5\delta t/4) : x \in V_0 \text{ and } \|x - v\| = 4\delta t\}$$

and let $\mathcal{B} = \{\mathrm{B}(v_i, 5\delta t/4)\}_{i=1}^M$ be a maximal disjoint subfamily of $\mathcal{C}$ (recall that M was defined in (3) of Section 2.3). Since $\delta \leq 1/17$ then for all $i = 1, \ldots M$

$$\|u - v\| + \|v - v_i\| \leq (1 - \delta)s + 4\delta t \leq (1 - \delta)t$$

and since t is good for u we conclude that for all i

$$\mathrm{S}(v_i, \delta t) \cap F \neq \emptyset.$$

So for $i = 1, \ldots, M$ choose $x_i \in \mathrm{S}(v_i, \delta t) \cap F$. Then if $i \neq j$

$$\|\mathrm{P}(x_i - x_j)\| \geq \delta t/2$$

and

$$\|\mathrm{P}(x_i - x_j)\| \leq 10\delta t.$$

Since t is good for u there is a $Y \in \mathrm{S}(u, 0)$ and a $W \in \mathrm{G}(n, m)$ such that

$$F \cap \mathrm{S}(u, t) \subset \mathrm{B}(Y + W, \delta t)$$

and for $x, y \in W$

$$K\|\mathrm{P}(x - y)\| \geq \|x - y\|.$$

Hence for all i and j

$$\begin{aligned} \|x_i - x_j\| &\leq 2K\delta t + K\|\mathrm{P}(x_i - x_j)\| \\ &\leq 2K[\delta + 5\delta]t =: \rho, \text{ say.} \end{aligned}$$

As $\rho \leq r_0 \leq r''$ then by (6) of Section 2.3.1 there is a $\nu \in \mathcal{M}_C^m(l, u)$ so that

$$\mathrm{F}_{R+3+\epsilon^{-1}}(\mu_{x_1,\rho}/\rho^m, \nu) < \alpha(m) l \epsilon^{m(m+3)}$$

and so, by Lemma 2.2.5, we can find for each i a $z_i \in [x_1 + \rho \mathrm{Spt}\, \nu]$ with

$$\|x_i - z_i\| \leq \rho\epsilon^m.$$

As $\rho\epsilon^m \leq \delta t/8$ we conclude that

$$\mathrm{P}(z_i) \in \mathrm{B}(v_i, 9\delta t/8)$$

and so if $i \neq j$ then

$$\|\mathrm{P}(z_i - z_j)\| \geq \delta t/4$$

and also

$$\|\mathrm{P}(z_i - z_j)\| \leq 10\delta t + 2\epsilon^m \rho \leq (41/4)\delta t.$$

In addition

$$\|z_i - z_j\| \leq \rho(1 + 2\epsilon^m).$$

Thus

$$\begin{aligned} \min_{i \neq j}\{\|\mathrm{P}(z_i - z_j)\|\} &\geq (\delta t/4)/[\rho(1 + 2\epsilon^m)] \max_{i,j}\{\|z_i - z_j\|\} \\ &= \delta\,(8K[\delta + 5\delta][1 + 2\epsilon^m])^{-1} \max_{i,j}\{\|z_i - z_j\|\} \\ &\geq \gamma \max_{i,j}\{\|z_i - z_j\|\}. \end{aligned}$$

I claim that v is in the convex hull of $\{\mathrm{P}z_1, \ldots, \mathrm{P}z_M\}$: For if it isn't then there is a unit vector $e \in V_0$ such that if

$$H^- := \{y \in V_0 : \langle y,\, e\rangle < 0\}$$

then

$$v + H^- \supset \operatorname{conv}\{\mathrm{P}z_1, \ldots, \mathrm{P}z_M\}.$$

But consider the point $v + 4\delta t e$: If $z \in v + H^-$ then $z = v + \zeta$ for some $\zeta \in H^-$. Thus, as $\zeta \in H^-$,

$$\begin{aligned} \|z - (v + 4\delta t e)\| = \|\zeta - 4\delta t e\| &\geq |\langle \zeta - 4\delta t e,\, e\rangle| \\ &\geq |4\delta t - \langle \zeta, e\rangle| \quad > \quad 4\delta t. \end{aligned}$$

Hence $\mathrm{B}(v + 4\delta t e, 5\delta t/4) \cap \mathrm{B}(v + H^-, 5\delta t/2) = \emptyset$ and so $\mathrm{B}(v + 4\delta t e, 5\delta t/4) \in \mathcal{C}$ which contradicts the maximality of $\mathcal{B}$ and thus the claim holds.

Hence $\rho^{-1}(v - x_1) \in \operatorname{conv}\{\rho^{-1}(\mathrm{P}z_1 - x_1), \ldots, \rho^{-1}(\mathrm{P}z_M - x_1)\}$ and for all i, $\rho^{-1}(z_i - x_1) \in \operatorname{Spt}\nu$. Thus, from (3) of Section 2.3, we can find a

$$z \in (\operatorname{Spt}\nu) \cap \mathrm{B}(\rho^{-1}(z_1 - x_1), R\rho^{-1}\max\|z_i - z_j\|)$$

$$\subset \operatorname{Spt}\nu \cap \mathrm{B}(\rho^{-1}(z_1 - x_1), R(1 + 2\epsilon^m))$$

such that

$$\mathrm{P}z \in \mathrm{B}(\rho^{-1}(v - \mathrm{P}x_1), \xi\rho^{-1}\min_{i\neq j}\|z_i - z_j\|).$$

Hence on setting $\zeta = x_1 + \rho z$ we conclude that

$$\zeta \in \mathrm{B}(z_1, R\rho(1 + 2\epsilon^m)) \cap [x_1 + \rho\operatorname{Spt}\nu]$$

and

$$\mathrm{P}\zeta \in \mathrm{B}(v, \xi(41/4)\delta t).$$

Thus as

$$(41/4)\xi\delta t \leq \delta s/2$$

and

$$\max_i\{\|x_i\|\} + \rho\epsilon^m + R\rho(1 + 2\epsilon^m) + \delta s/2 \leq r_0$$

we have that

$$\mathrm{B}(\zeta, \delta s/2) \subset \mathrm{S}(v, \delta s)\,(\subset [\mathrm{S}(u, s) \setminus F]).$$

Hence

$$\mu[\mathrm{S}(u, s) \setminus F] \ \geq \ \mu\mathrm{B}(\zeta, \delta s/2)$$

and as $\delta s/2 \in [\epsilon^m\rho, \rho]$, we can apply Lemma 2.2.8 to conclude that

$$\geq \ \alpha(m)l(1 - 3m\epsilon^m)(\delta s/2)^m$$

$$= \ \alpha(m)2^{-m}(1 - 3m\epsilon^m)L\delta^m(s/r_0)^m\mu\mathrm{B}(0, r_0)$$

and so as $s(u) \geq s > 3s(u)/4$

$$\begin{aligned}
\mu[\mathrm{S}(u, s(u)) \setminus F] &\geq \mu[\mathrm{S}(u, s) \setminus F] \\
&\geq \alpha(m) 2^{-m} (1 - 3m\epsilon^m) L \delta^m (s/r_0)^m \mu \mathrm{B}(0, r_0) \\
&\geq \alpha(m) 3^m 2^{-3m} L (1 - 3m\epsilon^m) \delta^m \left[\frac{s(u)}{r_0}\right]^m \mu \mathrm{B}(0, r_0).
\end{aligned}$$

But this implies that $u \in A_2$ which is impossible and so Case (1) cannot occur.

Case 2:[I] holds but [II] fails.

Hence either

(i) there is a $W \in \mathrm{G}(n, m)$ and $x_u \in \mathrm{S}(u, 0)$ such that

$$F \cap \mathrm{S}(u, s) \subset \mathrm{B}(x_u + W, \delta s)$$

and there are $x, y \in W$ with

$$K \|\mathrm{P}x - \mathrm{P}y\| < \|x - y\|.$$

Or

(ii) for all $W \in \mathrm{G}(n, m)$ and all $x_u \in \mathrm{S}(u, 0)$

$$[F \cap \mathrm{S}(u, s)] \setminus \mathrm{B}(x_u + W, \delta s) \neq \emptyset.$$

So suppose we have case *(i)* for some W and x_u. Then [Fed69, 1.7.3] enables us to find an orthonormal basis for W, $\{e_1, \ldots, e_m\}$, such that if $i \neq j$ then

$$\langle \mathrm{P}e_i, \mathrm{P}e_j \rangle = 0.$$

First let us observe that $\{\mathrm{P}e_1, \ldots, \mathrm{P}e_m\}$ is an orthogonal basis for V_0: For if it wasn't then we could find a unit vector $f \in V_0$ such that for all i, $\langle \mathrm{P}e_i, f \rangle = 0$. But then consider the vector $v = u + (1-\delta)sf$. If $\zeta \in (x_u + W) \cap \mathrm{S}(u,s)$ then

$$\begin{aligned}
\|\mathrm{P}\zeta - v\|^2 &= \|u + \mathrm{P}(\zeta - x_u) - u - (1-\delta)sf\|^2 \\
&= \|\mathrm{P}(\zeta - x_u)\|^2 + [(1-\delta)s]^2 \\
&\geq [(1-\delta)s]^2
\end{aligned}$$

but $F \cap \mathrm{S}(u,s) \subset \mathrm{B}(x_u + W, \delta s)$ and so if $z \in F \cap \mathrm{S}(u,s)$ then

$$\|\mathrm{P}z - v\| \geq (1 - \delta - \delta)s > \delta s$$

which contradicts [I]. Hence $\{\mathrm{P}e_1, \ldots, \mathrm{P}e_m\}$ is a basis of V_0 and so, in particular, $\|\mathrm{P}e_i\| \neq 0$ for all i.

Now observe that there is an i such that $\|\mathrm{P}e_i\| < 1/K$: For if $\|\mathrm{P}e_i\| \geq 1/K$ for all i and $x, y \in W$ are such that

$$K\|\mathrm{P}(x-y)\| < \|x-y\|$$

then

$$\begin{aligned}
\|\mathrm{P}(x-y)\|^2 &= \langle \mathrm{P}(x-y), \mathrm{P}(x-y) \rangle \\
&= \sum_i \langle x-y, e_i \rangle^2 \|\mathrm{P}e_i\|^2 \\
&\geq K^{-2} \sum_i \langle x-y, e_i \rangle^2 = (\|x-y\|/K)^2
\end{aligned}$$

which contradicts the definition of x and y. So we may suppose without loss of generality that $\|\mathrm{P}e_1\| < 1/K$.

Consider open cuboids in V_0 with sides parallel to $\mathrm{P}e_1, \ldots, \mathrm{P}e_m$ and with sidelength equal to $4s/(5Km)$ in the $\mathrm{P}e_1$ direction and s/m in all the others. Let $\mathcal{C}$ be a maximal disjoint family of such cuboids contained in $\mathrm{PB}(u, (5m-1)s/(5m))$. Then

$$\mathrm{card}\,(\mathcal{C}) \geq \left\lfloor \frac{2^m(5m-1)K}{4\sqrt{m}} \left(\frac{5m-1}{5\sqrt{m}}\right)^{m-1} \right\rfloor \geq 2^{m-3}Km^{-m/2}(5m-1)^m 5^{1-m}.$$

Suppose that $C \in \mathcal{C}$ and c is the centre of C then there is an $x_C \in F \cap \mathrm{S}(u, s)$ such that

$$\|\mathrm{P}x_C - c\| \leq \delta s$$

and so, as $\delta < 2/(5Km)$, $\mathrm{P}x_C \in C$. Consider the family of balls in $\mathbf{R}^n$ given by

$$\mathcal{B} = \{\mathrm{B}(x_C, (5m)^{-1}s) : C \in \mathcal{C}\}.$$

I claim that this is a disjoint family. In order to verify this we need to show that if x, x' are distinct centres of balls in $\mathcal{B}$ then $\|x - x'\| > 2s/(5m)$. So suppose that x and x' are two such distinct centres and let c and c' be the centres of the corresponding cuboids in $\mathcal{C}$. Notice that c and c' are also distinct.

Since $F \cap \mathrm{S}(u, s) \subset \mathrm{B}(x_u + W, \delta s)$ we can find X and X' in $x_u + W$ such that

$$\|x - x'\| \geq \|X - X'\| \geq \|x - x'\| - 2\delta s$$

and

$$\max\{\|\mathrm{P}(X - x)\|, \|\mathrm{P}(X' - x')\|\} \leq \delta s.$$

Hence

$$\max\{\|\mathrm{P}X - c\|, \|\mathrm{P}X' - c'\|\} \leq (\delta + \delta)s < 2s/(5Km)$$

and thus $\mathrm{P}X$ and $\mathrm{P}X'$ lie in different cuboids.

As c and c' are the centres of distinct cuboids in $\mathcal{C}$ there is an i such that

$$|\langle c - c', \mathrm{P}e_i/\|\mathrm{P}e_i\|\rangle| > 0.$$

If this $i \geq 2$ then

$$|\langle c - c', \mathrm{P}e_i/\|\mathrm{P}e_i\|\rangle| \geq s/m$$

and hence

$$\|\mathrm{P}(x - x')\| \geq (m^{-1} - 2\delta)s.$$

Thus

$$\|x - x'\| \geq (m^{-1} - 2\delta)s > 2s/(5m)$$

and we are done.

If $i = 1$ then

$$|\langle c - c', \mathrm{P}e_1/\|\mathrm{P}e_1\|\rangle| \geq 4s/(5Km)$$

and so

$$|\langle \mathrm{P}(X - X'), \mathrm{P}e_1/\|\mathrm{P}e_1\|\rangle| \geq [4(5Km)^{-1} - 2(\delta + \delta)]s$$

but

$$\begin{aligned} |\langle \mathrm{P}(X - X'), \mathrm{P}e_1/\|\mathrm{P}e_1\|\rangle| &= |\langle X - X', e_1\rangle|\|\mathrm{P}e_1\| \\ &< |\langle X - X', e_1\rangle|/K \end{aligned}$$

and thus

$$\begin{aligned} \|X - X'\| \geq |\langle X - X', e_1\rangle| &\geq K[4(5Km)^{-1} - 2(\delta + \delta)]s \\ &> 2s/(5m) \end{aligned}$$

hence

$$\|x - x'\| > 2s/(5m)$$

as required. Thus $\mathcal{B}$ is a disjoint collection of balls with centres in F and as for all $C \in \mathcal{C}$

$$\|x_C\| + s/(5m) \le r_0$$

we can conclude that all balls in $\mathcal{B}$ are contained in $\mathrm{S}(u, s)$. Thus

$$\begin{aligned}
\mu\mathrm{S}(u,s) &\ge \sum_{B \in \mathcal{B}} \mu(B) \\
&\ge \alpha(m) l (s/(5m))^m \operatorname{card}(\mathcal{B}) \\
&\ge \alpha(m) l (5m)^{-m} 2^{m-3} K m^{-m/2} (5m-1)^m 5^{1-m} s^m \\
&\ge \alpha(m)(2/5)^{m-1} m^{-m/2} LK (s/r_0)^m \mu\mathrm{B}(0, r_0)
\end{aligned}$$

and so

$$\mu\mathrm{S}^{\circ}(u, s(u)) \ge 2^{-1}\alpha(m) 5^{-m} m^{-m/2} LK \left[\frac{s(u)}{r_0}\right]^m \mu\mathrm{B}(0, r_0)$$

which implies that $u \in A_1$ which is impossible. Hence *(ii)* must hold. So for $3s(u)/4 \le s < s(u)$ we have that [I] holds for s at u and for all $W \in \mathrm{G}(n, m)$ and all $x_u \in \mathrm{S}(u, 0)$

$$[F \cap \mathrm{S}(u,s)] \setminus \mathrm{B}(x_u + W, \delta s) \ne \emptyset.$$

Hence, in particular, for all $x_u \in \mathrm{S}(u, 0)$

$$[F \cap \mathrm{S}(u,s)] \setminus \mathrm{B}(x_u + V_0, \delta s) \ne \emptyset$$

and so there are an x_s and y_s in $\mathrm{S}(u,s) \cap F$ such that

$$\|\mathrm{P}^{\perp}(x_s - y_s)\| \ge \delta s.$$

Thus, as $F \cap \mathrm{S}(u, s(u))$ is compact and contains $F \cap \mathrm{S}(u, s)$ for $s < s(u)$, we conclude that there exist x and y in $F \cap \mathrm{S}(u, s(u))$ with

$$\|\mathrm{P}^{\perp}(x - y)\| \geq \delta s.$$

Hence, as $s(u)$ is good for u (Lemma 2.3.2), we may use Lemma 2.3.3 to deduce that

$$\operatorname{diam}\left[\mathrm{P}^{\perp}\left(F \cap \mathrm{S}(u, s(u))\right)\right] \leq 2K(1+\delta)s(u)$$

and so $u \in A_3$ as required. ∎

Lemma 2.3.6 *Let*

$$\eta := (1 + 3m\epsilon^m)\left[1 - 4^{m-1}\epsilon^m \Lambda_2^{-1}\right]^{-1} - 1$$

and suppose that

$$1 \leq T \leq \epsilon^{(2/(3m))-1}.$$

Then there is a $u \in A_3$ and a Borel set J contained in $V_0 \cap \mathrm{B}(0, r_0)$ such that

$$V_0 \cap \mathrm{B}(u, Ts(u)) \subset J \subset V_0 \cap \mathrm{B}(u, 4Ts(u)),$$

$$\mu[\mathrm{P}^{-1}(J) \cap \mathrm{B}(0, r_0)] \leq \chi(1+\eta)\mathcal{H}^m(J)$$

and if $0 < \theta < [2/(3m)]^{m+1}$ then

$$\mathcal{H}^m\left[\mathrm{B}(\partial_{V_0} J, \theta s(u)) \cap V_0\right] \leq c(m)\theta^{1/(m+1)}[Ts(u)]^m$$

(where $c(m)$ is the constant from Lemma 2.2.4 .)

Proof: Consider

$$\mathcal{C} := \{\mathrm{PB}(u, s(u)) : u \in A_1\} \cup \{\mathrm{PB}(u, Ts(u)) : u \in A_3\}$$

which is a cover of $A_1 \cup A_3$. As $4Ts(u) \leq \alpha r_0$ it follows that for all $u \in A_1 \cup A_3$

$$\mathrm{PB}(u, 4Ts(u)) \subset V_0 \cap \mathrm{B}(0, r_0).$$

By Lemma 2.2.4 we can find a disjoint subcollection, $\mathcal{J}$ of Borel sets contained in V_0, which may be written as a disjoint union, $\mathcal{J}_1 \cup \mathcal{J}_3$, such that

1. $A_1 \subset \cup\mathcal{J}_1$ and $A_3 \subset \cup\mathcal{J}_3$,

2. for all $J \in \mathcal{J}_1$ there is a $u \in A_1$ such that

$$\mathrm{B}(u, s(u)) \cap V_0 \subset J \subset \mathrm{B}(u, 4s(u)) \cap V_0$$

and for all $J \in \mathcal{J}_3$ there is a $u \in A_3$ such that

$$\mathrm{B}(u, Ts(u)) \cap V_0 \subset J \subset \mathrm{B}(u, 4Ts(u)) \cap V_0,$$

3. for all $J \in \mathcal{J}_1$ if $0 < \theta < [2/(3m)]^{m+1}$ and if u is as determined in (2) then

$$\mathcal{H}^m \left[\mathrm{B}(\partial_{V_0} J, \theta s(u)) \cap V_0\right] \leq c(m)\theta^{1/(m+1)}[s(u)]^m.$$

and for all $J \in \mathcal{J}_3$ if $0 < \theta < [2/(3m)]^{m+1}$ and if u is as determined in (2) then

$$\mathcal{H}^m \left[\mathrm{B}(\partial_{V_0} J, \theta Ts(u)) \cap V_0\right] \leq c(m)\theta^{1/(m+1)}[Ts(u)]^m.$$

Thus $\mathcal{H}^m(\cup\mathcal{J}) \geq \mathcal{H}^m(A_1 \cup A_3)$. If for all $J \in \mathcal{J}$, $\mu[\mathrm{P}^{-1}(J) \cap \mathrm{B}(0, r_0)] > \chi(1+\eta)\mathcal{H}^m J$ then as

$$\begin{aligned} \sum \mu[\mathrm{P}^{-1}(J) \cap \mathrm{B}(0, r_0)] &\leq \mu\mathrm{B}(0, r_0) \\ &\leq \alpha(m)\chi(1+3m\epsilon^m)r_0^m \end{aligned}$$

we find that

$$\begin{aligned} \alpha(m)\chi(1+3m\epsilon^m)r_0^m &> \sum \chi(1+\eta)\mathcal{H}^m J \\ &\geq \chi(1+\eta)\mathcal{H}^m(A_1 \cup A_3) \end{aligned}$$

which, by our estimate for the size of A_2 (Lemma 2.3.4), is

$$\begin{aligned} &\geq \alpha(m)\chi(1+\eta)r_0^m[1 - 4^{m-1}\epsilon^m\Lambda_2^{-1}] \\ &= \alpha(m)\chi(1+3m\epsilon^m)r_0^m \text{ — a contradiction.} \end{aligned}$$

Hence there is a $J \in \mathcal{J}$ such that

$$\mu[\mathrm{P}^{-1}(J) \cap \mathrm{B}(0, r_0)] \leq \chi(1+\eta)\mathcal{H}^m(J).$$

If $J \in \mathcal{J}_1$ and u is the associated point of A_1 then we find that

$$\begin{aligned} \alpha(m)\chi(1+\eta)[4s(u)]^m \geq \chi(1+\eta)\mathcal{H}^m(J) &\geq \mu[\mathrm{P}^{-1}(J) \cap \mathrm{B}(0, r_0)] \\ &\geq \mu\mathrm{S}^0(u, s(u)) \end{aligned}$$

since $u \in A_1$

$$\begin{aligned} &\geq \Lambda_2[s(u)/r_0]^m \mu\mathrm{B}(0, r_0) \\ &\geq \alpha(m)\chi(1-3m\epsilon^m)\Lambda_2 s(u)^m \end{aligned}$$

but then

$$(1+\eta)4^m \geq (1-3m\epsilon^m)\Lambda_1$$

which is impossible and so $J \in \mathcal{J}_3$ which implies the Lemma. ∎

Lemma 2.3.7 *Suppose that* $1 \le T \le \epsilon^{(2/(3m))-1}$. *Then there are a* $u \in A_3$, $X \in F \cap \mathrm{S}(u, \delta s(u))$, $\nu \in \mathrm{Tan}_S(\mu, X)\,(\subset \mathcal{M}^m_C(l,u))$ *and a closed set* $I \subset V_0$ *such that:*

1. $\mathrm{F}_{R+3+\epsilon^{-1}}\left(\frac{\mu_{X,s(u)}}{s(u)^m},\, \nu\right) < \alpha(m) l \epsilon^{m(m+3)}$,

