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# Realizing Caustics in Acoustic Fields

S Moeini, G Memoli, S Subramanian

School of Engineering and Informatics, University of Sussex, Brighton BN1 9QT, UK

E-mail: s.moeini@sussex.ac.uk

**Abstract.** We present a method to realise a wide class of caustics in acoustic fields created by a set of phase holograms realized with metasurfaces. Given the desired caustic shape, we discuss how to identify the phase distribution along the metasurfaces and describe the calculations needed for determining the pressure field in the vicinity of the caustic. The results of this work can be used in realising acoustic traps controllable with acoustic metamaterials.

## 1. Introduction

The problem of creating a desired acoustic field from the waves going through a single metasurface has been addressed by many [1], with solutions ranging from ray-tracing [2] to back-propagation [3]. However, common inverse methods fail when the desired shape is closed — e.g. an ellipsoid — because they result in infinite wave amplitudes at its boundary.

In this paper, we approach this issue using the theory of differentiable maps, inspired by the seminal works of Arnold [4]. We consider the desired closed shape as a “caustic” — an envelope of rays described by a diffraction integral that has a number of stationary points. In electromagnetics, similar problems have been studied before, using a geometric optical approach, extended with wave asymptotics near the caustic [5–9].

Using the concept of quantified metasurfaces presented in [10] and the theoretical approach developed in this paper, one can realise acoustic field traps of controllable extent and geometry.

For the purpose of levitation of small objects, or any other application where the pressure distribution plays an important role, the ray picture must be complemented by calculations of the sound field along the ray trajectories: on the caustic, beyond it and in the sound shadow.

## 2. Wave propagation

A desired shape of the acoustic field is a particular solution of the Helmholtz equation:  $\nabla^2 u + k^2 n^2 u = 0$ , where  $u$  is the wave,  $k$  is the wavenumber and  $n$  is the relative index of refraction ( $n = c_0/c$ , where  $c$  is the local speed of sound and  $c_0$  is the one in the reference medium, e.g. air). In the approximation of geometric acoustics valid for large wavenumbers  $k$  and smooth variations of  $n$  on the scale of the wavelength [11], the solution  $u$  can be asymptotically represented as a superposition of the oscillating terms:  $u = \sum_{m=0}^{\infty} (ik)^{-m} U_m e^{ik\psi}$ , where the phase function  $\psi$  satisfies the eikonal equation  $(\nabla\psi)^2 = n^2$ , and the zero-order amplitude satisfies the transport equation,  $2\nabla U_0 \cdot \nabla\psi + U_0 \nabla^2\psi = 0$ , e.g. [12]. From these equations, by using the method of characteristics, one can derive the Hamiltonian-type dynamic equations

$$\frac{d\vec{r}}{d\tau} = \vec{p} = \nabla\psi, \quad \frac{d\vec{p}}{d\tau} = \frac{1}{2}\nabla n^2(\vec{r}), \quad (1)$$



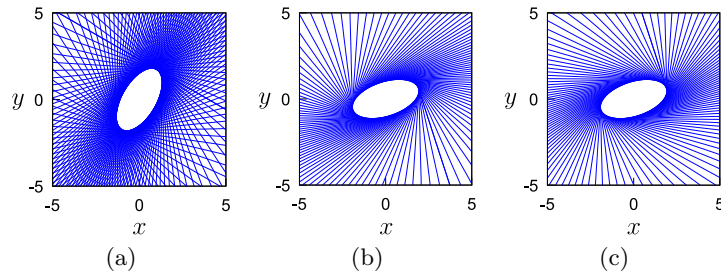


Figure 1: Elliptic caustic shape [a], Rotated semi-rays originating at a set of points at the edges of the domain and stopping at the caustic curve, for  $t \in (-\infty, 0]$  [b] and  $t \in [0, +\infty)$  [c].

which define the ray trajectories in the geometric acoustics approximation.

### 3. Realizable Caustic Shapes

Without compromising the generality of our theory, we restrict it to the situation when all rays are parallel to the  $xy$ -plane (this restriction can be removed later). We orient the coordinate axes so that the  $y$ -axis is vertical, so that the problem of creating a caustic in three dimensions reduces to a set of independent two-dimensional problems, each describing an open/closed curve in every plane  $z = \text{const}$ .

Showing which 3D shapes are representable in this way is beyond the scope of this work, but it can be shown that it is sufficient that, in each plane, the path curvature must not change sign. However, the paths laying in separate parallel planes (e.g.  $z = z_1$  and  $z = z_2$ ,  $z_1 \neq z_2$ ) may have different signs of curvature, i.e. the resulting three-dimensional caustic shape may have positive and negative curvature along different directions.

A possible generalization of this plane-by-plane shape construction could be in introducing three-dimensional curvilinear coordinates  $(u, v, w)$ , such that in every surface  $w = \text{const}$ , the curves  $u = \text{const}$  are straight lines (and therefore can be represented by rays propagating in straight lines). The curves  $v = \text{const}$  on the same surface do not necessarily need to be straight lines. A well-known example of a curved three-dimensional surface that satisfies this property is the one-sheeted hyperboloid, which means that the Cartesian coordinates can be replaced, e.g. by the hyperboloidal coordinates.

