MEDA

5th ERME Topic Conference

Mathematics Education in the Digital Age

5-7 September 2018, Copenhagen, Denmark

PROCEEDINGS

Edited by: Hans-Georg Weigand, Alison Clark-Wilson, Ana Donevska-Todorova, Eleonora Faggiano, Niels Grønbæk and Jana Trgalova

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Edited by

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Introduction

As technology and digital resources have become ubiquitous in mathematics education research and practice, it is time to examine the particular ways that digital technology is affecting the different knowledge domains. It is clear that the discerning use of technology requires a deeper understanding of how the mathematics shapes and is shaped by the technology. This prompts a rethinking of curriculum hierarchies and closer examination of the relationship between technological and non-technological approaches.

The CERME conferences in the last 20 years have revealed how research on technology has evolved from its early focus on interactions between mathematics and students, to involve a broader dialectic with theories and, more recently, aspects relating to resource and task design alongside the concepts of teachers' professional knowledge and practice.

Inspired by the contributions to the Thematic Working Groups 15 and 16 in the last CERME 10 in Dublin, which highlighted the diversity of current research and its overlaps with other TWG themes, the ERME Topic Conference 5 MEDA: *Mathematics Education in the Digital Age* is an interdisciplinary, multifaceted collaboration that brings together participants who would normally attend a range of CERME Thematic Working Groups to provide the opportunity for further in-depth discussion and debate.

The conference draws together the following three themes:

Theme 1: Mathematics teacher education and professional development in the digital age

Theme 1 focuses on how the digital world has impacted on mathematics teacher education (pre-service and in-service); professional development and teachers' professional growth; teachers' professional development practices, collaboration and communities of practice; models and programmes of professional development (contents, methods, and impacts); and the professional development of teacher educators and academic researchers.

Theme 2: Mathematics curriculum development and task design in the digital age

Theme 2 addresses issues related to digital curriculum materials, resources (including those using digital technology) and e-textbooks with a focus on their design, appropriation, use, and wider dissemination.

Theme 3: Theoretical perspectives and methodologies/approaches for researching mathematics education in the digital age

Theme 3 addresses how different theories shape research in this field and how they can possibly be combined in a synergic way to address the complexity of teaching and learning processes with digital technology, by providing a particular insight into: the relationship between uses of technology and the development of students' mathematical knowledge; the role of teachers in a technology-rich environment; and teachers' professional development needs, for example, concerning task design.

Cross-theme Relationships

Whilst these three themes were initially chosen to support more focused work during the conference, we were acutely aware of the overlaps and relationships between all three. Consequently, the conference programme was scheduled such that all participants could hear talks and work on ideas that concern all of the themes. This was to ensure that the ERME "three Cs" or cooperation, collaboration and communication were the golden threads through the conference!

We thank Dame Professor Celia Hoyles, UCL Institute of Education, London, for giving the plenary talk – Mathematics Education in the Digital Age: Promise and Reality.

We especially thank Niels Grønbæk from the University of Copenhagen and his whole local organising team for hosting the conference and giving all participants the opportunity to come to the beautiful city of Copenhagen.

Hans-Georg Weigand

Alison Clark-Wilson

Ana Donevska-Todorova

Eleonora Faggiano

Jana Trgalova

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Accepted papers

Designing mathematics learning activities in e-environments

Giovannina Albano [1], Umberto Dello Iacono [1], Giuseppe Fiorentino [2] and Maria Polo [3]

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This paper describes the use of a systemic model to design mathematics learning activities in advanced technological environments. The model comprises four roles involved in the learning process which can be played by different actors along the process. Technology is one of these.

Keywords: Instructional Design, Digital Storytelling, e-Learning, Tetrahedron Model, Technology Enhanced Learning

INTRODUCTION AND THEORETICAL FRAMEWORK

We present the ongoing design of a digital interactive storytelling framework that we are developing within an Italian research project [1]. In literature we find different ways of using storytelling in mathematics (Zazkis & Liljedahl, 2009; Zan, 2011), here we use it as a narrative framework for immersive role-playing activities. The project aims to devise a methodology for designing digital interactive storytelling in mathematics (DIST-M) (Albano, Dello Iacono & Fiorentino, 2016), based on a Vygotskian (Vygotsky, 1980) and discursive (Sfard, 2001) approach to mathematics learning, where learners are engaged in social discursive activities while constructing their knowledge. This approach takes the motivational benefits of the storytelling and considers the learner not as a mere listener but as a character of the story interacting with it and with other characters.

In Albano, Dello Iacono & Fiorentino (2016) we report some empirical data concerning a pilot study made with a prototype DIST-M. This paper is focused on the analysis of the work done so far, providing the theoretical framework for the design and rethinking of the model. The analysis follows the tetrahedron model framework (Albano, Faggiano & Mammana, 2013; Albano, 2017) which includes the classical entities of the didactic triangle (Chevallard, 1989) – *Mathematics* (or just M in the following) is some mathematical knowledge to teach/learn; *Student* (S) who is expected to learn *Mathematics*; *Tutor* (T), who is supposed to help *Student* learn *Mathematics* – and adds a new one: *Author* (A), who is in charge of planning, developing and managing the whole learning process. In this way, as shown in Figure 1, the new vertex (*Author*) adds three new (triangular) faces to the classical didactic triangle. In this model, technology is both internal and external to the learning system.

Technology inside the tetrahedron represents the set of digital tools chosen by all actors with an explicit didactic purpose. Technology outside the tetrahedron is the pervasive technology which everyone daily uses.

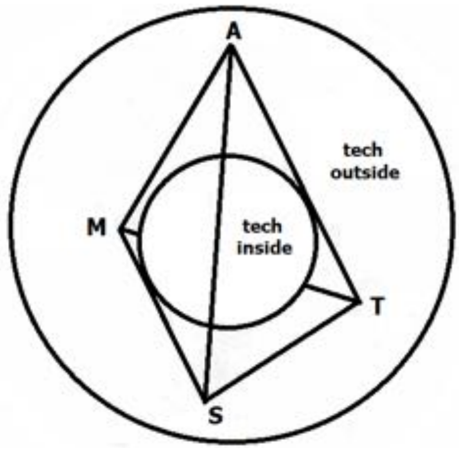

Figure 1: The tetrahedron model

THE CASE STUDY

The ongoing DIST-M prototype aims to introduce students to algebraic modelling, reasoning and proving using the following mathematical problem (Iannece & Romano, 2008; Mellone & Tortora, 2015): given four consecutive natural numbers, show that the difference between the product of the first and the last ones and the product of the second and the third ones is always 2. The problem can be generalized considering four consecutive odd (or even) natural numbers or, more generally, taking four consecutive items from an arithmetic progression of given ratio k. In the last case, for instance, students should prove that the above calculation always yields $2k_2$. The problem can also be used to foster students' thinking on mathematical key concepts such as the meaning of "consecutive" numbers or the density of rational numbers in R. The DIST-M supports students with individual and collaborative learning activities (LA) (Weinberger *et al*., 2009) about algebraic thinking, conjecturing and proving. The LAs are shown in Figure 2 and will be described in the following. The LAs in dashed boxes in Figure 2 are optional or alternatives.

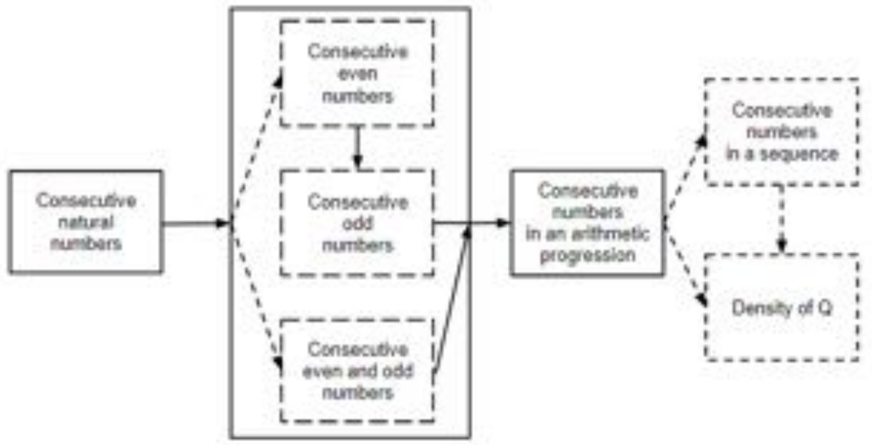

Figure 2: The learning activities' flow (dashed lines indicate optional LAs)

THE DESIGN PROCESS

The tetrahedron model foresees four vertices: *Author*, *Tutor*, *Student* and *Mathematics* which do not denote fixed entities (such as the teacher, the tutor or students) but active *Roles* that can be played by different *Actors* in different situations.

Within the *MST* (*Mathematics-Student-Tutor*) face of the tetrahedron, the interactions among the three vertices occur by means of digital storytelling. In fact, *Tutor* is embedded as a character of the story while *Mathematics* naturally arises from the plot. Several e-tools have been chosen and carefully configured to support the expected interactions. *Tutor* interacts with *Student* (all the engaged students) orchestrating discussions towards general arguments and algebraic proofs. In this context, *Tutor* behaves as an expert according to Vygotskian approach (Vygotsky, 1980). This interaction occurs with a Forum to exploit the intrinsic asynchronicity of written communication in a literate register and provide flexible time for adequate reactions. In the ongoing classroom pilot, *Tutor*'s role is played by the teacher, in cooperation with *Author*'s mathematics education researcher in charge of mathematical content and competence. Both actors are needed since the teacher knows students' mathematical background, their attitude and feelings, while the researcher is expert in managing mathematical discussions aimed at teaching objectives. We are now collecting notes and reflections from the first pilot to draft a guideline handbook as a scaffolding tool for teachers willing to use DIST-M resources in their classes.

Within the *AMT* (*Author-Mathematics-Tutor*) face, all vertices are involved in mediating *Mathematics* for *Student*. *Author* designs the learning path according to didactical objectives and plans when *Tutor* should intervene. However, the learning path can still develop in different directions according to contingencies such as the curriculum of the class and students' competences. In fact, some steps, shown with dashed lines in Figures 2 and 3, may be optional and supervised by *Tutor* managing *Mathematics* on the fly.

On the same *AMT* face of the tetrahedron, specific mathematical e-tools such as Spreadsheets and Computer Algebra Systems, when carefully used and dosed, can play an important role as semiotic mediators of *Mathematics* content.

Within the *AMS* (*Author-Mathematics-Student*) face, the central role is played by *Mathematics* and its didactic transposition. It is natural to think that *Author* arranges *Mathematics* for *Student* but, sometimes, also students may act as *Author* creating *Mathematics*. This happens when they produce conjectures, (counter)examples, etc. Moreover, the absence of *Tutor* allows to think at this face as the preferred situation for social interactions, where the whole group of engaged students acts as *Author* of *Mathematics* for their peers.

Within the *AST* (*Author-Student-Tutor*) face, we may place the classical (asymmetric) interaction between teacher and *Student*. In our case, the interaction goes from *Author*'s design of the activities, including the schedule of each phase, to *Tutor*'s scaffolding interventions.

THE ROLE OF TECHNOLOGY

During the design phase, *Author* may also consider technology as a *Tutor*, designing automatic scaffolding activities (for instance, by means of adaptive feedback or scaffolding questions). Moreover, as it happened in our pilot, due consideration to *Tutor*'s reports may suggest modifications or fine tuning of tasks and e-tools. For instance, in one case, students abandoned the chosen chat of the learning environment in favor of the more familiar WhatsApp.

As shown in Figure 3, a generic LA consists in various tasks, some of which are optional (shown in dashed lines) and taken according to *Tutor*'s decisions. The tetrahedron model helps thinking about which e-tool is better suited for the interaction needs of different actors.

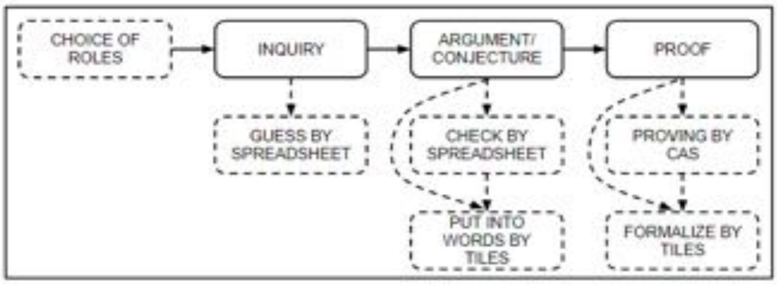

Figure 3: The tasks (dashed lines indicate optional ones)

Mathematics is always proposed to *Student* as a fundamental part of the story. Mathematical problems occur along the way and *Student*'s interventions are essential to carry on with the story. Sometimes *Mathematics* is just displayed on the screen, in other cases it is embedded into a mathematical e-tool. This happens when students are equipped with GeoGebra, benefitting from it's Spreadsheet and CAS (Computer Algebra System) views. Interactions with these (rather unusual) views are integrated within the story, providing valuable e-tools to enhance and support mathematic activities. The Spreadsheet view provides an effective way to explore, conjecture and formalize relationships among numbers. The CAS view, used by the end of the story, provides an interesting way to prove (or disprove) mathematical conjectures. So we may support the whole *Mathematics* involved, from numbers to algebra, with sound mathematical e-tools. It is well known that spreadsheets are valuable semiotic mediators (Bartolini Bussi & Mariotti, 2008); within our framework they allow a natural switch from numbers to algebra and hypothesis testing. In fact, cells, with their actual values and their (symbolic) references within formulae, provide a natural setting where "numbers" and "algebra" coexist and can be directly experienced by *Student*. By defining and dragging formulae, students can explore a wide range of cases and witness the synthesis of a single (algebraic) formula capturing and explaining regularities of (large) sets of data. Moreover, students will increase their confidence with spreadsheets, an important (and greatly underexploited) problem posing and solving tool. They will also face its peculiarities, among which the use of the equal sign (which has very little in common with both mathematics and programming) and the powerful (but rather unsetting) relative cell reference. They will also get in touch, probably for the first time, with a Computer Algebra System, witnessing its application in areas commonly believed out or reach of computers, such as theorem proving. So students will also gain some real transversal competencies (soft skills) with important problem-solving tools whose reach goes far beyond the proposed activity.

A SHORT DESCRIPTION OF THE TASKS

The task *Choice of roles* (Figure 3) pushes students towards a real collaboration (Weinberger et al., 2009) and can be framed in the *AST* face. This can be an optional task after the first LA if *Tutor* allows students to change their previously chosen roles.

With tasks *Inquiry, Argument/Conjecture* and *Proof*, *Student* is respectively asked to investigate on his own on the proposed question, conjecture and argue with his mates and, finally, develop a proof with the support of *Tutor*. *Tutor* may decide to extend these tasks with some technology enhanced ones, as shown in Figure 3.

The design of the task *Inquiry* has been done assuming the *AMS* view. In fact, *Author* designs the activity flow, planning all requirements and choosing the best (internal or external) e-tool for each one. In particular, *Author* typically chooses a Chat for informal communication steps, since it is closer to *Student*'s habits and a Forum for steps requiring higher mathematical communicative registers. Here is where *Mathematics* comes into play with mathematical content and argumentative competence, which is the main focus of DIST-M. The output of this task is a “digital resource” which is also the input of the following one. In this way, *Student*, besides acting as a peer, also produces "original" arguments and, in this sense, is also an *Author*. Also the task *Argument/conjecture* has been designed taking into account the *AMS* view. This task foresees the comparison of all conjectures and arguments found by the students fostering the convergence towards a common and agreed version. In doing that,, *Mathematics* is present as argumentative competence (Sfard, 2001). *Author* is also present as a role assumed by students when *Mathematics* content is produced by *Student* as in task 2. This task also shares with the *AST* view when students act as *Tutor* for their mates.

The task *Proof* consists in a Moodle Forum, where students are asked to agree on a shared proof; it aimed at fostering the development of more literate mathematical statements. The task can be thought as interrelation between all three *MST* vertices. In fact, mathematical discussions arise from asking *Student* to prove previously stated conjectures. In our mathematical problem, involving consecutive natural numbers, *Tutor*, acting as one of the characters of the story, mediates the discussion among the students with the aim of gradually guiding them, first to the identification of any four consecutive natural numbers with n, $n+1$, $n+2$ and $n+3$ and then towards the construction of the proof:

$$(n + 1)\,(n + 2) - n\,(n + 3) = 2, \text{ for each natural number } n.$$

Tutor manages the discussion and drives *Student* towards the construction of formal proofs. *Mathematics* is naturally involved in conjecturing and demonstrative competence. *Student* builds his own proof and, when its quality is high, he naturally acts as a *Tutor* for his mates. In reality, *Student* can even act as *Author* when he finds "novel" proofs. In this case, this task also shares with the *AMT* face.

Within the optional tasks *Guess by spreadsheet* and *Check by spreadsheet,* the inquiry and the production of conjectures are supported by spreadsheets (as a GeoGebra view) integrated within the story and the learning environment (Moodle). The explorative path is supported by a progression of worksheets preloaded with quadruples of numbers exploring cases of increasing generality (from consecutive natural numbers, to consecutive even or odd numbers, to consecutive items of an arithmetic progression). In all cases, students are always free to add more rows and explore relationships with formulae as in Figure 4.

	A	B	C	D	E	F
1	6	7	8	9		2
2	15	16	17	18		=C2*B2 - D2*A2
3						

Figure 4: The spreadsheet tool

By writing the correct formula (the product of the second and third number minus the product of the first and the fourth one, as in Figure 4) for one of the quadruples and dragging it, *Student* may check that the result is always the same (2 in the simplest case of Figure 4). The writing of the formula, with its cells references actually anticipates the *Mathematics* generalization through algebra. Moreover, with this tool *Student* can also explore and model the concept of "consecutive" in many ways.

The optional task *Proving by CAS* integrates a Computer Algebra System engine (again a GeoGebra view capable of performing symbolic calculations) within the story and the learning environment to support the production of algebraic proofs. This task scaffolds the construction of algebraic proofs. In fact, having correctly represented the four "consecutive numbers" (this is where *Mathematics* and algebra takeover takes place), it allows *Student* to explicitly compute the expected algebraic relation. For instance, Figure 5 shows how the student can write the general formula for two kinds of "consecutive numbers" and compute the resulting identities.

1	(n+1)*(n+2)-n*(n+3) → **2**
2	(n+k)*(n+2k)-n*(n+3k) → **2 k^2**

Figure 5: using the CAS tool

So *Student*, providing some mathematical knowledge, may interact with *Mathematics* by means of the CAS, using some of its embedded mathematical knowledge.

Looking at the relations among the vertices of the tetrahedron, we point out that the role of *Tutor* comes into play twice and is played by different actors. First, being an optional task, *Tutor* decides to start it or not, according to *Student*'s knowledge. Second, the CAS engine with its feedback may assume the *Tutor* role. From this point of view, the task lies within the *MST* face. However, *Author* designed this task to support *Student*'s proof building attempts by choosing the most appropriate tools and their configurations; the task thus also shares with the *AMS* face.

The optional tasks *Put into words by tiles* and *Formalize by tiles* respectively allow *Student* to build argumentations and proofs, using some sentence-tiles tiles designed by *Author* (Albano & Dello Iacono, 2017), as shown in Figure 6.

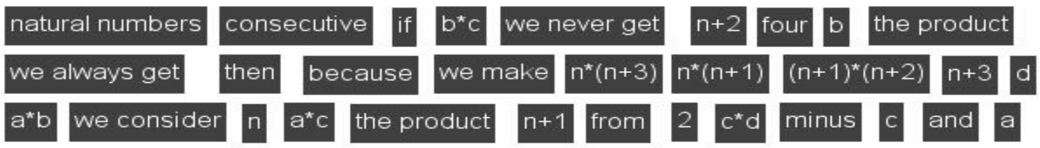

Figure 6: Some sentence-tiles for the given problem

Both tasks, on one hand, act as Tutor scaffolding argumentation and proof development; on the other hand, lead Student to reflect on the mathematical concepts referred by the cards. So, Mathematics is at stake both as competence in proof and argumentation, and as mathematical content in each sentence-tail. Student, when building his sentence becomes the Author of that proof.

CONCLUSIONS AND FUTURE WORK

This paper provides the theoretical framework for the analysis and design choices performed in Albano, Dello Iacono & Fiorentino (2016). It also provides a better insight on how technology can be further exploited within the model. A DIST-M (Digital Interactive Storytelling in Mathematics), as described in this paper, is now being tested with about 60 14/15-year-old students in two classroom pilot studies. Such pilots are providing interesting research data and useful suggestions for a redesign of the activities, allowing us to make small design adjustments in nearly real time. We are also working on another outcome of the project: a set of guidelines for teachers willing to adopt a DIST-M for teaching mathematics.

NOTES

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A didactical tetrahedron supporting co-disciplinary design, development and analysis of mathematical e-learning situations

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In this paper we propose a comprehensive model, conceived as a heuristic to support a co-disciplinary approach to the design, development and analysis of the didactical system, in particular in the case of mathematical e-learning situations. It has been developed by expanding the classical didactical triangle into a tetrahedron, and including within it a mediatory sphere whose intersection points with the tetrahedron can shed some light on the impact of technology within the didactical system. We explain how the model could address the need to take into account, in a co-disciplinary mode, different theoretical and empirical perspectives, within and beyond mathematics education.

Keywords: e-learning, didactical tetrahedron, co-disciplinary approach

INTRODUCTION

At a first level, this paper focuses on the interactions between students and teachers with the content and with digital resources, in the context of mathematical e-learning situations. First of all, we need to specify what we mean by e-learning. There is no doubt that we move into a web based environment, but the Web has changed throughout the years. The three web generations can be described by the following verbs: read, write/communicate, collaborate, each of them intended to be added to the previous ones (Hussain, 2012; Miranda et al., 2014). This means that educational technology has allowed the learners more and more engaged, from a passive role to an active one and finally to a social one. Just as knowledge was delivered in the era of e-learning 1.0, then it was co-constructed by the learners with the advent of e-learning 2.0 and now it is socially constructed by communities of learners. Note that the main difference between the 2.0 and 3.0 eras depends on the kinds of interaction among learners: in the former case, learners can write resources and share them, in the latter case, learners can collaborate in writing resources. Moreover, we point out a further feature of the Web 3.0, that is mobility: nowadays we can access technology anywhere and anyhow, e.g. by any device. Some contend that e-learning 3.0 should also be considered “intelligent”, as well as “collaborative” (Rubens et al., 2014), but in this paper we neglect the subject on artificial intelligence. Herein, we will draw our attention to the teaching/learning process which occurs in an e-learning 3.0 environment, taking into consideration general purposes teaching platforms, integrated with social apps, eventually added to online mathematical software. Note that the most popular platform, Moodle, is already available in mobile version, and it can be used in mobile learning together with other social apps, such as Whatsapp.

From now on, with the term mathematical e-learning we will refer to this kind of teaching/learning process, focused on mathematics education, concerning both a distance and blended setting. In order to study mathematical e-learning situations we believe that we need a comprehensive model which could firstly take into account research results on the use of tools in mathematics education, without disregarding input coming from other research fields beyond it. We consider worth of note, for instance, that in the General Didactic research field it has been underlined that learning is a process that can occur only with teaching mediation: thanks to the interaction with didactic mediators, that facilitate the transition from the specific experience to the generalization of it, pupils organize and conceptualize their own experience during the learning process. In particular, the key role of every learning activity is played by a system of didactic mediators: the educational action makes use of functional multiples mediators that follow each other. In this perspective what becomes important to understand, in the case of mathematical e-learning situations, is what does change if in the mediators' system there are also digital tools. As far as mathematics education is concerned, instead, tools were used long before new technology entered the classroom and have always played an important role. Here, the term "tools" is used in a broad meaning as means, incorporated in mathematical activities. Moreover, it is noteworthy that tools always have affordances and impose constraints on the user, and that teachers need to understand and to be aware of the implications of the use of tools in mathematics classrooms. And this is true, in particular, in every teaching/learning situation involving new technologies, and thus also in e-learning situations.

In this paper we intend to propose a comprehensive model which expands the classical didactical triangle into a tetrahedron. Indeed, in order to become aware of the impact of technology on the relationships between the Teacher (T), the Students (S) and the Mathematics (M) in a e-learning situation, as will be explained better below, we believe it is important to consider also the Designer (D). A fundamental characteristic of the model we are going to present is that, unlike what happened in other studies, the technology is not a vertex, but it is embedded in a mediatory sphere immersed in the tetrahedron, whose vertices are T, S, M and D.

At a second level, according to results coming from a recent study (Faggiano *et al*., 2017) involving educationalists and experts in mathematics education, we argue that every didactical intervention is shaped by a complex space/time device in which knowledge is consolidated and conceptualization is fine tuned. The interactions between teacher and student allow a sort of alignment between the student's experiences and the scientific knowledge. It has been made possible also thanks to the presence of artefacts/mediators with structural and structured role with respect to the mediation between teacher and student, student and knowledge, but also teacher and knowledge. These considerations call for the need to study mathematical teaching/learning processes from a wider perspective, considering and taking advantage of results coming from other research fields, such as general didactic or the educational technology. For this purpose, we adopted a co-disciplinary perspective

(Blanchard-Laville, 2000) in which the prefix "co", which means "with", is about evoking the construction of a co-thinking research space fostered by a certain empathic understanding and acquaintance with the points of view of the other researchers about the same object of study.

FROM THE DIDACTICAL TRIANGLE TO THE TETRAHEDRON

The didactical triangle can be considered as a heuristic that identifies the fundamental components of any didactic system: teacher, student and content. The idea to expand the didactical triangle to a tetrahedron in order to consider the role of technological artefacts in mathematics education is not new (see e.g. Tall, 1986). In particular, Rezat and Sträßer (2012) proposed a socio-cultural tetrahedron, in which the fourth vertex is the mediating artefact. It offers an important representation of the complexity of the system that affords, in particular, a level of detailed reflection on the didactical role of the tasks. Our approach starts from the assumption that the full exploitation of e-learning environments requires a well design didactical intervention, not only in terms of contents and tasks (didactical transposition) to be arranged and eventually included in a platform, but also of the environment to be set up, the structure of the teaching/learning activities to be organized, the technology to be selected, the methodologies through which the interactions can be allowed and fostered and so on (didactical engineering). We argue, indeed, that an e-learning platform has a role of aggregator and that the true and meaningful sense of the didactical action lies in the complex system architecture of the learning environment rather than only in the content materials. For this reason, in the attempt to model the situations, we contend the need to introduce as a new vertex the designer (D). This allows us to highlight the role, performed mainly through a-priori rather than situational choices, of a further actor or, more precisely, "scriptwriter", in charge of that complex designing activity. In an ordinary situation it is often the teacher who assumes, on one hand the role of the designer, when he/she is involved in the selection and/or design of the resources, the construction of the tasks and the planning of the activities, and on the other the role of teacher/tutor during the development of the teaching/learning process. In more complex situations, however, it could also be the case that a collective entity, with different professional skills, needs to act as designer, while a (eventually further) collective entity acts as teacher/tutor during the development of the activities.

As far as mathematical e-learning is concerned, Borba, Clarkson, and Gadanidis (2013) already noted the importance of teamwork inside the collective designer. They claimed that the low design and pedagogical quality of online interactive mathematics contents can be avoided by the simultaneous work of various experts, such as mathematics educators and human-computer designers, which can take into account and integrate both didactic objectives and design principles. We argue that, a co-disciplinary team, especially involving educationalist, can take care not only of the design of the content materials but also of all the other choices which impact on the teaching/learning process: the comparison, the discussion, the co-thoughts that can occur among the different experts can affect decisions about the whole didactical architecture with respect to a fixed didactic goal. This point of view also seems to be highlighted by Schoenfeld (2009) who wishes for a synergy between educational researchers and educational designers. He claims that the richness of the designer enables the creation of varied scenario of pedagogical expectations concerning knowledge, of professional or ideological beliefs, of implicit philosophies that provides an enrichment of the e-environment and carries out robust and well-engineered products made available to the targeted learners.

THE CO-DISCIPLINARY PERSPECTIVE

In light of the above, following Albano *et al*. (2013), we assume that the didactic system concerning mathematical e-learning situations can be modelled in a systemic way by a tetrahedron, which includes: some mathematical knowledge, that is Mathematics (M), someone who is expected to learn M, that is the Student (S), the Teacher/Tutor (T) and the Designer (D), in charge of planning, developing and managing the didactic organization. As a matter of fact, the present tetrahedron is an extended version of the cited, in particular with respect to the last vertex (that we have decided here to name "Designer" instead of "Author"): herein, indeed, we aim at assuming a wider approach considering the authoring of the content materials as part of a more complex role that is the one of the designer as described above.

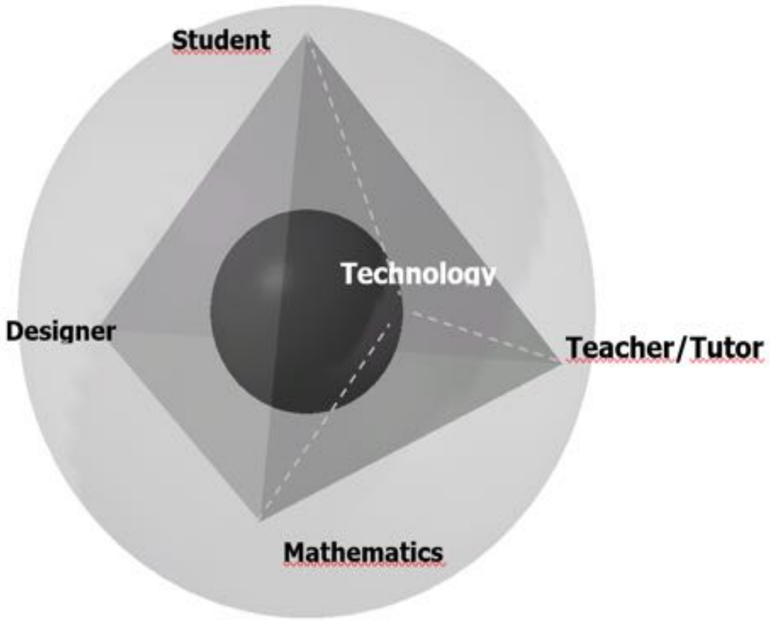

Figure 1: the didactical tetrahedron for mathematical e-learning

We claim that this model acknowledges the need to design, develop and analyse mathematical e-learning in a co-disciplinary approach, hence the tetrahedron with the internal mediatory sphere have to be considered in the whole complexity. However, an insight on each of the four faces allows us to focus on the various aspects on which the model can shed light, taking in account different perspectives on the interactions among the actors, with respect to the integration of technology in the teaching/learning situation.

The basis of the model still remains the classical didactical triangle students-teacher-mathematics, that is referred by the face STM. According to Rézeau (2001) teacher's action could be seen as a continuous balance/mediation between the didactical disciplinary oriented processes and the educational processes. The latters are intertwined with the formers and the teacher's expertise consists of the ability to realize this balance which varies according to the context, between scaffolding and fading, between autonomy and support. The intersection point between the mediatory sphere and this face of the tetrahedron is the technology, seen as a mediatory tool. The view of this face represents the focus on the didactical action and on the role of the teacher/tutor as "arranger".

In the perspective of DTM face, attention is focused on the processes, where design and enacting are intertwined, in which the Designer interacts with resources selecting, adapting, revising and reorganising them, by means of an a-priori analysis. It is worthwhile noting that these processes are ongoing processes which continue in usage. This face, hence, depicts the instrumented mediated activity of planning the mathematical experience. From this point of view, the Designer looks for resources, plans the activities, chooses the e-tools and defines the educational setting. The instrumental genesis takes place, that is the Designer defines how to use the artifact (instrumentalization) and at the same time the affordances and constraints of the particular chosen e-tools influence the design of the activities.

The view from the SDM face allows us to focus on the mediating role of technology with respect to the the role of the Designer in organizing the learning settings for the Student to learn Mathematics. From this perspective it can be useful to consider the Geiger's (2006) distinction of the four metaphors to describe the degree of sophistication in which technology can mediate learning: Technology as master, where the student is subservient to the technology and the relationship is induced by technological or mathematical dependence; Technology as servant, where the technology is subservient to the student, typically used as a reliable timesaving replacement for mental, or pen and paper computations; Technology as partner, where the technology is used creatively to boost student empowerment, treating the technology almost as a surrogate human partner; Technology as extension of self, where users draw on their technological expertise as an integral part of their mathematical thinking.

Looking at the face STD, the focus is on the classical relationship between the Teacher/Tutor and the Student, planned by the Designer. It can be represented by the word "conversation", referring to the Conversational Framework (Laurillard, 2001). The Designer models the learning experience as iterative interactions among two participants (e.g. Teacher and Student) at two levels, practice and communication, connecting the two levels by means of adaptation and reflection. This means that we can think of the whole face STD as cycles of designing for learning (D plans activities, also with the use of e-tools), doing for learning (S interacting with the e-environment), communicating for learning (starting from the practice, reflection and discussion between S and T and among students), tuning for learning (D perfects the design with respect to T feedbacks). The previous cycles including peer learning, assuming both the participants are students.

DISCUSSION AND CONCLUSIONS

The didactical tetrahedron proposed above can operate as a heuristic not only acknowledging the need to analyse the relationships among the actors, but also drawing attention to the didactical system supporting the design, the development and the analysis of mathematical e-learning situations according to a co-disciplinary approach. An example can be given considering the Theory of Semiotic Mediation (Bartolini Bussi and Mariotti, 2008). From this perspective: the view of the STM face focuses on the role of the teacher within the process, which consists of fostering the social evolution of the emergent personal signs, coming from the artefact-use, into shared mathematical signs, through the orchestration of meaningful discussions; the view of the DTM face focuses on the design of both materials and activity phases, in order to foster the unfolding of the semiotic potential of the artefact in use and the construction of mathematical meanings through the guided evolution of signs, performed by the Designer; the view of the SDM face allows us to focus on the artefact sign production provoked by the use of technology thanks to the task that has been set up by the Designer; finally, it is with a co-disciplinary perspective that the analysis of the teaching-learning process can be broaden considering the STD face, taking into account some contributions coming beyond mathematics education thanks to which we can also focus on the didactical elements influencing the teaching-learning practices.

As a further example we can refer to a study in which the model has been applied to define, tune and analyse the design of a Digital Interactive Storytelling in Mathematics (Albano *et al*., 2018). One of the main features of the model is its systemic view of the actors involved, which has allowed us to reflect on the learning process in a non linear way, differing from the initial mode of learning in e-environments. Anyway, the model can be used also to conceive the actors in terms of played roles rather than of persons. In fact, this has led to thinking of them in a dynamic way, so imaging in some cases technology (suitably chosen from the internal sphere and shaped according to the intended use) as Tutor scaffolding specific learning goals. Analogously, the student can play the role of the Designer, who produces the resources needed to make the activities progress, or the role of the Tutor, in terms of expert among peers. This example refers to the design phase, but further work is on-going in order to use the model for analysing the output in terms of learning.

The design of teaching/learning activities in e-learning 3.0 environment is a complex work. In fact, such activities generally foresee the use of various e-tools, some general purposes and some domain specific, which should be pedagogically integrated among them, on the basis of the didactical objectives of the teaching/learning activity. The tetrahedron by Rezat & Sträßer (2012) that generalizes the classical didactic triangle including the "Artifact" as a new vertex, has, in our view, the merit to recognizes that the connections represented by the classical didactical triangle require mediation. It can be seen as embedded in our model if we consider the face STM connected with the tangent point of the inside sphere of technology. However, we believe that the socio-didactical tetrahedron is not completely suitable to take care of the complexity of mathematical e-learning teaching-learning activities. This complexity is especially intrinsic in the non simultaneity of time and spaces of any interactions which requires a didactical orchestration not comparable to the face-to-face case. For this reason, we have considered the proposal of a new specific tetrahedron for mathematical e-learning worthwhile in which the vertex D is brought to the fore.

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Processes that a Community of Inquiry Undergo towards Developing Mathematics Lessons with Technology

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The present research main goal is to examine processes that a community of inquiry undergo within a PD program in which middle school mathematics teachers develop mathematics lessons with technology. The research used interviews, audio-recorded discussions, written lesson plans and reflections. Data were analyzed using constant comparison method following Goos and Bennison (2008) and Jaworski (2006). The research findings indicated that to plan and implement a mathematics lesson with technology, the participating teachers went through a series of community processes that included choosing, discussing, suggesting, deciding, commenting, critical alignment, implementing and reflecting. These processes resulted in a shared repertoire of resources.

Keywords: professional development, community of inquiry, mathematics lessons with technology.

INTRODUCTION

Professional development of mathematics teachers

Researchers are interested in PD programs that aim to develop teachers' integration of technology. Thomas and Palmer (2014) argued that PD practice is best constructed around a supportive community of inquiry (CoI) that gives teachers the opportunity to observe, practice, and reflect on the use of digital technology in the classroom (Goos, 2014). Doing so, Thomas and Palmer (2014) suggested a PD design that involves organizing a small heterogeneous group of teachers, each one presenting in turn a prepared lesson incorporating technology. The lesson becomes the center of community discussion and reflection. This PD design is close to the 'inquiry cycle' design suggested by Jaworski (2008). The present research used the 'inquiry cycle' (Jaworski, 2008). This cycle includes the following processes: planning, acting & observing, reflecting and analysing, giving feedback. This cycle provides teachers with opportunities to experiment, make mistakes, discuss, and negotiate. In fact, the participants in an 'inquiry cycle' work as members of a Community of inquiry (CoI) to develop their professional knowledge.

Mathematics teachers' knowledge

Shulman (1987) proposed a framework for professional knowledge that includes seven domains of teaching knowledge, where the category that revolutionized researchers' thinking was pedagogical content knowledge (PCK). Ball and colleagues (Ball et al., 2008) noted that Shulman’s categorization was theoretically rather than empirically based, and proposed a model that focuses on Mathematics Knowledge for Teaching (MKT). Shulman's PCK and Balls' MKT influenced a new theoretical framework, the pedagogical technology knowledge (PTK) framework, proposed by Thomas and Palmer (2014) for teachers' knowledge concerning the integration of technology in classroom practice. Thomas and Palmer (2014) maintained that several teacher factors combine to produce PTK, including the MKT factor, which relates to pedagogical and mathematical content knowledge. In the present research, one goal is to verify teachers' knowledge that develop in a CoI.

Development of teaching through CoI

The framework was developed based on Lave and Wenger's (1991) theories of community of practice. Lave and Wenger put the knowledge in the practice and argue that it can be interpreted in a school context. Teachers may develop their practices of teaching as part of the community of teachers within the school.

Wenger (1998) suggested that learning is developed through three modes of belonging: (a) *engagement* or mutual participation in joint tasks, (b) *imagination,* which is the willingness to explore and try new things, then reflect on how these relate to other practices, and (c) *alignment*, which is the convergence of a common focus, cause, or interest. Seeking to look critically at what alternatives may be available, a CoI is able to think and act differently (Jaworski, 2003). In such an inquiry approach, the idea of critical alignment is central to that of CoI (Jaworski, 2006). Jaworski (2006) and Goos and Bennison (2008) have described three dimensions or practices of CoI by which the community develops: *mutual engagement* of participants, *negotiation* of a joint enterprise, and development of a *shared repertoire* of resources for creating meaning.

Shifting from community of practice to CoI provides a reflective development of teaching (Wells, 1999). A feature of a CoI that distinguishes it from a community of practice is the importance attached to meta-knowing by reflecting on what is being or has been constructed, and on the tools and practices involved in the process (Wells, 1999). In the present study, we used the CoI (Jaworski, 2006) and the 'inquiry cycle' (Jaworski, 2008) designs to engage middle school mathematics teachers in teaching mathematics with technology. The present study aims to describe processes, which are observed in the CoI. Another aim is to verify the roles of the community members in developing the technology-based mathematics lessons. A third aim is to verify teachers' knowledge that develops in CoI.

Research question

1. What are the processes that middle school mathematics teachers who collaborate in the frame of a CoI undergo to develop technology-based lessons?
2. What are the roles of the community members in their collaboration to develop technology-based lessons?
3. What teachers' knowledge develops in the frame of a CoI who develops technology-based lessons?

METHODOLOGY

Research context and participants

The PD program was conducted in the academic year 2017-2018. It aimed to help middle school mathematics teachers make effective use of technology in their classrooms. It is based on CoI practices (Goos & Bennison, 2008; Jaworski, 2006), specifically it uses the 'inquiry cycle' suggested by (Jaworski, 2008). The participants in this PD program are mathematics middle school teachers. They meet once every two weeks, for two academic hours, during six months (10 meetings). In the PD program, teachers work in groups and engage in three inquiry cycles. Each cycle includes: Planning an ICT lesson within their groups, implementing and recording the planned lesson (acting) in their classrooms, observing the implemented lesson, reflecting on and analyzing this implemented lesson, and giving, as individuals, feedback about (suggesting improvements to) the plan of the lesson. The participants videotaped the lessons and discussed them in their group during the PD sessions. Specifically, for this study, we chose one of the groups that participated in the program and which constituted of five teachers: Marwa, Reem, Mosab, Salsabil and Manar (pseudonyms). Marwa and Reem completed last year their B.Ed. in mathematics education with honour degree, Mosab is ICT guide, Salsabil is Mathematics teacher guide and Manar is an experienced teacher (more than 15 years seniority). We chose this group as a focus group due to the group unique combination (varied expertise).

Data collection tools

The research used interviews, audio-recorded discussions, written lesson plans and reflections. In more detail, we interviewed each participating teacher individually at the end of the PD program. The interview was semi-structured. An example of a question in the interview: “What was the role of the other mathematics teachers in your group in writing your lesson plan that integrates technology?” The participants’ discussions at every meeting regarding the planning of their lessons, and the implementation of their lesson plans in the mathematics classroom were audio-recorded. The participating teachers prepared three written lesson plans that integrated technology in mathematics teaching as well as written reflections. The reflections were of two types: individual and collective reflections. The reflections were written after the implementation of each lesson.

Data analysis

Data were analyzed using the constant comparison method. Part of the analysis was deductive, following the categories described in Goos and Bennison (2008) and Jaworski (2006). These categories relate to practices of community of inquiry (mutual engagement, negotiation and shared repertoire). In addition, part of the analysis was inductive, searching for themes and categories related to the roles of the pre-service teachers in the CoI and related to teachers' knowledge developed in the frame of CoI processes.

FINDINGS

The series of processes that the community undergo while preparing a technology-based lesson

When preparing a mathematics lesson with technology, the participating teachers went through a series of processes as a community. This series included mutual engagement of the participants and negotiation regarding choosing the topic of the lesson, discussing its general plan, deciding upon the roles of each member in the group regarding the writing of the lesson, suggesting activities for each part of the lesson by the member responsible for it, commenting on the activities, and critical alignment regarding the final content and structure of the activity. These processes resulted in a shared repertoire of resources for the participants, where this repertoire included the components of the planned lessons: the built activities, the teaching strategies and the technological tools used in the activities.