2. *there is a*

$$Y \in \mathrm{Spt}\,\nu \cap \mathrm{P}^{-1}\left[\mathrm{B}(0, 1+\delta+\epsilon^m)\right] \cap \mathrm{X}(0, 2+\epsilon^m, 2K(1+2\delta), V_0)$$

such that

$$\|\mathrm{P}^{\perp} Y\| \ge \delta/2 - \epsilon^m,$$

3. *for all* $w \in V_0 \cap \mathrm{B}(0, 5T)$

$$\mathrm{P}^{-1}[\mathrm{B}(w, (5T+\delta)(1-\delta)^{-1}\delta + \epsilon^m)] \cap \mathrm{Spt}\,\nu \cap \mathrm{X}(0, 2+\epsilon^m, 2K(1+2\delta), V_0) \ne \emptyset,$$

4. $\mathrm{B}(0, T(1-(2\epsilon^m)^{m+1})) \cap V_0 \subset I \subset \mathrm{B}(0, 5T) \cap V_0$,

5. $\nu[\mathrm{P}^{-1}(I) \cap \mathrm{B}(0, 30K\epsilon^{(2/(3m))-1})] \le \chi(1+\eta)(\mathcal{H}^m(I) + 2c(m)\epsilon^m \epsilon^{(2/(3m))-1}) + \alpha(m) l \epsilon^{2m}$,

6. *for all* $0 < \theta \le [3m]^{-(m+1)}$

$$\mathcal{H}^m\left[\mathrm{B}(\partial_{V_0} I, \theta T) \cap V_0\right] \le c(m)[(2\epsilon^m)^{m+1} + \theta]^{1/(m+1)} T^m.$$

Proof: Fix $1 \le T \le \epsilon^{(2/(3m))-1}$. From Lemma 2.3.6 we can find a $u \in A_3$ and a Borel set $J \subset V_0$ such that

$$\mathrm{PB}(u, Ts(u)) \subset J \subset \mathrm{PB}(u, 4Ts(u)),$$

$$\mu[\mathrm{P}^{-1}(J) \cap \mathrm{B}(0, r_0)] \leq \chi(1+\eta)\mathcal{H}^m(J)$$

and if $0 < \theta \leq [2/(3m)]^{m+1}$ then

$$\mathcal{H}^m\left[\mathrm{B}(\partial_{V_0} J, \theta T s(u)) \cap V_0\right] \leq c(m)\theta^{1/(m+1)} T^m.$$

Since $u \in A_3$ we can find a $y, y' \in F \cap \mathrm{S}(u, s(u))$ such that

$$\|\mathrm{P}^{\perp}(y - y')\| \geq \delta s(u).$$

Thus, as $s(u)$ is good for u (Lemma 2.3.2), $F \cap \mathrm{S}(u, \delta s(u)) \neq \emptyset$ and so we can find an $X \in F \cap \mathrm{S}(u, \delta s(u))$ such that

$$\max\{\|\mathrm{P}^{\perp}(y - X)\|, \|\mathrm{P}^{\perp}(y' - X)\|\} \geq \delta s(u)/2.$$

We may assume without loss of generality that

$$\|\mathrm{P}^{\perp}(y - X)\| \geq s(u)/2$$

and, as $(5T + \delta)(1-\delta)^{-1}s(u) \leq \alpha r_0 \leq r_0 - \|u\|$, we may use Lemma 2.3.3 to conclude that there is a $t \in \mathrm{S}(u, 0)$ such that

$$F \cap \mathrm{P}^{-1}\left[\mathrm{B}(u, (5T+\delta)(1-\delta)^{-1}s(u))\right] \subset \mathrm{X}(t, s(u)/2, 2K(1+2\delta), V_0).$$

Hence

$$F \cap \mathrm{P}^{-1}\left[\mathrm{B}(u, (5T+\delta)(1-\delta)^{-1}s(u))\right] \subset \mathrm{X}(X, 2s(u), 2K(1+2\delta), V_0).$$

As $s(u) < r_0$ we may use (6) of Section 2.3.1 to find a

$$\nu \in \mathrm{Tan}_S(\mu, X)\,(\subset \mathcal{M}_C^m(l, u))$$

such that

$$\mathrm{F}_{R+3+\epsilon^{-1}}\left(\frac{\mu_{X,s(u)}}{s(u)^m},\,\nu\right) < \alpha(m)l\epsilon^{m(m+3)}$$

which is (1) of the Lemma.

If $z \in F \cap \mathrm{P}^{-1}\left[\mathrm{B}(u,(5T+\delta)(1-\delta)^{-1}s(u))\right]$ then

$$\|z - X\| \leq ((5T+\delta)(1-\delta)^{-1}s(u) + 2\epsilon r_0\delta/\delta \leq 3 + \epsilon^{-1}$$

and so we may use Lemma 2.2.5 to conclude that there is a $\zeta \in \operatorname{Spt}\nu$ such that

$$\|\zeta - (z-X)/s(u)\| \leq \epsilon^m.$$

In particular, as $y \in \mathrm{P}^{-1}\mathrm{B}(u,s(u)) \cap F \cap \mathrm{X}(X,2s(u),2K(1+2\delta),V_0)$, there is a $Y \in \operatorname{Spt}\nu \cap \mathrm{P}^{-1}\left[\mathrm{B}(0,1+\delta+\epsilon^m)\right] \cap \mathrm{X}(0,2+\epsilon^m,2K(1+2\delta),V_0)$ such that

$$\|Y - (y-X)/s(u)\| \leq \epsilon^m.$$

Hence by our estimate for y and as $0 \in \operatorname{Spt}\nu$ (since $\nu \in \operatorname{Tan}_S(\mu,x)$) we have

$$\operatorname{diam}\left[\mathrm{P}^{\perp}\left(\operatorname{Spt}\nu \cap \mathrm{P}^{-1}(\mathrm{B}(0,1+\delta+\epsilon^m)) \cap \mathrm{X}(0,2+\epsilon^m,2K(1+2\delta),V_0)\right)\right] \geq \frac{\delta}{2} - \epsilon^m$$

which verifies (2).

Suppose $w \in \mathrm{B}(0,5T) \cap V_0$ and let $v = s(u)w + \mathrm{P}X$ and so $v \in \mathrm{B}(u,(5T+\delta)s(u)) \cap V_0$. As $(5T+\delta)(1-\delta)^{-1}s(u)$ is good for u it follows (by [I]) that there is a

$$z \in F \cap \mathrm{P}^{-1}\mathrm{B}(v,(5T+\delta)(1-\delta)^{-1}\delta s(u)).$$

Hence we may use Lemma 2.2.5 again to conclude that there is a $\zeta \in \operatorname{Spt}\nu$ with

$$\|\zeta - (z-X)/s(u)\| \leq \epsilon^m.$$

But $\|v - \mathrm{P}z\| \leq (5T + \delta)(1 - \delta)^{-1}\delta s(u)$ and so

$$\|w - \mathrm{P}\zeta\| \leq (5T + \delta)(1 - \delta)^{-1}\delta + \epsilon^m.$$

On observing that $z \in \mathrm{X}(X, 2s(u), 2K(1 + 2\delta), V_0)$ we conclude that

$$\zeta \in \mathrm{X}(0, 2 + \epsilon^m, 2K(1 + 2\delta), V_0)$$

and so (3) holds.

Let

$$I := \mathrm{Clos}\left\{(x - \mathrm{P}X)/s(u) : \, x \in J \setminus \mathrm{B}(\partial_{V_0} J, (2\epsilon^m)^{m+1} T s(u))\right\}$$

and so

$$\mathrm{B}(I, \epsilon^{m(m+1)} T) \subset \{(x - \mathrm{P}X)/s(u) : \, x \in J\}$$

and, for $\theta > 0$

$$\mathrm{B}(\partial_{V_0} I, \theta T) \subset \mathrm{B}(\partial_{V_0} \{(x - \mathrm{P}X)/s(u) : \, x \in J\}, (\theta + (2\epsilon^m)^{m+1})T).$$

Moreover from the definition of J we have that

$$\mathcal{H}^m(I) \geq [s(u)]^{-m} \mathcal{H}^m(J) - 2c(m)\epsilon^m T^m$$

and, since $\epsilon^m < [3m]^{-(m+1)}$, for $0 < \theta < [3m]^{-(m+1)}$ we have

$$\mathcal{H}^m[\mathrm{B}(\partial_{V_0} I, \theta T) \cap V_0] \leq c(m) \left[(2\epsilon^m)^{m+1} + \theta\right]^{1/(m+1)} T^m$$

which verifies (6).

Since

$$\mathrm{B}(u, Ts(u)) \cap V_0 \subset J \subset \mathrm{B}(u, 4Ts(u)) \cap V_0$$

we conclude that

$$\mathrm{B}(0, T(1-(2\epsilon^m)^{m+1}-\delta)) \cap V_0 \subset I \subset \mathrm{B}(0, 5T) \cap V_0$$

which is (4).

It only remains to verify (5). Since

$$\max\{30K\epsilon^{(2/(3m))-1} + \epsilon^{m(m+1)}T,\ 5T\} \le R + 3 + \epsilon^{-1}$$

we may use Lemma 1.3.4 (with $E = \mathrm{P}^{-1}(I) \cap \mathrm{B}(0, 30K\epsilon^{(2/(3m))-1})$ and $\tau = \epsilon^{m(m+1)}T$) to conclude that

$$\nu\left[\mathrm{P}^{-1}(I) \cap \mathrm{B}(0, 30K\epsilon^{(2/(3m))-1})\right]$$

$$\le \frac{\mu\left[\mathrm{P}^{-1}(J) \cap \mathrm{B}(X, s(u)(30K\epsilon^{(2/(3m))-1} + \epsilon^{m(m+1)}T))\right]}{[s(u)]^{-m}} + \frac{\alpha(m)l\epsilon^{m(m+3)}}{\epsilon^{m(m+1)}T}$$

which as $\|X\| + s(u)(30K\epsilon^{(2/(3m))-1} + \epsilon^{m(m+1)}T) \le r_0$

$$\le [s(u)]^{-m}\mu\left[\mathrm{P}^{-1}(J) \cap \mathrm{B}(0, r_0)\right] + \frac{\alpha(m)l\epsilon^{m(m+3)}}{\epsilon^{m(m+1)}T}$$

which by the definition of J is

$$\le \chi(1+\eta)[s(u)]^{-m}\mathcal{H}^m(J) + \alpha(m)l\epsilon^{2m}/T$$

$$\le \chi(1+\eta)\left[\mathcal{H}^m(I) + 2c(m)\epsilon^m\epsilon^{(2/(3m))-1^m}\right] + \alpha(m)l\epsilon^{2m}$$

verifying (5) as required. ∎

2.3.2 Properties of $B^{(1)}$ independent of ϵ

Lemma 2.3.8 *For all $T \ge 1$ there is an $x \in B^{(2)}$, $\nu \in \mathrm{Tan}_S(\mu, x)$ and $V \in \mathrm{G}(n,m)$ such that*

1. *there is a $Y \in \operatorname{Spt}\nu \cap \mathrm{X}(0, 2, 2K(1+2\delta), V) \cap \mathrm{P}_V^{-1}(\mathrm{B}(0,1))$ with $\|\mathrm{P}_V^{\perp}Y\| \ge \delta/2$,*

2. $V \cap \mathrm{B}(0,5T) \subset \mathrm{P}_V[\operatorname{Spt}\nu \cap \mathrm{X}(0,2,2K(1+2\delta),V)]$,

3. for all Borel sets C contained in $V \cap \operatorname{int}(\mathrm{B}(0,T))$

$$\nu[\mathrm{P}_V^{-1}(C)] = \chi\mathcal{H}^m(C),$$

4. $\operatorname{Spt}\nu \cap \mathrm{P}_V^{-1}[\operatorname{int}(\mathrm{B}(0,T))] \subset \mathrm{X}(0,2,2K(1+2\delta),V)$.

Proof: Fix $T \geq 1$ and suppose we have a sequence of positive real numbers $\Xi_i \to 0$ and sequences $x_i \in B^{(2)}$, $\nu_i \in \operatorname{Tan}_S(\mu,x_i)\,(\subset \mathcal{M}_C^m(l,u))$ and $V_i \in \mathrm{G}(n,m)$ such that

(i) the points $x_i \to x \in B^{(2)}$, $\nu_i \to \nu$ and so, by the definition of $B^{(1)}\left(\supset B^{(2)}\right)$, $\nu \in \operatorname{Tan}_S(\mu,x) \subset \mathcal{M}_C^m(l,u)$,

(ii) the m-planes $V_i \to V$ and if $\mathrm{P}_i := \mathrm{P}_{V_i}$, $\mathrm{P}_i^\perp := \mathrm{P}_{V_i}^\perp$ then we can find a $Y_i \in \operatorname{Spt}\nu_i \cap \mathrm{P}_i^{-1}[\mathrm{B}(0,1+\Xi_i)] \cap \mathrm{X}(0,2+\Xi_i,2K(1+2\delta),V)$ with $\|\mathrm{P}_i^\perp Y_i\| \geq \delta/2 - \Xi_i$ and $Y_i \to Y$,

(iii) there is a compact set $I_i \subset V_i$ such that $I_i \to I$ in the Hausdorff metric (denoted by d_H) and

$$\mathrm{B}(0,T(1-\Xi_i)) \cap V \subset I_i \subset \mathrm{B}(0,4T+\Xi_i) \cap V,$$

$$\nu_i[\mathrm{P}_i^{-1}(I_i) \cap \mathrm{B}(0,Xi_i^{-1})] \leq \chi\mathcal{H}^m(I_i) + \Xi_i$$

and if $0 < \theta < [3m]^{-(m+1)}$ then

$$\mathcal{H}^m[\mathrm{B}(\partial_i I_i,\theta T) \cap V_i] \leq c(m)\,[\Xi_i+\theta]^{1/(m+1)}\,T^m$$

(where $\partial_i := \partial_{V_i}$.)

(iv) For all $v \in V_i \cap \mathrm{B}(0, 5T)$ there is a $\zeta \in \operatorname{Spt} \nu_i \cap \mathrm{X}(0, 2+\Xi_i, 2K(1+2w), V_i)$ such that

$$\|\mathrm{P}_i(\zeta) - v\| \leq \Xi_i T.$$

Then I claim that ν, x, Y and V would satisfy the Lemma. (We shall verify the existence of such sequences Ξ_i, ν_i, x_i, I_i and V_i later.)

From Lemma 2.2.14 (with $L = T$) we conclude that ν and Y satisfy (1) and (2).

It is clear that I (defined in (iii) above) satisfies

$$\mathrm{B}(0, T) \cap V \subset I \subset \mathrm{B}(0, 5T) \cap V.$$

Suppose that $0 < \delta \leq [3m]^{-(m+1)}$ and j is chosen so that for all $i > j$

$$I \subset \mathrm{B}(I_i, \delta T) \text{ and } I_i \subset \mathrm{B}(I, \delta T).$$

Thus $I \subset V \cap \mathrm{B}(I_i, \delta T)$ and, clearly,

$$\mathcal{H}^m(V \cap \mathrm{B}(I_i, \delta T)) \leq \mathcal{H}^m(V_i \cap \mathrm{B}(I_i, \delta T)).$$

Hence

$$\begin{aligned}
\mathcal{H}^m(I) \leq \mathcal{H}^m(V \cap \mathrm{B}(I_i, \delta T)) &\leq \mathcal{H}^m(V_i \cap \mathrm{B}(I_i, \delta T)) \\
&= \mathcal{H}^m\left[(I_i \cup \mathrm{B}(\partial_i I_i, \delta T)) \cap V_i\right] \\
&\leq \mathcal{H}^m(I_i) + c(m)\left[\Xi_i + \delta\right]^{1/(m+1)} T^m.
\end{aligned}$$

Now observe that as $I_i \subset V_i \cap \mathrm{B}(I, \delta T)$ we have

$$\mathcal{H}^m(I_i) \leq \mathcal{H}^m(V_i \cap \mathrm{B}(I, \delta T)) \leq \mathcal{H}^m(V \cap \mathrm{B}(I, \delta T)).$$

Hence on sending i to infinity we deduce that

$$\mathcal{H}^m(I) \leq \liminf_{i\to\infty} \mathcal{H}^m(I_i) + c(m)\delta^{1/(m+1)}T^m$$

and

$$\limsup_{i\to\infty} \mathcal{H}^m(I_i) \leq \mathcal{H}^m(V \cap \mathrm{B}(I, \delta T)).$$

But $I = \cap_{\delta>0}\mathrm{B}(I, \delta T)$ and so sending δ to zero gives

$$\mathcal{H}^m(I) = \lim_{i\to\infty} \mathcal{H}^m(I_i).$$

Now observe that if i and δ are such that $I \subset \mathrm{B}(I_i, \delta T)$ and if $x \in I \setminus \mathrm{B}(\partial_V I, \alpha T)$ (for some $\alpha > \delta$) then

$$V \cap \mathrm{B}(x, \alpha T) \subset I \subset \mathrm{B}(I_i, \delta T)$$

and so $x \in I_i$ and

$$\mathrm{d}(x, \partial_i I_i) > [\alpha - \delta]T$$

which means that

$$I \setminus \mathrm{B}(\partial_V I, \alpha T) \subset \mathrm{B}(I_i, \delta T) \setminus \mathrm{B}(\partial_i I, (\alpha - \delta)T).$$

Fix $0 < \delta < 1$ and choose j such that if $i \geq j$ then

$$d_H(I, I_i) \leq \delta T, \qquad \mathcal{H}^m(I_i) \leq \mathcal{H}^m(I) + \delta,$$

$$\Xi_i > 6\delta^{-1}T \qquad \text{and} \qquad \mathrm{F}_{7\delta^{-1}T}(\nu_i, \nu) < \delta^2 T$$

and if $x \in \mathrm{B}(0, 6\delta^{-1}T)$ then

$$\|\mathrm{P}_i(x) - \mathrm{P}_V(x)\| < \delta T/2.$$

If $x \in \mathrm{P}_V^{-1}\left[I \setminus \mathrm{B}(\partial_V I, 2\delta T)\right] \cap \mathrm{B}(0, 6\delta^{-1}T)$ then

$$\begin{aligned} \|\mathrm{P}_i(x) - \mathrm{P}_i\mathrm{P}_V(x)\| &\leq \|\mathrm{P}_i(x) - \mathrm{P}_V(x)\| + \|\mathrm{P}_V(x) - \mathrm{P}_i\mathrm{P}_V(x)\| \\ &< \delta T/2 + \delta T/2 = \delta T. \end{aligned}$$

Thus

$$\mathrm{P}_V^{-1}\left[I \setminus \mathrm{B}(\partial_V I, 2\delta T)\right] \cap \mathrm{B}(0, 6\delta^{-1}T) \subset \mathrm{P}_i^{-1}(I_i) \cap \mathrm{B}(0, 6\delta^{-1}T)$$

and so we may use Lemma 1.3.4 with $\tau = \delta T$ and $E = \mathrm{P}_V^{-1}\left[I \setminus \mathrm{B}(\partial_V I, 3\delta T)\right] \cap$ $\mathrm{B}(0, 6\delta^{-1}T)$ to deduce that

$$\begin{aligned} &\nu\left[\mathrm{P}_V^{-1}\left[I \setminus \mathrm{B}(\partial_V I, 3\delta T)\right] \cap \mathrm{B}(0, 5\delta^{-1}T)\right] \\ &\leq \nu_i\left[\mathrm{P}_V^{-1}\left[I \setminus \mathrm{B}(\partial_V I, 2\delta T)\right] \cap \mathrm{B}(0, 6\delta^{-1}T)\right] + \mathrm{F}_{7\delta^{-1}T}(\nu_i, \nu) \\ &\leq \nu_i\left[\mathrm{P}_i^{-1}(I_i) \cap \mathrm{B}(0, 6\delta^{-1}T)\right] + \delta \end{aligned}$$

as $\Xi_i^{-1} > 6\delta^{-1}T$ we may use (iii) to deduce

$$\begin{aligned} &\leq \chi\mathcal{H}^m(I_i) + \Xi_i + \delta \\ &\leq \chi\mathcal{H}^m(I) + \Xi_i + (\chi + 1)\delta. \end{aligned}$$

Hence on sending i to infinity we find that

$$\nu\left[\mathrm{P}_V^{-1}\left[I \setminus \mathrm{B}(\partial_V I, 3\delta T)\right] \cap \mathrm{B}(0, 5\delta^{-1}T)\right] \leq \chi\mathcal{H}^m(I) + (\chi + 1)\delta$$

but δ was arbitrary and so we conclude that

$$\nu\left[\mathrm{P}_V^{-1}(\operatorname{int}_V I)\right] \leq \chi\mathcal{H}^m(I).$$

Now recall from Section 2.3.1(4) that as $\nu \in \mathrm{Tan}_S(\mu, x)$ and $x \in B^{(2)}$ then for ν-a.e. ζ

$$\overline{\mathrm{D}}_m(\nu, \zeta) \geq \chi.$$

Thus if $C \subset \mathrm{B}(0, 5T) \cap V$ is a Borel set then from Lemma 1.3.2 we can deduce that

$$\begin{aligned}
&\nu\left[\mathrm{P}_V^{-1}(C) \cap \mathrm{X}(0, 2, 2K(1+2\delta), V)\right] \\
&\qquad \geq \chi\mathcal{H}^m[\mathrm{P}_V^{-1}(C) \cap \operatorname{Spt}\nu \cap \mathrm{X}(0, 2, 2K(1+2\delta), V)]
\end{aligned}$$

which, by projecting back onto V, is

$$\geq \chi\mathcal{H}^m(C).$$

Hence if $C \subset \operatorname{int}(\mathrm{B}(0, T)) \cap V$ is a Borel set then

$$\begin{aligned}
\chi\mathcal{H}^m(C) &\leq \nu[\mathrm{P}_V^{-1}(C)] \\
&\leq \nu[\mathrm{P}_V^{-1}(I)] - \nu[\mathrm{P}_V^{-1}(I \setminus C) \cap \mathrm{X}(0, 2, 2K(1+2\delta), V)]
\end{aligned}$$

which by the preceeding

$$\begin{aligned}
&\leq \chi\mathcal{H}^m(I) - \chi[\mathcal{H}^m(I) - \mathcal{H}^m(C)] \\
&= \chi\mathcal{H}^m(C).
\end{aligned}$$

Thus for all Borel sets $C \subset \operatorname{int}(\mathrm{B}(0, T)) \cap V$

$$\nu[\mathrm{P}_V^{-1}(C)] = \chi\mathcal{H}^m(C)$$

which is (3) of the Lemma.

