SHARED CONTROL FOR TELEOPERATION

USING A LIE GROUP APPROACH

Brian Hunter

Submitted in part fulfilment of the requirements for the Degree of Doctor of Philosophy of the University of London

1996

SHARED CONTROL FOR TELEOPERATION

USING A LIE GROUP APPROACH

Abstract

Shared control is a technique to provide interactive autonomy in a telerobotic task, replacing the requirement for pure teleoperation where the operator's intervention is unnecessary or even undesirable. In this thesis, a geometrically correct theory of shared control for teleoperation is developed using differential geometry. The autonomous function proposed is force control. In shared control, the workspace is commonly partitioned into a "position domain" and a "force domain". This computational process requires the use of a metric. In the context of manifolds, these are known as Riemannian metrics. The switching matrix is shown to be equivalent to a filter which embodies a Riemannian metric form. However, since the metric form is non-invariant, it is shown that the metric form must undergo a transformation if the measurement reference frame is moved. If the transformation is not made, then the switching matrix fails to produce correct results in the new measurement frame. Alternatively, the switching matrix can be viewed as a misinterpretation of a projection operator. Again, the projection operator needs to be transformed correctly if the measurement reference frame is moved. Many robot control architectures preclude the implementation of robust force control. However, a compliant device mounted between the robot wrist and the workpiece can be a good alternative in lieu of explicit force control. In this form of shared control, force and displacement are regulated by control of displacement only. The geometry of compliant devices is examined in the context of shared control and a geometrically correct scheme for shared control is derived. This scheme follows naturally from a theoretical analysis of stiffness and potential energy. This thesis unifies some recent results formulated for robotic hybrid position / force control under the modern framework of differential geometry and Lie groups.

Contents

Contents (continued)

Contents (continued)

Contents (continued)

List of Figures

List of Figures (continued)

List of Tables

"and for the sake of all things in general let us recall to mind that nothing can be known concerning the things of this world without the power of geometry ... "

Roger Bacon

Acknowledgements

This thesis was undertaken with the support of Telerobotic Systems, AEA Technology. I would like to thank C.J.H. Watson and M.H. Brown for giving their support to my registration at University College London.

I am very grateful to my supervisors - David Broome of UCL and Siva of AEA - and wish to thank them for giving me the freedom to explore ideas under their guidance and for their patience.

I would like to thank Patrick Fischer, Paul Elosegui, Ross McAree and Ron Daniel for many fruitful technical discussions over the years. I am indebted to Dr. J.M. Selig of Southbank University, Dr. J. Gilby of Sira Ltd and Dr. A. Greig of UCL for their critical review of my work.

Most of all I would like to thank my wife, Anne-Marie, and my parents, Bill and Jean, for their emotional support and for their belief in me, without which this thesis would surely not have been completed.

Chapter 1

Introduction

1.1 Overview of the Chapter

In this chapter, some basic concepts behind teleoperation and shared control are explained. A review of previous work on the implementation of shared control is also presented. Finally, the main contributions of this dissertation are put forward.

1.2 Teleoperation

The term "teleoperation" is used to describe mechanical activities performed by mechanical devices at a remote site under remote control. The remotely performed mechanical actions are usually associated with the normal work function of the human arm and hand. Thus teleoperation extends the manipulative capabilities of the human arm and hand to remote, physically difficult, or dangerous environments (Bejczy 1980a).

The first teleoperator systems were developed in the 1940s to allow an operator to handle radioactive materials from a workroom separated from the radioactive environment by a concrete wall. The operator observed the work scene through viewing ports in the wall. The development of teleoperator devices for handling radioactive materials culminated in the introduction of bilateral master-slave manipulator systems. In these very successful systems, the slave arm at the remote site is mechanically or electrically coupled to the geometrically identical or similar master arm and thus follows the motion of the master arm. But the coupling between the master and slave arms is two-way: inertial or external forces on the slave arm can backdrive the master arm. Hence the operator holding the master arm can feel forces acting on the slave arm. This is an essential requirement for dextrous control of remote manipulators, since general purpose manipulation consists of a series of well-controlled contacts or "collisions" between the handling device and the objects (Bejczy 1980a).

Master-slave teleoperator technology has been expanding to accommodate new telemanipulation requirements in space, under the sea, in nuclear facilities, and in other frontiers of science. This is reflected in a NASA study that described a teleoperator as

" a robotic device having video and/or other sensors, manipulator arms, and some mobility capability, which is remotely controlled over a telecommunication channel by a human operator. This human operator can be a direct in-the-loop controller who observes a video display of the teleoperator and, with joystick or analog device, continuously controls the position of the teleoperator vehicle, its arm, or its sensor orientation. Alternatively, the teleoperator can employ a computer, endowed with a modicum of artificial intelligence, capable of executing simple control functions automatically through local force or proximity sensing: in this case, the remote human operator shares and trades control with the computer" (Bejczy 1980a)

The bilateral force reflecting master-slave manipulator is a successful example of where a kinaesthetic coupling between operator and remote manipulator has been established. However, the establishment of this type of coupling is not constrained to geometrically similar master-slave systems; it is possible to establish a kinaesthetic coupling via a "universal" force reflecting input device - in fact, this can be viewed as a generalization of the technique (Bejczy 1980b).

1.2.1 Force-Reflecting Input Devices

The force-reflecting input device serves a general purpose in that it does not have any geometric or kinematic correspondence with the mechanical arm it controls and from which it is backdriven. The position control relation between this device and a mechanical arm is established through real-time mathematical transformation of joint variables measured at both the input device and the mechanical arm. Likewise, the forces and torques sensed at the base of the end effector are resolved into appropriate input device joint drives through real-time mathematical transformations to give the operator's hand the same force-torque "feeling" that is felt by the end effector on the remote mechanical arm (Bejczy 1980a).

An outline specification for a high fidelity force-reflecting input device was first proposed by Goertz (1964). Although his work was aimed primarily at conventional master-slave manipulator systems, the specification is consistent with aspects of more recent specifications devised for modern force-reflecting input devices. He noted that the master-slave systems had the same frequency response in both the directions but that this was inconsistent with the capabilities of the human operator; the human motor system was capable of generating frequencies of around 10Hz but the nervous system could detect much higher transient frequencies through the hand. Goertz (1964) proposed that a manipulator that was more consistent with the operator's capabilities might require a bandwidth of between 100Hz to 1kHz .

The issue of input device specification was raised more recently by McAffee and Fiorini (1991) and Fischer et al (1990). These specifications arose from an attempt to formalize the design of desk-top input devices that were kinematically dissimilar from the slave arms that they were controlling. These new devices have potential for excellent performance compared to conventional master-slave manipulators, especially where space constraints exist, since they are kinematically optimized for the human operator interface and not matched to the slave.

Development work undertaken at the Jet Propulsion Laboratory (JPL) raised many of the important issues associated with the design and control of desk-top force-reflecting input devices. The JPL Force Reflecting Hand Controller (FRHC) was under development for much of the early 1980s (Bejczy and Handlykken 1981) culminating in a flight prototype version at the end of the decade (McAffee et al 1990). More recently, AEA have produced an input device based on the Stewart Platform geometry, the *Bilateral Stewart Platform* (BSP) (Fischer 1993).

McAffee and Fiorini (1991) gives a summary of a suitable specification for a force-reflecting input device as follows:

"Highly intuitive operation:

- *provides full 6-dof articulation to specify a unique spatial position and orientation,*
- *provides excellent kinaesthetic feedback to the human operator to produce the same physiological motor sensations as if performing the task in person,*
- *configures the remote reference frame so that it coincides with the operator's own body reference,*
- *assures that spatial transformations are transparent to the operator,*
- *makes use of human eye-hand co-ordination so that commands can be given almost instinctively by the operator,*
- *allows a shorter learning time.*

Universal (generalized) applicability:

- *provides a common interface to control dissimilar remote systems,*
- *accommodates many control modes and allows for system performance adjustments.*

Good 6-dof position and orientation resolution:

- *guarantees accurate sensing of position and orientation commands from the operator,*
- *assures that the mechanism has a minimum of backlash.*

High fidelity force feedback:

- *faithfully reproduces remote forces and torques,*
- *generates crisp and distinguishable force and torque cues.*

Good mechanical design:

- *provides the mechanical stiffness necessary for a large system bandwidth,*
- *provides the simple kinematic structure necessary for fast kinematic and dynamic models,*
- *provides mechanically decoupled joints, simplifying control algorithms,*
- *assures good backdriveability by minimizing friction and inertia for higher force resolution and lower operator fatigue."*

1.2.2 Control Methodology Issues

Typically a force-reflecting input device is not a kinematically similar mechanism to the manipulator arm that it is controlling and is designed to utilize a more generalized form of control, the so-called resolved motion control (Whitney 1969). In this case the manipulator motion is specified in terms of a trajectory in the Cartesian workspace and is conceptually easy for the operator to use.

Resolved motion control is so-named because the required motion in the Cartesian workspace is resolved into a sequence of six joint angle commands for the manipulator joint servo drivers. Resolved motion control references the position and orientation of the manipulator's gripper, requiring three positions to fix the position of the gripper in three dimensional space and three rotations to fix the orientation of the gripper (Whitney 1969).

A common approach for resolved motion control is to use two three-axis joysticks to realize the three position commands and three orientation commands. However, a six degree of freedom input device integrates the function of two joysticks into one unit, enabling one-handed operation. The forward kinematic solution is a function of the input device mechanism design. For example, a numerical solution is required for the parallel BSP mechanism (Fischer 1993). The mapping that transforms Cartesian commands to joint commands is the inverse kinematics of the manipulator. This needs to be calculated at every sampling time. The design of the robot wrist can greatly simplify the computational burden associated with solving the inverse kinematics of the arm; the three wrist joint axes should intersect at a point (Craig 1986)).

Force-reflecting operation can be achieved with position-position control (Goertz et al 1961). This has the been the classic method for bilateral control, employed very successfully for master-slave manipulators for many years (Goertz et al 1966). This control mode is very simple since it involves no direct measurement of forces (Siva 1985). It is essentially two unilateral position servomechanisms connected back-to-back and the position error made common to both servo systems (Raimondi 1976). The position-position servo-system essentially requires backdriveability between the master and slave servo-drives (Siva 1992). Therefore, this method (in its basic form) is unsuitable for inherently non-backdriveable servo-systems. An example is a hydraulic servo-position system, controlled by a flow controlled servo-valve (Mosher 1960). In this case, the situation can be remedied by using a special pressure control servo-valve or by backdriving the master-arm (using a signal derived from the pressure differential across the actuator) and relying on the closed-loop position control to backdrive the slave (Wilson 1975). It should be noted that friction in either the master or slave will affect the success of this scheme.

Position-position control can also be employed on dissimilar input devices by considering position errors in Cartesian space rather than in joint space (Kim 1991). A non-backdriveable servo-system (such as a robot joint-servo using a high reduction gearbox) can use compliance control in order to achieve backdriveability. A major advantage here is that the position/orientation command from the input device can be perturbed directly; the input device itself does not need to be backdriven. This is because there is no absolute correspondence to maintain between the input device position and the manipulator position. Therefore, friction in the input device is irrelevant to the success of the technique.

Force-reflecting operation can be achieved with force-position control (Handlykken and Turner 1980). This mode of control is configured by essentially implementing two control loops. The first loop is a position control loop which serves to send position commands to the robot from an input device. The second loop is a force control loop which serves to implement force commands to the input device from the robot. This scheme has been successfully implemented for master-slave servo systems where a measurement of joint torque is required (Bicker 1990). Force-position control can also be employed on dissimilar input devices by considering measurement of forces in Cartesian space rather than in joint space. A wrist-mounted force sensor is used to measure end-point forces and torques in the tool coordinate frame. These measurements are then transformed from the tool coordinate frame to the world coordinate frame using a transformation matrix derived from the orientation of the manipulator wrist. The force measurement is then transformed into input device joint space, via the transpose of the input device Jacobean and used to backdrive the input device thus establishing the kinaesthetic coupling to the operator (Bejczy and Salisbury 1983).

In a typical telemanipulation system, the manipulator is under closed loop position control. The manipulator is typically stiff and small errors between the actual and the commanded position can give rise to undesired large contact forces and torques (Hannaford 1989). The same problem arises with automatic force control of manipulators and, although many approaches have been tried, the problem of oscillatory contact with a rigid environment persists (Eppinger and Steering 1987). Force-position control has been implemented on the JPL FRHC (Handlykken and Turner 1980). The stability problem meant that force gains had to be turned down. This meant that the maximum force ratio attainable without causing instability was only approximately 10:1. This meant that only 1N force could be felt for a 10N force on the manipulator (Kim 1991). This is the trade-off that appears to exists when only a simple force-position loop is implemented.

This situation can be alleviated by adding compliance and damping to the stiff robot. Active compliance and damping, emulating a programmable mechanical passive spring and damper for each Cartesian axis, can be implemented by first low pass filtering the force-torque signal from the wrist-mounted force sensor and then feeding back the low pass filtered signal to the position/orientation command signal from the input device. When implemented into a system incorporating a bilateral input device, the approach is called *shared compliant control* (Kim et al 1992). There are two parameters to control: compliance (or its inverse, stiffness) and damping. The compliance of the active spring is proportional to the force feedback gain K. The damping of the active damper is proportional to T/K, where T is the time constant of the first-order low pass filter. If a pure gain is used instead of the low pass filter, a spring with no damper is realized. If an integrator is used instead of the low pass filter, a damper with no spring is realized (Kim 1990).

The problem of instability can be reduced by introducing damping into the system via the input device (Fischer 1993). However, there is a trade-off because high levels of damping on the input device can be fatiguing for the operator. Other approaches to combat instability are force signal frequency shaping and digital compensation (Fischer 1993).

1.3 Shared Control Enhancement

1.3.1 Concepts and Classification

In shared control, control of the six degrees of freedom of the taskspace is shared with computer control algorithms referenced to a sensor or to some other world model information (Bejczy 1980a). Shared control is designed to inject autonomy into a telerobotic task, replacing the requirement for pure teleoperation in those situations where the operator's intervention is unnecessary or even undesirable.

It is appropriate to produce a rigorous definition and classification of sub-types. The terminology given in Yoerger and Slotine (1987) is adopted. Two basic forms of shared control are identified - serial and parallel.

In serial form, the control of the manipulator is *switched* in series between the operator and the autonomous function. In parallel form, the human operator and the autonomous function jointly execute the task (Yoerger and Slotine 1987).

The parallel form can be sub-divided further to two forms (Yokokohji et al 1993). These forms are termed combined and non-combined. In the combined form, the autonomous control is mixed with the operator control. An example is collision avoidance, where the operator's command is modified is some way, perhaps by proximity sensors, to avoid contact. A second example, though more subtle, is shared compliant control (SCC); again the operators command is modified by information from a sensor - in this case a force-torque sensor.

In the non-combined form, there is no mixing on any degree of freedom, but there is a mix of operator and autonomous control across the available degrees of freedom. If the autonomous control is a force control action, then we have a form of shared control that is analogous with the classic robotic *hybrid position / force control.* (Raibert and Craig 1981). The difference is that position control is from a human operator rather than from a robot trajectory generator.

1.3.2 Review of Previous Shared Control Designs

Shared control has been implemented on a number of teleoperator systems for use in space. The ROTEX experiment (Hirzinger et al 1992) was designed to test a number of concepts involved in the implementation of a partly autonomous robot system with extensive ground control capabilities for the European Space Agency. The experiment featured a small, six-axis robot (working volume around $1m^3$) moving inside a space-lab rack integrated into the US space shuttle. Its gripper was provided with a number of sensors, including a 6-axis force-torque sensor. The experiment has successfully flown in space on spacelab mission D2 on shuttle flight STS 55 from April 26 to May 6, 1993 (Hirzinger et al 1993). A parallel - non-combined form of shared control was used.

Shared control was used in the telerobotics validation experiments and demonstrations for the Space Station Freedom program (Backes et al 1993). The Johnson Space Centre in Houston acted as the local ground site and the JPL Supervisory Telerobotics (STELER) laboratory in Pasadena acted as the remote site. Operator control stations were supported at both the local ground site and the remote site, the remote site allowing teleoperation and shared control where time delays were of an acceptable order. The robot was a Robotics Research K1207 and the Modular Telerobot Task Execution System (MOTES) provided the remote site execution capability. The input device was a JPL/Salisbury Model C Hand Controller (Backes et al 1993). The control of each task space could be shared between a position control mode (via an input device) and a compliant control mode -termed as *"force nulling"* (Bejczy 1988). This form of shared control is parallel- non-combined. Force control was accomplished with a force control loop closed around an inner position control loop (Backes 1990). The force nulling mode was achieved by producing a position setpoint for the selected degree of freedom based on integration of the sensed force or torque in that direction. Stein (1993) detailed a more general force mode for the JPL Advanced Teloperation (ATOP) Laboratory which referenced a setpoint force value. A non-zero setpoint caused the robot to attempt to maintain a contact force, whilst a zero setpoint yielded a control action that avoided contact; the latter action is the same as force nulling. Stein (1993) also added a feature to the system whereby the shared control could be configured in either the world or the tool frame. The shared control capability was built into high level task primitives that could be selected by the operator from a menu.

Hayati and Venkataraman (1989) attempted to define a generic shared system architecture for software implementation. This was based on switching matrices to segment input vectors in an appropriate fashion. This was similar in concept to the switching matrices used in early implementations of hybrid position / force control (Raibert and Craig 1981).

The use of a switching matrix is best explained using an example. Consider Figure 1-1.

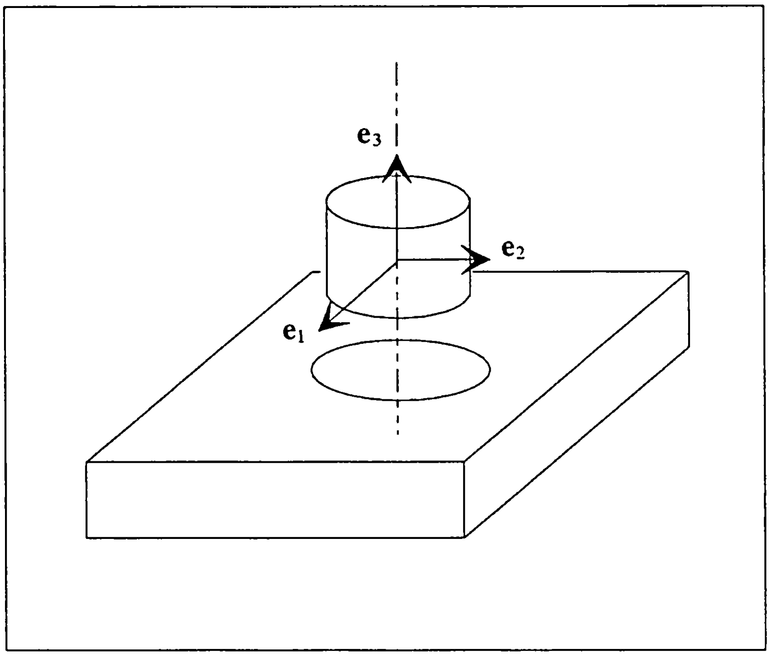

Figure 1-1: Peg Insertion Task

Figure 1-1 represents a simple peg insertion task. The velocity and force at the robot end effector are typically described by two vectors in end effector space.

$$\mathbf{v}^T = [v_1 \quad v_2 \quad v_3 \quad \omega_1 \quad \omega_2 \quad \omega_3] \tag{1.1}$$

$$\mathbf{f}^T = [f_1 \quad f_2 \quad f_3 \quad m_1 \quad m_2 \quad m_3] \tag{1.2}$$

The task is to insert the peg into the hole. Velocities v_3 and ω_3 and forces f_1, f_2, m_1 and m_2 must be controlled in order to perform the task. The end effector space is divided into a "position domain" and a "force domain". A switching matrix $\mathbf{S}$ and its "complement" $[\mathbf{I}-\mathbf{S}]$ are used to separate the directions in end effector space in which force and position are controlled according to the following laws given in (1.3) and (1.4). The directions constitute the velocity and force trajectories that will be followed to perform the task (West and Asada 1985).

$$\mathbf{Sv} = \mathbf{0} \tag{1.3}$$

$$[\mathbf{I}-\mathbf{S}]\mathbf{f} = \mathbf{0} \tag{1.4}$$

where

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.5}$$

If vectors $\mathbf{v}$ and $\mathbf{f}$ satisfy (1.3) and (1.4), then the instantaneous work is zero

$$\mathbf{fv} + \mathbf{mw} = \mathbf{0} \tag{1.6}$$

Traditionally in shared control, the velocity command from the operator and the force at the end effector is processed using a switching matrix to ensure that (1.6) is satisfied.

It can be shown that the switching matrix does in fact embody a **metric** or measure. In order for the switching matrix to qualify as a proper physical law, it must always produce correct results, even if the measurement reference frame is moved. It can be shown that this quality is a function of how the metric is transformed.

1.4 Main Contributions

In this dissertation, the focus is the theory of a parallel - non-combined form of shared control, analogous to robotic hybrid position / force control.

The main contributions of this dissertation are put forward as follows:

- using an argument based strictly on modern differential geometry, the switching matrix is shown to be equivalent to a filter which embodies a Riemannian metric form. However, since the metric form is non-invariant, it is shown that the metric form must undergo a transformation if the measurement reference frame is moved. If the transformation is not made, then the switching matrix fails to produce correct results in the new measurement frame,

- geometrically correct filters for shared control are given that are suitable for inclusion into software and are demonstrated using a test harness,

- the switching matrix is shown to be a misinterpretation of a projection operator,

- a correct theoretical description of shared control using a compliant mechanism is presented,

- geometrically correct transformations for compliance control are given that are suitable for inclusion into software and are demonstrated using a test harness,

- this work unifies recent results on robotic hybrid position / force control into a consistent description based on modern differential geometry and Lie groups.

1.5 Comparison to Other Work

The closest works in concept to this dissertation are Lipkin (1985) and Loncaric (1985).

Lipkin (1985) was the first to recognize that there was a flaw in the use of a switching matrix for robotic hybrid position / force control. His argument was based on the theory of screws (Ball 1900). Lipkin also derived correct filters for robotic hybrid position / force control. This dissertation differs in the following respects:

- the focus is on shared control rather than robotic hybrid position / force control,

- the switching matrix is shown to be a misinterpretation of a projection operator,

- an argument is formulated using the principles of modern differential geometry.

Loncaric (1985) studied the implementation of compliance programming for robotics from a modern perspective. This dissertation differs in the following respects:

- the focus is on shared control rather than on compliance programming,

- the connection is made that the switching matrix is associated with a non-invariant Riemannian metric,

- the switching matrix is shown to be a misinterpretation of a projection operator,

- the transformations required for invariant filtering are identified,

- algorithms for the implementation of geometrically correct schemes are developed.

Chapter 2

Manifolds and Groups

2.1 Overview of the Chapter

Differential geometry provides an excellent, modern tool for discussing the issues of shared control. The theory generalizes our familiar ideas about curves and surfaces to arbitrary dimensional objects called manifolds. However, the mathematical concepts are abstract and require considerable interpretation before they can be usefully employed to solve a practical engineering problem. Therefore, rather than just referring the reader to the appropriate references, a clear and concise interpretation of the theory is presented here.

This Chapter is in six main sections. The first section details the basic mathematical concepts behind modern differential geometry. The set of all rigid body displacements forms a group and this motivates a discussion on the theory of groups in the remaining sections.

2.2 Manifolds

Before a manifold can be defined, some basic definitions are required.

Definition 2.1 Topological Space

Let X be any set and $\Gamma = \{U_i | i \in I\}$ denote a certain collection of subsets of X. The pair (X, Γ) is a topological space if Γ satisfies the following requirements:

(i) $\varnothing, X \in \Gamma$

(ii) if J is any (may be infinite) subcollection of I, the family $\{U_j | j \in J\}$ satisfies $\cup_{j \in J} U_j \in \Gamma$.

(iii) if K is any finite subcollection of I, the family $\{U_k | k \in K\}$ satisfies $\cap_{k \in K} U_k \in \Gamma$.

X alone is often called a **topological space**. The U_i are called the open sets and Γ is said to give a topology to X (Nakahara 1990).

Definition 2.2 Neighbourhood

Suppose Γ gives a topology to X. N is a **neighbourhood** of a point $x \in X$ if N is a subset of X and N contains some (at least one) open set U_i to which X belongs.

Definition 2.3 Hausdorff space

A topological space (X,Γ) is a **Hausdorff space** if, for an arbitrary pair of distinct points $x,x' \in X$, there always exist neighbourhoods U_x of x and $U_{x'}$ of x' such that $U_x \cap U_{x'} = \varnothing$. (Nakahara 1990).

Physical space is a topological space under a "sphere" topology. The "sphere" topology is generated by the interior of spheres of arbitrary radius and arbitrary centre. The topology consists of all such open sets together with arbitrary unions and finite intersections.

Physical space is a Hausdorff space since any two distinct points can be encompassed by non-overlapping spheres of sufficiently small radius.

An equivalence relation is now introduced under which geometrical objects are classified according to whether it is possible to deform one object to the other by continuous deformation.

Definition 2.4 Homeomorphisms

Let X_1 and X_2 be topological spaces. A map $f: X_1 \to X_2$ is a **homeomorphism** if it is continuous and has an inverse $f^{-1}: X_2 \to X_1$ which is also continuous. If there exists a homeomorphism between X_1 and X_2, X_1 is said to be homeomorphic to X_2 and vice versa (Nakahara 1990).

Definition 2.5 Manifold

M is an m-dimensional differentiable **manifold** if

(i) M is a Hausdorff space,

(ii) M is provided with a family of pairs $\{(U_i \varphi_i)\}$,

(iii) $\{U_i\}$ is a family of open sets which covers M, that is $\cup_i U_i = M$. φ_i is a homeomorphism from U_i onto an open subset U_i' of $\Re^m$,

(iv) given U_i and U_j such that $U_i \cap U_j \neq \varnothing$, the map $\varphi_{ij} = \varphi_i \varphi_j^{-1}$ from $\varphi_j(U_i \cap U_j)$ to $\varphi_i(U_i \cap U_j)$ is infinitely differentiable.

The pair (U_i, φ_i) is called a chart while the whole family $\{(U_i \varphi_i)\}$ is called an atlas.

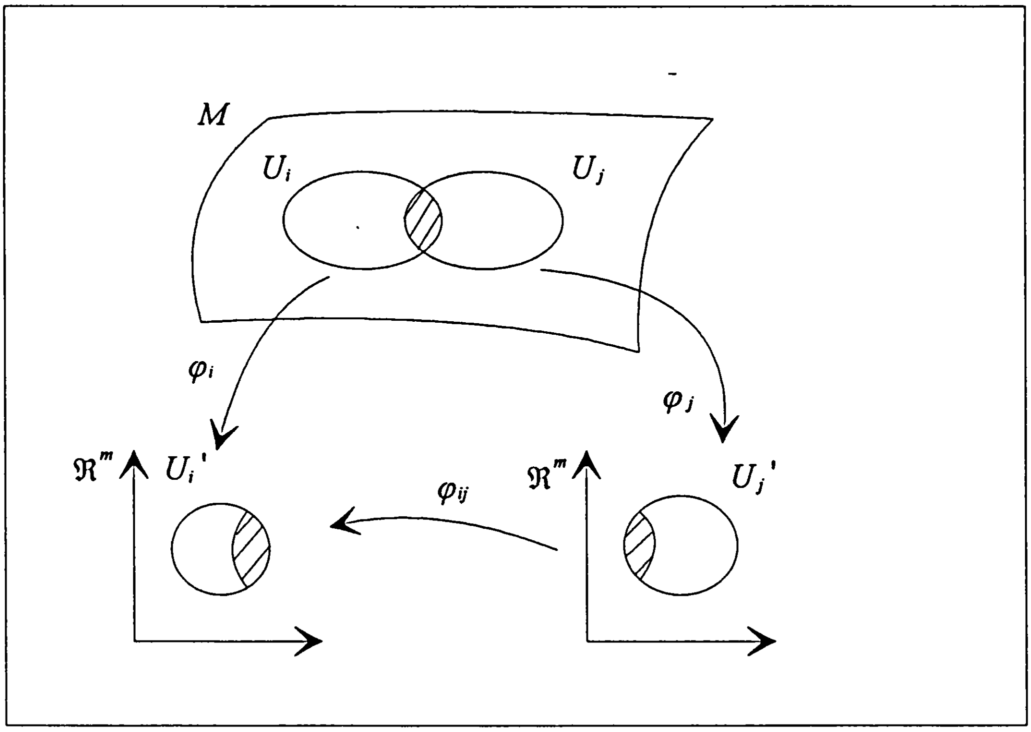

Figure 2-1: Differential manifold (adapted from Nakahara 1990)

φ is the coordinate function. φ is represented by m functions $\{x^1(p)\ldots x^m(p)\}$. The set $\{x^m(p)\}$ is called the coordinate.

The coordinate function corresponds to the assignment of a frame in U_i.

