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A Note on the Evolution of the Whitney Sphere Along Mean Curvature Flow

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Abstract
We study the evolution of the Whitney sphere along the Lagrangian mean curvature flow. We show that equivariant Lagrangian spheres in $\mathbb{C}^n$ satisfying mild geometric assumptions collapse to a point in finite time and the tangent flows converge to a Lagrangian plane with multiplicity two.

Keywords Mean curvature flow · Lagrangian · Whitney sphere

Mathematics Subject Classification 53E10 · 53D12

1 Introduction

The Whitney sphere is the immersion $F : \mathbb{S}^n \to \mathbb{R}^{2n}$ given by

$$F(x_1, \ldots, x_{n+1}) = \frac{1}{1 + x_{n+1}^2}(x_1, x_1 x_{n+1}, \ldots, x_n, x_n x_{n+1}).$$

This immersion is Lagrangian, i.e., $F^*\omega = 0$, where ω is the standard symplectic form on $\mathbb{R}^{2n}$. From the point of view of topology, the Whitney sphere is interesting since it has the best topological behavior: namely, it fails to be embedded only at the north and south pole where it has a transversal double point. An well known result of Gromov asserts that there are no embedded Lagrangian spheres in $\mathbb{C}^n$. On the geometry side, this immersion can be characterized by many geometric rigidity properties, see [3,12].

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In this sense, the Whitney sphere plays the role of totally umbilical hypersurfaces in $\mathbb{R}^n$ in the class of Lagrangian submanifolds.

Another interesting aspect of the Whitney sphere is that it appears as a limit surface under Lagrangian mean curvature flow of some well-behaved Lagrangian submanifolds in $\mathbb{R}^4$. Recall that the mean curvature flow (MCF) of an immersion $F_0 : M^k \to \mathbb{R}^m$ is a map $F : M \to [0, T] \to \mathbb{R}^m$ such that $F(x, 0) = F_0$ and satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} F = H,$$

where H is the mean curvature vector of M^n. It was shown by Smoczyk that the Lagrangian condition is preserved by MCF when the ambient space is a Kähler-Einstein manifold. The Lagrangian mean curvature flow gained a lot of interest recently as a potential tool to find minimal Lagrangian (special Lagrangian) in a given homology class or Hamiltonian isotopy class of a Calabi–Yau manifold. Special Lagrangian submanifolds have the remarkable property of being area minimizing by means of calibration arguments. The classical approach of minimizing area in a given class, however, does not seem very effective to find smooth special Lagrangian as shown by Schoen and Wolfson in [13].

Ideally, one could hope that the evolution of well behaved Lagrangian submanifolds along mean curvature flow to converge to special Lagrangians. In a series of works, A. Neves showed that finite time singularities are unavoidable in the Lagrangian mean curvature flow in general, see [8,10]. It is constructed in [8] a non-compact zero Maslov class Lagrangian in $\mathbb{R}^4$ with bounded Lagrangian angle and in the same Hamiltonian isotopy class of a Lagrangian plane that nevertheless develops a singularity in finite time. At the singular time the limit surface pictures like a connect sum of a smooth Lagrangian (diffeomorphic to a Lagrangian plane) with a Whitney sphere. Such construction were later generalized to 4-dimensional Calabi–Yau manifolds, see [10].

There are very few results regarding the evolution of compact Lagrangian submanifolds in $\mathbb{C}^n$. Motivate by this, we investigate the evolution of the Whitney sphere along mean curvature flow. Despite its many geometric properties, it is not a self-similar solution of the flow. By exploiting its rotationally symmetries, one can reduce its mean curvature flow to a flow about curves in the plane. As a particular case of our main result we prove

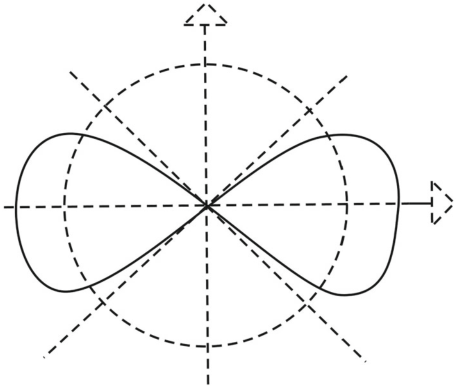
Let $F : \mathbb{S}^n \times [0, T) \to \mathbb{C}^n$ be the maximal existence mean curvature flow of the Whitney sphere. Then $F_T(x) = \{0\}$ for every $x \in \mathbb{S}^n$. The tangent flow at the origin is a Lagrangian plane with multiplicity two.