The roles of the group members in processes the community undergo

What influenced the roles of the group members were their previous experiences. Those who were more expert in technology were responsible for choosing appropriate technological tools to be used by the students in carrying out the activities. Those who had more seniority in teaching were responsible for determining the time and order of the activities. It could be argued that the group activity started from a repertoire that included the expertise of each of the members.

One of the members of the group, Reem, described the division of labour between the group members, saying: "Salsabil and Manar have more experience than the rest in teaching mathematics, so they told us what the students should learn in the mathematical topic, the topics that they learned before and the topic that they learn after. Salsabil suggested how we should sequence the activities". Reem continued: "Mosab is more experienced than the rest of the members in technology, so he looked for applets for the lesson 'The area of the parallelogram'. We afterwards decided which applets to use in the lesson".

The members of the group also took care of an alternative plan if something went wrong with the technological tool. One of the members, Manar, prepared such without-technology lesson plan. This role was given to Manar because she is an experienced teacher who teaches usually without technology.

The fixation of the group members' roles

In this group, the members had fixed roles. One of the members, Marwa, described the community members' roles in building the activities in the following way: "Salsabil and Mosab took the introduction of the lesson and the summary. This happened in the preparation of the three lessons. Everyone in the group took a specific part of the lesson and was responsible for its preparation. It is right that they sent us what they prepared, but it was their role to take care of the introduction and the summary. We gave comments for each other, and only after correcting according to the comments, the lesson part was adopted by all of us". So, the fixation of the roles of the group members did not indicate the lessening of negotiation in the group.

Although the roles of the members were fixed, the members considered their roles as complementing each other. This complementing was strengthened by the similar beliefs of the group members. Marwa said: "We were different regarding the knowledge that we possess. Some had more content-knowledge, some had more pedagogical-knowledge, while others had more technological-knowledge. In spite of this difference, we had similar beliefs and goals regarding the integration of technology in the mathematics classroom. These similar beliefs and goals made us utilize our differences in preparing the lesson plan".

Community processes that led to the development of the pedagogical mathematical knowledge

The discussions in the community developed the members' mathematical pedagogical knowledge. For example, in the preparation phase of the first lesson – the equation of the line that passes through two points, Reem suggested starting the lesson by drawing two points and asking the students about the number of the straight lines that pass through the two points. Manar did not agree with this suggestion and suggested drawing a straight line, assigning various points on it, finding the ratio of the difference of the y-coordinates to the difference of the x-coordinates of every two points and demonstrating that this ratio is fixed for any straight line. Reem agreed with her that her suggestion would advance the students to discovering one important feature towards finding the equation of a straight line.

In the preparation phase of the second lesson – the triangle area, Reem showed the other members an activity that she prepared for the students to explore the triangle area topic. All the members were engaged in suggesting some modifications. Manar suggested shortening this activity because it would take all the lesson time. She also suggested that they first give the students questions about the triangle altitude topic (the previous topic) to make them ready for the area topic. On the other hand, Salsabil suggested that there is no need for one question in the activity. This question requested the students to draw from a triangle's vertex a parallel line to the opposite edge and drag the altitude along the parallel line, observing all the time the length of this altitude". Manar did not agree with Salsabil's suggestion, arguing that this question is needed to take care of all types of triangles including obtuse-angle triangles. After discussion, the group members agreed to keep the question.

Community processes that led to the development of the technological pedagogical mathematical knowledge

In choosing a lesson to teach using technological tools, this group of pre-service teachers discussed the type of technological tools that could be integrated in each lesson: applets, GeoGebra, spreadsheets, videos, PowerPoint presentations, etc. Manar claimed that there is not technological tool that could help the students in discovering the rules of adding or multiplying positive and negative numbers. Mosab, on the other hand, suggested using Excel for that matter. He showed the group how to do that. Manar did not accept his suggestion, but when Mosab and Salsabil prepared an appropriate worksheet, she agreed that the worksheet could be a good start for the topic, but the students need to discuss the spreadsheets results in order to understand conceptually the addition and multiplication of positive and negative numbers.

In choosing an applet to use in a lesson, technological pedagogical knowledge played a role. In the third lesson – the area of the parallelogram, the applet that the pre-service teachers found at the beginning showed one altitude of the parallelogram. Salsabil said that they should look for another applet that shows the two altitudes of the parallelogram, which would help the students arrive at a generalization. Reem and Manar did not agree with her, saying that two altitudes would make the mathematical situation difficult for the students.

In addition to the above, the role of technology differed in the three lessons. This change was due to the reflection and discussion that the teachers performed after the implementation of each lesson. These reflections in which the members were engaged, helped them negotiate their further planning, which resulted in their critical alignment regarding the lesson plan.

To elaborate more about the role of technology in the lessons, in the first lesson, ‘finding the equation of the line that passes through two points’, the technology was an applet that enables the student to draw a line through two points, to change one of the points and watch the equation of the line changes. The applet chosen by the group did not have a functional role for the learnt topic; i.e. it did not help the student to see the method of finding the equation of a line that passes through two given points. This applet only enabled the student to see a line that passes through two points.

Reflecting and discussing the first lesson, One of the group members, Manar, suggested to give the students directions how to work with technological tool, so that the students do not waste the time lesson because they have not the sufficient knowledge to work with the tool. In addition, the group members aligned towards using tools with different functionalities. So, in the second lesson, ‘the area of the triangle’, they used an NCTM applet for the area of the triangle. Through this applet, the student could manipulate the triangle, add the lengths of the base and its altitude to a table, and the applet finds the triangle area. This applet is functional to the triangle area topic and the students are expected to arrive at the formula of the area by working with the applet.

Reflecting and discussing the second lesson, the group members aligned towards using an applet with similar functionalities. So, in the third lesson, ‘the area of the parallelogram’, the teachers decided to use an applet similar to the one in the second lesson, but here they themselves built an appropriate applet using GeoGebra. They decided to use a GeoGebra applet taking care of technical consideration; i.e. to have an applet that works offline. Also, in the third lesson, the worksheet was electronic.

Also, the group used a long PowerPoint presentation in the introduction of first lesson, which made it difficult to proceed as planned in the exploration part of the lesson. The engagement in reflection, discussion and negotiation after the lesson made the group align towards a shorter presentation in the second and third lessons. Finally, the group aligned towards not using a similar video for summary in the second and third lessons, as they found that it was not meaningful to the students' learning. The used video repeated the steps exactly as the students did, without emphasizing the main processes, and thus it did not work as a summary for the lesson. Instead, they used a short and animated PowerPoint presentation.

DISCUSSION

The present study intended to describe the processes a CoI undergo within a PD program, that enabled the participating teachers develop mathematics lessons with technology. The research results indicated that to plan and implement a mathematics lesson with technology, the participating teachers went through a series of processes as a community. These processes included: choosing the topic of the lesson, discussing its general plan, deciding upon the roles of each member in the group regarding the writing of the lesson, suggesting activities for each part of the lesson by the member responsible for it, commenting on the activities, critical alignment regarding the final content and form of the activity, implementing the activity in the classroom and reflecting individually and collectively on this implementation. These processes indicate that the teachers acted as community of inquiry (Goos & Bennison, 2008; Jaworski, 2006): engagement, negotiation and critical alignment in their intention to work on their joint enterprise that is developing technology based mathematical lessons. Furthermore, these processes resulted in a shared repertoire for the participants: the planned lessons that included the designed activities, the teaching strategies and the technological tools used in the activities. It could be argued that all the participating teachers developed as collective and as individuals as a result of participating in the community of inquiry which agrees with Jaworski's (2003) argument, that an individual's development of mathematics teaching practice "is most effective when it takes place in a supportive community through which knowledge can develop and be evaluated critically" (p. 252). This supportive community was special in the case of the community that the present research describes. In more detail, the different expertise of the members supported their community practices and the development of each one of them in the different expertise fields needed to develop and implement technology-based lessons. It should be noted that the different expertise of the members resulted in fixation in their community roles throughout the PD program, but this fixation did not hinder their development of the technology-based lessons.

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Designing mathematical tasks to promote students' online interaction

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In this paper we aim at identifying the tasks, posted online by a teacher, that occasion more students' interactions than others, namely threads for which a larger number of students posted more comments on line with respect to other threads, where fewer students posted less comments. Being explorative, our study considers a varied set of tasks in terms of topic, length and being procedural or conceptual. To analyse the data, we resort to the methods provided by the most recent advances in network analysis. The results allow us to pinpoint a tentative list of characteristics of tasks that promote online interaction, and at the same time we identify subgroups of tasks that attract students with different preferences and behaviour.

Keywords: community analysis, mathematical tasks, network analysis, online math forums, students interaction.

INTRODUCTION AND BACKGROUND

As van de Sande (2011) observes, school mathematics learning is usually featured as an in-class part plus an homework part, and recently students do not only use textbooks, class notes, peers, parents and tutors, in order to master the content they are trying to learn outside school, but they also seek help in open, free, on line math forums. An underlying inquiry-based teaching pedagogy characterises such forums (van de Sande, 2011), where many-to-many interactions are often allowed. These forums represent a form of computer-mediated discourse, which is distinct from speaking and writing (Herring, 2011): we can see a change from traditional, formal, private, one-to-one student-initiated help seeking, to a new, informal, public activity between people who share some interest in the subject domain but are otherwise unconnected. Several characteristics of both mathematics and communication emerge, that are unique to this kind of interaction (see van de Sande, 2011): (i) gestures and looks have to be conveyed in alternative ways, (ii) formal mathematics has to be communicated informally, (iii) contributions are directed towards anonymous and unknown recipients, (iv) a written record can be accessed later on time, and it can be revised.

Even if some out-of-school mathematics forums have thousands of members and receive hundreds of posts every day (van de Sande, 2011), researchers have observed that technology is underused in classroom mathematics teaching, and —when used— underexploited (Clark-Wilson *et al.*, 2011). Joubert (2013) argues that a reason for this is that the "grand challenges" themes on the use of technology in mathematics teaching and learning have not been addressed by researchers in mathematics education. According to Joubert (2013), in fact, the three "gran challenge" themes are: *orchestrating learning*—which aims at understanding and characterising the opportunities for teachers when technology is introduced into their classroom, and at exploring the use of tools to facilitate orchestration; *contextualising learning*—which focuses on how and to what extent technology provides new and different learning contexts for teaching and learning; and *connecting learners*—which concerns the issues and questions that arise from the increased connectedness of students. In her study, Joubert finds out that there is a need for more research on connecting learners, a theme that is strongly related to students' interaction in learning process, and thus to interactions theories in Mathematics Education. A special focus is on students' online interactions, distinguished from students' in class interactions. With Engelbrecht & Harding (2005), we observe that, in online math forums, mathematical knowledge is acquired by construction, not by transmission alone, and the process of knowledge acquisition is contained both internally, by what one already knows, and externally, by cultural artefacts such as *shared* language and notation. To this respect, Larsen & Liljedahl (2017) noted that a certain amount of *diversity* as well as a certain amount of *redundancy* are needed for interaction to take place in online forums. Diversity allows for novel actions and possibilities because it refers to the diversity among the participants in the forum, while redundancy allows for stability and coherence because it refers to the common ground among participants. Without redundancy, participants in a forum may not be able to communicate, but without diversity, they may never have anything to communicate about.

Studies like Larsen & Liljedahl's ones shed light on an important issue concerning the potential for online forums to be a case of learning where students are connected and share knowledge. Another important issue is the kind of math problem that is shared in online forums. As a matter of fact, indeed, we know that some threads generate lots of comments, posts and replies from students, while others get somehow ignored. If it is true that not necessarily the former ones are better than the latter ones, it is as well true that when the concern is promoting interaction, the former ones serve the purpose of a teacher better that the others. In our study, like Larsen & Liljedahl (2017), we apply the categories of diversity and redundancy to the threads and we understand redundancy as a feature of a task of connecting the students' knowledge. Diversity is a feature of the task consisting in appearing as new and challenging for the participants. A 100% redundant task is a task already seen by the students, while a 100% diverse task is so far from the students' knowledge that they are even unable to understand it. Such features of the tasks can be inferred a posteriori, from the data. We, thus, generated a varied set of tasks to be posted on line and we seek for evidence of interaction among the students.

METHODOLOGY

Which are the features of a task that promote online interaction? In order to answer to this research question, we consider the forum of a massive, open, online course (MOOC) realised at the Polytechnic of Milan in 2015. The name of the course is Pre-calculus and its contents reflect the mathematics that is needed to enter Engineering and Architecture courses at this university. Within the learning opportunities for high school students who aim at enrolling at the Polytechnic of Milan, there is a blended course which adopts an historical perspective in six in-presence lectures held by university professors, and an interactional perspective in five learning “weeks” delivered on line. For six subsequent school weeks, every Monday, the students came to university and attended a lecture on one of the topics of the Pre-calculus MOOC (arithmetic, logics, algebra, geometry, relations, and probability & statistics), while in the rest of the week they watched video lessons on the MOOC and interacted in the forum, since a secondary school teacher, called the *tutor* (who is also the third author of this paper), posted every day a new math task to discuss about. 30 different tasks are the object of the study and vary with respect to the topic (5 about arithmetic, 5 about logics, 5 algebra, 5 geometry, 5 relations and 5 probability), the nature (15 are procedural, 15 are conceptual), and the length (from few words to long paragraphs). Year after year, edition after edition, we noticed that some math problems occasion much more interaction than others, despite the time of the day, or the day in the week, in which they had been posted by the tutor. Moreover, the number of interactions does not seem to depend neither on the topic, nor on the order in which they are posted (which allows us to exclude variables like fatigue as the main reasons for that difference). In this paper, we consider the edition of the course that took place on January 2017 and that involved 35 high school students (17 years old). The online forum is the standard one provided by EdX, with a home page where new threads can be inserted by any participant, and the kind of interaction that is possible once a thread had been initiated is either to reply to, or to comment on a post.

We show some tasks that exemplify cases of many/few interactions occasioned.

Problem W1Q2: *Is the sum of three subsequent natural numbers divisible by 3? Justify your answer.* [arithmetic, conceptual, short]

Problem W1Q3: *Is the product of three subsequent natural numbers (greater than or equal to 1)* **always** *divisible by two? Is it* **always** *divisible by 4? Is it* **always** *divisible by 3? Is it* **always** *divisible by 6? Justify your answer.* [arithmetic, conceptual, long]

Problem W1Q4: *Order these numbers on the number line:* $\sqrt{2}\quad \frac{3}{4}\quad \frac{7}{3}\quad \sqrt{3}\quad \frac{9}{2}\quad \frac{56}{9}\quad 6.1\bar{2}\quad 64$

[arithmetic, procedural, short]

Problem W2Q4: *Solve these inequalities: (a)* $x^2>0$*, (b)* $x^2-2 \leq 0$*, (c)* $x^2+2x+5<0$*. Solve*

$$2+\frac{1}{x}<0$$
$$\frac{x^2+2x-3}{1-7x}\geq 0$$
$$\frac{2x-3}{x^2-25}\geq \frac{1}{x-5}+\frac{1}{x+5}$$

the following inequalities with fractions:

[algebra, procedural, long]

Problem W2Q5: *Write in symbols: (a) one third of the double of a number to which its half had been subtracted, (b) the half of the square root of three times the cube of* *an even number. Solve: (a) Charles says to Alison: "if you give me 2 euros, we have the same amount of money". Alison replies: "if you give me 2 euros, I have twice the amount of your money"; (b) twice my age is half of yours, and their sum is 20.* [algebra, conceptual, long]

Problem W3Q3: *Represent the following parts of the Cartesian plane and compute their perimeter and area: (a)* $x^2+y^2 \leq 9$ *(b)* $(x-1)^2 \leq 36$ *intersect* $(y+2)^2 \leq 25$ [geometry, conceptual, short]

Problem W4Q3: *There are elements in nature whose atoms tend to decade, emitting particles, generating other elements. The speed of decay varies from element to element and it is identified by the so called "halving time", namely the time that is necessary for the number of atoms of the element, which are initially present in a body, halves. Carbon 14 is a radioactive element that has an halving time of 5730 years. If today in a rock there are* 10^{23} *atoms of carbon 14, how many of them were present 2000 years ago? How many of them will be present 100 years ahead? 1000 years ahead?* 10^5 *years ahead?* [relations, conceptual, long]

Problem W5Q3: *In the Lotto game, five different numbers are extracted in each one of ten cities. Numbers range from 1 to 90. The Lotto outcome of Saturday, March 1 2014, had been this one:*

RUOTA	1° estr.	2° estr.	3° estr.	4° estr.	5° estr.
Bari	36	78	39	21	79
Cagliari	60	83	53	56	59
Firenze	21	2	5	90	61
Genova	52	13	38	58	85
Milano	27	69	19	32	5
Napoli	89	27	42	51	84
Palermo	81	15	9	25	36
Roma	7	89	41	75	27
Torino	54	63	29	2	43
Venezia	45	87	31	18	49
Nazionale	67	75	1	82	63

Which one of the following sequences would you bet on the next week?

a) 1 2 3 4 5

b) 13 7 45 36 72

c) 36 78 39 21 79

How many different sequences of 5 numbers is it possible to have? [probability, conceptual, long]

Each problem has been labelled as WxQy, where x is the respective week in the MOOC, while y represents the day when the task has been posted. For instance, problem W1Q2 is the second task posted on week 1 (arithmetic). Each problem corresponds to a separate thread in the forum and for each thread the students were invited to post their solutions and/or to comment the others' solutions.

In order to analyse the features of the tasks that promote students' interactions in the forum, we build a network in this way: each task is a *node* in the network and there exists a *link* between two tasks if the same student posted something in the two respective threads. The network is weighted because the more comments are posted by same students, the stronger the link between two nodes. The network is also undirected, because of the symmetry of the link.

The analysis of this network allows us to identify the tasks that are most central, by looking at their *weighted degree* and at their *betweenness*, but this also allows us to identify different "communities" of problems, which "attract" subgroups of students (see Newman, 2010). The weighted degree of a node is defined as the sum of the weights of the links of each node. The betweenness of a node *i* is related to the concept of distance among nodes and it measures how much "longer" would be to pass from node *j* to node *k* in case node *i* is removed from the network. In a network is it possible to identify clusters of nodes, called *communities*: they represent a partition of the set of nodes and community analysis is performed by computing a quantity that maximises the probability that a random walker would not get out from a community, if he starts moving from a node of the community (Newman, 2010). Moreover, we consider the *within-community degree*, which measures the extent to which a node is well connected to the other nodes in the community (see Guimera, Mossa, Turtschi & Amaral, 2005). It is equal to the difference between the degree of the node and the average degree of the nodes in the community, divided by the variance of the degree of the nodes in the community. If the within-community degree is positive, it means that the node is central in the community and it is connected to the other nodes within the same community. If it is negative, it means that the node is peripheral and it is connected to nodes outside the community. In our study, a community which has a high percentage of nodes that have negative within-community degree allows us to identify the tasks that attract students who prefer to interact a lot (namely, to publish many posts) on a limited set of tasks (i.e., the ones in that community), while the opposite situation is the one of tasks which attract students that interact a lot on many tasks.

DATA ANALYSIS

Figure 1 shows the network of problems posted in the forum. The weighted degree is represented by the radius of the node: the bigger the circle, the higher the degree. We notice that some of them are much more central than others: problem W1Q3 has the highest degree (1091), followed by W1Q2 (degree 936) and W3Q3 (757). Task W1Q3 has a conceptual nature and requires some reasoning and some general, abstract understanding of the concept of multiple numbers. It concerns arithmetic and is pretty long. W1Q2, which has the second highest degree, concerns arithmetic and general, abstract properties of natural numbers as well, but is shorter. W3Q3 is geometric, conceptual and pretty short. It may require a draw. Let us consider the five tasks with the lowest degree: W2Q1 and W2Q3 are procedural and about algebra, asking to compute the MCD and mcm of polynomials and sto simplify fractions; W3Q5 asked the definition of non-Euclidean geometries (and the students copy-pasted definitions taken from internet); W5Q4 asked to compute the probability of getting an 'ambo' in the Lotto game, and W5Q5 concerns the meaning of average in statistics. We notice that tasks W5Q4 and W5Q5 are conceptual and very far from the mathematical content the students are used to deal with in math classrooms in Italy.

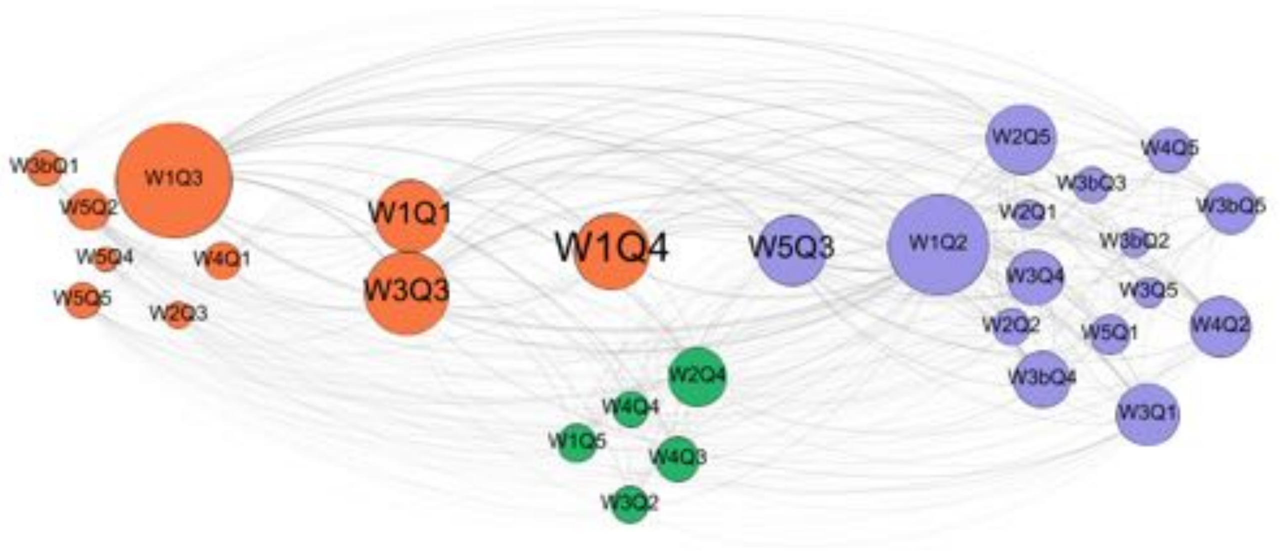

Figure 1: the network of 30 tasks in the online math forum.

High weighted degree for a task means that the comments to that task come from students who tend to comment a lot, so we can say that more interactive students have been more attracted by tasks concerning arithmetic (and analytic geometry), and requiring abstract reasoning or conjecturing. Students who are less interactive overall have been more attracted by tasks concerning probability and procedural algebra.

Betweenness in Figure 1 is represented by the size of the label of the node: the higher the size of the label, the higher its betweenness. If we consider betweenness as index of centrality, we see that problem W1Q3 has also the highest centrality (19), followed by W1Q1, W3Q3 and W5Q3, whose betweenness is 9. All the other problems have betweenness equal to 0. W1Q1 concerned arithmetics and logics, in that it asked to compute the number of students who do not play neither tennis not football, knowing that there are 400 students in the school, of which 150 play tennis, 100 play football and 30 play both. W5Q3 is about probability, it is conceptual and long.

High betweenness of a node means that it is necessary for a random walker to pass through the node in order to reach the other nodes, namely that the comments on those tasks come from students who commented at least once in many other nodes. While the degree measures how many times a student makes a comment, the betweenness measures the presence of a student in each thread. The fact that 26 over 30 tasks have betweenness equal to 0 means that the majority of students tend to post many times in the same task, instead of posting few times in almost all the tasks. This makes it interesting to analyse how they cluster around subgroups of tasks that attract subgroups of students. Tasks form a community if a student tends to interact on the tasks of that sort more than with other tasks.

In Figure 1 communities are identified by colours and three communities of tasks have been found. Community 1 (C1) (green) is made of W2Q4, W4Q3 and other three tasks.

In C2 (purple) we find W1Q2, W2Q5, W3Q1, W5Q3 and other eleven tasks. This community is the biggest one.

In C3 (orange) we find W1Q3, W1Q4, W3Q3 and other 9 problems. This is the community which has the most central problems in the network.

Within-community degree of each node in the network can be either positive or negative. C1 has 40% of nodes with positive within-community degree (tasks W4Q2 and W2Q4), and 60% of nodes with negative within-community degree. This cluster of tasks attracts the students who interact and post comments on few tasks, but they post relatively many comments. In a sense, we would say that this community of tasks attracts rather *focused, selective* students.

In C2, 47% of tasks have positive within-community degree (among them, W1Q2) and 53% have negative one (among them, W2Q1 and W3Q5). The tasks in this community attract those students who tend to make comments to many tasks, few comments per task. In a sense, we would say that this community attracts rather *superficial* students.

In C3, only 31.5% of tasks have positive within-community degree, while 68.5% of them are more connected to tasks outside the community. Since this is the community of tasks with highest weighted degree, we can say that it is made of tasks that attract the most interactive students: they interact a lot on the tasks of this community, and they interact a lot also on the tasks outside the cluster. We would say that this set of tasks allows us to identify the most *active* students.

CONCLUSIONS

Students' online interaction may help us to shed light on their engagement in math forums —even if engagement does not have to be confused with 'simple' interaction, namely with posting/commenting. The features of the tasks on which students tend to interact more than others are: the topic (e.g. arithmetic and geometry provoke more interaction than probability and algebra), and their conceptual nature. Being short or long seem not to have an effect on interaction. We also aimed at identifying tasks that attract students with different tendencies. We identified three kinds of students' behaviour in online forums: i) focused, specialised students who tend to comment a lot on selected, few tasks; ii) students that more superficially tend to comment a few on many tasks; and iii) very active students, who like some tasks more than others (and comment a lot on them), but who also comment on the other tasks. We further comment that these results confirm well-known findings in literature, however we remark that the methodology employed (i.e., network analysis) is potentially applicable to analyse forums with small to huge numbers of students and threads like the ones presented van de Sande’s (2011) study. Future work may address a larger sample performing finer grain analysis.

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Educating Pre-Service Teachers in Metacognitive Activities

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The present research examines the influence of pre-service teachers' professional development (PD) program in metacognitive skills on their practice of these skills in a mobile technologies environment. Twenty-four pre-service teachers participated in the PD program. The data was collected from the pre-service teachers' texts for activities solutions, as well as from their Edmodo discussion. We analyzed the data using the constant comparative method. The research results indicate that at the beginning, the pre-service teachers did not use such skills, but, because of the preparation, they started using these skills as learners, where this use utilized the mobile technologies. In a later phase, the pre-service teachers used these skills as teachers to encourage their students to use metacognitive skills collaboratively.

Keywords: metacognitive skills, pre-service teachers, mathematics teachers' education, mobile technologies.

INTRODUCTION

Researchers are interested in the metacognitive aspect because of its relationship with other aspects as the cognitive aspect (Gavelek & Raphael, 1985) and the social and affective aspects (Daher, Anabousy & Jabarin, 2018). Belet and Guven (2011) claim that metacognition makes students aware of their learning. This awareness supports the internalization of what one learns and makes him/her consider carefully how to solve problems posed in the classroom. These advantages of metacognition for students' learning make it necessary that teacher education colleges attempt to prepare pre-service teachers, so that they develop their knowledge of metacognition for teaching. This development is expected to develop also their perceptions of metacognition in teaching and learning. In the PD program that the present study accompanies, we intended to develop the metacognitive skills of mathematics pre-service teachers as learners and teachers of mathematics.

LITERATURE REVIEW

Researchers looked at metacognition as cognition about cognition or knowledge about knowledge (Flavell, 1976; Panaoura, Philippou & Christou, 2003). Flavell (1976) was the first to use the term 'metacognition' to refer to the individual's awareness, consideration and control of his or her own cognitive processes and strategies. Since then, a variety of definitions has been given to the term of metacognition. Du Toit and Kotze (2009) argue that the various definitions of metacognitive processes in the literature, including that of Schoenfeld (1992), emphasize the monitoring and regulation of cognitive processes. Moreover, Gavelek and Raphael (1985) argue that metacognition involves promoting effective understanding through adjusting the cognitive processes involved in the activity. Furthermore, Panaoura et al. (2003) say that it coordinates cognition, affecting it and, as a result, affecting students' academic success.

Researchers pointed out, that metacognition is comprised of two different components connected to each other. Veenman et al. (2006) argue that the most common distinction in metacognition distinguishes between metacognitive knowledge and metacognitive skills. On the one hand, Flavell (1999) defines metacognitive knowledge as the knowledge or beliefs about the factors that act and interact to affect the course and outcome of cognitive enterprises: person, the task and the strategy. On the other hand, metacognitive skills involve planning, monitoring, evaluating and regulating the processes leading to achieving goals. Davidson and Steinberg (1998) described a theoretical framework that includes the following metacognitive skills: encoding, representation, decomposition, planning, selecting strategy, monitoring, evaluating and suggesting other strategies. In the present study, we focused on metacognitive skills and utilized the previous framework to introduce metacognition to our pre-service teachers.

In addition, researchers suggested ways to encourage students to use metacognitive processes (e.g., Spiller & Ferguson, 2011). Schoenfeld (1992) describes ways for students to practice monitoring and evaluating their performance on math problems. For example, pause frequently during problem solving to ask themselves questions such as “What am I doing right now?” Spiller and Ferguson (2011) say that if we want students to use metacognitive processes, we need to encourage them to consider the nature and sequence of their own thinking processes. Chauhan and Singh (2014) say that as students become more skilled at using metacognitive strategies, they become confident and more independent as learners. In the present research, we wanted to educate mathematics pre-service teachers for using metacognitive processes, as learners and as teachers, through utilizing mobile technologies and collaborative learning.

Mobile technologies in mathematics education

Mobile technologies in general have been used in the mathematics classroom for more than a decade now. Advantages of using mobile technologies in education encourage teachers' use of these technologies, where various reasons encourage teachers to use them in their teaching (Daher & Baya'a, 2012; Ng & Nicholas, 2012). Ng and Nicholas (2012) reported that teachers are interested in mobile technologies for their professional development and because these technologies raise students' motivation to learn. In addition, these mobile technologies influence positively students’ behavior and emotions. Daher and Baya'a (2012) found that mobile technologies could be utilized as proper strategies for mathematics measurements and investigation in solving real life problems. These positive influences and utilizations of mobile technologies make us encourage our pre-service teachers to use them in their teaching. In the present research, we encouraged them to use the mobile technologies in their metacognitive processes, especially as strategies for solving real life mathematical problems.

Research question

How would mathematics pre-service teachers develop their metacognitive learning and teaching skills as a result of one-year preparation?

METHODOLOGY

Research context and participants

This PD program was held for a full academic year 2016-2017. Twenty-four pre-service teachers participated in the PD program. They were in their third academic year majoring in teaching mathematics and computer science in middle schools. Two of the authors, who were the pedagogical supervisors of these pre-service teachers, accompanied them in two middle schools in the frame of the practical training. Our preparing of the pre-service teachers in metacognitive skills was based on the work of Davidson and Steinberg (1998) (See above), with special emphasis on using mobile technologies for solution strategies. In addition, special attention was given for collaborative learning among the pre-service teachers' groups and their students. To achieve this goal, the pre-service teachers utilized the forums in Edmodo – an educational social network site, to discuss their preparation and implementation of the mathematics activities. The role of Edmodo discussions was crucial since all the disagreements, negotiation and alignment occurred in these forums. To read the detailed description of the PD phases, including the role of Edmodo discussions, see Daher, Baya'a, Jaber and Anabousy (2018).

Data collection and analysis

The data tools were the pre-service teachers' texts for the solutions of the activities that we requested them to carry out using metacognitive processes. In addition, we used the pre-service teachers' discussion texts in Edmodo forums.

To analyze the texts, we used inductive and deductive qualitative content analysis. Content analysis is a process designed to condense raw data into categories or themes based on valid inference and interpretation that use inductive reasoning. Deductive reasoning can also be used with the goal of generating concepts or variables from theory (Patton, 2002). Using the deductive reasoning we looked for themes related to the metacognitive skills from the work of Davidson and Steinberg (1998). Using the inductive reasoning, we tried to find out if additional metacognitive skills not given in the literature were described by the pre-service teachers.

FINDINGS

The findings report the participating pre-service teachers' metacognitive activity when solving authentic mathematical problems during the one-year preparation. First, they practiced using metacognitive skills, as learners, to solve an authentic mathematical problem prepared by their pedagogical supervisors. Second, they used metacognitive skills, as teachers, in the preparation of authentic mathematical activities that use mobile devices. Finally, it reports the pre-service teachers' metacognitive processes when implementing the prepared activities in the middle school with their students.

The pre-service teachers' metacognitive activity as learners

At the beginning and in the first activity, when the pre-service teachers were requested to build a plan for measuring the height of a tree, they did not mention any technological tools that would help them in measuring the tree height. They suggested using a stick, but could not elaborate on the process of the solution. While, in the following activities, for example, when given an authentic problem about a computer engineer from a village in the suburbs who was hired to work for a Hi-tech company in the city. The participating pre-service teachers were requested to help the engineer find the most efficient way to get to work. They suggested a plan that demonstrated their awareness of the metacognitive processes needed for such plan. Following is their suggestion in which they used the terms of the metacognitive processes.

Encoding of the givens – We can use Google Maps to identify the locations of the village and the city, and to measure the distance between them.

Representation of the givens – After presenting the locations on Google Maps, we prepare a table of the various measurements of the variables that could contribute to the efficiency of each way of transportation.

Decomposition of the problem – Depending on Google Maps representations, we can identify various roads of transportation.

Planning the solution strategies – After finding data about each road of transportation, we give weight for each road to determine its efficiency, and finally decide on the best road.

Selecting and implementing strategy – We will suggest to the engineer several roads that utilize Google Maps and the Mobile application "Waze". This would help provide data on each road, such as distance, time, toll payment, traffic jam, etc. These data determine the efficiency of the road. This mobile application would facilitate the obtaining of the data.

Monitoring of the plan – We advise the engineer to travel using more than one suggested road. Doing that, the engineer needs to keep collecting data, using a mobile application like "Waze", to keep calculating the efficiency.

Evaluating the solutions – To look after the measurement and compare between the efficiencies of the different roads, the engineer could register the collected data in a mobile spreadsheet. This application facilitates the evaluation of the efficiency of the transportation roads.

Suggesting other strategies or mobile applications – Finally, we advise the engineer to keep tracking of new strategies/applications that could improve the accuracy of the measurements in order to get better assessment of the efficiency of the transportation roads.

The pre-service teachers' metacognitive activity as teachers

We will describe an activity that shows the pre-service teachers' actions when preparing a metacognitive mathematical activity for their students. The whole activity occurred in Edmodo.

The activity: We want to plan an activity that encourages the use of metacognitive processes of middle school students regarding the following mathematical problem: A landowner needs to calculate the costs of tiling a wall in a building that includes the entrance to that building. Help the landowner with the calculations?

As a first step, the pre-service teachers discussed the task in Edmodo utilizing the metacognitive framework of Davidson and Steinberg (1998). Doing that, they discussed which questions they need to ask the middle school students in each phase of metacognitive processes. This discussion led them to adopt the following questions for students at each phase:

Encoding of the givens – Appropriate questions are: Which givens are present in the problem? Which givens are needed to compute the costs of tiling?

Representation of the givens – What ways do we have to represent the givens in the problem? What ways do we have to represent the givens needed to compute the costs of tiling?

Decomposition of the problem – After you represented the problem givens, how do you suggest that we decompose it?

Planning the solution strategies – How would you find the needed givens? How many ways or strategies are there? What are the differences between these strategies?

Selecting and implementing strategy – How can we implement each of the strategies that we identified? Which mobile applications can help us implement the planned strategy? What are the advantages and disadvantages of each mobile application?

Monitoring of the plan – How can we assert that the implementation of the plan is effective?

Evaluating the solutions – How can we evaluate that the implementation of the strategy is effective? How can the mobile application which we used help us in this evaluation?

Suggesting other strategies – How can we assert that the strategy which we used is more effective than other strategies? How can we assert that the mobile application which we used is more effective than other mobile applications?

The pre-service teachers' metacognitive processes when performing the prepared activity and their teaching this activity to middle school students

To ensure that they know how to guide the students' metacognitive processes, the pre-service teachers went through the whole series of metacognitive processes for solving the problem by themselves. Doing so, they suggested that the mobile applications 'Photo Ruler' and 'Smart Measure' would help them to implement their strategies. Some of them decided, in order to facilitate the implementation phase, to prepare user guides for using these applets and uploaded them to the Edmodo. In the monitoring phase, they looked at the measurements which each application gave. Evaluating the performance of each application, they found that the 'Photo Ruler' gave relative non-realistic measurements, while the 'Smart Measure' helped in calculating the scale of measurements from the same location, so they could convert the relational measurements to actual realistic ones.

The pre-service teachers pointed at the advantages of the collaborative learning to the metacognitive processes of the group regarding the solution of mathematical problems. They said that this collaborative learning contributes positively to all the metacognitive processes because every participant critically evaluates the suggestions of the rest of the participants.

The pre-service teachers worked with their students in the training school, in the Edmodo context and in the classroom context. They posed the questions which they agreed upon beforehand, and other questions that were raised as a result of the students' metacognitive processes related to the solving of the problem. The students were encouraged to manage their learning, to ask questions about this learning and to regulate it.

All the previous sequence of metacognitive processes was accompanied by reflections and discussions in Edmodo environment, which facilitated the success of these processes.

After the preparation and implementation of this activity, the pre-service teachers were required to work in groups of 4-5 members to design more activities of this type. Finally, each group of pre-service teachers was requested to choose an activity of those they designed earlier by themselves, and to implement it with a group of students.

DISCUSSION

Educating pre-service teachers for new practices has attracted the attention of educational researchers for its influence on teachers’ practice as college students and as future teachers. In the present research, we wanted to examine the influence of pre-service teachers’ preparation in metacognitive skills on their practice of these skills in a mobile technologies environment. The research results indicate that at the beginning, the pre-service teachers did not use such skills, but, as a result of the preparation, they started to use these skills as learners, where this use utilized the mobile technologies. In a later phase, the participating pre-service teachers used these skills as teachers to design activities and encourage their students to use metacognitive skills while performing them. These results indicate that metacognitive skills for learning and teaching could be learned and adopted by teachers, which agrees with other studies in mathematics education that examined the influence of education on teachers’ knowledge and practice. For example, Agyei and Voogt (2012) found that as a result of working collaboratively to design and develop technological solutions for authentic problems they face in teaching mathematics during their in-school training, a group of pre-service teachers developed their TPACK. We should note that the roles of the mobile technologies as tools for problem solving were influenced by metacognitive questions that the pre-service teachers asked in each metacognitive phase.

To conclude, the present research demonstrated that it is possible to educate mathematics teachers to use metacognitive processes. This education would affect positively their students' use of metacognitive processes (Du Toit & Kotze, 2009), which would result in deeper cognitive processes of the students (Gavelek & Raphael, 1985). Moreover, to succeed in this education, the pre-service teachers need to solve activities that emphasize metacognitive skills, to design such activities, to implement them with students, to discuss their practices, and to reflect on the whole sequence of their metacognitive processes.

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Instrumental genesis of a preservice mathematics teacher: instrumented actions for perpendicular line construction in a dynamic geometry environment

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This paper focuses on the instrumented actions of a preservice mathematics teacher while working on perpendicular line construction in a dynamic geometry environment. Data is collected through task-based interviews and were analysed from the perspective of instrumental genesis. A thematic analysis of the data revealed that the participant referred to four major strategies with the use of specific tools of GeoGebra to construct perpendicular line going through a point on a line that is provided. More precisely, instrumentation schemes including (*i*) *angle with given size tool,* (*ii*) *perpendicular line and the tangent tools,* (*iii*) *parallel line and perpendicular bisector tools, and* (*iv*) *the reflection tool have been used for achieving the construction task.*

Keywords: geometrical construction, dynamic geometry environment, instrumental genesis, preservice mathematics teacher, task based interview.

INTRODUCTION

In geometry education, geometrical representations enable learners to develop mathematical ideas about concepts. In particular, *drawing* and *figure* are highlighted as two important geometrical representation types in which "drawing refers to the material entity while figure refers to theoretical object" (Laborde, 1993, p. 49). Constructing geometrical figures in a dynamic geometry environment (DGE) provides learners with feedback about invariants of geometrical objects by manipulating spatio-graphical aspects of the representation (Arzarello, Olivero, Paola, & Robutti, 2002). In this sense, DGEs are considered having an important role in (re)inventing characteristic properties of the figures and also in the meaning-making of mathematical concepts. To create such a context, it is essential to design tasks that are based on Euclidean geometry in a DGE which bring about the difference between *drawing* and *figure*. Figure in a DGE can be regarded as *construction* in the Euclidean sense and preserves its invariants under dragging, which is defined as *robustness* (Laborde, 2005). Implementing mathematical ideas for making a robust construction in a DGE requires integration of appropriate tools into construction strategies.

In this work, we focus on construction strategies and aim to elaborate how GeoGebra could be used to shape the user's thinking processes while working on perpendicular line construction. This is particularly crucial in teacher education since working on construction tasks in a DGE require a combination of mathematical knowledge of Euclidean geometry as well as technology knowledge of DGEs, which are core competencies for teaching and learning mathematics. Along this direction, we consider a research question: what are the observed instrumented actions of a preservice mathematics teacher whilst working on geometric construction problems in GeoGebra?

THEORETICAL FRAMEWORK

In order to explore a preservice mathematics teacher's instrumented actions while working on construction problems with the use of GeoGebra, we chose to use instrumental genesis as a theoretical perspective (Artigue, 2002). The instrumental approach points out the difference between the terms 'artefact' and 'instrument'. In this approach, the process through which an artefact becomes an instrument is called instrumental genesis, which is a twofold process including instrumentalisation and instrumentation. The former is directed towards the artefact and concerns the way in which the artefact is shaped by the user. The latter is directed towards the user and concerns "the development and appropriation of schemes of instrumented action which progressively take shape as techniques that permit an effective response to given tasks" (Artigue, 2002, p. 250). For the aim of this study, the latter concept is of crucial importance. The word 'instrument' is used in a psychological sense, which is developed by the user through mental schemes for use in specific tasks (Hoyles, Noss, & Kent, 2004). Basically, an artefact is initially not meaningful to the user until he or she develops associated schemes of instrumented action to use the artefact for achieving a task, and effectively turning the artefact into a useful mathematical instrument. Instrumentation schemes consist of technical and conceptual components. In this paper, following the stance of Drijvers, Godino, Font, and Trouche (2013), techniques are considered "as the observable part of the students' work on solving a given type of tasks (i.e., a set of organized gestures)" and schemes "as the cognitive foundations of these techniques that are not directly observable, but can be inferred from the regularities and patterns in students' activities" (p. 27). From the instrumental genesis perspective, this study will focus on the preservice mathematics teacher's development or appropriation of instrumentation schemes and related conceptual components and techniques used while working on construction problems in GeoGebra.