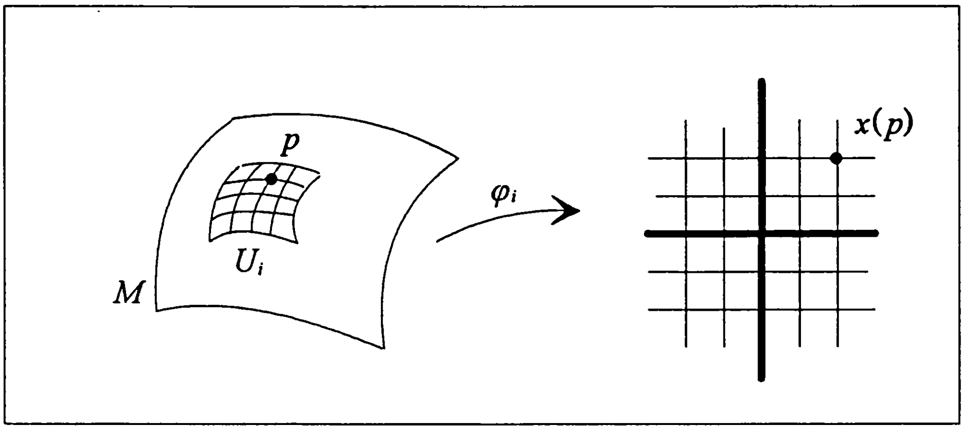

Figure 2-2: Coordinate function (adapted from Spivak 1979)

The dimension of the manifold M is the dimension of the space $\Re^m$.

$$\varphi_i : U_i \to U_i',\ U_i' \in \Re^m \tag{2.1}$$

In three dimensional space, the surface of a unit sphere can be defined as

$$x^2 + y^2 + z^2 = 1 \tag{2.2}$$

This is a two dimensional manifold, and the sphere is known as a two-sphere. The two sphere has a higher dimensional analogue. For example, the three-sphere is a three dimensional manifold in four dimensional space (Samuel et al 1991).

The formal mathematical generalization of "size" on a manifold is known as a metric. If the metric is constant over the manifold, then the manifold is said to be ***flat.***

Definition 2.6 Physical space

It is now possible to formally define **physical space** as a flat, orientable, 3 dimensional differential manifold, denoted by E.

E is assumed to be orientated, namely only a right handed orthonormal basis will be considered.

The set $\{x^1(p),\ \ x^2(p),\ \ x^3(p)\}$ is called the coordinate. The normal (Euclidean) metric on E is the distance between points $\mathbf{x}$ and $\mathbf{y}$.

$$\|\mathbf{x},\mathbf{y}\|_2 = \left((\mathbf{x}-\mathbf{y})^T(\mathbf{x}-\mathbf{y})\right)^{1/2} \tag{2.3}$$

There is a family of Euclidean metrics on E, parameterized by the choice of length scale.

Definition 2.7 Rigid body

A **rigid body** $B \subset E$ is an open subset of E, the differentiable structure of E naturally inducing one on B. B is said to be an open submanifold of E.

The formal definition of a submanifold is postponed until Chapter 3. The boundary of a rigid body can be included but topological features such as edges can mean the result is not a manifold.

2.3 Groups

Definition 2.8 $SE(3)$

Every rigid body displacement is an (internal) distance-preserving map, known as an isometry. The set of all isometries of E forms a group, the special Euclidean group, denoted $SE(3)$.

The special Euclidean group is the set of all maps

$$f:\mathbf{x}\mapsto \mathbf{A}\mathbf{x}+\mathbf{d} \qquad \mathbf{A}\in SO(3),\mathbf{d}\in T(3) \tag{2.4}$$

$SO(3)$ denotes a subgroup of the set of orthogonal groups, with determinant $+1$, called rotations. $T(3)$ denotes the group of all translations. Subgroups of the orthogonal group with determinant -1 (called reflections) are not used.

Definition 2.9 Group

A **group** G is a set $g_1 \ldots g_n \in G$ together with an operation, called group multiplication ($\circ$) such that

(i) $g_i \in G, g_j \in G \Rightarrow g_i \circ g_j \in G$ (closure)

(ii) $g_i \circ (g_j \circ g_k) = (g_i \circ g_j) \circ g_k$ (associativity)

(iii) $g_I \circ g_i = g_i \circ g_I$ for all g_i (existence of identity)

(iv) $g_k \circ g_l = g_l \circ g_k = g_I$ (unique inverse)

It is convenient to represent an element of $SE(3)$ in matrix form:

$$g = \begin{bmatrix} \mathbf{A} & \mathbf{d} \\ \mathbf{0} & 1 \end{bmatrix} \tag{2.5}$$

This representation is widely used in the robotics literature and is referred to as a homogeneous transformation (Paul 1981). This representation is necessary since displacements of $\Re^n$ cannot be represented by $n \times n$ matrix transformations. This inconvenience is removed by embedding $\Re^n$ in $\Re^{n+1}$ as the n dimensional hyperplane H. Linear transformations of $\Re^{n+1}$ exist that perform rigid displacements in the hyperplane $H: x_{n+1} = 1$ (McCarthy 1990).

To highlight the distinction, the set of $(n+1) \times (n+1)$ homogeneous transforms is denoted $H(n+1)$ rather than $SE(n)$.

Theorem 2-1

$g \in H(4)$ fulfils the properties of a group.

Proof of Theorem 2.1

Consider $g_1 = \begin{bmatrix} \mathbf{A}_1 & \mathbf{d}_1 \\ \mathbf{0} & 1 \end{bmatrix}$ and $g_2 = \begin{bmatrix} \mathbf{A}_2 & \mathbf{d}_2 \\ \mathbf{0} & 1 \end{bmatrix}$. If group multiplication is taken as matrix multiplication, then

(i) (closure)

$$g_1 \circ g_2 = \begin{bmatrix} \mathbf{A}_1 & \mathbf{d}_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{A}_2 & \mathbf{d}_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1\mathbf{A}_2 & \mathbf{A}_1\mathbf{d}_2 + \mathbf{d}_1 \\ 0 & 1 \end{bmatrix} \in H(4)$$

(ii) (associativity)

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{d}_1 \\ 0 & 1 \end{bmatrix} \left[\begin{bmatrix} \mathbf{A}_2 & \mathbf{d}_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{A}_3 & \mathbf{d}_3 \\ 0 & 1 \end{bmatrix} \right] = \left[\begin{bmatrix} \mathbf{A}_1 & \mathbf{d}_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{A}_2 & \mathbf{d}_2 \\ 0 & 1 \end{bmatrix} \right] \begin{bmatrix} \mathbf{A}_3 & \mathbf{d}_3 \\ 0 & 1 \end{bmatrix}$$

(iii) (existence of an identity). Consider $g_2 = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}$

$$g_1 \circ g_2 = \begin{bmatrix} \mathbf{A}_1 & \mathbf{d}_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{d}_1 \\ 0 & 1 \end{bmatrix}$$

(iv) (unique inverse). Consider $g_2 = \begin{bmatrix} \mathbf{A}_1^{\mathrm{T}} & -\mathbf{A}_1^{\mathrm{T}}\mathbf{d}_1 \\ 0 & 1 \end{bmatrix}$

$$g_1 \circ g_2 = \begin{bmatrix} \mathbf{A}_1 & \mathbf{d}_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{A}_1^{\mathrm{T}} & -\mathbf{A}_1^{\mathrm{T}}\mathbf{d}_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & 1 \end{bmatrix}$$

It is important to note that the group is not **abelian** (i.e. non commutative). This means that $g_1 \circ g_2 \neq g_2 \circ g_1$ in general.

One further definition will be required:

Definition 2.10 Diffeomorphism

A **diffeomorphism** is a infinitely (C^∞) differentiable homeomorphism.

Diffeomorphisms classify spaces into equivalence classes according to whether it is possible to deform one space to another smoothly (Nakahara 1990). The set of diffeomorphisms $f: M \to M$ is a group denoted by $Diff(M)$.

2.4 Lie Groups

A Lie group is a manifold on which group multiplications, product and inverse, are defined.

Definition 2.11 Lie group

A **Lie group** G is a differentiable manifold which is endowed with a group structure such that all group operations

(i) $\cdot : G \times G \to G$ by $(g_1, g_2) \mapsto g_1 \cdot g_2$

(ii) $^{-1} : G \to G$ by $g \mapsto g^{-1}$

are differentiable (Nakahara 1990).

$SE(3)$ has the structure of a Lie group.

Since a rigid body displacement is determined by six continuous parameters - three rotations and three translations, $SE(3)$ has the structure of a six-dimensional manifold. This is called the configuation manifold of the group (Samuel et al 1991). Each point on the configuration manifold corresponds to a rigid body displacement, see Figure 2-3.

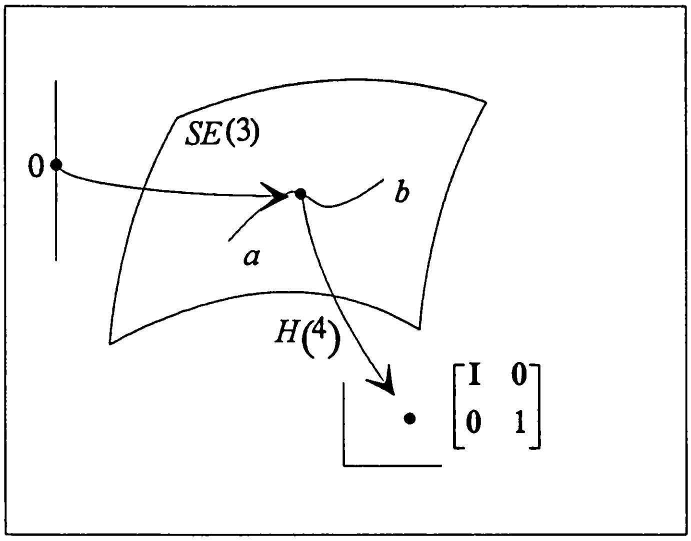

Figure 2-3: Configuration Manifold of $SE(3)$

2.5 The Classical Groups

It is useful to put $SE(3)$ in the context of the classical groups. Given a restriction to real matrices, the most comprehensive linear matrix group is the general linear group, denoted $GL(n)$. The real special linear group $SL(n)$ is obtained by the restriction that matrices have determinant $+1$. The set of orthogonal matrices forms the orthogonal group $O(n)$, while the set of orthogonal matrices of determinant $+1$ forms the special orthogonal group $SO(n)$. $O(n)$

consists of two disconnected pieces, with $SO(n)$ occurring as a subgroup (Wybourne 1974).

$SO(n)$ leaves invariant the symmetric bilinear form

$$(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n} x^i y^i \equiv \mathbf{x}^T \mathbf{y} \tag{2.6}$$

$GL(n)$ is a Lie group where the product of elements is simply the matrix multiplication and the inverse is given by the matrix inverse. $SL(n)$ and $SO(n)$ are Lie subgroups of $GL(n)$.

Next consider the skew symmetric bilinear form in a (necessarily even) $2n$ dimensional vector space

$$\begin{aligned}(\mathbf{x},\mathbf{y}) &= (x^1 y^{n+1} - x^{n+1} y^1) + (x^2 y^{n+2} - x^{n+2} y^2) + \ldots + (x^n y^{2n} - x^{2n} y^n) \\ &= \mathbf{x}^T \mathbf{J} \mathbf{y}\end{aligned} \tag{2.7}$$

with $\mathbf{J} = \begin{bmatrix} 0 & \mathbf{I}_n \\ -\mathbf{I}_n & 0 \end{bmatrix}$ (Sattinger and Weaver 1986).

Matrices which leave this form invariant satisfy $\mathbf{A}^T \mathbf{J} \mathbf{A} = \mathbf{J}$ and constitute the non-compact symplectic group $Sp(2n)$.

Before $SE(3)$ can be classified, some further definitions are required.

Definition 2.12 Product group

If G_1 and G_2 are two groups, the **product group** $G = G_1 \times G_2$ is the set of pairs (g_1, g_2), g_1 in G_1 and g_2 in G_2 with the group law

$$(g_1, g_2)(g_1' g_2') = (g_1 g_1', g_2 g_2') \tag{2.8}$$

Definition 2.13 Semi-direct product

if $\forall g_1 \in G_1$, $\forall g_2 \in G_2$, $g_2 \xrightarrow{g_1} {}^{g_1}g_2$, the **semi-direct product** of G_2 by G_1 is the set of pairs (g_1, g_2) with the following group law (Normand 1980)

$$(g_1, g_2)(g_1' g_2') = (g_1 g_1', g_2\, {}^{g_1}g_2') \tag{2.9}$$

where

$$g_2\, {}^{g_1}g_2' \equiv g_1 g_2' + g_2 \tag{2.10}$$

It is now possible to classify $SE(3)$ as a semi-direct product of the abelian invariant subgroup of translations $T(3)$ by the classical subgroup $SO(3)$.

2.6 The Rotation Group $SO(3)$

It is easy to see why $\mathbf{A} \in SO(3)$ must be an orthogonal matrix. Consider the Euclidean metric if a rigid body undergoes a displacement:

$$\begin{aligned}\|\mathbf{x}, \mathbf{y}\|_2 &= \|(\mathbf{Ax} + \mathbf{d}), (\mathbf{Ay} + \mathbf{d})\|_2 \\ &= \left((\mathbf{x} - \mathbf{y})^T \mathbf{A}^T \mathbf{A} (\mathbf{x} - \mathbf{y})\right)^{1/2}\end{aligned} \tag{2.11}$$

For the metric to be invariant (i.e. for internal points to preserve their distances), then $\mathbf{A}^T\mathbf{A} = \mathbf{I}$. Therefore $\mathbf{A} \in SO(3)$ must be orthogonal (McCarthy 1990).

Every rotation $\mathbf{A} \in SO(3)$ can be parameterized by an axis of rotation $\mathbf{n}$ and the angle θ of rotation about this axis: $\mathbf{A} = (\mathbf{n}, \theta)$. It should be emphasized that this parameterization is intrinsic i.e. independent of any choice of basis. The axis requires two angles for its specification, so three parameters are needed to specify a general rotation; $SO(3)$ is a three parameter group (Sattinger and Weaver 1986).

The range of angle used in any parameterization is an important issue (Altmann 1986). It is shown later that it is important that the neighbourhood of the identity of $SO(3)$ (the zero angle) be contained in the range. For this reason, the range is normally represented, for some angle θ : $-\pi < \theta \leq \pi$. There are a number of choices for the parameterization. Here a rotation is taken in terms of rotations about a right handed orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

$$\mathbf{A}(\mathbf{e}_1, \theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \tag{2.12}$$

$$\mathbf{A}(\mathbf{e}_2, \theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \tag{2.13}$$

$$\mathbf{A}(\mathbf{e}_3,\theta)=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix} \quad (2.14)$$

This representation is weak in the sense that rotations do not commute but it has certain properties in the neighbourhood of the identity which makes it useful (Altmann 1986).

2.6.1 The Configuration Manifold Structure of $SO(3)$

To define the configuration manifold structure of $SO(3)$, symbol $\theta\mathbf{n}$ is introduced to represent a single vector parameter - a vector parallel to the rotation axis with modulus equal to the rotation angle (Altmann 1986). Then $\mathbf{A}(\theta\,\mathbf{n})$ is called the parametric point. By continuously varying the direction of $\mathbf{n}$ and angle θ, the parametric point traces a three ball D^3 of radius π. This defines the configuration manifold structure and is sometimes referred to as the parametric ball of $SO(3)$ (Samuel et al 1991).

Diametrically opposite points on the surface of the parametric ball D^3 are not distinguished from each other. One can imagine that each antipodal point is linked in some manner to its podal point in such a way that when a parametric point reaches the surface of the ball, it jumps back from it to the corresponding podal point.

The identity of $SO(3)$ is parameterized by the vector $(0\mathbf{n})$ i.e. the centre of the parametric ball. The motivation behind the choice of range is now clear - the identity and its neighbourhood are well away from the surface of D^3 and its eccentric topology (Altmann 1986).

In the next chapter, the infinitesimal properties of Lie groups are studied, that is, the properties of the group near the identity element. This leads in a natural way to the important concepts of the infinitesimal generator and Lie algebra. These concepts allow a formal description of the velocity of a rigid body.

Chapter 3

Calculus on Manifolds

3.1 Overview of the Chapter

In this Chapter, the calculus on manifolds is developed. This leads to the important concept of a tangent space on a manifold. The theory is linked at each stage to $SE(3)$ and the infinitesimal operators for the subgroups are derived. The Lie algebra for the Lie group $SE(3)$ is examined and is shown to be a semi-direct sum of the Lie algebra associated with $T(3)$ and $SO(3)$.

3.2 Tangent Vector

Before a tangent vector can be defined, some preliminary definitions are required.

Definition 3.1 Open curve

An **open curve** in an m-dimensional manifold M is a map $c: (a,b) \to M$ where (a,b) is an open interval such that $a < 0 < b$ (Nakahara 1990).

Referring to Figure 3-1, the number $a\,(b)$ may be $-\infty\,(+\infty)$ and 0 is included in the interval. On a chart (U,φ), a curve $c(t)$ has the coordinate presentation

$$x = \varphi\ c \qquad : \qquad \Re \to \Re^m \tag{3.1}$$

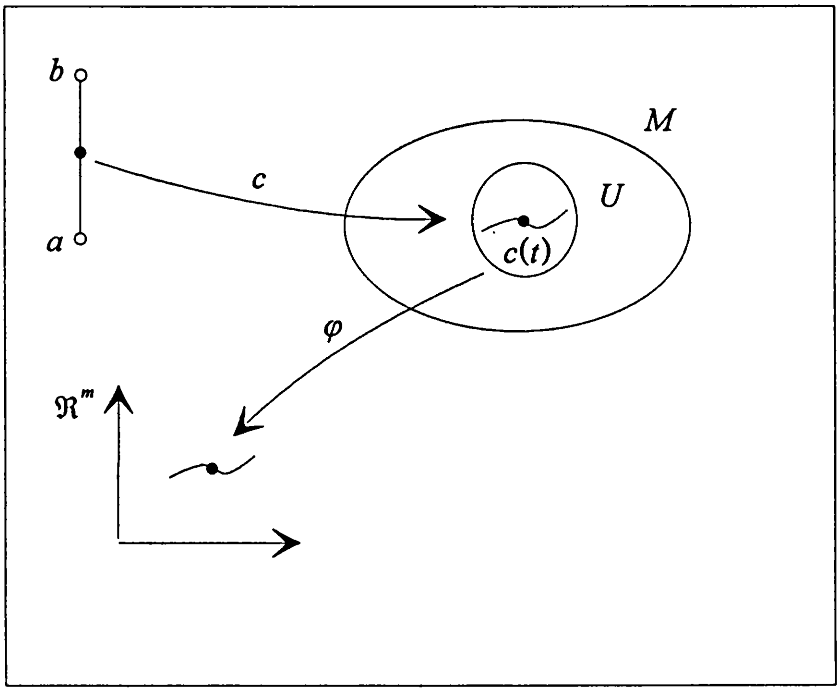

Figure 3-1: Open curve (adapted from Nakahara 1990)

Definition 3.2 Function

A **function** f on M is a smooth map from M to $\Re$ (Nakahara 1990).

On a chart (U,φ), the coordinate presentation of f is given by

$$f(p) = f\varphi^{-1}(x) \qquad : \qquad \Re^m \to \Re \tag{3.2}$$

which is real valued function of m variables.

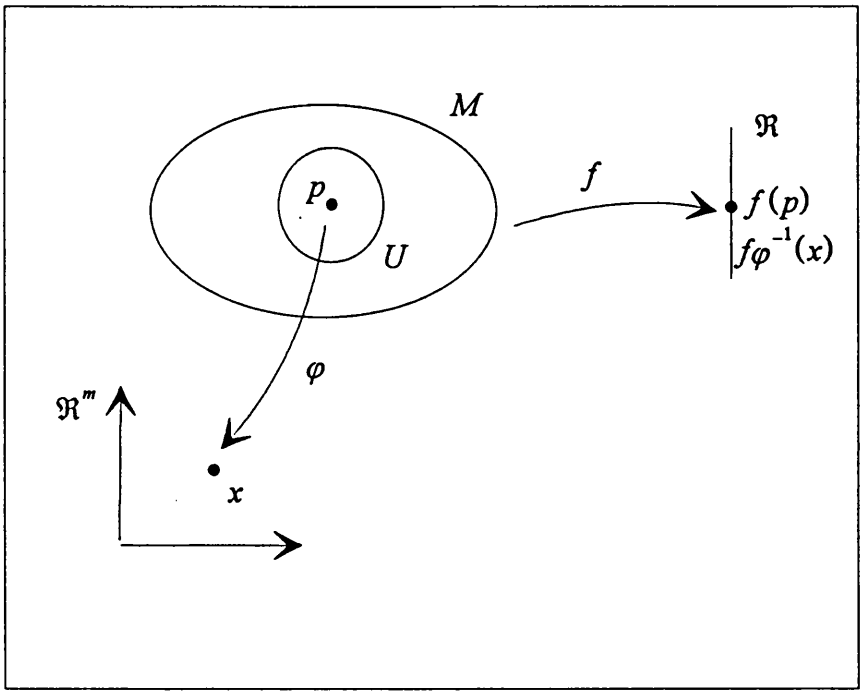

Figure 3-2: Function (adapted from Nakahara 1990)

Definition 3.3 Tangent Vector

To define a tangent vector, a curve $c:(a,b) \to M$ and a function $f: M \to \Re$ are required, where (a,b) is an open curve containing $t = 0$. A **tangent vector** at $c(0)$ is a directional derivative of a function $f(c(t))$ along the curve $c(t)$ at $t = 0$.

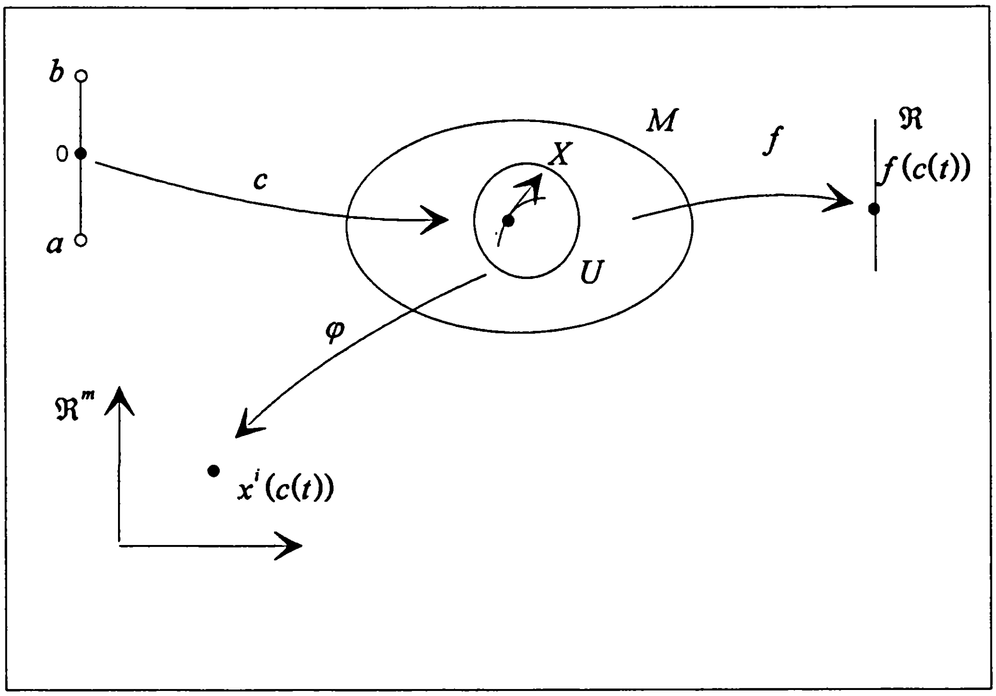

Figure 3-3: Tangent Vector (adapted from Nakahara 1990)

Definition 3.4 Tangent Space

All the tangent vectors at p form a vector space called the **tangent space** of M at p, denoted T_pM (Nakahara 1990). The tangent space of an m- dimensional manifold is m- dimensional.

For the configuration manifold of $SE(3)$, the tangent space at p comprises all the vectors tangent to $SE(3)$ at p, giving it the structure of a real six dimensional vector space, $\Re^6$ (Samuel et al 1991). The vectors of this vector space describe the velocities of a rigid body. The six elements $x^1 \ldots x^6$ correspond to the six velocity elements of a rigid body. The selection of a suitable chart φ is discussed in Chapter 4.

3.3 Infinitesimal Operator

Consider the simple case of a one parameter group in one variable x. Referring to Figure 3-4, the transformation $x' = f(x;a)$ takes all points of the space from their initial position x to the final position x'.

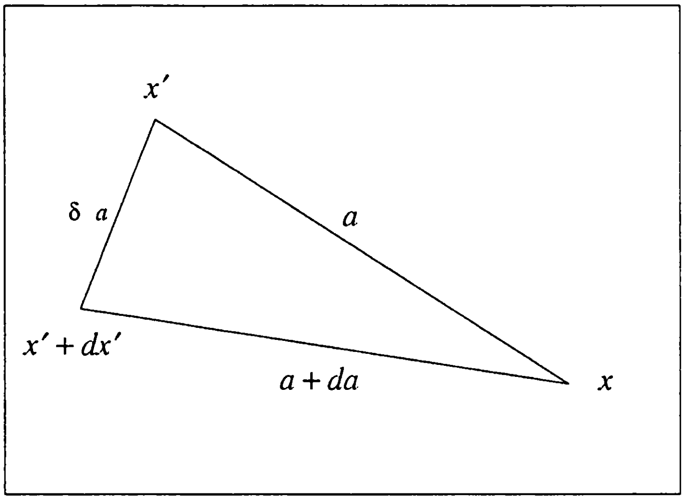

Figure 3-4: Infinitesimal Operator (adapted from Wybourne 1974)

The neighbouring parameter value $a + da$ will take the point x to $x' + dx'$ (if f is an analytic function of a). Thus there are two alternative paths from x to $x' + dx'$. Either $x' + dx' = f(x; a + da)$ or $x' + dx' = f(x'; \delta a)$ (Wybourne 1974).

Expanding the latter

$$dx' = \left. \frac{\partial f(x'; \delta a)}{\partial a} \right|_{a=0} \cdot \delta a \tag{3.3}$$

In the general case of m dimension and r parameters

$$(dx)' = \sum_{\sigma} \frac{\partial f^{i}(\mathbf{x}';\delta\ \mathbf{a})}{\partial a^{\sigma}}\bigg|_{\mathbf{a}=0} \cdot \delta\ a^{\sigma} \qquad i = 1 \ldots m \qquad \sigma = 1 \ldots r \tag{3.4}$$

or $$d\mathbf{x}' = \mathbf{U}^{i}_{\sigma}(\mathbf{x}) \cdot \delta\ a^{\sigma} \tag{3.5}$$

The infinitesimal transformation $\mathbf{x}' \to \mathbf{x}' + d\mathbf{x}'$ induces

$$f(\mathbf{x}') \to f(\mathbf{x}') + df(\mathbf{x}') \tag{3.6}$$

$$\begin{aligned} df(\mathbf{x}') &= \frac{\partial\ f}{(\partial\ x^{i})'}(dx^{i})' \\ &= \frac{\partial\ f}{(\partial\ x^{i})'}\mathbf{U}^{i}_{\sigma}\delta\ a^{\sigma} \end{aligned}$$

$$df(\mathbf{x}') = \delta\ a^{\sigma}\mathbf{X}_{\sigma}(f) \tag{3.7}$$

where $\mathbf{X}_{\sigma} = \mathbf{U}^{i}_{\sigma} \dfrac{\partial}{(\partial\ x^{i})'}$

It is now possible to define the infinitesimal operator of the group as

$$\mathbf{U}^{i}_{\sigma} \frac{\partial}{\partial\ x^{i}} \tag{3.8}$$

3.4 Infinitesimal Operator of $T(3)$

In the case of $T(3)$, and expressing matrices in the hyperplane

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & a_2 \\ 0 & 0 & 1 & a_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix} \tag{3.9}$$

$$\begin{bmatrix} x_1' + dx_1' \\ x_2' + dx_2' \\ x_3' + dx_3' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \delta a_1 \\ 0 & 1 & 0 & \delta a_2 \\ 0 & 0 & 1 & \delta a_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ 1 \end{bmatrix} \tag{3.10}$$

Therefore,

$$x_1' + dx_1' = x_1' + \delta a_1 \tag{3.11}$$

$$x_2' + dx_2' = x_2' + \delta a_2 \tag{3.12}$$

$$x_3' + dx_3' = x_3' + \delta a_3 \tag{3.13}$$

Taking derivatives

$$\frac{d}{d\,\delta a_1}(x_1' + \delta a_1) = 1 \tag{3.14}$$

and similarly for the other derivatives. Therefore

$$\begin{bmatrix} dx_1' \\ dx_2' \\ dx_3' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta a_1 \\ \delta a_2 \\ \delta a_3 \end{bmatrix} \tag{3.15}$$

So, for this group $\mathbf{U}_\sigma^i = \mathbf{I}$. Therefore, denoting the elements of the infinitesimal operator of the group by P_1, P_2 and P_3:

$$P_1 = \frac{\partial}{\partial x_1},\ P_2 = \frac{\partial}{\partial x_2} \text{ and } P_3 = \frac{\partial}{\partial x_3} \tag{3.16}$$

3.5 Infinitesimal Operator of $SO(3)$

Infinitesimal rotations are produced as follows:

$$\mathbf{A}(\mathbf{n}, d\theta) = \mathbf{I} + \delta A(\mathbf{n}, \theta) \tag{3.17}$$

where $\delta A(\mathbf{n}, \theta)$ is a matrix that has all its elements in the neighbourhood of zero.