If there is an $x \in \mathrm{P}_V^{-1}[\operatorname{int}\mathrm{B}(0, T)] \setminus \mathrm{X}(0, 2, 2K(1+2\delta), V)$ such that $x \in \operatorname{Spt}\nu$ then we can find an $r > 0$ such that $\mathrm{P}_V\mathrm{B}(x, r) \subset \operatorname{int}\mathrm{B}(0, T)$, $\nu\mathrm{B}(x, r) > 0$ and

$$\mathrm{B}(x, r) \cap \mathrm{X}(0, 2, 2K(1+2\delta), V) = \emptyset$$

but then

$$\chi\mathcal{H}^m(\mathrm{B}(\mathrm{P}_V x, r) \cap V) = \nu\left[\mathrm{P}_V^{-1}(\mathrm{B}(\mathrm{P}_V x, r))\right]$$

$$
\begin{aligned}
&= \nu\left[\mathrm{P}_V^{-1}(\mathrm{B}(\mathrm{P}_V x, r)) \cap \mathrm{X}(0,2,2K(1+2\delta),V)\right] \\
&\qquad + \nu\left[\mathrm{P}_V^{-1}(\mathrm{B}(\mathrm{P}_V x, r)) \setminus \mathrm{X}(0,2,2K(1+2\delta),V)\right] \\
&\geq \chi\mathcal{H}^m(\mathrm{B}(\mathrm{P}_V x, r) \cap V)) + \nu[\mathrm{B}(x,r)] \\
&> \chi\mathcal{H}^m(\mathrm{B}(\mathrm{P}_V x, r) \cap V) \text{ — a contradiction}
\end{aligned}
$$

and so ν satisfies (4).

It remains to show that we can find sequences which satisfy (i) through to (iv). In order to achieve this it suffices to choose a sequence of positive ϵ tending to zero and use Lemma 2.3.7 to find associated sequences of measures, points, planes and sets. Upon noting that any sequence of measures $(\omega_i) \subset \mathcal{M}_C^m(l,u)$ possesses a convergent subsequence (this is an application of Lemma 1.3.3 together with the uniform upper density estimate on the measures ω_i) and that, by compactness, any sequence of points x_i in $B^{(2)}$ possesses a convergent subsequence and similarly for $V_i \in \mathrm{G}(n,m)$ and compact sets $I_i \subset \mathrm{B}(0,5T)$ we deduce that we can, indeed, find a sequence satisfying (i) through (iv). (Ξ_i is chosen to be the maximum of all the appropriate error terms in Lemma 2.3.7.) Hence the Lemma holds. ∎

Lemma 2.3.9 *There is an $X \in B^{(2)}$, $\omega \in \mathrm{Tan}_S(\mu, X)$ and $W \in \mathrm{G}(n,m)$ such that*

1. $\mathrm{Spt}\,\omega \subset \mathrm{X}(0,2,2K(1+2\delta),W)$,
2. $W = \mathrm{P}_W[\mathrm{Spt}\,\omega]$,
3. *for all Borel sets $I \subset W$*

$$\omega[\mathrm{P}_W^{-1}(I)] = \chi\mathcal{H}^m(I),$$

4. there is a $Y \in \operatorname{Spt}\omega \cap \mathrm{P}_W^{-1}(\mathrm{B}(0,1)) \cap \mathrm{X}(0,2,2K(1+2\delta),W)$ *with*

$$\|\mathrm{P}_W^{\perp} Y\| \geq \delta/2,$$

5. for ω*-a.e.* ζ, $\overline{\mathrm{D}}_m(\omega,\zeta) \geq \chi$,

6. ω *is* m*-rectifiable.*

Proof: From Observation 8 we may find for all $T \geq 1$ an $x_T \in B^{(1)}$, $\omega_T \in \operatorname{Tan}_S(\mu, x_T)$, $W_T \in \mathrm{G}(n,m)$ and a $Y_T \in \operatorname{Spt}\omega_T \cap \mathrm{P}_W^{-1}(\mathrm{B}(0,1)) \cap \mathrm{X}(0,2,2K(1+2\delta),W_T)$ such that

(i) $\|\mathrm{P}_{W_T}^{\perp} Y_T\| \geq \delta/2$,

ii) $W_T \cap \mathrm{B}(0,5T) \subset \mathrm{P}_{W_T}[\operatorname{Spt}\omega_T \cap \mathrm{X}(0,2,2K(1+2\delta),W_T)]$,

(iii) for all Borel sets I contained in $\operatorname{int}(\mathrm{B}(0,T)) \cap W_T$

$$\omega_T[\mathrm{P}_{W_T}^{-1}(I)] = \chi\mathcal{H}^m(I),$$

(iv) $\operatorname{Spt}\omega_T \cap \mathrm{P}_{W_T}^{-1}[\operatorname{int}(\mathrm{B}(0,T))] \subset \mathrm{X}(0,2,2K(1+2\delta),W_T)$.

By repeated use of compactness and application of [Mat95, Theorem 1.23] we may find a sequence $T(i) \to \infty$ such that

(v) $x_i := x_{T(i)} \to X \in B^{(1)}$,

(vi) $\omega_i := \omega_{T(i)} \to \omega$ which, from Section 2.3(2), is in $\operatorname{Tan}_S(\mu, X)$,

(vii) $W_i := W_{T(i)} \to W \in \mathrm{G}(n,m)$.

(viii) $Y_i := Y_{T(i)} \to Y \in \mathrm{P}_W^{-1}(\mathrm{B}(0,1)) \cap \mathrm{X}(0,2,2K(1+2\delta),W)$.

Let $\mathrm{P}_i := \mathrm{P}_{W_{T(i)}}$ and $\mathrm{P}_i^\perp := \mathrm{P}_{W_{T(i)}}^\perp$. From Lemma 2.2.14 we may immediately deduce that $Y \in \operatorname{Spt}\omega \cap \mathrm{P}_W^{-1}(\mathrm{B}(0,1)) \cap \mathrm{X}(0,2,2K(1+2\delta),W)$, $\|\mathrm{P}_W^\perp Y\| \geq \delta/2$ and

$$\mathrm{P}_W[\operatorname{Spt}\nu \cap \mathrm{X}(0,2,2K(1+2\delta),W)] \supset W.$$

Hence (2) and (4) of the Lemma hold.

Since $\omega \in \operatorname{Tan}_S(\mu, X)$ for some $X \in B^{(2)}$ we know from Section 2.3.1(4) that for ω-a.e. ζ

$$\overline{\mathrm{D}}_m(\omega,\zeta) \geq \chi$$

and so (5) holds. Hence if $C \subset W$ is a Borel set then as $\mathrm{P}_W[\operatorname{Spt}\nu \cap \mathrm{X}(0,2,2K(1+2\delta),W)] \supset W$ we deduce from Lemma 1.3.2 that

$$\begin{aligned}
\omega[\mathrm{P}_W^{-1}(C) \cap \mathrm{X}(0,2,2K(1+2\delta),W)] & \\
&\geq \chi\mathcal{H}^m[\mathrm{P}_W^{-1}(C) \cap \mathrm{X}(0,2,2K(1+2\delta),W) \cap \operatorname{Spt}\omega] \\
&\geq \chi\mathcal{H}^m(C).
\end{aligned}$$

Hence in order to verify (3) it is sufficient to show that for all $T \geq 1$

$$\omega[\mathrm{P}_W^{-1}(\mathrm{B}(0,T))] \leq \alpha(m)\chi T^m.$$

Since then if $C \subset W$ we deduce that, for all $T \geq 1$,

$$\begin{aligned}
\chi\mathcal{H}^m[C \cap \mathrm{B}(0,T)] &\leq \omega[\mathrm{P}_W^{-1}(\mathrm{B}(0,T) \cap C)] \\
&\leq \omega[\mathrm{P}_W^{-1}\mathrm{B}(0,T)] - \omega[\mathrm{P}_W^{-1}(\mathrm{B}(0,T) \setminus (C \cap \mathrm{B}(0,T))] \\
&\leq \alpha(m)\chi T^m - \chi[\alpha(m)T^m - \mathcal{H}^m(C \cap \mathrm{B}(0,T))] \\
&= \chi\mathcal{H}^m(C \cap \mathrm{B}(0,T))
\end{aligned}$$

and (3) then follows on sending T to infinity. So fix $T \geq 1$ and $0 < \Xi < 1/(2T)$. Choose i so large that

$$\mathrm{F}_{1+1/\Xi}(\omega_i, \omega) \leq \Xi^2 T,$$

$$T(i) > T$$

and

$$\mathrm{P}_W^{-1}[\mathrm{B}(0,T)] \cap \mathrm{B}(0,1/\Xi)$$

is a subset of

$$\mathrm{P}_i^{-1}[\mathrm{B}(0,T(1+\Xi))] \cap \mathrm{B}(0,1/\Xi).$$

Then

$$\omega[\mathrm{P}_i^{-1}[\mathrm{B}(0,T)] \cap \mathrm{B}(0,1/\Xi)] \leq \omega[\mathrm{P}_i^{-1}[\mathrm{B}(0,T(1+\Xi))] \cap \mathrm{B}(0,1/\Xi)]$$

and so as Lemma 1.3.4 implies that

$$\omega[\mathrm{P}_i^{-1}[\mathrm{B}(0,T(1+\Xi))] \cap \mathrm{B}(0,1/\Xi)] \leq \omega_i[\mathrm{P}_i^{-1}[\mathrm{B}(0,T(1+2\Xi))] \cap \mathrm{B}(0,1/\Xi)] + \Xi$$

which, from *(iii)* above

$$= \alpha(m)\chi T^m (1+2\Xi)^m + \Xi$$

and so we may conclude that

$$\omega[\mathrm{P}_i^{-1}[\mathrm{B}(0,T)] \cap \mathrm{B}(0,1/\Xi)] \leq \alpha(m)\chi T^m (1+2\Xi)^m + \Xi$$

and as Ξ was arbitrary we deduce that

$$\omega[\mathrm{P}_i^{-1}(\mathrm{B}(0,T))] \leq \alpha(m)\chi T^m.$$

Thus (3) holds.

The fact that (3) holds together with the earlier note that

$$W = \mathrm{P}_W[\operatorname{Spt}\omega \cap \mathrm{X}(0, 2, 2K(1 + 2\delta), W)]$$

implies (using an identical technique to that used in Lemma 2.3.8) that

$$\operatorname{Spt}\omega \subset \mathrm{X}(0, 2, 2K(1 + 2\delta), W)$$

and so (1) holds.

It remains only to verify (6) but as $\omega \in \operatorname{Tan}_S(\mu, X) \subset \mathcal{M}_C^m(l, u)$ and we have already verified (1),(2),(3) and (5) of the Lemma then we may use Lemma 2.2.12 to deduce that ω is m-rectifiable as required. ∎

2.3.3 Deriving a contradiction

We are now able to find a contradiction: Let ω be the measure whose existence is guaranteed by Lemma 2.3.9 and let $X \in B^{(2)}$, $Y \in \mathbf{R}^n$ and $W \in \mathrm{G}(n, m)$ be as in Lemma 2.3.9. Since $\omega \in \operatorname{Tan}_S(\mu, X)$ we know that $0 \in \operatorname{Spt}\omega$ and as $Y \in \operatorname{Spt}\omega$ has $\|\mathrm{P}_W^{\perp} Y\| \geq \delta/2$ we conclude that

$$\operatorname{diam}(\mathrm{P}_W^{\perp}\operatorname{Spt}\omega) \geq \delta/2\,(> 0).$$

In addition ω is m-rectifiable,

$$W = \mathrm{P}_W[\operatorname{Spt}\omega],$$

and for ω-a.e. $x \in \operatorname{Spt}\omega$

$$\overline{\mathrm{D}}_m(\omega, x) \geq \chi.$$

Hence we may apply Lemma 2.2.13 to conclude that there is a Borel set $B \subset W$ with

$$\omega[\mathrm{P}_W^{-1} B] > \chi \mathcal{H}^m(B)$$

but this contradicts the definition of ω. Thus no such measure ω can exist and so our original measure μ must be m-rectifiable as claimed.

Chapter 3

Examples

In this chapter we shall present examples which illustrate some of the relationships between various tools for investigating the local structure of a measure.

3.1 Average densities and tangent measure distributions

It is well known that $\mathrm{D}_m(\mu, x)$ can only exist in general if m is an integer and μ is rectifiable (see [Pre87, Mat95] or Theorem 1.5.2). In an attempt to define a notion of density which exists for a wider class of measures and for, possibly, non-integer values of m, Bedford and Fisher [BF92] introduced the notion of average density: For μ a measure on $\mathbf{R}^n$ and $0 \leq m \leq n$ the upper average density of μ at x, $\overline{\mathrm{D}}^2_m(\mu, x)$, is defined by

$$\overline{\mathrm{D}}^2_m(\mu, x) := \limsup_{r\to 0} \frac{1}{-\log r} \int_r^1 \frac{\mu\mathrm{B}(x,t)}{\alpha(m)t^m} \frac{dt}{t}$$

and the lower average density of μ at x, $\underline{\mathrm{D}}^2_m(\mu, x)$ is defined to be

$$\underline{\mathrm{D}}^2_m(\mu, x) := \liminf_{r\to 0} \frac{1}{-\log r} \int_r^1 \frac{\mu\mathrm{B}(x,t)}{\alpha(m)t^m} \frac{dt}{t}.$$

If $\underline{\mathrm{D}}^2_m(\mu, x) = \overline{\mathrm{D}}^2_m(\mu, x)$ then the common value is denoted by $\mathrm{D}^2_m(\mu, x)$ and is called the average density of μ at x. It is elementary to see that

$$\underline{\mathrm{D}}_m(\mu, x) \le \underline{\mathrm{D}}^2_m(\mu, x) \le \overline{\mathrm{D}}^2_m(\mu, x) \le \overline{\mathrm{D}}_m(\mu, x). \tag{3.1}$$

In their paper [BF92] Bedford and Fisher showed that average density exists (is positive and finite) for the usual 1/3-Cantor set if m is taken to be $\log 2/\log 3$. They also showed that for Hyperbolic Cantor sets there is an m such that the average m-density exists. Since then their results have been extended to cover other classes of measures. For example in [Gra93] it is shown that average density exists for all self-similar measures for an appropriately chosen value of m.

In a slightly different direction a recent paper of Falconer and Springer [FS95] showed that if for μ-a.e. x

$$0 < \underline{\mathrm{D}}^2_m(\mu, x) = \overline{\mathrm{D}}_m(\mu, x) < \infty$$

then m is an integer. This has recently been improved by Marstrand [Mar94] to the following: If for μ-a.e. x

$$0 < \overline{\mathrm{D}}^2_m(\mu, x) = \overline{\mathrm{D}}_m(\mu, x) < \infty$$

then m is an integer. It was conjectured by Springer [Spr93] that equality of average density and upper density almost everywhere may imply rectifiability: The example of Section 3.2 shows that this is not true.

In [Ban92] Bandt introduced the notion of tangent measure distributions. These are probability measures defined on the tangent measures of a measure μ. They reflect how often a particular tangent measure appears as one looks at the blowups of μ in the vicinity of a point $x \in \mathbf{R}^n$. The definition we shall present here will only be for normalisations of the form r^m and we shall assume that both the upper and lower m-densities are positive and finite: For a more general approach see [Mör95]. Before we define tangent measure distributions it will be useful to recall some of the general theory of (Borel) probability measures on $\mathcal{M}(\mathbf{R}^n)$.

Given a sequence, P_i, of Borel probability measures on $\mathcal{M}(\mathbf{R}^n)$ we say that $P_i \to P$, a probability measure, if for all $h: \mathcal{M}(\mathbf{R}^n) \to \mathbf{R}$ which are continuous and bounded we have

$$\int_{\mathcal{M}} h(\mu)\, dP_i(\mu) \to \int_{\mathcal{M}} h(\mu)\, dP(\mu) \text{ as } i \to \infty.$$

This is the usual definition of weak convergence. The following Theorem states some useful equivalent definitions of convergence.

Theorem 3.1.1 *If P and P_i $(i \in \mathbf{N})$ are probability measures on $\mathcal{M}$ then the following are equivalent:*

1. $P_i \to P$,

2. *for all continuous functions $\theta : \mathbf{R}^n \to [0, \infty)$ with compact support*

$$\int_{\mathcal{M}} \exp\left[-\mu(\theta)\right]\, dP_i(\mu) \to \int_{\mathcal{M}} \exp\left[-\mu(\theta)\right]\, dP(\mu),$$

3. *for all continuous functions* $h : \mathbf{R} \to [0, \infty)$ *which are bounded and for all continuous functions,* $\theta : \mathbf{R}^n \to [0, \infty)$, *with compact support we have*

$$\int_{\mathcal{M}} h\left[-\mu(\theta)\right]\, dP_i(\mu) \to \int_{\mathcal{M}} h\left[-\mu(\theta)\right]\, dP(\mu),$$

4. *for all continuous functions* $h : \mathbf{R} \to [0, \infty)$ *which are bounded and for all continuous functions,* $\theta : \mathbf{R}^n \to [0, \infty)$, *with compact support and* $\operatorname{lip}(\theta) \leq 1$ *we have*

$$\int_{\mathcal{M}} h\left[-\mu(\theta)\right]\, dP_i(\mu) \to \int_{\mathcal{M}} h\left[-\mu(\theta)\right]\, dP(\mu).$$

Proof: The equivalence of (1) and (2) is shown in [Kal76, Theorem 4.2]. It is clear that (3) implies (4) and (3) implies (2). It suffices only to verify that (1) implies (3) and (4) implies (3).

(1)⇒(3). This reduces to checking that functions, $H : \mathcal{M} \to [0, \infty)$, of the form $H(\mu) = h(\mu(\theta))$ are continuous and bounded where $h : \mathbf{R} \to [0, \infty)$ is continuous and bounded and $\theta : \mathbf{R}^n \to [0, \infty)$ is continuous and has compact support. This is clear (from the definition of convergence on $\mathcal{M}$).

(4)⇒(3). This follows from the fact that if $f : \mathbf{R}^n \to [0, \infty)$ has support in $\mathrm{B}(0, R)$ ($R > 0$) then for all $\epsilon > 0$ we can find a Lipschitz function $g : \mathrm{B}(0, 2R) \to [0, \infty)$ such that for all $x \in \mathbf{R}^n$, $0 \leq (g - f)(x) \leq \epsilon$. ∎

Another useful result concerning convergence is the following:

Theorem 3.1.2 *Suppose that* P *and* P_i $(i \in \mathbf{N})$ *are probability measures on* $\mathcal{M}$ *then the following are equivalent:*

1. $P_i \to P$,

2. *for all closed sets* $\mathcal{F} \subset \mathcal{M}$, $\limsup_{i\to\infty} P_i(\mathcal{F}) \leq P(\mathcal{F})$,

3. *for all open sets* $\mathcal{G} \subset \mathcal{M}$, $\liminf_{i\to\infty} P_i(\mathcal{G}) \geq P(\mathcal{G})$.

Proof: See [Par67, Chapter 3]. ∎

We are now in a position to define tangent measure distributions: Suppose that μ is a Borel measure on $\mathbf{R}^n$ and there is a $0 \leq m \leq n$ such that for μ-a.e. x

$$0 < \underline{\mathrm{D}}_m(\mu, x) \leq \overline{\mathrm{D}}_m(\mu, x) < \infty.$$

Define, for $0 < r < 1$ and for x satisfying the above density estimates a probability measure, $\Theta_r(\mu, x)$, on $\mathcal{M}(\mathbf{R}^n)$ by

$$\Theta_r(\mu, x)(\mathcal{A}) := \frac{1}{-\log r} \int_r^1 I_{\mathcal{A}}\left(\frac{\mu_{x,t}}{t^m}\right) \frac{dt}{t} \quad \text{for } \mathcal{A} \subset \mathcal{M}$$

(where $I_{\mathcal{A}}$ denotes the indicator function of the set $\mathcal{A}$.) A probability measure P on $\mathcal{M}$ is a tangent measure distribution of μ at x if there exists a sequence $r(i) \searrow 0$ such that

$$P = \lim_{i\to\infty} \Theta_{r(i)}(\mu, x).$$

The set of all tangent measure distributions of μ at x shall be denoted by $\mathcal{P}(\mu, x)$. For a measure $\nu \in \mathcal{M}$ we shall let Δ_ν denote the probability measure on $\mathcal{M}$ which is given by

$$\Delta_\nu(\mathcal{A}) := \begin{cases} 1 & \text{if } \nu \in \mathcal{A}, \\ 0 & \text{otherwise.} \end{cases}$$

We now list a few simple properties of tangent measure distributions.

Lemma 3.1.3 *Suppose that μ is a measure on $\mathbf{R}^n$ and there is a $0 \leq m \leq n$ such that for μ-a.e. x*

$$0 < \underline{\mathrm{D}}_m(\mu, x) \leq \overline{\mathrm{D}}_m(\mu, x) < \infty$$

then for μ-a.e. x the following hold:

1. $\mathcal{P}(\mu, x) \neq \emptyset$,

2. *if* $P \in \mathcal{P}(\mu, x)$ *then* $\mathrm{Spt}\,(P) \subset \mathrm{Tan}_S(\mu, x)$.

Proof: See [Mör95]. ∎

The following Lemma relates average densities with tangent measure distributions.

Lemma 3.1.4 *Suppose that μ is a measure on $\mathbf{R}^n$, $0 \leq m \leq n$ and $x \in \mathbf{R}^n$ are such that*

$$0 < \underline{\mathrm{D}}_m(\mu, x) \leq \overline{\mathrm{D}}_m(\mu, x) < \infty.$$

Then

$$\alpha(m)\underline{\mathrm{D}}^2_m(\mu, x) = \inf_{P \in \mathcal{P}(\mu,x)} \int \nu \mathrm{B}(0,1)\, dP(\nu)$$

and

$$\alpha(m)\overline{\mathrm{D}}^2_m(\mu, x) = \sup_{P \in \mathcal{P}(\mu,x)} \int \nu \mathrm{B}(0,1)\, dP(\nu).$$

Proof: Since $0 < \underline{\mathrm{D}}_m(\mu, x) \leq \overline{\mathrm{D}}_m(\mu, x) < \infty$ it follows that

$$\limsup_{r \searrow 0} \frac{\mu \mathrm{B}(x, 2r)}{\mu \mathrm{B}(x, r)} < \infty$$

and so

$$\mathrm{clos}\left\{t^{-m}\mu_{x,t} : 0 < t \leq 1\right\}$$

is a compact set and thus

$$\{\Theta_r(\mu,x):\, 0<r<1\}$$

is a uniformly tight set of probability measures on $\mathcal{M}(\mathbf{R}^n)$. Consequently, by Prohorov's Theorem [Dud89, Theorem 11.5.4], for any sequence $1>r(i)\searrow 0$ we can find a subsequence $r(i(j))\searrow 0$ such that $\Theta_{r(i(j))}(\mu,x)$ converges to a probability measure P on $\mathcal{M}(\mathbf{R}^n)$ (and P is necessarily an element of $\mathcal{P}(\mu,x)$.) Hence for all continuous and bounded functions $H:\mathcal{M}\to\mathbf{R}$ we have that

$$\inf_{P\in\mathcal{P}(\mu,x)}\int H(\nu)\,dP(\nu)=\liminf_{r\searrow 0}\frac{1}{-\log r}\int_r^1 H\left(\frac{\mu_{x,t}}{t^m}\right)\frac{dt}{t}$$

and

$$\sup_{P\in\mathcal{P}(\mu,x)}\int H(\nu)\,dP(\nu)=\limsup_{r\searrow 0}\frac{1}{-\log r}\int_r^1 H\left(\frac{\mu_{x,t}}{t^m}\right)\frac{dt}{t}.$$

Fix $0<\xi<1$ and consider $\gamma,\Gamma:\mathbf{R}^n\to[0,\infty)$ given by

$$\gamma(\zeta):=\max\{0,\,\xi-\operatorname{dist}(\zeta,\mathrm{B}(0,1-\xi))\},$$

$$\Gamma(\zeta):=\max\{0,\,\xi-\operatorname{dist}(\zeta,\mathrm{B}(0,1+\xi))\}.$$

Observe that $h,H:\mathcal{M}\to[0,\infty)$ defined by $h(\nu):=\xi^{-1}\nu(\gamma)$ and $H(\nu):=\xi^{-1}\nu(\Gamma)$ are both bounded and continuous.