METHODOLOGY

This paper is part of an on–going research project on the elaboration of preservice mathematics teachers' instrumental genesis on specific geometrical construction tasks. The project is designed as a holistic case study focusing on the participants' instrumentation processes, more specifically, it focuses on the preservice teachers' *emerging* utilisation schemes and related conceptual and technical elements (Drijvers et al., 2013) while they make use of GeoGebra. GeoGebra is open source dynamic mathematics software that has a geometry toolbar containing tools organised into several main groups (i.e. construction tools, line tools, circle tools, measurement tools, transformation tools) and each tool is briefly described which helps to orientate the users. In this paper, we will present pilot study results of a single task with a preservice mathematics teacher, Gonca (pseudonym), who is twenty-two years old and enrolled in a (lower secondary) mathematics teacher education program at a state university in central Turkey. Gonca was selected through a purposeful sampling method in which we considered her (i) geometry course performance, (ii) communication skills and (iii) background in the use of GeoGebra. Data is collected from a series of task–based interviews, which were recorded through a video-camera filming the participant's working environment, which includes a laptop with GeoGebra installed in front of Gonca. Screen recorder software was also used to capture the techniques Gonca employed in detail. All the collected data were analysed through thematic analysis techniques. Codes and themes that define schemes and conceptual elements were assigned by each researcher according to the main tool utilised by the participant and her reasoning behind it with reference to the used techniques. Final decisions were made based on a consensus in the follow-up meetings.

The Task

The task was designed to explore the participant's various strategies for constructing a perpendicular line while using functions and tools of GeoGebra. A GeoGebra file including a line (called *d*) through a point (called *A*) was prepared in Graphics window without the grids and axes. The question at stake was: *construct a line/line segment that is perpendicular with line d and goes through point A*. In order to explore the participant's use of different strategies, during the task implementation, each time when the participant finished a strategy, the interviewer removed the tool that had just been used by the participant during the finished strategy. For instance, if the participant used the *ray* tool to complete the task, then the interviewer removed the *ray tool* from the screen to see which strategy the participant would use without this tool. In this sense, by removing a tool each time we aim to give deeper insight into the extent of the participant's knowledge.

FINDINGS

Gonca used five strategies in total for completing the task. For the sake of brevity (due to page constraints), we will present four of her strategies in detail (other one was not included in here since it has a very similar way with the first strategy). We will discuss each strategy in detail below.

The First Strategy: The Use of Angle with Given Size Tool

The first strategy is based on the *angle with given size* tool. Before using this tool, she created a point (called B) and drew a circle with the centre of B and radius [BA]. She formed a 45-degree centre-angle and then tried to form a 45-degree BCA angle with the use of the *angle with given size* tool, but she could not locate the C point. She then deleted the sketches and created the second point (where the line would pass) with the use of the *angle with given size* tool inputting 90 degree (see Figure 1, where the red arrow shows the second point assigned by angle with given size tool). Then she constructed the perpendicular line. She justified this strategy by saying: "this line is perpendicular because I used the right angle".

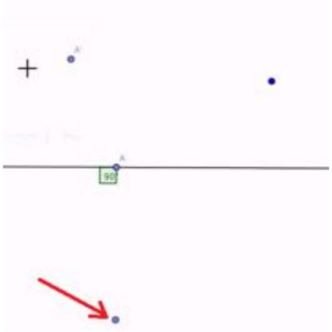


Figure 1: Finding the second point that the perpendicular line goes through

The Second Strategy: The Use of Perpendicular Line and Tangent Tools

After the first strategy, the interviewer removed the *angle with given size* tool and asked Gonca to complete the task with the remaining tools once again. This time, she focused on the relationships between a circle, its centre point and its chords. As a first step, she drew a circle with the centre A and then a chord (but she did not construct), which seemed to be parallel to the diameter. Next, she drew a line segment from the centre of the circle to the chord. This new line segment seemed as if it bisected the chord (Figure 2a). Although she could not express it in a clear way, she tried to employ her pre-knowledge, focusing on the fact that "two lines have a common perpendicular line if they are parallel", here she also stated that "a line segment from the centre to the chord is perpendicular to the chord". Then, she used the *angle measurement* tool and *dragging* test to check her claim and she noticed that her figure was not a robust construction, hence refuted her claim. After that, she constructed a circle through A with the centre point B and focused on properties of the tangent line. She used *the perpendicular line* tool to construct the tangent through A. With this, she completed her strategy (Figure 2b) by explaining the function of the *perpendicular line* tool. In the end, she noticed that she could have straightaway made use of the *perpendicular line* tool without creating a circle, which would have achieved the same thing.

As was done previously, the interviewer removed the *perpendicular line* tool from the GeoGebra interface and requested Gonca to consider the task with the remaining tools. She then completed the task with the same steps, but only used the *tangent* tool instead of the *perpendicular line* tool at the end.

(a) (b)

Figure 2: (a) Chord based strategy; (b) Tangent based strategy

The Third Strategy: the use of Parallel Line and Perpendicular Bisector Tools

The interviewer this time removed the *tangent* tool and requested Gonca to consider the task with the remaining tools. In her third strategy, she constructed a circle with the centre A. Then, she constructed a chord by using the *parallel line* tool which enabled her to construct a line parallel to the diameter chord of the circle (Figure 3a).

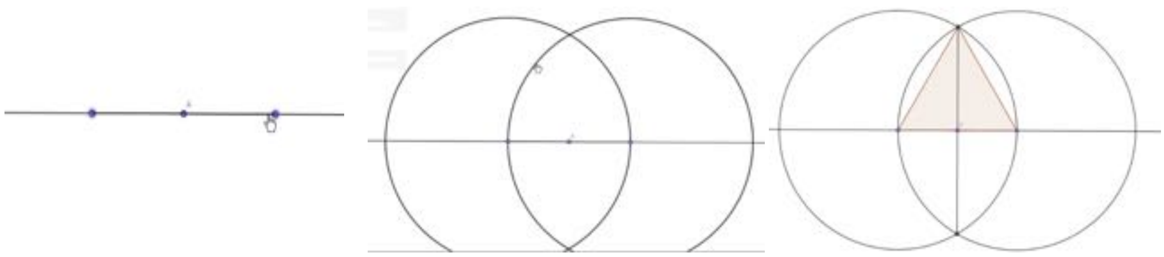
(a) (b)

Figure 3. (a) Construction of a chord parallel with the diameter chord, (b) Construction of the perpendicular bisector of the chord

To reason her initial steps, Gonca stated again that she first tried to develop a strategy based on her pre-knowledge that "two lines have a common perpendicular line if they are parallel". After that, she thought that she needed to construct the perpendicular bisector to the chord which then would also be the perpendicular bisector of the diameter chord. For this, she used the *intersect* tool and the *line segment* tool in order to mark the line segment. Finally, she used the *perpendicular bisector* tool and finished the strategy (Figure 3b). She then tested her figure's robustness by using the *dragging* test.

The Fourth Strategy: the use of the Reflection Tool

The interviewer removed the *parallel line* tool and the *perpendicular bisector* tool, and requested Gonca to consider the task with the remaining tools. Gonca, at first, aimed at adapting the strategy that she used earlier for the perpendicular bisector construction (constructing the common chord of intersecting circles) to its new strategy. Gonca then used the *point* and *reflect about point* tools to position the point A at the midpoint between two points on the line d. For this, Gonca firstly marked a point (called point B) on the line d and reflected point B about point A (Figure 4a) to construct point B' (aiming to situate point A as the midpoint) and through these points constructed the line segment BB' and then applied the perpendicular bisector construction strategy. During her adapted perpendicular bisector strategy, she constructed two intersecting circles whose centre points are B and B' respectively and whose radii measured BB' line segment by using the tool *circle passing through the centre and one point* (Figure 4b).



(a) (b) (c)

Figure 4. Gonca's reasoning steps with tools

In the next step, she marked intersection points of the circles with the use of the *intersect* tool and constructed the common chord of the circles with the use of the *line segment* tool. In the end, she used the *dragging* tool to test the robustness of her construction. Gonca justified her construction by stating: (1) the line segment goes through point A (perceptual inference), (2) the radii drawn from the central points to the intersection points of the circles by using the line segment tool create two congruent triangles (deductive justification), (3) these congruent triangles indicate the occurrence of a right angle on the basis of the fact that congruent triangles have equal angles (deductive justification) (Figure 4c).

In order to overview the participant's instrumentation schemes and related technical and conceptual elements regarding four strategies for completing perpendicular line task, we present the following table (Table 1) adapted from Drijvers et al. (2013).

Table 1: Instrumentation Schemes, Technical And Conceptual Components In The Process

Scheme	Technique	Conceptual Elements	Technical Elements
Angle Scheme	-Use angle with given size tool	-In order to construct a line, at least two points are required	-Click on, respectively, the *B* and *A* belonging to line *d* -Give measurement value 90^0 -Construct a right angle whose vertex is *A*
Tangent in Circle Scheme	-Use perpendicular line and tangent tool	-Tangent line is perpendicular to diameter	-Draw a circle going through *A* -Construct a tangent at *A* by using perpendicular line tool
Chord in Circle Scheme	-Use parallel line tool -Intersect tool -Perpendicular bisector tool	-Two lines have a common perpendicular line if they are parallel -Two parallel chords have a common perpendicular bisector	-Draw a circle with centre *A* -Draw a parallel line to diameter intersecting two points on the circle -Mark the intersection points -Highlight the chord with line segment tool -Construct perpendicular bisector line of chord through *A*
Perpendicular Bisector in Circle Scheme	-Use reflection tool -Draw two congruent intersecting circles	-Procedural knowledge about constructing perpendicular bisector with intersecting circles -Relationships between angles of equal triangles	-Construct two symmetrical points (B, B') at A -Draw two congruent circles whose centre points are B and B' and whose radii are line segment BB' -Draw a line segment between intersection points of circles

DISCUSSION AND CONCLUSIONS

The findings of the study revealed that the participant preservice teacher developed instrumentation schemes for perpendicular line construction with the use of GeoGebra and these schemes influenced the deductive steps in her strategies. Although she initially tended to justify her strategies according to her perceptual understanding, it has been become apparent that during the development of instrumentation schemes she extended her mathematical techniques, which also shaped her mathematical knowledge. She instrumented a number of tools in GeoGebra, such as *perpendicular bisector*, *tangent* and *symmetry* for achieving the perpendicular line construction task.

Gonca's construction processes indicate that she first tried to construct perpendicularity as an indirect invariant with the use of circles but she could not design a strategy based on a deductive reasoning process to complete the task (Baccaglini-Frank, Antonini, Leung, & Mariotti, 2017). At this point, her first strategies generally referred to her pre-procedural knowledge about using circles for construction problems instead of the use of deductive reasoning. Then, taking into account the affordances of various DGE tools, she constructed the direct invariant related to perpendicularity by utilising *angle with given size* and *perpendicular line tools.* After removing the tools that were already utilised by Gonca during the task, she was limited to construct the indirect invariant, therefore she started to engage in a deductive reasoning process by considering affordances of the *parallel line*, *perpendicular bisector* and *reflect about point* tools. Starting from this point, she finally adapted her previous procedural knowledge about perpendicular bisector construction to the task of perpendicular line construction. During the last strategy, she justified her steps with reference to both her perceptual knowledge and mathematical inferences obtained with the help of other DGE tools like *reflect about point*. In this sense, her instrumentation schemes on various DGE tools became an effective way to scaffold her deductive reasoning in construction tasks.

On the other hand, it was also seen that she initially formed soft constructions because her perceptual apprehension misled her geometrical conjectures. At this point, as Laborde (2005) states, robust construction tasks enable students to improve their theoretical perspective and also Arzarello et al. (2002) remark that the *dragging* test provides opportunities for students to see whether a figure preserve representations of invariant properties of a geometrical object as a robust construction. Similarly, by the means of the dragging test, Gonca started to distinguish the robust constructions based on the deductive approach from soft constructions that are generally based on her perceptual approach. Therefore, utilisation of the dragging test enabled her to reflect on how to develop valid strategies by considering mathematical affordances of the toolbar that was continuously limited by interviewers. At this point, her utilisation scheme about the *dragging* test provided development of further utilisation schemes about given DGE tools. To conclude, the participant's instrumented actions were observed with the use of particular tools including angle with given size tool, perpendicular line and the tangent tools, parallel line and perpendicular bisector tools, and the reflection tool for achieving the construction task.

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Design Principles for Resources and Tasks for Technology-Enhanced Teaching and Learning Mathematics

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ABSTRACT

This paper discusses the role of appropriate digitally-enhanced resources and tasks for learning and cognition in mathematics education. It bases on the cognitive load theory as a grand theoretical framing, lesson planning as an intermediate-level frame and a theory about modes of description and thinking of mathematical concepts as a domain-specific frame. The theoretical analysis resulted with three principles which we see important in the design of technology-supported resources and tasks. Further on, the paper offers a concrete design of a set of tasks for concepts in geometry in a dynamic geometry environment according to the suggested principles.

Keywords: mathematics education, learning and cognition, principles, design of resources and tasks, technology, dynamic geometry environment.

INTRODUCTION AND RELEVANCE

One of the three main themes at the fifth ERME Topic Conference on Mathematics Education in the Digital Age (MEDA) refers issues related to mathematics curriculum development and task design in the digital age. This paper is a contribution to this theme but may also be relevant for mathematics teacher education and their continuous professional development and to issues regarding development of theories for technology-enhanced designs. We begin with theoretical considerations about how are technology-enhanced resources and tasks in mathematics education defined in current literature. The paper then continues with a discussion about the present research state and our theoretical framing which enables formulating research questions. They are related to design principles that take into consideration the importance of technology-supported resources into facilitating learning and cognition having in mind empirical challenges for teachers and students. Finally we propose a design to illustrate and analyze the principals.

THEORETICAL BACKGROUND

Technology-Enhanced Resources and Tasks for Teaching and Learning Mathematics

With this paper we aim to increase teachers' awareness of the potentials of technology implementation and possible challenges and also to enable them to adapt existing designs according to the needs for learning and cognition of the students in their classrooms, facilitate in re-designs and eventually sustain and scale valuable resources. In order to achieve this, we have firstly considered the theoretical separation of *research on digital curriculum materials* from *research on instructional technology* (Pepin, Choppin, Ruthven & Sinclair, 2017). In our understanding, the first one refers to the deliberate attention to aims and content of teaching and learning mathematics, the roles of teachers and students and the educational potential of technologies, and the later one represents the broad literature about hardware and software designs and tools for educational purposes. This distinction cannot be perceived as rigorous, as the authors have also noticed that "clearly there is some overlap" (Pepin et al., 2017). Moreover, educational technology cannot lead to successful instruction by itself either. So, the first does not exclude the second one and vice versa. They rather coexist and complement each other. Therefore, we further use the terminology *technology-enhanced resources and tasks for the teaching and learning of mathematics* (inclosing substantial technology-based environments or micro-worlds, applets and digital manipulatives with accompanying instructional materials, tools with specifically created tasks for their usage). This understanding of the digitally-supported resources is in consistency with the definition *resources that afford or embed mathematical representations that teachers and learners can interact with by acting on objects in mathematical ways (Call of MEDA).*

State of Research Regarding Design Principals

In search of clarification about the place of task design within design research, Kieran et al., (2015) point out two key issues: (1) a distinction between *design as intention* and *design as implementation* (Ruthven, Laborde, Leach & Tiberghien, 2009) and (2) the status of an initial design of the set of tasks depending on the roles of theories and ways in which they are framed during the design process (Kieran et al., 2015, pp. 28 - 29). While design as intention focuses attention specifically on an innovative, hypothetical formulation of the design, design as implementation refers the integration of an actual teaching-learning designed sequence and its subsequent progressive refinement. Regarding the second central issue, resent theoretical frameworks and principles for task design in the research of mathematics education can be conceptualized as: *grand theoretical perspectives* (e.g., the cognitive-psychological, the constructivist, and the socio-constructivist), *intermediate-level frames* (e.g., the Theory of Didactical Situations, the Anthropological Theory of Didactics, Realistic Mathematics Education theory, Lesson Study, etc.), and *domain-specific frames* for task design research which deal with a particular mathematical field are more assorted than their intermediate-level counterparts (Kieran et al., 2015, p. 36). Prusak, Hershkowitz & Schwarz, (2013) have argued that (theoretically developed) five principals regarding a creation of: multiple solutions, collaborative situations, socio-cognitive conflicts, tools for checking hypotheses, and possibilities for reflection and evaluation on solutions may serve as principle-based research means for investigations of students' conceptual understanding and problem-solving through multimodal argumentation in elementary geometry.

We have undertaken a particular selection of a specific theoretical frame in each of the three major groups (by Kieran et al., 2015) and focus on their networking in the context of technology enhanced teaching and learning of geometry. The *cognitive load theory*, as a grand theoretical frame, has its validation in investigations of increased cognitive demand by large computations in arithmetic which have been widely researched and registered in literature (e.g. Weigand & Weth, 2002). Yet, little is known about how to facilitate the decrease of cognitive load when linking two or more representations of concepts in geometry. This is in particularly relevant to the context of technology enhanced teaching and learning, for example of congruence mappings, and we therefore use the theoretical framing of lesson planning to further ground the emergence of students' knowledge through solving exacting tasks. As a domain-specific frame, we refer to three modes of description and thinking of concepts in Geometry.

Three Modes of Description and Thinking of Concepts in Geometry

The theoretical construct about the existence of three modes of description and thinking is adapted from the research field in linear algebra (Hillel, 2000; Sierpinska, 2000) but here we transfer it to concepts in geometry. We refer to a *nested model* which emphasize three modes of thinking of concepts in linear algebra (Donevska-Todorova, 2018): the *arithmetic-algebraic*, the *geometric* and the *formal-structural mode*. This model has been exemplified for the group of all plane congruence mappings with the operation composition of transformations in a dynamic geometry environment from the semiotic perspective (Donevska-Todorova & Turgut, 2017). Here, we extend and deepen the analysis from a new perspective, the cognitive load theory and specify it with examples.

Potentials of Technology-Enhanced Resources and Tasks Examined Through the Cognitive Load Theory and Modes of Thinking

Resent literature (e.g., Sweller, Ayres & Kalyuga, 2011) describe *intrinsic* cognitive load depending on the difficulty of the content to be learned, *extraneous* cognitive load depending on the quality of the learning environment and "the levels of both intrinsic and extraneous cognitive load are determined by element of interactivity" (p. 58). "If the elements can be learned successively, rather than simultaneously because they do not interact, the intrinsic cognitive load will be low" (Sweller, 1994, p. 295). This is of high importance in designing tasks, teaching and learning sequences in digitally enhanced environments and digital curriculum materials and we perceive the supporting role in the reduction of extraneous cognitive load as crucial potential of technologies if *elements of interactivity* can be established. They may offer possibilities for a simultaneous learning of complex mathematical contents with a decreased total cognitive demand (if for example, there is an existence of elements of interactivity between the three modes of description and thinking).

RESEARCH QUESTIONS AND METHODOLOGY

Drawing upon the above theoretical analysis, we have identified three main challenges, which we formulate as the following three research questions:

(1) How to reduce the total cognitive load with the help of carefully designed technology-enhanced resources and tasks in geometry?

(2) How to integrate the three modes of description and thinking into technology-enhanced resources and tasks in dynamic geometry? How to embed elements of interactivity in the design?

(3) What could be design principles for technology-enhanced resources and tasks for the teaching and learning of mathematics with primary goal gaining mathematical knowledge and developing cognition?

The methodological approach involves consideration of relevant literature starting from more global towards more specific theoretical frames. We have initially taken into account general principles for mathematics education (e.g. principle of active learning, principle of adequate visualization, principle of variation of resources for learning) and wide-ranging principles for instructional design (e.g. simple-to-complex sequencing of tasks). Further, we have moved towards considering principles of specific theories, e.g. RME principles for designing online tasks (Drijvers, Boon, Doorman, Bokhove & Tacoma, 2013). Finally, we have narrowed the broad scope of approaches by founding our proposal on the triple-layer theoretical back-grounding suggested by Kieran et al., (2015) by adapting it to the purposes of investigations in geometry.

DESIGN PRINCIPLES FOR TECHNOLOGY-ENHANCED RESOURCES AND TASKS IN ELEMENTARY GEOMETRY

We continue the theoretical discussion from the perspective of teachers as 'actors' in the design and engaging in lesson planning. The above mentioned distinction of *research on digital curriculum materials* from *research on instructional technology* (Pepin, Choppin, Ruthven & Sinclair, 2017) seems to be related to two constructs, *purpose* and *utility,* correspondingly, suggested by Ainley, et al., (2006) aiming to offer a framework for task design that may resolve of what they called *a planning paradox*: "if teachers plan from tightly focused learning objectives, the tasks they set are likely to be unrewarding for the pupils, and mathematically impoverished. If teaching is planned around engaging tasks the pupils' activity may be far richer, but it is likely to be less focused and learning may be difficult to assess" (Ainley, et al., 2006, p. 24). Whilst the final form of presenting mathematics starts with definitions and aims towards theorems and proofs, the development or the discovery of mathematics and especially school mathematics usually has an opposite direction: experience through usage and engagement without mathematical rigor towards a conclusion later strengthened as a definition or other mathematical statement. Regarding the modes of description and thinking, this means that students firstly use either the algebraic or the geometric and seldom engage with both in a subsequent order. After a long-lasting process of learning they eventually develop the structural mode, although they have already used parts of an algebraic structure without being aware of it (e.g. one or more properties that define it, like associativity or commutativity of certain operation such as composition of congruence mappings). Thus, a connection of engagement and focus on mathematics (e.g. Ainley, Pratt & Hansen, 2006) in the design of technology supported resources and tasks is meaningful.

The above theoretical considerations allow framing three design principles for technology-enhanced resources and tasks for learning mathematics and developing cognition, which we present in the following Table 1 [1].

Design principles	Empirical rationale (through the theoretical frames)
Principle 1. Reduction of the total cognitive load by decrease of the extraneous cognitive load	Teacher-related challenge: Teachers' design capacities e.g. (un)awareness of the cognitive load by traditional and digital resources, possessing/lacking IT skills Student-related challenge: (Un)familiarity with technology enhanced learning, (in)ability to understand the role of the digital media Operational challenge: lacking digital equipment
Principle 2. Reduction of the total cognitive load by decrease of the intrinsic cognitive load by a careful and gradual integration of the three modes of description and thinking of mathematical concepts (in connection to element interactivity)	Teacher-related challenge: the level of mathematical knowledge, e.g. (un)awareness of the existence of all three modes and their connections Student-related challenge: integration of new with existing knowledge, e.g. developed geometric thinking but (in)ability to connect it with the algebraic counterpart of a particular concept Operational challenge: determine appropriate digital tools that can support all modes, i.e. detect effective tools
Principle 3. Connection of active engagement and focus on mathematical contents	Teacher-related challenge: Teachers' math knowledge, Resolving the planning paradox Student-related challenge: stayed focused during different purposeful activities Operational challenge: Methodical issues

Table 1: Overview of design principles for technology-enhanced resources and tasks for the teaching and learning mathematics with focus on cognition

The three principals explain the rationale for implementing technology and their meaningful and multifaceted usage in mathematics instruction. Certainly, there are challenges for teachers and students which we have also addressed in the right column of Table 1 (formed similarly as by Lo, Hew & Chen, 2017).

SUGGESTED DESIGN FOR CONGRUENCE MAPPINGS RELAYING ON THE PROPOSED PRINCIPLES

This section suggests a design of a set of tasks about congruence mappings in a DGE-environment e.g. in a pre-service teacher education course. It is only the introduction into the topic aiming at the understanding of a congruence mapping as a composition of at most three line reflections and finally understanding the meaning of the group structure of congruence mappings.

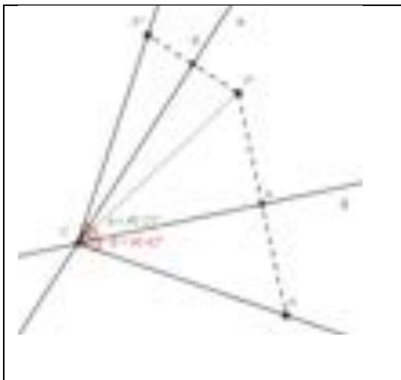
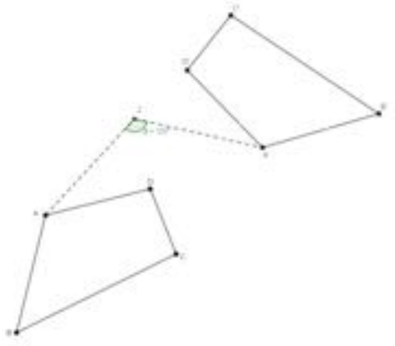
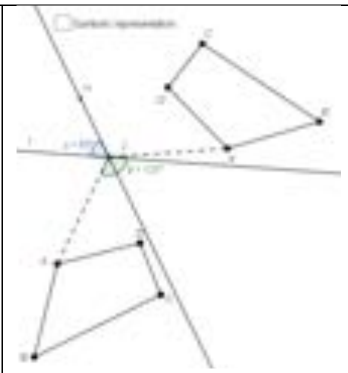
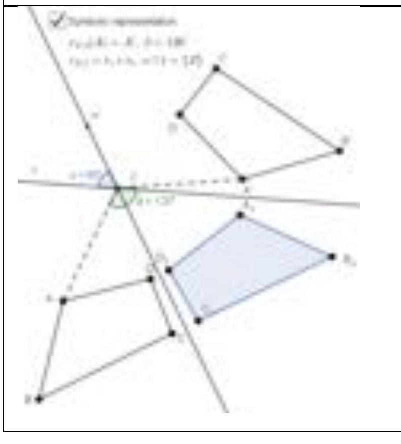
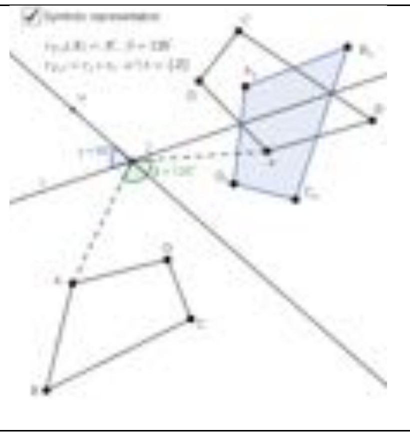
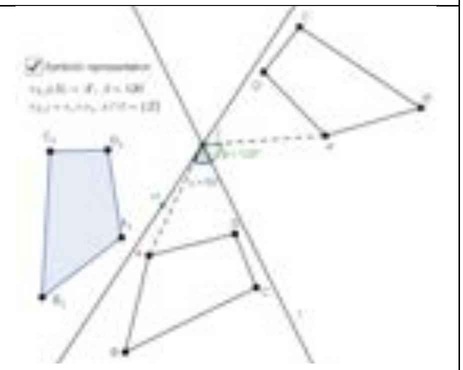
Task 1: What is the composition of two line reflections? Is the operation associative or commutative? (A design of intention and implementation for the composition of two line reflections which is a rotation, a translation or the identity congruence mapping depending on the position of both axes of reflection is suggested by Donevska-Todorova & Turgut, 2017).

Task 2: There are two plane geometric figures, e.g. quadrilaterals on the Figure below, given such that one is an image of the other one with respect to a particular congruence mapping in the plane.

a) Determine a congruence mapping that maps one of the quadrilaterals into the other one.

b) How can the congruence mapping in a) be represented as a composition of two line reflections? Explain and draw the axes of reflection.

c) Draw the ‘missing’ image quadrilateral obtained after the first line reflection.

		
a) Task 1. Composition of two line reflections with intersecting axes	b) Task 2. Center and angle of rotation	c) Task 2. Axes of both reflections
		

d) Task 2. Image quadrilateral and symbolic representation	e) Task 2. Invariance of the composition and the image figures under the drag mode for replacing the axes	f) Task 2. Invariance of the composition and the image figures under the drag mode for replacing the axes

Figure 1. Technology-enhanced task design as intention

While it may be easy to discover that a composition of two line reflections with intersecting axes at a point is a rotation (Task 1), the task in the opposite direction (Task 2) may be more requiring.

Analysis of the suggested technology-enhanced task design of intention through the principals

P1.	Design only in the geometry window with an integrated symbolic (algebraic) representation in it, e.g. see Fig 1.d), e) and f). Investigations of invariant properties through dragging.
P2.	First work only with the geometric mode - Fig.1b) and c), then include the arithmetic-algebraic - Fig.1d) and e) by including the checkbox and text tools from the menu as elements for interaction between the modes.
P3.	Extend the tasks with parallel or overlapping axes and prove that the composition is a translation for a vector, a rotation or the identity congruence mapping. Finalize with the formal-axiomatic mode of description- closure property of a group Fig.1a), d), e) and f).

Table 2. Analysis of the design of intention through the principals

Solving the problems in Tasks 1 and 2 takes place only in the geometry window (Principle 1), e.g. see Fig. 1. An increased cognitive load due to involvement of two or three modes may be reduced with a careful design for a gradual addition of a new mode of description (Principle 2). Current DGS allow such designs (e.g. Figure 1).

CONCLUSION

In this paper we have discussed technology-enhanced resources and tasks as designs of intention for teaching based on a cognitive load theory as a grand-theory, lesson planning as an intermediate-level frame and modes of description of concepts in geometry as a domain-specific theoretical frameworks. If one of the three main aspects of teachers' design capacities is "a set of design principles ("robust" and at the same time "flexible")" (Pepin, Choppin, Ruthven & Sinclair, 2017, p. 8), having the learning of mathematics and developing cognition in students as a goal of the design, then we propose such set of three principles (Table 1) which meets the third research question. Regarding the first research question about how to reduce the total cognitive load with designed technology-enhanced resources and tasks we suggest reduction of the extraneous and the intrinsic cognitive load by pointing elements of interactivity included in the design of intention (Table 2). This leads to answering the second research question that the elements of interactivity could be stimuli that allow switches from one into another mode of description and thinking without changing windows but by a simple drag or check of a text- or checkbox. Lastly, our further activities refer practical implementation of the created technology-enhances task design of intention referring the analyses through the principles related to lowering the cognitive load.

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Transitioning a Problem-Based Curriculum from Print to Digital: New Considerations for Task Design

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The purpose of this paper is to report on new considerations for task design of curriculum materials for a digital world. These considerations build upon a print problem-based curriculum that has evolved over 30 years. We offer a new design on problems presented in a digital medium.

Keywords: Task Design, Middle School Curriculum, Digital Technologies.

INTRODUCTION

In this paper, we discuss important task design considerations for problems in a problem-centered curriculum. In our work, we view problems that are embedded within sequences of mathematics problems to promote inquiry-based teaching and learning. Specifically, we report on the design considerations for problems in a print version of a problem-centered curriculum. We then revisit the design considerations for the problem-solving activities based on our work of transitioning from print to digital curriculum. This work is funded by two National Science Foundation projects in the United States. These projects investigate student learning and engagement through use of collaborative and individual spaces on a digital platform.

CHALLENGES OF PROBLEM-BASED CURRICULUM OVER TIME

The Connected Mathematics Project (CMP) at Michigan State University has worked over 30 years to design, develop, field-test, evaluate, and disseminate student and teacher materials for a middle school mathematics problem-centered curriculum, *Connected Mathematics*. The CMP curriculum development has been guided by a single mathematical standard:

> All students should be able to reason and communicate proficiently in mathematics. They should have knowledge of and skill in the use of the vocabulary, forms of representation, materials, tools, techniques, and intellectual methods of the discipline of mathematics, including the ability to define and solve problems with reason, insight, inventiveness, and technical proficiency (Lappan & Phillips, 2009, p. 4).

To accomplish this goal, problems must embody critical mathematical concepts and skills and have the potential to engage students in making sense of mathematics. Thus, each CMP Problem has some or all of the following characteristics:

- Embeds important, useful mathematics
- Promotes conceptual and procedural knowledge
- Builds on and connects to other important mathematical ideas
- Requires higher-level thinking, reasoning, and problem solving

- Provides multiple access points for students
- Engages students and promotes classroom discourse
- Allows for various solution strategies
- Creates an opportunity for teacher to assess student learning

From the onset in 1990, the design and development challenge of the CMP is to create an environment that supports students' mathematical development through the process of exploring, conjecturing, reasoning, communicating, and reflecting. Creating this classroom environment requires thoughtful attention to the strategies students use to solve the problem, to the embedded mathematics, and to connections to prior learnings (Figure 1).

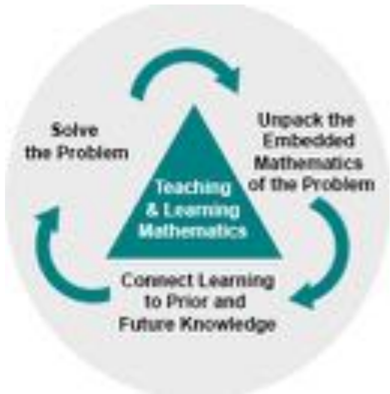


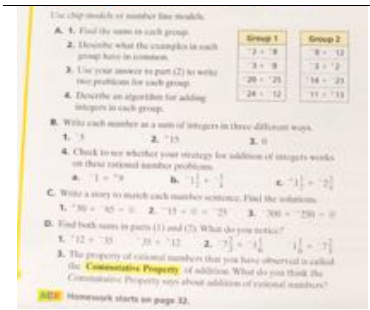
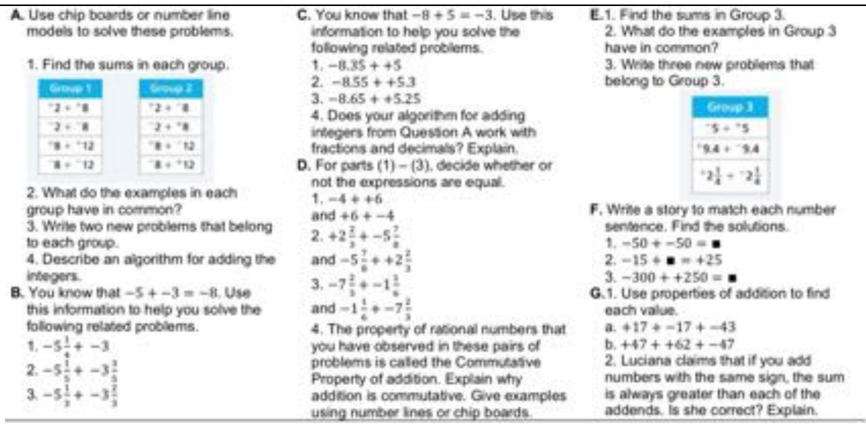
Figure 1: Teaching and learning mathematics in problem-centered classrooms.

The time required to develop a particular mathematical idea fully, the extent to which students grasp the mathematical subtlety of ideas, and the degree to which students reach useful closure of the idea being developed require careful attention to the mathematical challenge of the problem and the position of the problem within a carefully sequenced set of problems (Lappan & Phillips, 2009).

These design challenges were attented to in the unique design research development process of CMP which spans repetitive years of unit design, field trials, and data feedback cycles. Iterative feedback cycles focused on revisions to the curriculum units based on feedback from teachers and students across the country as well as mathematicians and educational researchers. Approximately 500 teachers in 55 trial sites around the country (and thousands of their students) were a significant part of the team of professionals that informed material development for CMP1, CMP2, and CMP3.

Even though extensive research on CMP shows that both student and teacher learning increases (e.g., Cai, Moyer, Hwang, Nie, & Garger, 2012; Reys, Reys, Lapan, Holliday, & Wasman, 2003) and CMP students' positive attitudes towards mathematics persisted through high school (Moyer, Robison, & Cai, 2018), the CMP authors continue to seek ways to enhance student and teacher learning. For example, as the CMP authors interacted with the field with the print curriculum, their knowledge of student understandings and teacher needs grew. Many of these new learnings found their way into problem tasks. In some cases, the tasks became longer and more nuanced. As an example, the following is a version of a problem from the *Accentuate the Negative* unit on addition of integers in CMP2 and CMP3 (see Figure 2).

Developing an Algorithm for Addition of Rational Numbers

Problem 2.1 from CMP2	Problem 2.1 from CMP3
	

A. Use chip boards or number line models to solve these problems.

1. Find the sums in each group.

Group 1	Group 2
$^{+}2 + {}^{+}8$	$^{+}2 + {}^{-}8$
$^{-}2 + {}^{-}8$	$^{-}2 + {}^{+}8$
$^{+}8 + {}^{+}12$	$^{+}8 + {}^{-}12$
$^{-}8 + {}^{-}12$	$^{-}8 + {}^{+}12$

2. What do the examples in each group have in common?
3. Write two new problems that belong to each group.
4. Describe an algorithm for adding the integers.

B. You know that $-5 + -3 = -8$. Use this information to help you solve the following related problems.

1. $-5\frac{1}{4} + -3$
2. $-5\frac{1}{5} + -3\frac{3}{5}$
3. $-5\frac{1}{3} + -3\frac{2}{3}$

C. You know that $-8 + 5 = -3$. Use this information to help you solve the following related problems.

1. $-8.35 + +5$
2. $-8.55 + +5.3$
3. $-8.65 + +5.25$
4. Does your algorithm for adding integers from Question A work with fractions and decimals? Explain.

D. For parts (1) – (3), decide whether or not the expressions are equal.

1. $-4 + +6$ and $+6 + -4$
2. $+2\frac{2}{3} + -5\frac{7}{8}$ and $-5\frac{7}{8} + +2\frac{2}{3}$
3. $-7\frac{2}{3} + -1\frac{1}{6}$ and $-1\frac{1}{6} + -7\frac{2}{3}$
4. The property of rational numbers that you have observed in these pairs of problems is called the Commutative Property of addition. Explain why addition is commutative. Give examples using number lines or chip boards.

E. 1. Find the sums in Group 3.
2. What do the examples in Group 3 have in common?
3. Write three new problems that belong to Group 3.

Group 3
$^{-}5 + {}^{+}5$
$^{+}9.4 + {}^{-}9.4$
$^{+}2\frac{1}{4} + {}^{-}2\frac{1}{4}$

F. Write a story to match each number sentence. Find the solutions.

1. $-50 + -50 = ■$
2. $-15 + ■ = +25$
3. $-300 + +250 = ■$

G. 1. Use properties of addition to find each value.
a. $+17 + -17 + -43$
b. $+47 + +62 + -47$
2. Luciana claims that if you add numbers with the same sign, the sum is always greater than each of the addends. Is she correct? Explain.

Figure 2: Evolution of a problem from CMP2 to CMP3 (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006; Lappan, Phillips, Fey, & Friel, 2014).

Transitioning to Digital Curriculum Materials

Building on prior research and development, the current NSF-funded projects explore two broad hypotheses about how a new technology environment can enhance student learning:

- Development and testing of a digital version of CMP will yield important insights into a variety of specific teaching and learning strategies made possible by technology rich educational environments.
- Use of curriculum materials delivered in a digital medium will produce significant improvement of student engagement and learning in diverse settings for middle grades mathematics instruction.

The purpose of one NSF project is to build collaborative and individual spaces on a digital platform to enhance student development, communication, and learning records of mathematics. The purpose of the second NSF project is to enhance student engagement and learning by redesigning problems and embedding them in the same digital platform.

Efforts to (re)design the mathematics problems in the digital curriculum focus on three aspects: (1) the Initial Challenge to contextualize and problematize the situation, (2) What If...? scenarios to surface the embedded mathematics of the problem, and (3) Now What Do You Know? to connect learning to prior and future knowledge. Additionally, the digital medium affords an opportunity for "just-in-time" supports that connect to each component. As shown in Figure 3, these aspects of the mathematics problem relate directly to the identified challenges described earlier.

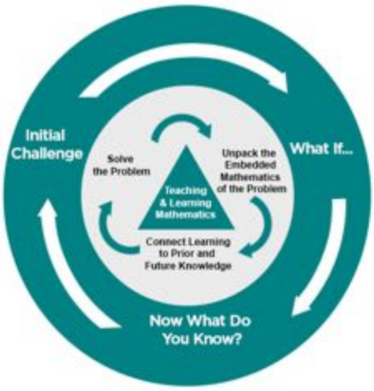


Figure 3: Redesigned problem structure and its connection to the task design challenges.

The new problem design format affords an environment that is visually less daunting with more succinct focus for students and teachers. For example, Table 1 shows the redesign of a problem on developing an algorithm for adding rational numbers that was discussed earlier (see Figure 2). The redesigned problem allows students to develop this algorithm efficiently as it builds on prior knowledge and experiences involving chip models and number lines. Through the problem and its three components, students can choose a model to help solve the problem or another approach such as a numeric strategy. This differs from the print material that directs learners by suggesting particular pathways for developing an algorithm, imposing a particular model, and finding specific sums of two numbers.

The "just-in-time" supports are provided to support students in their problem-solving pathways. Drawing on the affordances of learner-controlled scaffolding for inquiry-oriented mathematics classrooms (Edson, 2017), "just-in-time" supports or prompts are provided by curriculum authors and teachers that can help students solve an immediate difficulty, gain new knowledge, insight, or skill, or recall something that has been learned and forgotten. The premise of the prompt supports in a problem-centered environment is that if the students struggled unproductively in getting started in an open problem, they maintained ownership of the learning by using the supports without teacher intervention (Edson, 2016). In addition, empirical research has shown that students used the prompts as a mechanism to assess their group discussions or solution strategies without confirming their final answers with an external authority (Edson, 2016). Examples of "just-in-time" supports for Problem 1.3 are shown in Table 1.

Initial Challenge	*What If…?*	*Now What Do You Know?*
How can you predict whether the sum of two rational numbers is 0, positive, or negative?	What if you changed the order of the two numbers you are adding. Will this affect your answer? Explain. What if you have more than two addends?	What is an algorithm for adding rational numbers? How does this compare to your algorithm for adding whole numbers?

Just-in-Time Supports	Just-in-Time Supports	Just-in-Time Supports
• How could you model the example you have? Would a chip model be helpful? How would you use a number line to model your example? • Have you tried adding two positive numbers? Two negative numbers? One positive and one negative? Do you have a record of the examples you have tried? • [Teacher generated option]	• Study the patterns in the examples you recorded. • What patterns will help you add two numbers with the same sign? • What patterns help you add two numbers with different signs?” • [Teacher generated option]	• What is an algorithm for adding two numbers with the same sign? • What is an algorithm for adding two numbers with different signs? • [Teacher generated option]

Table 1: Redesign of Problem 2.1 of Accentuate the Negative.