For the transformation to preserve orthogonality

$$\begin{aligned} \mathbf{I} &= \mathbf{A}^T(\mathbf{n},\theta)\,\mathbf{A}(\mathbf{n},\theta) \\ &= (\mathbf{I} + \delta\mathbf{A}^T(\mathbf{n},\theta))(\mathbf{I} + \delta\mathbf{A}(\mathbf{n},\theta)) \end{aligned} \tag{3.18}$$

Ignoring second order terms,

$$\mathbf{I} = \mathbf{I} + \delta\mathbf{A}^T(\mathbf{n},\theta) + \delta\mathbf{A}(\mathbf{n},\theta) \tag{3.19}$$

Therefore

$$\delta\mathbf{A}^T(\mathbf{n},\theta) + \delta\mathbf{A}(\mathbf{n},\theta) = \mathbf{0} \tag{3.20}$$

Thus $\delta\mathbf{A}(\mathbf{n}\theta)$ must be a skew symmetric matrix with three components (Wybourne 1974).

$$\delta\mathbf{A}(\mathbf{n},\theta) = \begin{bmatrix} 0 & -\delta a_3 & \delta a_2 \\ \delta a_3 & 0 & -\delta a_1 \\ -\delta a_2 & \delta a_1 & 0 \end{bmatrix} \tag{3.21}$$

So

$$\begin{bmatrix} x_1' + dx_1' \\ x_2' + dx_2' \\ x_3' + dx_3' \end{bmatrix} = \begin{bmatrix} 1 & -\delta a_3 & \delta a_2 \\ \delta a_3 & 1 & -\delta a_1 \\ -\delta a_2 & \delta a_1 & 1 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} \tag{3.22}$$

$$x_1' + dx_1' = x_1' - \delta a_3 x_2' + \delta a_2 x_3' \tag{3.23}$$

$$x_2' + dx_2' = \delta a_3 x_1' + x_2' - \delta a_1 x_3' \tag{3.24}$$

$$x_3' + dx_3' = -\delta a_2 x_1' + \delta a_1 x_2' + x_3' \tag{3.25}$$

Taking partial derivatives

$$\frac{\partial}{\partial\,\delta a_3}(x_1' - \delta a_3 x_2' + \delta a_2 x_3') = -x_2' \qquad (3.26)$$

$$\frac{\partial}{\partial\,\delta a_3}(\delta a_3 x_1' + x_2' - \delta a_1 x_3') = x_1' \qquad (3.27)$$

$$\frac{\partial}{\partial\,\delta a_3}(-\delta a_2 x_1' + \delta a_1 x_2' + x_3') = 0 \qquad (3.28)$$

$$\frac{\partial}{\partial\,\delta a_2}(x_1' - \delta a_3 x_2' + \delta a_2 x_3') = x_3' \qquad (3.29)$$

$$\frac{\partial}{\partial\,\delta a_2}(\delta a_3 x_1' + x_2' - \delta a_1 x_3') = 0 \qquad (3.30)$$

$$\frac{\partial}{\partial\,\delta a_2}(-\delta a_2 x_1' + \delta a_1 x_2' + x_3') = -x_1' \qquad (3.31)$$

$$\frac{\partial}{\partial\,\delta a_1}(x_1' - \delta a_3 x_2' + \delta a_2 x_3') = 0 \qquad (3.32)$$

$$\frac{\partial}{\partial\,\delta a_1}(\delta a_3 x_1' + x_2' - \delta a_1 x_3') = -x_3' \qquad (3.33)$$

$$\frac{\partial}{\partial\,\delta a_1}(-\delta a_2 x_1' + \delta a_1 x_2' + x_3') = x_2' \qquad (3.34)$$

Therefore

$$\begin{bmatrix} dx_1' \\ dx_2' \\ dx_3' \end{bmatrix} = \begin{bmatrix} 0 & x_3' & -x_2' \\ -x_3' & 0 & x_1' \\ x_2' & -x_1' & 0 \end{bmatrix} \begin{bmatrix} \delta a_1 \\ \delta a_2 \\ \delta a_3 \end{bmatrix} \tag{3.35}$$

So, for this group

$$\mathbf{U}_\sigma^i = \begin{bmatrix} 0 & x_3' & -x_2' \\ -x_3' & 0 & x_1' \\ x_2' & -x_1' & 0 \end{bmatrix} \tag{3.36}$$

Therefore, denoting the elements of the infinitesimal operator of the group by R_1, R_2 and R_3:

$$R_1 = x_3' \frac{\partial}{\partial x_2} - x_2' \frac{\partial}{\partial x_3} \tag{3.37}$$

$$R_2 = -x_3' \frac{\partial}{\partial x_1} + x_1' \frac{\partial}{\partial x_3} \tag{3.38}$$

$$R_3 = x_2' \frac{\partial}{\partial x_1} - x_1' \frac{\partial}{\partial x_2} \tag{3.39}$$

3.6 Commutation Relations

A commutator is defined as follows:

$$[\mathbf{A},\mathbf{B}] = \mathbf{AB} - \mathbf{BA} \tag{3.40}$$

where [,] is known as the **Lie bracket**. Taking the infinitesimal operators for $SO(3)$:

$$\begin{aligned}[\mathbf{R}_1,\mathbf{R}_2] &= \mathbf{R}_1\mathbf{R}_2 - \mathbf{R}_2\mathbf{R}_1 \\ &= \left(x_3'\frac{\partial}{\partial x_2} - x_2'\frac{\partial}{\partial x_3}\right)\left(-x_3'\frac{\partial}{\partial x_1} + x_1'\frac{\partial}{\partial x_3}\right) - \left(-x_3'\frac{\partial}{\partial x_1} + x_1'\frac{\partial}{\partial x_3}\right)\left(x_3'\frac{\partial}{\partial x_2} - x_2'\frac{\partial}{\partial x_3}\right)\end{aligned} \tag{3.41}$$

Therefore

$$\begin{aligned}[\mathbf{R}_1,\mathbf{R}_2] &= x_2'\frac{\partial}{\partial x_1} - x_1'\frac{\partial}{\partial x_2} \\ &= \mathbf{R}_3\end{aligned} \tag{3.42}$$

Similarly, it is possible to compute the complete set commutation relations between R_1, R_2, R_3 and P_1, P_2 P_3. These relations are shown in the Table 3-1.

	$\mathbf{R}_1$	$\mathbf{R}_2$	$\mathbf{R}_3$	$\mathbf{P}_1$	$\mathbf{P}_2$	$\mathbf{P}_3$
$\mathbf{R}_1$	0	$\mathbf{R}_3$	$-\mathbf{R}_2$	0	$\mathbf{P}_3$	$-\mathbf{P}_2$
$\mathbf{R}_2$	$-\mathbf{R}_3$	0	$\mathbf{R}_1$	$-\mathbf{P}_3$	0	$\mathbf{P}_1$
$\mathbf{R}_3$	$\mathbf{R}_2$	$-\mathbf{R}_1$	0	$\mathbf{P}_2$	$-\mathbf{P}_1$	0
$\mathbf{P}_1$	0	$\mathbf{P}_3$	$-\mathbf{P}_2$	0	0	0
$\mathbf{P}_2$	$-\mathbf{P}_3$	0	$\mathbf{P}_1$	0	0	0
$\mathbf{P}_3$	$\mathbf{P}_2$	$-\mathbf{P}_1$	0	0	0	0

Table 3-1 Commutation relations

In general, the commutation relations can be expressed as

$$\left[\mathbf{R}_\sigma , \mathbf{R}_\rho\right] = c_{\sigma\rho}{}^{\tau}\, \mathbf{R}_\tau \qquad (3.43)$$

$$\left[\mathbf{R}_\sigma , \mathbf{P}_\rho\right] = c_{\sigma\rho}{}^{\tau}\, \mathbf{P}_\tau \qquad (3.44)$$

$$\left[\mathbf{P}_\sigma , \mathbf{P}_\rho\right] = 0 \qquad (3.45)$$

where $c_{\sigma\rho}{}^{\tau}$ are called the **structure constants** (Wybourne 1974).

The structure constants for each infinitesimal operator can be assembled into a matrix, $\mathbf{M}^{\tau}$. If the elements of $\mathbf{M}^{\tau}$ are set as follows

$$m_{ij}^{\tau} = -c_{ij}^{\tau} \tag{3.46}$$

then the matrix $\mathbf{M}^{\tau}$ is identical to the basis for the tangent vector.

3.7 Lie Algebra

Definition 3.5 Lie algebra

Let A be an r- dimensional vector space over field K in which the law of composition for vectors is such that to each pair of vectors $\mathbf{X}$ and $\mathbf{Y}$ there corresponds a vector $\mathbf{Z} = [\mathbf{X}, \mathbf{Y}]$ in such a way that

(i) $[\alpha\,\mathbf{X} + \beta\,\mathbf{Y}, \mathbf{Z}] = \alpha\,[\mathbf{X}, \mathbf{Z}] + \beta\,[\mathbf{Y}, \mathbf{Z}]$

(ii) $[\mathbf{X}, \mathbf{Y}] + [\mathbf{Y}, \mathbf{X}] = 0$

(iii) $[\mathbf{X}, [\mathbf{Y}, \mathbf{Z}]] + [\mathbf{Y}, [\mathbf{Z}, \mathbf{X}]] + [\mathbf{Z}, [\mathbf{X}, \mathbf{Y}]] = 0$ (Jacobi Identity)

for all $\alpha, \beta \in \mathrm{K}$ and all $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathbf{A}$.

A vector space $\mathbf{A}$ satisfying the above commutator relationships constitutes a **Lie algebra** (Wybourne 1974).

A given Lie algebra is said to be real if K is the field of real numbers and complex if K is the field of complex numbers. The Lie algebra associated with a Lie group is always real. For every Lie group there is a Lie algebra and for every Lie subgroup there is a subalgebra. The Lie algebra associated with a Lie group is denoted by the same letter as for the group, but in lowercase (Wybourne 1974).

The Lie algebra of $SE(3)$, denoted $se(3)$, is generated by the three infinitesimal rotation operators given in (3.37) to (3.39) and the three infinitesimal translation operators given in (3.16). For example, take $\mathbf{X}_1 = \mathbf{R}_1$, $\mathbf{Y} = \mathbf{R}_2$ and $\mathbf{Z} = \mathbf{P}_3$:

(i)
$$\begin{aligned}[][\alpha \mathbf{R}_1 + \beta \mathbf{R}_2, \mathbf{P}_3] &= [\alpha \mathbf{R}_1, \mathbf{P}_3] + [\beta \mathbf{R}_2, \mathbf{P}_3] \\ &= \alpha [\mathbf{R}_1, \mathbf{P}_3] + \beta [\mathbf{R}_2, \mathbf{P}_3]\end{aligned}$$

(ii)
$$\begin{aligned}[][\mathbf{R}_1, \mathbf{R}_2] + [\mathbf{R}_2, \mathbf{R}_1] &= \mathbf{R}_3 + (-\mathbf{R}_3) \\ &= 0\end{aligned}$$

(iii)
$$\begin{aligned}&[\mathbf{R}_1,[\mathbf{R}_2,\mathbf{P}_3]] + [\mathbf{R}_2,[\mathbf{P}_3,\mathbf{R}_1]] + [\mathbf{P}_3,[\mathbf{R}_1,\mathbf{R}_2]] \\ &= [\mathbf{R}_1,\mathbf{P}_1] + [\mathbf{R}_2,\mathbf{P}_2] + [\mathbf{P}_3,\mathbf{R}_3] = 0\end{aligned}$$

3.7.1 Lie subalgebras

A subset Ξ of a Lie algebra $\mathbf{A}$ is called a **subalgebra** of $\mathbf{A}$ if Ξ is a linear subspace of $\mathbf{A}$ and

$$[\mathbf{X}, \mathbf{Y}] \in \Xi \qquad \text{for any } (\mathbf{X}, \mathbf{Y} \in \Xi) \tag{3.47}$$

A subalgebra Ξ of $\mathbf{A}$ is said to **abelian** if

$$[\mathbf{X}, \mathbf{Y}] = 0 \qquad \text{for any } (\mathbf{X}, \mathbf{Y} \in \Xi) \tag{3.48}$$

The algebra $t(3)$ associated with the subgroup $T(3)$ is an abelian subalgebra of $se(3)$ (Wybourne 1974).

3.7.2 Ideals

A subset Ξ of $\mathbf{A}$ is said to form an **ideal** or invariant subalgebra of $\mathbf{A}$ if Ξ is a linear subspace of $\mathbf{A}$ and

$$[\mathbf{X}, \mathbf{Y}] \in \Xi \qquad \text{for any } (\mathbf{X}, \in \Xi, \mathbf{Y} \in \mathbf{A}) \tag{3.49}$$

If the algebra contains members that are not in the ideal, then the ideal is said to a **proper ideal**. In this case it is important to note that the identity element is always a member of the algebra. By restricting attention to proper ideals, the improper ideals formed by the whole algebra and by the subset containing the identity element are eliminated (Wybourne 1974).

Therefore, $t(3)$ forms a proper ideal of $se(3)$.

3.7.3 Simple and Semisimple Lie algebras

A Lie algebra is said to be **simple** if it contains no proper ideals. The algebra is said to be **semisimple** if it contains no abelian ideals except the subset containing the identity element (Wybourne 1974).

Although it is possible to assess $se(3)$ by inspection, it is useful to introduce a simple test for deciding if a Lie algebra is semisimple. The test is based on the Killing metric defined in terms of structure constants (Gilmore 1974).

Theorem 3.1   Cartan's test for a semisimple Lie algebra

A Lie algebra is semisimple if and only if

$$\det|g_{\sigma\lambda}| \neq 0 \tag{3.50}$$

where $g_{\sigma\lambda}$ is the symmetric Killing metric defined in terms of structure constants:

$$g_{\sigma\lambda} = c_{\sigma\rho}{}^{\tau} c_{\lambda\tau}{}^{\rho}$$

Proof of Theorem 3.1

see Wybourne (1974).

For $se(3)$, for example

$$\begin{aligned} g_{11} &= c_{12}{}^{3} c_{13}{}^{2} + c_{13}{}^{2} c_{12}{}^{3} + c_{15}{}^{6} c_{16}{}^{5} + c_{16}{}^{5} c_{15}{}^{6} \\ &= (1)(-1) + (-1)(1) + (1)(-1) + (-1)(1) \\ &= -4 \end{aligned} \tag{3.51}$$

Similarly $g_{22} = -4$ and $g_{33} = -4$. The remaining elements are zero. Therefore

$$g = \begin{bmatrix} -4\mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix} \tag{3.52}$$

Now $\det|g| = 0$, so the Lie algebra $se(3)$ is not semisimple.

However the Killing metric for $SO(3)$ is given by

$$g = [-2\mathbf{I}_3] \tag{3.53}$$

Now $\det|g| = -8$, so the Lie algebra $so(3)$ is semisimple.

3.7.4 Solveable Lie algebras

The derived algebra $\mathbf{A}^{(1)}$ of a Lie algebra $\mathbf{A}$ is formed by taking the set of all linear combinations of elements that can be expressed as commutations of the elements of $\mathbf{A}$

$$\mathbf{A}^{(1)} = [\mathbf{A}, \mathbf{A}] \tag{3.54}$$

It is possible to form a whole series of derived algebras. If the k th derived algebra $\mathbf{A}^{(k)} = [\mathbf{A}^{k-1}, \mathbf{A}^{k-1}]$, then the series

$$\mathbf{A}, \mathbf{A}^{(1)} \ldots \mathbf{A}^{(k)} \tag{3.55}$$

is called the derived series of the Lie algebra $\mathbf{A}$ (Wybourne 1974).

If for some positive integer k,

$$\mathbf{A}^{(k)} = 0 \tag{3.56}$$

the Lie algebra $\mathbf{A}$ is said to be a **solveable** Lie algebra. $SE(3)$ does not have a solveable Lie algebra but $T(3)$ does since $P^{(1)} = [P, P] = 0$.

3.7.5 Direct and Semidirect Sums

A Lie algebra $\mathbf{A}$ is a **direct sum** of Lie subalgebras

$$\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2 \oplus \ldots \oplus \mathbf{A}_n \tag{3.57}$$

if for every pair of subalgebras $\mathbf{A}_i, \mathbf{A}_j$

$$\mathbf{A}_i \cap \mathbf{A}_j = 0 \tag{3.58}$$

Any Lie algebra $\mathbf{A}$ can be written as a **semidirect sum**

$$\mathbf{A} = \mathbf{A}_1 \oplus_s \mathbf{A}_2 \tag{3.59}$$

of a solveable Lie algebra $\mathbf{A}_1$ and a semi-simple Lie algebra $\mathbf{A}_2$.

3.7.6 Classification of $se(3)$

The Lie algebra $se(3)$ cannot be expressed as a direct sum but can be expressed as a semidirect sum of the solveable Lie algebra $t(3)$ and the semi-simple Lie algebra $so(3)$:

$$se(3) = t(3) \oplus_s so(3) \tag{3.60}$$

3.8 Infinitesimal Generators

The basis for the tangent vector associated with the group $T(3)$ is

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{3.61}$$

$$\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{3.62}$$

$$\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{3.63}$$

These are known as the **infinitesimal generators** of the group since

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 \tag{3.64}$$

The bases for the tangent vector associated with the group $SO(3)$ can be found by considering (2.12) - (2.14) and (3.21) and noting that for an infinitesimal change, $\sin\theta \to \delta\theta$ and $\cos\theta \to 1$.

$$\mathbf{e}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{3.65}$$

$$\mathbf{e}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \tag{3.66}$$

$$\mathbf{e}_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.67}$$

Again, these are the infinitesimal generators of the group since

$$\mathbf{w} = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2 + \omega_3 \mathbf{e}_3 \tag{3.68}$$

3.9 Induced Maps on Manifolds

It is now possible to formally define a submanifold, which has been postponed from Chapter 2 until now. First some additional terminology is required.

A smooth map $f: M \to N$ naturally induces a map f_* called the differential map

$$f_*: T_p M \to T_{f(p)} N \tag{3.69}$$

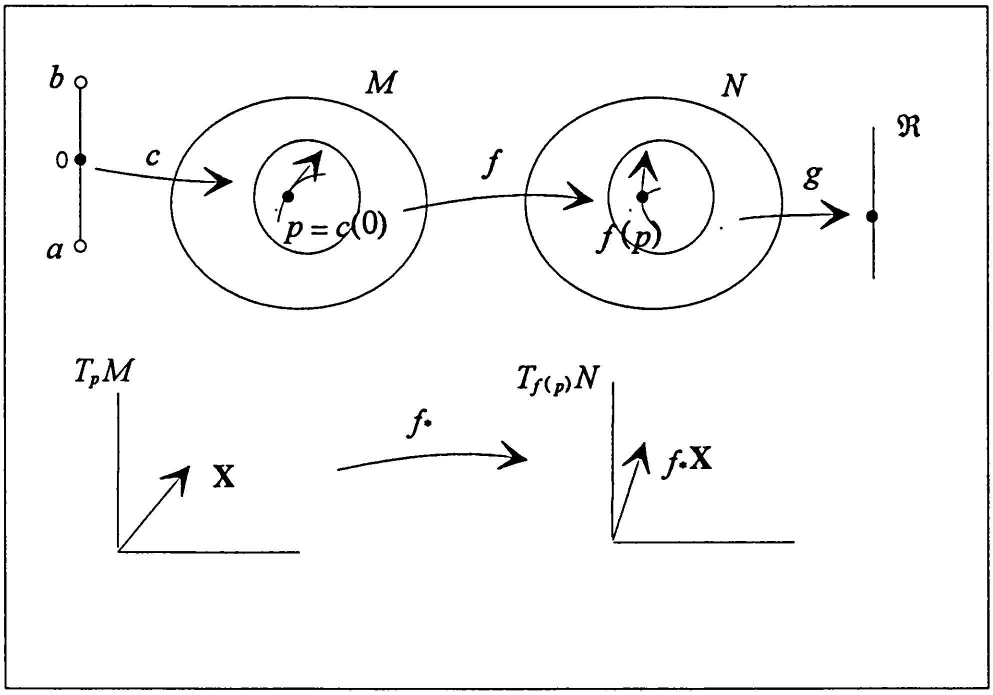

Figure 3-5 Induced map on manifolds (adapted from Nakahara 1990)

Definition 3.6 Submanifold

Let $f: M \to N$ be a smooth map and let $\dim M \leq \dim N$. The map f is called an immersion of M into N if $f_*: T_pM \to T_{f(p)}N$ is an injection (one-to-one map). The map f is called an embedding if f is an injection and an immersion. The image $f(M)$ is called a **submanifold** of N. $f(M)$ thus defined is diffeomorphic to M (Nakahara 1990).

Therefore a rigid body B is confirmed as an open submanifold of E without boundary of dimension three.

Chapter 4

Representation of the Lie algebra of the Special Euclidean Group

4.1 Overview of the Chapter

In this chapter the matrix representation of $se(3)$ is introduced. Representations in an inertial and body fixed frame are developed. The effect of a displacement of the reference frame leads to the important concept of the differential map. This has relevance in both fixed frame representations and moving frame representations. Lastly, an original review of the application of the theory of Lie groups in robotics is given, with particular reference to the exponential map.

4.2 Matrix representation of $se(3)$

For the group $SE(3)$, and using the map given in (2.4)

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{d} \tag{4.1}$$

Differentiating both sides of (4.1) with respect to time

Differentiating both sides of (4.1) with respect to time

$$\begin{aligned}\dot{\mathbf{x}}' &= \dot{\mathbf{A}}\mathbf{x} + \dot{\mathbf{d}} \\ &= \dot{\mathbf{A}}\mathbf{A}^T(\mathbf{x}' - \mathbf{d}) + \dot{\mathbf{d}}\end{aligned} \tag{4.2}$$

where $\dot{\mathbf{A}}\mathbf{A}^T$ is the angular velocity of the rigid body relative to an inertial frame, and $-\dot{\mathbf{A}}\mathbf{A}^T\mathbf{d} + \dot{\mathbf{d}}$ is the translational velocity of a point on the rigid body as it passes through the origin of the inertial frame. This is referred to as the **inertial representation** (Park and Brockett 1994).

In hyperplane matrix form

$$\begin{aligned}\begin{bmatrix}\dot{\mathbf{x}}' \\ 1\end{bmatrix} &= \begin{bmatrix}\dot{\mathbf{A}} & \dot{\mathbf{d}} \\ \mathbf{0} & 0\end{bmatrix}\begin{bmatrix}\mathbf{A}^T & -\mathbf{A}^T\mathbf{d} \\ \mathbf{0} & 1\end{bmatrix}\begin{bmatrix}\mathbf{x}' \\ 1\end{bmatrix} \\ &= \begin{bmatrix}\dot{\mathbf{A}}\mathbf{A}^T & -\dot{\mathbf{A}}\mathbf{A}^T\mathbf{d} + \dot{\mathbf{d}} \\ \mathbf{0} & 0\end{bmatrix}\begin{bmatrix}\mathbf{x}' \\ 1\end{bmatrix}\end{aligned} \tag{4.3}$$

Denoting

$$\dot{\mathbf{X}} = \begin{bmatrix}\dot{\mathbf{A}} & \dot{\mathbf{d}} \\ \mathbf{0} & 0\end{bmatrix} \text{ and } \mathbf{X}^{-1} = \begin{bmatrix}\mathbf{A}^T & -\mathbf{A}^T\mathbf{d} \\ \mathbf{0} & 1\end{bmatrix}$$

the inertial velocity representation can be written as $\dot{\mathbf{X}}\mathbf{X}^{-1}$. An alternative representation is the **body fixed representation** written as $\mathbf{X}^{-1}\dot{\mathbf{X}}$ (Park and Brockett 1994).

In this case

$$\begin{bmatrix} \dot{\mathbf{x}}' \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{A}^T & -\mathbf{A}^T\mathbf{d} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{A}} & \dot{\mathbf{d}} \\ \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}' \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{A}^T\dot{\mathbf{A}} & \mathbf{A}^T\dot{\mathbf{d}} \\ \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}' \\ 1 \end{bmatrix} \tag{4.4}$$

$\mathbf{A}^T\dot{\mathbf{A}}$ is the angular velocity of the rigid body relative to the instantaneous body frame and $\mathbf{A}^T\dot{\mathbf{d}}$ is the translational velocity of the rigid body relative to the instantaneous body frame. This representation is also known as a left invariant vector field on $SE(3)$ and is the more natural choice for analysis of the shared control of rigid bodies.

Taking the group $SO(3)$, the orthogonality condition is $\mathbf{A}^T\mathbf{A} = \mathbf{I}$. Differentiating both sides

$$\mathbf{A}^T\dot{\mathbf{A}} + \dot{\mathbf{A}}^T\mathbf{A} = \mathbf{0} \tag{4.5}$$

Therefore $\mathbf{A}^T\dot{\mathbf{A}}$ is a skew symmetric matrix, denoted Ω. This means that the combination $[\Omega \;\; \mathbf{A}^T\dot{\mathbf{d}}]$ which consists of a skew symmetric matrix and a vector, determines by its values at a certain instant, the velocity of all points at that instant. The bases for Ω are given in (3.65) - (3.67) and the bases for $\mathbf{A}^T\dot{\mathbf{d}}$ are given in (3.61) - (3.63). Therefore

$$\mathbf{X}^{-1}\dot{\mathbf{X}} = \begin{bmatrix} \Omega & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix} \tag{4.6}$$

where $\Omega = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$ (4.7)

$$\mathbf{v} = \mathbf{A}^T \dot{\mathbf{d}} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad (4.8)$$

The skew symmetric form is so useful that a special notation is used as follows

$$\lfloor \mathbf{x} \rfloor \equiv \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \quad (4.9)$$

Therefore

$$\hat{\mathbf{g}} = \mathbf{X}^{-1}\dot{\mathbf{X}} = \begin{bmatrix} \lfloor \mathbf{w} \rfloor & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix} \qquad \hat{\mathbf{g}} \in se(3) \quad (4.10)$$

It can be convenient to construct $\hat{\mathbf{g}}$ as a column vector of dimension six.

$$\hat{\mathbf{g}} = \begin{bmatrix} \mathbf{w} \\ \mathbf{v} \end{bmatrix}$$

This representation is sometimes referred to as a **screw**. This representation should be treated with some caution. To explain this, it is appropriate to be more specific about our definition of $\Re^6$. In this thesis, $\Re^6$ is used solely to denote the real six dimensional vector space and not a six dimensional Euclidean metric space. Since all real vector spaces of the same dimension are isomorphic, $se(3)$ is isomorphic to $\Re^6$ (Selig 1995). However, $se(3)$ is not a six dimensional Euclidean metric space. This distinction is often missed and can lead to erroneous use of metrics on the vector space.

4.3 Differential Map

The differential map is significant because its representation plays a crucial role in relating representations of vectors in $se(3)$ induced by different coordinate frames. Consider Figure 4-1.

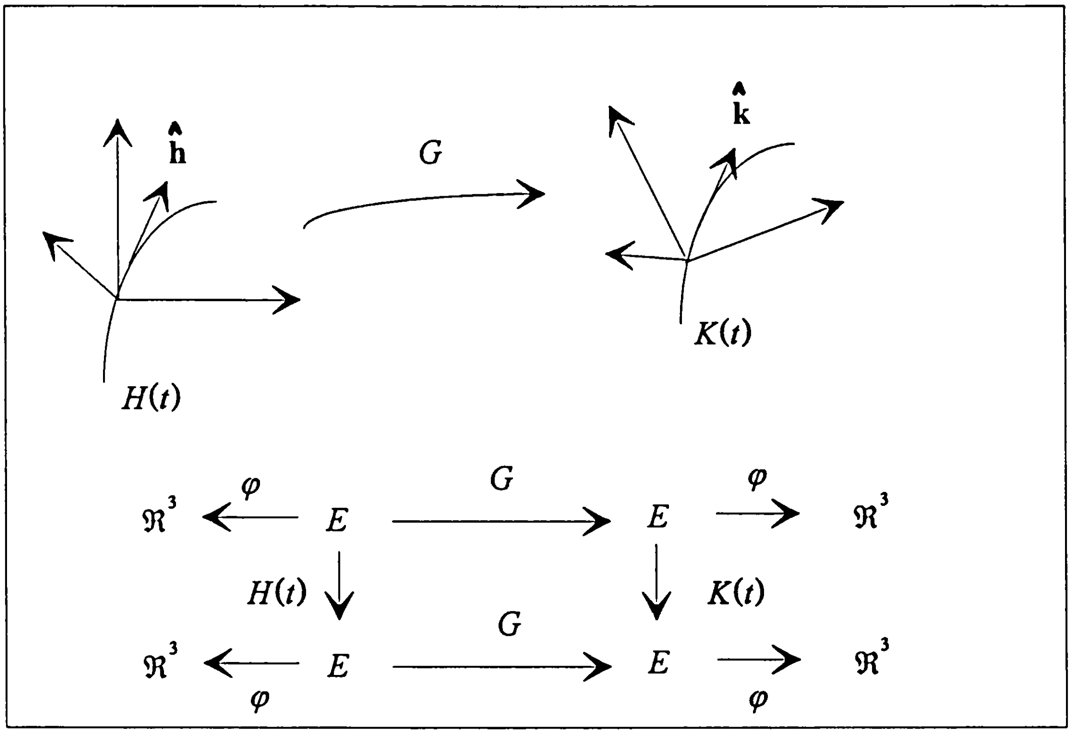

Figure 4-1: Active Transformation

$$K(t) = G\,H(t)\,G^{-1} \tag{4.11}$$

Any transformation of the form of (4.11) is known as a similarity transformation (Goldstein 1964).