A Lagrangian submanifold $L \subset \mathbb{C}^n$ is called *equivariant* if there exists a antipodal invariant curve $\gamma : I \to \mathbb{C}$ such that L can be written as

$$L = \{(\gamma(u)\, G_1(x), \ldots, \gamma(u)\, G_n(x)) \in \mathbb{C}^n \;:\; G : \mathbb{C}^{n-1} \to \mathbb{R}^n\},$$

where G is a the standard embedding of $\mathbb{C}^{n-1}$ in $\mathbb{R}^n$. Using spherical coordinates on $\mathbb{C}^n$, $(\cos(u)\, G(x), \sin(u))$, we check that the Whitney sphere is equivariant with

Fig. 1 Whitney sphere



associated curve $\gamma_0 : (0, 2\pi) \to \mathbb{R}^2$ given by:

$$\gamma_0(u) = \left(\frac{\sin(u)}{1 + \cos^2(u)}, \frac{\sin(u)\cos(u)}{1 + \cos^2(u)} \right).$$

The equivariant property is preserved by the mean curvature flow and the corresponding evolution equation for γ_t is

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \overrightarrow{k} - (n-1)\frac{\gamma^\perp}{|\gamma|^2}. \tag{1.1}$$

Here $\overrightarrow{k}$ denotes the curvature vector of γ, it is defined by $\overrightarrow{k} = \frac{1}{|\gamma'|}\frac{\mathrm{d}}{\mathrm{d}u}\frac{\gamma'}{|\gamma'|}$, and $\gamma^\perp$ denotes the normal projection of the position vector γ. This flow is known as the *equivariant flow*.

Definition 1.1 Let $\mathcal{C}$ be the set of antipodal invariant figure eight curves $\gamma : \mathbb{C}^1 \to \mathbb{C}$ with only one self-intersection which is transversal and located at the origin.

Definition 1.2 Let Ω_α be the antipodal invariant region in $\mathbb{R}^2$ bounded by two lines through the origin with angle between them equal to α (Fig. 1).

Theorem 1.3 *Let γ be a curve in $\mathcal{C}$ satisfying at least one of the following assumptions:*

(i) $\{\gamma\} \cap \mathbb{C}^1(R)$ *has at most* 4 *points for every* $R > 0$*;*
(ii) $\{\gamma\} \subset \Omega_{\frac{\pi}{n}}$.

If $\{\gamma_t\}_{t\in[0,T)}$ *is the maximal equivariant flow of* γ*, then* $\gamma_T = \{0\}$*. Moreover, the tangent flow at the origin is a line with multiplicity two.*

Remark 1.4 The assumptions in Theorem 1.3 are sharp. In Sect. 3 we construct for every $\alpha > \frac{\pi}{2}$ a curve $\gamma \in \mathcal{C}$ such that $\{\gamma\} \subset \Omega_\alpha$ that develops a non-trivial singularity at the origin along the equivariant flow (1.1) when $n = 2$.

The proof of Theorem 1.3 follows closely the ideas in [8,9] where it is shown that singularities for the mean curvature flow of monotone Lagrangian submanifolds in $\mathbb{R}^4$ are modeled on area minimizing cones.

2 Preliminaries

Let L be a Lagrangian submanifold in $\mathbb{C}^n$. This implies that $\omega|_L = 0$, where $\omega = \sum_{i=1}^n \frac{\sqrt{-1}}{2} \mathrm{d}z_i \wedge \overline{\mathrm{d}z_i}$ is the standard symplectic form on $\mathbb{C}^n$. Let Ω be the complex valued n-form given by

$$\Omega = \mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_n.$$

A standard computation implies that

$$\Omega|_L = e^{i\theta} vol_L. \tag{2.1}$$

The multivalued function θ is called the *Lagrangian angle* of L. If θ is a single valued function, then L is called *zero-Maslov class*. If $\theta = \theta_0$, then L is calibrated by $\mathrm{Re}(e^{-i\theta_0}\Omega)$ and hence area-minimizing. In this case, L is called *special Lagrangian*. More generally, the Lagrangian angle and the geometry of L are related through $\overrightarrow{H} = J(\nabla\theta)$. Recall also the Liouville one form given by

$$\lambda = \sum_{i=1}^{n} x_i \mathrm{d}y_i - y_i \mathrm{d}x_i.$$

One can check that $\mathrm{d}\lambda = \omega$. In particular, $[\lambda] \in H_1(L)$. When $[\lambda] = c[\mathrm{d}\theta]$ for some $c \in \mathbb{R}$, then L is said to be a *monotone Lagrangian*.

Let L be a equivariant Lagrangian submanifold in $\mathbb{R}^{2n}$. Hence, there exists a regular curve γ in $\mathbb{R}^2$ such that

$$L = \left\{ (\gamma\, G_1, \ldots, \gamma\, G_n) \in \mathbb{R}^{2n}, \sum_{i=1}^{n} G_i^2 = 1. \right\} \tag{2.2}$$

After choosing a parametrization of γ we have

$$\Omega_L := \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n \bigg|_L = e^{i\theta} \mathrm{vol}_L = \frac{\gamma'}{|\gamma'|} \cdot \left(\frac{\gamma}{|\gamma|}\right)^{n-1} \mathrm{vol}_L, \tag{2.3}$$

where $z \cdot w$ denotes the standard multiplication of complex numbers; here we consider γ as complex valued function. The Lagrangian angle relates to the geometry of L.