We find that

$$\begin{aligned}
\inf_{P\in\mathcal{P}(\mu,x)}\int \nu\mathrm{B}(0,1)\,dP(\nu) &\le \inf_{P\in\mathcal{P}(\mu,x)}\int H(\nu)\,dP(\nu)\\
&= \liminf_{r\searrow 0}\frac{1}{-\log r}\int_r^1 H\left(\frac{\mu_{x,t}}{t^m}\right)\frac{dt}{t}\\
&\le \liminf_{r\searrow 0}\frac{1}{-\log r}\int_r^1 \frac{\mu\mathrm{B}(x,(1+\xi)t)}{t^m}\frac{dt}{t}\\
&= \alpha(m)(1+\xi)^m\underline{\mathrm{D}}^2_m(\mu,x).
\end{aligned}$$

Similarly we find that

$$\begin{aligned}\inf_{P\in\mathcal{P}(\mu,x)}\int \nu\mathrm{B}(0,1)\,dP(\nu) &\geq \inf_{P\in\mathcal{P}(\mu,x)}\int h(\nu)\,dP(\nu)\\ &= \liminf_{r\searrow 0}\frac{1}{-\log r}\int_r^1 h\left(\frac{\mu_{x,t}}{t^m}\right)\frac{dt}{t}\\ &\geq \liminf_{r\searrow 0}\frac{1}{-\log r}\int_r^1 \frac{\mu\mathrm{B}(x,(1-\xi)t)}{t^m}\frac{dt}{t}\\ &= \alpha(m)(1-\xi)^m\underline{\mathrm{D}}^2_m(\mu,x).\end{aligned}$$

Hence, as ξ was arbitrary,

$$\alpha(m)\underline{\mathrm{D}}^2_m(\mu,x) = \inf_{P\in\mathcal{P}(\mu,x)}\int \nu\mathrm{B}(0,1)\,dP(\nu).$$

The result for $\overline{\mathrm{D}}^2_m(\mu,x)$ follows in an identical manner. ■

3.2 On the example of Dickinson.

In this section I reconsider the example constructed in [Dic39] which was originally designed as an example of an unrectifiable 1-set with lower 1-density equal to a 1/2 and upper 1-density equal to 1, almost everywhere. I show that a slightly modified version of it, which is also unrectifiable, possesses average 1-density equal to upper 1-density and has a unique tangent measure distribution almost everywhere. This example also illustrates in a simple way the necessity of a convexity condition in Theorem 2.1.1.

3.2.1 Construction

Suppose that (n_k) is a sequence of positive integers and $(\xi_k), (\delta_k), (\gamma_k)$ are sequences of positive real numbers and let

$$N_k := \prod_{i=1}^{k} 2n_i.$$

Additionally suppose that

1. $n_k \nearrow \infty$ and $\sum [n_k]^{-1} < \infty$,

2. $\delta_1 < 1$ and $[\delta_k]^{-1} \sum_{i \geq k+1} \delta_i \searrow 0$,

3. $N_k \delta_k \to 0$ and $N_{k+1} \delta_k \to \infty$,

4. $\gamma_k [\delta_k N_k]^{-1} \to 0$ and $\sum_k \gamma_{2k} = \sum_k \gamma_{2k-1} = \infty$,

5. $\xi_k > [2n_{k+1}]^{-1}$, $\sum_k \xi_k < \infty$ and
$$\frac{\sum_{i=1}^{k} - \log \xi_i}{\log N_k} \to 0.$$

For examples of such sequences one may consider $n_k := 2^{2^k}$, $\delta_k := k^{-1/2} N_k^{-1}$, $\xi_k := 2^{-k}$ and $\gamma_k := k^{-3/4}$.

Let

$$\Sigma := \prod_{k=1}^{\infty} \{0, \ldots, 2n_k - 1\}$$

and equip Σ with the usual product topology. Observe that Σ with this topology forms a compact topological space.

Define $m : \Sigma \to [0, 1]$ by

$$m((\eta_1, \eta_2, \ldots)) := \sum_{k=1}^{\infty} \eta_k / N_k.$$

Define $d : \Sigma \to \mathbf{R}$ by

$$d(\eta_1, \eta_2, \dots) := \sum_{k=1}^{\infty} \delta_k (-1)^{\eta_k}.$$

Let $D : \Sigma \to \mathbf{R}^2$ be given by

$$D(\eta) := (m(\eta), d(\eta)).$$

This map D is clearly invertible and it is also continuous: For if V is a neighbourhood of a point, $x = D(\eta) \in E$ say, then we can find a $\delta > 0$ such that $V \supset \{y : \|y - x\| < \delta\}$. If k is chosen such that $\sum_{i \geq k+1} (N_i^{-1} + 2\delta_i) < \delta$ then for the open set (in Σ) $\eta|_k := \{\sigma \in \Sigma : \sigma_1 = \eta_i \text{ for } i = 1 \dots k\}$ we find that $D(\eta|_k) \subset \{y : \|y - x\| < \delta\}$ and so the claim holds.

Since Σ is a compact space and D is continuous we conclude that E is a compact set.

By considering the orthogonal projection of E onto the x-axis one sees that $\mathcal{H}^1(E) \geq 1$. For $\eta \in \Sigma$ and $k \geq 1$ let

$$C(\eta, k) := \Bigg\{ (s,t) : s \in \left[\sum_{i \leq k} \frac{\eta_i}{N_i}, [N_k]^{-1} + \sum_{i \leq k} \frac{\eta_i}{N_i} \right] \text{ and }$$
$$t \in \left[\sum_{i \leq k} (-1)^{\eta_i} \delta_i - \sum_{i \geq k+1} \delta_i, \sum_{i \leq k} (-1)^{\eta_i} \delta_i + \sum_{i \geq k+1} \delta_i \right] \Bigg\}.$$

Let

$$\mathcal{C}_k := \{ C(\eta, k) : \eta \in \Sigma \}$$

and observe that $\operatorname{card}(\mathcal{C}_k) = N_k$ and for all k, $E \subset \cup \mathcal{C}_k$. Since the diameter of a set $C \in \mathcal{C}_k$ is no more than $[N_k]^{-1}[1 + 2N_k \sum_{i \geq k+1} \delta_i]$ we conclude that for all $\delta > 0$ we can find a K such that if $k \geq K$ then $\mathcal{C}_k$ is a δ-cover of E and so

$$\begin{aligned}\mathcal{H}^1_\delta(E) &\leq N_k \times [N_k]^{-1}\left[1 + 2N_k \sum_{i \geq k+1} \delta_i\right] \\ &= \left[1 + 2N_k \sum_{i \geq k+1} \delta_i\right].\end{aligned}$$

Hence on sending k to infinity we conclude that $\mathcal{H}^1_\delta(E) \leq 1$ which in view of our earlier lower bound on $\mathcal{H}^1(E)$ implies that $\mathcal{H}^1(E) = 1$. It is also straightforward to verify that if C is in $\mathcal{C}_k$ then $\mathcal{H}^1(E \cap C) = [N_k]^{-1}$.

Define the measure μ to be $\mathcal{H}^1 \lfloor_E$. The measure μ is Borel regular and is a probability measure. Moreover E is the support of μ.

3.2.2 Properties of μ

Lemma 3.2.1 *For all $x \in E$*

$$1/6 \leq \underline{\mathrm{D}}_1(\mu, x) \leq \overline{\mathrm{D}}_1(\mu, x) \leq 1.$$

Proof: The upper bound follows immediately from the observation that for all $x \in E$ and $r > 0$, $\mu\mathrm{B}(x, r) \leq 2r$.

For the lower bound suppose that $\eta \in \Sigma$ and fix $0 < r < 1$. Choose $k \geq 1$ so that

$$[N_k]^{-1} \leq r < [N_{k-1}]^{-1}.$$

There are two cases we shall consider

Case 1 : $[N_k]^{-1} \leq r < 3[N_k]^{-1}$

In this situation we have that

$$\mu[\mathrm{B}(D(\eta), r)] \geq [N_k]^{-1} - 2\delta_k$$

and so

$$\frac{\mu[\mathrm{B}(D(\eta),r)]}{2r} \geq \frac{[N_k]^{-1}-2\delta_k}{6[N_k]^{-1}} = \frac{1}{6}-\frac{1}{3}N_k\delta_k.$$

Case 2 : $3[N_k]^{-1} \leq r < [N_{k-1}]^{-1}$

In this case we have that

$$\mu[\mathrm{B}(D(\eta),r)] \geq [N_k]^{-1}\left[rN_k-2\right]-2\delta_k$$

and hence

$$\begin{aligned}\frac{\mu[\mathrm{B}(D(\eta),r)]}{2r} &\geq [N_k]^{-1}\left[\frac{N_k}{2}-\frac{1}{r}\right]-\frac{\delta_k}{r}\\ &\geq \frac{1}{2}-\frac{1}{N_kr}-\frac{\delta_k}{r}\end{aligned}$$

but $N_kr \geq 3$ and so

$$\begin{aligned}&\geq \frac{1}{2}-\frac{1}{3}-3N_k\delta_k\\ &= \frac{1}{6}-3N_k\delta_k.\end{aligned}$$

In either case we have that

$$\frac{\mu[\mathrm{B}(D(\eta),r)]}{2r} \geq \frac{1}{6}-3N_k\delta_k.$$

Moreover as $\eta \in \Sigma$ (and hence $x \in E$) was arbitrary and as $N_k\delta_k \to 0$ we conclude that for all $x \in E$

$$\underline{\mathrm{D}}_1(\mu,x) \geq \frac{1}{6}$$

as required. ∎

Let us now calculate the tangent measures of μ for μ-a.e. x. Define measures κ_d, κ_u by

$$\kappa_d := \mathcal{H}^1\lfloor_{\{(\alpha,0):\alpha\geq 0\}}+\mathcal{H}^1\lfloor_{\{(\alpha,1):\alpha\leq 0\}}$$

and

$$\kappa_u := \mathcal{H}^1 \lfloor_{\{(\alpha,-1):\alpha\leq 0\}} + \mathcal{H}^1 \lfloor_{\{(\alpha,0):\alpha\geq 0\}}$$

and also let

$$\lambda := \mathcal{H}^1 \lfloor_{\{(\alpha,0):\alpha\in\mathbf{R}\}},$$

$$\lambda_l := \mathcal{H}^1 \lfloor_{\{(\alpha,0):\alpha\leq 0\}},$$

and

$$\lambda_r := \mathcal{H}^1 \lfloor_{\{(\alpha,0):\alpha\geq 0\}},$$

It is straightforward to verify that

$$\mathcal{N} := \left\{ r^{-1}\omega_{\zeta,r} : \omega \in \{\kappa_d, \kappa_u, \lambda, \lambda_l, \lambda_r\},\ \zeta \in \operatorname{Spt}\omega \text{ and } r > 0 \right\}$$

is a closed set and that, in fact,

$$\mathcal{N} = \operatorname{clos}\left\{ r^{-1}\omega_{\zeta,r} : \omega \in \{\kappa_d, \kappa_u\},\ \zeta \in \operatorname{Spt}\omega \text{ and } r > 0 \right\}.$$

Lemma 3.2.2 *For μ-a.e. x, $\mathcal{N} \subset \operatorname{Tan}_S(\mu, x)$.*

Proof: In view of the shift invariance of standardised tangent measures and the fact that $\operatorname{Tan}_S(\mu, x)$ is a closed set for μ-a.e. x it suffices for us to show that for μ-a.e. x

$$\{\kappa_d,\ \kappa_u\} \subset \operatorname{Tan}_S(\mu, x).$$

We shall in fact only verify that for μ-a.e. x, $\kappa_u \in \operatorname{Tan}_S(\mu, x)$ as an entirely similar method will enable us to deduce the same result for κ_d.

For $\gamma \in (0,1)$ and $k \geq 1$ let

$$I_k(\gamma) := \{0, 1, \ldots, \lceil \gamma n_k \rceil\}$$

and observe that for $k \geq 1$

$$\begin{aligned}
\mathcal{H}^1 \left\{x \in E : 0 \neq [D^{-1}(x)]_k \text{ is even and } [D^{-1}(x)]_{k+1} \in I_{k+1}(\gamma_{k+1})\right\} \\
&= \left(\frac{n_k - 1}{2n_k}\right) \frac{\lceil \gamma_{k+1} n_{k+1} \rceil}{2n_{k+1}} \\
&\geq (1 - 1/n_k)\gamma_{k+1}/4.
\end{aligned}$$

Hence

$$\sum_k \mathcal{H}^1 \left\{x \in E : 0 \neq [D^{-1}(x)]_{2k} \text{ is even and } [D^{-1}(x)]_{2k+1} \in I_{2k+1}(\gamma_{2k+1})\right\}$$

$$\geq \sum_k (1 - 1/n_{2k})\gamma_{2k+1}/4 = \infty.$$

Hence we may use Borel-Cantelli and the independence of the above events to conclude that for μ-a.e. x we can find a sequence $k(i) \nearrow \infty$ such that for all i if $x = D(\eta)$ then

(i) $0 \neq \eta_{k(i)}$ which is even,

(ii) $\eta_{k(i)+1} \in I_{k(i)+1}(\gamma_{k(i)+1})$ ($I_{k(i)+1}$ for short.)

So fix such an x and sequence $k(i)$. In order to show $\kappa_u \in \mathrm{Tan}_S(\mu, x)$ we need to find a sequence $r(i) \searrow 0$ such that for all $R \geq 4$ and $\epsilon > 0$ there is an M so that for $i \geq M$

$$\mathrm{F}_R(r(i)^{-1}\mu_{x,r(i)},\, \kappa_u) \leq \epsilon.$$

Let $r(i) := 2\delta_{k(i)}$ then I claim that $r(i)^{-1}\mu_{x,r(i)}$ is such a sequence. Fix $R \geq 4$ and $\epsilon > 0$ and choose M such that for $i \geq M$

$$[2N_{k(i)}\delta_{k(i)}]^{-1} > 2R,$$

$$\delta_{k(i)}N_{k(i)+1} \geq 8R/\epsilon$$

and

$$\frac{\gamma_{k(i)+1}}{2\delta_{k(i)}N_{k(i)}} + \frac{1}{\delta_{k(i)}N_{k(i)+1}} \leq \epsilon/2.$$

If $x = (x_1, x_2)$ then let $\bar{x}^i = (\bar{x}_1^i, x_2) := (x_1 - m(\eta_1, \ldots, \eta_{k(i)}, 0, 0, \ldots), x_2)$ and let

$$\nu_i := \mathcal{H}^1 \lfloor_{\{(\alpha,0):\alpha \geq -\bar{x}_1^i/r(i)\}} + \mathcal{H}^1 \lfloor_{\{(\alpha,-1):\alpha \leq -\bar{x}_1^i/r(i)\}}.$$

Observe that

$$\begin{aligned}
\mathrm{F}_R(\kappa_u,\, \nu_i) &\leq 2R\|\bar{x}_1^i/r(i)\| \\
&\leq 2R\frac{\lceil \gamma_{k(i)+1} n_{k(i)+1} \rceil + 1}{2\delta_{k(i)}N_{k(i)+1}} \\
&\leq \epsilon/2.
\end{aligned}$$

Thus it suffices to show that

$$\mathrm{F}_R(r(i)^{-1}\mu_{x,r(i)},\, \nu_i) \leq \epsilon/2.$$

Recall from Section 3.2.1 the definition of $\mathcal{C}_{k(i)+1}$ and for $C \in \mathcal{C}_{k(i)+1}$ define

$$C_{x,r} := \{(y - x)/r : y \in C\}.$$

Let

$$\mathcal{F} := \left\{ C_{x,r(i)} :\, C \in \mathcal{C}_{k(i)+1} \text{ and } C \cap \mathrm{B}(x, Rr(i)) \neq \emptyset \right\}.$$

Notice that if $F \in \mathcal{F}$ then

$$\operatorname{diam}(F) \leq [r(i)N_{k(i)+1}]^{-1} \left(1 + 2N_{k(i)+1} \sum_{j \geq k(i)+2} \delta_j \right) < 2[r(i)N_{k(i)+1}]^{-1} =: d$$

and

$$r(i)^{-1}\mu_{x,r(i)}(F) = \nu_i(F) = [r(i)N_{k(i)+1}]^{-1}.$$

If $F \neq G \in \mathcal{F}$ then

$$r(i)^{-1}\mu_{x,r(i)}(F \cap G) = \nu_i(F \cap G) = 0.$$

Finally observe that

$$\left[\text{Spt }\left(r(i)^{-1}\mu_{x,r(i)} + \nu_i\right) \cap \mathrm{B}(0,R)\right] \subset \cup\mathcal{F}$$

and so we may apply Lemma 1.3.5 (with $\mathcal{A} = \mathcal{F}$ and $\xi = 0$) to conclude that

$$\begin{aligned} \mathrm{F}_R(r(i)^{-1}\mu_{x,r(i)}, \nu_i) &\leq d[\nu_i + r(i)^{-1}\mu_{x,r(i)}](\mathrm{B}(0,R)) \\ &\leq 4dR \\ &= 8R[r(i)N_{k(i)+1}]^{-1} \\ &\leq \epsilon/2. \end{aligned}$$

Hence the Lemma follows. ■

An immediate consequence of this result is that μ is purely 1-unrectifiable; this follows since for μ-almost every point, μ possesses tangent measures which are not flat (see Theorem 1.5.2).

Lemma 3.2.3 *For all $0 < \epsilon < 1$ and $R \geq 1$ there is a $K \geq 1$ such that for all $\eta \in \Sigma$ if $r < (8RN_K)^{-1}$ then there is a $\nu \in \mathcal{N}$ such that*

$$F_R\left(\frac{\mu_{D(\eta),r}}{r}, \nu\right) \leq \epsilon.$$

If, moreover, there is a $k \geq K$ such that $(8RN_{k+1})^{-1} \leq r < (8RN_k)^{-1}$ and η_{k+1} is such that

$$\min\left\{\frac{\eta_{k+1}}{rN_{k+1}}, \frac{|2n_{k+1}\eta_{k+1} - 1|}{rN_{k+1}}\right\} > R$$

then ν may be taken to be λ.

Proof: Fix $0 < \epsilon < 1$, $R \geq 1$ and $\eta \in \Sigma$ and let $x := m(\eta)$. Choose K such that for all $k \geq K$

$$\sum_{i \geq k+1} \delta_i \leq \delta_k/2 \qquad \text{and} \qquad \delta_k N_k \leq \epsilon/(128R^2).$$

Now choose $k \geq K$ such that

$$(8RN_{k+1})^{-1} \leq r < (8RN_k)^{-1}.$$

For $\sigma \in \Sigma$ and $k \geq 1$ let

$$D(\sigma, k) := \left\{ (s,t) : s \in \left[\sum_{i \leq k+2} \frac{\sigma_i}{N_i}, [N_{k+2}]^{-1} + \sum_{i \leq k+2} \frac{\sigma_i}{N_i} \right] \text{ and} \right.$$

$$\left. t \in \left[\sum_{i \leq k} (-1)^{\sigma_i} \delta_i - \sum_{i \geq k+1} \delta_i, \sum_{i \leq k} (-1)^{\sigma_i} \delta_i + \sum_{i \geq k+1} \delta_i \right] \right\}$$

and let

$$\mathcal{D}_k := \{ D(\sigma, k) : \sigma \in \Sigma \}.$$

Thus $\mathcal{D}_k$ consists of N_{k+2} strips of width $[N_{k+2}]^{-1}$ and height $2 \sum_{i \geq k+1} \delta_i \leq 3\delta_{k+1}$ which cover E (the support of μ.) Let

$$\mathcal{F} := \left\{ C_{D(\eta),r} : C \in \mathcal{D}_k \text{ and } C \cap \mathrm{B}(D(\eta), Rr) \neq \emptyset \right\}.$$

Observe that

(i) Spt $\left(r^{-1} \mu_{D(\eta),r} \right) \cap \mathrm{B}(0, R) \subset \cup \mathcal{F}$,

(ii) if $F \in \mathcal{F}$ then

$$\mathrm{diam}\,(F) \leq [rN_{k+2} + 3\delta k + 1]^{-1} < 4\delta_{k+1}/r =: d, \text{ say,}$$

(iii) if $F \in \mathcal{F}$ then $r^{-1}\mu_{D(\eta),r}(F) = [rN_{k+2}]^{-1}$ and if $F \neq G \in \mathcal{F}$ then $r^{-1}\mu_{D(\eta),r}(F \cap G) = 0$.

Thus if we can find a measure ν such that

(iv) Spt $(\nu) \cap \mathrm{B}(0, R) \subset \cup\mathcal{F}$,

(v) if $F \in \mathcal{F}$ then $\nu(F) = [rN_{k+2}]^{-1}$,

(vi) if $F \neq G \in \mathcal{F}$ then $\nu(F \cap G) = 0$

(vii) $\nu\mathrm{B}(0, R) \leq 2R$.