In the digital medium, the problem appears differently than on print. Figure 4 shows the Initial Challenge for Problem 1.3 of the *Moving Straight Ahead* unit as seen on the digital platform. Here, students navigate through each component of the new design.

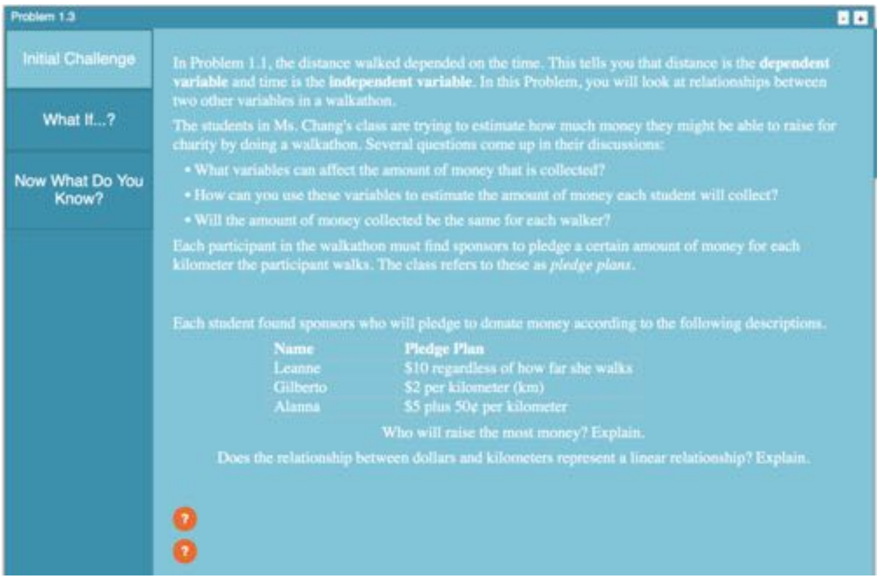
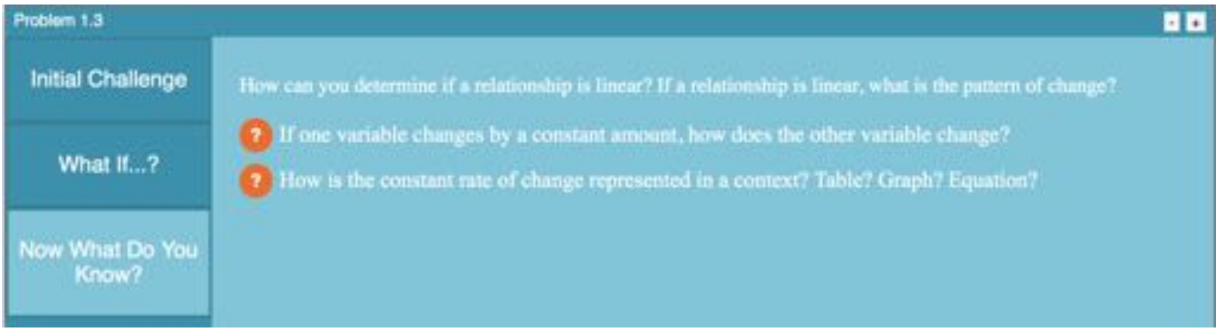


Figure 4: The Initial Challenge of a problem from the *Moving Straight Ahead* unit.

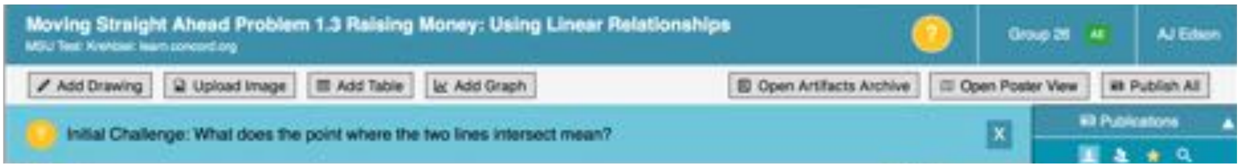
Student selected supports would be immediately available to each and every student and their groups as they explore problems. Students could select supports by clicking on buttons to reveal the prompts. The teachers could modify or create new supports. Another option is that teachers can generate and/or release prompts. While planning or enacting problems, teachers can assess the needs of their students, and release prompts to the entire class, select groups of students, or individuals. Not all students would access these supports unless the teacher released them.

For example, Figure 5 shows the student-selected supports (in orange) and the teacher-released supports (in yellow) when they have been activated by students in the digital medium. Also shown in Figure 5 is how the teacher can write the prompts and release the prompts to students. Note that teachers are notified when students activate the teacher-released supports.



a. Activated student-selected supports on the student digital platform.

b. Teacher space for generating teacher-released supports on the teacher digital platform.



c. Activated teacher-released support on the student digital platform.

Figure 5: Example of student-selected supports and teacher-released supports for Problem 1.3 of the *Moving Straight Ahead* unit.

Figure 6 shows a screenshot of the platform for a student with individual and collaborative spaces: Figure 4 is embedded in the upper left-hand corner, collaborative drawing, table, and graph tools are in the middle and the column to the right allows students to share their work with other students and the teacher.

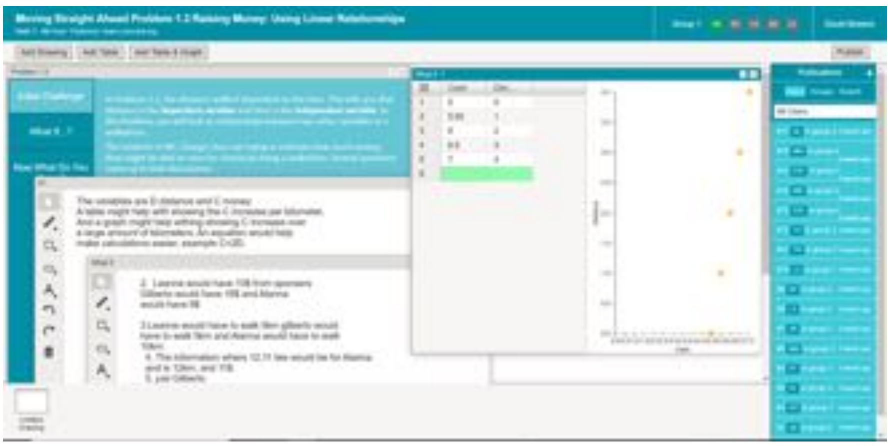


Figure 6: Sample student screenshot for Pb 1.3 of the *Moving Straight Ahead* unit.

CONCLUSION: NEW DESIGN CONSIDERATIONS

The design considerations that exist for print curriculum are still relevant for digital curriculum. These include: (a) identifying the important ideas and their related concepts and procedures; (b) designing a sequence of tasks to develop understanding of the idea; (c) organizing the sequences into coherent, connected curriculum; (d) balancing open and closed tasks; making effective transitions among representations and generalizations; (e) addressing student difficulties and ill-formed conceptions; (f) deciding when to go for closure of an idea or algorithm; (g) staying with an idea long enough for long-term retention; (h) balancing skill and concept development; (i) determining the kinds of practice and reflection needed to ensure a desired degree of automaticity with algorithms and reasoning; (j) writing for both students and teachers; and (k) meeting the needs of diverse learners (Lappan & Phillips, 2009).

The move to a digital platform that incorporates collaborative and individual learning spaces is not without risk, especially when problems are redesigned. New questions on task design considerations arise for developing digital curriculum:

- Will the nuances of the understandings emerge so that students form solid conceptual and procedural foundations?
- Does the redesign depend on the location of the problem in the learning progression? How effective is it?
- How do the "just-in-time" supports vary within and across a sequence of problems? When are they needed? Who is using them? How often?
- How much and what kind of new teacher support is needed? How does the teacher customize the environment?
- Does the new digital environment promote learning? When? Under what settings?

Since we are working with a well-researched and widely implemented print curriculum, we are building on learning progressions and teacher supports that are effective across the middle school curriculum. This provides a basis for the research project to study how use of the redesigned problems and "just-in-time" supports in a digital platform can promote an inquiry-based classroom environment where students are engaged in making sense of mathematics.

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Reflection as a mechanism to explain changes in teachers' identity: The case of Yosef

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This research, follows a professional-development that encourages the participating teachers to integrate technology in their lessons. A modified theoretical framework enables us to identify and explain the changes in the participating teachers' identity. This is demonstrated as we focus on the discourse of one participating teacher.

Keywords: Technology, commognition, reflection, identity, professional-development.

INTRODUCTION

For many years the integration of technology in teaching and learning has been the subject of research in the field of mathematical education. Although, in general, studies illuminate the integration of technology in a positive light, there is still consensus that the integration of technology is far from fulfilling its potential. We now understand that the key factor that can implement or limit the integration of technology in the math class is the teacher, who is required to change his practice (Hegedus et al., 2016). Therefore, it seems evident, that to promote technology integration we must first support teachers.

The research presented in this paper is looking closely at an in-school professional development tackling the challenge of encouraging teachers to use technology in class. The PD was planned according to the following guiding principles: (1) The teacher is an adult-learner; therefore, the PD should be planned to be relevant to the teacher's authentic professional needs; (2) Only outcomes emerging from those needs may lead to long-term changes in her practice of teaching; (3) At present, in the Israeli high school math education system the teacher is the one that decides which technology to integrate in her class and in what manner; (4) Integrating technology into teaching is a process that should encourage the teacher to reflect upon her practice. The uniqueness of the PD that stems from these principles, is that it gives teachers an opportunity to define their pedagogical needs, and it also provides them with resources of time and professional support to find a technological solution that meets these needs. This research intent to find out whether such a short-term PD may demonstrate detectable changes in the discourse of the participating teachers regarding their practice.

THEORETICAL FRAMEWORKS

The theoretical components of this work are inserted in two utility levels. At the first level, we use an adaptation of a section from the "UNESCO ICT Competency Framework for Teachers" (UNESCO, 2011). This framework focuses on mapping the three stages of teachers' development in the six complementary competencies that teachers should master for supporting their students as they become "collaborative, problem-solving, creative learners through using ICT so they will be effective citizens and members of the workforce" (pp. 3). In addition, the framework elaborates the operational meaning of each developmental stage regarding each of the six aspects of the teachers' work, which are: Understanding ICT in education, Curriculum and assessment, Pedagogy, ICT, Organization and administration, Teacher professional learning. In order to perform a preliminary analysis of the teachers' conduct in the PD, we chose three of the six aspects stated above (ICT, Pedagogy, Organization and administration), fine-tuned their definitions and elaborated the three stages in a manner that is relevant to our research as follows (Table 1):

level / category	poor	basic	advanced
Literacy of ICT	The teacher is technologically illiterate.	The teacher can use basic technology tools such as power point presentations, static visualizations or ready-made apps.	The teacher is able to create various technological tools using available resources.
Ability to conduct Technology based activities in class	The teacher is barely familiar with school technological facilities	The teacher can operate technology from the teacher's position	The teacher can conduct a whole lesson with students using technology
Pedagogy of ICT	Teaching with no technological aid in or out-side class.	Adding technology supported activities without changing pedagogy	Performing appropriate mutual adaptation of technological tools and teaching mathematics

Table 1: Revised framework for determining teachers' developmental level

The commognitive theory of learning (Sfard, 2008) is the second component which our research relies upon. It is our overarching framework which defines the basic concepts with which we intend to establish our research. The commognitive paradigm adopts the Vigotskyan tenet saying that every human skill is a product of a process of individualization of collective activities. In particular, thinking is a unique human skill emerging from the collective activity of communication and we refer to thinking as communicating with oneself. The term commognition is a constant reminder of the identification of thinking with communication. According to Sfard (2008), a discourse is a well-defined form of communication characterized by its keywords, visual mediators, routines and endorsed narratives. In addition, learning is the process of becoming a proficient interlocutor in an established discourse, and the detection of learning is the detection of changes in one's participation in that discourse. The commognitive theory is offering an additional perspective on learning by using the concept of *identity* of a person, which is operationally defined as *the collection of all reified, meaningful and endorsable stories that are told about that person* (Sfard & Prusak, 2005). The *identity* can be either *actual*, as the stories are told about a person at present, or, it can be *designated*, as the stories are told about the expected future of that person. Those stories, being told about a person, by either first, or a third person, are according to Sfard and Prusak, the missing link that enable us to reveal the mechanism through which a community effects the learning of its members. Therefore, we may look at learning as narrowing the gap between the actual identity and the designated one.

For the purpose of our present research some adaptations to the commognitive theory are required, since the object which is in the center of the discourse in the PD is not mathematics but rather the practice of the participating teachers, and the participating learners are not young students, but rather adult in-service teachers. Therefore, in this research we suggest the concept *professional-identity* as the collection of all reified, meaningful and endorsable stories told about the practice of a teacher. In addition, we would like to recruit the concept of *reflection* to assist us with talking about the way those stories are created. The concept reflection has few references in the context of teachers' professional development, where it is characterized as a tool that helps teachers turn their experience into knowledge (McAlpine & Weston, 2002; Clarke, 2000) and may or may not have implications for the teachers' practice (Ricks, 2011). Converting reflection into the commognitive framework we shall define it as *the* ***action*** *of telling the stories that eventually will constitute a teacher's professional-identity* (in first, second or third person). Therefore, according to this definition, when teachers in the PD, are communicating or participating in a discourse about their practice and by doing that they are telling stories that are, or will become, reified, meaningful and endorsable, we may say now that they are engaged in the action of reflection.

METHODOLOGY

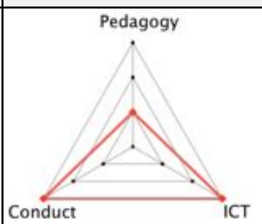
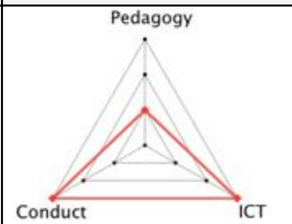
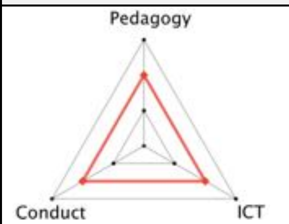
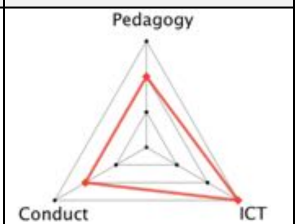
The research is following a PD that was conducted during the 2016-2017 school year in a public middle-high school in a major city in central Israel. Seven teachers took part in the PD, most of them teach mathematics in both middle and high school. In addition, most of the teachers have significant experience using technology, since teaching mathematics is their second career after working in high-tech professions for some time. There were 10 meetings of 90 minutes each, to the PD and they were two weeks apart to allow teachers to experience the integration of technology in their lessons between each two meetings. The main idea of the PD, as reported in the introduction, was to offer the participating teachers resources of time and professional support to allow them to work with their peers and come-up with technological solutions to their authentic pedagogical needs, and in addition, to let them have opportunities to share their experience with their peers and instructor.

Each PD meeting except the opening and closing sessions was conducted according to the same pattern: At the beginning each participant reported to the group what she achieved in the past time since the previous meeting, in addition, she reported what are her working plans for the present meeting. The main part of the meeting was dedicated to team work of planning the next lesson, and in the last part of the meeting, again, each participant or team reported to the group what did they achieve in this meeting. The teachers' duties for credit were to attend all the meetings, to enact three lessons with technology and to submit a written report of a given pattern regarding each lesson.

Having delineated both the theoretical frameworks and the PD being investigated, we may now translate our research goal to a research question as follows: *Can we point at changes in the stories teachers tell relating to their professional identity in the PD? If we do report changes, what can we infer from them regarding the teachers' professional identity?*

In order to answer these questions, the following data was collected: The instructor of the PD (the first author of this paper) took field notes documenting both the planning and the execution of each PD meeting and the aftermath reflection after each meeting. All the meetings were audio-recorded. All the written products of the participating teachers were collected including lesson plans, reflection reports and computer files. Two of the teachers' lessons were video-taped, and four of the participating teachers were interviewed few weeks after the PD ended by an external interviewer that was not familiar with the details of the PD.

The data analysis is composed of two phases: The first phase was to use the instructor's field-notes and the participating-teachers' written reports in order to form for each of them two web-diagrams according to the revised three components based on the UNESCO ICT Competency Framework for Teachers as shown in Table 1. The two diagrams summarize the initial and final levels of each developmental category. When the level could not be determined clearly, a middle point was selected. For lack of space the three categories were marked as "ICT", "Conduct" and "Pedagogy" as it is ordered in Table 1. Table 2 presents the initial and final web-diagram of four of the seven teachers that participated in the PD.

	Yosef	Simon	Dina	Sara
Initial				

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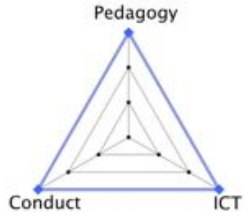
<td>final</td>
<td>
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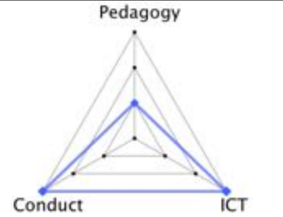
<td>
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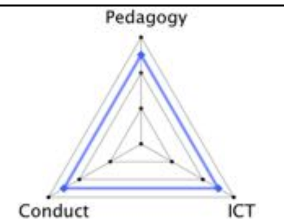
<td>
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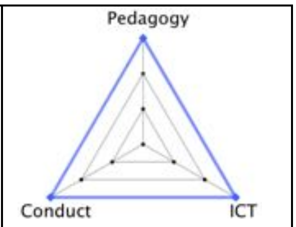
<td>
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Table 2: teachers' initial and final profiles

The second phase of data analysis focused on one of the seven teachers that according to the profiles depicted above, demonstrated a salient change. We decided that Yosef was the most promising candidate for this purpose, since he was the only teacher that moved two levels up in one category. For a deeper analysis of Yosef's discourse, we thoroughly reviewed all the audio recording that documented Yosef's participation in the discourse of the PD meetings, all Yosef's written products and his interview. The findings, reported ahead, are the evidence we selected in order to support our story of the change we detected in Yosef's professional identity.

FINDINGS

This chapter presents Yosef's story, as we see it, throughout the PD. Yosef is a novice mathematics teacher. He became a teacher after working in the high-tech industry for many years. His students are low-achieving students in both middle and high-school grades. In this section we unfold Yosef's professional identity as it appears to us, supported by evidence from the beginning, middle and the end of the PD.

Beginning

In the beginning of the PD Yosef had several opportunities to share his point of view regarding his practice of teaching with the assembly. We hereby analyze two of the episodes, we believe demonstrate in the best manner the stories that Yosef told during the beginning of the PD and that portray parts of his professional identity at that time. The following is taken from Yosef's oral report to the assembly in the second meeting:

> "I'm teaching the so-called non-calculus this year, "three units" (the lowest level). When I teach arithmetic sequences $a_{n+1} = a_n + d$, the students, many of them, have a problem of comprehension, what is this n? they have no idea. The fact that I'm saying, (they have) a problem, that does not mean I'm not explaining ... fine, even after two lessons of thorough explanation to small groups of students, there is still a problem of comprehension."

In the above quoted episode Yosef is describing the problematic situation of his students. Not only do they have difficulties understanding mathematics, this difficulty is persistent even when Yosef is explaining the difficult mathematics with deep intention. Let us emphasize, that it was Yosef's choice to share this experience with his peers and instructor at the PD. Therefore, we believe this story is a significant and representative story from the collection of stories that form Yosef's professional identity. In addition, we suggest that the way Yosef portrays his students, as totally incompetent, reflects implicitly his sense of impotence as their teacher.

Yosef's first written report presents a different angle of what we believe is the same phenomena. In the first lesson of integrating technology, Yosef decided to let his students experience an inquiry-based activity, in which they were instructed to infer the characteristics of each parameter in the linear-function f(x)=ax+b. Yosef reported that his students failed to accomplish the task, and in the reflection part of his report, he wrote: "It is a critical problem regarding the aptness of inquiry-based activities to this specific population". Yosef chose to refer to his students as "population", and their inability to perform as expected, is due to the qualities of the "population". We interpret Yosef's choice of vocabulary as signifying the irrelevance of this "population" being his students and the irrelevance of any possible action he may take as their teacher, as if it was a predestination, and there is nothing he can do to better their learning. He continues his reported reflection: "The attitude in this setting (inquiry-based task) is totally different and it requires higher level competencies. The question is whether this is to the benefit of the students, or not". Yosef presents the question whether inquiry-based activities are appropriate for low-achieving students, as a theoretical issue that may have an absolute answer regardless of Yosef's actions. We may infer that although he is their teacher, he does not feel that he has, or may ever have, a positive impact on his students' learning.

Although, this partial depiction of Yosef's professional identity presents him as a teacher who does not think that he has any true responsibility for his students' learning nor is he able to change their poor qualities as learners, we must state that he is the one that chose to join the PD and in addition, he chose to let his students experience an inquiry-based activity. We believe that those choices are a sign that Yosef was willing to try and be guided into a new route that will enable him to tell new stories that may re-define his professional identity.

Middle

In the second technology enactment Yosef planned three consecutive lessons. In the first two, he decided to use technology from the teacher's position in order to demonstrate graphic representations of parabolas and let his students participate by referring to this demonstration. At the end of the second lesson, Yosef showed a "face" constructed from parabolas and instructed the students to reconstruct the "face" at home using their cell-phones. According to the report, the third lesson was disappointing, since the attendance was low and most of the students failed to complete their homework. Nevertheless, Yosef reported this whole experience as more successful than the previous one. He said that now he understands that inquiry-based approach that lets the students read written instructions and figure out independently mathematical ideas as they use the computers by themselves, was too demanding for his students. This reflection is presenting a new and different story that replaces the one that was told in the beginning of the PD. Now, it is not the students that are incompetent for learning while participating in an inquiry-based activity, it is the activity which is unsuitable for the students. This seemingly minor change can open the door for bigger changes to come, because it shows signs to Yosef's shifting the responsibility for the favourable outcome of the lessons from the students to himself. Looking deeply into Yosef's second written report, we found some utterances that support this claim. For example: When he describes one of the moves he conducted during the first of the three lessons he said: "Maybe here, I made a slight mistake". On the one hand, we may say that this utterance can be interpreted as a critical view of Yosef towards himself. But, when we broaden our perspective, a story that may make sense of it, is that Yosef sees himself as someone that can make mistakes, therefore he can also better his performance and do good. Again, the impotency that characterized Yosef's view of himself in the beginning, is replaced, after his second report, by a more optimistic view, as Yosef sees himself as a significant figure to his students' learning.

Towards the end of the PD

Yosef decided to enact the third technology-implemented lesson in a class of low-achieving 7th graders. The subject was addition and subtraction of negative-numbers. In his oral report to the assembly, Yosef explained his choice to integrate an interactive game played using smart-phones (kahoot): "I needed it, in order to break the…course of the lesson… so it won't be too long. Letting them know that we are going to play kahoot at the end, it gives them some kind of motivation to work". The game is used here to reach a pedagogical necessity of keeping the students motivated and positively active for the whole lesson. Looking at all the data, this is the first time, Yosef refers to the integration of technology as addressing a pedagogical necessity. This may further support the story Yosef started telling previously: The students have difficulties and it is the teacher who helps them to overcome those difficulties. As before Yosef is critical regarding his decisions. In the reflection part of the third written report he says:

> "Although I tried giving very simple exercises in the kahoot, such that are repeating the principles in their simplest form, even good students, relative to this group, made mistakes derived from their attempts to give quick answers. It seems that the attempt to give quick answers, made them overlook the procedure".

After identifying the problem, Yosef suggest a solution: "Questions about the procedure should be added to the kahoot", and he elaborates several possible questions. In general, it seems that Yosef is looking both on the advantages and disadvantages of his pedagogical choices, and he is willing to learn from his experience and suggest solutions to the new problem that arise. Another sign to the change in Yosef's stories is his referring to some of his students as "good students". Therefore, we may claim that the same students, that in the beginning had persistent difficulties, are now described as good students that may make mistakes.

A new dimension that was not a part of Yosef's directly told stories, appeared towards the end of the PD as he started taking an active and meaningful part in the teachers' discourse in the PD. His oral reports evoke the other teaches to ask questions and to suggest ideas to answer Yosef's queries. In the last PD meeting Yosef presented a new self-made Geogebra applet that visually defines the trigonometric relations in a right triangle. His applet and presentation were followed by positive and encouraging responses from his fellow teachers. We believe that although it is only implicitly implied, we may add to the stories that constitute Yosef's professional-identity a story that describes him as a teacher that contributes to his community of peers and that community contributes to his practice.

DISCUSSION

The findings depicted above show that participation in this specific PD for the integration of technology may result in significant changes in the stories that constitute one teacher's professional identity. Interpreting Yosef's stories, we may claim that his professional identity took a meaningful turn from an impotent teacher that has no impact on his students' learning, to a teacher that is sensitive to his students' needs and is able to attend those needs with pedagogical and technological means.

Looking back at the PD, we cannot say that specific pedagogical or technological tools were taught, nevertheless, we showed that the resources of time and support, that let the participants work and reflect in a supportive community, may yield a significant change in a teacher's professional identity.

After looking at the changes that can evolve in a teacher's professional identity it seems inevitable to ask, how can we explain that while participating in the same PD, Yosef underwent such impressive progress but looking at Simon (see Table 2) we detected no change at all. We believe that this query can be answered if we further dwell on the stories that form all the teachers' *designated identity*. As stated in the theoretical introduction, learning may be looked at, as narrowing the gap between the actual and designated identities. If there is no such gap, we would expect no learning. Therefore, explaining the differences between the performance of different teachers will entail looking at their designated identities, and this is beyond the scope of this paper.

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Design principles supported by the collaborative design of mathematical digital resources within a CoI

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In this contribution we report on the collaborative design of mathematical digital resources within a community of interest in a specific socio-technical environment established in the framework of the MC Squared European project. We bring to the fore that the design of these resources requires technological, pedagogical and mathematical knowledge. Through the analysis of the process of design of one such resource, we show how the professional knowledge, the teachers' practice and the socio-technical environment have oriented the design choices.

Keywords: Collaborative design, digital resources, professional knowledge, elementary algebra; creative mathematical thinking.

INTRODCTION

Collaborative design reported in this paper took place within the MC Squared (MC2) project (http://www.mc2-project.eu/), in a socio-technical environment, called C-book technology, enabling to design digital resources for mathematics teaching and learning, called c-books (c for creative). The c-books were produced within four communities of interest (CoI) (Fischer, 2001), French, Greek, Spanish and British, in order to develop creative mathematical thinking (CMT) in their users. Through the analysis of one particular c-book, designed within the French CoI, we attempt to enlighten the role of the designers' professional knowledge, of practices of the mathematical teachers involved, and of the digital environment in the resource design. We start by presenting our theoretical and methodological framework. Then we continue by the presentation and analysis of a case study constituted of one c-book, before concluding.

THEORETICAL FRAMEWORK

Communities of practice and communities of interest

A *community of practice* (CoP) (Wenger, 1998) gathers people who share in its core a common identity, domain of knowledge and practice, whereas *communities of interest* (CoIs) "*bring together stakeholders from different CoPs to solve a particular (design) problem of common concern*" (Fischer, 2001).

According to Fischer, the diversity of members of a CoI makes the latter having a greater creative potential comparing to a CoP. Thus, in the MC2 project, CoIs were constituted for designing c-books; they gathered people from different professional worlds, such as researchers, teachers, software designers, artists, etc., with different activity systems (Engeström, 1987), different identities, knowledge and practice, to solve a (design) problem.

Activity theory

Collaborative design is a human activity. The activity theory (Engeström, 1987) is a lens for describing and understanding the context of this design.

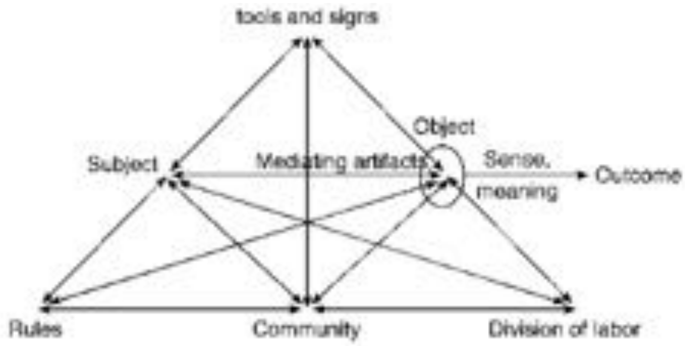

Figure 1. The structure of an activity system (Engeström, 1987, p. 78)

Engeström's model of activity structure (Fig. 1) allows us to bring to the fore the context of the collaborative design through the diverse entities that constitute the activity: subject, object, community, rules, division of labor, and tools and signs considered as artefacts.

Nevertheless, the collaborative design is studied from the perspective of the designers' knowledge and practice to highlight their impact on the design choices. We therefore chose the TPACK model in order to infer teachers' professional knowledge informing the c-book design.

The TPACK model

The TPACK (Technological Pedagogical And Content Knowledge) model (Mishra and Koehler, 2006) allows to model teachers' knowledge and to identify more precisely their knowledge of technology (TK), pedagogy (PK), content (CK), mathematics in our case, and the intersections of these three areas. One of the interests of this model is the special focus on knowledge related to technology for teaching.

Creative mathematical thinking (CMT)

Let us recall that the MC2 project aimed at the development of a technology enabling to design resources with a potential to foster creative mathematical thinking. The CMT was defined within the project as the ability to generate ideas in the process of problem solving displaying *fluency* (ability to generate many ideas), *flexibility* (ability to generate different categories of ideas), *originality* (ability to generate new and unique ideas that others are not likely to generate), and *elaboration* (ability to redefine a problem to create others by changing one or more aspects). *Social* and *affective* aspects were deemed as important factors stimulating CMT.

Our research question is: To what extent professional knowledge of the designers (referring to TPACK), their practice, as well as the socio-technological environment, have oriented the choice of activities, tasks, feedbacks and artefacts embedded in the c-book they designed?

CONTEXT AND METHODOLOGY

Our methodology relies on a case study, i.e., a collaborative design of a particular c-book that we analyze through the lens of our theoretical framework to answer the above mentioned research question. We have chosen to study the "Elementary algebra" c-book. The c-book had been designed in two phases. First, a CoI produced a first version of the c-book within the MC2 project, which was then re-designed by the associated *DEA* CoP comprising three secondary school mathematics teachers and one teacher educator with the aim of using the c-book with their students.

Context of the design within the CoI

Referring to the activity system, the design of the "Elementary algebra" c-book took place within the French CoI (*community*), which was composed of thirteen members with varied professional background, including researchers, mathematics teachers, teacher educators, and educational technology developers, and with a convergent interest in mathematics or mathematics education. The members share a socio-constructivist vision of mathematics learning rooted in the French didactic tradition of teaching and learning mathematics (Artigue, 2016). The *subject* of the c-book design activity was a group of six CoI members, closely related to mathematics education, engaged in the c-book design (called a sub-CoI, Fig. 2).

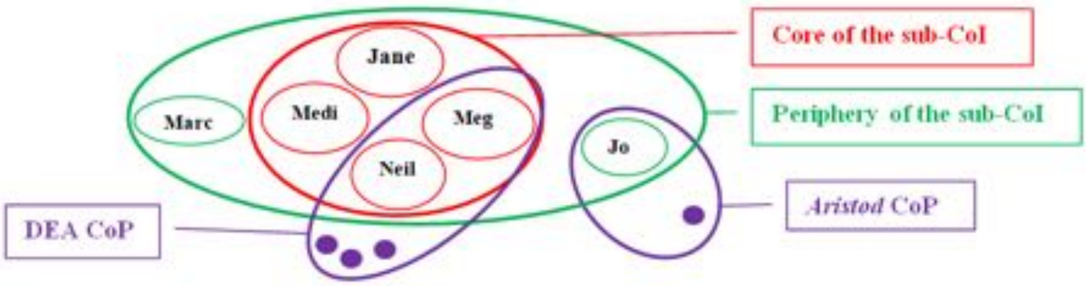

Figure 2. Sub-CoI engaged in the design of the "Elementary algebra" c-book

Four members were at the core of the sub-CoI with specific roles assigned from the beginning of the design: Meg, teacher educator, and Medi, post-doctorate, were the main designers, Jane, teacher educator and researcher, was moderator and Neil, teacher and PhD student, was secretary and observer. Two members were at the periphery of the sub-CoI, they intervened on request for a precise task (*division of labor)*: Jo, developer of educational technology, was asked to collaborate for technical support and Marc, teacher educator and researcher, intervened to design specific widgets with GeoGebra. The two associated CoPs were *Aristod* whose members, including Jo, develop educational software and the *Dynamic elementary algebra* (*DEA*) group whose members reflect on and develop resources for teaching elementary algebra.

One *rule* that the sub-CoI adopted for the design of this c-book was to use at least two widgets developed by the *Aristod* CoP. Indeed, the design of the c-book was an opportunity to reflect on the use of these widgets in mathematics classes. The *artefacts*, namely the C-book technology and widgets developed by Aristod, constituted the socio-technical environment in development along the project. The c-book designed to enhance the CMT potential of the students and develop early algebraic thinking was the *object* of the activity.

Methodology

The data were collected mostly via CoICode[1], a communication tool dedicated to the design of c-books with one workspace per c-book in the form of a threaded forum and used by the CoI members. Other data come from minutes of manifold meetings (CoI, CoP, etc.), emails and versions of the designed c-book. We have analysed the structure, some of the activities, tasks, feedback and artefacts embedded in the final version of the c-book designed by the sub-CoI and then seek for the major modifications made by the mathematics teachers of the *DEA* CoP before and after the experiment in the classroom with seven and eight grades' students. Correlation of the data allowed us to infer professional knowledge and practice that have oriented the designers' choices.

ANALYSIS OF SOME DESIGN CHOICES

Structure of the c-book

The c-book content is organized around three activities: "Pattern and generalisation", "Calculation programs", and "Equations". The purpose of the activity "Pattern and generalisation" is to find how to determine the number of cubes composing any figure of the sequence. The goals of the activity on calculation programs are to enable students to work on procedural and structural aspects of algebraic expressions and to feel the limits of calculations in order to show the need of algebra for proving (e.g., proving that the program A: $5(x + 3) – x – 10$ and the program B: $4x + 5$ are equivalent). Finally, the aim of the activity on equations is to allow students to reflect on the transformation rules used for solving an equation with one unknown using a digital artefact, a balance.

From the collected data we note that the choice of starting the c-book with an activity on patterns and generalisation was suggested by Meg who considers such an activity as a first step in the development of algebraic thinking for Grade 5 and 6 students and the calculation programs as a second step to enable the students to acknowledge the power of algebra as a tool for proving (PCK). The designers think that the balance widget embedded in the environment can help some students (Grade 6, 7 and 8) to better understand the transformation rules of equations (TPACK). In addition, the order of these three activities is related to the practice of the mathematics teachers of the *DEA* CoP.

Tension between constructivism and behaviourism

"Pattern and generalisation" activity starts, page 1, by inviting students to determine the number of cubes for any figure in the pattern, the first terms of the pattern given

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[1] CoICode is a communication environment; part of the C-book technology, offering a workspace within which members of a CoI engaged in a c-book design can communicate.

(Fig. 3) and write their answer in a box. The question is open and lets students free to take initiatives (constructivist approach).

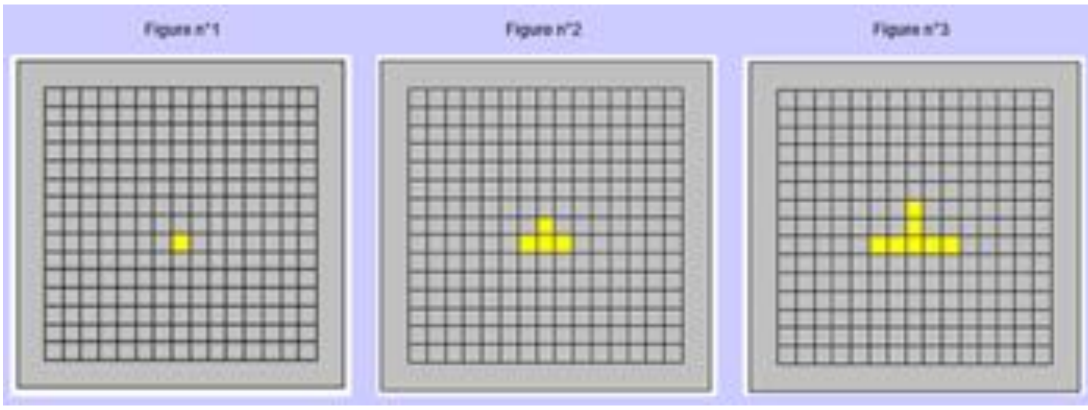

Figure 3. The first three figures of a sequence (adapted from Mason, 1996, p. 84)

However, the environment does not provide feedback. Hence, on page 2, the designers invite the students to determine how many cubes are needed for 4th, 5th and 10th figures with appropriate feedback.

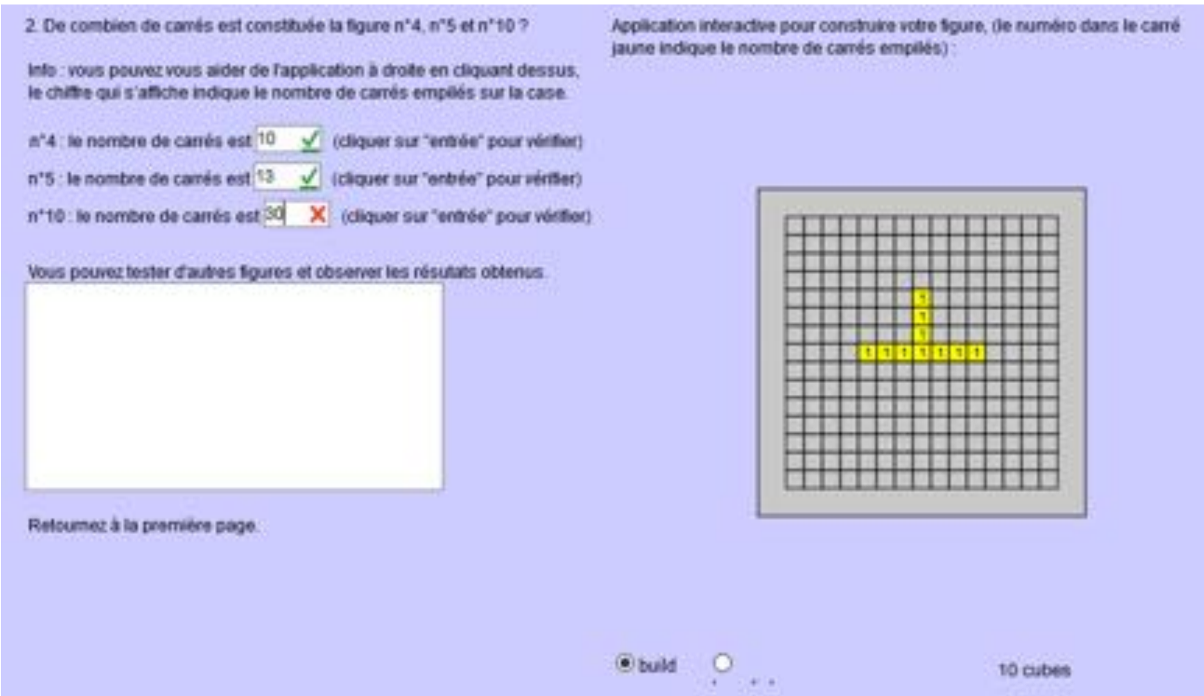

Figure 4. Screenshot of "Pattern and generalisation", page 2

They also provide Building blocks widget, which can enable students to represent all the figures of the sequence they want, with the automatic display of the number of squares/cubes (Fig. 4), in order to help them generalising. The students are thus guided toward the solution and their freedom is limited (behaviourist approach).

We note that the open question, intended to propose a problem to solve to students (constructivist approach) is followed by closed questions that guide the students (behaviourist approach). This choice is due to the constraints of the environment that cannot provide appropriate feedback to an open question. Hence, despite the designers' constructivist culture, the limits of the technological environment yield a resource underpinned by a behaviourist approach.

The choice of artefacts and feedback

The analysis of the modifications of the c-book shows that with the time, the designers employed more widgets and began to diversify the type of feedback that is offered by the environment such as positive reinforcement, answers or advices (TK and TCK, Fig. 5).

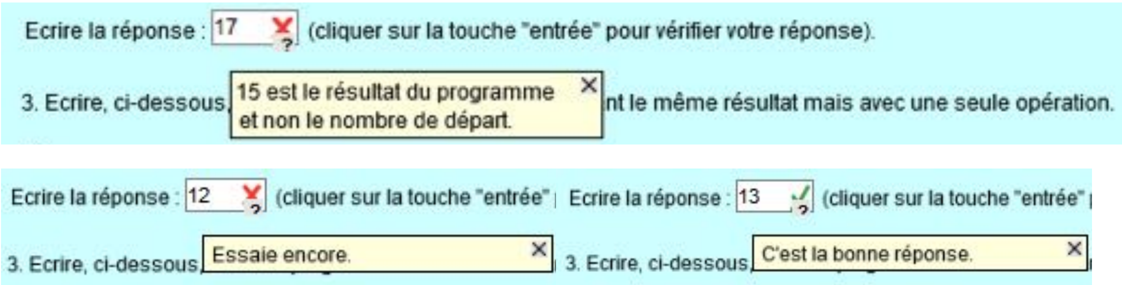

Figure 5. Examples of feedback depending of the answers

In the final version of the c-book, we can see that the designers have created new artefact using GeoGebra and integrated some widgets provided by the environment (e.g. Balance, Building Blocks, Algebra Trees). For example, a GeoGebra widget was designed by Marc to enable students to build the first terms of the pattern, which was then improved by Neil (after a time of appropriation, Fig. 6), or GeoGebra spreadsheets were embedded for the calculation programs in order to help students to find program formulae and then to generalise.