If L_t is the mean curvature flow starting at L, then L_t shares the same rotational symmetries of L, i.e., $L_t = \{\gamma_t G_1, \ldots, \gamma_t G_n) : G = (G_1, \ldots, G_n) \in \mathbb{C}^{n-1}\}$. Moreover,

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \overrightarrow{k} - (n-1)\frac{\gamma^{\perp}}{|\gamma|^2}. \tag{2.4}$$

Although the term $\frac{\gamma^{\perp}}{|\gamma|^2}$ is not well defined at the origin the quantity has its meaning even when a curve goes through the origin as we can see below.

Lemma 2.1 *Let $\gamma : [-a, a] \to \mathbb{R}^2$ a smooth regular curve such that $\gamma(0) = 0$. Then*

$$\lim_{s\to 0} \frac{\gamma^{\perp}}{|\gamma|^2}(s) = \frac{1}{2}\overrightarrow{k}(0).$$

Proof Let us write the left hand side as

$$\frac{\gamma^{\perp}}{|\gamma|^2}(s) = \frac{1}{|\gamma|^2}\left\langle \gamma, i\frac{\gamma'}{|\gamma'|}\right\rangle i\frac{\gamma'}{|\gamma'|} = \frac{s^2}{|\gamma|^2}\left\langle \frac{\gamma - s\gamma'(0)}{s^2}, i\frac{\gamma'}{|\gamma'|}\right\rangle i\frac{\gamma'}{|\gamma'|}.$$

Using that $\lim_{s\to 0}\frac{\gamma(s)}{s} = \gamma'(0)$ and applying the L'Hopital's rule twice, we obtain

$$\lim_{s\to 0} \frac{\gamma^{\perp}}{|\gamma|^2}(s) = \frac{1}{2}\frac{1}{|\gamma'(0)|^2}\left\langle \gamma''(0), i\frac{\gamma'(0)}{|\gamma'|(0)}\right\rangle i\frac{\gamma'(0)}{|\gamma'(0)|} = \frac{1}{2}\overrightarrow{k}(0).$$

□

Proposition 2.2 (Neves [10]) *Let $\gamma_{i,t} : [-a, a] \to \mathbb{R}^2$, $i = 1, 2$ and $0 \le t \le T$, smooth regular curves satisfying*

(1) $\gamma_{i,t}(-s) = -\gamma_{i,t}(s)$ *for all* $0 \le t \le T$ *and for every* $s \in [-a, a]$.
(2) *The curve* $\gamma_{i,t}$, $i = 1, 2$, *solves the equation*

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \overrightarrow{k} - (n-1)\frac{\gamma^{\perp}}{|\gamma|^2}.$$

(3) $\gamma_{1,0} \cap \gamma_{2,0} = \{0\}$ *(non-tangential intersection) and* $\partial\gamma_{1,t} \cap \gamma_{2,t} = \partial\gamma_{2,t} \cap \gamma_{1,t} = \emptyset$ *for all* t.

Then for all $0 \le t \le T$ *we have* $\gamma_{1,t} \cap \gamma_{2,t} = \{0\}$.

Proof It suffices to restrict to what happens near the origin since the proposition follows from the standard maximum principle applied to the first time of tangential intersection.

First notice that $\gamma_{i,t}$ can be written as a graph on $[-\delta, \delta]$ for some $\delta > 0$. Hence, $\gamma_{i,t}(s) = (s, f_{i,t}(s))$ and we define $h_{i,t}(s) = \frac{f_{i,t}(s)}{s}$. Let's check that $h_{i,t}(s)$ is smooth: if $s \neq 0$, then

$$h'(s) = \frac{f's - f}{s^2} \quad \text{and} \quad h''(s) = \frac{(f''s + f' - f')s^2 - (f's - f)2s}{s^4} = \frac{f''}{s} + 2\frac{f - f's}{s^3}. \tag{2.5}$$

Since $f(0) = 0$ and $f''(0) = 0$ (item (1)), we can apply L'Hopital's rule to show that α' and α'' in (2.5) have a limit when $s \to 0$. Hence, h is twice differentiable.

Finally we consider the function $u_t(s) = h_{1,t} - h_{2,t}$. Notice that $u_0 > 0$ by assumption (3) and $u_t(s) = u_t(-s)$. Recall that in the case of a graph $\gamma(s) = (s, f(s))$ we have

$$\gamma' = (1, f'), \quad \nu = \frac{(f', -1)}{\sqrt{1+(f')^2}} \quad \text{and} \quad \overrightarrow{k} = -\frac{f''}{(1+(f')^2)^{\frac{3}{2}}}\nu.$$

Besides,

$$\frac{\gamma^{\perp}}{|\gamma|^2} = \frac{sf' - f}{s^2 + f^2}\frac{1}{\sqrt{1+(f')^2}}\nu.$$

Therefore, the equation $\frac{\mathrm{d}\gamma}{\mathrm{d}t}^{\perp} = \overrightarrow{k} - (n-1)\frac{z^{\perp}}{|z|^2}$ implies

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{f''}{1+(f')^2} + (n-1)\left(\arctan\frac{f}{s}\right)'.$$