If G has matrix representation

$$G = \begin{bmatrix} \mathbf{A}_1 & \mathbf{d}_1 \\ \mathbf{0} & 1 \end{bmatrix},\; G^{-1} = \begin{bmatrix} \mathbf{A}_1^T & -\mathbf{A}_1^T\mathbf{d}_1 \\ \mathbf{0} & 1 \end{bmatrix} \tag{4.12}$$

and $$H = \begin{bmatrix} \mathbf{A}_2 & \mathbf{d}_2 \\ \mathbf{0} & 1 \end{bmatrix} \tag{4.13}$$

The similarity transform is

$$\begin{aligned} K &= \begin{bmatrix} \mathbf{A}_1 & \mathbf{d}_1 \\ \mathbf{0} & 1 \end{bmatrix}\begin{bmatrix} \mathbf{A}_2 & \mathbf{d}_2 \\ \mathbf{0} & 1 \end{bmatrix}\begin{bmatrix} \mathbf{A}_1^T & -\mathbf{A}_1^T\mathbf{d}_1 \\ \mathbf{0} & 1 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{A}_1\mathbf{A}_2\mathbf{A}_1^T & -\mathbf{A}_1\mathbf{A}_2\mathbf{A}_1^T\mathbf{d}_1 + \mathbf{A}_1\mathbf{d}_2 + \mathbf{d}_1 \\ \mathbf{0} & 1 \end{bmatrix} \end{aligned} \tag{4.14}$$

$K \rightarrow G\,H\,G^{-1}$ is called a conjugation map. If $\hat{\mathbf{k}}$ is the tangent vector of $K(t)$ and $\hat{\mathbf{h}}$ is the tangent vector of $H(t)$, then

$$\hat{\mathbf{k}} = \hat{\mathbf{A}}_{\bullet}\hat{\mathbf{h}} \tag{4.15}$$

where $\hat{\mathbf{A}}_* \in \left\{ f_*: T_p M \rightarrow T_{f(p)} M \right\}$ is the differential of the conjugation map at the identity.

$$\hat{\mathbf{k}} = \begin{bmatrix} \mathbf{A} & \mathbf{d} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \lfloor \mathbf{w} \rfloor & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{A}^T & -\mathbf{A}^T \mathbf{d} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{A} \lfloor \mathbf{w} \rfloor \mathbf{A}^T & -\mathbf{A} \lfloor \mathbf{w} \rfloor \mathbf{d} + \mathbf{A} \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix} \tag{4.16}$$

Using the identities

1. $\mathbf{A} \lfloor \mathbf{w} \rfloor \mathbf{A}^T = \lfloor \mathbf{A} \mathbf{w} \rfloor$ (see A2.1) (4.17)

2. $-\lfloor \mathbf{A} \mathbf{w} \rfloor \mathbf{d} = \lfloor \mathbf{d} \rfloor \mathbf{A} \mathbf{w}$ (by inspection) (4.18)

$$\hat{\mathbf{k}} = \begin{bmatrix} \lfloor \mathbf{A} \mathbf{w} \rfloor & \lfloor \mathbf{d} \rfloor \mathbf{A} \mathbf{w} + \mathbf{A} \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix} \tag{4.19}$$

Expressed as a six vector

$$\hat{\mathbf{k}} = \begin{bmatrix} \mathbf{A} \mathbf{w} \\ \lfloor \mathbf{d} \rfloor \mathbf{A} \mathbf{w} + \mathbf{A} \mathbf{v} \end{bmatrix} \tag{4.20}$$

Re-arranging to produce an operator form

$$\hat{\mathbf{k}} = \begin{bmatrix} \mathbf{A} & 0 \\ \lfloor \mathbf{d} \rfloor \mathbf{A} & \mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{v} \end{bmatrix} \tag{4.21}$$

So $$\hat{\mathbf{A}}_{*} = \begin{bmatrix} \mathbf{A} & 0 \\ \lfloor \mathbf{d} \rfloor \mathbf{A} & \mathbf{A} \end{bmatrix} \tag{4.22}$$

This is referred to as the adjoint representation of the Lie algebra. Here a so-called active point of view is taken and assumes that G is transforming the tangent vector. An alternative point of view is the passive point of view where G is assumed to transform the reference frame and not the vector. This point of view will be considered next. Consider Figure 4.2.

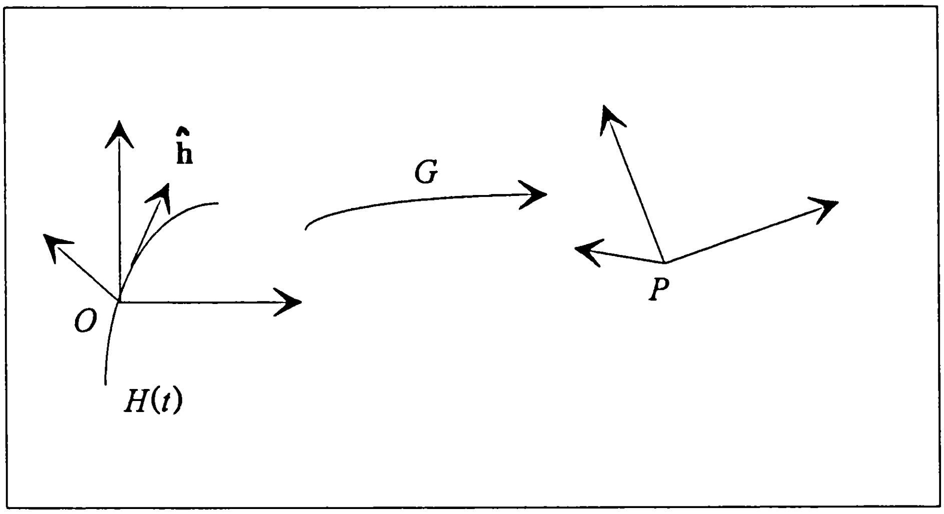

Figure 4-2: Passive Transformation

The tangent vector $\hat{\mathbf{h}}$ represented in reference frame O becomes

$$\hat{\mathbf{h}}_P = \mathbf{A}_*^{-1}\hat{\mathbf{h}}_O \tag{4.23}$$

when represented in reference frame P. Now

$$\begin{bmatrix} \mathbf{A}^T & 0 \\ -\mathbf{A}^T\lfloor \mathbf{d} \rfloor & \mathbf{A}^T \end{bmatrix}\begin{bmatrix} \mathbf{A} & 0 \\ \lfloor \mathbf{d} \rfloor \mathbf{A} & \mathbf{A} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^T\mathbf{A} & 0 \\ -\mathbf{A}^T\lfloor \mathbf{d} \rfloor \mathbf{A} + \mathbf{A}^T\lfloor \mathbf{d} \rfloor \mathbf{A} & \mathbf{A}^T\mathbf{A} \end{bmatrix} = \mathbf{I} \tag{4.24}$$

Therefore,

$$\hat{\mathbf{A}}_*^{-1} = \begin{bmatrix} \mathbf{A}^T & 0 \\ -\mathbf{A}^T\lfloor \mathbf{d} \rfloor & \mathbf{A}^T \end{bmatrix} \tag{4.25}$$

4.4 Differential Map in a Moving Reference Frame

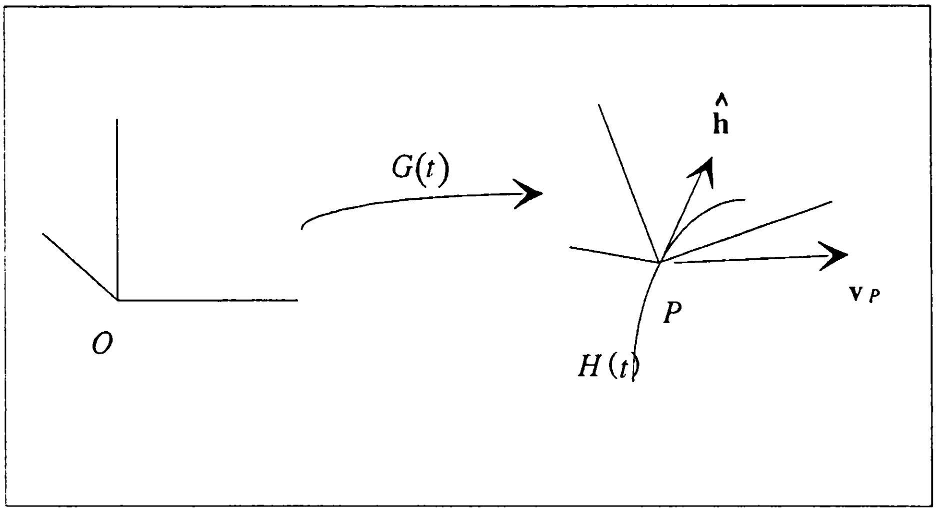

Figure 4-3: Moving Reference Frame

Consider Figure 4-3 where $\mathbf{v}_P$ is the velocity of frame P with respect to inertial frame O. The derivative of a moving tangent vector is formed by transforming the tangent vector to the inertial frame, differentiating, and then transforming back to the moving frame.

$$\begin{aligned}\frac{d}{dt}\hat{\mathbf{h}} &= \hat{\mathbf{A}}_*^{-1}\frac{d'}{dt}\left(\hat{\mathbf{A}}_*\hat{\mathbf{h}}\right) \\ &= \frac{d'}{dt}\hat{\mathbf{h}} + \hat{\mathbf{A}}_*^{-1}\left(\frac{d'}{dt}\hat{\mathbf{A}}_*\right)\hat{\mathbf{h}}\end{aligned} \tag{4.26}$$

where $\frac{d}{dt}$ denotes absolute differentiation and $\frac{d'}{dt}$ denotes apparent or component-wise differentiation (Featherstone 1987).

$$\frac{d'}{dt}\hat{\mathbf{A}}_* = \begin{bmatrix} \frac{d'}{dt}\mathbf{A} & 0 \\ \frac{d'}{dt}\lfloor\mathbf{d}\rfloor\mathbf{A} & \frac{d'}{dt}\mathbf{A}\end{bmatrix} \tag{4.27}$$

Using the identities

1. $\frac{d'}{dt}\mathbf{A} = \lfloor\mathbf{w}\rfloor\mathbf{A}$ (see A2.2) (4.28)

2. $\frac{d'}{dt}\lfloor\mathbf{d}\rfloor\mathbf{A} = \lfloor\mathbf{v}_O\rfloor\mathbf{A} + \lfloor\mathbf{w}\rfloor\lfloor\mathbf{d}\rfloor\mathbf{A}$ (see A2.3) (4.29)

$$\frac{d'}{dt}\hat{\mathbf{A}}_* = \begin{bmatrix} \lfloor \mathbf{w} \rfloor \mathbf{A} & 0 \\ \lfloor \mathbf{v}_O \rfloor \mathbf{A} + \lfloor \mathbf{w} \rfloor \lfloor \mathbf{d} \rfloor \mathbf{A} & \lfloor \mathbf{w} \rfloor \mathbf{A} \end{bmatrix} = \begin{bmatrix} \lfloor \mathbf{w} \rfloor & 0 \\ \lfloor \mathbf{v}_O \rfloor & \lfloor \mathbf{w} \rfloor \end{bmatrix} \begin{bmatrix} \mathbf{A} & 0 \\ \lfloor \mathbf{d} \rfloor \mathbf{A} & \mathbf{A} \end{bmatrix} \tag{4.30}$$

The differential map operator is denoted $\mathbf{ad}_{\hat{\mathbf{g}}}$. Therefore

$$\mathbf{ad}_{\hat{\mathbf{g}}}{}^{O} = \begin{bmatrix} \lfloor \mathbf{w} \rfloor & 0 \\ \lfloor \mathbf{v}_O \rfloor & \lfloor \mathbf{w} \rfloor \end{bmatrix} \tag{4.31}$$

Therefore

$$\frac{d}{dt}\hat{\mathbf{h}} = \frac{d'}{dt}\hat{\mathbf{h}} + \hat{\mathbf{A}}_*^{-1}\mathbf{ad}_{\hat{\mathbf{g}}}{}^{O}\hat{\mathbf{A}}_*\hat{\mathbf{h}} \tag{4.32}$$

4.5 Lie Bracket

It can be shown that $\mathbf{ad}_{\hat{\mathbf{g}}}$ is given by the Lie Bracket (Price 1977).

$$[\hat{\mathbf{g}}_1, \hat{\mathbf{g}}_2] = \begin{bmatrix} \lfloor \mathbf{w}_1 \rfloor & \mathbf{v}_1 \\ \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} \lfloor \mathbf{w}_2 \rfloor & \mathbf{v}_2 \\ \mathbf{0} & 0 \end{bmatrix} - \begin{bmatrix} \lfloor \mathbf{w}_2 \rfloor & \mathbf{v}_2 \\ \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} \lfloor \mathbf{w}_1 \rfloor & \mathbf{v}_1 \\ \mathbf{0} & 0 \end{bmatrix} \tag{4.33}$$

Using the identity (by inspection)

1. $\lfloor \mathbf{w}_1 \rfloor \lfloor \mathbf{w}_2 \rfloor - \lfloor \mathbf{w}_2 \rfloor \lfloor \mathbf{w}_1 \rfloor = \lfloor \lfloor \mathbf{w}_1 \rfloor \mathbf{w}_2 \rfloor$ (4.34)

Therefore

$$[\hat{\mathbf{g}}_1, \hat{\mathbf{g}}_2] = \begin{bmatrix} \lfloor \lfloor \mathbf{w}_1 \rfloor \mathbf{w}_2 \rfloor & \lfloor \mathbf{w}_1 \rfloor \mathbf{v}_2 - \lfloor \mathbf{w}_2 \rfloor \mathbf{v}_1 \\ \mathbf{0} & 0 \end{bmatrix} \tag{4.35}$$

or represented as a six vector and using the identity (by inspection)

$$-\lfloor \mathbf{w}_2 \rfloor \mathbf{v}_1 = \lfloor \mathbf{v}_1 \rfloor \mathbf{w}_2 \tag{4.36}$$

we have

$$[\hat{\mathbf{g}}_1, \hat{\mathbf{g}}_2] = \begin{bmatrix} \lfloor \mathbf{w}_1 \rfloor \mathbf{w}_2 \\ \lfloor \mathbf{v}_1 \rfloor \mathbf{w}_2 - \lfloor \mathbf{w}_1 \rfloor \mathbf{v}_2 \end{bmatrix} \tag{4.37}$$

or

$$\begin{aligned} [\hat{\mathbf{g}}_1, \hat{\mathbf{g}}_2] &= \begin{bmatrix} \lfloor \mathbf{w}_1 \rfloor & \mathbf{0} \\ \lfloor \mathbf{v}_1 \rfloor & \lfloor \mathbf{w}_1 \rfloor \end{bmatrix} \begin{bmatrix} \mathbf{w}_2 \\ \mathbf{v}_2 \end{bmatrix} \\ &= \mathbf{ad}_{\hat{\mathbf{g}}}(\hat{\mathbf{g}}_1)(\hat{\mathbf{g}}_2) \end{aligned} \tag{4.38}$$

4.6 Exponential Map

The following differential equation holds for inertial velocity representation:

$$\dot{G} = \begin{bmatrix} \lfloor \mathbf{w} \rfloor & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{d} \\ \mathbf{0} & 1 \end{bmatrix} = \hat{\mathbf{g}} G(t) \tag{4.39}$$

The solution is

$$G(t) = \exp(t\,\hat{\mathbf{g}}) \tag{4.40}$$

which assumes that the fixed and moving frames coincide at the instant $t = 0$. i.e $G(0) = \mathbf{I}$. Therefore a chart in the neighbourhood of the identity in $SE(3)$ can be obtained from a basis in $se(3)$ through the exponential map:

$$\text{EXP:} \quad se(3) \rightarrow SE(3)$$

Integral curves (see Figure 4-4) describe motion in E given by

$$\int EXP(\hat{\mathbf{g}}t) \quad \in SE(3) \tag{4.41}$$

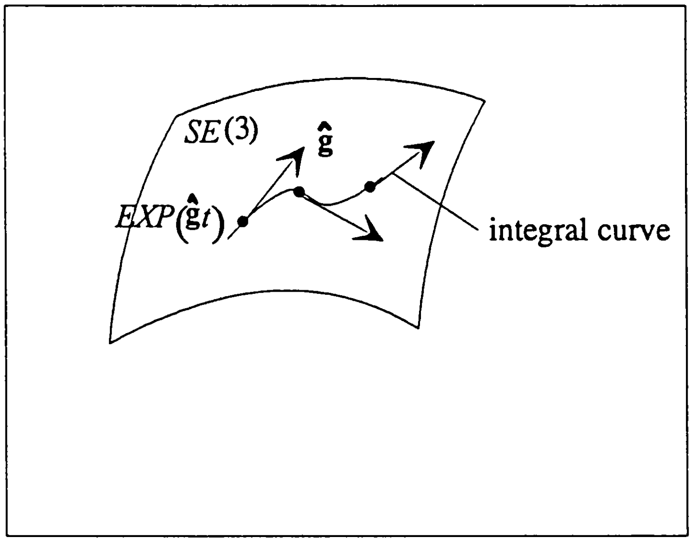

Figure 4-4: Integral Curve

The rotational component ω of $\hat{\mathbf{g}}$ can be interpreted as the curl of $U(\hat{\mathbf{g}})$ where $U(\hat{\mathbf{g}})$ is a vector field on E defined by $\hat{\mathbf{g}}$. The translational component $\mathbf{v}$ of $\hat{\mathbf{g}}$ only has intrinsic meaning if the direction is parallel to the axis of rotation. Perpendicular components of $\mathbf{v}$ depend on the choice of the chart origin in E (Loncaric 1985).

So the exponentiation map gives a chart in the neighbourhood of the identity in $SE(3)$. In fact, the map is a diffeomorphism of $\mathbf{0}$ in $se(3)$ and a neighbourhood of the identity in $SE(3)$.

$$\text{Let } \hat{\mathbf{g}} = \begin{bmatrix} \lfloor \mathbf{w} \rfloor & \mathbf{v} \\ 0 & 0 \end{bmatrix} \qquad \lfloor \mathbf{w} \rfloor \in so(3) \text{ and } \mathbf{v} \in t(3) \tag{4.42}$$

Then the exponential of $\hat{\mathbf{g}}$ is (Park and Bobrow 1995)

$$EXP(\hat{\mathbf{g}}) = \begin{bmatrix} e^{\lfloor \mathbf{w} \rfloor} & \mathbf{A}\mathbf{v} \\ \mathbf{0} & 1 \end{bmatrix} \tag{4.43}$$

where

$$e^{\lfloor \mathbf{w} \rfloor} = \mathbf{I} + \frac{\sin\|\mathbf{w}\|}{\|\mathbf{w}\|} \lfloor \mathbf{w} \rfloor + \frac{1-\cos\|\mathbf{w}\|}{\|\mathbf{w}\|^2} \lfloor \mathbf{w} \rfloor^2 \tag{4.44}$$

$$\mathbf{A} = \mathbf{I} + \frac{1-\cos\|\mathbf{w}\|}{\|\mathbf{w}\|^2} \lfloor \mathbf{w} \rfloor + \frac{\|\mathbf{w}\| - \sin\|\mathbf{w}\|}{\|\mathbf{w}\|^3} \lfloor \mathbf{w} \rfloor^2 \tag{4.45}$$

and

$$\|\mathbf{w}\|^2 = \omega_1^2 + \omega_2^2 + \omega_3^2 \tag{4.46}$$

Matrix representations can be derived easily. For example, $e^{\lfloor \mathbf{w}\theta \rfloor} \in SO(3)$ with $\|\mathbf{w}\| = 1$ can be expressed as follows (see A2.4 for proof):

$$e^{\lfloor \mathbf{w}\theta \rfloor} = \begin{bmatrix} \omega_1^2 v + c & \omega_1\omega_2 v - \omega_3 s & \omega_1\omega_3 v + \omega_2 s \\ \omega_1\omega_2 v + \omega_3 s & \omega_2^2 v + c & \omega_2\omega_3 v - \omega_1 s \\ \omega_1\omega_3 v - \omega_2 s & \omega_2\omega_3 v + \omega_1 s & \omega_3^2 v + c \end{bmatrix} \tag{4.47}$$

where $s \equiv \sin(\theta)$, $c \equiv \cos(\theta)$ and $v \equiv 1 - \cos(\theta)$.

If we let $G_{\mathbf{A}}(0)$ represent the initial configuration of a rigid body relative to a frame $\mathbf{A}$, then the final configuration still with respect to frame $\mathbf{A}$ is given by:

$$G_{\mathbf{A}}(\theta) = e^{\hat{\mathbf{g}}\theta} G_{\mathbf{A}}(0) \tag{4.48}$$

Thus the exponential map gives a relative motion of a rigid body.

The exponential map is extremely important in the geometry of robotics and manipulation. An example of how it can be used is given in the next section.

4.7 The Product of Exponentials Formula

The product of exponentials formula gives an expression for the forward kinematics map of an open-chain robot in terms of relative transformations between adjacent link frames.

Consider Figure 4-5. The joint manifold structure arises as follows. A suitable chart is taken where ϑ_j denotes the joint rotation angle for the j th revolute joint of a robot. This is termed multi-parameter motion (compared to a single parameter motion denoted by $G \in H(4)$) If the motion is not limited by mechanical stops, ϑ_j can take on all values in the interval $(-\pi\ \pi]$. Therefore, the j th joint space manifold is a circle, denoted S^1 (Burdick 1989).

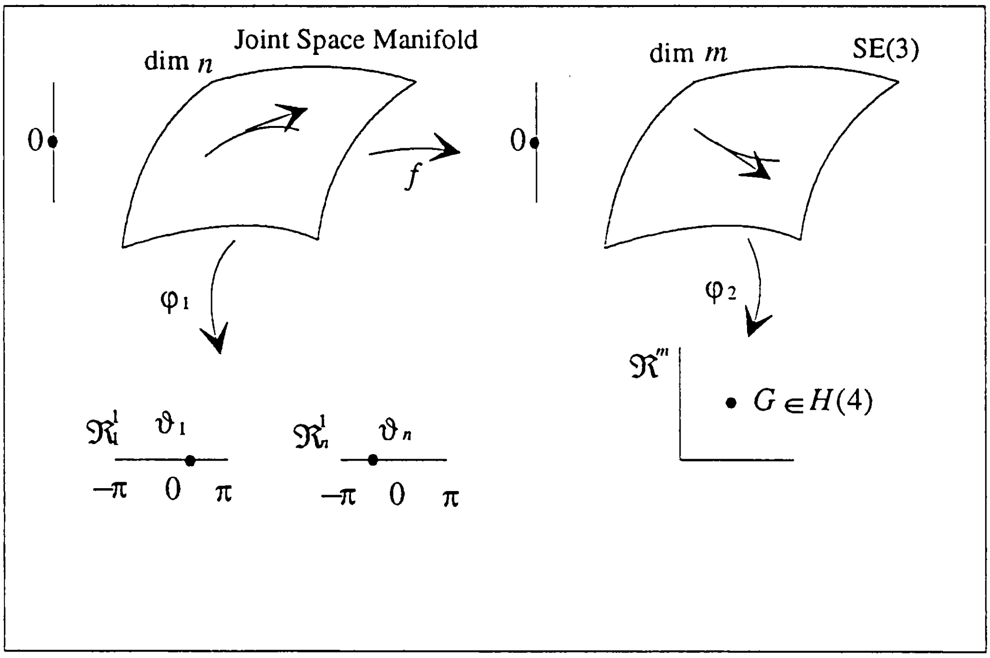

Figure 4-5: Mapping from Joint Space to $SE(3)$

The joint space manifold of a n-revolute robot is a product space formed by the n times product of the individual joint manifolds.

$$T^n = S^1 \times \ldots \times S^1 \tag{4.49}$$

where T^n is an n-torus. For example, a 2R planar robot has a 2-torus joint space manifold, see Figure 4-6.

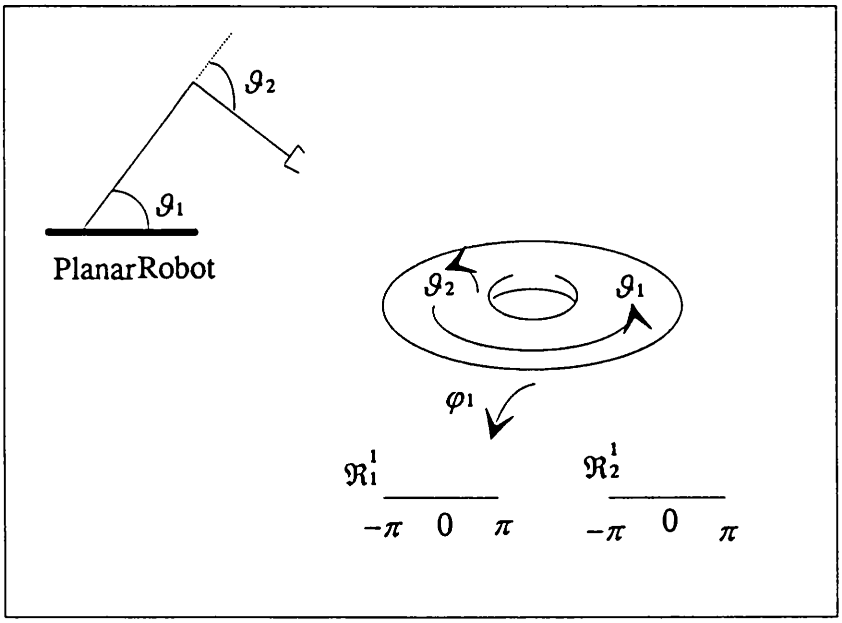

Figure 4-6: Joint Space Manifold for a 2R Planar Robot

The forward kinematics map is given by

$$f: \; T^n \rightarrow SE(3) \tag{4.50}$$

Take a 2R planar robot for example. If the first joint is fixed, then

$$f(\theta_2) = e^{\hat{\mathbf{g}}_2\theta_2} f(\mathbf{0}) \tag{4.51}$$

where $f(\mathbf{0})$ represents the rigid body transformation from the tool frame to the base frame at the reference configuration $\theta_1 = \theta_2 = 0$.

$\hat{\mathbf{g}}_2$ represents a screw corresponding to rotation around the second joint:

$$\hat{\mathbf{g}}_2 = \begin{bmatrix} \mathbf{w}_2 \\ -\lfloor \mathbf{d} \rfloor \mathbf{w}_2 \end{bmatrix} \tag{4.52}$$

where $\mathbf{w}_2$ is a unit vector in the direction of the screw axis and $\mathbf{d}$ is the displacement from the base frame to any point on the screw axis (Paden and Sastry 1988). Similarly, if θ_2 is fixed and θ_1 is moved:

$$f(\theta_1 \theta_2) = e^{\hat{\mathbf{g}}_1\theta_1} f(\theta_2) \tag{4.53}$$

where $\hat{\mathbf{g}}_1$ is the screw associated with the first joint. Therefore

$$f(\theta_1 \theta_2) = e^{\hat{\mathbf{g}}_1\theta_1} e^{\hat{\mathbf{g}}_2\theta_2} f(\mathbf{0}) \tag{4.54}$$

This process can be generalized to find the forward kinematics map for an arbitrary n-revolute robot:

$$f(\theta) = e^{\hat{\mathbf{g}}_1\theta_1} \cdots e^{\hat{\mathbf{g}}_n\theta_n} f(\mathbf{0}) \tag{4.55}$$

This is known as the product of exponentials (POE) formula for the robot forward kinematics map (Paden and Sastry 1988) (Brockett 1988). One advantage that the POE formula has over the Denavit-Hartenberg parameters is that the latter are extremely sensitive to small kinematic variations when neighbouring joint axes are nearly parallel. On the other hand, $\hat{\mathbf{g}}_i$ in the POE formula vary smoothly with variations in the joint axes (Park and Bobrow 1995).

The mapping of the tangent spaces induced by f :

$$f_*: T_p M \to T_{f(p)} M \tag{4.56}$$

is known as the **jacobean** of the mapping f. In robotics, the spatial manipulator jacobean, denoted $\mathbf{J}$, is thought of as a linear transformation relating joint velocities to the generalized Cartesian velocities of the end effector, expressed with respect to some reference frame.

One of the advantages of the POE formula is the compact expression for the jacobean. Suppose $\theta(t)$ is a trajectory in joint space, and $f(t)$ is the corresponding trajectory in $SE(3)$ as given by the forward kinematics map. The generalized velocity of the tool frame with respect to the base reference frame is given by $\dot{f}(t)f^{-1}(t)$.

Therefore (Park and Pack 1991):

$$\dot{f}(t)f^{-1}(t) = \hat{\mathbf{g}}_1\dot{\theta}_1 + e^{\hat{\mathbf{g}}_1\theta_1}\hat{\mathbf{g}}_2 e^{-\hat{\mathbf{g}}_1\theta_1}\dot{\theta}_2 + \ldots \tag{4.57}$$

This can be expressed in conventional matrix representation, since $\dot{f}(t)f^{-1}(t)$ is a matrix in $se(3)$ and the right hand side can be re-arranged as a linear transformation of the joint velocity vector $(\dot{\theta}_1(t)\ldots\dot{\theta}_n(t))$ by the $6\times n$ matrix $\mathbf{J}(\theta)$.

