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What is teaching with variation and is it relevant to teaching and learning mathematics in England?

Laurie Jacques

UCL Institute of Education, UK

This paper considers the literature to understand a teaching approach used by mathematics teachers in Shanghai, known as teaching with variation or *bianshi*. What the bianshi framework involves is explored; also what impact bianshi has had on learning mathematics in Shanghai and how relevant this approach might be to primary classrooms in England. Bianshi involves generalising from examples using *conceptual* and *procedural* variation for concept development. There is some evidence to suggest that bianshi has a positive impact on mathematics learning in parts of China and is complementary to *constructivist* principles and thus potentially transferable to classrooms in England.

Keywords: procedural variation; conceptual variation; Shanghai; bianshi; primary; generalisation

Introduction

In recent years, the Department for Education for England (DfE), through the channels of the National Centre for Excellence in the Teaching of Mathematics (NCETM) and the Maths Hubs network, have promoted mathematics teaching practices observed in Shanghai, China (Department for Education, 2016). 'Expert teachers' from Shanghai have provided show-case lessons for primary teachers from England in both China and in English primary schools. One feature of these lessons, of interest to the author, is the attention the expert teachers pay to the deliberate variation of problems and examples as their lessons unfold. This is called bianshi jiaoxue ("teaching with variation"). The use of 'variation' is just one part of the reform agenda in England however, this paper will argue that particular attention to this might indicate how well the desired reform can be achieved.

This paper considers the theoretical framework for bianshi in order to understand the implications for its use in the afore-mentioned English context and contributes to the author's doctoral research that seeks to answer the following research question:

> *What changes and what stays the same when primary teachers incorporate pedagogical practices associated with promoting learning from variation?*

The paper attempts to address the following questions:

1. How is 'teaching with variation' evident in Shanghai, China?
2. What evidence suggests that 'teaching with variation' has a positive impact on teaching and learning mathematics?

3. What evidence suggests that 'teaching with variation' is a practice that could be transferable to primary classrooms in England?

Teaching with Variation in Shanghai – The Bianshi Framework

Worldwide interest in the mathematics classroom practices in East Asian countries has arisen as a result of international student achievement tests such as TIMSS 2011 (Mullis, Martin, Foy, & Aurora, 2012) and PISA (Organisation for Economic Co-operation and Development, 2014), which have revealed the repeated success of Chinese pupils in the mathematics element of these tests. Numerous studies continue to examine Chinese mathematics instruction in an attempt to draw out the features of teaching and learning mathematics that might contribute to this 'superior' performance (L. Gu, Yang, & He, 2015) and the presence of bianshi has been noted (Clarke, Keitel, & Yoshinori, 2006). Bianshi is based on the Chinese maxim "*only by comparing can one distinguish*" (F. Gu, Huang, & Gu, 2017). It involves the architect of learning (teacher or textbook author) devising opportunities for learners to distinguish variant and invariant properties of a mathematical object (concept or procedure) in order to gain a deeper understanding of a mathematical concept or process. A longitudinal study conducted in the region of Qingpu in the 1980s and 1990s in Shanghai observed two types of variation used by Chinese teachers and defined them as *conceptual* and *procedural* variation (L. Gu, Huang, & Marton, 2004). Such "indigenous" approaches are strongly evident in task design in China (Sun, 2013).

Conceptual variation

Conceptual variation is concerned with experiencing a concept from multiple perspectives which contributes to deeper understanding of a concept. The teacher provides examples of a concept by offering deliberately varied contexts or representations. E.g. to understand the meaning of 'three' as a quantity, one should experience threes of different objects, sounds, movements, in different arrangements as well as how it relates to 2 and 4. These combined experiences all contribute to a deeper understanding of its mathematical structure. Examples such as these are handled carefully and frequently by Chinese teachers (Cai & Nie, 2007). In a show-case Year 4 lesson in England in 2017, an expert teacher from Shanghai compared three-quarters with one-quarter using a number line, a bar model and an area model and used them to orchestrate discussions to lead the pupils to draw out the generalisation "to compare two fractions they must be of the same whole".(See fig. 1).

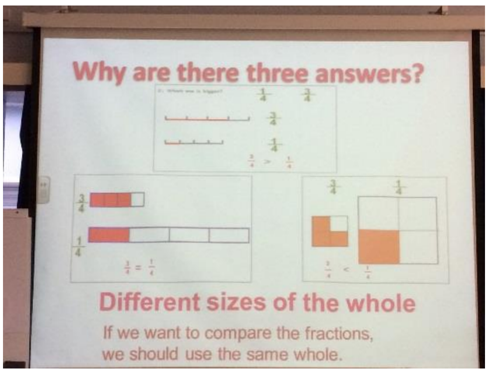

Fig 1. Representations for comparing two fractions each of a different whole.