Then we may use Lemma 1.3.5 to deduce that

$$\begin{aligned} \mathrm{F}_R(r^{-1}\mu_{D(\eta),r},\, \nu) &\leq d\left(r^{-1}\mu_{D(\eta),r} + \nu\right)[\mathrm{B}(0, r)] \\ &\leq 4Rd = 16R\delta_{k+1}r^{-1} \\ &\leq 128R^2\delta_{k+1}N_{k+1} \leq \epsilon. \end{aligned}$$

Hence it only remains to show that we can find such a $\nu \in \mathcal{N}$. Let

$$\begin{aligned} \omega := \mathcal{H}^1 \lfloor_{\{(\alpha,0):\exists F\in\mathcal{F} \text{ with } (\alpha,0)\in F\}} + \mathcal{H}^1 \lfloor_{\{(\alpha,2\delta_k/r):\exists F\in\mathcal{F} \text{ with } (\alpha,2\delta_k/r)\in F\}} \\ + \mathcal{H}^1 \lfloor_{\{(\alpha,-2\delta_k/r):\exists F\in\mathcal{F} \text{ with } (\alpha,-2\delta_k/r)\in F\}} \end{aligned}$$

and observe that, as $[rN_k]^{-1} \geq 8R$, at least one of

$$\{(\alpha, 2\delta_k/r) : \exists F \in \mathcal{F} \text{ with } (\alpha, 2\delta_k/r) \in F\},$$

$$\{(\alpha, -2\delta_k/r) : \exists F \in \mathcal{F} \text{ with } (\alpha, -2\delta_k/r) \in F\}$$

is empty. In addition,

$$\{(\alpha, 0) : \exists F \in \mathcal{F} \text{ with } (\alpha, 0) \in F\} \neq \emptyset.$$

Hence $\mathrm{B}(0,R) \cap \operatorname{Spt} \omega$ consists of at most two disjoint line segments. In addition these line segments intersect the boundary of $\mathrm{B}(0,R)$. Finally observe that, as $r^{-1}[2\delta_k - 3\delta_{k+1}] > 0$, each $F \in \mathcal{F}$ has an intersects exactly one line segment in a positive length. Hence we can find a $\nu \in \mathcal{N}$ such that $\mathrm{F}_R(\omega,\nu) = 0$. Clearly for all $F \in \mathcal{F}$ we have that $\nu(F) = [rN_{k+2}]^{-1}$ and if $F \neq G \in \mathcal{F}$ then $\nu(F \cap G) = 0$. From the definition of ω we can see that

$$\mathrm{B}(0,R) \cap \operatorname{Spt} \nu \subset \cup \mathcal{F}$$

and clearly $\nu(\mathrm{B}(0,R)) \leq 2R$. Thus **(iv)** to **(vii)** are satisfied and we are done.

If, in addition,

$$\min\left\{\frac{\eta_{k+1}}{rN_{k+1}}, \frac{|2n_{k+1}\eta_{k+1} - 1|}{rN_{k+1}}\right\} > R$$

then we conclude that the support of ω consists of just one line segment which includes

$$\{(\alpha,0) : |\alpha| \leq R\}$$

and hence we may take ν to be λ as required. ∎

Corollary 3.2.4 *For μ-a.e. x*

$$\operatorname{Tan}_S(\mu,x) = \mathcal{N}.$$

Proof: This follows immediately from the preceding two Lemmas together with Lemma 1.4.6. ∎

Corollary 3.2.5 *For μ-a.e. x,*

$$\underline{\mathrm{D}}_1(\mu,x) = 1/2$$

and

$$\overline{\mathrm{D}}_1(\mu, x) = 1.$$

Proof: From Lemma 3.2.2 we know that for μ-a.e. x

$$\lambda_l \in \mathrm{Tan}_S(\mu, x).$$

Upon observing that

$$\mathrm{D}_1(\lambda_l, (0,0)) = 1/2$$

we immediately conclude from Corollary 1.4.4 that for μ-a.e. x

$$\underline{\mathrm{D}}_1(\mu, x) \leq 1/2.$$

However as for all $\nu \in \mathcal{N}$ and all $\zeta \in \mathrm{Spt}\,\nu$

$$\underline{\mathrm{D}}_1(\nu, \zeta) \geq 1/2$$

we deduce from Lemma 3.2.3 that for μ-a.e. x

$$\underline{\mathrm{D}}_1(\mu, x) = 1/2.$$

For the upper density just observe, again from Lemma 3.2.2, that for μ-a.e. x, $\lambda \in \mathrm{Tan}_S(\mu, x)$ and

$$\overline{\mathrm{D}}_1(\lambda, 0) = 1$$

hence, again from Corollary 1.4.4, we conclude that

$$\overline{\mathrm{D}}_1(\mu, x) \geq 1$$

and this together with the result of Lemma 3.2.1 implies the conclusion. ∎

In many respects this example can be considered as only just failing to be rectifiable. The presence of 'broken' lines (for example κ_u) as tangent measures easily implies that the convexity condition, (2), of Theorem 2.1.1 fails to hold. (Just consider the projection of the support of κ_u onto the y-axis.) If $\gamma_{2,1}$ denotes the normalised Haar measure on $\mathrm{G}(2,1)$ (normalised so as $\gamma_{2,1}(\mathrm{G}(2,1)) = 1$) then it is a straightforward calculation to verify that for all $\nu \in \mathcal{N}$

$$\gamma_{2,1}\left\{V \in \mathrm{G}(2,1) : \mathrm{P}_V\left[\mathrm{Spt}\,\nu\right] \text{ is not convex}\right\} \leq 1/2.$$

This suggests the following conjecture:

Conjecture 1 *If μ is a Borel regular locally finite measure on the plane and for μ-a.e. x*

1. $0 < \underline{\mathrm{D}}_1(\mu, x) \leq \overline{\mathrm{D}}_1(\mu, x) < \infty$,

2. *for all $\nu \in \mathrm{Tan}_S(\mu, x)$ we have*

$$\gamma_{2,1}\left\{V \in \mathrm{G}(2,1) : \mathrm{P}_V\left[\mathrm{Spt}\,\nu\right] \text{ is convex}\right\} > 1/2.$$

Then μ is 1-rectifiable.

(There are, of course, natural higher dimensional generalisations of this conjecture.) Notice that if (1) and (2) hold for μ at x then there is a $\xi > 0$ such that the following, apparently stronger, statement holds:

(2') for all $\nu \in \mathrm{Tan}_S(\mu, x)$ we have

$$\gamma_{2,1}\left\{V \in \mathrm{G}(2,1) : \mathrm{P}_V\left[\mathrm{Spt}\,\nu\right] \text{ is convex}\right\} \geq (1+\xi)/2.$$

We are now in a position to calculate the tangent measure distributions of μ for μ-a.e. x.

Theorem 3.2.6 *For μ-a.e. x*

$$\mathcal{P}(\mu, x) = \{\Delta_\lambda\}.$$

Proof: We need to show that for μ-a.e. x

$$\Theta_r(\mu, x) \to \Delta_\lambda \text{ as } r \to 0.$$

From Theorem 3.1.1 it suffices to show that for μ-a.e. x if $\theta : \mathbf{R}^n \to [0, \infty)$ has compact support and $\operatorname{lip}(\theta) \leq 1$ then

$$\frac{1}{-\log r} \int_r^1 E\left(-\frac{\mu_{x,t}}{t}(\theta)\right) \frac{dt}{t} \to E(-\lambda(\theta))$$

where $E(y) := \exp(y)$. Thus it suffices to show that for all $R \in \mathbf{N}$ and for μ-a.e. x if $\theta : \mathbf{R}^n \to [0, \infty)$ is such that $\operatorname{Spt}(\theta) \subset \mathrm{B}(0, R)$ and $\operatorname{lip} \theta \leq 1$ then

$$\frac{1}{-\log r} \int_r^1 E\left(-\frac{\mu_{x,t}}{t}(\theta)\right) \frac{dt}{t} \to E(-\lambda(\theta)).$$

Fix an $R \in \mathbf{N}$ and recall from Subsection 3.2.1 that the sequence $\xi_k \searrow 0$ was defined so that

$$\sum_k \xi_k < \infty, \qquad \xi_k > (2n_{k+1})^{-1}$$

and

$$\frac{\sum_{i=1}^k - \log \xi_i}{\log N_k} \to 0.$$

Observe that

$$\begin{aligned} P(\eta_{k+1} \in \{0, \ldots, \lceil \xi_k n_{k+1} \rceil - 1\} \cup \{2n_{k+1} - \lceil \xi_k n_{k+1} \rceil - 1, \ldots, 2n_{k+1} - 1\}) \\ \leq \; 2(\gamma_k + n_{k+1}^{-1}) \end{aligned}$$

and thus

$$\sum_k P(\eta_{k+1} \in \{0,\ldots,\lceil \xi_k n_{k+1}\rceil - 1\} \cup \{2n_{k+1} - \lceil \xi_k n_{k+1}\rceil, \ldots, 2n_{k+1} - 1\}) < \infty.$$

Hence by the Borel-Cantelli Lemma we deduce that for P-a.e. η there is a $K' \in \mathbf{N}$ such that for all $k \geq K'$

$$\eta_{k+1} \in \{\lceil \xi_k n_{k+1}\rceil, \ldots, 2n_{k+1} - \lceil \xi_k n_{k+1}\rceil - 2\} =: J_{k+1}, \text{ say.}$$

Fix such an $\eta \in \Sigma$ and K', let $x := m(\eta)$ and fix $0 < \epsilon < 1$. Let $\theta : \mathbf{R}^n \to [0, \infty)$ have $\operatorname{lip}(\theta) \leq 1$ and $\operatorname{Spt}(\theta) \subset \mathrm{B}(0, R)$. Choose $\delta > 0$ such that if $|y - \lambda(\theta)| \leq \delta$ then $|E(-y) - E(-\lambda(\theta))| \leq \epsilon/3$. Observe that if $k \geq K'$ and $t > 0$ are such that $(8RN_{k+1})^{-1} \leq t < \xi_k (8RN_k)^{-1}$ then

$$\min\left\{\frac{\eta_{k+1}}{tN_{k+1}}, \frac{|2n_{k+1} - \eta_k - 1|}{tN_{k+1}}\right\} > R.$$

Hence, by Lemma 3.2.3, it follows that we can find $K'' \geq K'$ such that for all $k \geq K''$ if $(8RN_{k+1})^{-1} \leq t < \xi_k (8RN_k)^{-1}$ then

$$\mathrm{F}_R\left(t^{-1}\mu_{x,t}, \lambda\right) \leq \delta$$

and so, in particular,

$$|t^{-1}\mu_{x,t}(\theta) - \lambda(\theta)| \leq \delta$$

and thus

$$|E(-t^{-1}\mu_{x,t}(\theta)) - E(-\lambda(\theta))| \leq \epsilon/3.$$

For $i \in \mathbf{N}$ let $a_i := (8RN_i)^{-1}$ — this will make some of the calculations appear a little less unwieldy. Finally, choose $K \geq K''$ such that for all $k \geq K$

$$\frac{\log \xi_{K''} + \log a_{K''}}{-\log \xi_k a_k} \leq \frac{\epsilon}{3[1 + E(-\lambda(\theta))]}$$

and

$$\frac{\sum_{i=K''+1}^{k} -\log \xi_i}{-\log \xi_k a_k} \leq \frac{\epsilon}{3[1+E(-\lambda(\theta))]}.$$

Fix $0 < r < \xi_K (8RN_K)^{-1}$ and let us estimate

$$\left| \frac{1}{-\log r} \int_r^1 E(-t^{-1}\mu_{x,t}(\theta)) \frac{dt}{t} - E(-\lambda(\theta)) \right|.$$

Choose k such that

$$(8RN_{k+1})^{-1} \leq r < \xi_k (8RN_k)^{-1}.$$

Observe that for all $y \geq 0$, $|E(-y)| \leq 1$ and so we may estimate that

$$\begin{aligned}
&\left| \frac{1}{-\log r} \int_r^1 E(t^{-1}\mu_{x,t}(\theta)) \frac{dt}{t} - E(-\lambda(\theta)) \right| \\
&\leq \left| \frac{1}{-\log r} \int_r^{\xi_{K''} a_{K''}} E(-t^{-1}\mu_{x,t}(\theta)) \frac{dt}{t} - E(-\lambda(\theta)) \right| + \\
&\qquad \frac{\log \xi_{K''} + \log a_{K''}}{\log r} \\
&\leq \left| \frac{1}{-\log r} \left(\int_r^{\xi_k a_k} + \sum_{i=K''}^{k-1} \int_{a_{i+1}}^{\xi_i a_i} \right) E(-t^{-1}\mu_{x,t}(\theta)) \frac{dt}{t} - E(-\lambda(\theta)) \right| + \\
&\qquad \frac{\epsilon}{3[1+E(-\lambda(\theta))]} + \frac{\sum_{i=K''+1}^{k} -\log \xi_i}{-\log r} \\
&\leq \left| \frac{1}{-\log r} \left(\int_r^{\xi_k a_k} + \sum_{i=K''}^{k-1} \int_{a_{i+1}}^{\xi_i a_i} \right) E(-t^{-1}\mu_{x,t}(\theta)) \frac{dt}{t} - E(-\lambda(\theta)) \right| + \\
&\qquad \frac{2\epsilon}{3[1+E(-\lambda(\theta))]}.
\end{aligned}$$

However for $K'' \leq i \leq k$ and $a_{i+1} \leq t < \xi_i a_i$ we have that

$$E(-\lambda(\theta)) - \epsilon/3 \leq E(-t^{-1}\mu_{x,t}(\theta)) \leq E(-\lambda(\theta)) + \epsilon/3$$

and since

$$\left(1 - \frac{2\epsilon}{3[1+E(-\lambda(\theta))]}\right) \leq \frac{1}{-\log r} \left(\int_r^{\xi_k a_k} + \sum_{i=K''}^{k-1} \int_{a_{i+1}}^{\xi_i a_i} \right) \frac{dt}{t} \leq 1$$

we conclude that

$$\left|\frac{1}{-\log r}\left(\int_r^{\xi_k a_k} + \sum_{i=K''}^{k-1}\int_{a_{i+1}}^{\xi_i a_i}\right) E(-t^{-1}\mu_{x,t}(\theta))\frac{dt}{t} - E(-\lambda(\theta))\right|$$
$$\leq \frac{\epsilon}{3} + \frac{2\epsilon}{3[1+E(-\lambda(\theta))]}E(-\lambda(\theta)).$$

Hence upon combining these estimates we deduce that

$$\left|\frac{1}{-\log r}\int_r^1 E(t^{-1}\mu_{x,t}(\theta))\frac{dt}{t} - E(-\lambda(\theta))\right| \leq \epsilon$$

as required. ∎

Theorem 3.2.7 *For μ-a.e. x,*

$$\mathrm{D}_1^2(\mu, x) = \overline{\mathrm{D}}_1(\mu, x) = 1.$$

Proof: It is easy to calculate that for μ-a.e. x

$$\begin{aligned}\inf_{P\in\mathcal{P}(\mu,x)}\int \nu\mathrm{B}(0,1)\,dP(\nu) &= 2\\ &= \sup_{P\in\mathcal{P}(\mu,x)}\int \nu\mathrm{B}(0,1)\,dP(\nu).\end{aligned}$$

Hence from Lemma 3.1.4 we deduce that

$$\mathrm{D}_1^2(\mu, x) = 1$$

as required. ∎

3.3 On an example of Besicovitch.

The class of examples we shall construct in this section is based upon an example given by Besicovitch [Bes28, §11] and it will illustrate some of the properties of the concepts introduced at the start of this chapter.

Throughout this section we shall work in $\mathbf{R}^2$. For $B = \mathrm{B}(z,r) \subset \mathbf{R}^2$, $j \in \mathbf{N}$ and $i \in \{1, \ldots, j\}$ define

$$D(B,i,j) := \mathrm{B}\left(z + r(1 - 1/j)(\cos(2\pi i/j), \sin(2\pi i/j)), r/j\right).$$

Thus $D(B,i,j)$ is a disc of radius r/j contained in B which touches the boundary of B.

Let (n_i) be a sequence of integers with $n_1 \geq 4$ and $n_i \nearrow \infty$. Let $\Sigma := \prod_{i=1}^{\infty}\{1, \ldots, n_i\}$ be the code space defined by this sequence and let it be endowed with the usual (discrete) product topology. For $j \in \mathbf{N}$ let $\Sigma_j := \prod_{i=1}^{j}\{1, \ldots, n_i\}$ and set $\Sigma_0 := \{()\}$ (the set consisting of the empty sequence). If $\eta \in \Sigma \cup \Sigma_j$ then η_i will denote the value of the i^{th}-coordinate of η if this makes sense. For $j \in \mathbf{N}$, $i \leq j$ and $\eta \in \Sigma \cup \Sigma_j$ let

$$\eta|_i := (\eta_1, \ldots, \eta_i).$$

Finally let $\eta|_0 := ()$ (the empty sequence). Define a probability measure on Σ by setting κ to be the measure which satisfies: For $\eta \in \Sigma$ and $j \in \mathbf{N} \cup \{0\}$

$$\kappa\left(\{\sigma \in \Sigma : \sigma|_j = \eta|_j\}\right) = \prod_{i=1}^{j} n_i^{-1}$$

(with the convention that the product is defined to be 1 when $j = 0$.)

Define a map m_0 from Σ_0 to the subsets of the plane by

$$m_0\left(()\right) := \mathrm{B}(0, 1/2).$$

Now for $j \in \mathbf{N}$ define a map m_j from Σ_j to the subsets of the plane inductively as follows: For $(\eta_1, \ldots, \eta_j) \in \Sigma_j$

$$m_j(\eta_1, \ldots, \eta_j) := D\left(m_{j-1}(\eta|_{j-1}), \eta_j, n_j\right).$$

Thus $m_j(\eta_1, \ldots, \eta_j)$ determines a disc contained in $m_{j-1}(\eta|_{j-1})$ of radius n_j^{-1} times that of $m_{j-1}(\eta|_{j-1})$. Hence

$$\operatorname{diam}[m_j(\eta_1, \ldots, \eta_j)] = \prod_{i=1}^{j} n_i^{-1} = \kappa(\{\sigma \in \Sigma : \sigma|_j = (\eta_1, \ldots, \eta_j)\}).$$

Define collections of discs, $\mathcal{F}_j$, for $j \in \{0\} \cup \mathbf{N}$ by

$$\mathcal{F}_j := \{m_j(\sigma) : \sigma \in \Sigma_j\}$$

and define sets E_j by

$$E_j := \cup \mathcal{F}_j.$$

Notice that the E_j are compact, non-empty sets and $E_j \supset E_{j+1}$. Hence we may define a non-empty, compact set E by

$$E := \bigcap_{j \in \mathbf{N}} E_j.$$

For $\eta \in \Sigma$ we can observe that $\operatorname{diam}[m_j(\eta|_j)] \to 0$ as $j \to \infty$ and hence we may define a map $m : \Sigma \to E$ by setting $m(\eta)$ to be the unique point contained in

$$\bigcap_{j \in \mathbf{N}} m_j(\eta|_j).$$

Moreover this map is clearly invertible and continuous.

It is possible to show that $\mathcal{H}^1(E) = 1$ by a method identical to that used by Besicovitch in [Bes28, §11]. However this result is unnecessary for our purposes and is omitted here. Instead let us define a measure μ whose support is E. Set $\mu := m_{\#}\kappa$, that is for $A \subset \mathbf{R}^2$ define

$$\mu(A) := \kappa(\{\eta \in \Sigma : m(\eta) \in A\}).$$

Since m is continuous it follows that μ is a Borel regular measure on the plane and, clearly, μ is finite. Observe that for $j \in \mathbf{N}$ we have that if $D \in \mathcal{F}_j$ then

$$\mu(D) = \prod_{i=1}^{j} n_i^{-1}.$$

Before proceeding further it shall be useful to make a few simple geometric observations about the set E. If we set $\rho_0 = 1/2$ and define for $j \in \mathbf{N}$

$$\rho_j := \frac{1}{2}\left(\prod_{i=1}^{j} n_i\right)^{-1}$$

then we find that if x_1 and x_2 are the centres of two disjoint discs from $\mathcal{F}_j$ such that both the discs lie in the same disc from $\mathcal{F}_{j-1}$ and they are separated by $t-1$ discs in $\mathcal{F}_j$ for some $1 \le t \le \lfloor \frac{n_j}{2} \rfloor$ then

$$\|x_1 - x_2\| = 2\rho_{j-1}\left(1 - n_j^{-1}\right)\sin\left(\frac{\pi t}{n_j}\right). \tag{3.2}$$

Consequently we find that the minimum distance between disjoint discs in $\mathcal{F}_j$ is given by

$$d_j := 2\rho_{j-1}\left(1 - n_j^{-1}\right)\sin\left(\frac{\pi}{n_j}\right) - 2\rho_j$$

and, as $n_1 \ge 4$,

$$2\left(\frac{1}{2}3\sqrt{2} - 1\right)\rho_j \le d_j \le 2(\pi - 1)\rho_j \tag{3.3}$$

and, since $n_j \nearrow \infty$,

$$\lim_{j\to\infty} \frac{d_j}{\rho_j} = 2(\pi - 1).$$

Fix $j \in \mathbf{N}$ and suppose that $B \in \mathcal{F}_j$. Let $y \in \mathbf{R}^2$ and suppose that $r > 0$. I claim that

$$\mu\left[\partial \mathrm{B}(y, r) \cap B\right] = 0. \tag{3.4}$$

For choose $J \in \mathbf{N}$ such that if $i \geq J$ and $D \in \mathcal{F}_i$ then $n_i \geq 1/100$ and

$$\mathcal{H}^1[\partial\mathrm{B}(y,r) \cap D] \leq (11/10)\mathrm{diam}\,[\partial\mathrm{B}(y,r) \cap D].$$

Then if $i \geq J$ and $D \in \mathcal{F}_i$ we find that

$$\begin{aligned} \mathrm{card}\,\{C \in \mathcal{F}_{i+1} : C \cap \partial\mathrm{B}(y,r) \neq \emptyset \text{ and } C \subset D\} &\leq 2\sqrt{n}_{i+1} + (2/5)n_{i+1} \\ &\leq (3/5)n_{i+1}. \end{aligned}$$

Thus if $k \in \mathbf{N}$ and $k > J + j$ then

$$\mathrm{card}\,\{D \in \mathcal{F}_k : D \cap \partial\mathrm{B}(y,r) \neq \emptyset \text{ and } D \subset B\} \leq \prod_{i=j}^{J+j} n_i \times \prod_{i=J+j+1}^{k} (3n_i/5)$$

and so

$$\begin{aligned} \mu(B \cap \partial\mathrm{B}(y,r)) &\leq \left(\prod_{i=j}^{J+j} n_i\right) \times \left(\prod_{i=J+j+1}^{k} 3n_i/5\right) \times \left(\prod_{i=1}^{k} n_i\right)^{-1} \\ &\to 0 \text{ as } k \to \infty \end{aligned}$$

as required.

3.3.1 Properties of our measure

Our first task is to estimate the upper and lower 1-densities of points in our set E.