Now we have a sensible notation, we can go on to examine the issues of shared control. An important issue is the use of metrics on vector spaces. This will be examined in the next chapter.

Chapter 5

Riemannian Metrics

5.1 Overview of the Chapter

In this chapter, the crucial concept of Riemannian metrics is examined and shown to be a generalization of the inner product at each point on a manifold. Metrics will be used in the next chapter in the derivation of filters for shared control. A metric that is invariant to changes in reference frame is defined as an $\hat{\mathbf{A}}_{\bullet}$ - invariant metric. The general form of the $\hat{\mathbf{A}}_{\bullet}$ - invariant metric for rigid bodies is derived. An important left invariant Riemannian metric is examined - the kinetic energy metric.

5.2 Dual Vector

Before metrics can be examined, some further concepts are required. Let $\mathbf{V} = \mathrm{V}(n, \mathrm{K})$ be a vector space with basis $\mathbf{e}_1 \ldots \mathbf{e}_n$. In simple linear algebra, a dual vector space $\mathrm{V}^*(n, \mathrm{K})$ has a basis $\mathbf{e}^{*1} \ldots \mathbf{e}^{*n}$ such that

$$\mathbf{e}^{*i}(\mathbf{e}_j) = \delta^i_j \tag{5.1}$$

where $\delta^i_j = \begin{cases} 1 & if\ i = j \\ 0 & if\ i \neq j \end{cases}$

is known as the Kronecker delta. In the context of manifolds, since T_pM is a vector space, there exists a dual vector space to T_pM, whose element is a linear function from T_pM to $\Re$. The dual space is called the **cotangent space** at p, denoted T_p^*M.

Definition 5.1 Dual Vector

An element $\hat{\mathbf{h}}^*: T_pM \to \Re$ is called a **dual vector** or cotangent vector (Nakahara 1990).

Therefore, a dual vector is a linear object that maps a vector to a scalar. This may be generalized to multilinear objects called tensors, which map several vectors to a scalar.

Definition 5.2 Tensor Field

A tensor of type (q,r) is a map that maps q dual vectors and r vectors to $\Re$. A **tensor field** of type (q,r) is defined as a smooth assignment of an element of the set of type (q,r) tensors at each point $p \in M$. A tensor field of type (q,r) is denoted $\Gamma^q_{r,p}$.

For example, consider a tensor which produces the following mapping at a point:

$$T_pM \to \Re \tag{5.2}$$

In this case, the tensor must be a dual vector.

Consider a tensor which produces the following mapping at a point:

$$T_p^* M \rightarrow \Re \tag{5.3}$$

In this case, the tensor must be a vector.

5.3 Riemannian Metric

A Riemannian metric on a manifold is a smooth assignment of an inner product to the tangent space at each point on the manifold (Park and Brockett 1994).

Definition 5.3 Riemannian Metric

Let M be a manifold. A **Riemannian metric** g on M is a type $(0,2)$ tensor field on M which satisfies the following axioms at each point $p \in M$:

(i) $$g_p(\hat{\mathbf{h}}, \hat{\mathbf{k}}) = g_p(\hat{\mathbf{k}}, \hat{\mathbf{h}}) \qquad \hat{\mathbf{h}}, \hat{\mathbf{k}} \in T_p M \tag{5.4}$$

(ii) $$g_p(\hat{\mathbf{h}}, \hat{\mathbf{h}}) \geq 0 \tag{5.5}$$

where the equality of (5.5) holds only when $\hat{\mathbf{h}} = 0$.

Here $g_p = g|_p$. g_p is a symmetric positive definite bilinear form (Nakahara 1990). If a manifold M admits a Riemannian metric g, the pair (M, g) is called a Riemannian manifold.

There exists an isomorphism between T_pM and T_p^*M but no natural isomorphism. A metric gives a distinguished isomorphism:

$$g_p : T_pM \to T_p^*M \tag{5.6}$$

$$\hat{\mathbf{h}} \mapsto g_p\left(\hat{\mathbf{h}},\ \right) \tag{5.7}$$

Definition 5.4 Inner Product

An **inner product**, denoted $\langle\ ,\ \rangle$ is defined as

$$\left\langle \hat{\mathbf{h}}, \hat{\mathbf{k}} \right\rangle = g_p\left(\hat{\mathbf{h}}, \hat{\mathbf{k}}\right) \qquad \hat{\mathbf{h}}, \hat{\mathbf{k}} \in T_pM \tag{5.8}$$

where $g_p\left(\hat{\mathbf{h}},\ \right)$ is associated with a mapping to the cotangent space.

Given a basis for the tangent space, an inner product can be written in terms of matrices:

$$\left\langle \hat{\mathbf{h}}, \hat{\mathbf{k}} \right\rangle = \hat{\mathbf{h}}^T \mathbf{Q}\, \hat{\mathbf{k}} \tag{5.9}$$

where $\mathbf{Q}$ is a symmetric matrix. Non-degeneracy of the inner product is equivalent to the condition that

$$\det(\mathbf{Q}) \neq 0 \tag{5.10}$$

Note that if $\mathbf{Q}$ is positive-definite, then an associated norm can be defined.

5.4 Isometry

The term isometry has been used without proper definition. First some preliminaries.

A smooth map $f: M \rightarrow N$ which induces a differential map f_*:

$$f_*: T_pM \rightarrow T_{f(p)}N \tag{5.11}$$

also induces a map f^*:

$$f^*: T_{f(p)}{}^*N \rightarrow T_p{}^*M \tag{5.12}$$

called the pullback map.

Definition 5.5 Isometry

Let (M,g) be a Riemannian manifold. A diffeomorphism $f: M \rightarrow M$ is an **isometry** if it preserves the metric

$$f^* g_{f(p)} = g_p \tag{5.13}$$

i.e. $$g_{f(p)}\left(f_*\hat{\mathbf{h}}, f_*\hat{\mathbf{k}}\right) = g_p\left(\hat{\mathbf{h}}, \hat{\mathbf{k}}\right) \qquad \hat{\mathbf{h}}, \hat{\mathbf{k}} \in T_pM \tag{5.14}$$

If the metric induced by the left invariant form is the same as that induced by the right invariant form, the metric is said to be invariant (Price 1977). From Section 4.2, $SE(3)$ does not have identical left and right invariant forms and therefore there is no invariant metric on $SE(3)$ (Loncaric 1985). However, $SO(3)$ does have an invariant metric form.

In the case of rigid body motions, the following mappings hold:

$$\hat{\mathbf{A}}_*: \quad \hat{\mathbf{h}} \rightarrow \hat{\mathbf{k}} \tag{5.15}$$

$$\left(\hat{\mathbf{A}}_*\right)^T: \quad \hat{\mathbf{h}}^T \rightarrow \hat{\mathbf{k}}^T \tag{5.16}$$

In matrix form

$$\hat{\mathbf{A}}_* = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \lfloor \mathbf{d} \rfloor \mathbf{A} & \mathbf{A} \end{bmatrix} \tag{5.17}$$

$$\left(\hat{\mathbf{A}}_*\right)^T = \begin{bmatrix} \mathbf{A}^T & -\mathbf{A}^T \lfloor \mathbf{d} \rfloor \\ \mathbf{0} & \mathbf{A}^T \end{bmatrix} \tag{5.18}$$

The operator $\left(\hat{\mathbf{A}}_*\right)^T$ can be represented from consideration of (5.19):

$$\hat{\mathbf{k}}^T = \left[(\mathbf{A}\mathbf{w})^T \quad \left(\lfloor \mathbf{d} \rfloor \mathbf{A}\mathbf{w} + \mathbf{A}\mathbf{v}\right)^T \right] \qquad \hat{\mathbf{k}} \in se(3) \tag{5.19}$$

Therefore

$$\hat{\mathbf{k}}^T = \left[\mathbf{w}^T \quad \mathbf{v}^T \right] \begin{bmatrix} \mathbf{A}^T & -\mathbf{A}^T \lfloor \mathbf{d} \rfloor \\ \mathbf{0} & \mathbf{A}^T \end{bmatrix} \tag{5.20}$$

Therefore

$$\left(\hat{\mathbf{A}}_*\right)^T = \begin{bmatrix} \mathbf{A}^T & -\mathbf{A}^T\lfloor \mathbf{d} \rfloor \\ \mathbf{0} & \mathbf{A}^T \end{bmatrix} \tag{5.21}$$

5.5 $\hat{\mathbf{A}}_*$ - invariant Riemannian Metric

Every rigid body displacement is an isometry. Therefore,

$$\left\langle \hat{\mathbf{A}}_*\hat{\mathbf{h}}, \hat{\mathbf{A}}_*\hat{\mathbf{k}} \right\rangle_p = \left\langle \hat{\mathbf{h}}, \hat{\mathbf{k}} \right\rangle_p \tag{5.22}$$

This equation gives a basis for determining the form of an $\hat{\mathbf{A}}_*$ - invariant Riemannian metric.

Denoting $\mathbf{Q}$ in the form

$$\mathbf{Q} = \begin{bmatrix} \mathbf{G}_1 & \mathbf{G}_2 \\ \mathbf{G}_2^T & \mathbf{G}_3 \end{bmatrix} \tag{5.23}$$

Then $\left\langle \hat{\mathbf{A}}_*\hat{\mathbf{h}}, \hat{\mathbf{A}}_*\hat{\mathbf{k}} \right\rangle_p$ can be represented in matrix form as

$$\begin{bmatrix} \mathbf{w}_1^T & \mathbf{v}_1^T \end{bmatrix} \begin{bmatrix} \mathbf{A}^T & -\mathbf{A}^T\lfloor \mathbf{d} \rfloor \\ \mathbf{0} & \mathbf{A}^T \end{bmatrix} \begin{bmatrix} \mathbf{G}_1 & \mathbf{G}_2 \\ \mathbf{G}_2^T & \mathbf{G}_3 \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \lfloor \mathbf{d} \rfloor \mathbf{A} & \mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{w}_2 \\ \mathbf{v}_2 \end{bmatrix} \tag{5.24}$$

Therefore

$$\left\langle \hat{\mathbf{A}}_{\bullet}\hat{\mathbf{h}}, \hat{\mathbf{A}}_{\bullet}\hat{\mathbf{k}} \right\rangle_p = \hat{\mathbf{h}}^T \left(\hat{\mathbf{A}}_{\bullet}\right)^T \mathbf{Q}\, \hat{\mathbf{A}}_{\bullet}\, \hat{\mathbf{k}} \tag{5.25}$$

Therefore the metric is " $\hat{\mathbf{A}}_{\bullet}$ **- invariant"** , if

$$\mathbf{Q} = \left(\hat{\mathbf{A}}_{\bullet}\right)^T \mathbf{Q}\, \hat{\mathbf{A}}_{\bullet} \tag{5.26}$$

Therefore

$$\begin{bmatrix} \mathbf{G}_1 & \mathbf{G}_2 \\ \mathbf{G}_2^T & \mathbf{G}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{A}^T & -\mathbf{A}^T\lfloor \mathbf{d} \rfloor \\ \mathbf{0} & \mathbf{A}^T \end{bmatrix} \begin{bmatrix} \mathbf{G}_1 & \mathbf{G}_2 \\ \mathbf{G}_2^T & \mathbf{G}_3 \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \lfloor \mathbf{d} \rfloor \mathbf{A} & \mathbf{A} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{A}^T\left(\mathbf{G}_1 - \lfloor \mathbf{d} \rfloor \mathbf{G}_2^T + \mathbf{G}_2 \lfloor \mathbf{d} \rfloor - \lfloor \mathbf{d} \rfloor \mathbf{G}_3 \lfloor \mathbf{d} \rfloor\right)\mathbf{A} & \mathbf{A}^T\left(\mathbf{G}_2 - \lfloor \mathbf{d} \rfloor \mathbf{G}_3\right)\mathbf{A} \\ \mathbf{A}^T\left(\mathbf{G}_2^T + \mathbf{G}_3 \lfloor \mathbf{d} \rfloor\right)\mathbf{A} & \mathbf{A}^T\mathbf{G}_3\mathbf{A} \end{bmatrix} \tag{5.27}$$

For any general $\mathbf{A}$

$$\mathbf{G}_3 = \mathbf{A}^T\mathbf{G}_3\mathbf{A} \Rightarrow \mathbf{G}_3 = c\mathbf{I} \tag{5.28}$$

where c is a scalar.

For any general $\mathbf{A}$

$$\mathbf{G}_2 = \mathbf{A}^T\left(\mathbf{G}_2 - \lfloor \mathbf{d} \rfloor \mathbf{G}_3\right)\mathbf{A} \Rightarrow \mathbf{G}_3 = \mathbf{0}, \mathbf{G}_2 = b\mathbf{I} \tag{5.29}$$

where b is a scalar.

For any general $\mathbf{A}$

$$\mathbf{G}_1 = \mathbf{A}^T\left(\mathbf{G}_1 - \lfloor \mathbf{d} \rfloor b\mathbf{I} + b\mathbf{I}\lfloor \mathbf{d} \rfloor\right)\mathbf{A} \Rightarrow \mathbf{G}_1 = a\mathbf{I} \tag{5.30}$$

where a is a scalar.

Therefore, the most general form of $\mathbf{Q}$ is

$$\mathbf{Q} = \begin{bmatrix} a\mathbf{I} & b\mathbf{I} \\ b\mathbf{I} & \mathbf{0} \end{bmatrix} \tag{5.31}$$

Similarly the metric should be invariant if there is a moving reference frame.

Therefore

$$\frac{d'}{dt}\left[\left(\hat{\mathbf{A}}_{*}\right)^T \mathbf{Q}\, \hat{\mathbf{A}}_{*}\right] = \mathbf{0} \tag{5.32}$$

Therefore

$$\left(\hat{\mathbf{A}}_*\right)^T \mathbf{Q}\frac{d'}{dt}\left(\hat{\mathbf{A}}_*\right)+\frac{d'}{dt}\left(\left(\hat{\mathbf{A}}_*\right)^T\right)\mathbf{Q}\hat{\mathbf{A}}_* = \mathbf{0} \tag{5.33}$$

$$\left(\hat{\mathbf{A}}_*\right)^T \mathbf{Q}\left(\mathbf{ad}_G\hat{\mathbf{A}}_*\right)+\left(\left(\hat{\mathbf{A}}_*\right)^T \mathbf{ad}_G{}^T\right)\mathbf{Q}\hat{\mathbf{A}}_* = \mathbf{0} \tag{5.34}$$

$$\left(\hat{\mathbf{A}}_*\right)^T \left(\mathbf{Q}\,\mathbf{ad}_G + \mathbf{ad}_G{}^T\mathbf{Q}\right)\hat{\mathbf{A}}_* = \mathbf{0} \tag{5.35}$$

So

$$\mathbf{Q}\,\mathbf{ad}_G + \mathbf{ad}_G{}^T\mathbf{Q} = \mathbf{0} \tag{5.36}$$

Consider $\mathbf{ad}_G\hat{\mathbf{h}}$

$$\begin{aligned}\mathbf{ad}_G\hat{\mathbf{h}} &= \begin{bmatrix}\lfloor \mathbf{w} \rfloor & 0\\ \lfloor \mathbf{v}_O \rfloor & \lfloor \mathbf{w} \rfloor\end{bmatrix}\begin{bmatrix}\mathbf{w}_1\\ \mathbf{v}_1\end{bmatrix}\\ &= \begin{bmatrix}\lfloor \mathbf{w} \rfloor \mathbf{w}_1\\ \lfloor \mathbf{v}_O \rfloor \mathbf{w}_1 + \lfloor \mathbf{w} \rfloor \mathbf{v}_1\end{bmatrix}\end{aligned} \tag{5.37}$$

Therefore

$$\hat{\mathbf{h}}^T\mathbf{ad}_G{}^T = \begin{bmatrix}\left(\lfloor \mathbf{w} \rfloor \mathbf{w}_1\right)^T & \left(\lfloor \mathbf{v}_O \rfloor \mathbf{w}_1 + \lfloor \mathbf{w} \rfloor \mathbf{v}_1\right)^T\end{bmatrix} \tag{5.38}$$

Therefore

$$\hat{\mathbf{h}}^T \mathbf{ad}_G{}^T = \begin{bmatrix} \mathbf{w}_1 & \mathbf{v}_1 \end{bmatrix} \begin{bmatrix} \lfloor \mathbf{w} \rfloor^T & \lfloor \mathbf{v}_O \rfloor^T \\ \mathbf{0} & \lfloor \mathbf{w} \rfloor^T \end{bmatrix} \tag{5.39}$$

Therefore

$$\mathbf{ad}_G{}^T = \begin{bmatrix} \lfloor \mathbf{w} \rfloor^T & \lfloor \mathbf{v}_O \rfloor^T \\ \mathbf{0} & \lfloor \mathbf{w} \rfloor^T \end{bmatrix} = \begin{bmatrix} -\lfloor \mathbf{w} \rfloor & -\lfloor \mathbf{v}_O \rfloor \\ \mathbf{0} & -\lfloor \mathbf{w} \rfloor \end{bmatrix} \tag{5.40}$$

So

$$\begin{bmatrix} \mathbf{G}_1 & \mathbf{G}_2 \\ \mathbf{G}_2{}^T & \mathbf{G}_3 \end{bmatrix} \begin{bmatrix} \lfloor \mathbf{w} \rfloor & \mathbf{0} \\ \lfloor \mathbf{v}_O \rfloor & \lfloor \mathbf{w} \rfloor \end{bmatrix} + \begin{bmatrix} -\lfloor \mathbf{w} \rfloor & -\lfloor \mathbf{v}_O \rfloor \\ \mathbf{0} & -\lfloor \mathbf{w} \rfloor \end{bmatrix} \begin{bmatrix} \mathbf{G}_1 & \mathbf{G}_2 \\ \mathbf{G}_2{}^T & \mathbf{G}_3 \end{bmatrix} = \mathbf{0} \tag{5.41}$$

Therefore

$$\begin{bmatrix} \mathbf{G}_1 \lfloor \mathbf{w} \rfloor + \mathbf{G}_2 \lfloor \mathbf{v}_O \rfloor - \lfloor \mathbf{w} \rfloor \mathbf{G}_1 - \lfloor \mathbf{v}_O \rfloor \mathbf{G}_2{}^T & \mathbf{G}_2 \lfloor \mathbf{w} \rfloor - \lfloor \mathbf{w} \rfloor \mathbf{G}_2 - \lfloor \mathbf{v}_O \rfloor \mathbf{G}_3 \\ \mathbf{G}_2{}^T \lfloor \mathbf{w} \rfloor + \mathbf{G}_3 \lfloor \mathbf{v}_O \rfloor - \lfloor \mathbf{w} \rfloor \mathbf{G}_2{}^T & \mathbf{G}_3 \lfloor \mathbf{w} \rfloor - \lfloor \mathbf{w} \rfloor \mathbf{G}_3 \end{bmatrix} = \mathbf{0}$$

(5.42)

$$\mathbf{G}_3 \lfloor \mathbf{w} \rfloor - \lfloor \mathbf{w} \rfloor \mathbf{G}_3 = \mathbf{0} \Rightarrow \mathbf{G}_3 = c\mathbf{I} \tag{5.43}$$

where c is a scalar.

$$\mathbf{G}_2\lfloor \mathbf{w} \rfloor - \lfloor \mathbf{w} \rfloor \mathbf{G}_2 - \lfloor \mathbf{v}_O \rfloor \mathbf{G}_3 = \mathbf{0} \Rightarrow \mathbf{G}_3 = \mathbf{0}, \mathbf{G}_2 = b\mathbf{I} \tag{5.44}$$

where b is a scalar.

$$\mathbf{G}_1\lfloor \mathbf{w} \rfloor + b\mathbf{I}\lfloor \mathbf{v}_O \rfloor - \lfloor \mathbf{w} \rfloor \mathbf{G}_1 - \lfloor \mathbf{v}_O \rfloor b\mathbf{I} = \mathbf{0} \Rightarrow \mathbf{G}_1 = a\mathbf{I} \tag{5.45}$$

where a is a scalar.

Therefore, the general form of the metric given in (5.31) is confirmed.

In Chapter 3, the Killing metric form was introduced.

$$\langle\, , \rangle_K = \begin{bmatrix} -4\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{5.46}$$

This corresponds to $a = -4, b = 0$. However, this metric form is degenerate on $SE(3)$. An alternative is to set $a = 0, b = 1/2$. This is known as the hyperbolic metric form (Loncaric 1985).

$$\langle\, , \rangle_H = \begin{bmatrix} \mathbf{0} & 1/2\,\mathbf{I} \\ 1/2\,\mathbf{I} & \mathbf{0} \end{bmatrix} \tag{5.47}$$

The hyperbolic metric form is non-degenerate on $SE(3)$ but is not positive-definite. (Note the term 'hyperbolic' arises since a non-singular indefinite space is referred to as a hyperbolic space.) In fact, any linear combination of the invariant metric form is also an invariant metric form but none are positive definite **and** non-degenerate. Therefore, there is no *natural* choice of metric form on $se(3)$.

The Riemannian metric provides an isomorphism

$$Q: se(3) \rightarrow se^*(3) \tag{5.48}$$

where elements of $se^*(3)$ can be represented as

$$\mathbf{M} = \begin{bmatrix} \mathbf{j} \\ \mathbf{p} \end{bmatrix} \tag{5.49}$$

where $\mathbf{j}$ and $\mathbf{p}$ are angular and linear momentum respectively. As required, momentum gives the mapping:

$$\hat{\mathbf{h}}^*: se(3) \rightarrow \Re \qquad \hat{\mathbf{h}}^* \in se^*(3) \tag{5.50}$$

It should be noted that $\hat{\mathbf{h}}^* \in se^*(3)$ transforms differently compared to $\hat{\mathbf{h}} \in se(3)$. The operator for the dual vector is given by

$$\begin{bmatrix} \mathbf{A} & \lfloor \mathbf{d} \rfloor \mathbf{A} \\ \mathbf{0} & \mathbf{A} \end{bmatrix} \tag{5.51}$$

Elements $\mathbf{w}$ and $\mathbf{p}$ that transform in the same way are sometimes referred to as free vectors and $\mathbf{v}$ and $\mathbf{j}$ as line vectors (Hunt 1978).

5.6 Kinetic Energy Metric

The kinetic energy of a rigid body is a left invariant Riemannian metric associated with a left invariant tangent vector (Arnold 1978), denoted $\langle\, , \rangle_K$. Before the metric can be examined, a definition is required:

Definition 5.6 Kinetic Energy

Let M be a Riemannian manifold. The quadratic form on each tangent space

$$K = 1/2\langle \mathbf{v}, \mathbf{v}\rangle \qquad \mathbf{v} \in se(3) \tag{5.52}$$

is called the **kinetic energy** (Arnold 1978).

The kinetic energy of a rigid body at the centre of mass is given by

$$K = 1/2\left(\mathbf{w}^T\,\Pi_m \mathbf{w} + m\mathbf{v}^T\mathbf{v}\right) \tag{5.53}$$

where m is the mass of the rigid body and Π_m is the inertia tensor of the rigid body about the centre of mass, relative to a frame at the centre of mass.

Therefore

$$K = 1/2\begin{bmatrix}\mathbf{w}^T & \mathbf{v}^T\end{bmatrix}\begin{bmatrix}\Pi_m & \mathbf{0} \\ \mathbf{0} & m\mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{w} \\ \mathbf{v}\end{bmatrix} \tag{5.54}$$

Therefore

$$K = 1/2\,\hat{\mathbf{h}}^T\,\mathbf{Q}\,\hat{\mathbf{h}} \tag{5.55}$$

where the generalized moment of inertia, $\mathbf{Q}^x$ is given by

$$\mathbf{Q}^x = \begin{bmatrix} \Pi_m & \mathbf{0} \\ \mathbf{0} & m\mathbf{I} \end{bmatrix} \tag{5.56}$$

The same body has a more complicated generalized moment of inertia when viewed from another frame (Selig 1989). In particular, if y is x translated by t, the off-diagonal blocks of $\mathbf{Q}$ are antisymmetric matrices:

$$\mathbf{Q}^y = \begin{bmatrix} \Pi_m - m\lfloor\mathbf{t}\rfloor^2 & m\lfloor\mathbf{t}\rfloor \\ -m\lfloor\mathbf{t}\rfloor & m\mathbf{I} \end{bmatrix} \tag{5.57}$$

The generalized moment of inertia is not an $\hat{\mathbf{A}}_*$ - invariant metric as defined in Section 5.5; the matrix $\mathbf{Q}$ needs to be transformed with a shift in the measurement frame:

$$\mathbf{Q} \rightarrow \left(\hat{\mathbf{A}}_*\right)^T \mathbf{Q}\,\hat{\mathbf{A}}_* \tag{5.58}$$

5.7 Orthogonality and Reciprocity

'Strict' **orthogonality** is only defined on a Euclidean metric space, E^n. If

$$\mathbf{a}\cdot\mathbf{b}=0 \qquad \mathbf{a},\mathbf{b}\in E^n \tag{5.59}$$

where $\cdot$ denotes dot product, then $\mathbf{a}$ and $\mathbf{b}$ are orthogonal.

However, invariant 'notional' orthogonality, or **reciprocity**, can be defined for elements in the Lie algebra of $SE(3)$ using the hyperbolic metric. If

$$\hat{\mathbf{h}}^T\mathbf{Q}\,\hat{\mathbf{k}}=0 \qquad \hat{\mathbf{h}},\hat{\mathbf{k}}\in se(3) \tag{5.60}$$

where $\mathbf{Q}$ is $\langle\ ,\ \rangle_H$, then $\hat{\mathbf{h}}$ and $\hat{\mathbf{k}}$ are reciprocal.

If $2\langle\ ,\ \rangle_H$ is used in (5.60) then the equation is referred to as a reciprocal product. A reciprocal product between a tangent vector and a dual vector is given in matrix form by:

$$\left(\hat{\mathbf{h}}^*\right)^T\hat{\mathbf{k}}=0 \qquad \hat{\mathbf{h}}^*\in se^*(3),\hat{\mathbf{k}}\in se(3) \tag{5.61}$$

If (5.61) holds then then $\hat{\mathbf{h}}^*$ and $\hat{\mathbf{k}}$ are reciprocal.

5.8 Validation of the Reciprocal Product

A simple check is made to show that the reciprocal product is invariant to changes in measurement frame.

If two elements of $se(3)$ are reciprocal via $2\langle\, ,\, \rangle_H$, then under a change of reference frame, the reciprocal product becomes:

$$\begin{bmatrix} \mathbf{w}^T & \mathbf{v}^T \end{bmatrix}_2 \begin{bmatrix} \mathbf{A}^T & -\mathbf{A}^T\lfloor \mathbf{d} \rfloor \\ 0 & \mathbf{A}^T \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{A} & 0 \\ \lfloor \mathbf{d} \rfloor \mathbf{A} & \mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{v} \end{bmatrix}_1 \tag{5.62}$$

Simplifying

$$\begin{bmatrix} \mathbf{w}^T & \mathbf{v}^T \end{bmatrix}_2 \begin{bmatrix} -\mathbf{A}^T\lfloor \mathbf{d} \rfloor \mathbf{A} + \mathbf{A}^T\lfloor \mathbf{d} \rfloor \mathbf{A} & \mathbf{A}^T\mathbf{A} \\ \mathbf{A}^T\mathbf{A} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{v} \end{bmatrix}_1 \tag{5.63}$$

which gives

$$\begin{bmatrix} \mathbf{w}^T & \mathbf{v}^T \end{bmatrix}_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{v} \end{bmatrix}_1 \tag{5.64}$$

Therefore, under a change of reference frame, the two elements will still be reciprocal.

Similarly it can be easily shown that reciprocity between a tangent vector and a dual vector is invariant to changes in reference frame. If an element of $se(3)$ is reciprocal to an element of $se^*(3)$, then under a change of reference frame, the reciprocal product becomes

$$\begin{bmatrix} \mathbf{j}^T & \mathbf{p}^T \end{bmatrix} \begin{bmatrix} \mathbf{A}^T & \mathbf{0} \\ -\mathbf{A}^T \lfloor \mathbf{d} \rfloor & \mathbf{A}^T \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \lfloor \mathbf{d} \rfloor \mathbf{A} & \mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{v} \end{bmatrix} \tag{5.65}$$

Simplifying

$$\begin{bmatrix} \mathbf{j}^T & \mathbf{p}^T \end{bmatrix} \begin{bmatrix} \mathbf{A}^T \mathbf{A} & 0 \\ -\mathbf{A}^T \lfloor \mathbf{d} \rfloor \mathbf{A} + \mathbf{A}^T \lfloor \mathbf{d} \rfloor \mathbf{A} & \mathbf{A}^T \mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{v} \end{bmatrix} \tag{5.66}$$

which gives

$$\begin{bmatrix} \mathbf{j}^T & \mathbf{p}^T \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{v} \end{bmatrix} \tag{5.67}$$

Therefore, under a change of reference frame, the two elements will still be reciprocal.

Chapter 6

Invariant Filtering for Shared Control

6.1 Overview of the Chapter

In this Chapter, a proper framework is outlined for implementing shared control in a geometrically correct way. A simple partitioning of $se(3)$ is considered first. Traditionally this has been done using a switching matrix. This is shown to be equivalent to a filter which embodies the metric form $\langle\, , \rangle_I$. If the filter is to be successful when the reference frame is moved, then the metric form has to be transformed correctly.