Standard computations imply that $h_{i,t} = \frac{f_{i,t}}{s}$ satisfies

$$\frac{\mathrm{d}\,h_{i,t}}{\mathrm{d}t} = \frac{h''_{i,t}}{1+(s\,h'_{i,t} + h_{i,t})^2} + \frac{h'_{i,t}}{s}\frac{2}{1+(s\,h'_{i,t}+h_{i,t})^2} + (n-1)\frac{h'_{i,t}}{s}\frac{1}{1+h^2_{i,t}}.$$

Now we proceed to find the equation for $\frac{\mathrm{d}u_t}{\mathrm{d}t}$. Using that $\frac{h_{i,t}}{s}$ is also smooth, one can checked that

$$\frac{\mathrm{d}u_t}{\mathrm{d}t} = C_1^2 u''_t + C_2 u'_t + C_3 u_t + C_4^2\frac{u'_t}{s},$$

where each C_k is a smooth and bounded function. By item (3), the function $u_{t=0}$ is strictly positive since γ_1 and γ_2 have a non-tangential intersection at the origin.

Suppose T_1 is the first time where u_t has a zero say at s_0. Hence, s_0 is a minimum point as $u_{T_1} \geq 0$. We consider the function $v_t = u_t e^{-Ct} + \varepsilon(t - T_1)$ where C is very large and ε is a very small positive number. So at (s_0, T_1) we have

$$0 \geq \frac{\mathrm{d}v_t}{\mathrm{d}t}(s_0, T_1) = \frac{\mathrm{d}u_t}{\mathrm{d}t}(s_0, T_1)e^{-CT_1} + \varepsilon \geq \varepsilon + C_4^2\frac{u'_t(s_0)}{s_0}e^{-CT_1}.$$

We used in the equality part that $u_{T_1}(s_)) = 0$ and that $u'_{T_1}(s_0) = 0$ and $u''_{T_1}(s_)) \leq 0$ since s_0 is a minimum point for u_{T_1}. If $s_0 \neq 0$ then the second term in the right hand side is zero and we get a contradiction. If $s_0 = 0$ then that term is just $u''_t(0)e^{-CT_1}$ by the L'Hopital's rule, hence, non-negative and we obtain a contradiction again. □

Corollary 2.3 *The set $\mathcal{C}$ is preserved by the equivariant flow. Moreover, if $\gamma \in \mathcal{C}$ satisfies item (i) (respectively, item (ii)) in Theorem 1.3, then so does γ_t.*

Proof The symmetries of the curve γ are preserved by the equivariant flow, hence γ_t is also antipodal invariant. Proposition 2.2 guarantees that the only self intersection of γ_t is at the origin and it is transversal for every t. Hence, $\mathcal{C}$ is preserved by the equivariant flow. Moreover, Proposition 2.2 also implies that $\gamma_t(s)$ can only intersect the line $\overrightarrow{se^{i\beta}}_{s\in\mathbb{R}}$, with $\frac{\pi}{n} < \beta < \pi - \frac{\pi}{n}$, only at the origin for every $t \in [0, T)$. Therefore, if $\{\gamma\} \subset \Omega_{\frac{\pi}{n}}$, then $\{\gamma_t\} \subset \Omega_{\frac{\pi}{n}}$ also. Finally, by Theorem 1.3 in [2], the number of intersections between $\{\gamma\}$ and $\mathbb{C}^1(R)$ is non-increasing along the flow. □

Lemma 2.4 *If $\gamma \in \Omega_{\frac{\pi}{n}}$, then for every $t > 0$ there exists $\delta_t > 0$ such that $\{\gamma_t\} \subset \Omega_{\frac{\pi}{n}-\delta_t}$.*

Proof Since $\gamma \in \mathcal{C}$ is antipodal invariant and passes through the origin, one can check that $\lim_{s\to 0} \frac{\gamma^{\perp}}{|\gamma|^2}(s) = 0$, where $\gamma(s)$ is a local parametrization of γ with $\gamma(s) = -\gamma(-s)$. By Lemma 2.1, we have that $\overrightarrow{k}(z_0) = \overrightarrow{k}(-z_0) = 0$, where $\gamma(z_0) = \gamma(-z_0) = 0$. Consequently, $\overrightarrow{H}(z_0) = \overrightarrow{H}(-z_0) = 0$. This implies that z_0 and $-z_0$ are critical points of the Lagrangian angle θ_L. It can be check easily that they correspond to local minimum and local maximum critical points. The strong maximum principle applied to $\frac{\mathrm{d}}{\mathrm{d}t}\theta = \Delta\theta$ implies that $\theta_t(z_0) < \theta(z_0)$ and $\theta_t(-z_0) > \theta(-z_0)$. □

Let us use Area(γ) to denote the area enclosed by $\gamma \in \mathcal{C}$. By the Stokes' theorem we have that $\text{Area}(\gamma_t) = -\frac{1}{2}\int_{\gamma_t}\langle\gamma_t, \nu\rangle \mathrm{d}_{\gamma_t}$, where ν is the unit outward normal vector of γ.