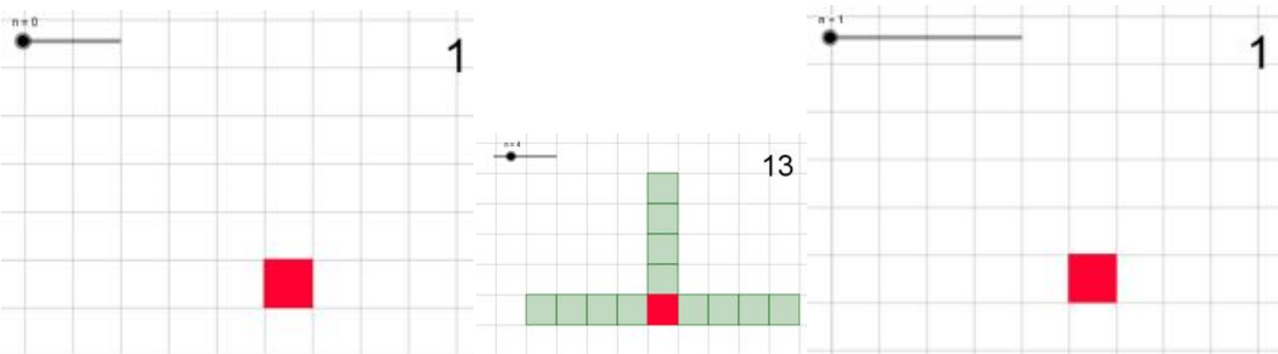

Figure 6. The widget designed by Marc (n = 0, 1 square and n = 4, 13 squares), then adapted by Neil (n = 1, 1 square); n the number of the figure

The choice of artefacts was based on the designers' TPACK. The teachers from the *DEA* CoP have chosen to add the Algebra Trees widget in some pages of the "calculation programs" unit, to enable students to work on the structural aspect of algebraic expressions. Such a progressive diversification of artefacts used in the c-book shows that the instrumental genesis (Rabardel, 2002) of the C-book environment by the designers needed time.

We also note that they have decided to add an introduction to help teachers to use the c-book, explaining the content of the digital book. Then, after the experiment with the students, the teachers added three pages in the introduction to accompany students' instrumental genesis related to three artefacts (epsilonwriter, spreadsheets, Algebra Trees) proposed in the c-book.

The development of CMT

Two widgets designed and developed by the *Aristod* CoP, namely epsilonchat, a chat tool, and TQuiz, a serious game tool, have been integrated into the c-book. The designers have also implemented some pictures. These choices show an attempt of the designers to improve the CMT potential of the c-book by enhancing social aspects via the epsilonchat widget as well as affective aspects through the integration of pictures or TQuiz widgets.

CONCLUSION

The analysis of the "Elementary algebra" c-book shows that the teachers' practice has influenced the order of the activities and their PCK and TPACK impacted the activities and artefacts present in the c-book. We showed a tension between the constructivist background of the designers and the limits of the C-book digital environment, which did not enable to build a rich (a-didactical) milieu but obliged the designers to guide students in their solving process through closed questions and feedback thus conferring behaviourist character to the c-book. We saw that the instrumental genesis related to the environment is necessary at three stages: first in the designers themselves in order to better understand the affordances of the C-book environment and take better profit from it in the design, then in the teachers using the c-book and finally in the students. We also brought to the fore that the will to develop the CMT in the users led the designers to embed a serious game and pictures to enhance affective aspects, and a chat tool to enable social interactions between the users.

ACKNOWLEDGEMENT

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How the word ‘mathematical’ influences students’ responses to explanation tasks in a dynamic mathematics software environment

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Task design is a central issue in mathematics education, not least in relation to digital technology. This paper reports how a small but significant change in wording affects students’ explanatory responses. The study is comparative and involves 229 10th grade students working on tasks designed for a dynamic mathematics software environment. The findings indicate that inclusion of the word ‘mathematical’ prompted students to use algebraic symbols and algebraic arguments, to a higher degree.

Keywords: task design, dynamic mathematics software, explanation task.

INTRODUCTION

Task design within mathematics education has been an important issue for decades, and the increased availability of different kinds of technology in mathematics classrooms has made this issue even more important. Designing tasks that utilize the affordances provided by digital technologies is recognized in the literature as being a complex and subtle process (Joubert, 2017).

In relation to Dynamic Mathematics Software (DMS), the literature emphasizes the possibility of visualizing and linking various representations of mathematical objects, particularly in relation to functions and graphs (Hegedus et al., 2017). With DMS it is possible to make direct manipulation of dynamically linked representations of functions, e.g. algebraic and graphic representations (Drijvers, 2003). However, there is an identified risk that students, while working on a task in a DMS environment, will only relate to the empirical/visual objects obtained on the screen without reflecting on the mathematics involved (Drijvers, 2003; Joubert, 2017). Hence, one important issue to consider in the design of tasks for DMS environments is how to formulate tasks that encourage students to move from the empirical/visual to the mathematical/systematic field (Joubert, 2017).

Several studies, focusing on task design in DMS environments, emphasize the importance of asking students to explain their empirical findings (Leung, 2011). According to Leung, “A meaningful mathematical explorative task should be one that involves conjecturing and explanation activities.” (p. 328). The issue of student explanations in the teaching and learning of mathematics has been a focus of research literature for several decades (e.g. Dreyfus, 1999; Silver, 1994). Silver (1994) acknowledge the challenge for students to provide explanations in writing, and suggests “…the need for explanations, especially written explanations, to become a more prevalent feature of school mathematics instruction.” (p. 315).

Moreover, Sierpinska (2004) discusses the importance of ‘task problematization’, and pinpoints that small differences in the formulation of tasks might have a significant impact on students’ responses. In line with this, we found, in a previous study, that the wording is crucial in the formulation of questions where students are asked for explanations (Brunström & Fahlgren, 2015). Particularly, we found that students’ responses tend to be superficial and more descriptive than explanatory. The result from the study prompted us to further investigate how small but potentially significant changes in wording might influence students’ explanatory responses in a DMS environment.

The change in task wording investigated in the study reported in this paper involves moving from asking students simply for an explanation to asking them for a ‘mathematical’ explanation. The explanation tasks in question are embedded in a task sequence developed for a DMS environment with the aim of developing students’ awareness of some of the connections between the standard form of quadratic function $f(x) = ax^2 + bx + c$ and the corresponding graphical representation and quadratic equation. The research question will be presented in detail later.

THE PRACTICE OF EXPLANATION

In (digitalized) mathematics classrooms, technology provides feedback in response to students’ action with the environment (Joubert, 2017). Using the terminology introduced by Noss and Hoyles, Joubert argues that for students to move from the ‘pragmatic/empirical’ to the ‘mathematical/systematic’ field, the students must go beyond just reporting what they have seen (Joubert, 2017). The literature suggests asking students for explanations as a way to encourage students’ movement between these fields (e.g. Dreyfus, 1999).

However, the literature recognizes the challenge for students to provide mathematical explanations. According to Dreyfus (1999), students have rarely learned what counts as a satisfactory explanation. Moreover, Levenson (2013) argues that the properties of a task as well as the mathematical concepts under consideration affect the features of the explanations used. In a study investigating students’ conceptions of the qualities of mathematical explanations, Healy and Hoyles (2000) found that many students preferred explanations described in everyday narratives. For these students “empirical data convince whereas words and pictures, but not algebra, explain.” (p. 415). However, the study showed that although students predominantly used narrative explanations they were aware of their limitations, and they thought that to receive a good mark, explanations should include algebraic arguments (Healy & Hoyles, 2000).

Researchers refer to ‘expository writing’ as “writing which is intended to describe and explain mathematical ideas” (Shield & Galbraith, 1998, p. 29). The main idea is that by communicating their mathematical thinking in writing, students improve their mathematical understanding (Santos & Semana, 2015; Shield & Galbraith, 1998). In the study reported by Santos and Semana (2015), students used several types of representations in their written expositions, e.g. verbal language, iconic representations, numerical and/or algebraic symbols.

These ideas lead us to formulate more precise research questions: What impact, if any, has inclusion of the word 'mathematical' on students' responses when asked to explain observations made in a DMS environment on: (a) the forms of representation employed, and (b) the characteristics of explanations used?

METHOD

Research Setting

In total, 229 tenth grade students at a secondary school in Sweden participated in the study, conducted during a year-long school development project with the overarching aim to develop and test sequences of tasks designed for a DMS environment. The aim of the particular task sequence was to introduce graphical representations of quadratic functions written in the standard form $f(x) = ax^2 + bx + c$, and the corresponding quadratic equation. The students had previously worked with linear functions, linear equations and with solving quadratic equations algebraically. In Sweden, the predominant method for solving a quadratic equation of the form $ax^2 + bx + c = 0$, is first to reduce it to the equivalent quadratic equation $x^2 + px + q = 0$, and then to apply the so called *pq*-formula.

Material

In total, the task sequence includes three embedded explanation tasks formulated in the following two versions: "Explain why…" and "Give a mathematical explanation why…". Besides the word 'mathematical', these phrases also differ grammatically (i.e. 'explain' vs. 'explanation'). However, we suggest that this has an insignificant impact on student responses. The two versions are labeled U and M for Unspecified and Mathematical explanation respectively. Due to limitation of space, this paper only reports on two of the tasks; Task 1c and Task 3c (see Figure 1).

Task 1

(a) Investigate, by dragging the slider *c*, in what way the value of *c* alters the graph. Describe in your own words.

(b) The value of the constant *c* can be found in the coordinate system. How?

(c) Explain why/Give a mathematical explanation why the value of *c* can be found in this way.

Task 3

(a) Solve the quadratic equation $x^2 - 4x + 3 = 0$ algebraically (using pen and paper).

(b) Set the sliders so that the graph of the function $f(x) = x^2 - 4x + 3$ is shown. The solutions to the corresponding quadratic equation, $x^2 - 4x + 3 = 0$, can be found in the coordinate system. How?

(c) Explain why/Give a mathematical explanation why the solutions to the equation can be found in this way.

Figure 1: Two tasks including a request for explanation (subtask c)

Data Collection

The empirical data for each task consist of the written responses from students. Only the teachers were told that there were two versions of the task sequence. Each student received one or other version, distributed at random in each class. However, not all students provided answers to all of the tasks. The number of student responses on Task 1c are 109 (version U) and 100 (version M). The corresponding numbers for Task 3c are 102 (version U) and 99 (version M).

Data Analysis

The analysis process was conducted in several phases. Initially, only Task 1c was analysed, which resulted in preliminary results that were presented as a poster at the 13th International Conference on Technology in Mathematics Teaching (Fahlgren & Brunström, 2017). This analysis provided insights into what kind of results we could get from the empirical material, and thus, how to continue the analysis process.

The initial analysis was followed by a more structured content analysis. Student responses were inspected and compared to identify a basic set of elements of explanation which could be used to summarise the content of any response. This made it possible to create a manual used to code all responses in terms of the presence or absence of each of the explanation elements and representation types.

The Coding Manual

While the categories of representation type are the same for all tasks, based on predefined general patterns of use of verbal and algebraic representation (Santos & Semana, 2015), most of the categories of explanation elements are task specific (as suggested in Levenson, 2013). The later ones were developed inductively through analysis of the substantive content of students' responses to the specific tasks. To illustrate and clarify the categorization, exemplars of student responses in each category, including representation type, are presented for one of the tasks (Task 3c).

Representation type

In this study, the types of representation were divided into four categories. Students' responses were classified as "Verbal only" (V) even if they included single letter coefficients or variables. In student responses classified as "Verbal with elements of Algebraic symbols" (VeA) formulas or other algebraic symbols are just included without being evaluated or manipulated in some way. Hence, the categories "Algebraic symbols only" (A) and "Verbal and Algebraic symbols" (VA) are the only categories where students really use algebraic symbols (even if not always in an appropriate way). No student responses included a graph, although some made reference to graphs (See elements B and H in Table 1 below).

Explanation elements

Due to limitation of space, we choose one task (Task 3c) to present as exemplary of the analysis process in terms of identified elements of explanation (see Table 1) making up a particular response.

Code	Explanation element
A	Express that '$y = 0$' (where the solutions occur)
B	Relates '$y = 0$' to intersection with the x-axis

C	Express that $f(x) = 0$ corresponds to $y = 0$
D	Referring to two solutions/values of x
F	Referring to the *pq*-formula
G	Verifying the solution, e.g. inserting the values 1 and 3 in the equation
H	Referring to the DMS feedback

Table 1: The categories of explanation element in Task 3c

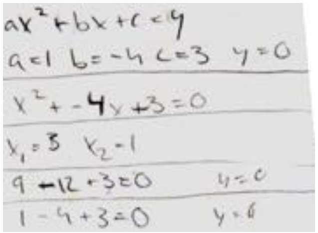
Below are exemplars of six student responses and the suggested categorization for each of them is shown in Table 2:

S1: "Because that is where y is 0"

S2: "$f(x) = x^2 – 4x + 3 = 0$, hence $y = 0$, where x is the correct answer. That is, where the line intersects the x-axis (where $y = 0$)"

S3: "I use the *pq*-formula. That is the easiest way."

S4:



S5: "We inserted the formula $x^2 – 4x + 3$, and received the solution"

S6: "Because we have got 2 x values."

Student response	Elements of explanation							Representation type			
	A	B	C	D	F	G	H	V	VeA	VA	A
S1	1							1			
S2	1	1	1							1	
S3					1			1			
S4	1					1					1
S5							1		1		
S6				1				1			

Table 2: The categorisation of student responses S1 to S6 on Task 3c

RESULTS AND ANALYSIS

This section provides the results from the comparison between two groups of students; Group U and Group M, the ones answering Versions U and M respectively. First, the results concerning representation type for the two tasks are introduced. Then, the results related to explanation elements are presented for each task separately.

Representation type

The results indicate that the task formulation including "mathematical" (i.e. Version M) prompts more students to use algebraic symbols in their explanations (see Table 3).

In particular, merging the two categories where students really use algebraic symbols (A and VA) the tendency becomes clear for both tasks, with all differences statistically significant (Task 1c: $p<0.001$ and Task 3c: $p<0.01$).

Task	Version	A	VA	A or VA
Task 1c	U	0.0%	6.4%	6,4%
	M	10.0%	20.0%	30.0%
Task 3c	U	1.0%	7.8%	8.8%
	M	14.1%	8.1%	22.2%

Table 3: The proportion of student responses using algebraic symbols (A), verbal and algebraic symbols (VA), and one or the other of these

Explanation elements

Task 1c

The results indicate some differences between the groups. Compared to Group U, Group M were, as an element of their explanation:

- less inclined to repeat their answer to the previous subtask (29.0% vs 45.9%)
- less inclined to refer to the feedback from the DMS environment (11.0% vs 26.6%)
- more inclined to use the fact that $x = 0$ when the graph intersects the y-axis (14.0% vs 6.4%)
- more inclined to use linear analogy (64.0% vs 44.0%)

Concerning the categories ‘Repeating the answers from the previous subtask’ and ‘Referring to the DMS feedback’, in several cases these elements of explanation were combined with other more relevant elements. Therefore, it is interesting to investigate the proportion of student responses using one or both of these explanation elements *only*. In this analysis, a significant difference ($p<0.001$) between the groups emerged (32.1% for group U vs 6.0% for group M).

The explanation element using the fact that $x = 0$ when the graph intersects the y-axis focuses on an algebraic expression, and in this aspect aligns with several other explanation elements; ‘c is the constant term’, ‘c is independent of x’, ‘c is independent of a and/or b’, and ‘solves for c’. A further analysis, looking at responses including one or several of these categories, revealed a significant difference ($p<0.001$) between the groups (50.0% in group M vs 28.4% in group U).

Task 3c

There are some differences between the groups in Task 3c. Compared to Group U, Group M were, as an element of explanation:

- more inclined to refer to the *pq*-formula (Category F; 19.2% vs 2.9%; $p<0.001$)
- more inclined to verify the solution by inserting the values 1 and 3 into the equation (Category G; 12.1% vs 2.0%; $p<0.01$)
- less inclined to refer to DMS feedback (Category H; 7.1% vs 11.8%)

The first two categories involve, although in different ways, references to the equation elaborated on in subtask a). That is, students using these explanation elements are referring back to the algebraic subtask a) rather than the graphical subtask b). Concerning the category 'Referring to the DMS feedback', the tendency is the same as in Task 1c. The difference between the groups in Task 3c appears somewhat clearer when looking at responses with this element *only* (10.8% in group U vs 5.1% in group M).

DISCUSSION

As stated in the Introduction, one overarching goal of letting students work in DMS environments is to encourage their movement from the empirical/visual to the mathematical/systematic field (Joubert, 2017). One way of doing this, is to prompt students to explain, in writing, what they notice when interacting with the technology (e.g. Leung, 2011). By comparing student responses from two versions of explanation tasks, this study sought to investigate whether a small but significant change in task wording influences students' explanatory responses in a DMS environment. The results show that there were significant differences between the groups, both in relation to type of representation and explanation elements used.

Students asked for a 'mathematical' explanation, used algebraic symbols and algebraic arguments to a higher degree. Moreover, they were more likely to use linear analogy, and hence to utilize their prior knowledge in mathematics. In contrast, the other group of students were more inclined to use the feedback from the DMS environment as an element of explanation and/or to repeat the answer to the previous subtask. In both these cases, student responses were descriptive based on reporting visual information rather than explanatory based on accounting for that information, which indicate that the students still are in the empirical/visual field.

Taken together, the findings in this study indicate that the word 'mathematical' signals a request for an algebraic explanation. Consequently, this small change in wording might enhance students' movement from the empirical/visual to the mathematical/systematic field.

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Robots as Mathematical Objects- and Actions-to-Think-With

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We investigated how robots might be used to design experiences that support the development of key mathematical proficiencies. Specifically, we sought to understand how the situated, movement-focused, and problem-driven spaces opened up by programming robots might enhance specific core mathematical competencies and how, in turn, those competencies might enable and amplify general mathematical understanding. To gain insight we video recorded children in Grades 4-6 as they engaged in programming robots. We selected three video clips that illustrate aspects of children using robots as objects and actions-to-think-with about the number line and a mutable grid. Our data illustrates how well-structured encounters with programming robots can support developing multiple understandings of the number.

Keywords: Elementary, programming robots, number line, grids, mathematical understanding

First diagnosed with a learning disability in mathematics in Kindergarten, Sara's first five years of mathematics instruction were focused almost entirely on basic numeration. Despite these efforts, when observed mid-year in Grade 4, Sara was still struggling with simple arithmetic. Although provided with continuous access to plastic counters, base-10 blocks, and number charts, we observed that she could not reliably respond to addition statements such as "9 + 2 = __."

Yet, a few days after that observation, when engaged in building and programming robots, Sara demonstrated herself fluidly capable of directing a device to trace out different polygons marked on the classroom floor – an achievement that required abilities to decompose shapes, interpret movement through number, format instructions in logical sequences, and systematically analyze relationships between expected and actual results. Most surprising to her teachers, her approximations and extrapolations while using robots were consistently precise, suggesting good number sense, strong estimation skills, and sound proportional reasoning.

Naturally, her teachers were taken aback at the disconnection between Sara's facility with mathematical competencies across the two settings, prompting them to wonder if and how her robotics-based competencies might support her mathematics learning. We wanted to better understand how robots might be used to design experiences that support the development of key mathematical proficiencies. Specifically, we investigated how engaging with programming robots might promote learners' facility with general mathematics by focusing on how this technology supports the development of two specific "objects" and two specific "actions." That is, we sought to understand how the situated, movement-focused, and problem-driven spaces opened up by programming robots might enhance specific core mathematical competencies and how, in turn, those competencies might enable and amplify general mathematical understanding and achievement. Regarding the two "objects" that serve as the foci of this work, we studied: (1) number line, engaged as a device to develop and connect diverse interpretations of "number"; (2) mutable grids, utilized to parse space, pinpoint locations, interpret motion, structure movement, etc. As for the two actions, we examined: (1) modulating between decomposing and recomposing (of procedures, operations, concepts, etc.); (2) isolating and systematically varying of elements of situations that can be mathematically modelled.

We might redescribe the above objects as *mathematical constructs* and the actions as *mathematical strategies*. Each has been selected for its ubiquity in school mathematics and its "natural" presence in most introductory coding setting. Our thesis is that well-structured coding tasks, within which learners engage with concepts pragmatically rather than didactically, can powerfully support mathematics learning if properly bridged. Thus our novel approach is to target four distinct instances of mathematizing that underlie and give rise to discrete mathematics topics (e.g., number, geometry, algebra, measurement, etc.), rather than approaching school mathematics as a compilation of such topics. We next situate the proposed research historically, conceptually, and methodologically.

SOME HISTORICAL CONTEXT: AN EVOLUTION OF "MATHEMATICAL UNDERSTANDING"

Since the inception of modern schooling, elementary-level mathematics has focused on developing learners' understandings of *number* and *shape.* However, while those two emphases have remained steady over centuries, perspectives on what it means to *understand* them have evolved greatly.

For example, understanding number and shape was originally framed in terms of mastery of facts and skills, consistent with a newly industrialized society's need for efficiency and accuracy in the workplace. This procedural attitude towards understanding was especially evident in the sorts of examples and illustrations that were used to support learner sense-making. Where and when conceptual meaning was explored – through, for example, images, applications, or instantiations – they tended to be paper based, teacher led, and otherwise structured for a passive learner.

Such structures were heavily criticized by educational reformers in the late 20th century, oriented by a distinction between "instrumental" and "relational" understandings (Skemp, 1976). Reformers argued that an instrumental or procedural focus might support automaticity, but it limited capacities for solving problems, linking concepts, and extending ideas. They thus advocated for an elaborated definition of understanding, one that was also attentive to the relational and conceptual – that is, to connections across concepts, big ideas, and so on. The call was soon translated into such classroom emphases as "active learning," "problem posing/solving," and "manipulatives" (see, e.g., NCTM, 1980).

Use of manipulatives – that is, of artefacts and object-based tasks that are designed to channel learners' attentions to key properties of mathematical concepts – proved particularly popular, especially in the elementary school classroom. An array of manipulatives soon appeared, ranging from items that could be used across many topics (e.g., interlocking cubes for counting, measurement, and making shapes) to tools that were designed to support specific concepts (e.g., base-10 blocks for place value and arithmetic operations). Now as an integral part of school culture, they tend to be seen and used as concrete instantiations of concepts, evidenced by Wikipedia's "Manipulative (mathematics education)" entry:

> Mathematical manipulatives are frequently used in the first step of teaching mathematical concepts, that of concrete representation. The second and third steps are representational and abstract, respectively. (emphasis added; accessed 2017 November 10).

While popular, this tendency to regard *manipulatives as concrete representations* of concepts was demonstrated as problematical two decades ago. Uttal, Scudder and Deloache (1997) reported on a series of experiments that suggested the use of manipulatives may set up a dual-representation system, one system being the manipulatives and the other being the symbols or operations intended to be represented. Students frequently became confident working within the manipulative system, but did not see the connection to the symbolic system. Furthermore, learners who were able to translate between systems did more and harder work than those who operated purely within the symbol system.

These critiques helped to highlight how the popular assumption that manipulatives are concrete representations of concepts had eclipsed (and continues to overshadow) an earlier rational for incorporating artefacts into mathematics learning. As articulated by Piaget (1954), and since elaborated by many mathematics education researchers with interests in the bodily basis of understanding (see de Freitas & Sinclair, 2014), the main reason for using manipulatives is not to concretize a concept nor to excavate the ideas built into objects, but *to move*. That is, one's senses of shape, quantity, proportion, and so on have more to do with *structured acts of moving* than with *acts of moving structures.* From this perspective, the main purpose of a manipulative is not to (re)present mathematical concepts, but to mould the learner's motions, in the process occasioning opportunities for learners to expand and interweave their repertoires of mathematically relevant structures. Departing even further from the sensibilities presented in the Wikipedia entry, cited above, the realization that bodily experience is a basis of mathematical meaning does not entail rigid instructional sequences (e.g., from concrete to pictorial to abstract representations). Rather, and as demonstrated by Sara's coding, well-structured situations that invite and compel learners to link and fuse diverse action-, image-, and symbol-based encounters may contribute to profound competence.

Against this backdrop, devices such as programmable robots could constitute a new sort of manipulative tool. As Sara's actions illustrate, they can be used to support activities that compel learners to integrate and extend diverse representations. We also see in them possibilities for extending learners' senses of their own physical beings by displacing experience in both space and time. Instances of such displacements include controlling objects at a distance, receiving immediate feedback on one's decisions, and repeating and varying actions with precision.

THEORETICAL FRAMING

We structure our thinking around Papert's (1980) notion of "*objects-to-think-with*," a term he coined to describe seminal models that learners draw upon when encountering new situations, particularly when engaged in activities with heavy reasoning demands such as coding. Papert's personal *objects-to-think-with* were gears. He played with them as a child and they later provided images and dispositions that enabled him to see mathematical situations in specific structured ways, including conceptions of ratios and proportions. Unlike a tool with a defined purpose (e.g., a compass that is used to draw circles), an *object-to-think-with* is a device that can be applied across many situations and application as it channels attention, invites comparison, and enables analysis. Essentially, *objects-to-think-with* are both spatial and dynamic as they emerge from human actions (on screens, robots, or with manipulatives). Coupled to the idea of *objects-to-think-with*, we propose the notion of *actions-to-think-with*. We envision these as strategies that are useful across applications and topics – in effect, means to engage with situations when suitable *actions-to-think-with* are either unavailable or insufficient to the task at hand.

Our first *object-to-think-with* is the **number line** – which, as cognitive scientists (e.g., Lakoff & Núñez, 2000) and mathematicians (e.g., Mazur, 2003) alike have argued, is integral to virtually all arithmetic and algebraic concepts studied at middle and secondary school levels. It is also a construct that can be used to link multiple interpretations of number, the four most fundamental of which Lakoff and Núñez identified as *counts* (addressing "How many?"), as *sizes* (addressing "How big?" or "How much?"), as *lengths* (addressing "How far?"), and as *locations* (addressing "Where?"). The contrast between Sara's struggles in math class and her competence with robots illustrates the importance of the principles at work here. In math class, number was strictly a *count* of discrete objects, and for whatever reason it proved insufficient for her. When she was coding robots, number was used to model *counts* (e.g., of wheel turns), *sizes* (e.g., of angles), *lengths* (e.g., of polygon sides), and *locations* (e.g., starting points), with the last two of these interpretations dominating across tasks.

The number line is actually a specific instance of **mutable grids**, our second category of *objects-to-think-with*. A grid is much more than a device to parse space; it is a means to structure perception. Returning to the example of programming robots, when a user imposes a grid to parse space and track motion, robots become proxies for body-syntonic knowing, thereby linking the movements of the robot with experiences of the user's own body movements. Most coding environments require users to express elements and relations in terms of an underlying grid, whether that grid is specified through coordinates or simply used as a navigational device on which objects can be moved (forward, right turn, etc.). One particular grid, the Cartesian coordinate system, underlies a significant portion of secondary school mathematics related to functions.

Similar rectangular grids are also used as models in elementary school contexts such as multiplication (e.g., array and area models) and measurement (where 1-, 2- and 3-dimensional grids enable the computation of length, area, and volume). Our hypothesis is that by engaging in coding experiences in which grids are available as devices to interpret space, learners may develop more "dynamic" dispositions towards these *objects-to-think-with*, engaging with them not just as static forms but as flexible tools to interpret space and movement.

We further suspect that number lines and mutable grids afford opportunities for **modulating between decomposing and recomposing**, the first of our *actions-to-think-with*. Much of school mathematics is devoted to acts of decomposing (e.g., expressing numbers in expanded form, or as products of prime factors) and recomposing (e.g., interpreting a function as a completed curve, vs. a collection of points). In addition to its obvious relevance to geometry (Duval, 2006), decomposing–recomposing is invoked in early number, fractions, proportional reasoning, and measurement. It also arises centrally as a geometric interpretation of factoring polynomials, as evident in the algebra tiles model frequently used in secondary school mathematics. Decomposing–recomposing is especially prominent in the spatial reasoning literature, where it includes the disposition to see a diagram or other form either in terms of its parts and as a whole, as suited to the needs of a situation (see Davis *et al*., 2015). The focus on parts or on wholes has been described, respectively, as involving successive processing (integrating things into a temporal or serial order) and simultaneous processing (integrating things into gestalts) (Männamaa, Kikas, Peets, & Palu, 2012). The former has been associated with higher success in third-grade children's success in solving complex mathematics problems (Clements & Sarama, 2011), and those temporal and serial components are main reasons that we hypothesize that coding settings might present powerful mathematics learning occasions. Coding is, at its core, about decomposing global processes into elements and recomposing those elements into global processes.

Our second *action-to-think-with*, **systematic varying**, is a powerful complement of decomposing and recomposing. It arises from Marton's (2018) Variation Theory of Learning, the core principle of which is that perception and understanding can be strongly enabled when difference is experienced against a background of sameness (versus sameness experienced against a background of difference). In operational terms, the role and character of one element in a situation can often become more evident when it is varied systematically while all others are held constant. One frequently encountered use of this approach is around high-school study of the roles of coefficients in equations, such as $ax^2 + bx + c = 0$. Varying more than one co-efficient at once, or varying a single co-efficient haphazardly, is not likely to sponsor much insight; but focusing on a single co-efficient and being deliberate about how it is varied can be a powerful route to insight. Every topic in school mathematics can be approached through systematic varying, but (in our experience at least), it is a rare strategy in classrooms. Much in contrast, in coding settings, systematic varying is common. It can be especially prevalent in early stages of new projects, as users might begin with quasi-random guessing-and-testing, but typically shift quickly toward more structured approaches of examining relationships between lines of code and their consequences.

METHODOLOGY: DESIGN BASED RESEARCH

Our findings presented are mid-way of a three-year design-based research (DBR) study. DBR (see McKenney & Reeves, 2012) enables us to conduct an emergent, multifaceted study of an intervention (i.e., teaching coding and mathematics with robotics) in naturalistic settings (i.e., elementary classrooms). Following Hoadley (2004), we see DBR as a methodology in which participants attempt to understand the world by/while working to change it. It entails an adaptive attitude, multiple methods, and numerous data sources, collected within an iterative structure that enables and compels participants to be responsive to contingent and emergent circumstances.

For the past 1½ years we participated in and observed the designing, teaching and learning of weekly robotics classes in Grades 4–6 in a Calgary school. The site was chosen for several reasons, the main ones of which are stable student and teacher populations to facilitate tracking and tracing of varied elements. The school serves more than 600 students with diverse learning needs across Grades 2–12. The school has a commitment to flexibility in curricula, scheduling, pedagogy, and student clustering, as it strives to adapt structures to meet learner needs. Additionally, one afternoon each week is set aside for collaborative professional learning. Consequently, it offers an ideal space for implementing and honing novel academic foci, such as coding.

Our research focused on Grades 4–6, and involved 6 teachers and approximately 70 students. Most students in these grades have developed the necessary physical dexterity to construct robots, along with the conceptual understandings of mathematical elements (e.g., distances, angles, binary operations, logical operators) that are necessary to engage in reasonably sophisticated modeling and coding activities. We collected weekly video-recordings of the robotics teaching activities, and the learning activities of about 20 children in each grade, the equivalent of one full class. Video data was vital for providing rich illustrative examples and it permits us to 'slow down' to identify integrated/nested processes of learning, and to study learning in action. Using ongoing interpretive video analysis (Knoblauch & Schnettler, 2012), we selected clips that illustrate aspects of children using robots as objects and actions-to-think-with.

FINDINGS

Examples of using robots as 'objects- and actions-to-think' about the number line.

Figure 1: How far?

In Figure 1, two Grade 4 boys are measuring how far the robot travels with one rotation of the wheel. Sequential activities had them measure the distance for various numbers of wheel rotations (systematic varying). The robot provided dynamic actions for thinking about number as measurement of movement along a straight line.

Figure 2: How many? How long?

In Figure 2, the Grade 5 children are tasked with racing their robots to the wall. They have to come as close as possible without touching the wall. The robot did not travel quite far enough with 17-wheel rotations. The children played with numbers between 17 and 17.4. The number of wheel rotations was not a whole number, so the task required children to think of number as continuous rather than discrete. The robot provided dynamic actions for varying distance travelled with wheel rotations as well as opportunities for de/recomposing appropriate robot travel.

Example of using robots as 'objects- and actions- to-think' about mutable grids

Figure 3: How many? How long?

In Figure 3, a Grade 4 boy and girl are programming their robot to travel around a pentagon. This pair recomposed and decomposed the robots movements into a series of forward moves and right turns – e.g. movement along a plane. The sequential program for the robot indicates that they were not viewing the pentagon as a whole (5 times a forward and a turn). Each segment was a dynamic encounter with how many wheel rotations correspond to how long is the side, how big is the angle of the turn.

SUMMARY

Summing up, each of the above *objects*- and *actions-to-think-with* is commonly encountered in coding tasks. Each can also be used to support efforts to model a range of significant mathematical phenomena – but, in our experience, it is rare that they are explicitly encountered as competencies that might be deliberately developed and exercised across topics. Our data illustrates that (1) well-structured encounters with programming robots can support these competencies in ways that traditional images, applications, and manipulatives cannot, and (2) once developed, carefully considered classroom strategies can make them available for the learning of mathematics.

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Creating examples as a way to examine mathematical concepts' definitions

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Using examples supports the construction of concept images and concept definitions of mathematical concepts. Examples can also serve to examine relationships and connections between different mathematical concepts that learners sometimes make during the learning process of different mathematical concepts. This paper focuses on the interactions between teachers and definitions of mathematical concepts, as they manifested when working with online mathematical tasks that require learners to create examples that meet a specified set of conditions. These interactions required the teachers to re-examine the definitions of mathematical concepts and their boundaries. We will describe how the automatic formative assessment platform enables, and sometimes forces, reiteration and fine tuning of mathematical concepts' definitions, and examine possible impact on the learning process of both teachers and students.

Keywords*: definitions, concepts, automatic formative assessment.*

BACKGROUND

"I can't understand anything in general unless I'm carrying along in my mind a specific example and watching it go" (Feynman, 1985, p. 244). In mathematics, examples are an essential part of many theories of learning processes. The connection between examples and concepts was described in Vinner (1983) that conceptualized concept image as a mental image that is connected to the concepts in the mind, determined by the examples that are connected to the concept. Examples are also used to illustrate and communicate concepts between teachers and learners, and offer some insight about mathematical concepts and relations between concepts. A key feature of examples is that they are chosen from a range of possibilities (Watson & Mason 2005, p. 238) and it is vital that learners appreciate that range. Various mathematicians have written about the importance of examples in appreciating and understanding mathematical ideas and in solving mathematical problems (e.g. Pólya, Hilbert, Halmos, Davis, & Feynman). Whenever a mathematician encounters a statement that is not immediately obvious, he thinks of a particular example. When a conjecture arises, one practice is to seek a counter example or to use an example perceived as generic to see how the conjecture can be proven. The relationship between definitions and examples in mathematics is complex. Sometimes an example is given when a concept is used, while on other occasions a formal definition is required and afterwards the example is given as an illustration of the definition. Research indicates that students' comprehension of the concept is made up of a collection of examples, which form a concept image. Vinner (1983) indicates the connections between understanding the concept image and the concept definition. It seems that having a wide variety of examples for a concept allows for a better understanding of its definition. With the development of technology, students have the opportunity to create multiple examples with the aid of dynamic geometry environments and other types of technological tools. The development of technology presents new opportunities to exemplify using different means. With technology we can exemplify quicker and easier and can access the exemplification automatically (Sweller, 2013). Submissions of various sample sizes to geometrical questions with multiple solutions were analysed to study the loci constructed by their solutions (Leung & Lee, 2013). Other studies suggest analyzing mathematical and didactic characteristics in the presentation of examples by students (Olsher, Yerushalmy, & Chazan, 2016), suggesting this automatics predefined analysis could assist teachers in performing real-time decisions in the classroom based upon student data, thus performing formative assessment (Black & Wiliam, 1998). When we assess an example, we want to assess if the example fits the conditions of the task. In most cases, we can assess other features that the student was not explicitly asked to meet. For example, in geometry tasks, we sometimes want to relate the orientation of the shapes, or we want to recognize some extreme cases of submissions. With the development of technology, we can assess this feature automatically (Olsher et al., 2016). The research questions in this paper are: (1) How does the use of a system for automatic formative assessment enable and encourage discourse on definitions of mathematical concepts? And (2) How do tasks invite a discussion about the definitions of concepts?

METHODOLOGY

Research setting

The setting for this research was a professional development program (PD) for in-service teachers aimed at instructing and supporting the implementation of the STEP platform (Olsher et al., 2016) in classrooms. The PD included four face to face meetings (total of 30 hours), that included theoretical representation of formative assessment in mathematics, and use of the platform as students and teachers. The participants were also expected to use the platform in their classrooms between the meetings. Each classroom enactment was documented with a questionnaire. In addition, following the first enactments, each teacher had a discussion about their implementation in the classroom with the PD instructor (first author).

Participants

The participants in the PD were 22 teachers, teaching mathematics in Israeli secondary and high-schools. The teachers teaching experience ranges between 2 and 25 years. The teachers are all certified teachers; some hold MA degrees in Mathematics education, while others studied for BSc. or BEng. Degree in computer engineering or electrical engineering, worked in that profession for several years, and then participated in programs for career retraining aimed at enhancing the number and abilities of Israeli Mathematics teachers.

Data sources and analysis

The data sources include: (1) the students' submissions of solutions. (2) Teachers answers to the questionnaire, (3) Reflections, that were collected in discussions with the PD instructor post-enactment, (4) Field notes made by the two researchers during the PD meetings, and (5) Video recordings of the PD meetings. Analysis of the enactment of the assessment tasks in the classroom focused on discussions about mathematical definitions. First, we identified the episodes in which mathematical definitions were addressed by the teacher. These episodes occurred either in the classroom or during the PD sessions. Next, we categorized the episodes. Trouche (2004) uses the term "instrumental orchestration" to describe didactic configurations and the way that they are being exploited in the classroom, and also suggests them as a construct that could "give birth to new instrument systems" (ibid, p. 304). In the reflected lessons in this study, this framework is suitable to describe the way the teacher works with the students' answers, and suggest "new instrument systems" that help to the teacher and the student to think again about mathematical definitions of concepts. The "new instrument systems" embodies the practice of mathematical definition and concepts as they were enacted by the teachers. Our analysis process was iterative, fitting each relevant episode into a specific category that have specific characteristics of teachers practice with the definitions of mathematical concepts. The four different categories that were identified will be described in the following section.

FINDINGS

We identified four different instrumental systems of dealing with mathematical concepts' definitions: (1) Addressing the definition of a mathematical concept during the activity, (2) Resolving conflicts between definitions of a mathematical concept, (3) Establishing an inclusion relation between mathematical concepts through their definitions, and (4) Differentiation of characteristics into subcategories. In this section, we will describe each of these new systems that we categorized, and give an example of one of the episodes that were identified from this category.

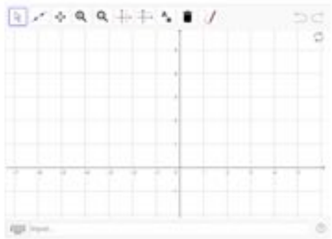
Addressing the definition of a mathematical concept during the activity

Definitions of mathematical concepts lie at the heart of many tasks that require giving an example that fits certain constraints. Yet, sometimes the definitions that are available to learners are not appropriate for the task at hand. These cases, in which a definition that was taught, or the general definition, is not sufficient to determine whether the examples fits the definitions, requires the teacher to address the definition of the mathematical concept during the activity. The instrumental system is that of the teacher revisiting the definition of a concept, which is part of the task at hand. The learners dealing with the task demonstrate uncertainty in terms of the definition, and the teacher addresses this uncertainty in referring the students to or leading the student's way towards the definition of the relevant mathematical concept. For example, the definition of a tangent is that is parallel to the Y axis. This does not fit to the regular definition of tangent, since the function has no derivative at the point in which the tangent is parallel to Y axis. As manifested during the classwork on the following task (figure 1). One of the teachers, Sapir (pseudonym) gave this task (Figure 1) to her 12th grade students as homework. While preparing the next lesson, Sapir encountered a wide variety of answers submitted by her students (Figure 2), demonstrating various instances of tangents, not necessarily aligned with the definition (e.g. Figure 2a where tangent and asymptote might be mixed up by the student).

Task 1

Below are two characteristics of a function and three characteristics of the tangents to points on that function. Is there a function that has all of the 5 characteristics on the list below? Yes No

- There is a tangent with more than one tangency point with the function.
- There is a tangent that crosses the function in another point.
- There is a tangent that is parallel to the Y-axis.
- The function is not continuous.
- The function does not have any extremum points.



If such a function exists, using the dynamic diagram below, sketch a function like that. If there is no function that can satisfy all of these 5 items, mark a set of the maximum number of characteristics that could exist in a single function, and sketch a function for which all of those characteristics hold.

Figure 1. Task requiring student examples of functions and tangents meeting certain conditions

In the following lesson, Sapir presented the student answers to the class, and initiated a discussion with the students. One of the students asked, “How we can calculate the tangent if we cannot find the derivative of the function in this point?” Other students attempted to think how this could be performed and to find a concrete definition for this case. Sapir asked the students to find an appropriate definition of a tangent that would include this case, thus addressing the finer points of the definition, not necessarily clear before the task addressed them. Later, during the PD, Sapir stated that she also thought about it when she assigned the task to the students, but she could not find a good definition and she performed an online search, to be better prepared for such.

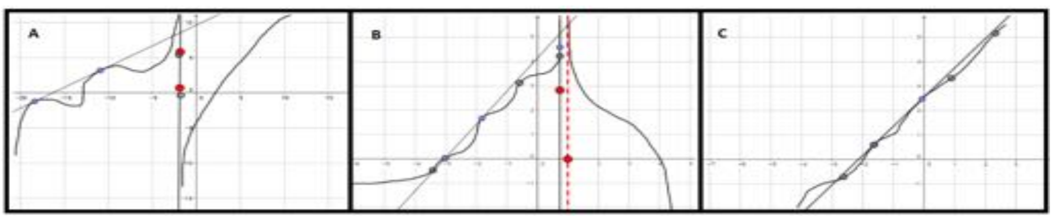


Figure 2. Example of submission of students

This example presents an episode in which the students completed the task, while not emphasizing whether their constructed tangents fit the familiar definition of the concept. The definition of the concept at hand, the tangent, was not fully clear to both the teacher and the students. The collective example space of the students in the class as shown by the STEP platform, followed by the description in the following lesson raised the question about the need for a better, clearer, definition.

Resolving conflicts between definitions of a mathematical concept

Mathematical concepts could sometimes be defined in different ways. Having different definitions for one mathematical concept, might lead to inconsistencies when students deal with the concept. These cases, in which there are multiple definitions for a concept, require the teacher to negotiate between the different definitions in order to clarify the relevant characteristics of the concept that they wish to emphasize. The instrumental system is that of the teacher choosing between the different definitions in order to elicit and treat the inconsistencies that might rise. The learners dealing with the task follow a certain definition, while not attending to characteristics that derive from a different definition. For example, the definition of an extremum point of a function. As manifested in Anna's classwork on the following task (Figure 3). The task requires students to draw a graph of a function, which has one line that is tangent to the function at two different points. Anna, the teacher, focused on examples in which the tangent was a horizontal line, but the tangency point was not an extremum in her opinion. In the PD session following the enactment, we discussed this mathematical concept. Anna said that the red dot (which was asked for in the question as a tangent point (Figure 3) is not an extremum. The PD instructor asked her to specify the reason for her opinion, and Anna answered that the derivative after the point is zero (for $x > x_0$ $f'(x) = 0$), so for this point, the function does not have an extremum.