Procedural variation

Procedural variation is used by teachers to promote conceptual understanding of a mathematical object through the use of 'problem sets' (Sun, 2011). In her analysis of Chinese textbooks, Sun (2011) categorised procedural variation 'problem sets' in the following way:

I. Varying one condition in a problem to become aware of the variant and invariant relationships. (one problem multiple changes, OPMC)
II. Varying the approach to solve a single problem. (one problem multiple solutions, (OPMS)
III. Varying the problems that use a single approach to solve a variety of problems. (multiple problems one solution, MPOS)

The use of conceptual and procedural variation offers the learner a 'space of relations' from which learners can abstract generalisations that contribute to the building of a comprehensive structure of a mathematical object (Sun, 2011). Sun only focuses her attention on the presence of problems that are chosen with procedural variation. She does not describe how teachers design and promote learning by using varied problems. It is the author's view that how these problems unfold is of central importance to the bianshi framework. L. Gu et al. (2004) provide further constructs within the bianshi framework in relation to procedural variation.

Pudian, anchoring point and potential distance

In procedural variation, the order in which a teacher chooses to introduce a set of problems is called *pudian* (L. Gu et al., 2004). This sequence offers pupils experience of steps that carefully unfold the intended concept. The first step that is chosen is familiar to the pupils and defined as the *anchoring point of knowledge* (L. Gu et al., 2004). Teachers consider how to bridge the gap between the *anchoring point* and the intended new learning which is defined as the *potential distance* (L. Gu et al., 2004). Teachers must understand how to vary the steps of the pudian to create 'proper learning distances' (Ding, Jones, Mei, & Sikko, 2016).

The impact of bianshi on teaching and learning mathematics in Shanghai

Bianshi has sparked interest with several researchers from western cultures as a possible explanation of why Chinese learners achieve so much better than their international counterparts in comparative tests (Clarke et al., 2006). It is not argued that bianshi is solely responsible for the high performance in these international comparison tests, however a number of studies have evaluated the effectiveness of variation in Chinese classrooms, concluding positive effects on pupil learning. L. Gu et al. (2004); Bao et al (2003) (cited in Shao, Fan, Huang, Ding, and Li (2013)) and F. Gu et al. (2017). Other studies are reported in Mandarin making the evidence difficult to assess for non-Mandarin-speakers.

The findings from a large longitudinal study conducted in the poor Shanghai district of Qingpu (L. Gu et al. (2004); (F. Gu et al., 2017)) shed further light. As part of educational reform in China in the late 1970s, when standards of achievement in mathematics were generally poor in this district, Lingyuan Gu and colleagues began exploring the use of variation in classrooms with a small number of experimental schools. The number of experimental schools increased and by 1986 the pass rate for entrance to the junior high schools had risen to 85% from 16% in 1979 (L. Gu et al., 2015). As a result, the reformed teaching approaches were extended to the whole of China. Other features of Chinese mathematics classrooms such as coherence (Wang & Murphy, 2004); teacher dominated lessons (Mok, 2006) and profound teacher knowledge (Ma, 1999) might also contribute to their success but in each case these features can be referred back to the presence of deliberate variation.

Some research on the use of deliberate variation in secondary mathematics classrooms in England has shown that promoting learning from variation "is a powerful design strategy for producing exercises that encourage learners to engage with mathematical structure, to generalize and to conceptualize, even when doing apparently mundane questions…" but "…knowing more about its impact on learning is going to take more experimentation and longer immersion" (Watson & Mason, 2006, p. 108).

Bianshi and constructivism

Learning theories related to constructivism are widely and implicitly used as the principles of teaching and learning mathematics in the US, UK and mainland Europe. Over the last 50 years the works of Piaget, Bruner and Vygostsky have influenced mathematics educators keen to apply research findings to improve the teaching and learning of mathematics. Constructivism is based on new learning forming as a result of personal experiences that build on prior knowledge: learners construct knowledge on the basis of links with previous knowledge. However, the principles for teaching and learning mathematics in China have been exposed to influences over millennia; none more so than those derived from their Confucian heritage (Shao et al., 2013). Since the turn of the new millennium, mathematics curriculum reform in China has begun to incorporate classroom practices heavily influenced by the US National Council of Teachers of Mathematics (NCTM) standards, which promotes pupils' learning using constructivist principles (Clements & Battista, 1990). However, bianshi has remained a feature despite this reform. So, if it has been possible to map constructivist principles to practices that make use of bianshi in China, then one must consider if it is possible to map bianshi to practices that make use of constructivist principles in western cultures.