The case of shared control with constrained motion is considered next. The critical relationship in this case is shown to be the reciprocal product between elements of $se(3)$ and $se^*(3)$.

The switching matrix is re-examined next and shown to be a misinterpretation of a projection operator which is a proper geometric entity. The necessary transformations for the use of the projection operator are derived.

6.2 Partitioning of Vector Spaces

Partitioning of vectors can be achieved using a projection operator which is derived directly from the basis of the vector space and the basis of the dual vector space (Selig 1995). This will be examined in a later section. First, partitioning is achieved using a left pseudo inverse. This is convenient because the formulation gives the partition in a form which clearly embodies a metric.

Consider the tangent space of $SE(3)$ at the identity. This is shown below in Figure 6-1.

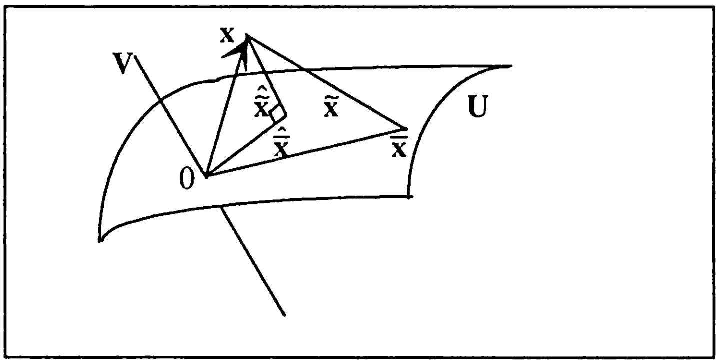

Figure 6-1: Partition of the Tangent Space

Assume a partition is required into two subspaces $\mathbf{U}$ and $\mathbf{V}$ such that

$$\mathbf{U} \oplus \mathbf{V} = se(3) \qquad \text{and} \qquad \mathbf{U} \cap \mathbf{V} = \mathbf{0} \tag{6.1}$$

Suppose there is a basis $\{\bar{\mathbf{e}}_1 \ldots \bar{\mathbf{e}}_k\}$ for $\mathbf{U}$ and a basis $\{\tilde{\mathbf{e}}_{k+1} \ldots \tilde{\mathbf{e}}_6\}$ for $\mathbf{V}$.

Now consider a vector $\mathbf{x} \in se(3)$. It is always possible to write

$$\mathbf{x} = \bar{a}_1 \bar{\mathbf{e}}_1 + \quad + \bar{a}_k \bar{\mathbf{e}}_k + \tilde{a}_{k+1} \tilde{\mathbf{e}}_{k+1} + \quad + \tilde{a}_6 \tilde{\mathbf{e}}_6 \tag{6.2}$$

Let $\mathbf{M}_{\mathrm{U}}$ be the matrix whose columns are $\{\bar{\mathbf{e}}_1 \ldots \bar{\mathbf{e}}_k\}$, so that

$$\bar{\mathbf{x}} = \mathbf{M}_{\mathrm{U}} \bar{\mathbf{a}}_k \tag{6.3}$$

where

$$\bar{\mathbf{a}}_k = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} \tag{6.4}$$

This follows since

$$\mathbf{M}_{\mathrm{U}} \bar{\mathbf{a}}_k = \begin{bmatrix} \bar{\mathbf{e}}_1 & \ldots & \bar{\mathbf{e}}_k \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} \tag{6.5}$$

or

$$\mathbf{M}_{\mathrm{U}} \bar{\mathbf{a}}_k = a_1 \bar{\mathbf{e}}_1 + \ldots + a_k \bar{\mathbf{e}}_k \qquad (a_i \text{ are scalars}) \tag{6.6}$$

With reference to Figure 6-1, $\mathbf{x}$ can be viewed as the vector sum of a unique $\hat{\bar{\mathbf{x}}} \in \mathbf{U}$ and $\hat{\tilde{\mathbf{x}}} \in \mathbf{V}$, or as the vector sum of a general $\bar{\mathbf{x}} \in \mathbf{U}$ and $\tilde{\mathbf{x}} \notin \mathbf{V}$. Note that $\hat{\tilde{\mathbf{x}}}$ is a minimum $\tilde{\mathbf{x}}$ on $\mathbf{V}$.

Define an error term, $\mathbf{e}$, where

$$\|\mathbf{e}\|^2 = \|\mathbf{M}_U \overline{\mathbf{a}}_k - \mathbf{x}\|^2 \tag{6.7}$$

if $\mathbf{x} \in \mathbf{U}$, then

$$\begin{aligned} \|\mathbf{e}\|^2 &= \|\mathbf{M}_U \overline{\mathbf{a}}_k - \mathbf{x}\|^2 \\ &= 0 \end{aligned}$$

More generally,

$$\|\mathbf{e}\|^2 = \|\mathbf{M}_U \overline{\mathbf{a}}_k - \mathbf{x}\|^2 > 0 \tag{6.8}$$

In Section 6.3, for an equation of the form

$$\mathbf{M}_U \overline{\mathbf{a}}_k = \mathbf{x} \tag{6.9}$$

where $\mathbf{M}_U$ has full column rank, the error term is shown to be minimized in a least squares sense by a choice of $\overline{\mathbf{a}}_k$, denoted $\hat{\overline{\mathbf{a}}}_k$, where

$$\hat{\overline{\mathbf{a}}}_k = \mathbf{M}_U^{+} \mathbf{x} \tag{6.10}$$

where $\mathbf{M}_U^{+}$ denotes the left-pseudo inverse of $\mathbf{M}_U$. The matrix $\mathbf{M}_U$ has full column rank since its columns are basis vectors of $\mathbf{U}$. Re-call that

$$\overline{\mathbf{x}} = \mathbf{M}_U \overline{\mathbf{a}}_k \tag{6.11}$$

Therefore

$$\hat{\bar{\mathbf{x}}} = \mathbf{M}_{\mathrm{U}} \hat{\bar{\mathbf{a}}}_k \tag{6.12}$$

Substituting (6.10) into (6.12) gives

$$\hat{\bar{\mathbf{x}}} = \mathbf{M}_{\mathrm{U}} \mathbf{M}_{\mathrm{U}}^{+} \mathbf{x} \tag{6.13}$$

An similar argument follows for $\tilde{\mathbf{x}}$ giving

$$\hat{\tilde{\mathbf{x}}} = \mathbf{M}_{\mathrm{V}} \mathbf{M}_{\mathrm{V}}^{+} \mathbf{x} \tag{6.14}$$

Now

$$\mathbf{x} = \hat{\bar{\mathbf{x}}} + \hat{\tilde{\mathbf{x}}} \tag{6.15}$$

Therefore

$$\mathbf{x} = \mathbf{M}_{\mathrm{U}} \mathbf{M}_{\mathrm{U}}^{+} \mathbf{x} + \mathbf{M}_{\mathrm{V}} \mathbf{M}_{\mathrm{V}}^{+} \mathbf{x} \tag{6.16}$$

6.3 Left Pseudo-Inverse

For partitioning a real vector space, an explicit formula is required for left pseudo-inverse in (6.10). It is shown in this section that the solution requires a choice of metric.

A formula for left pseudo inverse arises from consideration of an ordinary least squares problem. An ordinary least squares problem is characterized by

$$\mathbf{Ax} = \mathbf{b} \tag{6.17}$$

where $\mathbf{A}$ is $m \times n$.

The best choice $\breve{\mathbf{x}}$ is the one that minimizes the norm of the error vector (Strang 1986), where the error vector is defined as

$$\mathbf{e} = \mathbf{Ax} - \mathbf{b} \tag{6.18}$$

The norm of the error vector is given by

$$\| \mathbf{Ax} - \mathbf{b} \|^2 = (\mathbf{Ax} - \mathbf{b})^T \mathbf{Q} (\mathbf{Ax} - \mathbf{b}) \tag{6.19}$$

where $\mathbf{Q}$ is the matrix form of a Riemannian metric.

Expanding

$$\| \mathbf{Ax} - \mathbf{b} \|^2 = (\mathbf{Ax})^T \mathbf{Q} (\mathbf{Ax}) - 2(\mathbf{Ax})^T \mathbf{Qb} + \mathbf{b}^T \mathbf{Qb} \tag{6.20}$$

The term $\mathbf{b}^T\mathbf{Qb}$ will not affect any minimization and can be ignored. Define a function

$$P(\mathbf{x}) = 1/2\,(\mathbf{Ax})^T \mathbf{Q} (\mathbf{Ax}) - (\mathbf{Ax})^T \mathbf{Qb} \tag{6.21}$$

Define

$$\underline{\mathbf{M}} \equiv \mathbf{A}^T \mathbf{Q} \mathbf{A} \tag{6.22}$$

$$\underline{\mathbf{n}} \equiv \mathbf{A}^T \mathbf{Q} \mathbf{b} \tag{6.23}$$

Therefore

$$P(\mathbf{x}) = 1/2\, \mathbf{x}^T \underline{\mathbf{M}} \mathbf{x} - \mathbf{x}^T \underline{\mathbf{n}} \tag{6.24}$$

If $\underline{\mathbf{M}}$ is positive definite, then $P(\mathbf{x})$ is minimized at the point

$$\breve{\mathbf{x}} = \underline{\mathbf{M}}^{-1} \underline{\mathbf{n}} \tag{6.25}$$

(see A2.5 for proof).

Substituting back

$$\breve{\mathbf{x}} = \left(\mathbf{A}^T \mathbf{Q} \mathbf{A}\right)^{-1} \mathbf{A}^T \mathbf{Q} \mathbf{b} \tag{6.26}$$

is the least squares solution to (6.17).

If (6.26) is written as

$$\breve{\mathbf{x}} = \mathbf{A}^{+}\mathbf{b} \tag{6.27}$$

then it is clear that the formula for left pseudo-inverse when $\mathbf{A}$ has full rank is given by

$$\mathbf{A}^{+} = \left(\mathbf{A}^{T}\mathbf{Q}\mathbf{A}\right)^{-1}\mathbf{A}^{T}\mathbf{Q} \tag{6.28}$$

6.4 Conceptual Design for a Shared Controller

A simple conceptual form of shared control is given in Figure 6-2.

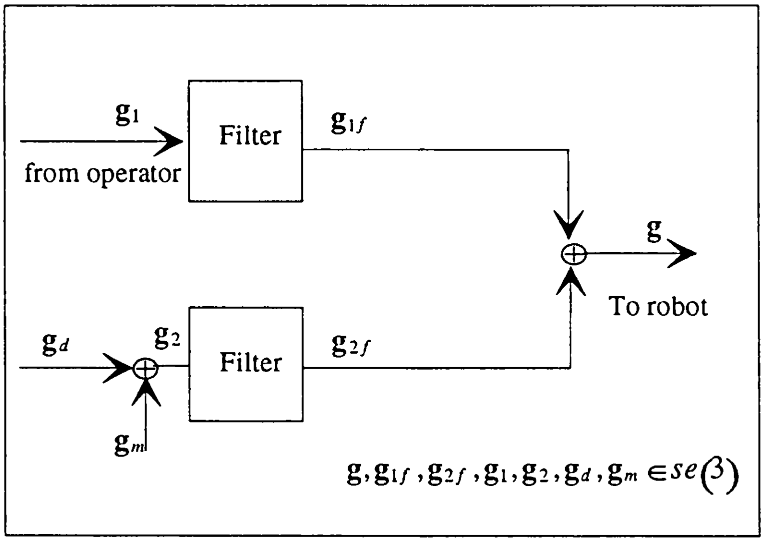

Figure 6-2: Conceptual Design for Simple Shared Controller

In this case, there is a simple partition of $se(3)$ such that

$$\mathbf{U} \oplus \mathbf{V} = se(3), \mathbf{U} \cap \mathbf{V} = \mathbf{0} \tag{6.29}$$

From (6.16)

$$\hat{\mathbf{g}} = \mathbf{M}_U \mathbf{M}_U^{+} \hat{\mathbf{g}} + \mathbf{M}_V \mathbf{M}_V^{+} \hat{\mathbf{g}} \tag{6.30}$$

If a definition is made

$$\hat{\mathbf{g}}_f \equiv \mathbf{M}_U \mathbf{M}_U^{+} \hat{\mathbf{g}} \tag{6.31}$$

then the formula for left pseudo inverse gives the following expression:

$$\hat{\mathbf{g}}_f = \mathbf{M}_U \left(\mathbf{M}_U^T \mathbf{Q} \mathbf{M}_U \right)^{-1} \mathbf{M}_U^T \mathbf{Q} \hat{\mathbf{g}} \tag{6.32}$$

Now a switching matrix is defined as $\mathbf{M}_U \mathbf{M}_U^T$. Therefore, for a switching matrix

$$\hat{\mathbf{g}}_f = \mathbf{M}_U \mathbf{M}_U^T \hat{\mathbf{g}} \tag{6.33}$$

Comparing (6.32) and (6.33) yields for a switching matrix

$$\mathbf{M}_U^T = \left(\mathbf{M}_U^T \mathbf{Q} \mathbf{M}_U \right)^{-1} \mathbf{M}_U^T \mathbf{Q} \tag{6.34}$$

This is true in general if $\mathbf{Q} = \mathbf{I}$, since

$$\left(\mathbf{M_U}^T\mathbf{I}\mathbf{M_U}\right)^{-1}\mathbf{M_U}^T\mathbf{I} = \mathbf{I}^{-1}\mathbf{M_U}^T = \mathbf{M_U}^T \tag{6.35}$$

Therefore, it is now possible to conclude that the switching matrix is associated with $\langle\, , \rangle_I$., where

$$\langle\, , \rangle_I = \begin{bmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{bmatrix} \tag{6.36}$$

6.5 Design for Invariant Filtering

There is no unique way to partition $\hat{\mathbf{g}} \in se(3)$ because there is no natural positive definite metric on $se(3)$ (Loncaric 1985). **Any** symmetric positive definite metric can be used but the metric has to be transformed correctly with a change in reference frame (Lipkin and Duffy 1988) (Doty et al 1993). For example, if $\langle\, , \rangle_I$ is used, then the transformation is

$$\mathbf{Q} \rightarrow \left(\hat{\mathbf{A}}_*\right)^T \mathbf{Q}\, \hat{\mathbf{A}}_* = \left(\hat{\mathbf{A}}_*\right)^T \hat{\mathbf{A}}_* \tag{6.37}$$

6.6 Shared Control and Constrained Motion

To re-cap, a dual vector is a linear function:

$$\hat{\mathbf{h}}^* : T_p M \rightarrow \Re \tag{6.38}$$

Momentum, denoted $\mathbf{M}$, is an element of the cotangent space to the Lie algebra $se(3)$, denoted $se^*(3)$.

$$\mathbf{M} = \begin{bmatrix} \mathbf{j} \\ \mathbf{p} \end{bmatrix} \quad \in se^*(3) \qquad (6.39)$$

where $\mathbf{j}$ and $\mathbf{p}$ are the angular and linear momentum.

Under a rigid body displacement in $SE(3)$, the momentum will change. The derivative of the momentum is given by

$$\dot{\mathbf{M}} = \begin{bmatrix} \mathbf{m} \\ \mathbf{f} \end{bmatrix} \qquad (6.40)$$

where $\mathbf{f}$ denotes linear force and $\mathbf{m}$ denotes moment (collectively referred to as a force vector). This vector will lie in the tangent space to $se^*(3)$, but since this is a finite dimensional vector space, its tangent space is naturally isomorphic to the space itself. Hence a force vector is an element of $se^*(3)$ (Selig 1995):

$$\begin{bmatrix} \mathbf{m} \\ \mathbf{f} \end{bmatrix} \quad \in se^*(3) \qquad (6.41)$$

This force vector is sometimes referred to as a wrench.

At this point, it should be emphasized that velocities and forces lie in different (dual) vector spaces. A proper mathematical approach is required for discussing forces and velocities together in a geometrically correct way. This is described in the following sections.

6.7 Symplectic Vector Space

First some definitions are required.

Definition 6.1 Dual Vector Space

Let M be an n - dimensional manifold. A dual vector on the tangent space to M at a point p is called the dual vector to M at p. The set of all dual vectors to M at p forms an n - dimensional vector space, dual to the tangent space $T_p M$, called the **dual vector space**, denoted $T_p^* M$.

Definition 6.2 Cotangent Bundle

The union of $T_p^* M$ at all points $p \in M$ is called the **cotangent bundle** of M and is denoted $T^* M$ (Arnold 1978).

The set $T^* M$ has a natural structure of a manifold of dimension $2n$. Now a choice of chart gives a point in the tangent space a representation $\mathbf{p}$ of dimension n. Similarly the choice of chart gives a point in the dual vector space a representation $\mathbf{q}$ of dimension n. Together the $2n$ representation $(\mathbf{p}, \mathbf{q}) \in \Re^{2n}$ characterizes $T^* M$ (Arnold 1978).

A symplectic linear structure on $\Re^{2n}$ can be defined by a form known as the skew scalar product. Some definitions are required.

Definition 6.3 Skew Scalar Product

A symplectic linear structure on $\Re^{2n}$ is a non-degenerate bilinear skew symmetric form, called the **skew scalar product**, denoted $[[\ ,\ ]]$

$$[[\xi,\eta]] \ = \ -[[\xi,\eta]] \qquad\qquad \xi,\eta \in \Re^{2n}$$

Definition 6.4 Symplectic Vector Space

The space $\Re^{2n}$ together with the symplectic structure $[[\ ,\ ]]$ is called a **symplectic vector space** (Arnold 1978).

Definition 6.5 Symplectic Group

A linear transformation $S: \Re^{2n} \rightarrow \Re^{2n}$ of the symplectic space $\Re^{2n}$ to itself is called symplectic if it preserves the skew scalar product

$$[[S\xi, S\eta]] \ = \ [[\xi,\eta]] \qquad\qquad \forall\xi,\eta \in \Re^{2n}$$

The set of all symplectic transformations of $\Re^{2n}$ is called the **symplectic group** $Sp(2n)$. (This group has already been discussed in Section 2.5.)

Definition 6.6 Skew Orthogonal

Two vectors $\xi,\eta \in \Re^{2n}$ are **skew orthogonal** if

$$[[\xi,\eta]] \ = \ 0$$

Definition 6.7 Langrangian Plane

A k - dimensional plane of a symplectic space is a null (or isotropic) plane if it is skew orthogonal to itself, i.e. if the skew scalar product of any two vectors in the plane is equal to zero. If $k = n$ the null plane is called a **Lagrangian plane** (Arnold 1978).

In this case, the 12-dimensional representation is given by

$$\left(\hat{\mathbf{h}}, \hat{\mathbf{h}}^*\right) \in \Re^{12} \tag{6.42}$$

A partially constrained rigid body can always be characterized by a Langrangian plane in $\Re^{12}$ by encapsulating a reciprocal relationship into the definition of the skew scalar product :

$$\left[\left[\left(\hat{\mathbf{h}}_1, \hat{\mathbf{h}}_1^*\right), \left(\hat{\mathbf{h}}_2, \hat{\mathbf{h}}_2^*\right)\right]\right] = \hat{\mathbf{h}}_1^* \hat{\mathbf{h}}_2 - \hat{\mathbf{h}}_2^* \hat{\mathbf{h}}_1 \tag{6.43}$$

It is tempting to assert that a partially constrained rigid body can be characterized by the subspace of $\hat{\mathbf{h}}$ and the subspace of $\hat{\mathbf{h}}^*$ which simply sum to span $\Re^6$ (Vukobratovic and Stojic 1995). However, from a geometric viewpoint, this assertion would be incorrect since the two belong to different vector spaces. A geometrically correct description is as follows: a partially constrained rigid body is characterized by a Langrangian plane in $\Re^{12}$ where the symplectic structure of $\Re^{12}$ is defined by the skew scalar product given in (6.43) (Loncaric 1985).

6.8 Constraints

A proper geometric description of constraint is required. The technique is best explained using an example. First a definition is required.

Definition 6.8 Holonomic Constraint

Consider a system described by n generalized coordinates $\mathbf{q}_1 \ldots \mathbf{q}_n$. Suppose there are m independent constraint equations of the form

$$\phi_j(\mathbf{q}_1 \cdots \mathbf{q}_n, t) = 0 \qquad j = 1 \ldots m \tag{6.44}$$

Constraints of this form are known as **holonomic constraints** (Greenwood 1988).

Consider the rigid body in Figure 6-3 with a holonomic constraint which does not vary with time (known as a scleronomic constraint).

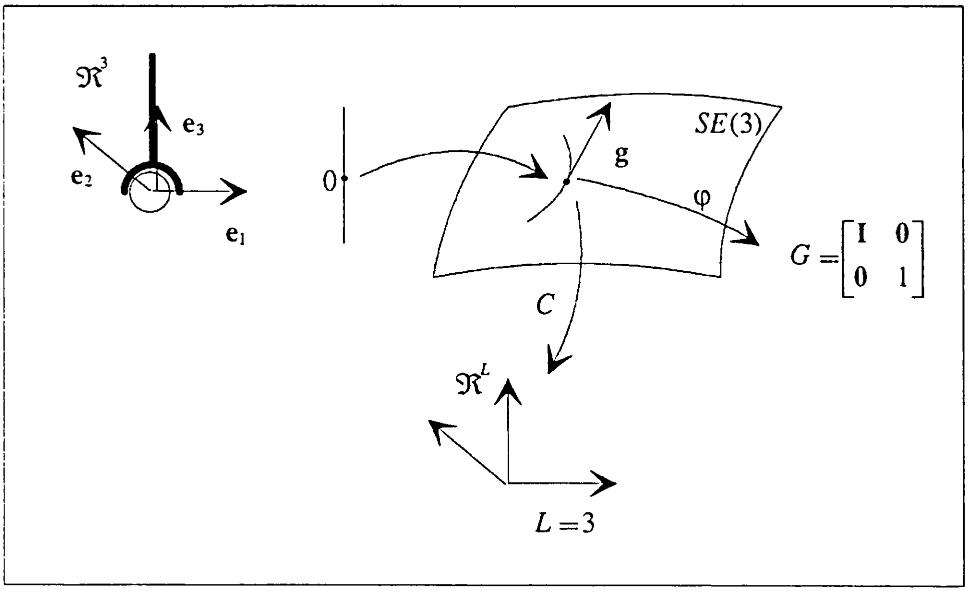

Figure 6-3: Partially Constrained Rigid Body

Assume that the constraint surface passes through the identity (so that the tangent vector can be defined) and that the surface is smooth (Gibson 1995).

The equation of constraint is

$$C(G) = [\mathbf{d}] = 0 \tag{6.45}$$

This defines the so-called C-surface (Mason 1981). Consider the mapping of tangent vector spaces induced by C: $f_*: T_p M \to T_{C(p)} M$. Any small displacement generated by $\hat{\mathbf{g}}$ must lie in the kernel of f_*.

In matrix form, f_* is associated with a $L \times 6$ matrix, denoted $\mathbf{dC}$

$$[\mathbf{dC}][\hat{\mathbf{g}}] = \hat{\mathbf{g}}_C \tag{6.46}$$

where $\hat{\mathbf{g}}_C$ denotes an element of the tangent space of $\Re^L$. The matrix $\mathbf{dC}$ comprises of L dual vectors. Any small displacement generated by $\hat{\mathbf{g}}$ must lie in kernel of $\mathbf{dC}$, denoted $\ker(\mathbf{dC})$. So the basis of $\ker(\mathbf{dC})$ gives the basis for directions of free motion. In this case this is

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.47}$$

$\ker(\mathbf{dC})$ gives the first element in the Lagrangian plane which characterizes the partially constrained rigid body. The second element in the Lagrangian plane is the space of wrenches that *annihilate* $\ker(\mathbf{dC})$ (Selig 1995). This follows from the principle of virtual work:

$$\begin{bmatrix} \mathbf{m}^T & \mathbf{f}^T \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{v} \end{bmatrix} \rightarrow 0 \tag{6.48}$$

for a partially constrained rigid body where the constraint forces do no work. Equation (6.48) is a reciprocal product.

In this example, the basis of the space of wrenches that annihilate $\ker(\mathbf{dC})$ is given by

$$\mathbf{e}_1^* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2^* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3^* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{6.49}$$

6.9 Conceptual Scheme for Constrained Motion

A conceptual form of shared control for constrained motion is given in Figure 6-4.

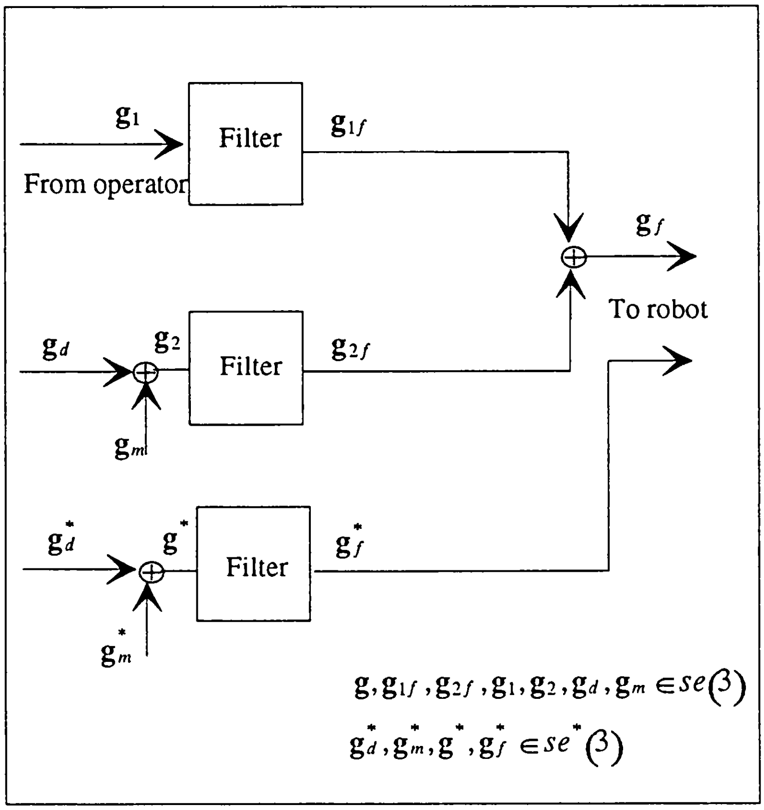

Figure 6-4: Conceptual Scheme for Constrained Motion

It is tempting to assert that the use of switching matrices can be used as the filters in Figure 6-4, especially if one considers the example given in Section 6.8. However, the fact that the switching matrix does work in this example is an "accident" due to the fact that the switching matrix happens to produce a partition which results in a reciprocal product with the reference frame centered as shown. When this situation occurs, the origin of the reference frame is sited at a so-called "centre-of-compliance" (Mason 1981).

If one considers a shift in the reference frame from this "centre of compliance", then the naive use of the switching matrix on the transformed vectors will not produce correct results. This will be validated in the next section.

If the reference frame is shifted then, using the method outlined in Section 6.5, the respective filters can be transformed so that elements are filtered correctly.

Although $se(3)$ is isomorphic to $\Re^6$, it is not a Euclidean metric space. Therefore, complements of $se(3)$ can be defined as only **locally** orthogonal to each other. Similarly complements of $se^*(3)$ can be defined as locally orthogonal to each other. Note that it would be incorrect to assert that complements of $se(3)$ and complements of $se^*(3)$ are orthogonal to each other because the complements lie in different vector spaces (Duffy 1990).

6.10 Validation of Invariant Filtering

An example of a shared mode task is examined which involves constrained motion. Using a test harness, both the filters derived in Section 6.5 and the switching matrix are tested. The task is the peg in the hole task introduced in Section 1.3.2. Consider a unit shift in origin in the direction shown in Figure 6-5. This example is deliberately the same as used in Lipkin and Duffy (1988) in order to show consistency of results.