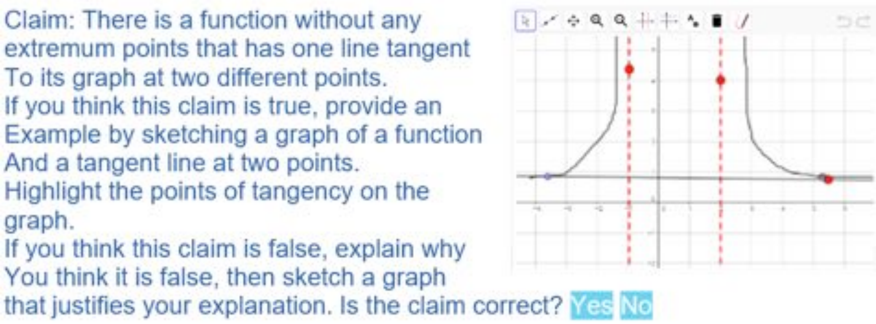


Figure 3: example of concept of extremum point

This led Anna to explore the definition of an extreme point. According to the definition in one mathematics book local extremum exists if there is an environment of x_0 so that all $x, f(x_0) \le f(x)$ ($f(x_0) \ge f(x)$). On the other hand, in the book that Anna taught from there was an examination of extremum points according to the variance of the derivative (if the previous derivative rises and then decreases). A closer examination of the mathematical definition of extremum points in the textbooks revealed an interesting phenomenon: Different definitions appeared in different books. This could change the decision that a horizontal line does have extrema points. The first setting allows a maximum for a fixed function, while the second setting does not allow this. This example presents an episode in which the students completed a task, but the constructed mathematical object did not fit with the definition that the teacher was familiar with. Furthermore, the constructed object did fit an alternative definition that appeared in another mathematical textbook. The use of an automatic analysis of student answers could easily make these conflicting definitions episodes more accessible, as the technological platform requires an algorithm to automatically categorize the different answers: and this algorithm would comply with the chosen definition, when they are conflicting.

Establishing an inclusion relation between mathematical concepts

Inclusion relations between mathematical concepts appears in various mathematical strands, such as numbers, functions and geometry. These relations lay out the hierarchy between different mathematical concepts: which concept is a specific, special case of a more general concept. Unfolding these hierarchies requires the teacher to differentiate between definitions, clarifying what are the relevant characteristics that induced the inclusion relation between the concepts. The instrumental system is that of the teacher choosing tasks that reveal incorrect inclusion relations assumed by the students, and then sorting out the different concepts and the relations between them. For example, regarding inclusion relations of special types of quadrilaterals, the teacher Pnina, assigned the following task to Primary school students. The teacher did not introduce the task to the students, rather she let the students discuss and ask each other questions regarding the task.

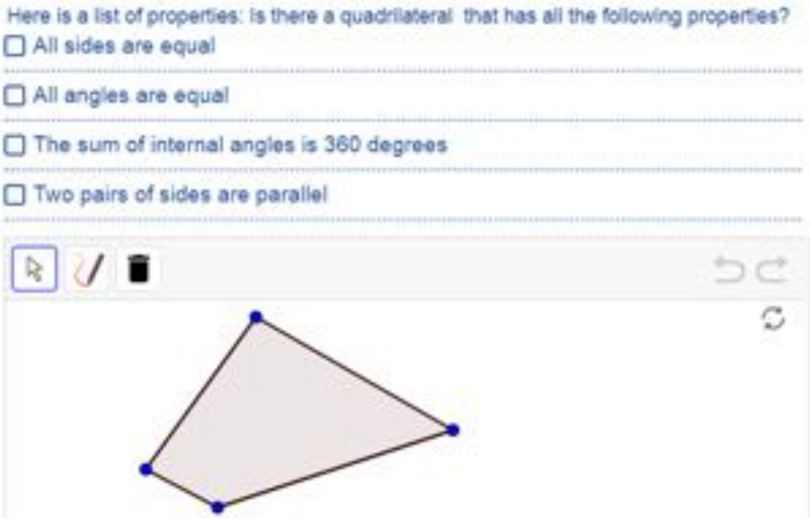


Figure 4: a task in inclusion relations between types of quadrilaterals

This task is part of a longer activity regarding inclusion relations between quadrilaterals. The students had to choose as many statements as they could, and drag the quadrilaterals points so that it matched the chosen statements. Some students began to ask questions about the relations between the statements in the task and definitions of types of quadrilaterals. For example, one student asked “if all the sides are equal than it is square, right?” Pnina did not answer the question and waited until the students submitted the task. Afterwards she reviewed all the examples in the task and the students had the opportunity to ask about definitions and the relation between the definition and the statements that appeared in the task. Some students revealed their thinking about some of the mathematical objects, such as “a square is not a parallelogram”, or “if all the angles are equal it must be a square”. The students discussed each other's claims, and with subtle guidance of Pnina, concluded the definitions of these mathematical concepts and the relations between them. This example presents an episode in which the students completed a task, and in the discussion after the task, they concentrated on definitions and on inclusion relations. The use of an automatic analysis of students answers, could easily connect between the statements in the task and the student's’ submissions, the discussion in the class raised question and statement about inclusion relation between quadrilateral that reflect their concept images about these concepts.

Differentiation of characteristics into subcategories

Differentiation of characteristics into subcategories is a process that splits a category into subcategories. In these cases, a general definition or category is too wide and contains more than one subcategory that emphasizes some important criteria. The instrumental system is that of the teacher revisiting the category of a concept and splitting the examples that fit into this category to sub-categories. For example, Alon’s use of the task “Claim: There are functions that have one line tangent to their graph at two different points, If you think this claim is true, provide three examples by sketching a graph of a function and a tangent line at two points If not, explain” (Figure 1). Alon assigned this task to his students as homework.

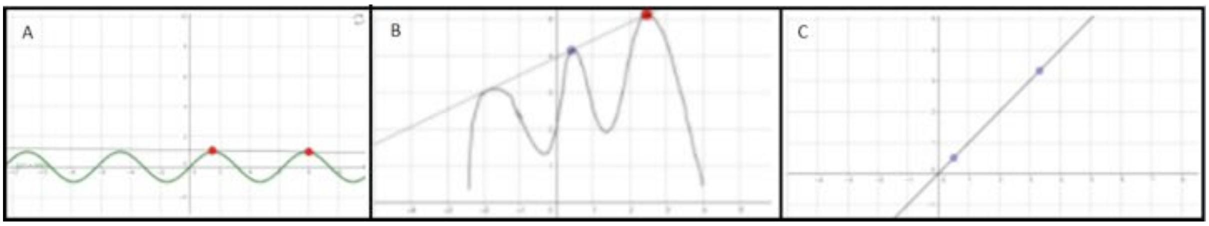


Figure 5: three different submissions that fit the automatic characteristics

This led to a discussion about the difference between the right and left submissions and between two different types of infinite points of tangent to line. In figure, 5 A, there is a symbolic function f(x)=sin(x). The tangent line represents the line “y=1”. This tangent has infinite countable points at every point x=/2 +k, k Z. In the figure windows, only four of them appear. In figure 5 C we can see a linear function in a symbolic representation in which the function and the tangents converge. In this example, there are non-countable, infinite tangents points. These three different categories are all using the filter “more than 2 tangent points”, which is the general category. The differentiation between them raises two questions about definitions. The first was about the definition of tangents, in which one student claimed that in figure 5 C this is not a tangent point because the lines (of the function and the tangent line) are convergent and so they cannot be tangential.

DISCUSSION

In the findings, we presented some examples of discourse about definitions of concepts during the use of the formative assessment platform. The focus of the tasks on examples, and specifically the opportunity to interact with extreme cases facilitates the opportunity to address the definitions of mathematical concepts, thus enhancing the concept image (Vinner, 1983). The linkage to automatic assessment of the examples and the requirement to specifically program an algorithm for the identification of the mathematical concepts in a precise fashion adds another point of interaction with the definitions, especially when there are several conflicting definitions for a single concept. These new instrument systems (Trouche, 2004) enable teachers and students to communicate and interact with mathematical definitions of concepts. Finally, it is not enough to address the student answers and define the algorithm for the automatic assessment. The tasks should also be open ended, enabling different possible correct and incorrect answers in order show different exemplifications, possibly challenging the definition, as suggested by Olsher et al. (2016). The tasks included the automatic assessments in them, i.e. there are characteristics that were used to classify the submissions. This classification was the basis for discussions challenging or elaborating about the definitions of the mathematical concepts.

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From the Classroom into the Online Space: Meeting the Needs of Developmental Mathematics Students

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The work reported in this paper seeks to design online courses for developmental mathematics students by transforming two existing face-to-face (f2f) courses into online instructional contexts. Particular consideration has been given to what is sufficient and necessary with regard to course design and curriculum modification, and the instructional shifts that address the particular mathematical, socio-emotional, and institutional needs of developmental mathematics students. This paper reports on preliminary analysis of instant messaging chat transcripts and interaction within documents in an online collaborative work space.

Keywords: developmental mathematics, online course design, design-based implementation research.

INTRODUCTION

In the United States, college-level mathematics is a gatekeeper for hundreds of thousands of students. Every year, nearly 60% of students entering two-year colleges, and approximately 40% of undergraduates entering four-year universities, are deemed not ready for college-level mathematics and are placed into remedial, also called developmental, mathematics (Bailey, Jeong, & Cho, 2010). Approximately 80% of these students do not successfully complete any college-level mathematics courses within three years of this placement (Bailey et al., 2010), preventing approximately half a million U.S. students per cohort from achieving their college and career goals.

Increasing numbers of developmental mathematics students choose to enroll in online courses, particularly in two-year colleges (Ashby, Sadera, & McNary, 2011); however, little research has addressed their experiences and outcomes. While characterizing developmental mathematics students as a monolithic population is problematic, research suggests that they tend to be older than their college peers, are more likely to be managing work, family, and school simultaneously, are more underprepared mathematically, and are more likely to have low self-efficacy with respect to mathematics learning (Ashby et al., 2011)—all of which are factors that impact student outcomes in online learning environments. Indeed, recent studies show that failure and withdrawal rates are sharply higher in online developmental mathematics courses than in equivalent f2f courses (Jaggars, Edgecomb, & Stacey, 2013). The work reported here seeks to design online courses for the needs of and conditions impacting online developmental mathematics students by translating two highly effective f2f developmental mathematics courses into online instructional contexts.

Carnegie Math Pathways

The Carnegie Math Pathways (CMP) are alternatives to the traditional developmental course sequence and are designed to address the needs of developmental-level students while also reducing a multiple-term course pathway to a single term, streamlining the course pathway to and through college-level mathematics credit. They include two course sequences: Statway, which focuses on statistical reasoning, and Quantway, which focuses on quantitative reasoning. Launched in 2011-12, the CMP network now includes over 80 colleges and universities in the U.S. that are implementing the CMP courses, and over 27 000 students have enrolled in either Statway or Quantway. On average across all six years, approximately 52% of community college students successfully completed Statway, while Quantway students had a 63% weighted average success—triple the success of the typical approach (Huang, 2018).

The Statway and Quantway curricula are designed to engage students in context- and data-rich tasks that are relevant to students' lives. In the f2f setting, students work in collaborative groups and instructors employ pedagogical practices that promote students' productive struggle with substantive mathematical tasks and make explicit connections between key mathematical concepts and ideas through discussion of students' reasoning on those tasks (Hiebert & Grouws, 2007; Merseth, 2011). Additionally, research-based interventions that target a growth mindset and other socio-emotional constructs relevant for mathematics learning and engagement (Dweck, Walton, & Cohen, 2014) are embedded in the curriculum and pedagogy of the Pathways courses.

To provide the large population of online developmental mathematics students with a rich and effective learning experience that produces similar outcomes to the f2f courses, in 2017 the CMP network launched an effort to develop online versions of Statway and Quantway. This study focuses on the design process to transition the f2f CMP courses into the online setting, and seeks to identify what is sufficient and necessary with regard to course design and curriculum modification, and the instructional shifts needed to address the particular mathematical, socio-emotional, and institutional needs of the online student population.

REVIEW OF RELEVANT LITERATURE

To preserve the pedagogical approach of the f2f CMP courses in the online setting, we drew upon the Community of Inquiry (CoI) model by Garrison et al., (2010). The model proposes three components to help develop critical thinking and inquiry in an online course: cognitive presence, social presence, and teaching presence.

Cognitive presence is the extent to which students could construct and apply knowledge as a result of engaging in dialogue in the classroom community. It involves students engaging in productive struggle and collaborative engagement in problem solving to optimize learning. The design seeks to provide cognitively challenging tasks that are related to real life and are a mathematical stretch for students, in a collaborative environment (Garrison, 2016; Zakaria et al., 2013).

Course design that supports effective synchronous or asynchronous collaboration is critical to develop social presence, or a student's online presence. It is crucial for promoting a student's sense of belonging as well as supporting mathematical discourse. To promote engagement, course design must intentionally include opportunities for learners to articulate their thinking with others during problem solving in order to help them make sense of what they are learning and to foster connections with others (Dejarnette & González, 2016; Warren, 2014).

Teaching presence is vital for effectively supporting student engagement in collaborative inquiry in an online class (Garrison et al., 2010; Warren, 2017). Teaching presence requires an instructor's intentional, regular, and reliable presence throughout the course, supporting stimulating, productive, and safe engagement so that students achieve high levels of social and cognitive presence.

Research outlining essential design components of online courses more generally was also considered, particularly with respect to how elements of course design could create conditions for the CoI model to flourish. For example, space in the online course for discussion and idea exchange needs to be supplemented with high teaching presence to engender purposeful interaction (Jaggars & Xu, 2016).

Our implementation framework was developed to articulate the core design and instructional principles of CMP courses. Grounded in the CoI model and further informed by practitioner input, the implementation framework provides instructors and course designers with statements of the core principles that serve to guide the transition from f2f to the online space. In this paper, we focus on three of our design and instructional principles: Always Welcoming, Collaborative, and Interactive.

METHODS AND ANALYTICAL APPROACH

As the goals of this study are both to advance knowledge regarding developmental mathematics students' engagement and learning in the online setting and to directly impact students' engagement and learning in this context through design, this study employs a developmental, or design-based implementation research, approach (Fishman et al., 2013). Specifically, this study is interventionist, iterative, and improvement-oriented (Akker et al., 2006). This work starts from the premise that the intervention is accountable to actual users and contexts of use; namely, students, instructors, and classrooms across the wide variety of institutional settings found in the CMP network. Additionally, the design process, rooted in theory and research on students' online mathematics learning and engagement, is necessarily iterative as we seek to understand "what works, for whom, and under what conditions" and to apply that knowledge to continuously improve student and instructor outcomes (Bryk et al., 2015). We used grounded theory to examine data from the iterative process of design and implementation, identifying emergent themes (e.g., opportunities for productive struggle), developing alternate justifiable interpretations of the data, and returning to the data to develop consensus interpretations of the evidence for and conceptualization of the emergent themes.

Study Sites and Data Sources

In Fall 2017 and Spring 2018, multiple versions of CMP online courses were piloted at different institutions. Such institutions implemented delivery models ranging from fully asynchronous to a mix of synchronous and asynchronous aspects, across differing LMSs. This paper reports on preliminary data analysis from instant messaging (IM) transcripts and student-content and student-teacher interaction from two of these sites that employed contrasting implementation designs. Site 1 and site 2 are both small 2-year colleges in the American Midwest that offer f2f and online courses. At site 1, we looked at data from a Quantway class of 17 students; the instructor was an experienced mathematics educator who teaches exclusively online. A site 2, we looked at data from a Statway class of 22 students; the instructor was an experienced mathematics educator with no online teaching experience. The students in both classes were placed into developmental mathematics.

In this study, we report only on transcripts of synchronous IM chats (from site 1) and Google documents (from site 2) that capture asynchronous collaboration on mathematical tasks. Beyond this report, these data will be supplemented with analysis of online discussion forums, LMS analytics (e.g., login behavior, page views, assignment submissions), student course outcomes, brief interviews with instructors and students, and instructor and student surveys from both of the sites.

Site 1 delivers Quantway with a mix of synchronous and asynchronous aspects; that is, students work through some parts of the course in groups and with the instructor, while working individually through other parts. Each IM session involves the instructor and a small group of students working through a lesson. Transcripts of IM chats are downloaded by the instructor and made available to the class. In addition to the IM chat, students further individually and asynchronously engage with content in the form of two online homework components, one of which is preparatory, while the other is auxiliary, to a core lesson. Student-student interaction occurs synchronously within the IM chat and asynchronously within the class's online discussion space.

Site 2 delivers Statway asynchronously; that is, students in groups and individually work through course material at different times. Student engagement with the course content entails working individually through part of a lesson, working collaboratively with group members through the remainder of that lesson, and working individually through a series of aligned formative assignments. Student-student interaction took place asynchronously both within lessons (through the collaboration feature of Google docs) and in the LMS discussion space.

We analyzed interactions from 20 synchronous IM chats in one online spring 2018 section of Quantway at site 1. The discourse was examined for mathematics discussion pertaining to productive struggle; that is, identifying both actual and missed productive struggle opportunities. Additionally, we noted actual and missed instances of both community development and teaching presence. From site 2, we analyzed three completed Statway lessons (as enageged with by students collaboratively in Gooogle docs) and noted instructor modifications, student-student interaction, and instances of teaching presence.

PRELIMINARY FINDINGS AND DISCUSSION

Broadly speaking, preliminary analysis indicates that some aspects of core Pathways pedagogy were effectively translated into the online contexts at both sites.

At site 1, the purpose of the first synchronous IM session is to establish chat norms and begin developing a community. After the first session, the format is consistent. First, students were asked if they had questions about past work. In these data, there were no instances where a student raised a question. Students were then told to read the lesson text and responded to a series of questions. Students would post an answer in the IM and the instructor would comment. Most answers were numerical. The mean number of turns for the instructor (81.1) and students (28.2) over the course of 20 sessions suggests that the IM chats were primarily instructor led.

At the beginning of most IM chats, students inquired about others or commented about something previously mentioned. The instructor did not participate in these interactions. These instances are viewed as missed opportunities to heighten teaching presence and to strengthen the community of the online class.

At least once a lesson, the instructor noticed a productive struggle opportunity and broke with the lesson to ask questions to either help clarify what was being learned or to remedy any mathematical concept not fully understood. For example, in one IM session, a student is unsure about the difference between relative and absolute changes and the instructor uses a different example to allow the student to consider the concepts afresh.

There were also a few instances (about one in ten) in which a productive struggle opportunity was not exploited by the instructor. For example, in one IM session, a student had trouble transforming decimals into percentages and the instructor provides a definition and three examples. A possible explanation is that the instructor was focused on the lesson and what the next steps were. This may be something that online instruction training could mitigate. In most instances, such opportunities were noticed and exploited by the instructor with redirection questions to others, engendering student-led discovery.

From site 2, we looked for adaptations to lesson content flow as delivered in an asynchronous Statway class of 22 students. Students engaged with lessons in two different ways: preparatory content that students worked through individually and asynchronously, and content that students worked through collaboratively and asynchronously.

In a f2f environment, a discussion section that begins each lesson allows the instructor to highlight potential student misconceptions to smooth any struggle surrounding the ensuing main mathematical concepts. With the transition to the online space, the instructor at site 2 supplemented such lesson content with screencast tutorials, which included calculator use examples. These are embedded in the Google doc lessons, and their placement was informed by both f2f Pathways teaching experience, and as suggested in the lesson's accompanying instructor notes.

In one instance, the instructor embeds a screencast tutorial in the first question of the lesson. The screencast tutorial intervention comprises a description of a shown normal curve, conceptual connections to information stated in the preceding lesson introduction, and an example showing *Z*-score and region area calculation.

In the collaborative section of a lesson, we note that the instructor has supplemented the lesson document with comments, question prompts, or reframing statements to engender fuller student comprehension. In one instance, the instructor prompted a student to consider any further implications of a stated answer. We also note student-student interaction in the comment function, a feature present in Google docs. In one instance, perhaps indicative of a developing sense of group belonging, a student-student exchange culminates in a sharing of a student-sourced resource.

CONCLUSION

The preliminary data analysis suggests that these early attempts at translating the core principles of the Pathways f2f courses into online settings were inconsistent with respect to certain key elements, notably effective support of productive struggle, teaching presence, and social belonging. However, these findings suggest specific norms and routines that could improve the effectiveness of the implementation of the Pathways courses online, as well as improvements regarding professional development about online instruction and course design.

Specifically, the data from site 1 reveals that the instructor did not consistently notice and take up the opportunities for productive struggle afforded by the curricular tasks. However, the instances in which opportunities were taken up indicate that synchronous IM chat is a format from which rich and effective mathematics struggle can be leveraged. Professional development that explicitly exposes faculty to examples of instructor-student interaction that make salient these opportunities, and strategies necessary to take advantage of them, would support more effective implementation of the curricular tasks and of students' productive struggle. From the same data, instructor noticing of non-mathematical questions at the beginning and throughout the IM chat sessions is inconsistent. Supporting faculty to implement an instructional routine in which initial exchanges in IMs, or other instructor-student interactions are not related to mathematics, could help instructors establish a relational foundation from which sense of belonging and teaching presence could thrive.

The analysis of site 2 indicates a need for modification of the lesson flow to mitigate the effects of student misconceptions that would typically be addressed in a f2f class discussion. Establishing a norm in which video screencasts are used to prepare students to engage in tasks, i.e., a problem launch, could both heighten teaching presence and pave the way for productive struggle with mathematical content. An alternative approach here may be for instructor-facilitated asynchronous discussions to expose and address misconceptions. Further, there is indication that student-student asynchronous interaction is under-utilized. Establishing a norm that encourages students to show work/thinking in the comments could help build student-student connection. Additionally, instructor-led asynchronous interaction, either in response to an answer/question or unprompted, would enhance teaching presence.

In further analyses of the data corpus, we intend to build upon the research and findings detailed here, extending analyses to include attention to enhanced and specific instructional strategies surrounding noticing and responding to productive struggle instances, in addition to supporting both a sense of belonging and teaching presence. The complete study will inform curricula and instructional considerations when transitioning into the online space, and by emphasizing productive mathematical struggle and student sense of belonging, contribute much needed insight into how to effectively support online developmental mathematics students.

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Task quality vs. task quantity. A dialog-based review system to ensure a certain quality of tasks the MathCityMap web community

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Authentic tasks require realistic objects and questions, which can for example be realized through outdoor mathematics. MathCityMap takes up this idea of outdoor mathematics through the creation of math trails, which lead to places where interesting mathematical problems can be observed and solved. MathCityMap bases on an app and a web portal in which every registered user is allowed to create and publish own tasks. Through a constantly growing community and the claim of a certain quality of the published material, the system bases on a multistep review process and several criteria for published tasks. The paper presents the steps of the review process, defines the underlying criteria and how they are communicated, and discusses the consequences of a review system for users and their professional development.

Keywords: modelling, task design, stepped hints, feedback, mobile devices.

INTRODUCTION

Especially through integrating realistic tasks in mathematics school lessons, modelling and authentic tasks play an important role in this context (e.g. Borromeo Ferri, Greefrath & Kaiser, 2013). Modelling means – apart from other processes – to translate real and authentic contexts into mathematical models and vice versa. Following the definition of Vos (2015), the authenticity of a task is given if (1) the task is created in an “out-of-school” origin and (2) the task has a “certification” (p. 108). Nevertheless, these types of tasks are often proceeded inside the classroom with help of a picture and/or text information. This means that a mathematical problem referring to an authentic object is in many cases adapted to the educational context. Here, the authenticity in the sense of a certification is obviously not guaranteed.

Taking up this issue, one can observe a trend in doing outdoor mathematics through running so called math trails. The idea of math trails, meaning a route which leads to special locations where mathematics can be observed, is already some decades old. In the 1980s, the first documented math trails were created in Melbourne, Australia by Blane and Clarke (Blane & Clarke, 1984). Nevertheless, the original intention was not to teach mathematics or modelling competence in the educational context, but to popularize mathematics in society. In 2012, the MathCityMap (MCM) project was funded at Goethe University in Frankfurt, Germany and led the idea of math trails into the educational context with help of new technologies (Ludwig, Jesberg & Weiß, 2013). In the following, we will present the project and focus on the basic review system, which is an important feature of the project in terms of quality aspects and the professional development of teachers as task designers.

THEORETICAL FRAMEWORK

Review Processes

Reviewing process is a common way to guarantee quality in science, literature or music. Even commercial reviews, often based on user opinions, are getting more helpful when mixed with expert reviews (Connors, Mudambi, & Schuff, 2011). In academia, peer or expert review is standard. It "is the process by which experts in some discipline comment on the quality of the works of others in that discipline." (Price & Flach, 2017, p. 70). It is a common way of guaranteeing quality of academic papers and material produced by different authors (Price & Flach, 2017). Also in growing web communities, which allow users to produce and publish material, re-viewing processes are necessary.

Wikipedia is an example of an online platform with over 40 million articles that anyone can edit (Brandes & Lerner, 2007). This amount of articles and authors does not allow for a review of every edit and asks for a complex review process. Review and protection elements used by Wikipedia are the storage of elder pages in case of edits, and the distribution of roles, e.g. reviewer or administrator, which allow particular actions on Wikipedia (Ferschke, 2014). This enables experienced users to give feedback on the edits of "normal" users in so-called flagged revisions (Ferschke, 2014). Further, "[t]he editorial review was intended to minimize the risk of vandalism and improve the accuracy and overall quality of the articles by having experienced Wikipedia authors approve revisions before they go public." (ibid., p. 33).

GeoGebraTube, a platform with online material for the dynamic mathematics soft-ware GeoGebra, serves as an example of reviewing material in the context of math-ematics education. The tool makes it possible to create and access material e.g. worksheets, for the software. Currently, about one million files are available. In terms of quality, GeoGebraTube counts on editorial review, which rates excellent materials, and on user review (Gassner & Hohenwarter, 2012).

These two examples show the conflict of providing quality and quantity in a growing content creating web community. Especially in the educational context, openness in terms of the creation of material has to consider "limitations in the verification of learning outcomes" (Camilleri, Ehlers & Pawlowski, 2014, p. 39).

We take up this issue and bring it into the context of the MathCityMap (MCM) project, which asks for a review system for mathematical tasks. The problem of quality and quantity leads to the following research question: Which impact does the MathCityMap review system have on the quantity and quality of published tasks?

To answer this question, we present the MCM project and its review process on a theoretical basis. Afterwards, the analysis of a successfully reviewed task will bring first results on the research question and a basis for further studies.

IMPLEMENTATION

The MathCityMap project

The MCM project is a digital tool to facilitate the integration of math trails in the educational context. Through GPS-coordinates on a math trail map, it leads to places where interesting mathematical problems can be found and solved (Ludwig et al., 2013). The map and trail data are available in the MCM app, which navigates to these spots, gives direct feedback on entered solutions, and offers hints. In the MCM web portal (www.mathcitymap.eu), one can access published MCM tasks and trails, spotted in many different countries all over the world (Figure 1).

Task: Table Mountain's Monument (Cape Town, South Africa)

Definition of the task: Calculate the mass of the stone monument. Give the result in kg. 1 cm^3 of granite weighs 2,6 g.

Figure 1: MCM Task on Table Mountain's Monument

The web portal also enables to create one's own tasks and trails. These tasks can either be used for own purpose, or can be published and shared with all registered MCM users. Although the MCM project benefits from a growing community and a growing number of material, the published tasks have to meet the MCM standards. To guarantee this, all tasks have to go through a review process before publishing.

The MathCityMap review process

The MCM review process, established in Oct 2016 for published tasks, is based on four steps (see Figure 2).

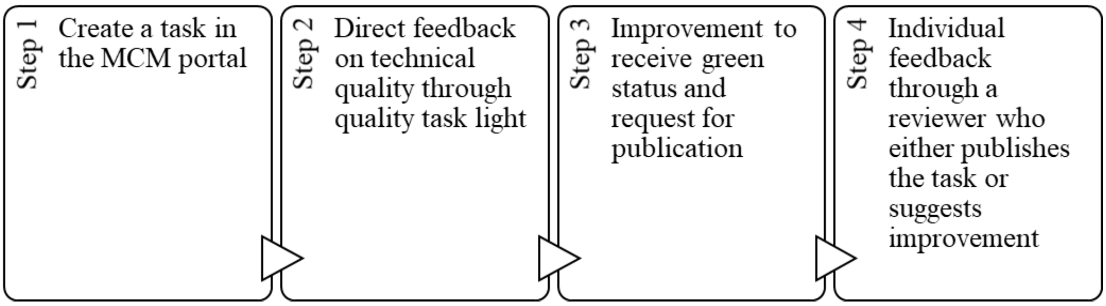

Figure 2: The MCM review process in four steps

Step 1: The registered author creates a task in the MCM portal.

Step 2: A task light gives direct feedback (red, yellow and green) on the completeness and fulfillment of technical criteria. In the example in Figure 3, a task fulfils all "orange" technical criteria, and all but one "green" criterion. The task cannot be submitted into the review process without improvement. Since many review processes solely included technical issues, to draw attention on didactical feedback, traffic light system was included, which is helpful and timesaving for users and reviewers.

Step 3: In case all criteria of the green level are fulfilled, it is possible to submit the task into the review process.

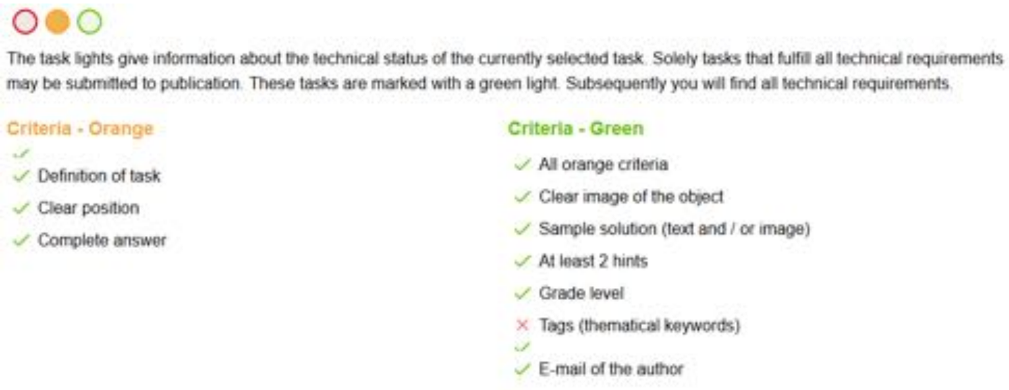

Figure 3: Example of a feedback given by the task light system

Step 4: In the review process, the task is checked in terms of the appropriateness for a published MCM task. This is done for each language by selected experts with experience in the creation of MCM tasks and running of MCM trails. After the review, the task is either published in the portal, or the author receives feedback on how to revise the task via e-mail. All review feedback for a task is stored in its review log.

This process guarantees that only appropriate tasks that correspond with the math trail idea and the MCM concept are published and shared with the community. The MCM web portal consists of 2289 tasks (Feb 2018), out of which. 956 are public (42%), wherefrom the newest 651 (68%) went through the described review process.

Nevertheless, through a growing community and number of authors and reviewers, the system asks for a transparent review guideline in order to avoid arbitrariness.

Criteria for tasks in a MathCityMap math trail

A catalogue of criteria was developed, based on relevant literature and years of experience enabling authors to comprehend feedback and reviewers to give a transparent feedback on submitted tasks.

1. *Uniqueness*. To make clear which object is meant, every task should provide a picture that helps identify the object of the task and what the task is about.
2. *Attendance*. A task should be authentic, i.e., leaving the educational context and having a certification. Thus, the task can only be solved at the object location and its description should never be enough to solve it (Ludwig et al., 2013).
3. *Activity*. Physical activity has a positive effect on learning, implying the idea of embodied mathematics, i.e., mathematics can only be fully comprehended through an active experience (Tall, 2013). The task solver should therefore become active and do something in order to solve the task, e.g. measure and count.
4. *Multiple solutions*. Authentic and modelling tasks are characterized by the fact that they are solvable in different ways through the choice of a mathematical model. The task should therefore be solvable in various ways.
5. *Reality*. An important characteristic in this context is the connection of mathematics and emotions, interest and relevance for the students – aspects that

significantly correlate with performance (e.g. Tulis, 2010). The task should have meaningful relevance and not appear too artificial.

6. *Hints*. As Jesberg & Ludwig (2012) summarize, several studies come to the conclusion that stepped aids have a positive impact on learning performance, experience and communication (Jesberg & Ludwig, 2012). Therefore, every task should provide at least one hint in terms of solving the task.
7. *School math and tags*. The task should feature a connection to school math. Therefore, one can use tags with relevant key words and assign them to a grade.
8. *Solution formats*. The solution should be representable in one of the solution formats provided by MCM: interval, exact value and multiple choice.. Especially for modelling tasks, the interval seems very relevant as it enables to refrain from minor deviations in the solution, as through measuring differences or different mathematical models. In this format, one defines a green interval for correct solutions, and an orange interval for incorrect, but acceptable ones. Solution values that do not fit into these intervals receive the negative feedback and the player tis asked to retry.
9. *Tools*. The task should be solved without special and extraordinary tools apart from calculator, measuring tape etc.
10. *Sample solution*. One should provide a sample solution including measured data (only visible in the portal and in the solution PDF) for teachers in order to talk about the tasks in the following lessons and analyze typical errors.

The catalogue is formulated for single tasks as they are individually checked within the review process. Nevertheless, a math trail idea is a combination of different tasks that should harmonize as a trail. Therefore, the whole trail comes into the review process after every task of a trail went through it.

Communication

Apart from defining a catalogue of criteria, it obviously has to be communicated to active and future authors. One way is to present the criteria for MCM tasks during teacher training before the teachers create their own first tasks. On the MCM website, one can find a tutorial explaining MCM tasks criteria and best practice examples in the newsfeed category “Task of the Week” where already published tasks are analyzed in terms of the MCM criteria. A further step towards facilitation and transparency in the reviewing process is the idea of generic tasks. Common objects, such as stairs, offer the chance to easily and quickly transfer existing tasks to other locations.

An important part in the communication of criteria is the individual feedback within the MCM review process. Figure 4 shows an example of a task which passed through this process. First, the task was created in the web portal and reached green status in the task light system according to its technical quality. After the request for publication, the author received a feedback on the fulfillment of the MCM criteria. Through the picture, it fulfilled criterion 1. The measurements which have to be done on the object guarantee criteria 2 and 3. The area of the hexagon can be determined in multiple ways (criterion 4). Green light in the traffic light system guarantees that hints, school math and tags, as well as a sample solution (criteria 6, 7 and 10) are fulfilled. The task does not require special tools (criterion 9). Some information on the height of the bottom plate of the flowerpot had to be included (criterion 5 “Reality”). The answer type exact value initially defined was not adequate for measuring tasks as it does not allow multiple solutions and minor measuring deviations. Thus, the task needed further improvement (criterion 8 “Solution format”). After receiving feedback, the author was able to improve the task and can request for publication again.

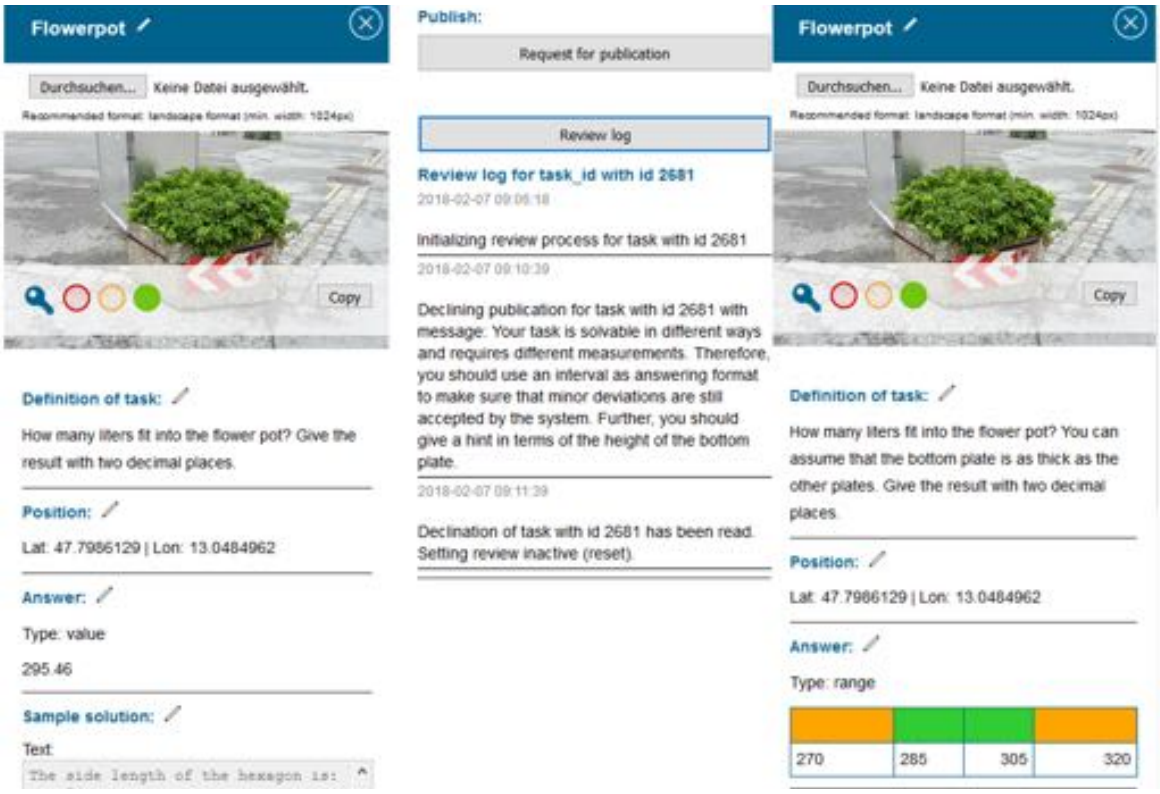

Figure 4: Example of a review process in the MCM web portal

Especially in regard of measuring tasks and solution formats, we can often observe problems with the definition of an adequate, didactical reflected interval. For example, a task asks for the determination of the weight of gravel which is needed to fill a circular area. The author created the task in respect of all criteria, even with an interval as solution format. Nevertheless, in the review process, the task expert did some errors in calculations with marginal measuring differences. He came to the conclusion that the interval might be too to let the students use different methods and to accept minor measuring errors. Therefore, he rejected the task with the hint to improve the interval. The author improved the task and afterwards it could be published. Thanks to the feedback of the task author, we can see in a qualitative way that this kind of feedback helped the teacher improve the task in terms of an authentic outdoor task:

> Thank you! For publishing the task and especially for the hints and corrections. […] Especially helpful was the hint in terms of the interval areas for the circular area […] (quoted from an e-mail; translated into English).

Apart from the benefits of a review system, users and reviewers face several consequences. As the statistics show, more than a half of all created tasks have not been published. We see two possible reasons. First, not all task creators want to publish or see a benefit in publishing and sharing own material. They regard MCM as a tool that they use for their individual needs. Second, the review process may hinder authors to revise their task after it was rejected. Even though our statistics and experience show that some authors are willing to revise their tasks up to three times and are thankful for the received feedback, we observe that 130 out of 743 review processes (17,5%) were interrupted after feedback was given. This observation is a small limitation of the MCM web portal in terms of public quantity, which can be accepted in advantage of the increased quality of published tasks. Nevertheless, the review system is a developing process to be optimized with the users' feedback and needs.

The whole process is a new situation for teachers through a definition of new roles. The teacher is not only the task user in this process, but also the task designer who asks for feedback. Through our dialogue-based review system, a potential contribution towards professional development can be realized (Jones & Pepin, 2016).

DISCUSSION

The paper gives an overview of the educational web portal of MCM, in which a growing community participates in creating and sharing material. For this purpose and in terms of the idea of authentic math trail tasks, a certain quality of the material must be guaranteed before it can be published. In this context, a review process is introduced and presented in terms of steps and fundamental criteria that allow transparency. The implementation shows that the review system can prove itself not only theoretically. Through an adequate and transparent communication of the underlying steps and criteria, users are willing to improve their tasks according to the standards. Especially for the "Solution Formats" criterion, we observe major improvements leading to adequate outdoor tasks. We also observe that less than every fifth task review does not end in a published task. The MCM review process is thus a successful example of quality standards in a digital educational community platform. However, such a system faces consequences, mostly in terms of quantity of tasks and the higher expenditure for users and reviewers. A possible future development could be a point-based system similar to StackOverflow, where users get points for good tasks and can also review as they have proven their experience within the system.

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The KOM framework's aids and tools competency in relation to digital technologies – a networking of theories perspective

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This paper focuses on the KOM framework for mathematical competencies and in particular its aids and tools competency and investigates its application to digital technologies used for mathematical learning. Very little has been done on networking the KOM's mathematics competencies framework to internationally established theories and theoretical constructs. Through an analysis of an authentic example involving digital technologies for teaching slope fields, we compare and combine the aids and tools competency with the approach of instrumental genesis and the notion of scheme. We claim that the networking of these theoretical perspectives facilitates our understanding of mathematical competencies in the digital age.

Keywords: competency, digital technology, instrumental genesis, scheme.

INTRODUCTION

Mathematics Education, in its 50-year (or so) life, has been focusing on developing learning theories and deriving teaching approaches and pedagogies that promote students' mathematical thinking and competencies (among various other crucial issues regarding mathematical learning). Living in the digital age, students are influenced by digital technologies used for mathematical learning and which are designed to aid their understanding and mathematical thinking (e.g. Noss & Hoyles, 1996; Monaghan, Trouche & Borwein, 2016). Students are challenged to use and apply their mathematical knowledge and competencies in their interactions with such digital technologies and resources (e.g. Geraniou & Jankvist, in review; Weigand, 2014). In this paper, we focus on how students' mathematical competencies relate to mathematical learning when using digital technologies. To achieve this, we use the Danish mathematics competencies framework, KOM (Niss & Jensen, 2002; Niss & Højgaard, 2011), which was adopted by OECD in the PISA framework (OECD, 2013; Lindenskov & Jankvist, 2013), and through a networking of theories approach, investigate how it compares to and combines with the approach of instrumental genesis (e.g. Trouche, 2005) embedding Vergnaud's (2009) notion of scheme.

Instrumental genesis involves the process of transforming digital tools into mathematical instruments, which become part of students' cognitive scheme (Vergnaud, 2009) and can be used to support students' learning of mathematical concepts (Artigue, 2002). We use an authentic example concerning the teaching of slope fields at the Danish master program of mathematics education to make our case. Namely that augmenting the KOM framework with the instrumental approach and its embedded heritage to the notion of scheme (Guin & Trouche, 1999) deepens our understanding of students' mathematical competencies in the digital era.