The term *scaffolding* is associated with Wood, Bruner, and Ross (1976) and involves the teacher providing suitable support for learning. The teachers' deliberate removal of scaffolding is defined as *fading* by van de Pol, Volman, and Beishuizen (2010). Scaffolding is akin to pudian (Ding et al., 2016) however F. Gu et al. (2017) argue that pudian devotes greater attention to the hierarchy of the steps i.e. to build relational understanding (Skemp, 1976) between each varied example. An untrained eye may perceive this as rote learning (or instrumental learning (Skemp, 1976). This has been referred to as "the paradox of the Chinese learner" (Huang & Leung, 2004) because such approaches, leading to success in international tests appear contradictory to constructivist theories of learning.

Similarities can be drawn between bianshi and the work of Dienes (1971) who suggested that for pupils to abstract and generalise mathematics they need to experience *perceptual* and *mathematical variability* both of which are akin to conceptual variation. Perceptual variability involves experiencing a mathematical structure in different observable situations to perceive its structural properties. Mathematical variability involves experiencing essential features of a mathematical concept being varied so that a generality of the concept can be achieved. Gattegno's (1971) four 'powers of mind' - the powers of *extraction*, *transformation*, *abstraction* and the power of *stressing and ignoring* – describe a processes that learners experience to make sense of mathematical examples presented to them. Such experiences are implicit in the design of problems using conceptual and procedural variation. The author argues that bianshi provides practical applications of the learning frameworks offered by Dienes (1971) and Gattegno (1971).

In Chinese text-books, new learning is stimulated from a familiar situation or context which are then used to develop the abstract concepts (Sun, Teresa B, & Loudes E, 2013). Using realistic starting points is also a feature of Realistic Mathematics Education (RME) (Freudenthal, 1973). In both bianshi and RME, the mathematiztion of realistic examples into abstract mathematical concepts is promoted. Both approaches also make use of pupils' own solution strategies through discussion led by the teacher (Schleppenbach, Perry, Miller, Sims, & Fang, 2007). Pupils' varied solutions provide a teaching tool for collective discussion and comparison in the *Theory of Didactical Situations* (TDS) (Brousseau & Balacheff, 1997). I argue that the use of pupil-generated solutions from authentic mathematical situations as featured in both the RME and TDS theoretical frameworks and developed within the constructivist paradigm, are what Sun (2011) describes as type-II procedural variation.

So what is it about bianshi that separates it from western practices? A number of studies have sought to explore what differences exist between Chinese and US Grade-6 pupils (Cai, 2000, 2004). These have shown that Chinese pupils generally outperform their US counterparts and that an indicator of this success is Chinese pupils' ability to represent word problems using algebraic generalisations (Cai, 2000). Interviews with a sample of teachers from the US and China, revealed that Chinese teachers had a clearly articulated teaching goal to lead learners to generalisations when solving problems but the US teachers were satisfied with any representation of the problem solution (Cai, 2004). This suggests that the abstraction to a generalisation, a feature of bianshi, may explain one key difference between practices.

Conclusion

In this paper I have sought to answer three afore-mentioned questions. Bianshi is an approach developed in Shanghai from Chinese "indigenous practice" (Sun, 2011) where teachers use pudian - sequences of problems - designed with *procedural variation* for pupils to build relational understanding (Skemp, 1976) of a mathematical process or concept. Chinese teachers also use *conceptual variation* to representation concepts in multiple ways and in varied contexts. In both cases the variations experienced are used to lead pupils to draw generalisations of the intended concepts; a goal strongly expressed by Chinese teachers (Cai & Hwang, 2002). Bianshi can have a positive impact on learning (L. Gu et al., 2015) and could be relevant to teaching mathematics in England because it is complementary to constructivist principles implicit in mathematics classrooms there. Little is written about how Chinese teachers come to use bianshi effectively and as such may present problems for the intended pedagogical reform in England. I conjecture that to teach with variation effectively, teachers in England will not only have to understand the features of the bianshi framework but also appreciate how pudian is carefully designed to lead pupils to draw generalisations of the intended mathematical concept being taught. The author's doctoral research will examine a sample of primary teachers' classroom practices in England as they learn to use the bianshi framework to promote learning from vaiationin mathematics. This will contribute to the knowledge of how bianshi becomes an accomplished classroom practice in cultures where variation is not an indigenous practice.

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