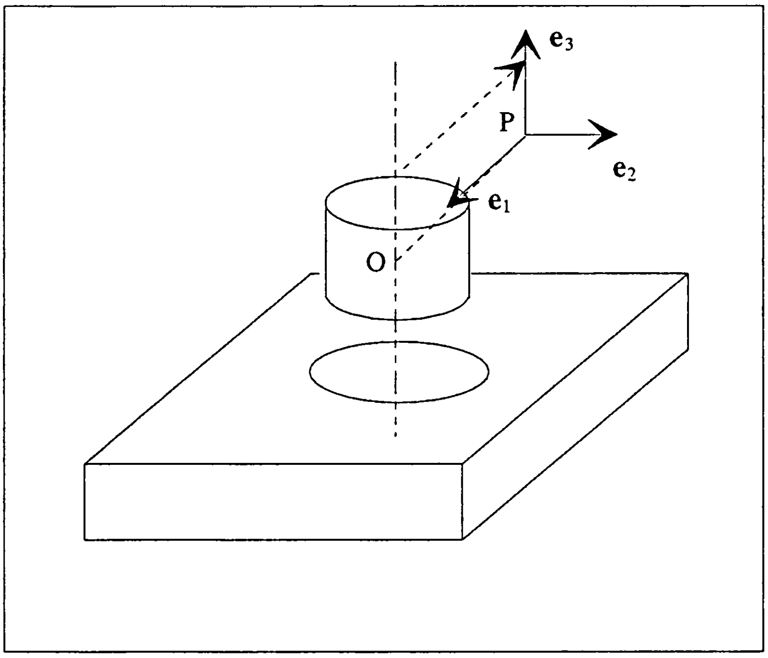

Figure 6-5: Shared Control Task with Displaced Reference Frame

There is a displacement of reference frame, so the passive viewpoint holds. Therefore, from (4.23)

$$\hat{\mathbf{h}}_P = \mathbf{A}_*^{-1}\hat{\mathbf{h}}_O \tag{6.50}$$

Therefore

$$\hat{\mathbf{h}}_P = \begin{bmatrix} \mathbf{A}^T & \mathbf{0} \\ -\mathbf{A}^T\lfloor \mathbf{d} \rfloor & \mathbf{A}^T \end{bmatrix} \hat{\mathbf{h}}_O \tag{6.51}$$

In this case

$$\mathbf{A}^T = \mathbf{I} \tag{6.52}$$

$$-\lfloor \mathbf{d} \rfloor = \begin{bmatrix} 0 & d_3 & -d_2 \\ -d_3 & 0 & d_1 \\ d_2 & -d_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & d_1 \\ 0 & -d_1 & 0 \end{bmatrix} \tag{6.53}$$

In coordinate frame O, the basis of $\hat{\mathbf{g}}_f$ is given by:

$$\mathbf{e}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{6.54}$$

and the basis of $\hat{\mathbf{g}}_f^*$ is given by

$$\mathbf{e}_1^* = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2^* = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3^* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{e}_4^* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{6.55}$$

Using a test harness to implement the invariant filters (see A1.2), a test element in coordinate frame O:

$$\hat{g}_O = \begin{bmatrix} 0 \\ \omega_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (6.56)$$

is filtered in coordinate frame P. The result of the filtering process (transformed back to O to help comparison) is

$$(\hat{g}_O)_f = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (6.57)$$

This result is correct. The command for ω_2 in (6.56) is inconsistent with the task characteristics and is correctly filtered out.

If this is repeated using a test harness for the switching matrix (see A1.1) then the result is

$$(\hat{g}_O)_f = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -d_1\omega_2 \end{bmatrix} \tag{6.58}$$

This result is incorrect- this would cause a command $v_3 = -d_1\omega_2$ to be sent to the robot, causing the peg to withdraw from the hole.

6.11 Projection Operators

An alternative approach to the partitioning of vector spaces is to use dual vectors as projection operators. This approach was suggested by Dr Selig (Selig 1995). Assume a partition is required into two subspaces $\mathbf{U}$ and $\mathbf{V}$ such that

$$\mathbf{U} \oplus \mathbf{V} = se(3) \qquad \text{and} \qquad \mathbf{U} \cap \mathbf{V} = \mathbf{0} \tag{6.59}$$

Suppose there is a basis $\{\bar{\mathbf{e}}_1 \ldots \bar{\mathbf{e}}_k\}$for $\mathbf{U}$ and a basis $\{\tilde{\mathbf{e}}_{k+1} \ldots \tilde{\mathbf{e}}_6\}$for $\mathbf{V}$. There is a dual basis for $se^*(3)$; $\{\bar{\mathbf{e}}_1^* \ldots \bar{\mathbf{e}}_k^*, \tilde{\mathbf{e}}_{k+1}^* \ldots \tilde{\mathbf{e}}_6^*\}$such that

$$\bar{\mathbf{e}}_i^*(\bar{\mathbf{e}}_j) = \delta_{ij} \qquad \tilde{\mathbf{e}}_i^*(\tilde{\mathbf{e}}_j) = \delta_{ij} \qquad \bar{\mathbf{e}}_i^*(\tilde{\mathbf{e}}_j) = \tilde{\mathbf{e}}_i^*(\bar{\mathbf{e}}_j) = 0 \tag{6.60}$$

Now consider a vector $\mathbf{x} \in se(3)$. It is always possible to write

$$\mathbf{x} = \overline{a}_1\overline{\mathbf{e}}_1 + \ldots + \overline{a}_k\overline{\mathbf{e}}_k + \widetilde{a}_{k+1}\widetilde{\mathbf{e}}_{k+1} + \ldots + \widetilde{a}_6\widetilde{\mathbf{e}}_6 \tag{6.61}$$

where the coefficients are given by:

$$\overline{a}_i = \overline{\mathbf{e}}_i^*(\mathbf{x}) \quad 1 \geq i \geq k \quad \text{and} \quad \widetilde{a}_i = \widetilde{\mathbf{e}}_i^*(\mathbf{x}), \quad k+1 \geq i \geq 6 \tag{6.62}$$

Projection operators can be constructed as follows. Let $\mathbf{M}_{\mathrm{U}}^*$ be the matrix whose rows are the transpose of $\{\overline{\mathbf{e}}_1^* \ldots \overline{\mathbf{e}}_k^*\}$. Therefore

$$\mathbf{M}_{\mathrm{U}}^*\mathbf{x} = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} \tag{6.63}$$

Next let $\mathbf{M}_{\mathrm{U}}$ be the matrix whose columns are $\{\overline{\mathbf{e}}_1 \ldots \overline{\mathbf{e}}_k\}$, so that

$$\mathbf{M}_{\mathrm{U}}\mathbf{M}_{\mathrm{U}}^*\mathbf{x} = a_1\overline{\mathbf{e}}_1 + \cdots a_k\overline{\mathbf{e}}_k \tag{6.64}$$

So the projection operator which projects vectors onto the subspace $\mathbf{U}$ is given by $\mathbf{M}_{\mathrm{U}}\mathbf{M}_{\mathrm{U}}^*$.

It is straightforward to find projection operators for $\mathbf{V}$, and, by a similar argument as given above, for $\mathbf{U}^*$ and $\mathbf{V}^*$ where

$$\mathbf{U}^* \oplus \mathbf{V}^* = se^*(3) \qquad \text{and} \qquad \mathbf{U}^* \cap \mathbf{V}^* = \mathbf{0} \tag{6.65}$$

The square projection operator is a proper geometric entity which can be misinterpreted as a switching or selection matrix in a particular measurement frame.

Given this construction, it is simple to see how this projection operator will transform with a change in reference frame. In frame O,

$$\mathbf{M}_{UO}\mathbf{M}^{*}_{UO}\mathbf{x}_{O} = \mathbf{x}_{fO} \tag{6.66}$$

where f denotes filtered. In a new frame P,

$$\mathbf{M}_{UP}\mathbf{M}^{*}_{UP}\mathbf{x}_{P} = \mathbf{x}_{fP} \tag{6.67}$$

But (6.67) can be written as

$$\mathbf{M}_{UP}\mathbf{M}^{*}_{UP}\hat{\mathbf{A}}_{*}^{-1}\mathbf{x}_{O} = \hat{\mathbf{A}}_{*}^{-1}\mathbf{x}_{fO} \tag{6.68}$$

Therefore

$$\hat{\mathbf{A}}_{*}\mathbf{M}_{UP}\mathbf{M}^{*}_{UP}\hat{\mathbf{A}}_{*}^{-1}\mathbf{x}_{O} = \mathbf{x}_{fO} \tag{6.69}$$

Comparing (6.69) and (6.66),

$$\mathbf{M}_{UO}\mathbf{M}^{*}_{UO} = \hat{\mathbf{A}}_{*}\mathbf{M}_{UP}\mathbf{M}^{*}_{UP}\hat{\mathbf{A}}_{*}^{-1} \tag{6.70}$$

Therefore

$$\mathbf{M}_{UP}\mathbf{M}_{UP}^{*} = \hat{\mathbf{A}}_{*}^{-1}\mathbf{M}_{UO}\mathbf{M}_{UO}^{*}\hat{\mathbf{A}}_{*} \tag{6.71}$$

This transformation is verified in A1.3.

6.12 Transformation of Velocity

The mapping of the tangent spaces induced by the forward kinematics map f :

$$f_{*}: T_{p}M \rightarrow T_{f(p)}M \tag{6.72}$$

is known as the jacobean of the mapping f. The spatial manipulator jacobean as defined on page 83 is denoted $\mathbf{J}$. For an non-redundant robot $n = m = 6$. In general

$$\hat{\mathbf{g}} = \mathbf{J}\,\dot{\Theta} \tag{6.73}$$

Referring to Figure 6-4, if there were no filters present, then the transformation would be expressed as given in (6.73). With a filter F present,

$$F\hat{\mathbf{g}} = F\mathbf{J}\,\dot{\Theta} \tag{6.74}$$

Therefore,

$$\dot{\Theta} = (F\mathbf{J})^{+}F\hat{\mathbf{g}} \tag{6.75}$$

Therefore

$$\dot{\Theta} = (F\mathbf{J})^{+}\hat{\mathbf{g}}_f \tag{6.76}$$

Early implementations of hybrid position / force control mistakenly implemented equation (6.76) as

$$\dot{\Theta} = (F\mathbf{J})^{-1}\hat{\mathbf{g}}_f \tag{6.77}$$

This was incorrect because in general $F\mathbf{J}$ is a singular matrix and does not have an inverse. This was first pointed out in Fisher and Mujtaba (1992). However, their alternative scheme featured a switching matrix and therefore cannot be considered as a complete and proper description in a geometric sense.

The appearance of the left pseudo inverse of the Jacobean gives the shared control scheme an analogy with the control of redundant manipulators (Burdick 1989). The general solution of (6.75) for a revolute robot is in fact

$$\dot{\Theta} = (F\mathbf{J})^{+}\hat{\mathbf{g}}_f + [1-(F\mathbf{J})^{+}(F\mathbf{J})]\mathbf{z} \tag{6.78}$$

where $\mathbf{z}$ is an arbitrary $n \times 1$ vector. Following Fisher and Mujtaba (1992), $\mathbf{z}$ is set to zero, and the so-called "minimum norm" solution is used, given in (6.76).

Chapter 7

Compliance and Shared Control

7.1 Overview of the Chapter

The implementation of an explicit force control scheme is commonly undertaken by closing a external force control loop around an inner position control loop. A wrist-mounted force sensor is typically used. It has been extensively reported that this type of scheme suffers from instability when contacting a stiff environment. The destabilizing mechanism is high gain combined with the use of a non-collocated sensor (Eppinger and Steering 1987) (Daniel et al 1993). Many robot control architectures preclude the implementation of robust force control. However, a compliant device mounted between the robot wrist and the workpiece can be a good alternative, in lieu of explicit force control.

In this Chapter, the geometry of compliant devices is examined in the context of shared control. In this form of shared control, force and displacement are regulated by control of displacement only. A geometrically correct scheme for shared control based on the use of a compliance is derived. This follows naturally from a theoretical analysis of stiffness and potential energy.

7.2 Stiffness

A compliant device is referred to as a generalized spring. A generalized spring is mathematically represented by a potential energy function V defined on $SE(3)$:

$$V: SE(3) \rightarrow \Re \tag{7.1}$$

For a **particle**, it is possible to define a force in terms of the gradient on V :

$$F = -grad\, V \tag{7.2}$$

Equation (7.2) assumes the existence of a natural positive definite metric. Since there is no natural positive definite metric on $SE(3)$, this definition cannot be used. Instead the force exerted by a generalized spring on the rigid body that the spring is attached is given by

$$F = -dV \tag{7.3}$$

where $F \in se^{*}(3)$.

Definition 7.1 Stiffness

At the identity of $SE(3)$, stiffness is a mapping which maps small displacements into forces:

$$f: se(3) \rightarrow se^{*}(3) \tag{7.4}$$

The required mapping is given by the Hessian d^2V at the identity (Loncaric 1985) **Stiffness** is defined as

$$\mathbf{K} \equiv d^2V\Big|_I$$

(Note that one can use left translation to move the region of interest to the identity.)

Stiffness $\mathbf{K}$ can be represented by a symmetric 6×6 matrix:

$$\mathbf{K} \mapsto \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix} \tag{7.5}$$

where $\mathbf{A} = \mathbf{A}^T$, $\mathbf{C} = \mathbf{C}^T$ and $\mathbf{B}$ are 3×3 matrices (Loncaric 1985).

7.3 Potential Energy

The force resulting from a small displacement is

$$\begin{bmatrix} \mathbf{m} \\ \mathbf{f} \end{bmatrix} = -\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{v} \end{bmatrix} \tag{7.6}$$

This relationship gives rise to an expression for potential energy stored in a generalized spring system. First a definition:

Definition 7.2 Potential Energy

Let M be a Riemannian manifold. A differentiable function U

$$U: M \rightarrow \Re \tag{7.7}$$

is called **potential energy** (Arnold 1978).

For a generalized spring system, potential energy can be expressed algebraically as

$$\begin{bmatrix} \mathbf{w}^T & \mathbf{v}^T \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{v} \end{bmatrix} > 0 \tag{7.8}$$

For calculation of potential energy under displacement of the measurement reference frame

$$\mathbf{K} \rightarrow (\hat{\mathbf{A}}_{\bullet})^T \mathbf{K} \hat{\mathbf{A}}_{\bullet} \tag{7.9}$$

This is analogous to the expression for the transformation of the matrix associated with the kinetic energy metric. The transformation ensures that the potential energy measure is invariant to changes in measurement reference frame.

7.4 Conceptual Scheme for Shared Control

A conceptual scheme for shared control follows naturally. In the scheme, an error wrench is produced by subtracting a sensed wrench from a desired wrench. Assuming $\mathbf{K}$ is invertible, then a controller with gain G can produce $\hat{\mathbf{g}}_1 \in se(3)$. The operator may control $\hat{\mathbf{g}}_2 \in se(3)$ where the bases of $\hat{\mathbf{g}}_1$ and $\hat{\mathbf{g}}_2$ span $se(3)$. This type of scheme has been referred to as *kinestatic control* (Griffis and Duffy 1990).

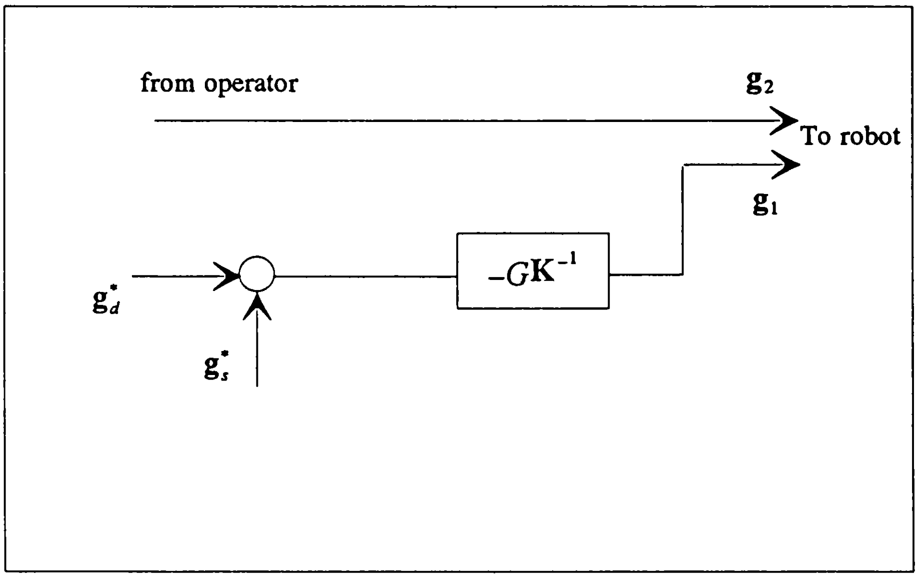

Figure 7-1: Conceptual Shared Control Scheme

A meaningful definition of 'centre of compliance' is the origin of the coordinate frame that decouples rotational and translational aspects of the matrix $\mathbf{K}^{-1}$ (Loncaric 1985).

There is an important assumption here that the stiffness of the robot is much greater than that of the generalized spring. However, conceptually, the stiffness of the robot could be taken into account with this scheme.

To validate the scheme, a test harness (given in A1.4) implements the following scheme:

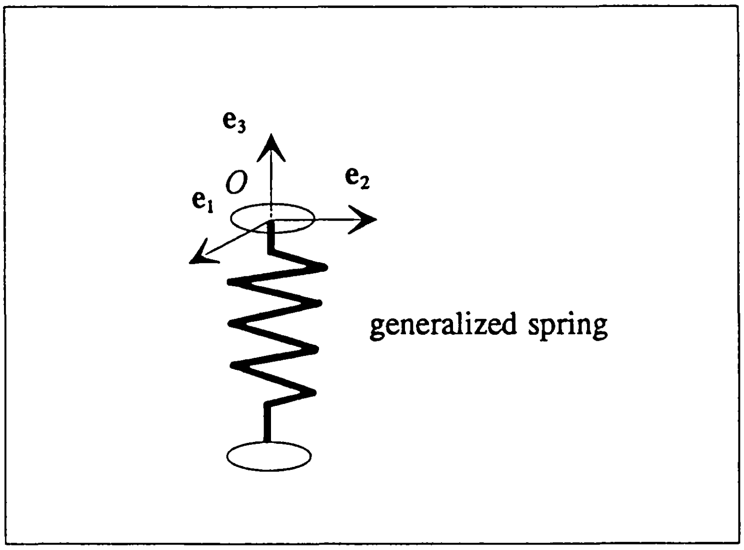

Figure 7-2: Generalized Spring

Defining $\mathbf{K}$ with $\mathbf{A}=\mathbf{I}$, $\mathbf{C}=\mathbf{I}$ and $\mathbf{B}=\mathbf{0}$, then if a desired wrench is given by

$$\hat{\mathbf{g}}^{*}=\begin{bmatrix}0\\0\\-m_3\\0\\0\\-f_3\end{bmatrix} \tag{7.10}$$

Then the required $\hat{g} \in se(3)$ is given by

$$\hat{g}_O = \begin{bmatrix} 0 \\ 0 \\ \omega_3 \\ 0 \\ 0 \\ v_3 \end{bmatrix} \tag{7.11}$$

Under the change of reference frame shown in Figure 7-3, the test harness gives the following result:

$$\hat{g}_P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & d_3 & -d_2 & 1 & 0 & 0 \\ -d_3 & 0 & d_1 & 0 & 1 & 0 \\ d_2 & -d_1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \omega_3 \\ 0 \\ 0 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \omega_3 \\ -d_2\omega_3 \\ d_1\omega_3 \\ v_3 \end{bmatrix} \tag{7.12}$$

If $d_2 = d_3 = 0$ then

$$\hat{g}_P = \begin{bmatrix} 0 \\ 0 \\ \omega_3 \\ 0 \\ d_1\omega_3 \\ v_3 \end{bmatrix} \tag{7.13}$$

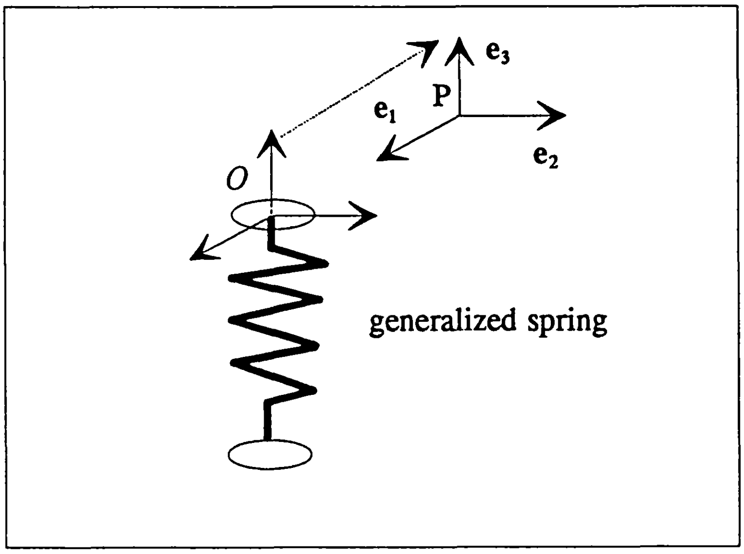

Figure 7-3: Generalized Spring with Shifted Reference Frame

If the reference frame of $\hat{\mathbf{g}} \in se(3)$ is moved from O to P then

$$\hat{\mathbf{g}}_P^{*} = -\mathbf{K}_P\,\hat{\mathbf{g}}_P \tag{7.14}$$

where (see A1.4)

$$\mathbf{K}_P = \left(\hat{\mathbf{A}}^{*}\right)^{-1}\mathbf{K}_O\,\hat{\mathbf{A}}. \tag{7.15}$$

and

$$\left(\hat{\mathbf{A}}^{*}\right)^{-1} = \begin{bmatrix} -\mathbf{A}^T\lfloor\mathbf{d}\rfloor & \mathbf{A}^T \\ \mathbf{A}^T & \mathbf{0} \end{bmatrix} \tag{7.16}$$

The example given above is simple but the theory is applicable to much more sophisticated compliant devices if they can be characterized by an invertible stiffness matrix (i.e. where $\mathbf{K}$ is symmetric positive definite).

Chapter 8

Conclusions

8.1 Overview of the Chapter

In this Chapter, the main contributions of this thesis are put forward and discussed. There are many theoretical questions which arise from a proper geometric discussion of shared control and teleoperation that have not been addressed in this thesis; some ideas for future research are presented.

8.2 Contributions of the Thesis

The geometry that underpins shared control (and hybrid position / force control) is complex and is not generally well understood. The geometric issues associated with hybrid position / force control have been raised by other authors using Ball's screw notation (Ball 1900) and results relating to the invariance of filter forms have been produced. However, the uptake of these results has been slow (Fisher and Mujtaba 1992). In some cases, the results have been openly challenged (Lawrence and Chapel 1994).

In this thesis, the theory of invariant shared control is presented from a different perspective using the theory of differential geometry and Lie groups. The advantage of this approach is that the theory is widely used; differential geometry is the basic tool that physicists use in relativity theory. This is the first presentation of an invariant shared control scheme from the perspective of modern differential geometry.

It has been shown that the switching matrix is equivalent to a filter which embodies a Riemannian metric form. Since there is no natural positive definite metric on $SE(3)$, the metric needs to be transformed correctly if the measurement reference frame is moved in order for the filter to work correctly. Alternatively, the switching matrix can be viewed as a misinterpretation of a projection operator. Again, the projection operator needs to be transformed correctly if the measurement reference frame is moved. This transformation is derived. The invariance properties of filter forms are checked using test harnesses.

The role of compliance in a shared control scheme is analysed. This leads naturally to a control scheme that has been termed kinestatic control. This is the first presentation of the kinestatic control scheme from the perspective of modern differential geometry.

8.3 Ideas for Future Research

A major theme in this thesis has been the unification of ideas under a consistent theory. There may be scope to continue this unification to cover other theoretical descriptions that relate to teleoperation. A particularly suitable candidate may be the theory of bond graphs. This has been a very successful theoretical tool in the analysis of energy flows in a bilateral teleoperation system (Siva 1985).

In the conceptual shared control scheme described in Chapter 6, the constraint manifold is assumed to be smooth. This greatly simplifies the resulting analysis. A challenging research objective would be to extend the theoretical description presented in this thesis to cover the case of the non-smooth constraint manifold. One approach may be to view the constraint manifold parametrically and then to analyse the singularities (Gibson 1995).

The theoretical analysis for the generalized spring given in Chapter 7 could be extended to cover other physically realizable systems - for example the spring plus damper. Also the theory has the potential to include the stiffness of the robot into the control scheme. This has interesting possibilities for shared control with flexible robots.

References

Altmann, S.L. (1986) Rotations, Quaternions, and Double Groups Oxford : Clarendon Press

Arnold, V.I. (1978) Mathematical Methods of Classical Mechanics New York : Springer-Verlag

Backes, P.G. (1990) UMI: An Interactive Supervisory and Shared Control System for Telerobotics Proceedings of IEEE International Conference on Robotics and Automation

Backes, P.G., Beahan, J., Bon, B. (1993) Interactive Command Building and Sequencing for Supervised Autonomy Proceedings of IEEE International Conference on Robotics and Automation

Ball, Sir R.S. (1900) Theory of Screws Cambridge : Cambridge University Press

Bejczy, A.K. (1980a) Sensors, Controls, and Man-Machine Interface for Advanced Teleoperation Science, Vol. 208

Bejczy, A.K. (1980b) Kinesthetic Coupling Between Operator and Remote Manipulator Proceedings of International Conference Tech Conf ASME Century 2 Vol. 1

Bejczy, A.K. (1988) Multi-Mode Manual Control in Telerobotics Proceedings of RoManSy

Bejczy, A.K., Handlykken, M. (1981) Experimental Results with a Six-Degree-Of-Freedom Force-Reflecting Hand Controller Proceedings of 17th Annual Conference on Manual Control

Bejczy, A.K., Salisbury, J.K. (1983) Controlling Remote Manipulators Through Kinesthetic Coupling Computers in Mechanical Engineering

Bicker, R. (1990) Ph.D. Thesis, Department of Mechanical Engineering, University of Newcastle Upon Tyne

Bottema, O., Roth, B. (1979) Theoretical Kinematics Amsterdam : North-Holland Publishing Company

Brockett, R.W. (1988) Robotic Manipulators and the Product of Exponentials Formula Proc. Int. Symp. Math. Theory of Networks and Systems

Burdick, J.W. (1989) On the Inverse Kinematics of Redundant Manipulators: Characterization of the Self-Motion Manifolds Proceedings of IEEE International Conference on Robotics and Automation

Craig, J.J. (1986) Introduction to Robotics Reading : Addison-Wesley Publishing Company

Daniel, R.W., Fischer, P.J., Hunter, B. (1993) A High Performance Parallel Input Device Proceedings of SPIE Telemanipulator Technology and Space Telerobotics Vol. 2057

Doty, K.L., Melchiorri, C., Bonivento, C. (1993) A Theory of Generalized Inverses Applied to Robotics The International Journal of Robotics Research, Vol.12, No.1

Duffy, J. (1990) Editorial: The Fallacy of Modern Hybrid Control that is Based on "Orthogonal Complements" of Twist and Wrench Spaces Journal of Robotic Systems 7(2)

Eppinger, S.D. Steering, W.P. (1987) Understanding Bandwidth Limitations in Robot Force Control Proceedings of IEEE International Conference on Robotics and Automation

Featherstone, R. (1987) Robot Dynamics Algorithms Boston : Kluwer Academic Publishers

Fischer, P.J., Daniel, R.W., Siva, K.V. (1990) Specification and Design of Input Devices for Teleoperation Proceedings of IEEE International Conference on Robotics and Automation

Fischer, P.J. (1993) Ph.D. Thesis, Department of Engineering Science, University of Oxford

Fisher, W.D., Mujtaba, M.S. (1992) Hybrid Position / Force Control: A Correct Formulation The International Journal of Robotics Research, Vol.11, No.4

Gibson, C.G. (1995) Personal Communication

Gilmore, R. (1974) Lie Groups, Lie Algebras, and Some of Their Applications New York : John Wiley & Sons

Goertz, R.C., Blomgren, R.A., Grimson, J.H. (1961) The ANL Model 3 Master-Slave Electric Manipulator - Its Design and Use in a Cave Proceedings of 9th Conference on Hot Labs and Equipment

Goertz, R.C. (1964) Manipulator Systems Development at ANL Proceedings of 12th Conference on Remote Systems Technology

Goertz, R.C., Grimson, J., Potts, C. (1966) ANL Mark E4A Electric Master Slave Manipulator Proceedings of 14th Conference on Remote Systems Technology

Goldstein, H. (1964) Classical Mechanics Reading : Addison-Wesley Publishing Company Inc.

Greenwood, D.T. (1988) Principles of Dynamics New Jersey : Prentice-Hall International Inc.