Lemma 2.5

$$\pi(T-t) \leq Area(\gamma_t) - Area(\gamma_T) \leq 3\pi(T-t).$$

Proof Let $\gamma_t(u)$ be a parametrization of γ_t. Using that $\nu = i\frac{\gamma_t'}{|\gamma_t'|}$, we have that $\text{Area}(\gamma_t) = -\frac{1}{2}\int_{\gamma_t}\langle\gamma_t, i\,\gamma_t'\rangle \mathrm{d}u$. Hence,

$$\begin{aligned}
\text{Area}'(t) &= -\frac{1}{2}\int_{\gamma_t}\Big(\langle\partial_t\gamma, i\,\gamma_t'\rangle + \langle\gamma, i\,(\partial_t\gamma)'\rangle\Big)\mathrm{d}u \\
&= -\frac{1}{2}\int_{\gamma_t}\Big(\langle\partial_t\gamma, i\,\gamma_t'\rangle + \langle\gamma_t, i\partial_t\gamma\rangle' - \langle\gamma', i\,\partial_t\gamma\rangle\Big)\mathrm{d}u \\
&= -\int_{\gamma_t}\langle\partial_t\gamma, i\,\gamma_t'\rangle\,\mathrm{d}u - \frac{1}{2}\int_{\gamma_t}\langle\gamma_t, i\partial_t\gamma\rangle'\,\mathrm{d}u = -\int_{\gamma_t}\langle\partial_t\gamma, \nu\rangle\,\mathrm{d}\gamma_t.
\end{aligned}$$

The last equality follows from the Fundamental Theorem of Calculus. Hence,

$$\text{Area}'(t) = -\int_{\gamma_t}\left\langle\overrightarrow{k} - (n-1)\frac{z^{\perp}}{|z|^2}, \nu\right\rangle \mathrm{d}_{\gamma_t} = -\int_{\gamma_t}\langle\overrightarrow{k}, \nu\rangle \mathrm{d}_{\gamma_t}.$$

The last equality follows from the Divergence Theorem applied to vector field $X = \frac{z}{|z|^2}$ and the fact that $z = 0$ is not in the interior of the region enclosed by γ_t. Combining the Gauss–Bonnet theorem and the fact that the exterior angle α_t of γ_t at the origin is in $[-\pi, \pi]$ we obtain

$$\int_{\gamma_t} \langle \overrightarrow{k}, -\nu \rangle \mathrm{d}_{\gamma_t} + \alpha_t = 2\pi \Longrightarrow \pi \leq \int_{\gamma_t} \langle \overrightarrow{k}, -\nu \rangle \mathrm{d}_{\gamma_t} \leq 3\pi.$$

Therefore, $-3\pi \leq \mathrm{Area}'(\gamma_t) \leq -\pi$. The Lemma now follows if we integrate this quantity from t to T. □

3 Proof of the Theorem

Let L_t be a solution of the mean curvature flow starting on a k-dimensional submanifold L in $\mathbb{R}^m$. Consider the backward heat kernel

$$\Phi_{x_0,T}(x,t) = \frac{1}{(4\pi(T-t))^{\frac{k}{2}}} e^{-\frac{|x-x_0|^2}{4(T-t)}}.$$

The following formula is known as the Huisken's monotonicity formula:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_{L_t} f_t \Phi_{x_0,T} \mathrm{d}\mathcal{H}^k \\ &= \int_{L_t} \left(\frac{\mathrm{d}}{\mathrm{d}t} f_t - \Delta f_t - \left| H - \frac{(x-x_0)^\perp}{2(T-t)} \right|^2 f_t \right) \Phi_{x_0,T} \mathrm{d}\mathcal{H}^k, \end{aligned} \tag{3.1}$$

where $d\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure.

Recall that if $\{L_t\}_{t\in[0,T)}$ is the Lagrangian mean curvature flow starting at L, then

$$L_s^\sigma = \sigma \left(L_{T+\frac{s}{\sigma^2}} - x_0 \right),$$

for $s \in [-T\lambda^2, 0)$, also satisfies the Lagrangian mean curvature flow and is referred as the tangent flow at x_0. The following is a restatement of Theorem 1.3:

Theorem 3.1 *Let γ be a curve in $\mathcal{C}$ which satisfies at least one of the following assumptions*

(i) $\{\gamma\} \cap \mathbb{C}^1(R)$ *has at most* 4 *points for every* $R > 0$*;*
(ii) $\{\gamma\} \subset \Omega_{\frac{\pi}{n}}$.

If $\{\gamma_t\}_{t\in[0,T)}$ is the maximal equivariant flow of γ, then $\gamma_T = \{0\}$. Moreover, the tangent flow at the origin is a line with multiplicity two.

Proof Let us prove first that if $z = 0$ is a singular point, then $\gamma_T = \{0\}$. Arguing by contradiction, we assume that $z = 0$ is a singular point for $\{\gamma_t\}_{0\leq t<T}$ and $\gamma_T \neq \{0\}$. Given $\sigma_i \to \infty$, let $\gamma_s^i = \sigma_i \gamma_{T+\frac{s}{\sigma_i^2}}$.

Lemma 3.2 *Let a and b real numbers such that $a < b < 0$. Then*

$$\lim_{i\to\infty}\int_a^b\int_{\gamma_s^i\cap A(\frac{1}{\eta},\eta,0)}\left(|\overrightarrow{k}|^2+|\gamma^\perp|^2\right)\mathrm{d}\mathcal{H}^1 ds=0,$$

where $A(\frac{1}{\eta},\eta,0)$ is an annulus centered at $z=0$ with inner and outer radius η and $\frac{1}{\eta}$, respectively.