Lemma 3.3.1 *For all $x \in E$*

$$\frac{1}{4\pi} \leq \underline{\mathrm{D}}_1(\mu, x) \leq \overline{\mathrm{D}}_1(\mu, x) \leq \frac{5}{3}.$$

Proof: Fix $0 < r < 2\rho_2$ and $x \in E$ and choose $j \in \mathbf{N}$ such that

$$2\rho_j \leq r < 2\rho_{j-1}.$$

Observe that, as inequality 3.2 implies that $d_{j-1} > r$, $\mathrm{B}(x,r)$ intersects only one disc from $\mathcal{F}_{j-1}$. We shall consider three cases:

Case 1. If there is a $t \in \{1, \ldots, \lfloor n_j/2 \rfloor\}$ such that

$$2\rho_{j-1}\left(1 - n_j^{-1}\right)\sin\left(\frac{\pi t}{n_j}\right) \leq r < 2\rho_{j-1}\left(1 - n_j^{-1}\right)\sin\left(\frac{\pi(t+1)}{n_j}\right)$$

then

$$2t - 1 \leq \mathrm{card}\,\{D \in \mathcal{F}_j : D \subset \mathrm{B}(x,r)\}$$

and

$$\mathrm{card}\,\{D \in \mathcal{F}_j : D \cap \mathrm{B}(x,r) \neq \emptyset\} \leq 2t + 3.$$

Thus

$$2(2t-1)\rho_j \leq \mu\mathrm{B}(x,r) \leq 2(2t+3)\rho_j$$

which implies that

$$\frac{2t-1}{2(n_j - 1)\sin[\pi(t+1)/n_j]} \leq \frac{\mu\mathrm{B}(x,r)}{2r} \leq \frac{2t+3}{2(n_j-1)\sin[\pi t/n_j]}.$$

This gives that for r in this range

$$\frac{1}{4\pi} \leq \frac{\mu\mathrm{B}(x,r)}{2r} \leq \frac{5}{3}.$$

Case 2. If $2\rho_j \leq r < 2\rho_{j-1}\left(1 - n_j^{-1}\right)\sin[\pi/n_j]$ then inequality 3.2 implies that $\mathrm{B}(x,r)$ intersects no more than 3 discs of $\mathcal{F}_j$ and so

$$2\rho_j \leq \mu\mathrm{B}(x,r) \leq 6\rho_j.$$

Thus

$$\frac{1}{2(n_j-1)\sin[\pi/n_j]} \leq \frac{\mu\mathrm{B}(x,r)}{2r} \leq \frac{3}{2}$$

and so

$$\frac{1}{2\pi} \leq \frac{\mu\mathrm{B}(x,r)}{2r} \leq \frac{3}{2}.$$

Case 3. Finally, if $2\rho_{j-1}\left(1-n_j^{-1}\right) \leq r < 2\rho_{j-1}$ then

$$\rho_{j-1} \leq \mu\mathrm{B}(x,r) \leq 2\rho_{j-1}$$

and so

$$\frac{1}{2} \leq \frac{\mu\mathrm{B}(x,r)}{2r} \leq \frac{n_j}{2(n_j-1)} \leq \frac{2}{3}.$$

Since one of these three cases must occur for any $0 < r < 2\rho_2$ we conclude that the Lemma holds. ∎

Our next lemma investigates the geometry of the support of μ and will save a lot of repetition later.

Lemma 3.3.2 *For all $R \geq 1$ and $C \geq 20R$ there is a $K \in \mathbf{N}$ such that for all $k \geq K$ and $\eta \in \Sigma$ if*

$$r \in [\rho_k/C, \rho_{k-1}/C]$$

then there is a unique disc $D \in \mathcal{F}_{k-1}$ with

$$D_{m(\eta),r} \cap \mathrm{B}(0,R) \neq \emptyset.$$

Moreover we can find a disjoint collection, $\mathcal{F}$ say, of $1 + \lceil 2Rr/(\pi\rho_k)\rceil$ discs contained in D such that

1. *there is a $B_0 = \mathrm{B}(y_0, \rho_k/r) \in \mathcal{F}$ which contains the origin,*

2. *for all* $B \in \mathcal{F}$ *there is a* $\sigma \in \Sigma_k$ *such that* $B = [m_k(\sigma)]_{m(\eta),r}$,

3. $\mathrm{B}(0,R) \cap \operatorname{Spt} \mu_{m(\eta),r} \subset \cup\mathcal{F}$,

4. *Let* L *be the line through the origin which makes an angle of* $(2\eta_k/n_k - (1/2))\pi$ *with the positive* x*-axis and let* Q *denote orthogonal projection onto* L *and* $\mathrm{Q}^{\perp}$ *denote orthogonal projection onto the line perpendicular to* L. *Then for any* $B = \mathrm{B}(y, \rho_k/r) \in \mathcal{F}$ *such that* $B \neq B_0$ *we find that*

$$\|\mathrm{Q}^{\perp}(y - y_0)\| \leq 2\frac{1}{n_k}\frac{\rho_k}{r}\left(\pi + \frac{2Rr}{\pi\rho_k}\right)^2 \tag{3.5}$$

and

$$|2\pi t\rho_k/r - \|\mathrm{Q}(y - y_0)\|\,| \leq 8\frac{1}{n_k}\frac{\rho_k}{r}\left(\pi + \frac{2Rr}{\rho_k}\right)^2 \tag{3.6}$$

where B *is* t *discs away from* B_0 *(that is* $t - 1$ *discs of* $\mathcal{F}$ *lie between* B *and* B_0.*)*

Proof: Fix C and R as described in the Lemma. Choose K such that for all $k \geq K$

$$n_{k-1} \geq 10$$

Fix $k \geq K$ and suppose that $\eta \in \Sigma$. Now consider $D = m_{k-1}(\eta|_{k-1})$ and observe that $0 \in D_{m(\eta),r}$. Hence

$$\mathrm{B}(0,R) \subset \mathrm{B}(D_{m(\eta),r}, R)$$

but

$$R \leq C/20 \leq \rho_{k-1}/(20r) < 2(\pi - 2)\rho_{k-1}/r$$

and so, as the separation of discs in $\mathcal{F}_{k-1}$ is at least

$$2\rho_{k-2}\left(1 - n_{k-1}^{-1}\right)\sin\left(\frac{\pi}{n_{k-1}}\right) - 2\rho_{k-1} \geq 2(\pi - 2)\rho_{k-1},$$

we conclude that $\mathrm{B}(0,R)$ intersects only one disc from $\mathcal{F}_{k-1}$.

Let

$$\mathcal{F} := \Bigg\{ B_{m(\eta),r} : B = m_k(\sigma) \text{ where } \sigma \in \Sigma_k, \sigma|_{k-1} = \eta|_{k-1} \text{ and}$$

$$\sigma_k \in \left\{ 1 \le i \le n_k : |i - \eta_k| \le \left(1 + \left\lceil \frac{2Rr}{\pi\rho_k} \right\rceil\right) \mod(n_k) \right\} \Bigg\}.$$

Thus $\mathcal{F}$ consists of B_0 together with $1 + \left\lceil \frac{2Rr}{\pi\rho_k} \right\rceil$ balls on either side of B_0. Observe that (1) and (2) of the lemma are clearly satisfied by $\mathcal{F}$. To verify (3) holds for $\mathcal{F}$ observe that if

$$\mathcal{C} := \{B_{m(\eta),r} : B \in \mathcal{F}_k \text{ and } B_{m(\eta),r} \cap \mathrm{B}(0,R) \neq \emptyset\}$$

then $\cup\mathcal{C} \supset \mathrm{B}(0,R) \cap \mathrm{Spt}\, \mu_{m(\eta),r}$ and

$$\mathrm{card}\,(\mathcal{C}) \le 1 + 2\left\lceil \frac{n_k \arcsin(Rr/\rho_{k-1})}{2\pi} \right\rceil.$$

However $Rr/\rho_{k-1} \le 1/20$ and so $\arcsin(Rr/\rho_{k-1}) \le 2Rr/\rho_{k-1}$ which gives

$$\mathrm{card}\,(\mathcal{C}) \le 1 + 2\left\lceil \frac{2Rr}{\pi\rho_k} \right\rceil.$$

Thus $\mathcal{C} \subset \mathcal{F}$ and so (3) holds.

It only remains to verify that for the L defined in (4) above both inequalities 3.5 and 3.6 hold. Since $0 \in B_0$ it is clear that

$$\|y_0\| \le \rho_k/r$$

and thus B_0 satisfies the inequalities. Suppose that $B = \mathrm{B}(y, \rho_k/r) \in \mathcal{F}$ and $B \neq B_0$ and choose $t \in \mathbf{N}$ such that $t-1$ discs of $\mathcal{F}$ lie between B and B_0. Notice that $t \le 1 + (2Rr)/(\pi\rho_k)$. From equation 3.2 we deduce that

$$\|y - y_0\| = 2(1 - n_k^{-1})\frac{\rho_{k-1}}{r}\sin\left(\frac{\pi t}{n_k}\right).$$

Hence we find that

$$\begin{aligned}
\|\mathrm{Q}^{\perp}(y-y_0)\| &= \|y-y_0\| \sin\left(\frac{\pi t}{n_k}\right) \\
&\leq 2(1-n_k^{-1})\frac{\rho_{k-1}}{r}\left(\frac{\pi t}{n_k}\right)^2 \\
&\leq 2\frac{1}{n_k}\frac{\rho_k}{r}\left(\pi+\frac{2Rr}{\pi\rho_k}\right)^2
\end{aligned}$$

as required.

Finally observe that

$$\begin{aligned}
\|\mathrm{Q}(y-y_0)\| &= \|y-y_0\| \cos(\pi t/n_k) \\
&= (1-n_k^{-1})\frac{\rho_{k-1}}{r}\sin\left(\frac{2\pi t}{n_k}\right)
\end{aligned}$$

as $n_k \geq 10$

$$\begin{aligned}
&\geq (1-n_k^{-1})\frac{\rho_{k-1}}{r}\frac{2\pi t}{n_k}\left[1-\frac{1}{6}\left(\frac{2\pi t}{n_k}\right)^2\right] \\
&\geq 2\pi t\frac{\rho_k}{r}\left[1-\frac{2}{n_k}\left(\pi+\frac{2Rr}{\rho_k}\right)\right]^2
\end{aligned}$$

and thus

$$\begin{aligned}
\left|\frac{2\pi t\rho_k}{r}-\|\mathrm{Q}(y-y_0)\|\right| &\leq 2\pi t\frac{\rho_k}{r}\left[1-\left[1-\frac{2}{n_k}\left(\pi+\frac{2Rr}{\rho_k}\right)\right]^2\right] \\
&\leq 8\pi t\frac{\rho_k}{rn_k}\left(\pi+\frac{2Rr}{\rho_k}\right)
\end{aligned}$$

but $t \leq 1+2Rr/(\pi\rho_k)$ and so

$$\leq 8\left(\frac{1}{n_k}\right)\left(\frac{\rho_k}{r}\right)\left(\pi+\frac{2Rr}{\rho_k}\right)^2$$

as required. ■

For a unit vector $\hat{e}$ which makes an angle $\theta \in [0,2\pi)$ with the x-axis let $\hat{e}^{\perp}$ be the unit vector perpendicular to $\hat{e}$ which makes an angle $\theta-\pi/2$ with the x-axis. Let

$$\nu_{\hat{e}} := \frac{1}{\pi} \sum_{i \in \mathbf{Z}} \mathcal{H}^1 \lfloor_{\partial \mathrm{B}\left(\hat{e}^{\perp} + 2\pi i \hat{e}, 1\right)}$$

and let

$$\mathcal{N} := \left\{ r^{-1} \omega_{\zeta, r} : \zeta \in \operatorname{Spt} \omega,\ r > 0 \text{ and } \omega = \nu_{\hat{e}} \text{ for some } \hat{e} \right\}.$$

Observe that $\mathcal{N}$ is not a closed set and set

$$\bar{\mathcal{N}} := \operatorname{clos} \mathcal{N} = \mathcal{N} \cup \left\{ \pi^{-1} \mathcal{H}^1 \lfloor_L \colon L \in \mathrm{G}(2,1) \right\}.$$

Our next Lemmas will enable us to describe the tangent measures of μ and will provide sufficient information to allow us determine the tangent measure distributions of μ.

Lemma 3.3.3 *For all $0 < \epsilon < 1$,$R \geq 1$ and all $C \geq 20R$ there is a $K \in \mathbf{N}$ such that for all $k \geq K$ and all $\eta \in \Sigma$ there is a $\nu \in \mathcal{N}$ such that if*

$$r \in [\rho_k / C, C \rho_k]$$

then

$$\mathrm{F}_R \left(r^{-1} \mu_{m(\eta), r},\ \nu \right) \leq \epsilon.$$

Proof: Fix R, ϵ and C as in the statement of the lemma. Choose K such that for all $k \geq K$

$$n_k \geq 264 C (\pi + 2RC)^2 (R + C) \epsilon^{-1}$$

and such that the conclusions of Lemma 3.3.2 hold for R and C as above. Fix $0 < r \leq C \rho_K$ and choose $k \geq K$ such that

$$\rho_k / C \leq r < C \rho_k.$$

Fix $\eta \in \Sigma$ and set $x := m(\eta)$. Let $\hat{e}$ be the unit vector which makes an angle $(2\eta_k n_k^{-1} - (1/2))\pi$ with the positive x-axis and let $\hat{f}$ be the unit vector which makes an angle $2\pi\eta_{k+1}/n_{k+1}$ with the positive x-axis. Define, for $t \in \mathbf{Z}$,

$$C_t := \partial \mathrm{B}\left(-\frac{\rho_k}{r}\hat{f} + 2\pi t\hat{e},\ \frac{\rho_k}{r}\right)$$

and set

$$\nu := \frac{r}{\pi\rho_k}\sum_{t\in\mathbf{Z}}\mathcal{H}^1\lfloor_{C_t}.$$

It is easy to see that $\nu \in \mathcal{N}$. We wish to estimate $\mathrm{F}_R(r^{-1}\mu_{x,r}, \nu)$. Let $\mathcal{F}$ be as given by Lemma 3.3.2 and for $B \in \mathcal{F}$ let

$$\mathcal{E}(B) := \{D_{x,r} : D \in \mathcal{F}_{k+1} \text{ and } D_{x,r} \subset B\}.$$

Observe that if $B \in \mathcal{F}$ then

$$\mathrm{diam}\,(B) = 2\frac{\rho_k}{r} = r^{-1}\mu_{x,r}(B)$$

and if $D \in \mathcal{E}(B)$ then $\mathrm{diam}\,(D) = 2\rho_{k+1}/r$. Now let

$$N := \left\{0, \dots, \left\lceil \frac{2Rr}{\pi\rho_k} \right\rceil\right\}$$

and define

$$\bar{\nu} := \frac{r}{\pi\rho_k}\sum_{|t|\in N}\mathcal{H}^1\lfloor_{C_t}.$$

Observe that

$$\mathrm{F}_R(\nu, \bar{\nu}) = 0$$

and so

$$\mathrm{F}_R(r^{-1}\mu_{x,r}, \nu) = \mathrm{F}_R(r^{-1}\mu_{x,r}, \bar{\nu}).$$

Let us index the $1 + 2\lceil (2Rr)/(\pi \rho_k) \rceil$ of the discs in $\mathcal{F}$ as follows: Let B_0 be the unique disc in $\mathcal{F}$ which contains the origin. For $|t| \in N \setminus \{0\}$ let B_t be the disc in $\mathcal{F}$ which is t discs away from B_0 (that is, there are $t-1$ discs between B_0 and B_t) and such that $\operatorname{dist}(B_t, C_t) < \operatorname{dist}(B_t, C_{-t})$. From (3) of Lemma 3.3.2 we conclude that

$$\mathrm{F}_R \left(\mu_{x,r} \lfloor_{\cup B_t}, \mu_{x,r} \right) = 0.$$

Thus if we let $\bar{\omega} := r^{-1} \mu_{x,r} \lfloor_{\cup B_t}$ then we have that

$$\mathrm{F}_R \left(r^{-1} \mu_{x,r}, \bar{\nu} \right) = \mathrm{F}_R \left(\bar{\omega}, \bar{\nu} \right).$$

Let y_t be the centre of B_t and let c_t be the centre of C_t (hence $c_t = -(\rho_k/r)\hat{f} + 2\pi t \hat{e}$.) Define a cover, $\mathcal{A}$, of $[\operatorname{Spt}(\bar{\omega} + \bar{\nu}) \cap \mathrm{B}(0, R)]$ as follows: For $|t| \in N$ and $m \in \{1, \ldots, n_{k+1}\}$ let

$$A(t,m) := \left\{ c_t + (\rho_k/r)(\cos\theta, \sin\theta) : \theta \in [2\pi(m - 1/2)n_{k+1}^{-1}, 2\pi(m + 1/2)n_{k+1}^{-1}) \right\}$$

and set

$$\mathcal{A} := \{ D(B_t, m, n_{k+1}) \cup A(t,m) : |t| \in N \text{ and } m \in \{1, \ldots, n_{k+1}\}.$$

Clearly

$$\cup \mathcal{A} \supset [\operatorname{Spt}(\bar{\omega} + \bar{\nu}) \cap \mathrm{B}(0, R)]$$

and $\mathcal{A}$ is a finite family. Also, if there are distinct $A, B \in \mathcal{A}$ with $A \cap B \neq \emptyset$ then this means that for some m, n, s and t

$$D(B_s, n, n_{k+1}) \cap A(t,m) \neq \emptyset$$

and from equation 3.4 we know that

$$\bar{\omega}[D(B_s, n, n_{k+1}) \cap A(t, m)] = 0.$$

Hence we can conclude from our density estimates (Lemma 3.3.1) that

$$\bar{\nu}[D(B_s, n, n_{k+1}) \cap A(t, m)] = 0.$$

Finally, observe that for all $A \in \mathcal{A}$

$$\bar{\omega}(A) = \bar{\nu}(A) \quad \left(= 2\frac{\rho_{k+1}}{r}\right).$$

Hence if d is such that for all $A \in \mathcal{A}$, $\operatorname{diam}(A) \leq d$ then we may use Lemma 1.3.5 to deduce that

$$\begin{aligned} \mathrm{F}_R\left(r_k^{-1}\mu_{x,r_k},\, \nu\right) \quad = \quad \mathrm{F}_R(\bar{\omega}, \bar{\nu}) &\leq d(\bar{\omega} + \bar{\nu})\mathrm{B}(0, R) \\ &\leq 4d\left(1 + 2\left\lceil \frac{2Rr}{\pi \rho_k} \right\rceil\right)\frac{\rho_k}{r} \\ &\leq 12d(R + \rho_k/r). \end{aligned}$$

So suppose that $A \in \mathcal{A}$ and so, for some suitable t and m,

$$A = D(B_t, m, n_{k+1}) \cup A(t, m).$$

Observe that

$$\operatorname{diam}(A(t, m)) \leq 2\pi\frac{\rho_{k+1}}{r}$$

and

$$\operatorname{diam}(D(B_t, m, n_{k+1})) = 2\frac{\rho_{k+1}}{r}.$$

Thus

$$\begin{aligned} \operatorname{diam}(A) &\leq \operatorname{diam}(A(t, m)) + \operatorname{diam}(D(B_t, m, n_{k+1})) + \|y_t - c_t\| \\ &\leq 3\pi\frac{\rho_{k+1}}{r} + \|y_t - c_t\|. \end{aligned}$$

Hence we need to estimate $\|y_t - c_t\|$ for $|t| \in N$. Recall from Lemma 3.3.2 that Q denotes orthogonal projection onto L and $\mathrm{Q}^\perp$ denotes orthogonal projection onto the line perpendicular to L.

Observe that

$$\|y_0 - c_0\| \leq 2\rho_{k+1}/r.$$

Now fix $t \in N \setminus \{0\}$ and consider $\|y_t - c_t\|$. We have that

$$\begin{aligned}
\|y_t - c_t\| &\leq \|y_t - c_t - y_0 + c_0 + y_0 - c_0\| \\
&\leq \|\mathrm{Q}(y_t - y_0) - \mathrm{Q}(c_t - c_0)\| + \|\mathrm{Q}^\perp(y_t - y_0) - \mathrm{Q}^\perp(c_t - c_0)\| \\
&\qquad + \|y_0 - c_0\| \\
&\leq \left| \frac{2\pi|t|\rho_k}{r} - \|\mathrm{Q}(y_t - y_0)\| \right| + \|\mathrm{Q}^\perp(y_t - y_0)\| + 2\frac{1}{n_{k+1}}\frac{\rho_k}{r}
\end{aligned}$$

but from (4) of Lemma 3.3.2 we may conclude that

$$\begin{aligned}
&\leq 10\frac{1}{n_k}\frac{\rho_k}{r}\left(\pi + \frac{2Rr}{\rho_k}\right)^2 + 2\frac{1}{n_{k+1}}\frac{\rho_k}{r} \\
&\leq 12\frac{1}{n_k}\frac{\rho_k}{r}\left(\pi + \frac{2Rr}{\rho_k}\right)^2.
\end{aligned}$$

Thus for all $A \in \mathcal{A}$

$$\mathrm{diam}\,(A) \leq 22\frac{1}{n_k}\frac{\rho_k}{r}\left(\pi + \frac{2Rr}{\rho_k}\right)^2.$$

Hence

$$\mathrm{F}_R(\bar{\omega}, \bar{\nu}) \leq 264\frac{1}{n_k}\frac{\rho_k}{r}\left(\pi + \frac{2Rr}{\rho_k}\right)^2\left(R + \frac{\rho_k}{r}\right)$$

but $\rho_k/C \leq r \leq C\rho_k$ and so

$$\begin{aligned}
&\leq \frac{264}{n_k}C(\pi + 2RC)^2(R + C) \\
&\leq \epsilon
\end{aligned}$$

and so, as

$$\mathrm{F}_R\left(r^{-1}\mu_{x,r},\,\nu\right) = \mathrm{F}_R(\bar{\omega},\,\bar{\nu}),$$

the Lemma follows. ∎

Lemma 3.3.4 *For μ-a.e. x* $\mathrm{Tan}_S(\mu) \supset \bar{\mathcal{N}}$.

Proof: Recall that for a unit vector $\hat{e}$ which makes an angle of θ with the positive x-axis

$$\nu_{\hat{e}} := \frac{1}{\pi}\sum_{i\in\mathbf{Z}} \mathcal{H}^1 \lfloor_{\partial\mathrm{B}\left(\hat{e}^\perp + 2\pi i\hat{e},1\right)}$$

where $\hat{e}^\perp$ makes an angle of $\theta - \pi/2$ with the positive x-axis.