Griffis, M., Duffy, J. (1990) Kinestatic Control: A Novel Theory for Simultaneously Regulating Force and Displacement 21st ASME Mechanisms Conference

Handlykken, M., Turner, T. (1980) Control System Analysis and Synthesis for a Six Degree-of-Freedom Universal Force-Reflecting Hand Controller Proceedings of 19th IEEE Conference on Decision and Control

Hannaford, B. (1989) Stability and Performance Tradeoffs in Bilateral Telemanipulation Proceedings of IEEE International Conference on Robotics and Automation

Hayati, S., Venkataraman, S.T. (1989) Design and Implementation of a Robot Control System with Traded and Shared Control Capability Proceedings of IEEE International Conference on Robotics and Automation

Hirzinger, G., Heindl, J., Landzettel, K., Brunner, B. (1992) Multisensory shared autonomy - a key issue in the space robot technology experiment ROTEX Proceedings of the IEEE / RSJ Conference on Intelligent Robots and Systems

Hirzinger, G., Brunner, B., Dietrich, J., Heindl, J. (1993) Sensor-Based Robotics - ROTEX and Its Telerobotic Features IEEE Transactions on Robotics and Automation Vol.9, No.5

Hunt, K.H. (1978) Kinematic Geometry of Mechanisms Oxford : Clarendon Press

Kim, W.S. (1990) A Stability Analysis of Shared Compliant Control Japan-USA Symposium on Flexible Automation

Kim, W.S. (1991) A New Scheme of Force Reflecting Control 5th Annual Workshop on Space Operations, Applications and Research (SOAR '91), NASA Conference Publication 3127, Vol. 1

Kim, W.S., Hannaford, B., Bejczy, A.K. (1992) Force Reflection and Shared Compliant Control in Operating Telemanipulators with Time Delay IEEE Transactions on Robotics and Automation, Vol. 8, No. 2

Lawrence, D.A., Chapel, J.D. (1994) Quantitative Control of Manipulator / Task Interaction IEEE Control Systems

Loncaric, J. (1985) Geometrical Analysis of Compliant Mechanisms in Robotics Ph.D. Thesis, Harvard University

Lipkin, H. (1985) Geometry and Mapping of Screws with Applications to the Hybrid Control of Robotic Manipulators Ph.D. Thesis, University of Florida

Lipkin, H., Duffy, J. (1988) Hybrid Twist and Wrench Control for a Robotic Manipulator Transactions of the ASME Vol. 110

Mason, M. (1981) Compliance and Force Control for Computer Controlled Manipulators IEEE Trans. Systems, Man and Cybernetics

McAffee, D., Snow, E., Townsend, W. (1990) A New Six-Degree-of-Freedom Force Reflecting Hand Controller for Space Telerobotics Proceedings of 24th Aerospace Mechanisms Symposium

McAffee, D., Fiorini, P. (1991) Hand Controller Design Requirements and Performance Issues in Telerobotics 5th International Conference on Advanced Robotics

McCarthy, J.M. (1990) An Introduction to Theoretical Kinematics Cambridge : The MIT Press

Mosher, R.S. (1960) An Electrohydraulic Bilateral Servomechanism Proceedings of 8th Conference on Hot Labs and Equipment

Nakahara, M. (1990) Geometry, Topology and Physics Bristol : Adam Hilger

Normand, J-M (1980) A Lie Group: Rotations in Quantum Mechanics Amsterdam : North-Holland Publishing Company

Paden, B., Sastry, S.S. (1988) Optimal Kinematic Design for a 6R Manipulator International Journal of Robotics Research, Vol.7

Park, F.C., Pack, D.J. (1991) Motion Control Using the Product of Exponentials Kinematic Equations Proceedings of IEEE International Conference on Robotics and Automation

Park, F.C., Brockett, R.W. (1994) Kinematic Dexterity of Robotic Mechanisms The International Journal of Robotics Research Vol.13, No.1

Park, F.C., Bobrow, J.E. (1995) Efficient Geometric Algorithms for Robot Kinematic Design Proceedings of IEEE International Conference on Robotics and Automation

Paul, R.P. (1981) Robot Manipulators: Programming, and Control Cambridge : The MIT Press

Price, J.F. (1977) Lie Groups and Compact Groups Cambridge : Cambridge University Press

Raibert, M.H., Craig, J.J. (1981) Hybrid Position / Force Control of Manipulators Transactions of ASME Journal of Dynamic Systems, Measurement and Control, Vol. 102

Raimondi, T. (1976) Remote Handling in the Joint European Torus (JET) Fusion Experiment Proceedings of 24th Conference on Remote Systems Technology

Samuel, A.E., McAree, P.R., Hunt, K.H. (1991) Unifying Screw Geometry and Matrix Transformations The International Journal of Robotics Research, Vol.10, No.5

Sattinger, D.H., Weaver, O.L. (1986) Lie Groups and Algebras with Applications to Physics, Geometry, and Mechanisms New York : Springer-Verlag

Selig, J.M. (1989) Robot Dynamics Using Screw Theory in Robotics: Applied Mathematics and Computational Aspects Oxford : Oxford University Press

Selig, J.M. (1995) Personal Communication

Siva, K.V. (1985) The Covariant Control System and Its Applications in Remote and Robotic Manipulations Ph.D. Thesis, University of London

Siva, K.V. (1992) Force Reflecting Telerobotic Systems IEE Lecture

Spivak, M. (1979) A Comprehensive Introduction to Differential Geometry Berkeley : Publish or Perish Inc

Stein, M.R. (1993) Portability Issues in Teleprogramming: A Case Study Proceedings of SPIE Telemanipulator Technology and Space Telerobotics Vol.2057

Strang, G. (1986) Introduction to Applied Mathematics Wellesley : Wellesley-Cambridge Press

Vukobratovic, M., Stojic, R. (1995) Historical Perspective of Hybrid Control in Robotics: Beginnings, Evolution, Criticism and Trends Mech. Mach. Theory Vol.30 No.4

West, H., Asada, H. (1985) A Method for the Design of Hybrid Position / Force Controllers for Manipulators Constrained by Contact with the Environment Proceedings of IEEE International Conference on Robotics and Automation

Whitney, D.E. (1969) Resolved Motion Rate Control of Manipulators and Human Protheses IEEE Transactions on Man-Machine Systems, Vol. MMS-10, No.2

Wilson, K.B. (1975) Servoarm - A Water Hydraulic Master-Slave Manipulator Proceedings of 23rd Conference on Remote Systems Technology

Wybourne, B.G. (1974) Classical Groups for Physicists New York : John Wiley & Sons

Yoerger, D.R., Slotine, J-J,E. (1987) Supervisory Control Architecture for Underwater Teleoperation Proceedings of IEEE International Conference on Robotics and Automation

Yokokohji, Y., Ogawa, A., Hasunuma, H., Yoshikawa, T. (1993) Operating Modes for Cooperating with Autonomous Functions in Intelligent Teleoperation Systems Proceedings of IEEE International Conference on Robotics and Automation

Appendix 1

A1.1 Switching Matrix Test Harness

For the peg-in-the-hole task described in Chapter 6, and using the relations $\bar{\mathbf{e}}_O \equiv \mathbf{M}_{UO}$ and $\bar{\mathbf{e}}_P \equiv \mathbf{M}_{UP}$

$$\bar{\mathbf{e}}_O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{A1.1}$$

Assuming $\mathbf{A} = \mathbf{I}$,

$$\mathbf{A}_*^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & d_3 & -d_2 & 1 & 0 & 0 \\ -d_3 & 0 & d_1 & 0 & 1 & 0 \\ d_2 & -d_1 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{A1.2}$$

Therefore

$$\bar{\mathbf{e}}_P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & d_3 & -d_2 & 1 & 0 & 0 \\ -d_3 & 0 & d_1 & 0 & 1 & 0 \\ d_2 & -d_1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ -d_2 & 0 \\ d_1 & 0 \\ 0 & 1 \end{bmatrix} \tag{A1.3}$$

Denoting the switching matrix by $\mathbf{S}$,

$$\mathbf{S} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ -d_2 & 0 \\ d_1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ -d_2 & 0 \\ d_1 & 0 \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -d_2 & d_1 & 0 \\ 0 & 0 & -d_2 & d_2^2 & d_2 d_1 & 0 \\ 0 & 0 & d_1 & -d_1 d_2 & d_1^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{A1.4}$$

Now the test vector in frame O is given by

$$\hat{\mathbf{g}}_O = \begin{bmatrix} 0 \\ \omega_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{A1.5}$$

This is transformed to frame P as follows

$$\hat{\mathbf{g}}_P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & d_3 & -d_2 & 1 & 0 & 0 \\ -d_3 & 0 & d_1 & 0 & 1 & 0 \\ d_2 & -d_1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \omega_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \omega_2 \\ 0 \\ d_3\omega_2 \\ 0 \\ -d_1\omega_2 \end{bmatrix} \qquad \text{(A1.6)}$$

Now filter the test vector in frame P using the transformed switching matrix

$$\hat{\mathbf{g}}_{Pf} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -d_2 & d_1 & 0 \\ 0 & 0 & -d_2 & d_2^2 & d_2 d_1 & 0 \\ 0 & 0 & d_1 & -d_1 d_2 & d_1^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \omega_2 \\ 0 \\ d_3\omega_2 \\ 0 \\ -d_1\omega_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -d_2 d_3\omega_2 \\ d_2^2 d_3\omega_2 \\ -d_1 d_2 d_3\omega_2 \\ -d_1\omega_2 \end{bmatrix} \qquad \text{(A1.7)}$$

If $d_2 = d_3 = 0$

$$\hat{\mathbf{g}}_{Pf} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -d_1\omega_2 \end{bmatrix} \qquad \text{(A1.8)}$$

Transforming back to frame O to help comparison,

$$\hat{\mathbf{g}}_{Of} = \mathbf{A}.\hat{\mathbf{g}}_{Pf} \tag{A1.9}$$

Therefore

$$\hat{\mathbf{g}}_{Of} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -d_3 & d_2 & 1 & 0 & 0 \\ d_3 & 0 & -d_1 & 0 & 1 & 0 \\ -d_2 & d_1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -d_1\omega_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -d_1\omega_2 \end{bmatrix} \tag{A1.10}$$

```
%
% MATLAB Script file to demonstrate a Switching matrix for shared control
%
% The example is a classic peg-in-the-hole task

% Assume initially that the origin is at O

% Denote ker(dC) by eO.
eO = [0 0;
     0 0;
     1 0;
     0 0;
     0 0;
     0 1];


% Now consider a change in origin from O to P. If we define
a = [-1 0 0]; % a unit displacement of the origin in -e1 direction

% some definitions
I3 = eye(3);
O3 = zeros(3);
A = I3;
d = [   0 -a(3)  a(2);
     a(3)    0 -a(1);
    -a(2)  a(1)    0;];
```

```
% derive adjoint action
invA_ = [A' O3;(A'* (-d)) I3];
A_ = [A O3;(d*A) A];

% Derive switching matrix
% S = e * e'

% transform eO
eP = invA_ * eO;

% derive filter matrix
SP = eP * eP';

% generate test vector in se(3) [0 1 0 0 0 0]' in frame O
gO = zeros(6,1);
gO(2) = 1;

% transform gO
gP = invA_ * gO;

% filter gP
gPf = SP * gP;

% transform gPf back
gOf = A_ * gPf

% gOf= [0 0 0 0 0 1] - incorrect
```

A1.2 Invariant Filter Test Harness

The invariant filter form in frame O, $\mathbf{F}_O$ is given by

$$\mathbf{F}_O = \bar{\mathbf{e}}_O \left(\bar{\mathbf{e}}_O^T \mathbf{Q}_O \bar{\mathbf{e}}_O \right)^{-1} \bar{\mathbf{e}}_O^T \mathbf{Q}_O \tag{A1.11}$$

Taking the metric form $\langle\, , \rangle_I$, in frame P

$$\mathbf{Q}_P = \left(\mathbf{A}_\bullet\right)^T \mathbf{Q}_O \mathbf{A}_\bullet = \left(\mathbf{A}_\bullet\right)^T \mathbf{A}_\bullet \tag{A1.12}$$

Therefore

$$\mathbf{Q}_P = \begin{bmatrix} 1 & 0 & 0 & 0 & d_3 & -d_2 \\ 0 & 1 & 0 & -d_3 & 0 & d_1 \\ 0 & 0 & 1 & d_2 & -d_1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -d_3 & d_2 & 1 & 0 & 0 \\ d_3 & 0 & -d_1 & 0 & 1 & 0 \\ -d_2 & d_1 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{A1.13}$$

Therefore

$$\mathbf{Q}_P = \begin{bmatrix} 1+d_3^2+d_2^2 & -d_2d_1 & -d_3d_1 & 0 & d_3 & -d_2 \\ -d_1d_2 & 1+d_3^2+d_1^2 & -d_3d_2 & -d_3 & 0 & d_1 \\ -d_1d_3 & -d_2d_3 & 1+d_2^2+d_1^2 & d_2 & -d_1 & 0 \\ 0 & -d_3 & d_2 & 1 & 0 & 0 \\ d_3 & 0 & -d_1 & 0 & 1 & 0 \\ -d_2 & d_1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(A1.14)

Using this result and simplifying,

$$\overline{\mathbf{e}}_P{}^T \mathbf{Q}_P \overline{\mathbf{e}}_P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{A1.15}$$

Therefore

$$\mathbf{F}_P = \overline{\mathbf{e}}_P \overline{\mathbf{e}}_P^T \mathbf{Q}_P \tag{A1.16}$$

Therefore

$$\mathbf{F}_P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -d_2 & 0 & 0 & 0 \\ 0 & 0 & d_1 & 0 & 0 & 0 \\ -d_2 & d_1 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{A1.17}$$

Now

$$\hat{\mathbf{g}}_{Pf} = \mathbf{F}_P \, \hat{\mathbf{g}}_P \tag{A1.18}$$

Therefore

$$\hat{\mathbf{g}}_{Pf} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -d_2 & 0 & 0 & 0 \\ 0 & 0 & d_1 & 0 & 0 & 0 \\ -d_2 & d_1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \omega_2 \\ 0 \\ d_3\omega_2 \\ 0 \\ -d_1\omega_2 \end{bmatrix} \tag{A1.19}$$

Therefore

$$\hat{\mathbf{g}}_{Pf} = \hat{\mathbf{g}}_{Of} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{A1.20}$$

```
%
% MATLAB Script file to demonstrate an Invariant filter for shared control
%
% The example is a classic peg-in-the-hole task

% Assume initially that the origin is at O

% Denote ker(dC) by eO.
eO = [0 0;
      0 0;
      1 0;
      0 0;
      0 0;
      0 1];


% Now consider a change in origin from O to P. If we define
a = [1 0 0]; % a unit displacement of the origin in -e1 direction

% some definitions
I3 = eye(3);
O3 = zeros(3);
A = I3;
d = [   0 -a(3)  a(2);
      a(3)    0 -a(1);
     -a(2)  a(1)    0;];
```

```
% derive adjoint action
invA_ = [A' O3;(A'* (-d)) A'];
A_ = [A O3;(d*A) A];
A_t = [A' (A'* (-d));O3 A'];

% Derive invariant filter
% F = e * inv(e'* Q * e) * e' * Q

% generate Q = I
Q = [I3 O3;O3 I3];

% transform eO
eP = invA_ * eO;

% transform Q
QT = A_t * Q * A_;

% derive filter matrix
FP = eP * inv(eP'* QT * eP) * eP' * QT;

% generate test vector in se(3) [0 1 0 0 0 0]'
gO = zeros(6,1);
gO(2) = 1;

% transform gO
gP = invA_ * gO;
```

```
%   filter gP
gPf = FP * gP;

%   transform gPf back
gOf = A_ * gPf

%   gOf= [0 0 0 0 0 0] as expected
```

A1.3 Test Harness for Projection Operator

The projection operator in frame P is given by

$$\mathbf{M}_{UP}\mathbf{M}_{UP}^{*} = \hat{\mathbf{A}}_{*}^{-1}\mathbf{M}_{UO}\mathbf{M}_{UO}^{*}\hat{\mathbf{A}}_{*} \qquad \text{(A1.21)}$$

For the peg in the hole example and assuming $\mathbf{A} = \mathbf{I}$:

$$\mathbf{M}_{UP}\mathbf{M}_{UP}^{*} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & d_3 & -d_2 & 1 & 0 & 0 \\ -d_3 & 0 & d_1 & 0 & 1 & 0 \\ d_2 & -d_1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -d_3 & d_2 & 1 & 0 & 0 \\ d_3 & 0 & -d_1 & 0 & 1 & 0 \\ -d_2 & d_1 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{(A1.22)}$$

Therefore

$$\mathbf{M}_{UP}\mathbf{M}_{UP}^{*} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -d_2 & 0 & 0 & 0 \\ 0 & 0 & d_1 & 0 & 0 & 0 \\ -d_2 & d_1 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{(A1.23)}$$

$$\hat{\mathbf{g}}_{Pf} = \mathbf{M}_{UP}\mathbf{M}^{\bullet}_{UP}\hat{\mathbf{g}}_{P} \tag{A1.24}$$

Therefore

$$\hat{\mathbf{g}}_{Pf} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -d_2 & 0 & 0 & 0 \\ 0 & 0 & d_1 & 0 & 0 & 0 \\ -d_2 & d_1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \omega_2 \\ 0 \\ d_3\omega_2 \\ 0 \\ -d_1\omega_2 \end{bmatrix} \tag{A1.25}$$

Therefore

$$\hat{\mathbf{g}}_{Pf} = \hat{\mathbf{g}}_{Of} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{A1.26}$$

Note that (A1.21) can be written as

$$\mathbf{M}_{UP}\mathbf{M}^{\bullet}_{UP} = \hat{\mathbf{A}}_{\bullet}^{-1}\mathbf{M}_{UO}\mathbf{M}^{\bullet}_{UO}\left(\hat{\mathbf{A}}^{\bullet}\right)^{T} \tag{A1.27}$$

which yields the equality

$$\hat{\mathbf{A}}_{\bullet} = \left(\hat{\mathbf{A}}^{\bullet}\right)^{T} \tag{A1.28}$$

```
%
% MATLAB Script file to demonstrate a Projection operator for shared control
%
% The example is a classic peg-in-the-hole task

% Assume initially that the origin is at O

% Denote ker(dC) by eO.
eO = [0 0;
      0 0;
      1 0;
      0 0;
      0 0;
      0 1];

estarO = [0 0;
          0 0;
          1 0;
          0 0;
          0 0;
          0 1];

% Now consider a change in origin from O to P. If we define
a = [-1 0 0]; % a unit displacement of the origin in -e1 direction
```

```
% some definitions
I3 = eye(3);
O3 = zeros(3);
A = I3;
d = [   0 -a(3)  a(2);
      a(3)    0 -a(1);
     -a(2)  a(1)    0;];

% derive adjoint action for se(3)
invA_ = [A' O3;(A'* (-d)) I3];
A_ = [A O3;(d*A) A];

% derive adjoint action for se*(3)
invA2_ = [A' (A'* (-d));O3 I3];
A2_ = [A (d*A);O3 A];
invA2_t = [A O3;(d*A) A];

% Derive projection matrix
% PM = M * Mstar
MO = eO;
MstarO=estarO';
PMO = MO * MstarO;

% generate test vector in se(3) [0 1 0 0 0 0]' in frame O
gO = zeros(6,1);
gO(2) = 1;
```

```
% filter gO
gOf = PMO * gO; % [0 0 0 0 0 0] as expected



% derive transformed projection matrix
% transform PMO
PMP = invA_ * PMO * A_;

% transform gO
gP = invA_ * gO;

% filter gP
gPf = PMP * gP;

% transform gPf back
gOf = A_ * gPf; % [0 0 0 0 0 0] as expected

% derive transformed projection matrix another way
% transform MO and MstarO
MP = invA_ * MO;
MstarP = MstarO * invA2_t;

% Derive projection matrix
% PM = M * Mstar
PMP = MP * MstarP;
```

```
% filter gP
gPf = PMP * gP;

% transform gPf back
gOf = A_ * gPf; % [0 0 0 0 0 0] as expected
```

A1.4 Test Harness for Generalized Spring

The required transformation for $\mathbf{K}$ is derived as follows:

$$\hat{\mathbf{g}}_O^{\bullet} = -\mathbf{K}_O \hat{\mathbf{g}}_O \tag{A1.29}$$

Under a passive transformation to frame P, we have

$$\left(\hat{\mathbf{A}}^{\bullet}\right)^{-1} \hat{\mathbf{g}}_O^{\bullet} = -\mathbf{K}_P \left(\mathbf{A}_{\bullet}\right)^{-1} \hat{\mathbf{g}}_O \tag{A1.30}$$

Therefore

$$\hat{\mathbf{g}}_O^{\bullet} = -\hat{\mathbf{A}}^{\bullet} \mathbf{K}_P \left(\mathbf{A}_{\bullet}\right)^{-1} \hat{\mathbf{g}}_O \tag{A1.31}$$

Comparing (A1.29) and (A1.31)

$$\mathbf{K}_O = \hat{\mathbf{A}}^{\bullet} \mathbf{K}_P \left(\mathbf{A}_{\bullet}\right)^{-1} \tag{A1.32}$$

Therefore

$$\mathbf{K}_P = \left(\hat{\mathbf{A}}^{\bullet}\right)^{-1} \mathbf{K}_O \mathbf{A}_{\bullet} \tag{A1.33}$$

```
%
% MATLAB Script file to investigate shared control via generalized springs
%

% Set up stiffness matrix

AK=[1 0 0;
   0 1 0;
   0 0 1];

BK=zeros(3,3);

CK=[1 0 0;
   0 1 0;
   0 0 1];

KO=[AK BK;
  BK'CK];

% Determine wrench for a given screw

gO= [0 0 1 0 0 1]';

fO= - KO * gO;

% Calculate potential energy inner product
PE= gO' * KO * gO;
```

```
% Now consider a change in origin from O to P. If we define
a = [-1 0 0]; % a displacement of the origin in -e1 direction

% some definitions
I3 = eye(3);
O3 = zeros(3);

A=I3;
d = [   0 -a(3)  a(2);
      a(3)    0 -a(1);
     -a(2)  a(1)    0];

% derive adjoint action for velocity (order: ang vel - vel)

A1_ = [A O3;(d*A) A];
invA1_ = [A' O3;(-A'* d) A'];
invA1_t = [A' (-A'* d);O3 A'];

% derive adjoint action for forces (order: moment - force)

A2_ = [A (d*A);O3 A];
invA2_ = [A' (-A'* d);O3 A'];

% Now check things still work correctly in new frame
KP=invA2_ * KO * A1_;
gP = invA1_ * gO;
```

```
fP = - KP * gP;
% switch back to O to compare
fO = A2_ * fP; % same as before

% Calculate potential energy inner product (transform K)
KT = invA1_t * KO * A1_;
PE= gP' * KT * gP; % same as before

% Now see if twist can be derived for a desired wrench
gO= - inv(KO) * fO;


% Now check things still work correctly in new frame
fP = invA2_ * fO;
gP = - inv(KP) * fP ;
% switch back to O to compare
gO = A1_ * gP


% gO=[0 0 1 0 0 1] as expected
```

Appendix 2

A2.1 Identity $\mathbf{A}\lfloor \mathbf{w} \rfloor \mathbf{A}^T = \lfloor \mathbf{A}\mathbf{w} \rfloor$

Proof (Loncaric 1985):

$\mathbf{A}$ is composed of orthonormal row vectors $\mathbf{a}_i$. Therefore,

$$\left(\mathbf{A}\lfloor \mathbf{w} \rfloor \mathbf{A}^T\right)_{ij} = \mathbf{a}_i\left(\mathbf{w} \times \mathbf{a}_j^T\right) = -\mathbf{w}\left(\mathbf{a}_i^T \times \mathbf{a}_j^T\right) \tag{A2.1}$$

This matrix is antisymmetric. Since column vectors $\mathbf{a}_i^T$ are orthonormal, non-zero entries are the components of $\mathbf{w}$ relative to the basis $\left\{\mathbf{a}_i^T\right\}$. Therefore

$$\mathbf{A}\lfloor \mathbf{w} \rfloor \mathbf{A}^T = \lfloor \mathbf{A}\mathbf{w} \rfloor \tag{A2.2}$$

A2.2 Identity $\dot{\mathbf{A}} = \lfloor \mathbf{w} \rfloor \mathbf{A}$

Proof:

$$\mathbf{A}\mathbf{A}^T(t) = \mathbf{I} \tag{A2.3}$$

Therefore

$$\dot{\mathbf{A}}\mathbf{A}^T + \mathbf{A}^T\dot{\mathbf{A}} = \mathbf{0} \tag{A2.4}$$

$$\dot{\mathbf{A}}\mathbf{A}^T = -\left[\dot{\mathbf{A}}\mathbf{A}^T\right]^T \tag{A2.5}$$

Now,

$$\dot{\mathbf{A}}\mathbf{A}^T \equiv \lfloor \mathbf{w} \rfloor \tag{A2.6}$$

where $\lfloor \ \rfloor$ denotes skew symmetric

Therefore,

$$\dot{\mathbf{A}}\mathbf{A}^T = -\lfloor \mathbf{w} \rfloor^T = \lfloor \mathbf{w} \rfloor \tag{A2.7}$$

Therefore

$$\dot{\mathbf{A}} = \lfloor \mathbf{w} \rfloor \mathbf{A} \tag{A2.8}$$

A2.3 Identity $\frac{d}{dt}\lfloor \mathbf{d} \rfloor \mathbf{A} = \lfloor \mathbf{v}_O \rfloor \mathbf{A} + \lfloor \mathbf{w} \rfloor \lfloor \mathbf{d} \rfloor \mathbf{A}$

Proof:

$$\frac{d}{dt}\left(\lfloor \mathbf{d} \rfloor \mathbf{A}\right) = \left\lfloor \frac{d}{dt}\mathbf{d} \right\rfloor \mathbf{A} + \lfloor \mathbf{d} \rfloor \frac{d}{dt}\mathbf{A} \tag{A2.9}$$

Now

$$\frac{d}{dt}\mathbf{d} = \mathbf{v}_P = \mathbf{v}_O - \lfloor \mathbf{d} \rfloor \mathbf{w} \tag{A2.10}$$

Therefore,

$$\frac{d}{dt}\left(\lfloor \mathbf{d} \rfloor \mathbf{A}\right) = \lfloor \mathbf{v}_O \rfloor \mathbf{A} - \lfloor \lfloor \mathbf{d} \rfloor \mathbf{w} \rfloor \mathbf{A} + \lfloor \mathbf{d} \rfloor \lfloor \mathbf{w} \rfloor \mathbf{A} \tag{A2.11}$$

Now, by inspection

$$-\lfloor \lfloor \mathbf{d} \rfloor \mathbf{w} \rfloor \mathbf{A} + \lfloor \mathbf{d} \rfloor \lfloor \mathbf{w} \rfloor \mathbf{A} = \lfloor \mathbf{w} \rfloor \lfloor \mathbf{d} \rfloor \mathbf{A} \tag{A2.12}$$

Therefore

$$\frac{d}{dt}\left(\lfloor \mathbf{d} \rfloor \mathbf{A}\right) = \lfloor \mathbf{v}_O \rfloor \mathbf{A} + \lfloor \mathbf{w} \rfloor \lfloor \mathbf{d} \rfloor \mathbf{A} \tag{A2.13}$$

A2.4 Matrix expression for $e^{\lfloor w\theta \rfloor}$

Using the identites:

$$s \equiv \sin(\|\mathbf{w}\|\theta), \ c \equiv \cos(\|\mathbf{w}\|\theta) \ \text{and} \ v \equiv 1-\cos(\|\mathbf{w}\|\theta) \qquad \text{(A2.14)}$$

Therefore,

$$\lfloor \mathbf{w} \rfloor \sin(\|\mathbf{w}\|\theta) = \begin{bmatrix} 0 & -s\omega_3 & s\omega_2 \\ s\omega_3 & 0 & -s\omega_1 \\ -s\omega_2 & s\omega_1 & 0 \end{bmatrix} \qquad \text{(A2.15)}$$

and

$$\lfloor \mathbf{w} \rfloor^2 (1-\cos(\|\mathbf{w}\|\theta)) = \begin{bmatrix} -v\omega_3^2 - v\omega_2^2 & v\omega_2\omega_1 & v\omega_3\omega_1 \\ v\omega_1\omega_2 & -v\omega_3^2 - v\omega_1^2 & v\omega_3\omega_2 \\ v\omega_1\omega_3 & v\omega_2\omega_3 & -v\omega_2^2 - v\omega_1^2 \end{bmatrix}$$

(A2.16)

Using the identity:

$$\|\mathbf{w}\|^2 = \omega_1^2 + \omega_2^2 + \omega_3^2 = 1 \qquad \text{(A2.17)}$$

Therefore

$$e^{\lfloor w\theta \rfloor} = \begin{bmatrix} \omega_1^2 v + c & \omega_1\omega_2 v - \omega_3 s & \omega_1\omega_3 v + \omega_2 s \\ \omega_1\omega_2 v + \omega_3 s & \omega_2^2 v + c & \omega_2\omega_3 v - \omega_1 s \\ \omega_1\omega_3 v - \omega_2 s & \omega_2\omega_3 v + \omega_1 s & \omega_3^2 v + c \end{bmatrix} \quad \text{(A2.18)}$$

A2.5 $P(\mathbf{x}) = 1/2\left(\mathbf{x}^T\mathbf{A}\mathbf{x} - \mathbf{x}^T\mathbf{b}\right)$ is minimized at $\breve{\mathbf{x}} = \mathbf{A}^{-1}\mathbf{b}$

Proof (Strang 1986):

Consider the following expression:

$$\frac{1}{2}\left(\mathbf{x} - \mathbf{A}^{-1}\mathbf{b}\right)^T \mathbf{A}\left(\mathbf{x} - \mathbf{A}^{-1}\mathbf{b}\right) - \frac{1}{2}\mathbf{b}^T\mathbf{A}^{-1}\mathbf{b} \tag{A2.19}$$

Expanding, we have

$$\frac{1}{2}\left(\mathbf{x}^T\mathbf{A}\mathbf{x} - \mathbf{x}^T\mathbf{A}\mathbf{A}^{-1}\mathbf{b} - \left(\mathbf{A}^{-1}\mathbf{b}\right)^T\mathbf{A}\mathbf{x} + \left(\mathbf{A}^{-1}\mathbf{b}\right)^T\mathbf{A}\mathbf{A}^{-1}\mathbf{b}\right) - \frac{1}{2}\mathbf{b}^T\mathbf{A}^{-1}\mathbf{b} \tag{A2.20}$$

Simplifying gives

$$\frac{1}{2}\left(\mathbf{x}^T\mathbf{A}\mathbf{x} - \mathbf{x}^T\mathbf{b} - \mathbf{b}^T\mathbf{x} + \mathbf{b}^T\mathbf{A}^{-1}\mathbf{b}\right) - \frac{1}{2}\mathbf{b}^T\mathbf{A}^{-1}\mathbf{b} \tag{A2.21}$$

This gives

$$\frac{1}{2}\left(\mathbf{x}^T\mathbf{A}\mathbf{x} - \mathbf{x}^T\mathbf{b}\right) \tag{A2.22}$$

So the minimum of (A2.22) is at

$$\breve{\mathbf{x}} = \mathbf{A}^{-1}\mathbf{b} \tag{A2.23}$$

which brings the first term in (A2.19) to zero.