### 4. An example: A caustic in a shape of twisted elliptic rod

Let us consider how to create a caustic in a shape of a twisted rod with elliptic cross section in every plane  $z = \text{const}$  (see figure 1). We assume that the rod is elongated in the  $z$ -axis direction. In every plane  $z = \text{const}$  the cross section of such a rod is an ellipse. We can define an ellipse parametrically by the radius-vector  $\vec{\rho}$  in the  $xy$ -plane as follows:

$$\vec{\rho}(\varphi, \alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \cdot \begin{pmatrix} a \cos(\varphi) \\ b \sin(\varphi) \end{pmatrix} = \begin{pmatrix} X(\varphi, \alpha) \\ Y(\varphi, \alpha) \end{pmatrix}, \quad (2)$$

where  $a$  and  $b$  being the two semi-axes of the ellipse and  $\alpha = \alpha(z)$  is the angle of rotation of the basic shape. The rotation matrix approach is general and can be used with any other parametrically defined shapes, in which the parameter  $\varphi$  is not necessarily the polar angle.

The equation for the tangent lines in the Cartesian coordinates can be derived by calculating  $\frac{dy}{dx} = \frac{\partial Y / \partial \varphi}{\partial X / \partial \varphi}$ . The rays that form the caustic can thus be defined parametrically as  $x = R_x(t, \varphi, \alpha) = X(\varphi, \alpha) + t \frac{\partial X}{\partial \varphi}$  and  $y = R_y(t, \varphi, \alpha) = Y(\varphi, \alpha) + t \frac{\partial Y}{\partial \varphi}$ . When the parameter  $t$

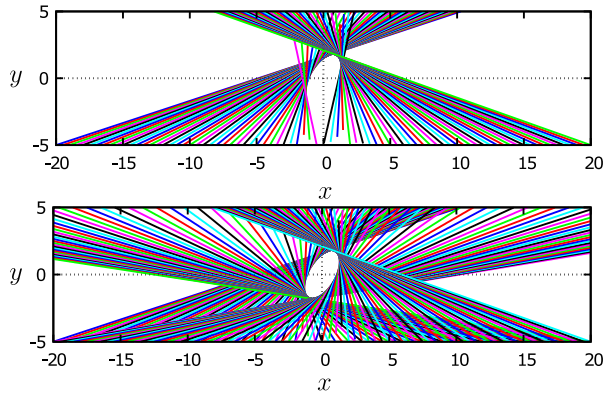


Figure 2: The caustic formed by a single line of sources located at  $y = y_0 = -5$ ,  $z = z_0$  [top panel] and a pair of such lines located at  $z = z_0$  and  $z = z_0 + \Delta z$  [bottom panel].

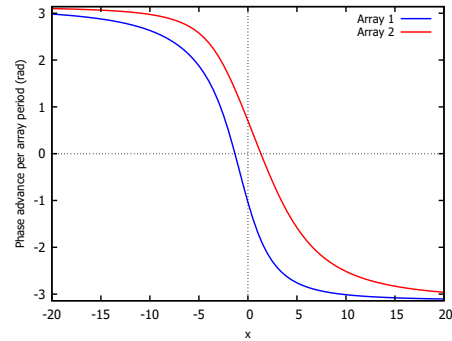


Figure 3: The phase advance per unit cell length for the two arrays realizing the caustic shown in figure 2(bottom).

varies in the range  $(-\infty, 0]$ , a set of semi-infinite rays is formed which is shown in figure 1(b). Another set of semi-rays is formed for  $t \in [0, +\infty)$  [figure 1(c)].

In practice, it would be desirable if the caustic could be created by sources located on just a single side of the domain where the metasurface is located, for example, just on the horizontal plane  $y = y_0 = \text{const}$ . For every point  $x = x_0$  on this line, there are two possible tangent rays that pass through this point [see figure 1(a)] so that, for physical reasons, one will have to select just one of these tangent rays. We select the one that propagates at an angle closer to the normal to the array, i.e. the one that requires smaller phase gradient. The resulting caustic image is shown in figure 2(top). Comparing this to figure 1, one can see that the restriction on the number of rays that can originate from a single point of the single metasurface reduces the set of realizable caustic shapes.

When working in the three dimensions, we can separate the two sets of rays and the respective sources along the  $z$  direction, placing them at  $z = z_0$  and  $z = z_0 + \Delta z$ . Although this setup is no longer planar, by spatially separating the sources and recovering the second set of rays, the caustic shape can be restored to the desired one. The projection of this caustic geometry on the plane  $z = z_0$  can be found in figure 2(bottom).