Proof Let L_s^i be the immersed Lagrangian sphere in $\mathbb{C}^2$ obtained via $L_s^i = (\gamma_s^i G_1,\ldots,\gamma_s^i G_n)$. It is proved in Lemma 5.4 in [8] that

$$\lim_{i\to\infty}\int_a^b\int_{L_s^i\cap B_R(0)}\left(|H|^2+|x^\perp|^2\right)\mathrm{d}\mathcal{H}^n(x)ds=0, \tag{3.2}$$

where H is the mean curvature vector of L_s^i. For the convenience of the reader let us recall the proof of this fact. It is a standard computation to check that the Lagrangian angle θ obeys the following evolution equation $\frac{\mathrm{d}}{\mathrm{d}t}\theta_{i,s}^2 = \Delta\theta_{i,s}^2 - 2|H|^2$. Applying (3.1) with $f_t=\theta_{i,s}^2$ and $f_t=1$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_{L_s^i}\theta_{i,s}^2\Phi\mathrm{d}\mathcal{H}^n=\int_{L_s^i}\left(-2|H|^2-\left|H-\frac{x^\perp}{2s}\right|^2\theta_{i,s}^2\right)\Phi\,\mathrm{d}\mathcal{H}^n \tag{3.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_{L_s^i}\Phi\mathrm{d}\mathcal{H}^n=\int_{L_s^i}-\left|H-\frac{x^\perp}{2s}\right|^2\Phi\,\mathrm{d}\mathcal{H}^n, \tag{3.4}$$

respectively. Integrating (3.3) from a to b gives

$$2\lim_{i\to\infty}\int_a^b\int_{L_s^i}|H|^2\,\Phi\,\mathrm{d}\mathcal{H}^n ds\le\lim_{i\to\infty}\int_{L_b^i}\theta_{i,b}^2\Phi\,\mathrm{d}\mathcal{H}^n-\lim_{i\to\infty}\int_{L_a^i}\theta_{i,a}^2\Phi\,\mathrm{d}\mathcal{H}^n=0.$$

The last inequality follows from the scale invariance and monotonicity of $\int_{L_t}\theta^2\,\Phi\mathrm{d}\mathcal{H}^n$. Similarly, we obtain

$$\lim_{i\to\infty}\int_a^b\int_{L_s^i}\left|H-\frac{x^\perp}{2s}\right|^2\Phi\,\mathrm{d}\mathcal{H}^n ds=\lim_{i\to\infty}\int_{L_b^i}\Phi\,\mathrm{d}\mathcal{H}^2-\lim_{i\to\infty}\int_{L_a^i}\Phi\,\mathrm{d}\mathcal{H}^n=0.$$

It follows from the triangular inequality that

$$\lim_{i\to\infty}\int_a^b\int_{L_s^i}\left|\frac{x^\perp}{2s}\right|^2\Phi\,\mathrm{d}\mathcal{H}^n\mathrm{d}s=0.$$

This completes the proof of (3.2). As $|H|^2 = |\overrightarrow{k} - (n-1)\frac{\gamma^\perp}{|\gamma|^2}|^2$ and $|x^\perp|^2 = |\gamma^\perp|^2$, we obtain for each $\eta > 0$ that

$$\lim_{i\to\infty}\int_a^b\int_{\gamma_s^i\cap A(\frac{1}{\eta},\eta,0)}\left(|\overrightarrow{k}|^2 + |\gamma^\perp|^2\right)\mathrm{d}\mathcal{H}^1\mathrm{d}s = 0.$$

□ From previous lemma it follows that for almost every $s \in (a, b)$ that

$$\lim_{i\to\infty}\int_{\gamma_s^i\cap A(\frac{1}{\eta},\eta,0)}\left(|\overrightarrow{k}|^2 + |\gamma^\perp|^2\right)\mathrm{d}\mathcal{H}^1 = 0.$$

This implies that γ_s^i converges to a union of lines in $C^{1,\frac{1}{2}}_{loc}(\mathbb{R}^2 - \{0\})$. In fact, each connected component of γ_s^i inside $B_R(0) - \{0\}$ converge to a line segment with multiplicity one since the convergence is in $C^{1,\frac{1}{2}}_{loc}(\mathbb{R}^2 - \{0\})$.

Assume first that γ satisfies **item i)**, then by Proposition 2.2 and Corollary 2.3, the curve γ_s^i in $B_R(0) - \{0\}$ has two embedded connected components. Hence, each converges to a line segment with multiplicity one in $B_R(0) - \{0\}$. Equivalently, in a neighborhood of the origin L_t is a union of two smooth embedded discs intersecting transversally at a interior point. Hence, each piece of L_s^i converges weakly to a plane with multiplicity one. Since $\gamma_T \neq \{0\}$, we can talk about the localized Gaussian density of each connected component of $L_t \cap B_r(0)$ computed at $(0, T)$ which will be very close to one. Applying White's Local Regularity Theorem, see localized version Theorem 5.6 in [4]), to each component of $L_t \cap B_r(0)$, we conclude that the origin is not a singularity of $\{L_t\}_{t\in[0,T)}$, contradiction.