NETWORKING THEORIES

Ideas concerning *networking of theories* have been around for a decade or so (Prediger, Bikner-Ahsbahs & Arzarello, 2008). Prediger and colleagues (2008) introduce a "scale" of networking strategies stretching from "ignoring other theories" to "unifying globally" (p. 170). We locate ourselves in the spectrum between these two outer poles. In this spectrum are the strategies for *coordinating and combining*, i.e. strategies mostly used for a networked understanding of an empirical phenomenon or piece of data, and the strategies for *synthesizing and integrating* (*locally*), which is when "theoretical approaches are coordinated carefully and in a reflected way [that] goes beyond understanding a special empirical phenomenon" (Bikner-Ahsbahs & Prediger, 2010, p. 496). Thus, *integrating locally* refers to the problem at hand; in our case digital technologies in relation to mathematical competencies.

The framework of the instrumental approach has proven well-suited for networking with other theoretical constructs (Drijvers, Godino, Font & Trouche, 2013); an argument for choosing it in relation to the KOM framework.

THE KOM FRAMEWORK AND ITS AIDS AND TOOLS COMPETENCY

The mathematical competencies framework, referred to as KOM, was published in Denmark in 2002 (Niss & Jensen, 2002). Since then it has been influential in mathematics programs at practically all educational levels in Denmark; not least in primary and secondary school, upper secondary school, and teacher education.

Niss and Højgaard (2011) define a *mathematical competency* as (an individual's) "…well-informed readiness to act appropriately in situations involving a certain type of mathematical challenge" (p. 49). The KOM framework operates with eight distinct, yet mutually related, mathematical competencies: mathematical thinking; problem tackling; modelling; reasoning; representing; symbols and formalism; communication; aids and tools. Each of these competencies consist of a producing side and an analytical side. The *aids and tools competency*, "consists of, on the one hand, *having knowledge of* the existence and properties of the diverse forms of relevant tools used in mathematics and having an insight into their *possibilities and limitations* in different sorts of contexts, and, on the other hand, being able to reflectively *use* such aids" (pp. 68-69, italics in original). It continues:

> Mathematics has always made use of diverse technical aids, both to represent and maintain mathematical entities and phenomena, and to deal with them, e.g. in relation to measurements and calculations. This is not just a reference to ICT, i.e. calculators and computers (including arithmetic programmes, graphic programmes, computer algebra and spreadsheets), but also to tables, slide rules, abacuses, rulers, compasses, protractors, logarithmic and normal distribution paper, etc. The competency is about being able to deal with and relate to such aids. […] Since each of these aids involves one or more types of mathematical representation, the aids and tools competency is closely linked to the representing competency. Furthermore, since using certain aids often involves submitting

to rather definite "rules" and rests on particular mathematical assumptions, the aids and tools competency is also linked to the symbol and formalism competency. […] (p. 69)

The *representing competency* firstly comprises being able to understand, i.e. decode, interpret, and distinguish between, as well as utilize different representations of mathematical objects, phenomena, problems, or situations (including symbolic, algebraic, visual, geometric, graphic, diagrammatic, tabular, verbal, or material representations). Secondly, it includes being able to understand the mutual relations between different representational forms of the same object, knowing about their strengths and weaknesses, and being able to choose and switch between them in given situations. The *symbols and formalism competency* deals with the ability to decode symbolic and formal language, translate back and forth between mathematical symbolism and natural language, and the ability to handle and utilize mathematical symbolism, including transforming symbolic expressions. Furthermore, it involves having an insight into the nature of the rules of formal mathematical systems and it focuses on the nature, role and meaning of symbols.

The prevalence of and the role that digital technologies play in the mathematics programs at all educational levels in Denmark in 2018 is significantly different than it was in 2002 when KOM was launched. Dynamic Geometry Software (DGS), such as *GeoGebra*, is extensively used in both primary and secondary level. Computer Algebra Systems (CAS), such as *Maple*, *TI-Nspire*, *WordMath*, etc., are an integral part of the upper secondary school mathematics programs—even mandatory at the final national written assessments. In relation to the various mathematics programs' reliance on the KOM framework and the escalated situation concerning digital technologies, there seems to be a need for providing a deepening of digital technology aspects of KOM's competencies descriptions and in particular the aids and tools competency. And this is not only from a practice perspective, but also from the perspective of doing research related to the use of technology in the mathematics programs of the Danish educational system—or any other educational system relying on competencies descriptions of mathematics.

THE INSTRUMENTAL APPROACH AND THE NOTION OF SCHEME

Drijvers et al. (2013) present the instrumental approach in terms of three dualities.

Firstly, the *artefact-instrument duality* describes the lengthy process of an artefact becoming an instrument in the hands of a user, which is referred to as *instrumental genesis*.

Secondly, the *instrumentation-instrumentalisation duality* concerns the relationship between the artefact and the user, i.e. how the user's knowledge directs the use of an artefact (instrumentalisation), and how a tool can shape and affect the user's thinking and actions (instrumentation). The process of instrumentation is closely connected to the digital tool serving an *epistemic purpose*, which means that it is used to create understanding or support learning within the user's cognitive system. By contrast, when a digital tool is to create a difference in the world external to the user, it is said to serve a *pragmatic purpose* (Artigue, 2002; Lagrange, 2005; Trouche, 2005). Digital tools serve of course both pragmatic and epistemic purposes, but any use which is only, or mainly, pragmatic is according to Artigue (2010) of little—or even negative—educational value.

Thirdly, the *scheme-technique duality* concerns "the relationships between thinking and gesture" (Drijvers et al., p. 26). From a practical perspective, techniques can be seen as "the observable part of the students' work on solving a given type of tasks (i.e. a set of organized gestures) and schemes as the cognitive foundations of these techniques that are not directly observable, but can be inferred from the regularities and patterns in students' activities" (ibid, p. 27). For Vergnaud (2009), concepts are psychological entities fundamentally related to actions. Vergnaud refers to this relation as a *scheme*, and it can be defined as implicit or explicit ways of organising behaviour, involving also the necessary knowledge to act meaningfully in certain situations. Hence, a scheme combines intentions and actions with conceptual knowledge. Furthermore, schemes enable us to understand the conceptualisation process by linking gestures and thoughts through the encountering of various situations. Conceptualisation here refers to the process in which learners develop concepts and make connections in their knowledge. Drijvers et al. (2013) define a scheme as "a more or less stable way to deal with specific situations or tasks, guided by developing knowledge" (p. 27). These three dualities can be used as analytical constructs in exploring how the use of artefacts, such as digital tools, can shape the learning (and teaching) of mathematics (e.g. Geraniou & Jankvist, in review).

AN AUTHENTIC EXAMPLE OF TWO DIGITAL APPROACHES

The first and third authors have both taught slope fields to the mathematics education students at the Aarhus University. Although the approach to teaching this topic has changed over the years, the aids and tools competency in connection to the representing and symbol and formalism competencies have always been in focus.

In 2009 and 2011 the approach was on learning how to programme a computer to create slope fields of simple differential equations. This was done both with the free CAS *Wiris* and with the DGS *GeoGebra*. The activities were based in the constructionist ideas of Papert (1980), i.e. assigning a particular pedagogical value to the development of one's own mathematical tools. In 2013, 2015 and 2017 the approach was changed to use *Wolfram Alpha* instead to simply call commands that plot the slope fields. The reason for this change is multi-faceted, but one of the main problems experienced with the first approach was that it simply was too much work and effort to create the string of code required to plot a slope field. The amount of knowledge about loops/sequences, and about how to plot vectors in a lattice that are needed in order to develop one's own slope field plot with tools like *Wiris* or *GeoGebra*, did not seem to be worthwhile. Rather it—in this specific case—moved the students' focus away from the numerical solutions of differential equations.

One example showcasing the differences between the two approaches is how to create a slope field showing solutions to the equation: $\frac{dy}{dx} = \sin(x)\sin(y)$ (figure 1).

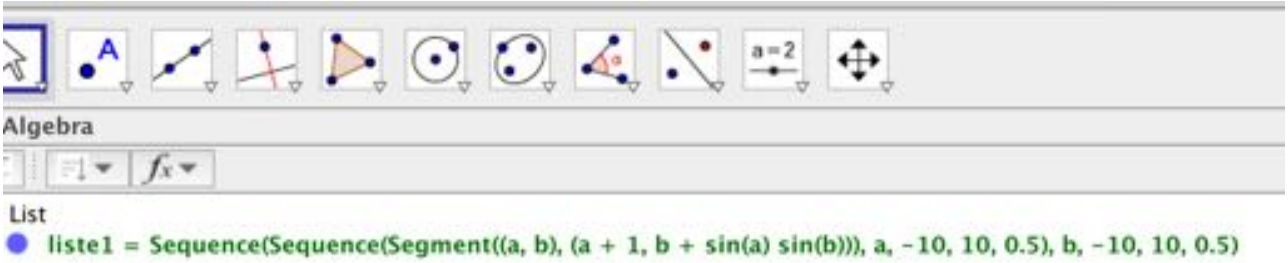

Figure 1. The command line in *GeoGebra* for a 'home made' slope field.

The resulting slope field looked rather coarse. On the contrary, if the differential equation is typed into *Wolfram Alpha*, you immediately get the stepwise solution as well as an illustration of the solution curves. In this case the image from *Wolfram Alpha* is more illustrative and detailed (see figure 2).

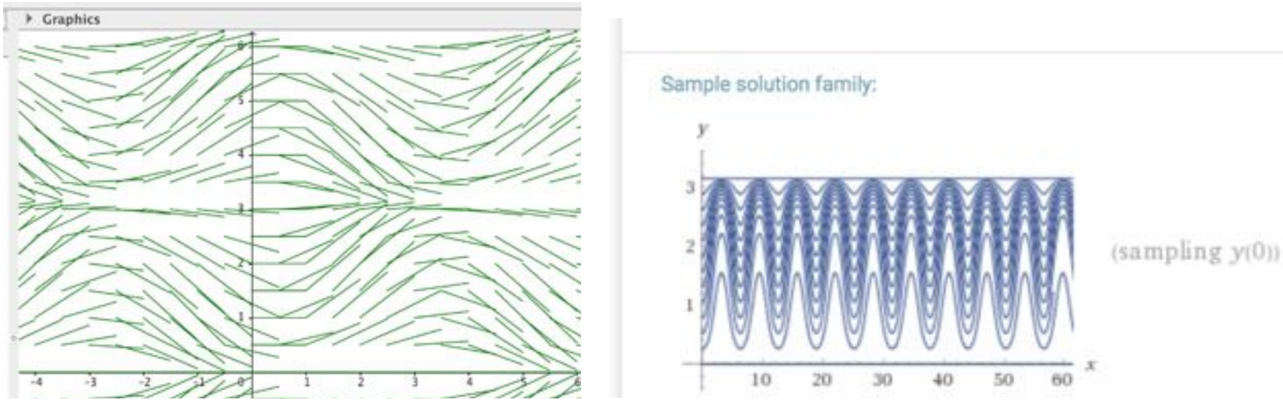

Figure 2. Coarse 'home made' *GeoGebra* slope field vs. smooth *Wolfram Alpha* plot.

ANALYSIS 1: THE COMPETENCIES FRAMEWORK

From a competencies perspective, the example above calls for students to apply their *aids and tools competency* to, firstly, know that there exists digital tools for constructing slope fields and that this can be beneficial in cases where differential equations cannot be solved analytically. Secondly, the aids and tools competency may come in play if students are to choose between the two different approaches laid out above, i.e. is it more beneficial, also from a learning point of view, to program your own plotter, or is it perfectly fine to use the already made app, e.g. that of *Wolfram Alpha*, knowing that it will black-box several of the underlying processes? The students also need to apply their *symbol and formalism competency* when having to use the notation and language of the digital tools—and when translating back and forth between the usual mathematical notation and this—to apply the slope field plotter (in both approaches). In a similar manner the students will need to activate—and it also results in a development of their—*representing competency* when interpreting the slope field plots of the two approaches. In this respect, the students must also be able to know what the strengths and weaknesses are of the different representations of the (family of) solutions to the differential equation in question.

ANALYSIS 2: INSTRUMENTAL APPROACH AND NOTION OF SCHEME

In terms of the *artefact-instrument duality*, we see that the choice of technology for visualising slope fields has consequences reaching further than just this specific task and topic. The resulting instrumental genesis leads to students' familiarity and control over the tool they use. And even though both tools are relevant from a mathematics education point of view, they are very different, and familiarity with each of these tools might influence further the learning of mathematics. The *instrumentation-instrumentalisation duality* as well as the *scheme-technique duality* can be used to look at the details in these differences. The case where *GeoGebra* is used to create slope fields clearly brings the tool to a use that might not be directly intended by the creators of *GeoGebra* and clearly pushes the software a little bit out of the usual scenario (for instance by creating a lattice as a sequence of sequences in order to place a line segment at each lattice point). This means, on the one hand, that the students will be required to instrumentalise *GeoGebra* and take control over it (and this can obviously benefit their future ability to use *GeoGebra*). On the other hand, the focus of the work with *GeoGebra* is on creating a lattice, and perhaps on controlling the length of the line elements (they can get very large or small—making the image incomprehensible). Hence, this use of *GeoGebra* is instrumentalising the students to focus more on the procedure of creating the slope field (deciding on a lattice, and "programming" a procedure for setting line segments from each point) than on the actual layout of the slope field. The instrumented techniques obtained might focus on a number of technical concerns that are of little relevance to understanding the involved mathematics, which might (this is a hypothetical analysis) pollute the students' scheme of differential equations and (numerical) solutions to such. Hence, in this case, the instructor had to pay special attention to bring in play the students' schemes of numerical and analytical solutions to differential equations in relation to the slope field plot.

The work with *Wolfram Alpha*, however, is focussed directly on the visuals of the slope fields, black-boxing everything leading to this image. Furthermore, the case of working with a differential equation and visualising the family of solutions does seem to be considered by the developers of *Wolfram Alpha*. Writing the differential equation into the system automatically gives access to the solution (including—in the premium version—a stepwise solution replicating a paper-and-pencil solution) as well as relevant visualisations of families of solutions. The students' instrumentation of *Wolfram Alpha* is thus almost salient. The instrumentalisation might go in different directions depending on the focus of the teaching and the abilities and preferences of the students. *Wolfram Alpha* allows for the development of a completely black-boxed trial and error technique, where the student simply tries various commands in the command field and sees if the input is somehow interpretable with regard to the task at hand. Such a technique might not lead to the development of a strong and relevant scheme for differential equations (for a related case, see Jankvist and Misfeldt, 2015). However, the tool allows students to investigate and explore mathematics without the technical barriers that were experienced when programming in *GeoGebra*. This may lead students more directly to consider families of solutions to differential equations, which should force them to activate their schemes related to e.g. what it means to be a solution to a differential equation as well as, say, the difference between numerical and analytic solutions to differential equations.

CONCLUDING REMARKS

As can be seen from the above analyses, the KOM framework offers a rather limited analysis in relation to the aids and tools competency, even when taking representing and symbols and formalism into account. It does, however, articulate students' needs to know about the digital tools' strengths and weaknesses, also in relation to specific mathematical representations, which the instrumental approach does not do explicitly. But the instrumental approach focusses more directly on the students' interactions with the digital tools, rather than merely addressing students' knowledge about these. The embedded notion of scheme, enables us to say something about the students' conceptual understanding, in this case in relation to differential equations and solutions of such. Hence, from our perspective, the networking of these perspectives appears both feasible and quite promising in relation to looking deeper into students' possession and development of mathematical competencies in the digital era—which we intend to do in our future research.

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A student's perception of CAS-related sociomathematical norms surrounding teacher change in the classroom

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In this paper, we investigate the use of CAS in Danish upper secondary school from a norms perspective. The construct of sociomathematical norms is used to understand the potential influence that CAS may have on the rules (and values) in the mathematics classroom. We focus on a situation of teacher change, where the different teachers enact different norm sets related to students' CAS use in their mathematical work. We zoom in on one student's perception of this, and find that the teachers' CAS policies have a direct influence on how students' navigate in their mathematical work. In particular, the alignment between teachers' endorsed and enacted norms as well as the extent to which the norms impose judgement on the students' behalf in relation to CAS use play a major role in this respect.

Keywords: CAS, CAS policy, socio-mathematical norms, teacher change.

INTRODUCTION

Computer Algebra Systems (CAS) play an important role in Danish upper secondary school, as is the case in several other countries. It has been argued for more than 15 years that the challenges posed by CAS to the organization of mathematics teaching is not one that has been called for by the school topic itself, but rather a consequence of the development of technologies for non-didactical purposes (Trouche, 2005). The didactical difficulties and possibilities of CAS use have been widely documented (e.g. Hoyles & Lagrange, 2010; Jankvist & Misfeldt, 2015; Weigand, 2014), but teachers are largely left to develop their own ideas and understanding of how to use CAS in teaching (e.g. Jankvist, Misfeldt & Marcussen, 2016). Hence, different teachers—at least in Denmark—work with these tools in different ways, enacting different values and norms about for example the subject of mathematics, its teaching, and the use of technology, and not least the relation between these.

In this paper, we ask how students experience and make sense of various teachers' different views on the role of CAS. In order to see this phenomenon more clearly, we investigate how one student—Emil—experienced and tried to navigate between various teachers' different CAS-related norms. More precisely, from a teacher focusing on the students' ability to work with paper and pencil to another teacher focusing on correct and efficient CAS use, and again to a third teacher who believed that the students should decide themselves when and when not to use CAS. We apply the term *CAS policy* to articulate teachers' expectations about students' CAS use in their mathematical work. We use the construct of sociomathematical norms to understand the influence of such CAS policies on students' learning and their possibilities to participate in the mathematical activities.

SOCIOMATHEMATICAL NORMS

Sociomathematical norms were observed and named by Yackel and Cobb (1996), who in a teaching situation noticed that aspects which could neither be described as purely mathematical norms nor purely as classroom social norms were in play. Yackel and Cobb defined sociomathematical norms as "normative aspects of mathematical discussions specific to students' mathematical activity" and describe the difference to social norms as:

> The understanding that students are expected to explain their solutions and their ways of thinking is a social norm, whereas the understanding of what counts as an acceptable mathematical explanation is a sociomathematical norm. Likewise, the understanding that when discussing a problem students should offer solutions different from those already contributed is a social norm, whereas the understanding of what constitutes mathematical difference is a sociomathematical norm. (Yackel & Cobb, 1996, p. 461)

Sociomathematical norms are negotiated between the students and the teacher, and may thus vary from classroom to classroom. This negotiation builds on already "taken-as-shared" perceptions within the classroom, and as such they are:

> … intrinsic aspects of the classroom's mathematical microculture. Nevertheless, although they are specific to mathematics, they cut across areas of mathematical content by dealing with mathematical qualities of solutions, such as their similarities and differences, sophistication, and efficiency. Additionally, they encompass ways of judging what counts as an acceptable mathematical explanation. (Yackel & Cobb, 1996, p. 474)

In the study described by Yackel and Cobb, a sociomathematical norm is negotiated where an acceptable mathematical explanation must describe actions performed on mathematical objects. Hence, explanations and justifications are themselves made the objects of reflection. According to Levenson, Tirosh and Tsamir (2009), there are three kinds—or aspects of—sociomathematical norms that should be taken into account: *teachers' endorsed norms*; *teachers' and students' enacted norms*; and *students' perceived norms*. Based on classroom studies, Levenson et al. noticed that even when the observed enacted norms were in agreement with the teachers' endorsed norms, students may not have the same perception of these norms.

EDUCATIONAL SETTING OF THE CASE STUDY

Denmark has three different types of upper secondary school programs: the classical stream, the technical stream, and the business stream. Danish upper secondary school is usually three years, and students may take mathematics at one of three levels (C, B or A), depending on the number of years they follow it, e.g. A-level is mathematics for all three years. CAS were introduced into the upper secondary streams in 2005 and is now a mandatory part of the national written assessments. For the technical stream, the ministerial orders for mathematics A-level state:

> The student works with CAS tools and other mathematical software, so that the student becomes familiar with syntax, terminology and application of at least one mathematical software. Over the course of the program, the digital tools may be increasingly applied for:

modelling; visualizations; geometric investigations; repetitive calculations; complex symbolic manipulations and calculations; numerical calculations; documentation and communication of results. (UVM, 2013)

In some Danish upper secondary schools, CAS are not introduced until after Christmas in the first year, i.e. the first semester follows a more traditional paper-and-pencil approach, while other schools provide students with a license to a given CAS-tool from day one. Even if schools do offer a CAS license from day one, individual teachers may still choose to wait until later in the first year. Furthermore, *how* exactly CAS is then introduced and used is often left entirely up to the individual teachers, subject of course to equally independent decisions and discourses of a given textbook system.

Our case stems from an A-level mathematics class at the technical stream; one which experienced four different mathematics teachers within their three years. We focus on the student Emil (see also Iversen, 2014; Iversen, Misfeldt & Jankvist, accepted) and his perception of the various teachers' CAS policies. Our case description and analysis solely builds on an interview with Emil and excerpts from his hand-in assignments, corrected and commented by the different teachers.

CASE: THE UPPER SECONDARY SCHOOL STUDENT EMIL

The first of Emil's four teachers, Teacher 0, was only in the class for a very short time at the beginning of the first year, for the reason of which we do not—and neither did our case student, Emil—take her influence on the class into account. Emil provided the following description of the Teachers 1, 2, and 3, and their approaches to the use of CAS (all quotations are translated from Danish):

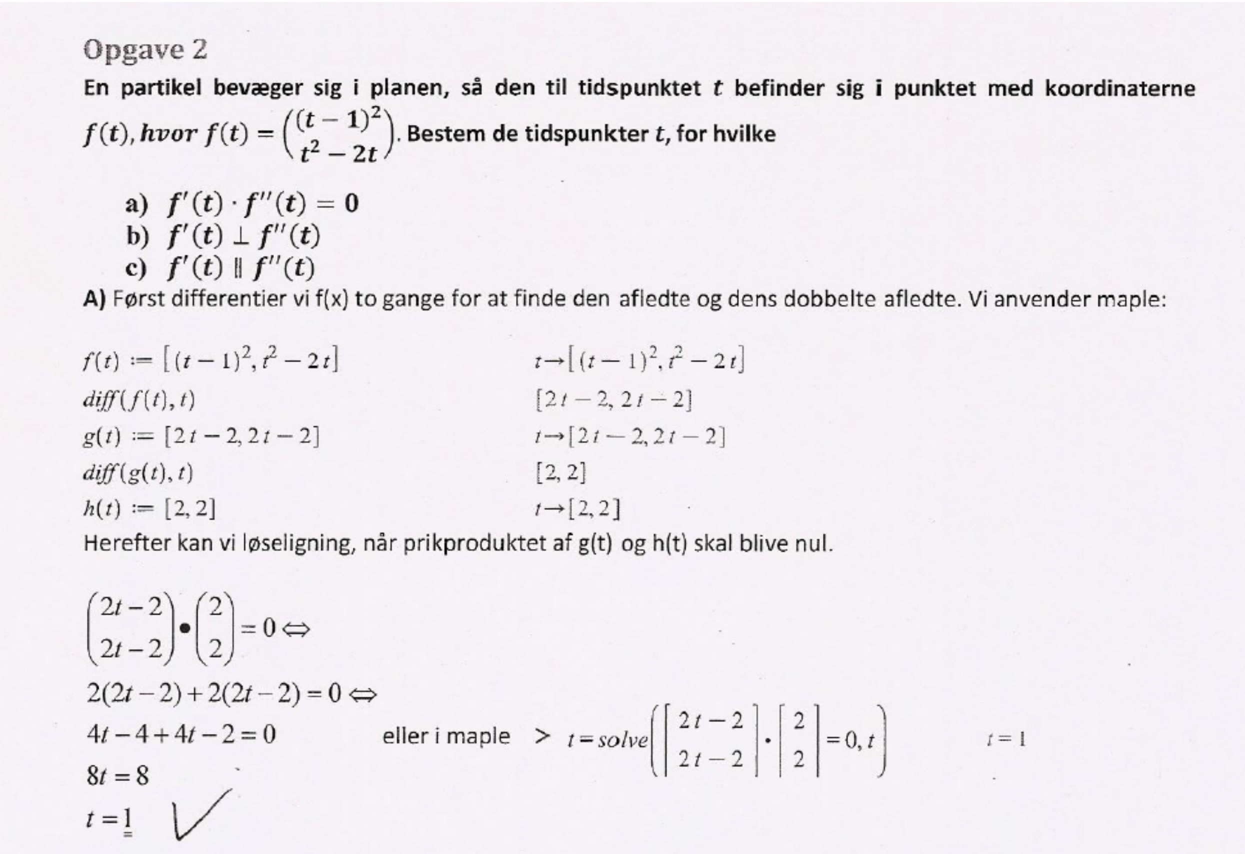
> Emil: The first teacher [1] made it very clear that the purpose of using sketches was that the teacher/reader should be able to see what was going on in our minds. He "punked" us about that; lots of sketches, and they had to be good, so he could follow our way of thinking. The next teacher [2] talked a lot about us using our tool in a correct manner; that now we were past the point, where we had to explain everything; that now we had to use it [i.e. CAS] and see that we can come from A to B faster; and that we had to *solve* tasks. The third teacher [3] has been a bit of a mix, saying we had to use the tools more limitedly, and that she also wanted us to be able to *do* some mathematics. (Emil, May 9, 2012)

Neither Teacher 0 nor Teacher 1 spend time introducing CAS to the students. Teacher 2 however did, more precisely to *Maple*. Emil explained:

> Emil: Yes, almost immediately we got this new software, *Maple*, for the computer. Then we spent a few lessons learning the basics about it. You can say that *Maple* can do more compared to the handheld calculator, because when we deal with stuff like, for example, rotation around a fixed axis, the calculator can't sketch this. The big change from the first teacher [1] to the second teacher [2] was going from doing everything by hand to having to solve everything on the computer now.

Iversen:	Everything? Like the hand-in assignments, or also during lessons or what?
Emil:	I would go so far as to say *everything*. […]
Emil:	So, while with the first [Teacher 1] it was like, say, if we had to isolate in relation to something, then we had to do it by hand, then [with Teacher 2] it was just “Solve” [in CAS]. You still had to write down a little about what you were doing, but you didn’t need the long steps of calculations, you could just use the computer now. (Emil, Sept. 9, 2011)

Figure 1 illustrates a hand-in assignment by Emil at the time of the teacher change to Teacher 2, i.e. at the beginning of the second year (August, 2010).


Opgave 2

En partikel bevæger sig i planen, så den til tidspunktet t befinder sig i punktet med koordinaterne $f(t)$, hvor $f(t) = \begin{pmatrix}(t-1)^2\\ t^2-2t\end{pmatrix}$. Bestem de tidspunkter t, for hvilke

a) $f'(t)\cdot f''(t) = 0$
b) $f'(t) \perp f''(t)$
c) $f'(t) \parallel f''(t)$

A) Først differentier vi f(x) to gange for at finde den afledte og dens dobbelte afledte. Vi anvender maple:

$f(t) := [(t-1)^2, t^2-2t]$	$t \rightarrow [(t-1)^2, t^2-2t]$
$diff(f(t), t)$	$[2t-2, 2t-2]$
$g(t) := [2t-2, 2t-2]$	$t \rightarrow [2t-2, 2t-2]$
$diff(g(t), t)$	$[2, 2]$
$h(t) := [2, 2]$	$t \rightarrow [2, 2]$

Herefter kan vi løseligning, når prikproduktet af g(t) og h(t) skal blive nul.

$$\begin{pmatrix}2t-2\\2t-2\end{pmatrix}\bullet\begin{pmatrix}2\\2\end{pmatrix} = 0 \Leftrightarrow$$
$$2(2t-2)+2(2t-2)=0 \Leftrightarrow$$
$$4t-4+4t-2=0$$

eller i maple $> t = solve\left(\begin{bmatrix}2t-2\\2t-2\end{bmatrix}\cdot\begin{bmatrix}2\\2\end{bmatrix}=0, t\right)$ $t=1$

$$8t=8$$
$$t=1$$

Figure 1: Homework assignment from Emil’s second year of upper secondary school. Notice the text at the bottom “eller i maple” meaning “or in *Maple*”.

The task reads: “A particle moves in the plane, so that it at time t is located in the point with coordinates $f(t)$, where $f(t)$=… Find those points of time t for which a)...; b)...; c)...” We are interested in Emil’s answer to question a), in which he used that the dot product of the functions $f'(t)$ and $f''(t)$ should be zero. Emil answered question a) of the task by first doing a rather long and—at least at this level of education—somewhat complex calculation (bottom, left hand side) and immediately after solving the same task with one line of *Maple* code. What is interesting is his need for including both solutions as his answer to the question. Email elaborated:

Iversen: Okay, so what did you think about the shift from having to do everything by hand and then having to use... [CAS]?

Emil: It was very intense. It felt much easier to get the computer to do it. You saved a lot of time. In a way you felt that you had been ridiculed; that you had to do so much with the first teacher [1], and then this intense shift… but the good thing always was—not to jump to the third teacher [3] [...]—yes, because with the first teacher [1], you always knew what to do. There was no question that you had to do everything by hand. With the second teacher [2] I knew that I had to do as much as possible by means of the computer [...] he didn't want to see the intermediate results. (Emil, Sept. 9, 2011)

As evident from the above, once Emil had figured out the CAS policy of Teacher 2, it became equally clear to him what was expected as it was previously with Teacher 1. But as hinted to with the saying of "not to jump to the third teacher…", something about Teacher 3's CAS policy was less clear to him:

Iversen: If we take the thing with *Maple* [...] you've really taken this to heart and used it a lot. In the hand-in assignments I've looked at, there are a couple of times where she [Teacher 3] comments something like "phew", when there are long expressions or long commands in *Maple* or something like that...

Emil: I also got some saying "I think I can almost follow what is going on" [...]

Iversen: Yes, and what exactly do you think about that?

Email: I know that my teacher [3] isn't very… well, I wouldn't call her old fashioned. That would be wrong. Maybe she just isn't very fond of using *Maple*. [...]

Iversen: What do you think about the impression it makes on a reader, for example, that your mathematical texts include these long command sequences? [...]

Emil: My mom thinks it looks advanced...

Iversen: So, not to over interpret, but could it be somewhat the same with your teacher [3], when she writes "phew" and "I think I can follow it" and so on?

Emil: It could be that it may seem a bit "overkill"—or with a Danish saying; "to shoot sparrows with cannons"—to use *Maple*. (Emil, May 9, 2012)

From lower secondary school, Emil was used to a traditional paper-and-pencil approach to the teaching and learning of mathematics. The first "shift" for Emil concerned having to hand in all mathematics assignments in electronic form in upper secondary school—a rule installed by Teacher 0. With Teacher 1, Emil again experienced a traditional approach, since the use of technology was limited to a text editor, e.g. *MS Word* and its equation functionalities, and software to draw sketches, e.g. *Graph*. With Teacher 1, the CAS functionalities of the handheld TI-89 were only to be used to check results obtained by paper-and-pencil methods.

With Teacher 2, however, Emil experienced a change in how to apply CAS in the hand-in assignments. Teacher 2 did not care about intermediate steps and detailed calculations. Rather he endorsed (or even required) that everything had to be done on the computer, and he considered correct use of *Maple* as a key competence for the students. Hence, the focus for evaluation of the students' written products changed from a focus on providing detailed algebraic calculations, to a focus on how to address the problem with the tool and resources in an efficient and correct manner. According to Emil, Teacher 2 expressed that the students should not explain every little detail of the calculations but instead focus on correct application of *Maple*.

Teacher 3 had a more liberal, balanced and almost unengaged approach to the use of CAS. Students could choose to use CAS, when they wanted to and when they could argue for its meaningfulness. Emil described this as a "mix" of the two previous teachers' policies, and he also suggested that Teacher 3 did not possess a deep knowledge of advanced CAS tools, and that she often considered it to be "overkill" to apply such tools. Even though this teacher had a liberal approach to CAS, we can see from her comments to some of Emil's written assignments that she did consider it interesting whether certain algebraic steps had been conducted with or without the aid of CAS (Iversen et al., accepted).

EMIL'S EXPERIENCE OF ENDORSED AND ENACTED NORMS

Overall, the norms, rules and regulations around CAS changed from suppressing CAS arguments (Teacher 1) to endorsing such arguments (Teacher 2), to a more balanced approach suggesting openness about whether or not CAS should be applied in a given situation, and a demand for active argumentation on the students' behalf as to why they used CAS for a given task (Teacher 3).

Emil experienced the sociomathematical norms endorsed by Teacher 1 as aligned with the enacted norms of Teacher 1. Hence, at the time of Teacher 1 there was some coherence between Emil's perceived norms and his enacted norms in relation to CAS use. However, a discrepancy occurred between Emil's experience of the endorsed norms of Teacher 1 and Teacher 2, since they appeared to have rather different views on the role of CAS in the teaching and learning of mathematics. Both Teacher 1 and Teacher 2, respectively, appeared to have their own endorsed and enacted norms aligned. For Emil, however, the new norm set of Teacher 2 challenged his perceived norms due to the teaching of Teacher 1. We see this from his hand-in assignment (cf. figure 1) at the time when Teacher 2 had recently taken over the class. Here Emil provided solutions that potentially could satisfy the endorsed and enacted norms of Teacher 1 as well as those of Teacher 2. While Teacher 1 emphasized cognitive strategies and algebraic skills, Teacher 2 was much more focused on efficient problem solving and correct use of *Maple*. Furthermore, the overall problem solving approach was highly valued by Teacher 2 who was less focused on detailed aspects of computation and argumentation.

In time, Emil aligned the problem solving in his assignments to the new norm set of Teacher 2. In fact, Emil expressed that although the shift from Teacher 1 to Teacher 2 was very “intense” in relation to the use and role of CAS, it was still clear to him what was expected of him—both from Teacher 1 and from Teacher 2. This, we believe, has to do with Emil’s experience of alignment between the endorsed and the enacted norms of Teacher 1 and Teacher 2, respectively. This appeared to make it easier for Emil to align his own perceived norms with his enacted norms.

Teacher 3 endorsed yet a new set of norms, which entailed that CAS should be used *when it makes sense* to use it in a given mathematical situation. As far as we can tell Teacher 3 also enacted norms according to this (Iversen, 2014). Still, Emil apparently found it difficult to decode the norm set of Teacher 3. But why is this, when an alignment of the endorsed and enacted norms of Teacher 1 and Teacher 2, respectively, seemed to make this easier for Emil? It appears that such an alignment may be thought of as a necessary although not sufficient condition. Hence, it makes sense to ask why the sociomathematical norms that Teacher 3 brought into the classroom were so difficult for Emil—a high-performing mathematics student—to perceive? One explanation may be that the norm sets of Teachers 1 and 2 in a sense were rather binary—never use CAS and always use CAS—while the norm set of Teacher 3 imposed upon the students to make an actual judgement of when it is needed and when it is not needed to use CAS. Performing such judgement is surely more demanding on the students and requires them to develop competences to do so. Eventually Emil did seem to make sense of Teacher 3’s sociomathematical norms as indicated by his statement that she did not appreciate when you “shoot sparrows with cannons”—which is Danish for “take not a musket to kill a butterfly”—meaning of course that there is no need to use a powerful CAS tool to do something which you could equally easy, or maybe even easier, do by hand.

FINAL REMARKS ON EXPERIENCED CAS POLICIES

We have witnessed how an upper secondary student had to learn to navigate between different teachers’ varying CAS-related norms. In particular, the alignment between teachers’ endorsed and enacted norms as well as the extent to which the norms imposed judgement on the students’ behalf played a major role in this respect.

Due to changing ministerial orders in relation to CAS use in Danish upper secondary school, and due to lack of alignment between these orders, textbook writes’ interpretations of the orders, local school policies and teachers’ own policies, students are bound to “feel like a fish out of water”. In order to better understand this problem of teachers’ CAS policies we applied the construct of sociomathematical norms. We find this construct to have been productive, not least due to the explanatory power of considering (students’ experiences of) teachers’ endorsed and enacted norms and the students’ own enacted and perceived norms. Furthermore, established theoretical constructs, such as that of sociomathematical norms, make up relevant lenses to understand how different teachers’ different CAS policies shape the experience of students’ participation in the classroom, not least in a situation of teacher change.

Finally, we propose that in order to account more deeply for situations regarding teachers' CAS policies, the use of sociomathematical norms could be augmented with theoretical constructs considering teachers' mathematics-related values and beliefs related to technology. Such theoretical *bricolage* considerations shall be part of our future endeavours.

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Conceptualization of function as covariation through the use of learning trajectories

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This study aims to identify levels of 11th grade students' conceptualization of function as covariation. The students worked with modelling tasks involving the use of the digital environment Casyopée which combines algebraic and geometrical representations of functions. The results indicated six hierarchical levels of thinking about function as covariation though the use of learning trajectories.

Keywords: function, covariation, learning trajectories, modelling tasks, Casyopée.

THEORETICAL FRAMEWORK

This paper reports classroom based research aiming to identify levels of 11th grade students' conceptualization of function as covariation. The students were engaged in modelling realistic problems through the use of the digital environment Casyopée, which involves interconnected representations and allows the manipulation of covarying quantities and the treatment of the corresponding functions.

The notion of function plays a predominant role in secondary education and can be conceived in two ways: (a) as a correspondence of two variables and (b) as a covariation, which is related to the understanding of the way in which dependent and independent variables change as well as to the coordination between these changes (Carlson et al., 2002). Recent studies connect directly the definition of function with the idea of covariation:

> A function, covariationally, is the conception of two quantities varying simultaneously such that there is an invariant relationship between their values that has the property that, in the person's conception, every value of one quantity determines exactly one value of the other (Thompson & Carlson, 2017, p. 444).

This definition is based on the person's conceptualization of function as covariation emphasizing the ways by which two quantities (corresponding to dependent and independent variables) covary in relation to each other. Carlson et al. (2002) studied the development of undergraduate student's thinking about the covariation of quantities in dynamic situations such as filling a bottle with water. Their research results led to a framework of five levels of covariation which were described through corresponding mental processes: *Dependence* (observation of changes in the two variables); *Directed change* (increase or decrease - with changes of the other); *Quantitative correlation* (coordination of the amount change of a variable with changes of the other); *Average rate* (correlation of the mean rate of change with uniform increases of the independent variable); and *Instantaneous rate of change* (correlation of the instantaneous rate with continuous increases of the independent variable). Therefore, the concept of covariation links functional relationships to the rate of change and is essential for understanding the fundamental concepts of advanced mathematics. The present study aims to contribute to the research literature related to students' understanding about function as covariation at the upper secondary education level since research at this level is rather limited.

Modelling tasks involving the use of digital tools have been indicated as providing rich opportunities for students to engage in functional thinking and therefore to interpret function as covariation (Lagrange & Psycharis, 2014; Psycharis, 2015). Lagrange (2014) describes the process of modelling a problem in Casyopée through a "modelling cycle" that includes four settings: (a) a physical object (e.g., paper), allowing students to experiment; (b) the dynamic figure resulting by modelling the dependencies in a digital tool (e.g., a dynamic rectangle in a Dynamic Geometry window); (c) the covarying magnitudes (e.g., the side length and the area of the rectangle); (d) the algebraic functions that model problem. In this approach, students' transition from experimentation with the physical object (quantities) to working with functions (variables) is mediated by working with covarying magnitudes and measurements, through the use of multiple representations such as algebraic notation, graphs and tables (Lagrange, 2014). In this study, we use realistic problems and specially designed digital tools for designing modelling activities to encourage students' transitions in the different settings of the modelling cycle.

Another strand of research that influenced this study concerns *learning trajectories* (Clements & Sarama, 2009), which include three essential elements: (a) a mathematical goal, (b) educational activities to achieve the goal, and (c) a description of the development of students' thinking as they are engaged with the activities. In this paper, we use the idea of trajectories to describe the progression of students' thinking about function as covariation. The trajectories define different layers of thinking from simple to more complex understandings. However, these levels do not indicate a unique sequence of stages from which all students pass in the same way. Students can move to different levels in both directions as their learning progresses depending on the difficulties they face (Clements & Sarama, 2014). Furthermore, in order to study the role of context and available resources in the learning process, we consider construction of knowledge about function as covariation as an abstraction process and we use the *Abstraction in Context* theory (AiC, Hershkowitz et al., 2001). According to AiC, the construction of mathematical knowledge in a specific context takes place through three epistemic actions: (a) *recognizing* a previous construction as relevant to the situation; (b) *building-with*: rebuilding existing knowledge to achieve a localized goal (e.g., the solution of the problem); and (c) *constructing* a new construct through the integration and consolidation of previous constructions. In this study, we use learning trajectories to identify levels of students' conceptualization of function as covariation and AiC to highlight the development of students' thinking within the learning trajectories.

THE DIGITAL ENVIRONMENT CASYOPÉE

Casyopée combines an Algebra window and a Dynamic Geometry window, which are interconnected. Students can create in Casyopée free or fixed geometric objects (e.g., points), define independent and dependent quantities as magnitudes (e.g., lengths and areas symbolized as c0, c1, c2, etc.) in "geometric calculations" tab and investigate their covariation.

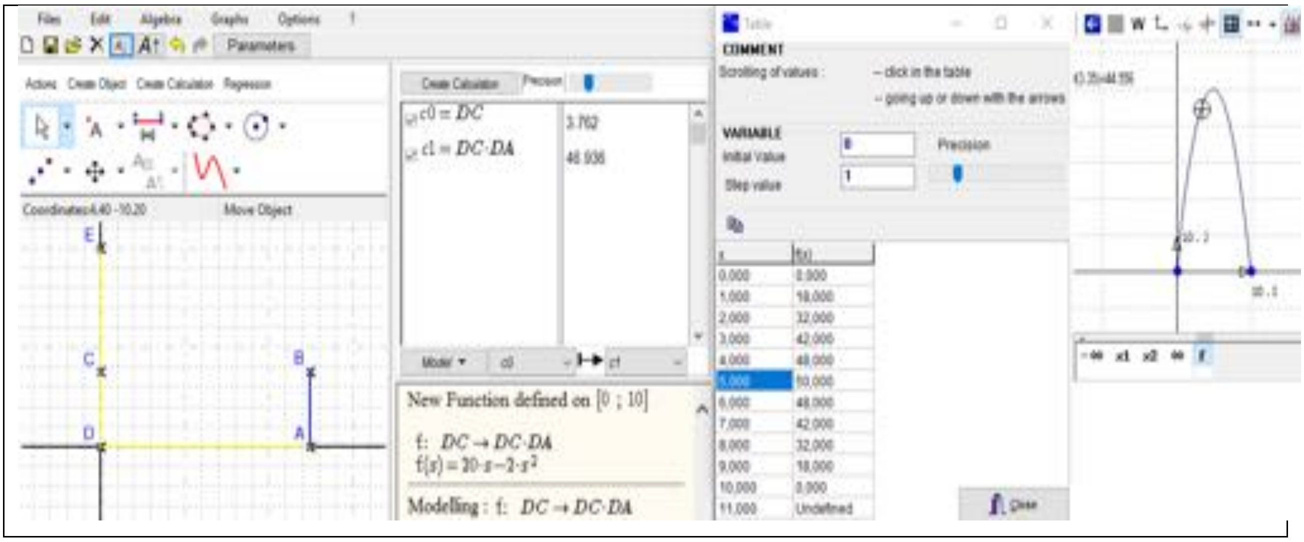

Figure 1: Dynamic Geometry, geometric calculations, table of values and graph.