In view of shift invariance (Lemma 1.4.3) and the fact that $\mathrm{Tan}_S(\mu, x)$ is a closed set it suffices to verify that for μ-a.e. x and all $\nu_{\hat{e}}$ we have

$$\nu_{\hat{e}} \in \mathrm{Tan}_S(\mu, x).$$

Moreover we can find a countable set, $\mathcal{S}$ say, of unit vectors, $\hat{e}$, of the form described above which is dense in the set of unit vectors directed into the upper half-plane. Hence we deduce that it suffices to show that for all $\hat{e} \in \mathcal{S}$ and for μ-a.e. x

$$\nu_{\hat{e}} \in \mathrm{Tan}_S(\mu, x).$$

Choose such a unit vector $\hat{e} \in \mathcal{S}$. Let $\nu := \nu_{\hat{e}}$ and observe that

$$\mathrm{Spt}\,\nu = \bigcup_{t\in\mathbf{Z}} \partial\mathrm{B}(\hat{e}^\perp + 2\pi t\hat{e}, 1).$$

Choose a sequence $1/4 \geq \gamma_j \searrow 0$ such that $\sum_j \gamma_{2j}\gamma_{2j+1} = \infty$. An application of the Borel-Cantelli Lemma shows that

$$\kappa\left(\left\{\eta \in \Sigma : \left|\left(\theta - \frac{\pi}{2}\right)\frac{n_{2j+i}}{2\pi} - \eta_{2j+i}\right| (\mathrm{mod}\ n_{2j+i}) \leq \gamma_{2j+i}\right.\right.$$

$$\text{for } i \in \{0,1\} \text{ infinitely often}\Big\}\Big) = 1.$$

So fix such an $\eta \in \Sigma$ and let $x = m(\eta)$. We can find a sequence of even integers $k(i) \nearrow \infty$ such that for $l = 0, 1$ and all $i \in \mathbf{N}$

$$\left|\left(\theta - \frac{\pi}{2}\right)\frac{n_{2k(i)+l}}{2\pi} - \eta_{2k(i)+l}\right| \left(\mathrm{mod}\ n_{2k(i)+l}\right) \leq \gamma_{2k(i)+l}.$$

In order to verify that $\nu \in \mathrm{Tan}_S(\mu, x)$ we need to find a sequence $r(i) \searrow 0$ such that for all $R \geq 1$ and all $0 < \epsilon < 1$ there is a $J \in \mathbf{N}$ such that for $i \geq J$

$$\mathrm{F}_R\left(r(i)^{-1}\mu_{x,r(i)},\, \nu\right) \leq \epsilon.$$

So fix such an R and ϵ. I claim that the sequence given by setting $r(i) := \rho_{k(i)}$ is such a sequence. For, by Lemma 3.3.3, we can find a $K \in \mathbf{N}$ such that for all $i \geq K$ there is a $\nu_i \in \mathcal{N}$ such that

$$\mathrm{F}_R(r(i)^{-1}\mu_{x,r(i)}, \nu_i) \leq \epsilon/2.$$

Moreover the radius of the circles which make up the support of ν_i is equal to $\rho_{k(i)}/r(i) = 1$. Hence we can find an (orientation preserving) isometry I_i of the plane which maps the support of ν_i onto the support of ν in such a way as to ensure that the circle in the support of ν_i which contains the origin is mapped onto $\partial\mathrm{B}(\hat{e}^{\perp}, 1)$. Moreover for all $A \subset \mathbf{R}^2$, $\nu(A) = \nu_i(I_i^{-1}(A))$ thus $\nu = (I_i)_{\#}\nu_i$. However ν_i is determined only by the values of $\eta_{k(i)}$ and $\eta_{k(i)+1}$ and, by definition, both $2\pi\eta_{k(i)}/n_{k(i)}$ and $2\pi\eta_{k(i)+1}/n_{k(i)+1}$ tend to $\left(\theta - \frac{\pi}{2}\right) (\mathrm{mod}\, 2\pi)$. Hence we conclude that $I_i \to I$ (the identity isometry) as $i \to \infty$ and thus we find that if $f\colon \mathbf{R}^n \to [0, \infty)$ is such that $\mathrm{Spt}\,(f) \subset \mathrm{B}(0, R)$ and $\mathrm{lip}\,(f) \leq 1$ then

$$\left|\int f\, d\nu_i - \int f\, d\nu\right| \;=\; \left|\int f \circ I_i\, d\nu - \int f\, d\nu\right|$$

$$\leq \quad R\|I_i - I\|\nu\mathrm{B}(0, R).$$

Hence

$$\mathrm{F}_R(\nu_i, \nu) \to 0 \text{ as } i \to \infty$$

and so we can find a $J \geq K$ such that for all $i \geq J$

$$\mathrm{F}_R(\nu_i, \nu) \leq \epsilon/2$$

and thus the Lemma holds. ■

We may immediately conclude from this Lemma that μ is purely 1-unrectifiable.

Lemma 3.3.5 *For all* $0 < \epsilon < 1$ *and* $R \geq 1$ *there is a* $C \geq 1$ *and a* $K \in \mathbf{N}$ *so that for all* $\eta \in \Sigma$ *and for all* $k \geq K$ *there is an* $L \in \mathrm{G}(2,1)$ *(which depends only on the value of* η_k*) such that if*

$$r \in [C\rho_k,\ \rho_{k-1}/C]$$

then

$$\mathrm{F}_R\left(r^{-1}\mu_{m(\eta),r},\ \pi^{-1}\mathcal{H}^1\lfloor_L\right) \leq \epsilon.$$

Proof: Fix ϵ and R as in the Lemma and choose $C \geq 20R$ such that

$$C \geq 6R\epsilon^{-1}.$$

Choose $K \in \mathbf{N}$ such that for all $k \geq K$ the conclusions of Lemma 3.3.2 hold for R and C and

$$n_k \geq 360\pi R^2\epsilon^{-1}.$$

Let $\eta \in \Sigma$ and set $x := m(\eta)$. Fix $k \geq K$ and suppose that $r \in [C\rho_k, \rho_{k-1}/C]$. Throughout this proof we shall use the notation of Lemma 3.3.2. In particular L denotes the line which is determined by Lemma 3.3.2 — notice that $L = L(\eta_k)$. As in the last Lemma observe that for all $B \in \mathcal{F}$

$$\operatorname{diam}(B) = 2\frac{\rho_k}{r} = r^{-1}\mu_{x,r}.$$

Let

$$N := \left\{0, \dots, \left\lceil \frac{2Rr}{\pi\rho_k} \right\rceil \right\}$$

and for $|t| \in N$ define

$$L(t) := \{z \in L : \|Qz - 2\pi t\rho_k\hat{e}/r\| \leq \pi\rho_k/r\}.$$

(where $\hat{e}$ is a unit vector along L.) Observe that $\{L(t)\}$ forms a cover of $L \cap \mathrm{B}(0, R)$ and for any $L(t)$, $\pi^{-1}\mathcal{H}^1\lfloor_L(L(t)) = 2\rho_k/r$. Now, for $|t| \in N \setminus \{0\}$, define $B_t = \mathrm{B}(y_t, \rho_k/r) \in \mathcal{F}$ to be the disc in $\mathcal{F}$ which is separated from B_0 by $t-1$ discs of $\mathcal{F}$ and chosen such that $\|y_t - 2\pi t\rho_k\hat{e}/r\| < \|y_t + 2\pi t\rho_k\hat{e}/r\|$. Recall from Lemma 3.3.2 that $\mathcal{F}$ covers $\mathrm{B}(0, R) \cap \operatorname{Spt}\mu_{x,r}$ and define a cover of $\mathrm{B}(0, R) \cap \operatorname{Spt}(\mu_{x,r} + \mathcal{H}^1\lfloor_L)$ by setting

$$\mathcal{C} := \{\operatorname{conv}(L(t) \cup B_t) : |t| \in N\}.$$

(recall that $\operatorname{conv}(A)$ denotes the closed convex hull of A.) Observe that for all $A \in \mathcal{C}$

$$r^{-1}\mu_{x,r}(A) = \pi^{-1}\mathcal{H}^1\lfloor_L(A)$$

and if $B \in \mathcal{C}$ is distinct from A then

$$\mu_{x,r}(A \cap B) = \mathcal{H}^1\lfloor_L(A \cap B) = 0.$$

Thus if d is chosen such that for all $A \in \mathcal{A}$, $\operatorname{diam}(A) \leq d$ then we may use Lemma 1.3.5 to deduce that

$$\begin{aligned}
\mathrm{F}_R\left(r^{-1}\mu_{x,r},\, \pi^{-1}\mathcal{H}^1\lfloor_L\right) &\leq d(r^{-1}\mu_{x,r} + \pi^{-1}\mathcal{H}^1\lfloor_L)\mathrm{B}(0,R) \\
&\leq d\left(\frac{2R}{\pi} + 4R + 6\frac{\rho_k}{r}\right) \\
&\leq d\left(\frac{2R}{\pi} + 4R + \frac{6}{C}\right) \\
&\leq 5Rd.
\end{aligned}$$

Thus let us find an upper bound for d. Fix $A \in \mathcal{C}$ and choose t such that $A = \operatorname{conv}(L(t) \cup B_t)$ and let y be the center of B_t and $c = 2\pi t\rho_k\hat{e}/r$. We find that

$$\operatorname{diam}(A) \leq 3\pi\frac{\rho_k}{r} + \|y - c\|.$$

However

$$\begin{aligned}
\|y - c\| &\leq \|y - c - y_0 + y_0\| \\
&\leq \|\mathrm{Q}(y - y_0) - \mathrm{Q}c\| + \|\mathrm{Q}^\perp(y - y_0) - \mathrm{Q}^\perp c\| + \|y_0\| \\
&= \left|\frac{2\pi|t|\rho_k}{r} - \|\mathrm{Q}(y - y_0)\|\right| + \|\mathrm{Q}^\perp(y - y_0)\| + 2\frac{1}{n_{k+1}}\frac{\rho_k}{r}
\end{aligned}$$

but from (4) of Lemma 3.3.2 we may conclude that

$$\begin{aligned}
&\leq 10\frac{1}{n_k}\frac{\rho_k}{r}\left(\pi + \frac{2Rr}{\rho_k}\right)^2 + 2\frac{1}{n_{k+1}}\frac{\rho_k}{r} \\
&\leq 12\frac{1}{n_k}\frac{\rho_k}{r}\left(\pi + \frac{2Rr}{\rho_k}\right)^2.
\end{aligned}$$

Thus

$$\operatorname{diam}(A) \leq 3\pi\frac{\rho_k}{r} + 12\frac{1}{n_k}\frac{\rho_k}{r}\left(\pi + \frac{2Rr}{\rho_k}\right)^2$$

and as $C\rho_k \leq r \leq \rho_{k-1}/C$ we conclude that

$$
\begin{aligned}
&\leq \frac{3\pi}{C} + 12\left(\frac{\pi}{C} + 2R\right)\left(\frac{\pi}{n_k} + \frac{2R}{C}\right) \\
&\leq 36R\left(\frac{\pi}{n_k} + \frac{3R}{C}\right).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathrm{F}_R\left(r^{-1}\mu_{x,r},\, \pi^{-1}\mathcal{H}^1\lfloor_L\right) &\leq 36R\left(\frac{\pi}{n_k} + \frac{3R}{C}\right) \times 5R \\
&\leq 180R^2\left(\frac{\pi}{n_k} + \frac{3R}{C}\right) \\
&\leq \epsilon
\end{aligned}
$$

as required. ∎

Corollary 3.3.6 *For all $\eta \in \Sigma$, $\mathrm{Tan}_S(\mu, m(\eta)) \subset \bar{\mathcal{N}}$.*

Proof: Fix $\eta \in \Sigma$. In view of Lemma 1.4.6 it suffices for us to show that for all $0 < \epsilon < 1$ and $R \geq 1$ there is an $s > 0$ such that if $0 < r < s$ then there is a $\nu \in \bar{\mathcal{N}}$ with

$$
\mathrm{F}_R(r^{-1}\mu_{m(\eta),r}, \nu) \leq \epsilon.
$$

But in order to ensure this, choose C and K such that the conclusion of Lemma 3.3.5 holds for $k \geq K$ and then choose $K' \geq K$ such that the conclusion of Lemma 3.3.3 holds for $k \geq K'$ and for the constant C. Then, if we set $s = \rho_{K'}/C$, we are done. ∎

Corollary 3.3.7 *For μ-a.e. x*

$$
\underline{\mathrm{D}}_1(\mu, x) = \frac{1}{\sqrt{4\pi^2 + 1} - 1} \text{ and } \overline{\mathrm{D}}_1(\mu, x) = \frac{1}{2}.
$$

Proof: This follows from calculating the bounds on the upper and lower densities of the tangent measures and applying Lemma 1.4.4. ∎

Lemma 3.3.8 *For all $\eta \in \Sigma$ if P is a tangent measure distribution of μ at $m(\eta)$ then*

$$\operatorname{Spt}(P) \subset \left\{\pi^{-1}\mathcal{H}^1\lfloor_V \colon V \in \mathrm{G}(2,1)\right\} =: \mathcal{L}, \textit{ say.}$$

Proof: Fix $1 > \epsilon > 0$ and $R \geq 1$ and choose $C \geq 1$ and $K \in \mathbf{N}$ such that for all $k \geq K$ if $r \in [C\rho_k, \rho_{k-1}/C]$ then for all $\eta \in \Sigma$ there is a $V \in \mathrm{G}(2,1)$ such that

$$\mathrm{F}_R\left(r^{-1}\mu_{m(\eta),r},\ \pi^{-1}\mathcal{H}^1\lfloor_V\right) \leq \epsilon.$$

(This is possible by Lemma 3.3.5.) Consider the closed set

$$\mathcal{L}(\epsilon, R) := \{\nu \in \mathcal{M} :\ \text{There is an } \omega \in \mathcal{L} \text{ with } \mathrm{F}_R(\nu,\omega) \leq \epsilon\}.$$

Fix $\eta \in \Sigma$ and set $x := m(\eta)$. We will show that

$$\lim_{r\searrow 0} [\Theta_r(\mu, x)]\,(\mathcal{L}(\epsilon, R)) = 1$$

as we may then deduce from Theorem 3.1.2 that for any $P \in \mathcal{P}(\mu, x)$

$$P\left(\mathcal{L}(\epsilon, R)\right) = 1.$$

Hence, as

$$\mathcal{L} = \bigcap_{l\in\mathbf{N}} \mathcal{L}(l^{-1}, l),$$

it follows that for any such P, $P(\mathcal{L}) = 1$ and so, as $\mathcal{L}$ is closed, $\operatorname{Spt}(P) \subset \mathcal{L}$ as required.

Observe that if, for some $j \geq K$, $t \in [C\rho_j, \rho_{j-1}/C]$ then $t^{-1}\mu_{x,t} \in \mathcal{L}(\epsilon, R)$. Thus suppose that $0 < r < \rho_{K-1}/C$ and choose k such that $r \in [\rho_k/C, \rho_{k-1}/C)$.

If $r \geq C\rho_k$ then

$$\begin{aligned}
[\Theta_r(\mu,x)]\,(\mathcal{L}(\epsilon,R)) &= \frac{1}{-\log r}\int_r^1 I_{\mathcal{L}(\epsilon,R)}\left(\frac{\mu_{x,t}}{t}\right)\frac{dt}{t} \\
&\geq \frac{1}{-\log r}\left[\log\frac{\rho_{k-1}}{C} - \log r + \sum_{j=K}^{k-1}\left(\log\frac{\rho_{j-1}}{C} - \log C\rho_j\right)\right] \\
&\geq 1 - \frac{2(k+1-K)\log C}{-\log r} \\
&\geq 1 - \frac{2(k+1-K)\log C}{\log C + \sum_{j=1}^{k-1}\log n_j}.
\end{aligned}$$

If $\rho_k/C \leq r < C\rho_k$ then a similar calculation gives

$$\begin{aligned}
[\Theta_r(\mu,x)]\,(\mathcal{L}(\epsilon,R)) &= \frac{1}{-\log r}\int_r^1 I_{\mathcal{L}(\epsilon,R)}\left(\frac{\mu_{x,t}}{t}\right)\frac{dt}{t} \\
&\geq \frac{1}{-\log r}\sum_{j=K}^{k}\left[\log\frac{\rho_{j-1}}{C} - \log C\rho_j\right] \\
&\geq \frac{-\rho_k}{-\log r} - \frac{2(k+1-K)\log C}{-\log r} \\
&\geq \frac{1}{1-(\log C/\log\rho_k)} - \frac{2(k+1-K)\log C}{-\log C + \sum_{j=1}^{k-1}\log n_j}.
\end{aligned}$$

Hence, as

$$\lim_{k\to\infty}\frac{k}{\sum_{j=1}^{k}\log n_j} = 0$$

we deduce that on sending r to zero (and hence k to infinity) that

$$[\Theta_r(\mu,x)]\,(\mathcal{L}(\epsilon,R)) \to 1$$

as required. ■

Corollary 3.3.9 *For μ-a.e. x*

$$\mathrm{D}_1^2(\mu,x) = \frac{1}{\pi}.$$

Proof: From Lemma 3.1.3 we know that for μ-a.e. x, $\mathcal{P}(\mu, x)$ is a non-empty set. By the preceding Theorem we can easily see that, for μ-a.e. x, if $P \in \mathcal{P}(\mu, x)$ and $\nu \in \mathrm{Spt}\,(P)$ then $\nu\mathrm{B}(0,1) = 2/\pi$. Thus, from Lemma 3.1.4, we immediately conclude that, for μ-a.e. x, $\underline{\mathrm{D}}_1^2(\mu, x) = \overline{\mathrm{D}}_1^2(\mu, x) = 1/\pi$ and so the claim holds. ∎

We shall now calculate the tangent measure distributions of μ — these will depend on how quickly the sequence n_k diverges. We shall first investigate what happens when n_k diverges quickly to infinity.

Lemma 3.3.10 *Suppose that*

$$\frac{\log n_k}{\sum_{i=1}^{k-1} \log n_i} \to \infty$$

and that $\eta \in \Sigma$, $1 > r(i) \searrow 0$ *and* $k(i) \nearrow \infty$ *are such that*

1. *for all* i, $\rho_{k(i)} \leq r(i) < \rho_{k(i)-1}$,
2. $2\pi\eta_{k(i)}/n_{k(i)} \to \alpha \in [0, 2\pi]$,
3. $2\pi\eta_{k(i)-1}/n_{k(i)-1} \to \beta \in [0, 2\pi]$,
4. $(\log \rho_{k(i)-1})/\log r(i) \to \gamma \in [0, 1]$.

For $\Xi \in [0, 2\pi]$ *let* $\lambda_\Xi := \pi^{-1}\mathcal{H}^1\lfloor_V$ *where* $V \in \mathrm{G}(2,1)$ *is the line which makes an angle* $\Xi - \pi/2$ *with the positive* x*-axis. Then*

$$\Theta_{r(i)}(\mu, m(\eta)) \to (1-\gamma)\Delta_{\lambda_\alpha} + \gamma\Delta_{\lambda_\beta}.$$

Proof: First notice that if $(\log n_k)/(\sum_{i=1}^{k-1} \log n_i) \to \infty$ then

$$\frac{\log \rho_k}{\log \rho_{k-1}} \to \infty.$$

Let $x := m(\eta)$ and, for convenience, let $E(y) := \exp(y)$. For $\sigma \in \Sigma$ let $L(\sigma_k)$ denote the line which makes an angle of $\pi(2\sigma_k/n_k - 1/2)$ with the positive x-axis. In view of Theorem 3.1.1 it suffices to show that for all $\theta : \mathbf{R}^n \to [0,\infty)$ for which $\operatorname{lip}(\theta) \leq 1$ and $\operatorname{Spt}(\theta)$ is compact we have

$$\frac{1}{-\log r(i)} \int_{r(i)}^{1} E\left(\frac{\mu_{x,t}(\theta)}{t}\right) \frac{dt}{t} \to (1-\gamma)E(-\lambda_\alpha(\theta)) + \gamma E(-\lambda_\beta(\theta)).$$

So fix such a θ and choose $R \geq 1$ such that $\operatorname{Spt}\theta \subset \mathrm{B}(0,R)$. Fix $0 < \epsilon < 1$. Choose $\delta > 0$ such that if $|y - \lambda_\alpha(\theta)| \leq \delta$ then $|E(-y) - E(-\lambda_\alpha(\theta))| \leq \epsilon/18$ and if $|y - \lambda_\beta(\theta)| \leq \delta$ then $|E(-y) - E(-\lambda_\beta(\theta))| \leq \epsilon/18$.

By Lemma 3.3.5 we can find C and $K' \in \mathbf{N}$ such that for all $k \geq K' - 1$ if $\sigma \in \Sigma$ then there is an $L = L(\sigma_k) \in \mathrm{G}(2,1)$ such that for all $t \in [C\rho_k,\, \rho_{k-1}/C]$

$$\mathrm{F}_R\left(t^{-1}\mu_{m(\sigma),t},\, \pi^{-1}\mathcal{H}^1 \lfloor_L\right) \leq \delta/2.$$

Hypotheses (2) and (3) also enable us to find a $K'' \geq K'$ such that for all $i \geq K''$,

$$\mathrm{F}_R\left(\pi^{-1}\mathcal{H}^1 \lfloor_{L(\eta_{k(i)})},\, \lambda_\alpha\right) \leq \delta/2$$

and

$$\mathrm{F}_R\left(\pi^{-1}\mathcal{H}^1 \lfloor_{L(\eta_{k(i)-1})},\, \lambda_\beta\right) \leq \delta/2.$$

Hence, if $i \geq K''$ then for $t \in [C\rho_{k(i)}, \rho_{k(i)-1}/C]$ we have

$$\mathrm{F}_R\left(t^{-1}\mu_{x,t},\, \lambda_\alpha\right) \leq \delta$$

and for $t \in [C\rho_{k(i)-1}, \rho_{k(i)-2}/C]$ we have

$$\mathrm{F}_R\left(t^{-1}\mu_{x,t},\, \lambda_\beta\right) \leq \delta.$$

Thus, in these instances, either

$$|E(-t^{-1}\mu_{x,t}(\theta) - E(-\lambda_\alpha(\theta))| \leq \epsilon/18$$

or

$$|E(-t^{-1}\mu_{x,t}(\theta) - E(-\lambda_\beta(\theta))| \leq \epsilon/18,$$

respectively.

We shall consider three separate cases:

Case 1: $0 < \gamma < 1$.