To handle other connected components of γ_s^i in $B_{4R}(0)$ we study the Lagrangian angle θ_s^i. Let β be a primitive of λ_L. It is proved in [9] that $\nabla\beta = J(x^\perp)$ and $\frac{\mathrm{d}}{\mathrm{d}t}\beta = \Delta\beta - 2\theta$. This implies that the function $u = \beta + 2(t - t_0)\theta$ satisfies $\frac{\mathrm{d}}{\mathrm{d}t} f(u) = \Delta f(u) - f''(u)|x^\perp + 2(t - t_0)H|^2$, where $f \in C_0^\infty(\mathbb{R})$. Plugging the function $f(u)$ in (3.1), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_{L_s^i} f(u_s^i)\Phi = -\int_{L_s^i}\left|H - \frac{x^\perp}{2s}\right|^2 f(u_s^i)\,\Phi\, + f''(u_s^i)\left|x^\perp + 2(s - s_0)H\right|^2\Phi.$$

Integrating this formula from -1 to s_0 and using (3.2) , we obtain

$$\lim_{i\to\infty}\int_{L_{s_0}^i\cap B_{4R}(0)} f(\beta_{s_0}^i)\Phi \;=\; \lim_{i\to\infty}\int_{L_{-1}^i\cap B_{4R}(0)} f(\beta_{-1}^i - 2(1 + s_0)\theta_{-1}^i)\Phi.$$

Let γ^i be a connected component of γ_s^i in $B_{4R}(0)$ that intersects $B_R(0)$ and does not passes through the origin. Since $|\nabla f(\beta_s^i)|$ is bounded, there exists a constant b_{s_0} such that $\lim_{i\to\infty} f(\beta_{s_0}^i) = f(b_{s_0})$. Similarly, $\lim_{i\to\infty} f(\beta_{-1}^i) = f(b_{-1})$. As before, γ_i converges in $C^{1,\frac{1}{2}}(\mathbb{R}^2 - \{0\})$ to lines $l_{\overrightarrow{v_1^s}}$ and $l_{\overrightarrow{v_2^s}}$ in the direction of the vectors $\overrightarrow{v_i^s}$. Moreover,

$$\lim_{i\to\infty}\int_{\gamma^i} f(\beta^i_{-1}-2(1+s_0)\theta^i_{-1})\Phi\, \mathrm{d}\mathcal{H}^1 = \sum_{i=1}^{2}\int_{l_{\vec{v_i}}} f(b_{-1}-2(1+s_0)\theta_i)\Phi\, \mathrm{d}\mathcal{H}^1.$$

Note that (2.3) implies that θ^i_s converge to a constant in each connected component of $\gamma^i_s\cap(B_R(0)-B_r(0))$. We claim that $\theta_1=\theta_2$. Otherwise, by choosing f with support near b_{s_0} and equal to 1 near b_{s_0}, we obtain

$$\sum_{i=1}^{2}\int_{l_{\vec{v_i}}\cap B_R(0)}\Phi\, \mathrm{d}\mathcal{H}^1 = \int_{l_{\vec{v_{i_0}}}\cap B_R(0)}\Phi\, \mathrm{d}\mathcal{H}^1,$$

contradiction.

Let us assume that γ satisfies item (ii). In this case, $\gamma^i_s\cap B_{4R}(0)$ has a connected component γ^i intersecting B_{2R} which converges in $C^{1,\frac{1}{2}}(B_R(0)-\{0\})$ to the lines γ_A and γ_B with multiplicity one. Moreover, θ^i_s converge to a constant θ_0 on each connected component of $\gamma^i\cap(B_R(0)-B_r(0))$. This implies that $\gamma_A=\gamma_B$ with the same orientation or the angle between γ_A and γ_B is $\frac{\pi}{n}$. The first case cannot happen since $I_2(\beta^i_s,\mathbb{C}^1(0,r))=0$, where $I_2(\cdot,\cdot)$ is the intersection number mod 2. The second case cannot happen since $\{\gamma_t\}\subset\Omega_{\frac{\pi}{n}-\delta_t}$ by Lemma 2.4. Hence, the origin is not a singularity if we assume that $\gamma_T\neq\{0\}$.

On the other hand, no singularities away from the origin occur. Indeed, in [11] Oaks complement the work of Angenent on singularities of equations of type $\frac{\mathrm{d}}{\mathrm{d}t}\gamma_t = V(\overrightarrow{T},k)\overrightarrow{N}$ by showing that near the singularity the curve γ_t must lose a self intersection. Since Proposition 2.2 asserts the only self intersection of γ_t is at the origin we are done.