With all this information we can define now the phases of the sources that have to be placed at the edges of the domain in order to create a given caustic shape. Let us consider, for example, the edge  $y = y_0 = -5$  in figure 2. A source placed at the point  $x = x_0$  must create the ray with the parameters defined by the following system of two equations for the unknowns  $t_0 < 0$  (or  $t_0 > 0$  for the other set of semi-rays) and  $\varphi_0$ :  $R_x(t_0, \varphi_0, \alpha) = x_0$ ,  $R_y(t_0, \varphi_0, \alpha) = y_0$ .

This ray propagates at an angle  $\theta$  with respect to the array (which is parallel to the  $x$ -axis), such that  $\tan \theta = \frac{\partial Y / \partial \varphi}{\partial X / \partial \varphi} \Big|_{\varphi = \varphi_0}$ . From the theory of phased antenna arrays, in order to emit the ray at this angle, the gradient of the sources' phase distribution  $\Phi(x)$  along the array line at this point has to be  $\frac{\partial \Phi(x)}{\partial x} \Big|_{x=x_0} = \frac{2\pi}{\lambda} \cos \theta$ , where  $\lambda$  is the sound wavelength. The phase distribution

$\Phi(x)$  is found, with accuracy up to an irrelevant constant initial phase, as  $\Phi(x) = \int \frac{\partial \Phi(x)}{\partial x} dx$ . The required phase distribution along the other edges can be determined analogously. In future studies, one may consider the minimum number of metasurfaces in order to realize a given caustic shape. In figure 3, the phase gradient along the two array lines realizing the caustic shown in figure 2(bottom) is plotted, for the case when the array element separation (the period

of the array) is  $\lambda/2$ .

## 5. Sound Field Calculation in the Caustic Region

In asymptotic sound field calculations the ray picture obtained under the geometric acoustics approximation plays a role of the “skeleton” supporting the propagating wave field “flesh” [12]. In simple  $A_2$ -type (fold) caustics considered here, there are always two rays intersecting at a given point, while in the shadow region there are no rays at all. When moving across the caustic from the “illuminated” side to the “dark” side, the number of rays that meet at a given observation point changes abruptly from 2 to 0. At the same time, the origin points of the two rays collapse when the observation point approaches the caustic.

In order to obtain a useful expression for the sound field in the caustic region, we can first represent the sound field of the two mentioned rays as  $u = U_1 e^{ik\psi_1} + U_2 e^{ik\psi_2}$ , where  $k = 2\pi/\lambda$ , and  $U_{1,2}$  and  $\psi_{1,2}$  are the ray amplitude and eikonal values, respectively, obtained with the geometric acoustics (eikonal) approach. Although the amplitudes  $U_{1,2}$  can be calculated with the help of the transport equation at most of the points within the domain, the eikonal approximation results in infinite amplitudes at the caustic. This can be corrected by introducing the uniform asymptotic expansion of the sound field near the caustic [12]:

$$u = \left[ A(\vec{\rho}) \tilde{I}(\tilde{\zeta}(\vec{\rho})) + (ik)^{-1} B(\vec{\rho}) \tilde{I}'(\tilde{\zeta}(\vec{\rho})) \right] e^{ik\chi(\vec{\rho})}, \quad (3)$$

where  $\tilde{I}'(\tilde{\zeta}) = \partial \tilde{I}(\tilde{\zeta}) / \partial \tilde{\zeta}$ , and  $\tilde{I}(\tilde{\zeta})$  is Airy-type diffraction integral:  $\tilde{I}(\tilde{\zeta}) = k^{1/6} \sqrt{2\pi} \text{Ai}(k^{2/3} \tilde{\zeta})$  and  $\tilde{I}'(\tilde{\zeta}) = k^{5/6} \sqrt{2\pi} \text{Ai}'(k^{2/3} \tilde{\zeta})$ , where  $\text{Ai}(\zeta) = \pi^{-1} \int_0^\infty \cos(\zeta t + t^3/3) dt$  is the standard Airy function. The functions  $\chi(\vec{\rho})$ ,  $\tilde{\zeta}(\vec{\rho})$ , and the amplitudes  $A(\vec{\rho})$  and  $B(\vec{\rho})$  in the asymptotic expansion can be expressed through  $\psi_{1,2}$  and  $U_{1,2}$  as follows:

$$\chi = \frac{\psi_1 + \psi_2}{2}, \quad \frac{2}{3}(-\tilde{\zeta})^{3/2} = \frac{\psi_1 - \psi_2}{2}, \quad A = \frac{(-\tilde{\zeta})^{1/4}(U_1 + iU_2)}{\sqrt{2}e^{i\pi/4}}, \quad B = \frac{(-\tilde{\zeta})^{-1/4}(U_1 - iU_2)}{\sqrt{2}e^{i\pi/4}}. \quad (4)$$

It can be shown that the amplitudes  $A$  and  $B$  remain finite at the caustic, despite that  $U_{1,2}$  diverge.

## 6. Conclusion

In this paper, a method for realizing caustics in acoustic field was presented. This method can be used to produce a wide class of caustics by calculation of the required phase distribution along the metasurface. The method was illustrated with an example.

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