Let $e := \epsilon[1 + E(-\lambda_\alpha) + E(-\lambda_\beta)]^{-1}/6$ and choose a $K''' \geq K''$ such that for all $i \geq K'''$

$$\gamma[1-e] \leq \frac{\log \rho_{k(i)-1}}{\log r(i)} \leq \frac{\log(\rho_{k(i)-1}/C)}{\log r(i)} \leq \gamma(1+e)$$

and

$$C\rho_{k(i)} \leq r_i < \rho_{k(i)-1}/C.$$

(We are using that $\log \rho_{i-1}/\log \rho_i \to 0$.) Finally choose $K \geq K'''$ such that for all $i \geq K$

$$\max\left\{-\frac{2\log C}{\log(\rho_{k(i)-1}/C)}, \frac{\log(\rho_{k(i)-2}/C)}{\log(\rho_{k(i)-1}/C)}\right\} \leq \epsilon/6.$$

We can now estimate that

$$\begin{aligned}
&\left|\frac{1}{|\log r(i)|}\int_{r(i)}^1 E\left(\frac{\mu_{x,t}(\theta)}{t}\right)\frac{dt}{t} - (1-\gamma)E(-\lambda_\alpha) - \gamma E(-\lambda_\beta)\right| \\
&\leq \left|\frac{1}{|\log r(i)|}\left(\int_{r_i}^{\frac{\rho_{k(i)-1}}{C}} + \int_{C\rho_{k(i)-1}}^{\frac{\rho_{k(i)-2}}{C}}\right) E\left(-\frac{\mu_{x,t}}{t}(\theta)\right)\frac{dt}{t} - (1-\gamma)E(-\lambda_\alpha) - \gamma E(-\lambda_\beta)\right| \\
&\qquad + \frac{2\log C}{-\log r(i)} + \frac{-\log(\rho_{k(i)-2}/C)}{-\log r(i)}
\end{aligned}$$

$$\leq \left| \frac{1}{-\log r(i)} \int_{r_i}^{\frac{\rho_{k(i)-1}}{C}} E\left(-\frac{\mu_{x,t}}{t}(\theta)\right) \frac{dt}{t} - (1-\gamma)E(-\lambda_\alpha) \right|$$
$$+ \left| \frac{1}{-\log r(i)} \int_{C\rho_{k(i)-1}}^{\frac{\rho_{k(i)-2}}{C}} E\left(-\frac{\mu_{x,t}}{t}(\theta)\right) \frac{dt}{t} - \gamma E(-\lambda_\beta) \right| + \epsilon/3.$$

Now for $r \in [r(i), \rho_{k(i)-1}/C]$ we know that

$$-(\epsilon/18) + E(-\lambda_\alpha(\theta)) \leq E\left(-\frac{\mu_{x,t}}{t}(\theta)\right) \leq E(-\lambda_\alpha(\theta)) + (\epsilon/18)$$

and so

$$\left[\frac{\log(\rho_{k(i)-1}/C)}{-\log r(i)} + 1\right][-(\epsilon/18) + E(-\lambda_\alpha(\theta))]$$
$$\leq \frac{1}{-\log r(i)} \int_{r_i}^{\frac{\rho_{k(i)-1}}{C}} E\left(-\frac{\mu_{x,t}}{t}(\theta)\right) \frac{dt}{t}$$
$$\leq \left[\frac{\log(\rho_{k(i)-1}/C)}{-\log r(i)} + 1\right][E(-\lambda_\alpha(\theta)) + (\epsilon/18)].$$

Hence

$$\left| \frac{1}{|\log r(i)|} \int_{r_i}^{\frac{\rho_{k(i)-1}}{C}} E\left(-\frac{\mu_{x,t}}{t}(\theta)\right) \frac{dt}{t} - (1-\gamma)E(-\lambda_\alpha) \right|$$
$$\leq \gamma e E(-\lambda_\alpha(\theta)) + (\epsilon/18)(1 + \gamma(1+e))$$
$$\leq \epsilon/3.$$

Similarly we calculate that

$$\left| \frac{1}{-\log r(i)} \int_{C\rho_{k(i)-1}}^{\frac{\rho_{k(i)-2}}{C}} E\left(-\frac{\mu_{x,t}}{t}(\theta)\right) \frac{dt}{t} - \gamma E(-\lambda_\beta) \right| \leq \epsilon/3.$$

Thus adding together we deduce that

$$\left| \frac{1}{-\log r(i)} \int_{r(i)}^{1} E\left(\frac{\mu_{x,t}(\theta)}{t}\right) \frac{dt}{t} - (1-\gamma)E(-\lambda_\alpha) - \gamma E(-\lambda_\beta) \right| \leq \epsilon$$

as required.

Case 2: $\gamma = 1$.

Let $a := 5\epsilon[1 + E(-\lambda_\beta(\theta))]^{-1}/6$ and choose $K \geq K''$ such that for all $i \geq K$

$$1 - a \leq \frac{\log C\rho_{k(i)-1}}{\log r(i)} \leq 1 + a$$

(and so $\frac{\log \rho_{k(i)-1}}{\log r(i)} \geq 1 - a$),

$$r(i) \geq C^{-1} \log \rho_{k(i)-1}$$

and

$$\frac{-\log(\rho_{k(i)-2}/C)}{-\log \rho_{k(i)-1}} \leq \epsilon/9.$$

We can calculate that

$$\begin{aligned}
&\left| \frac{1}{-\log r(i)} \int_{r(i)}^{1} E\left(\frac{\mu_{x,t}(\theta)}{t}\right) \frac{dt}{t} - E(-\lambda_\beta) \right| \\
&\leq \left| \frac{1}{-\log r(i)} \int_{C\rho_{k(i)-1}}^{\rho_{k(i)-2}/C} E\left(\frac{\mu_{x,t}(\theta)}{t}\right) \frac{dt}{t} - E(-\lambda_\beta) \right| \\
&\qquad + \left| \frac{\log C\rho_{k(i)-1} - \log r(i)}{-\log r(i)} \right| + \frac{\log(\rho_{k(i)-2}/C)}{\log r(i)} \\
&\leq \left| \frac{1}{-\log r(i)} \int_{C\rho_{k(i)-1}}^{\rho_{k(i)-2}/C} E\left(\frac{\mu_{x,t}(\theta)}{t}\right) \frac{dt}{t} - E(-\lambda_\beta) \right| + a + \epsilon/9.
\end{aligned}$$

Proceeding as in Case 1 we find that

$$\begin{aligned}
&\left| \frac{1}{-\log r(i)} \int_{C\rho_{k(i)-1}}^{\rho_{k(i)-2}/C} E\left(\frac{\mu_{x,t}(\theta)}{t}\right) \frac{dt}{t} - E(-\lambda_\beta) \right| \\
&\qquad \leq \epsilon/18 + aE(-\lambda_\beta).
\end{aligned}$$

Thus recombining we deduce that

$$\left| \frac{1}{-\log r(i)} \int_{r(i)}^{1} E\left(\frac{\mu_{x,t}(\theta)}{t}\right) \frac{dt}{t} - E(-\lambda_\beta) \right| \leq \epsilon$$

as required.

Case 3: $\gamma = 0$.

We may assume that either

(i) there is a $K''' \geq K''$ such that for all $i \geq K'''$ $r(i) \leq C\rho_{k(i)}$ which implies that $\log \rho_{k(i)} / \log r(i) \to 1$.

Or

(ii) there is a $K''' \geq K''$ such that for all $i \geq K'''$ $r(i) > C\rho_{k(i)}$.

In either case the procedure is the same as before.

For *(i)*: Let $a := 5\epsilon[1 + E(-\lambda_\alpha(\theta))]^{-1}/6$ and choose $K \geq K'''$ such that for all $i \geq K$

$$\frac{\log C\rho_{k(i)}}{\log r(i)} \geq 1 - a$$

and

$$\frac{\log(\rho_{k(i)-1}/C)}{\log r(i)} \leq \epsilon/9.$$

Then we may calculate that

$$\begin{aligned}
&\left| \frac{1}{-\log r(i)} \int_{r(i)}^{1} E\left(\frac{\mu_{x,t}(\theta)}{t}\right) \frac{dt}{t} - E(-\lambda_\alpha) \right| \\
\leq & \left| \frac{1}{-\log r(i)} \int_{C\rho_{k(i)}}^{\rho_{k(i)-1}/C} E\left(\frac{\mu_{x,t}(\theta)}{t}\right) \frac{dt}{t} - E(-\lambda_\alpha) \right| \\
& \qquad + \left| \frac{\log C\rho_{k(i)} - \log r(i)}{-\log r(i)} \right| + \frac{\log(\rho_{k(i)-1}/C)}{\log r(i)} \\
\leq & \left| \frac{1}{-\log r(i)} \int_{C\rho_{k(i)}}^{\rho_{k(i)-1}/C} E\left(\frac{\mu_{x,t}(\theta)}{t}\right) \frac{dt}{t} - E(-\lambda_\alpha) \right| + a + \epsilon/9.
\end{aligned}$$

Hence, as in Case (2) we find that

$$\begin{aligned}
&\left| \frac{1}{-\log r(i)} \int_{C\rho_{k(i)}}^{\rho_{k(i)-1}/C} E\left(\frac{\mu_{x,t}(\theta)}{t}\right) \frac{dt}{t} - E(-\lambda_\alpha) \right| \\
&\qquad \leq \epsilon/18 + aE(-\lambda_\beta).
\end{aligned}$$

Thus recombining we deduce that

$$\left| \frac{1}{-\log r(i)} \int_{r(i)}^{1} E\left(\frac{\mu_{x,t}(\theta)}{t} \right) \frac{dt}{t} - E(-\lambda_\alpha) \right| \leq \epsilon$$

as required.

It is straightforward (but tedious) to verify that *(ii)* follows in a similar manner. ∎

Corollary 3.3.11 *Suppose that* $\frac{\log n_k}{\sum_{i=1}^{k-1} \log n_i} \to \infty$. *Then for all* $x \in E$ *if* $P \in \mathcal{P}(\mu, x)$ *then there is a* $\gamma \in [0,1]$ *and* $V, W \in \mathrm{G}(2,1)$ *such that*

$$P := (1-\gamma)\Delta\lfloor_{\pi^{-1}\mathcal{H}^1\lfloor_V} + \gamma\Delta\lfloor_{\pi^{-1}\mathcal{H}^1\lfloor_W}.$$

In fact it is possible to show slightly more: For μ-a.e. x, $\mathcal{P}(\mu, x)$ consists of all possible such distributions. This follows from the Borel-Cantelli Lemma and some careful estimates.

If n_k tends to infinity slowly then the tangent measure distributions are unique: In order to see this we need the following results.

Lemma 3.3.12 *Let* (Ω, P) *be a probability space and suppose that* $X_i : \Omega \to$ $\mathbf{R}$ *is a sequence of independent random variables with zero mean for which there is a* $C \geq 0$ *such that for all* i, $|X_i| \leq C$. *Then*

$$P\left(\left\{ \omega : \lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} X_i(\omega) = 0 \right\} \right) = 1.$$

Proof: Define

$$S_n := \frac{1}{n} \sum_{i=1}^{n} X_i.$$

We wish to prove that

$$P\left(\lim_{n\to\infty} S_n = 0 \right) = 1.$$

In order to show this it suffices to verify that

$$\sum_{n=1}^{\infty} \int (S_n)^4 \, dP < \infty$$

as then, by the Monotone Convergence Theorem, we may deduce that

$$\int \left[\sum_{n=1}^{\infty} (S_n)^4 \right] dP < \infty$$

which in turn implies that

$$P\left(\sum_{n=1}^{\infty} S_n^4 < \infty \right) = 1$$

and thus $P\left(\lim_{n\to\infty} S_n^4 = 0\right) = 1$ which is equivalent to $P\left(\lim_{n\to\infty} S_n = 0\right) = 1$ as required.

Thus consider, for $n \in \mathbf{N}$ which is larger than 4,

$$\int S_n^4 \, dP = n^{-4} \int \left(\sum_{i=1}^{n} X_i \right)^4 dP.$$

Since the X_i are independent it follows that the only terms of the right hand expression which contribute to the integral are of the form

$$X_1^{\alpha_1} \dots X_n^{\alpha_n}$$

where $\alpha_i \neq 1$ for $i \in \{1, \dots, n\}$. Hence

$$\begin{aligned} \int S_n^4 \, dP &\leq n^{-4} \left[\sum_{i=1}^{n} \int X_i^4 \, dP + \frac{4!}{2!2!} \sum_{i<j} \int X_i^2 X_j^2 \, dP \right] \\ &\leq n^{-4} C^4 (n + 3n(n-1)) \\ &\leq 3n^{-2} C^4. \end{aligned}$$

Finally summing over n gives that

$$\sum_{i=1}^{\infty} \int S_n^4 \, dP < \infty$$

and so, in view of our earlier comments, the Lemma follows. ∎

Lemma 3.3.13 *Let (Ω, P) be a probability space. Suppose that $a_i \nearrow \infty$ is a sequence of positive real numbers with*

$$\limsup_{k\to\infty} \left(\frac{k a_k}{\sum_{i=1}^k a_i} \right) < \infty$$

and $X_i : \Omega \to \mathbf{R}$ is a sequence of independent, uniformly bounded random variables such that $\int X_i \, dP \to e \in \mathbf{R}$ as $i \to \infty$. Then

$$P\left(\lim_{k\to\infty} \frac{\sum_{i=1}^k a_i X_i}{\sum_{i=1}^k a_i} = e \right) = 1.$$

Proof: We shall only prove the Lemma in the case that $e > 0$. The other cases are similar. Let $E_i := \int X_i \, dP$ and set $S_k := \sum_{i=1}^k (X_i - E_i)$. Fix $0 < \epsilon < e$ and observe that from the last Lemma we can find for P-almost every ω a $k' \in \mathbf{N}$ such that for all $k > k'$

$$|k^{-1} S_k| \leq \epsilon/3.$$

We can also find a $k'' > k'$ such that for all $k > k''$, $|E_i - e| \leq \epsilon/3$. Finally we can find a $K \geq k''$ such that

$$|K^{-1} S_{k''}| \leq \epsilon/3.$$

Combining this we deduce that for P-almost every ω there is a $K \in \mathbf{N}$ such that for all $k \geq K$

$$|S_k - ke| \leq k\epsilon.$$

Now

$$\begin{aligned} \sum_{i=1}^k a_i X_i &= a_1 S_1 + \sum_{i=2}^k (S_i - S_{i-1}) a_i \\ &= a_k S_k + \sum_{i=1}^{k-1} (a_i - a_{i+1}) S_i. \end{aligned}$$

Thus, P-almost surely, there is a constant $C(K)$ such that

$$\begin{aligned}
&\sum_{i=1}^{k} a_i X_i \\
&\leq \; C(K) + (e+\epsilon)ka_k + (e-\epsilon)\sum_{i=K+1}^{k-1} i(a_i - a_{i+1}) \\
&= \; C(K) + (e+\epsilon)ka_k + (e-\epsilon)\left[(K+1)a_{K+1} + (k-1)a_k + \sum_{i=K+2}^{k-1} a_i\right]
\end{aligned}$$

which for some constant $C'(K)$

$$= \; C'(K) + 2\epsilon k a_k + (e-\epsilon)\sum_{i=K+2}^{k} a_i.$$

In a similar manner we find that, P-almost surely, there is some constant $C''(K)$ such that

$$\sum_{i=1}^{k} a_i X_i \geq C''(K) - 2\epsilon k a_k + (e+\epsilon)\sum_{i=K+2}^{k} a_i.$$

Hence, on rearranging and taking limits we find that

$$\liminf_{k\to\infty} \frac{\sum_{i=1}^{k} a_i X_i}{\sum_{i=1}^{k} a_i} - e \geq -\epsilon\left(2\limsup \frac{ka_k}{\sum_{i=1}^{k} a_i} - 1\right)$$

and

$$\limsup_{k\to\infty} \frac{\sum_{i=1}^{k} a_i X_i}{\sum_{i=1}^{k} a_i} - e \leq +\epsilon\left(2\limsup \frac{ka_k}{\sum_{i=1}^{k} a_i} - 1\right)$$

which, as ϵ was arbitrary, implies the result. ■

Finally we are in a position to calculate the tangent measure distributions for slowly increasing sequences n_k.

Theorem 3.3.14 *Suppose that*

$$\limsup_{k\to\infty}\left(\frac{k\log n_k}{\sum_{i=1}^{k}\log n_i}\right) < \infty$$

then for μ-a.e. x the tangent measure distributions of μ at x are unique and equal the probability measure on $\mathcal{M}$ which is uniformly distributed on $\mathcal{L} := \left\{\pi^{-1}\mathcal{H}^1\lfloor_L\colon L \in \mathrm{G}(2,1)\right\}$.

Proof: Fix $N \in \mathbf{N}$ and define for $l \in \{1, \dots, N\}$, $A_l := [(l-1)/N,\, l/N]$. Define a map $u : [0,1] \to \mathcal{L}$ by $u(s) := \pi^{-1}\mathcal{H}^1\lfloor_L$ where L is chosen such that it makes an angle of πs with the positive x-axis. Let λ be Lebesgue measure restricted to the unit interval and set

$$P := u_{\#}\lambda.$$

That is, for $\mathcal{A} \subset \mathcal{M}$, define

$$P(\mathcal{A}) := \lambda\left(\{s : u(s) \in \mathcal{A}\}\right).$$

We wish to show that for μ-a.e. x

$$\mathcal{P}(\mu, x) = \{P\}.$$

Equivalently we need to show that for μ-a.e. x

$$\Theta_r(\mu, x) \to P \text{ as } r \to 0.$$

Let, for $l \in \{1, \dots, N\}$, $\mathcal{F}_l := u(A_l)$ then $\mathcal{F}_l$ is closed and a subset of $\mathcal{L}$. Moreover

$$\mathcal{L} = \bigcup_l \mathcal{F}_l$$

and recall from Lemma 3.3.8 that for any tangent measure distribution, Q, of μ, $\mathrm{Spt}\,(Q) \subset \mathcal{L}$. Thus in order to show that P is the only tangent measure distribution for μ-a.e. x it suffices to verify that for all $N \in \mathbf{N}$ and $l \in \{1, \ldots, N\}$

$$\liminf_{r \to 0} [\Theta_r(\mu, x)] (\mathcal{F}_l) \geq 1/N$$

as we may then deduce (from Theorem 3.1.2) that for any $Q \in \mathcal{P}(\mu, x)$

$$Q(\mathcal{F}_l) \geq 1/N$$

which, since N and l were arbitrary, would imply that Q was uniformly distributed on $\mathcal{L}$ and hence equal to P.

Hence, in order to verify this, fix $\eta \in \Sigma$ and let $x := m(\eta)$. Fix $0 < \epsilon < 1$ and $R \geq 1$ and choose $C \geq 20R$ and $K \in \mathbf{N}$ such that for all $k \geq K$, if $t \in [C\rho_k, \rho_{k-1}/C]$ then there is a $V \in \mathrm{G}(2, 1)$ such that $\mathrm{F}_R(t^{-1}\mu_{x,t}, \pi^{-1}\mathcal{H}^1 \lfloor_V) \leq \epsilon$ (this is possible by Lemma 3.3.5). Also observe that V depends only on η_k. Thus we can define a sequence of independent, uniformly bounded random variables $X_i : \Sigma \to \mathbf{R}$ by

$$X_k(\eta) := \begin{cases} 1 & \text{if } V \in \mathcal{F}_l, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $\int X_i \, d\kappa \to 1/N$ as $i \to \infty$.

Define

$$\mathcal{F}_{l,R,\epsilon} := \left\{ \nu \in \mathcal{M}(\mathbf{R}^2) : \text{There is an } \omega \in \mathcal{F}_l \text{ with } \mathrm{F}_R(\nu, \omega) \leq \epsilon \right\}$$

and let

$$F := \left\{ s \in (0, 1] : s^{-1}\mu_{x,s} \in \mathcal{F}_{l,R,\epsilon} \right\}.$$

Fix $0 < r < C\rho_{K+1}$ and choose k such that $C\rho_k \leq r \leq C\rho_{k-1}$. Now let us estimate $[\Theta_r(\mu, x)] (\mathcal{F}_{l,R,\epsilon})$ for μ-a.e. x:

$$[\Theta_r(\mu, x)] (\mathcal{F}_{l,R,\epsilon}) \quad = \quad \frac{1}{-\log r} \int_r^1 I_F(t) \, \frac{dt}{t}$$

$$\begin{aligned}
&\geq \frac{1}{-\log r} \sum_{i: r \leq C\rho_i, i \geq K} X_i(\eta) \left[\log \frac{\rho_{i-1}}{C} - \log C\rho_i\right] \\
&= \frac{1}{-\log r} \sum_{i=K}^{k} X_i(\eta)[\log n_i - 2\log C] \\
&\geq \frac{\sum_{i=K}^{k} X_i(\eta)[\log n_i - 2\log C]}{-\log C + \sum_{i=1}^{k+1} \log n_i}.
\end{aligned}$$

Since, as noted in the proof of Lemma 3.3.8,

$$\lim_{k\to\infty} \frac{k}{\sum_{i=1}^{k} \log n_i} = 0$$

we deduce that

$$\begin{aligned}
&\liminf_{k\to\infty} \frac{\sum i = K^k X_i(\eta)[\log n_i - 2\log C]}{-\log C + \sum_{i=1}^{k} \log n_i} \\
&= \liminf_{k\to\infty} \frac{\sum_{i=1}^{k} X_i(\eta) \log n_i}{\sum_{i=1}^{k+1} \log n_i} \\
&\geq \liminf_{k\to\infty} \frac{\sum_{i=1}^{k} X_i(\eta) \log n_i}{\sum_{i=1}^{k} \log n_i} \left(1 - \limsup_{k\to\infty} \frac{\log n_{k+1}}{\sum_{i=1}^{k+1} \log n_i}\right)
\end{aligned}$$

which, by our hypothesis on the sequence n_k,

$$= \liminf_{k\to\infty} \frac{\sum_{i=1}^{k} X_i(\eta) \log n_i}{\sum_{i=1}^{k} \log n_i}.$$

However we may apply Lemma 3.3.13 (with $a_i = \log n_i$) to deduce that for κ-a.e. η,

$$\liminf_{k\to\infty} \frac{\sum_{i=1}^{k} X_i(\eta) \log n_i}{\sum_{i=1}^{k} \log n_i} = 1/N$$

and so we deduce that for μ-a.e. x

$$\liminf_{r\searrow 0} [\Theta_r(\mu, x)] (\mathcal{F}_{l,R,\epsilon}) \geq 1/N.$$

Hence as

$$\mathcal{F}_l = \bigcap_{i\in\mathbf{N}} \mathcal{F}_{l,i,1/i}$$

we deduce that

$$\liminf_{r \searrow 0} \left[\Theta_r(\mu, x)\right](\mathcal{F}_l) \geq 1/N$$

and so, from our earlier observations, the result follows. ∎

Thus we have calculated the tangent measure distributions of μ in two cases; when $n_k \nearrow \infty$ quickly and when $n_k \nearrow \infty$ slowly. We have not investigated the case when

$$\liminf_{k \to \infty} \left(\frac{k \log n_k}{\sum_{i=1}^{k} \log n_i} \right) < \infty$$

and yet

$$\limsup_{k \to \infty} \left(\frac{k \log n_k}{\sum_{i=1}^{k} \log n_i} \right) = \infty.$$

It seems likely that there would be a mixture of different types of tangent measure distribution — some with (finite) discrete supports and others supported by the whole of $\mathcal{L}$. This seems interesting but I feel that the actual calculations would be very similar to those presented here and little is to be gained by making them.

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