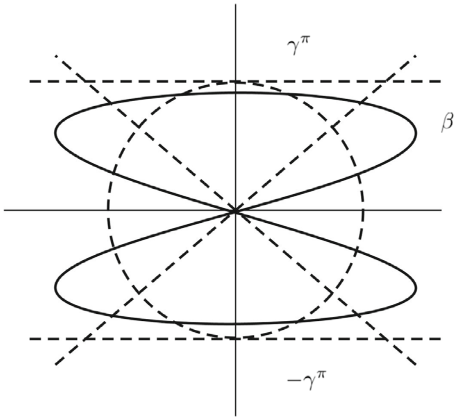
Now let us prove that the tangent flow at the singular point is a line through the origin with multiplicity 2. For this we choose a sequence of scale factors $\lambda_i\to+\infty$ and we set $\gamma^i_s=\lambda_i\gamma_{T+\frac{s}{\lambda_i^2}}$ defined in $[-T\lambda_i^2,0)$.

As discussed before γ^i_s converges in $C^{1,\frac{1}{2}}_{\mathrm{loc}}(\mathbb{R}^2-\{0\})$ to a union of two lines through the origin for almost every s fixed. Let us denote them by l_A and l_B. As $\mathrm{Area}(\gamma_t)$ is going to zero there exist a unique $t_i\in[0,T)$ for which $\mathrm{Area}(\gamma_{t_i})=\frac{1}{\lambda_i^2}$. This implies that $\mathrm{Area}(\gamma^i_{s^i_1})=1$, where s^i_1 is given by $s^i_1=-\lambda_i^2(T-t_i)$. Since $\pi(T-t)\le A(t)\le 3\pi(T-t)$ by Lemma 2.5, we obtain that $s^i_1\in[-\frac{1}{\pi},-\frac{1}{3\pi}]$. In particular, if $s^*=-\frac{1}{3\pi}$, then $\limsup_{i\to\infty}\mathrm{Area}(\gamma^i_{s^*})\le 1$. Therefore, $\gamma^i_{s^*}$ must converge to $2\gamma_A+2\gamma_B$ or $\gamma_A=\gamma_B$ since $\gamma^i_{s^*}$ is becoming non-compact enclosing bounded area.
The first case does not happen as it violates the assumptions (i) and (ii) as discussed above. □

The next example constructs equivariant Lagrangian spheres in $\mathbb{R}^4$ that do not collapse to a point along the mean curvature flow.

Example 3.3 Let γ_0 be the curve $\gamma^\alpha(u)=\sin(\frac{\pi u}{\alpha})^{-\frac{\alpha}{\pi}}(\cos(u),\sin(u))$ with $u\in\mathbb{R}$. The existence of a solution of the equivariant flow starting at γ^α is given in [8], let

Fig. 2 Curve β



us denote it by $\{\gamma_t\}_{t\in[0,T_\alpha)}$. It is shown in [8] that when $\alpha > \frac{\pi}{2}$, then $T_\alpha < \infty$ and γ_t develops a singularity at the origin. When $\alpha \in (0,\pi)$, then γ^α is contained in Ω_α and it is asymptotic to its boundary. Consider the region U_α in Ω_α that is bounded by $\{\gamma^\alpha\}\cup\{-\gamma^\alpha\}$. One can check that U_α has infinite area. Choose $\beta \in \mathcal{C}$ contained in U_α whose area enclosed, Area(β), is greater than $3\pi\, T_\alpha$. See Fig. 2 for the case $\alpha = \pi$. Let $\{\beta_t\}_{t\in[0,T)}$ be the solution of the equivariant flow starting at β. By the avoidance principle, β_t and γ_t do not intersect. Hence, $T < T_\alpha$. On the other hand, by Lemma 2.5 we have that $\text{Area}(\beta_T) \geq \text{Area}(\beta) - 3\pi T \geq 3\pi(T_\alpha - T) > 0$. Therefore, a non trivial singularity must occur at the origin.

Let us show that any Type II dilation of γ_t near the singularity converges to an eternal solution of curve shortening flow. As in Chapter 4 in [7], there exist for each $k > 0$, points $z_k \in \gamma_t(\mathbb{C}^1)$, $t_k \in [0, T - \frac{1}{k}]$, and scaling $\lambda_k > 0$ such that $\beta^k_s = \lambda_k(\gamma_{T+\frac{s}{\lambda_k^2}} - z_k)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}s}\beta^k_s = \overrightarrow{k}(\beta^k_s) - (n-1)\frac{(\beta^k_s + \lambda_k z_k)^\perp}{|\beta^k_s + \lambda_k z_k|^2},$$

where $s \in (a_k, b_k)$. Moreover, $\lim_{k\to\infty} a_k = -\infty$, $\lim_{k\to\infty} b_k = \infty$, and $0 < \lim_{k\to\infty} \sup_{(a_k,b_k)\times\mathbb{C}^1} |\overrightarrow{k}(\beta^k_s)| \leq C$. It is proved that β^k_s converge smoothly as $k \to \infty$ to a non-compact flow $(\beta_s)_{s\in\mathbb{R}}$. We claim that $\lim_{k\to\infty} \lambda_k z_k = \infty$. If not, then we could replace the points z_k by $z = 0$ and obtain the same conclusion. This is impossible since central dilations converge to lines. Therefore, as $k \to \infty$,

$$\frac{\mathrm{d}}{\mathrm{d}s}\beta_s = \overrightarrow{k}(\beta_s).$$

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