In addition, through the "automatic modelling" functionality students have the opportunity to check whether a function can be defined using two covarying magnitudes (e.g., c0=DC and c1=DC*DA, Fig. 1). If a function can be defined, its algebraic formula is automatically extracted in the Algebra window, otherwise appropriate information is provided. Finally, a function can be interpreted using different representations, such as the table of values and the graph.

METHODOLOGY

Method, framework, tasks and data collection

This research is characterized as a design research (Cobb et al., 2003) since there were two cycles of implementation, in two secondary schools in Athens. Before the study the students had been introduced to functions according to the mathematics curriculum, including the definition of function, monotonicity and extreme points. A series of three modelling tasks was implemented through the close collaboration between two mathematics teachers (one in each school) and one researcher who acted as participant observer in the classroom.

The three tasks were related to realistic optimization problems and their sequence was such that the covariation appeared from simple to more complex situations according to the expected learning trajectories. A priori we anticipated students' transition from the intuitive approach of covariation by experimenting with manipulatives to deeper conceptualization of covariation between magnitudes and further between variables. The first task (*Gutter Design*) required optimal gutter design to maximize water flow.

The design followed the modelling cycle including students' engagement in: (1) experimenting with the folding of a paper (10 cm X 20 cm), observing the covariations and expressing the algebraic relationship using a variable, (2) designing and exploring a dynamic model that models the problem in Casyopée, (3) experimenting with covarying magnitudes and (4) creating the function that models the problem and resolving it through the available representations. In the first school, 23 students worked in eight groups of three for 14 hours over four months. In the second school, 25 students worked in eight groups of three for six hours over three months. The data collected consists of video recordings and audio recordings (four groups). The data were fully transcribed for the analysis. In this article we analyze the data from the two focus groups from two schools (first school: group 1 – S1, S2, S3 and second school: group 2 – S4, S5) during the implementation of the *Gutter Design* (three hours at one occasion in each school).

Method of analysis

In the first phase of the analysis we coded categories of episodes (open coding, Strauss & Corbin, 1998) based on students' references to covarying magnitudes in the different settings of the modelling cycle. Then we analyzed qualitative elements (e.g., use of symbolism) in students' thinking about function as covariation, taking into account existing classifications (e.g., Carlson et al., 2002) and the expected learning trajectories from the *Gutter Design*. Using continuous comparisons, we traced the initial categorization of the episodes, taking into account students' conceptualizations of function as covariation from simple to complex ones. This resulted in the identification of six levels of students' conceptualization. In the second phase, we analyzed line by line the transcripts in every category of episodes with the help of AiC (recognizing, building-with, constructing) in order to describe students' thinking about function as covariation as an abstraction process. In this analysis we put our notes in brackets within the transcripts.

RESULTS

Level 1: Identifying dependencies

In this level, the students recognized the dependencies of the covarying quantities needed in order to model the problem (e.g., the side length and the cross-sectional area). In the beginning, the students were able to experiment with a paper model and later to model the problem in the software by creating a dynamic rectangle referring to the cross section of the gutter. Level 1 appeared at the first hour of experimentation of the students with (a) the paper model (physical object) and (b) the dynamic rectangle constructed in Casyopée, where they recognized the interdependence between the one side (e.g., DC) and the area of the rectangle ABCD in the Dynamic Geometry window (Fig. 1). For example, in group 1 (school 1) students were experimenting with the paper model with appropriate folds and observed the interdependence of the sides in order to maximize the amount of water passing through the cross-sectional area of the gutter.

1 S1: In order to maximize the water flow, we need to maximize the base, but up to a point. The bigger she gets the more water will pass, but the side walls will become smaller. We need both of them to find the...

2 S2: Aha! We need to maximize the area of this rectangle! [*the cross-section area*]

In this episode, students' experimentation with the paper model helped them to understand the situation. Using the model, S1 recognized the dependence of the two sides in order to maximize the amount of water and observes that both sides are needed to identify a new quantity (building-with) to work with for solving the problem. Finally, S2 constructs the identification of dependencies pointing that the required quantity is the rectangular area.

Level 2: Conceptualizing covarying quantities as magnitudes

The transition from quantities to magnitudes is a critical step in covariation towards more abstract conceptualizations. In this level, the students recognized that changing a magnitude causes a corresponding change to another magnitude. This level also includes cases where students linked the two magnitudes by recognizing that they are proportional. Level 2 episodes appeared during the first and the second teaching hours during an introductory whole class discussion as well as while the students were working with Casyopée and linked the changes between quantities (Dynamic Geometry window) to those between magnitudes (Geometric Calculation window, Fig. 1) (*"As long as one magnitude changes, the other changes too"*).

Level 3: Conceptualizing the direction of change

In this level, the students were able to describe the direction of changes between two magnitudes. Level 3 episodes appeared during the second hour while the students (a) were experimenting with specific values while folding the paper to determine the cross-section area, and (b) were observing the changing values of magnitudes in geometric calculations tab. Level 3 is more sophisticated than the previous level as the students emphasized the direction of change (*"As long as one magnitude decreases, the other decreases too"*). The selected episode (school 1) refers to the experimentation in the geometric calculations tab in Casyopée.

1 R: How did you construct the rectangle?

2 S1: Look at the area here. We see that the maximum area is 50 and as we change this value... [*the value of DC*].

3 S2: Ok. We selected point C, we constructed a parallel line and then we created it. We inserted the coordinates, y-coordinate is equal to zero, but x-coordinate is equal to AD.

4 S1: Ok. Here we cannot say that it is the maximum. We can see that if we change point C in this straight line [*on the segment DC*] the area continuously decreases and maximizes when it [*DC*] gets its maximum value.

5 S2: Look here [*in the geometric calculations tab*] it says 50 and we have the maximum value of segment DC. While we move down point C, we see that the area is decreasing too.

By moving the point C, the students correlated the changes in the length of the segment DC with the changes in the area of the rectangle ABCD in order to maximize the cross-section area of the gutter. As we see in the excerpt, the interconnected representations helped S2 to observe the covariation of measurements. S1 observed that changes on the length DC change also the area values (line 2, recognizing). Then, he was able to link specifically the two covarying magnitudes (line 4, building-with) to answer the problem. Finally, S2 conceptualized the direction of the change of the covarying magnitudes as an abstraction (line 5, constructing) stating that the area decreases as the length of DC decreases.

Level 4: Conceptualizing covarying magnitudes as variables

In the beginning of the third hour, students were able to observe that the pair of the covarying magnitudes can be considered as a pair of covarying variables. In this level, while the students were creating the function to model the problem (automatic modelling) they were able to observe that one variable can be considered as an independent variable and a second one as a dependent. This level is more sophisticated than the previous one as it emphasizes the functional relationship between the two variables (*"If we say DA*DC as dependent (the area) then the independent must be DC, because through its movement the area changes"*).

Level 5: Formalizing the covariation of variables

In this level, the students described the covariation of variables more formal through the use of algebraic symbolism. Conceptualizations of that kind emerged gradually during the third hour. In the beginning, the students conceived the changes of each variable separately and later they connected these variations formally. For example, group 1 students modelled the dynamic rectangle ABCD in Casyopée, defined the independent and dependent variables in automatic modelling and opened the table to examine the values. In the next excerpt, we see how they conceptualized function as covariation formally by observing the changes in each column of the table of values.

1 S1: From the table we see the maximum value [*of DC*], that at 5… [*the area is 50*]

2 S2: It shows the area for each value that x takes with the restrictions we set.

3 S1: If we change the step it shows us the area in relation to the side DC that changes by 0.5. We see that 5 remains the value of the side DC so as to have the maximum area. We notice that for the different values of x the area changes and reaches its maximum in DC [*equal to 5*]

4 S3: Wait. For the various values of x, the area changes and finds a maximum for x = 5 with the area equal to f (5) = 50.

The three students observe the variation of the side DC and its value that maximizes the area ABCD. By linking DC with column x of the table values (line 2), S2 helps S1 to conceptualize the variation of DC as the variation of the independent variable x (line 3, recognizing level 4). Then, S1 experiments with different values in the step of the table so as to determine the maximum area (line 3, building-with). Finally, S3 conceptualizes function as covariation by relating the changes in the two columns of the table as dependent and independent variables (line 3, constructing level 5).

Level 6: Formalizing covarying variables and connecting representations

In this level, the students described the function as covariation using algebraic terms as well as different representations of function. Level 6 appeared during the last experimental hour in both schools when the students had already created the required function in Casyopée using the automatic modelling functionality and answered the problem. For example, using the graph and the algebraic formula $f(x) = 20x - 2x^2$ provided in the Algebra window the students of group 2 (school 2) observed the changes of the two variables and identified that the maximum point is (5, 50). Challenged by the researcher, they were engaged in linking the table of values and the graph in order to explain how they got to their final answer. In doing so, they used primarily algebraic terms.

2 R: Using the graphs can you answer the problem? Which is the better folding?

3 S4: We will find out which point on the x axis has the maximum point and we will identify the coordinates of the peak. Look here, it changes! Look, I move the point and the area changes.

4 S5: We can get the same result, I mean we can find which x corresponds, which y corresponds here and we say that in the top, where the highest point lies, it is the largest and we have found the coordinates from both the graph and the table of values.

CONCLUSION

We used modelling activities to identify levels of students' conceptualization of function as covariation in their transitions in the different settings of the modelling cycle (Lagrange, 2014). The analysis revealed six original hierarchical levels, which inform existing research about the evolution of students' functional thinking in secondary education while working with digital tools combining algebraic and geometrical representations. Our study enriches existing levels of covariation by providing a more subtle categorization of students' thinking taking into account the specificities of the learning context and the rich available repertoire of tools and representations. The crucial role of tools can be highlighted as follows: (a) at level 1 and 2 the transition to the identification of dependencies and the conceptualization of covariation perceptually at the level of quantities was supported by the manipulation of the dynamic rectangle ABCD in the Dynamic Geometry window, (b) transition to levels 3 and 4 where the students moved to describing covariation as the direction of change and working with covarying magnitudes was facilitated by the use of geometric calculations functionality, and (c) further move to levels 5 and 6 indicated by the formal use of algebraic terms and the extended use of multiple representations of functions was promoted through the automatic modelling functionality (i.e. favoring definition of independent and dependent variables) and the availability of multiple and interconnected representations of Casyopée.

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Making Connections: Launching a Co-created Digital Mathematics Curriculum Network

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In this project we use technology as a tool to make the connections within school mathematics more visible. Drawing on complexity thinking, participatory curriculum design and non-linear models of mathematics learning, we re-conceptualize the curriculum as a dynamic, co-created, web-enabled network of nodes. We describe how we developed the nodes and connections using video data we gathered from problem solving forums held with mathematics educators. We share participants' thinking as they identify and connect concepts and processes related to algebraic thinking. We also describe how their building of physical models to relate the concepts enhanced how they thought about the connections in school mathematics and helped us refine the digital features of the network.

Keywords: curriculum design, technology, complexity thinking, algebraic thinking.

RATIONALE AND PURPOSE OF THE STUDY

In this project we explore technology as a tool for making and displaying the many connections among mathematical ideas within mathematics curricula. Connecting concepts, processes, and representations has been shown to increase awareness of underlying mathematics structures and concepts for students and teachers (Blomeke & Delaney, 2012; Johanning, 2010). We have found many curricula align with this literature in emphasizing the value of making connections. For instance, the "intertwinement" of concepts within and across mathematics domains is a key principle of Realistic Mathematics Education (Van den Heuvel-Panhuizen, 2010) and several provincial curriculum documents in Canada include statements such as "teachers are expected to weave together related expectations from different strands, as well as the relevant process expectations" (Ontario Ministry of Education, 2005, p. 8). And yet, we find the structure of print-based curriculum documents often obscures the connections that can be made. That is, curriculum documents and related resources tend to use a hierarchical, linear sequence of chapters organized by strand (i.e. geometry, algebra etc.). This structure offers little support to educators to connect mathematical ideas or to help students make these connections. Thus, in this project we explore ways to leverage technology to create a representation of mathematics curriculum as a network of connected ideas. This approach goes beyond merely placing a curriculum resource online but rather uses the affordances of digital technology to create a dynamic and responsive curriculum.

Working with mathematics educators across Canada, we re-conceptualize the mathematics curriculum as a dynamic, co-created and web-enabled network that teachers can use to plan their program. This *mathematics curriculum network* makes the connections within school mathematics highly visible by representing the elements as layers of interconnected nodes in a digital interface. When complete, the curriculum network will also include links to resources such as tasks, multiple representations, and samples of student thinking related to concepts in the network. The network is web-based so that educators, no matter their role or location, can view the network connections and contribute ideas. In this way, the network will continually emerge as nodes, connections and resources are added.

In this paper, which connects to the conference theme of "Mathematics curriculum development and task design in the digital age", we outline our theoretical framework and summarize the process of developing the network. Then, through an analysis of data gathered in this study, we explore the ways participants identify and connect concepts and processes related to algebraic thinking and consider how their dialogue and model building enabled us to enhance the network. A longer version of the paper with visual images of the network will be distributed at the conference.

THEORETICAL FRAMEWORK

The structure of our curriculum network is based on articulations of complexity thinking from curriculum studies (Doll, 2008; St. Julien, 2005), educational change (Lemke & Sabelli, 2008), and mathematics education research (Davis & Simmt, 2003). Two aspects of complex systems from this literature are central to our project. First, a complex system is a network of interconnected elements or nodes arranged in multiple, co-implicated layers (St. Julien, 2005). Networks are inherently more flexible than linear or hierarchical structures because their structure facilitates movement from node to node in a non-linear way, including skipping adjacent nodes if desired (Doll, 2008). Drawing on this aspect of complex systems, we envision school mathematics as a network of layers of interconnected nodes. An individual node might, for instance, represent a mathematics concept, which can be connected to other nodes in the network. A second aspect of complex systems related to our project is that these systems are generative and adaptive as a result of interactions between elements in the system (St. Julien, 2005). Similarly, our curriculum network has the capacity to grow and adapt and to reflect multiple and emergent approaches to mathematics since unlimited nodes and connections can be proposed. Moreover, the use of a digital interface means nodes and connections can be added with ease.

Participatory approaches to curriculum design are a second theoretical basis for our work. We concur with the view that participatory approaches to curriculum design focused on how teachers enact mandated curricula result in more relevant and robust teaching (Clandinin & Connelly, 1988; Cochran-Smith & Lytle, 2009). In addition, researchers who study how curriculum unfolds in mathematics classrooms highlight the value of teachers working together to create an enacted curriculum in response to their needs (Boaler, 2002; Breyfogle, McDuffie & Wohlhuter, 2010; Drake & Sherin, 2006). Our curriculum network acknowledges the situated work of teachers and enables their insights to be gathered and shared. Educators have contributed to the development of the initial network and can continue to do so in future iterations.

Non-linear models of mathematics learning also inform our work. These models recognize that students build their understanding in diverse ways. We are particularly drawn to learning pathways that are non-linear and that acknowledge that as learning takes place the anticipated pathway may be altered (e.g. Van den Heuvel-Panhuizen, 2008; Fosnot & Dolk, 2001). For example, Confrey et al. (2012) combine 18 research-based learning trajectories into a curriculum map for Grades 1 to 8 mathematics learning. The trajectories are linked to curriculum standards with each standard represented as a hexagon connected to other standards. This curriculum map provides guidance to educators without specifying paths to follow. These projects prompted our thinking in many ways. Do some concepts have more points of connection than others? In what ways do educators understand concepts and make connections among them? How can the dynamic nature of connections be displayed?

METHODOLOGY

In the pilot study for this project, we consulted with a technology provider to identify the digital tools best suited to create the network of nodes we envisioned. Features of the resulting design include that users/contributors can: start anywhere in the network and follow an existing path or choose to forge a new path; learn about a connection or node by clicking on the node or the connecting line to see explanations and resources; add nodes and connections; upload resources related to elements in the network; and bookmark paths and resources for future use. The digital interface tracks proposed changes to the network so researchers can analyse suggested changes before adjusting the network. Users/contributors can also provide comments about their experience with the network. During the pilot study we also developed a process, as described below, for engaging educators and gathering their ideas for nodes and connections.

We chose algebraic thinking as a starting point for the network as algebraic thinking is taught in elementary and secondary grades and can be viewed in many different ways. We wanted to ensure that teachers across grades would be able to contribute to the initial development of the network. In addition, algebra makes use of many representations and models that connect to one another and to other aspects of mathematics. We felt this characteristic might be beneficial for developing the features of the network.

Data Collection and Analysis

In the first phase of data collection, we held four data gathering forums of one to two hours in length. The forums took place in three Canadian provinces and included 16 mathematics educators. At each forum, after obtaining informed consent, we explained the project and then asked participants to work in groups of two or three on a problem-solving task intended to stimulate algebraic thinking. Participants made note of any mathematics concepts, processes or ideas that seemed salient as they worked on the task. After 20-30 minutes, we asked each group to use the materials provided (e.g.

straws, connectors, pipe cleaners, sticky labels, chart paper etc.) to create a physical model of the noted ideas and the ways the ideas might connect. Participants could include as many nodes and connections as they wished and name the nodes using their own terms. We explained that in our analysis of the data, we would bring together models from all the groups to create an initial set of nodes and connections for the network website. We video recorded each group in each forum, photographed the models they created, and gathered their notes and drawings. This process resulted in the creation of six different models focused on algebraic thinking.

We began our analysis of the data by focusing on the nodes. Because each group had chosen their own terms, we created codes to use for related participants' terms. For instance, we decided on 'conjecturing and verifying' to encompass participant node names such as "conjecture", "test conjecture", "trial and error", "verify", and "guess and check". Next, we listed all of the nodes in the models using these codes. We then analysed the connections among the nodes in each model. An example of a connection would be from a node labelled 'patterns' to a node labelled 'algebraic expressions'. We created a matrix of the proposed connections across the models and determined the frequency of each connection. This helped us determine the structure of the initial draft of the web-based network. We also worked with transcripts of the video recordings of the groups and conducted a content analysis to get a sense of the participants' algebraic thinking as they worked on the task, of how they identified mathematics concepts and processes, and of how they decided to connect the elements in their model.

After the web-based network was populated with the information from these participants, we entered the next phase of data gathering. In this phase, participants at a national mathematics education research meeting were invited to access the website and provide feedback on the nodes, connections and other features of the network. Participants could use the digital interface to propose new nodes and/or connections, suggest changes to the descriptions of the nodes and connections, suggest resources related to a node or connection, and comment on the features and utility of the emerging network. Information from these phases of the project helped us provide descriptive text to explain each node and connection. We are in the process of analysing these responses and will make changes to the network based on the participants' input. This will constitute the second iteration of the co-created network.

In the final phase of data collection, we will email a link to the second iteration of the network to the participants from the first phase of data gathering and asked them to provide comments and suggestions on the network and the features of the website.

The results from each phase of data collection will be described in the longer version of the paper and the latest iteration of the curriculum network will also be shared.

INITIAL OBSERVATIONS

In this section, we describe the nodes and connections proposed by the 16 participants in phase one. We also share some examples of the algebraic thinking of these participants as they collaborated to build their physical models and we summarize other contributions the participants made that helped us more fully use technology to display the network.

Proposed Nodes and Connections

Analysis of the participants' models led to the identification of ten initial nodes. Six of the nodes refer to mathematics concepts ('mathematical operations', 'patterns', 'table of values', 'algebraic expressions', 'relations/functions', and 'probability') while four nodes refer to processes ('communicating & collaborating'; 'representing'; 'conjecturing & verifying'; and 'comparing, connecting, reasoning & analysing'). Across the models, 31 distinct connections among the ten nodes were proposed, with many connections proposed by several groups. The most highly connected nodes were 'mathematical operations', 'patterns', 'table of values' and 'representing'. Analysis of the transcripts shows that groups chose different starting points for their models though they had all worked on the same problem-solving task. For instance, this excerpt is from a group that chose 'patterns' as the starting node.

Re: I feel like something like patterning is something that's so at the core of any of these types of problems, so almost like if that is a centre of something, right?

Gi: . . . So do we want the little connector of patterns in the middle? And then do we have things coming out of it?

Th: I think so. That's our core.

Another group settled on 'representing' as their starting node, after some discussion.

Ra: So maybe we should have, maybe we should start with representations. Patterns can go, can come somewhere else …

Ja: Yeah.

Ra: …because then we can have everything feeding into that a little bit, right?

Ja: Well, and the way I was thinking about it is that, like, representations . . . are sort of, like you could, I think you could do representations with everything.

Insights into Mathematical Thinking

Many examples of participants' mathematical thinking were evident in the data. For instance, participants spoke about 'algebraic expressions' in several different ways. One group emphasized that 'algebraic expressions' was at the centre of things and also discussed in what ways it connects to 'patterns'. They came to an agreement that these concepts went together as parts of the same node as "algebraic expressions is part of pattern". Another group thought about algebraic expressions differently. They discussed what might be encompassed in the term "algebraic representation" and how they thought this term might relate to equations. One participant urged "I mean all of it is algebraic representation" not just algebraic expressions or equations.

As another group worked on their model, they discussed the nature of conjecturing and verifying and, in particular, the role calculating might have in verifying. An excerpt of this conversation is provided.

Js: Testing the conjecture takes you back to calculation because that's how you're testing it, right? Okay **. . .** I've got comparing here.

Ad: Okay. So comparing, ah, so ah, we did that. So the comparing is happening when we connect.

Am: And it's connected to the verifying.

Ad: Connected.

Js: Yes.

Ad: So where was our verifying? I don't think we put it in here, did we?

Am: Verify – it's right here. . . .

Js: Waiting to be placed. That's my, my question, where does the verify go?

Ad: Oh! Gotcha. Test conjecture, you have to verify, okay.

Js: And when, did, did the verify come out of the test or did it come out of the cycling back through the calculation?

Augmenting the Network with Digital Affordances

Our analysis also suggests an intriguing dynamic between participants' mathematical thinking, their efforts to build physical models, and the ways their dialogue prompted our thinking as we refine the network and website. This was evident in several ways. An example of this dynamic is from the group that discussed the role of calculating in conjecturing and verifying. They wanted to show the iterative nature of this process in their model. During their discussion, one person suggested using strands of wire.

Js: And what I want, because I did jewellery [making] right, what I want is just flexible wire. And I want to be able to put flexible wire between two nodes every time we double back, so you'd see, like well there's one and there's another and there's another and there's another.

Ad: And another and another.

Js: And to demonstrate that that's a really strong link between those two.

Analysis of this passage and a few others, as well as our matrix of connections, suggested it was important to include the frequency or strength of connections in the network. To achieve this, we decided to vary the width of the connecting lines between nodes based on the frequency of the connections that were proposed by participants.

Another example of how the participants prompted our thinking was the way each group wrestled with how to locate the mathematics processes in their model. One group wrote seven processes on chart paper and described how they would wrap the paper around their model so the processes would apply to all elements of the network. A second group identified three processes they wanted to include throughout their model and did so by intertwining three different coloured pipe cleaners into a braid wound around each connection. This prompted us to think of ways to use colour to differentiate and emphasize mathematics processes in the digital model. Another participant, a secondary mathematics teacher, suggested using other digital affordances such as pattern, motion, sound, size, and proximity in order to include and emphasize different kinds of information in the network. Although we had already considered a few of the ideas that emerged, several other ideas have encouraged us to think boldly and creatively and to more fully employ digital affordances in the network. Analysis of the feedback from participants in the final data collection phase will be included in the longer version of the paper and will provide additional insights about the digital features in use.

CONCLUDING COMMENTS

In keeping with complexity thinking and participatory curriculum design, we have found the interactions in the data gathering sessions and through our analysis of the data to be very generative. These interactions and our response to them ensure that the network develops along dimensions that are meaningful to participants.

We have found that the process of engaging in a mathematics task and of trying to build a model using limited physical tools pushes the mathematical thinking of our participants, and creates ideal conditions for them to suggest ways that their physical model could exist more robustly in a technology environment. They recognize the complexity of the representations and see the affordances that technology can add so that the mathematics curriculum network can capture the complex and multifaceted connections in school mathematics in a way that is helpful for mathematics educators. Although aspects of the study may be context specific in that they pertain to the curriculum and resources in the provinces where the forums took place, we expect the curriculum network itself to be robust enough to adapt to many contexts and to contribute to non-linear ways of thinking about mathematics teaching and learning.

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An interactive text in Precalculus: students' response

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This paper discusses an implementation of an interactive text on functions' transformations in a pre-calculus course and students' feedback on this learning experience. Students' perceived ability to make sense of this topic is viewed through lenses of the media-milieu dialectics. Comparison of results obtained from different groups suggests that in many cases interactive texts are perceived by students as more engaging and efficient for learning than ordinary textbooks. Oftentimes, both prior oral instruction and autonomous study, consisting of going back-and-forth between interactive diagrams and written text, contribute to the milieu construction. From the milieu the meaning is derived and problems' solutions are obtained. Even then, this experience does not guarantee a development of mathematical curiosity.

Keywords: interactive text, media, milieu, meaning construction, functions' transformations

INTRODUCTION

Recent decades witness the growth of universities' enrolment and the increased diversity in students' backgrounds and levels of preparedness. Development of new teaching initiatives is a natural response to this situation. Among the keywords that describe desirable teaching and learning experiences are engagement, flexibility and challenge. Engagement includes having active participants, enthusiastic professors, meaningful material, connection between theory and practice. Flexibility applies to each scheduling of classes, architecture of learning spaces, teaching styles and evaluation. Meeting a challenge assumes an environment where it is safe to take a risk and overcome learning difficulties. Modern technologies allow developing learning environments that have a potential to support and enhance the above desirable features in the teaching of mathematics. One such example is the use of interactive tools embedded in a regular mathematical text in such a way that students can manipulate and interact with formulas and diagrams. The benefits of the use of interactive software (e.g. dynamic geometry and computer algebra systems) by itself have been studied by several researchers. While using interactive software, students find patterns, observe dynamic invariants and produce their own conjectures and interpretations. These experiences prepare them for further generalizations, formalizations and more elaborate mathematical ideas (see e.g. Kondratieva, 2013 and references there). De Villiers (1990) suggests that "experimental explorations of objects and ideas are essential for development of mathematical intuition." According to Freudenthal (1971), intuition and prior experience constitute an essential basis for growing elements of formal mathematical thinking, which "cannot be simply imposed on a learner in their final form". When the latter happens, the students may "experience 'epistemological anxiety' resulting from not being able to understand the meaning and purpose of the actions they perform, even if they receive high marks for their performance" (Wilensky, 1993). Similarly, Sfard (2003) suggests making meaning is one of the main learners' needs.

In this respect, it is important to note that "using technology to develop mathematical textbooks and tasks is an attempt to create new venues for engagement with mathematical meaning" (Naftaliev & Yerushalmy, 2017). The project described in this paper is one such recent attempt undertaken at Memorial University.

THEORETICAL FRAMEWORK

An interactive text is a resource for students' learning. Chevallard (2006, p.29) distinguishes between two types of resource, namely media and milieu. The media is defined as "any social system pretending to inform some group of people about natural and social worlds". In contrast, milieu is regarded as devoid of intention to give any answers and behaves like a fragment of nature. But it is the students' interaction with milieu that is identified as a critical condition for their learning. Wozniak (2017, p.45) elaborates on media-milieu dialectics in the process of inquiry:

> Experiments, observations are then used to create a milieu to test and validate the information provided by these media. … The outcome … is based on a dual assessment of the reliability level and reception quality of the information provided by a media: "Is it right?" and "Do I understand?" respectively. Accumulation and testing of resources contribute to the validation of the produced answer and the construction of a milieu is fed by the validation of information provided by the consulted media. Thus milieus providing feedback to media may combine and evolve into a larger milieu of the problem situation.

However, this scenario is not necessarily limited to a problem solving student activity. Even in the situation of lectures,

> this dialectics is possible: while the lecturer is of course, basically, acting as a medium, he may use the blackboard to create a milieu and let the students observe how he interacts with it… The main problem is …to what extent do students develop a critical and autonomous relationship to the "answers" found in the media (Grønbæk & Winsløw, 2015)

Like Grønbæk and Winsløw (2015), when designing an interactive text, we intend to

> provide students with media, which they could access on their own, and milieus in which they could both acquire and validate adequate relationships to the subject.

In order to evaluate the implementation of an interactive text, particularly the process of students' interactions with embedded technology while solving numerous problems within the text, we ask the following questions (Wozniak, 2017, p. 46): (1) Mesogenesis: Did the milieu evolve during the inquiry process? Has the media-milieus dialectics been used? How has the answer to the problem been validated? (2) Topogenesis: How and by whom is the milieu made? (3) Chronogenesis: How has the teacher managed the time of the inquiry process?

In order to collect students' feedback on interactive texts one can adapt the characteristics of effective teaching behaviours most important for students' learning and originally identified by Feldman (2007) in the context of face-to-face teaching. When reformulated in the context of interactive texts, the criteria related to the media quality are: (1) clarity of the written text explanation, (2) written text organization. The criteria related to embedded milieu are: (3) the interactive part being engaging, (4) text ability to lead students to conjectures and to answers to their questions. The criteria related to media-milieu dynamics are: (5) text usefulness for students' learning and for critically reviewing the meaning of the information given by the media, (6) text ability to motivate and generate further interest in the topic and (7) overall students' satisfaction with their experience of working with interactive texts.

We now describe a case of implementation of an interactive text and report on its perceived quality and success in view of the above criteria. The research question is: what can be said about the milieu-media interplay based on students' responses?

DESIGN, IMPLEMENTATION AND STUDENTS' REACTION

The interactive text

The mathematical theme for our interactive text was Functions' Transformations. This topic is part of a pre-calculus course offered for first year students at Memorial University. The course main aim is to prepare students for the calculus sequence and students are advised to take it based on their results in the mathematics placement test. About 1000 students take the course each year. The course is diverse in terms of mathematical topics and dense in terms of the amount of material covered in class in one semester. Approximately 30% of students enrolled in each semester fail the course. Many students need more time to study for grasping each topic. Having extra resources such as interactive texts could potentially improve the success rates.

The unit on Functions' Transformations is motivated by the following task: given two curves and knowing that one of the curves can be transformed into another by an elementary transformation such as translation, reflexion, stretch or a combination of them identify these transformations. This task could be relatively easy if only one transformation is involved. However, it could be a challenging task if several transformation are combined. By working with the interactive text students become familiar with the action of each elementary transformation. The minimum goal of this unit is for students to know how the graph of a function $y = F(x)$ will change if the algebraic formula is altered in the following way $y = aF(kx+h)+m$ for some real numbers a, k, h and m.

The text includes many interactive diagrams (ID) among which one finds different types: illustrating, guiding and elaborating (Naftaliev & Yerushalmy, 2017).

> Illustrating IDs demonstrate the objective of the activity to the reader, usually by offering a single representation and relatively simple actions, … for example, an illustrating ID might allow learners to manipulate rather than read a definition. (ibid, p.156)

For instance, the diagram in Figure 1 (left) shows a point reflected with respect to an axis of symmetry. The learner can change the slope of the axis as well as the position of the red point A, observing what happens to the blue point A'. The diagram emphasizes that the segment AA' is always perpendicular to the axis of symmetry that acts as a mirror in which point A' is a reflexion of A.

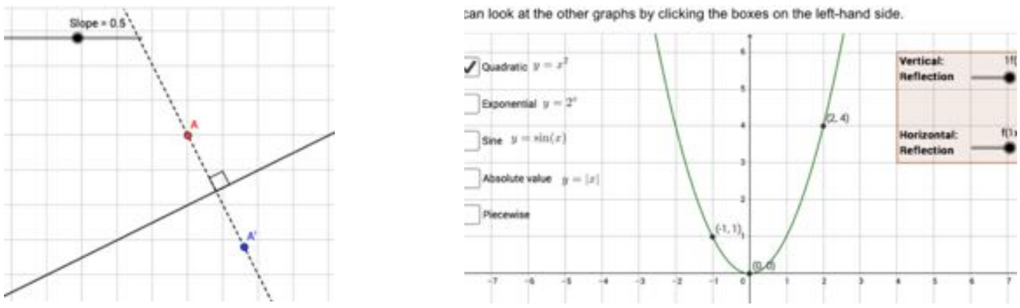

Figure 1: Interactive diagrams: axial reflexion of a point (left) and of a curve (right).

Once the idea of a point's reflexion is understood, the learner is invited to extend this notion to the case of a curve and observe what happens to graphs of various functions when they are reflected with respect to either x or y axis. The reflexion can be viewed point-wise, that is, the image of every point can be identified as a point on the reflected curve. In this approach the reader is led to the idea that the reflexion in the y-axis of the graph of $y = F(x)$ is related to the sign change of the x-variable: $y = F(-x)$, while the reflexion in the x-axis is related to the sign change of the y-variable: $(-y) = F(x)$, which is equivalent to $y = -F(x)$. The reader is also directed to the idea of symmetry and invariance of some curves w.r.t. the x-axis reflexion or a composition of two reflexions. This is achieved by using the guiding diagram which

> provides a means for learner exploration, but it is designed to also set boundaries for the available exploration options in such a way that it narrates the story to be learned by working on the task. Guiding IDs are designed to point students toward specific actions intended to support them in developing specific mathematical ideas, (p. 156)

in our case, the symmetry of graphs of even and odd functions (Figure 1, right).

Next, students are invited to solve a quiz involving a greater variety of related problems. To assist their progress they can use elaborating diagrams, which

> present occurrences relevant to the problem being explored while working on the task. They attempt to provide a means for students to engage in activities that lead to the formulation of a solution in different ways. (p. 156)

At this point, students could enter a function of their choice and graphically verify the predictions of prior algebraic analysis. It becomes especially valuable when all three elementary transformations studied each in a separate section are combined.

As a separate section, we had an entertaining exercise asking students to move the hands of Mark the Mathematician by using the control panel related to functions' transformations (Figure 2, left) to achieve one of the positions shown in Figure 2.

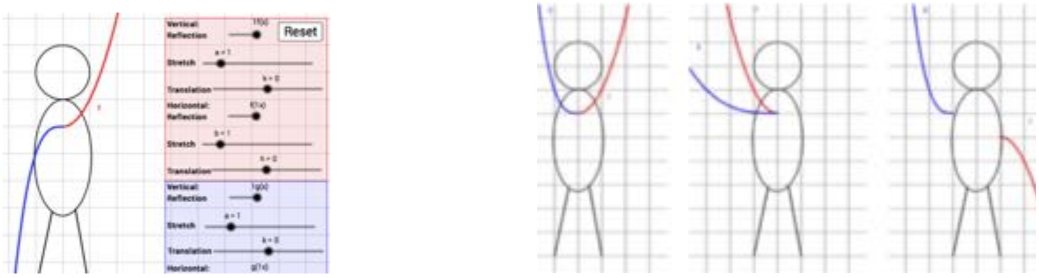

Figure 2: An entertaining exercise on functions' transformations.

The role of this exercise is to establish the connections between the terminology and the visual representation of the reflexion, shift, and stretch in either vertical or horizontal directions in an entertaining form, without any references to algebra.

Next section gives the details on the project implementation and students' feedback.

Two stages of implementation and students' reaction

The project involved three groups of students, a total of 355. All students were given one week to complete their work. They could form study groups or do it individually. Participation in the project warranted a bonus mark in their course. After completing all work students were asked to give feedback on the related learning experience. All students responded to a survey and selected students were also interviewed.

The project included two stages. In Stage One the interactive text was tested with Group 1 (N=25) and Group 2 (N=30), and in Stage Two – with Group 3 (N=300). Group 1 consisted of pre-service math teachers who already had a bachelor degree in mathematics and thus used the text for a review of the topic. Groups 2 and 3 consisted of pre-calculus students who basically learned the topic for the first time (some of them could have seen some of these ideas in the high school, but we did not assume that). Groups 1 and 2 were given very brief description of the project and no explanations of the mathematical topic. They were asked to read the text and follow all the instructions within the text to perform all exercises and quizzes. Thus, in Stage One of the project the students were responsible for building the milieu from reading and interacting with the software. Table 1 shows the average response for each group of students based on the Likert scale: 1=strongly disagree, 2=disagree, 3=neutral, 4=agree, 5=strongly agree. We see that students from both Group 1 and Group 2 found the text well-organized (line 2), however for Group 2 the explanations appeared to be a bit less clear than for Group 1 (line 1). In both groups the interactive components helped to clarify the ideas (line 6), however again Group 1 gave a bit higher mark than Group 2. Looking at these responses we may propose that in the case of students in Group 1, the milieu was constructed from the media presented in the text but was also informed by their previous experiences with mathematics. The interactive component helped them to make sense of the ideas that they once were familiar with in the past but perhaps forgot some of them and now tried to recall and upgrade their knowledge using the interactive text. Based on students' feedback this experience was more satisfactory for Group 1 than for Group 2 (line 8).

Statement	Group 1 (N=25)	Group 2 (N=30)	Group 3 (N=300)
1. The topics are well explained	4.29	3.6	4.08
2. The text is well organized	4.29	4.4	4.18
3. The interactive part is engaging	4.54	3.7	4.09
4. This interactive text helps me to answer some questions that would be harder to answer from just reading a book	4.7	3.2	4.19
5. This text is useful for study and understanding	4.33	3.5	4.27
6. The interactive components help to clarify ideas	4.58	4.1	4.08
7. This text made me curious about mathematics	3.41	2.8	3.31
8. Overall I am satisfied with the experience	4.25	3.4	4.20

Table 1: Students' feedback on their experience with interactive text.

Our philosophy in constructing the first version of the interactive text was the primacy of the written text over the interaction part. We assumed that students would read the discussion first and use the interactive part only when they felt it was necessary to clarify and make sense of some information. However we found that many students from Group 1 actually jumped ahead and played with the interactive components first. In doing that, they found the interactions more engaging (line 3) and praised the interactive text more in comparison with a textbook (line 4) in terms of finding answers to their questions. We also learned that students do not like to read long explanations, especially on the screen and they generally prefer to receive the information in a variety of modes besides the written one.

Based on the analysis of students' feedback in Stage One, we made the following changes: (1) We put more actions up front before discussion of theory in the interactive text. In particular, the activity in Figure 2 had been moved to the introduction. (2) We created videos to replace several written explanations. (3) We asked the instructor to give a brief mathematical introduction to the topic. These modifications were tested in Stage Two with a bigger group of pre-calculus students, Group 3 (N=300). The results are shown in last column of Table 1. In this scenario, the milieu was constructed from the information given by the instructor and some experimentations with interactive components before students read the text or watched videos. This seems to improve students' perception of the clarity and the quality of explanations (line 1). We also saw an improvement in students' opinion about the interactive text as being engaging (line 3), useful for study (line 5) and overall satisfaction (line 8).

CONCLUDING REMARKS

A positive outcome of this project is that 91% of students in Group 3 agreed with that the interactive text is useful for their study and understanding of the topic. This figure is almost the same (92%) for Group 1, but is an improvement compared to 67% of such responses in Group 2. We partly attribute this improvement to the three changes we made before working with Group 3. From students' responses we derive tentative answers to the questions such as how, when and by whom the milieu is created allowing students to interpret the information given in the media.

The following freshman's response confirms that milieu construction is a gradual process requiring both the instruction given in class and autonomous study at home:

> I have had moments like 'oh, this is related to that'. For example, I am in a lecture and I kind of understand what's happening; then maybe a couple of days later a lecture on the same topic, and I kind of understand what's going on; and then I get home and I apply myself what I have learned, and I really have time to compare and absorb it, then I have this aha moment; (…) or the next day in class, 'oh, yeh, I did it last night when I did my practice problem or my reading, it completely makes sense, I have it!'

An autonomous study from an interactive text may not be sufficient for a freshman; prior instruction significantly contributes in the milieu construction for this student:

> The instructor definitely has a big impact. Having somebody to walk you through the material is very useful. With this piece, it was some limited instruction before we used the product, but it was important for me.

Many students confirm that interactive diagrams help to clarify the meaning:

> In a book I do not see what exactly happens, here I can manipulate as much as I like. Through the interaction, I can find some answers if I got a question when I read the text. It helped to create the base knowledge (…) greater return than from reading the book.

We also find that students have different requirements and expectations about the interactive text compare to a book. For example, "reading on the screen is different from reading in a book. I would prefer a bullets list format. Especially on a tablet."

Finally, because interactive texts give students an opportunity to engage with mathematical meaning, to progress at own pace and to meet challenges "in the comfort of your own home", we were hoping that this experience would generate some curiosity and interest in the study of mathematics. And indeed, about 39% of students in each group agreed with that proposition. But although some students found that "it made it interesting as I got to play around with the questions to help me better understand the topics", this item received the overall lowest mark (see also line 7 in Table 1). As stated by an interviewed student, "It did not made me crazy curious, but it was somewhat interesting; and being a non-math person I thought, 'this is not bad!' … I guess, everybody has their own preferences and inclinations". This situation shows that even if students are engaged and extract mathematical meaning from the created milieu, they still could remain indifferent towards further exploration of the subject. However, this issue might be a topic for a different paper.

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Realizing students' ability to use technology with silent video tasks

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Teachers who encounter difficulties when implementing technology in their classes often hesitate to give it another try. They expect too many technical problems to emerge, reducing time spent on learning mathematics. Still, if a task requires only short class-time they might try. Four mathematics teachers in different upper sec-ondary schools in Iceland assigned a silent video task in their classes in fall 2017. Such tasks consist of asking students to record their commentary to a short silent mathematics film. Teachers were positively surprised by their students' technologi-cal abilities. Nevertheless, because the task was not directly related to final test preparation, they expressed that they would unlikely use it again. In this paper, a review of technology and silent video activities and related research will be outlined.

Keywords: silent video task, upper secondary school, in-service mathematics teachers, use of new technologies in class, formative assessment.

INTRODUCTION

Leung and Bolite-Frant (2015) ask for the design of tasks that encourage discourses for mathematical knowledge mediated by tools in the classroom. Silent video tasks are an example of such tasks. Not only do they encourage classroom discussions about mathematics, they also seem to open teachers' eyes to the fact that students are capable of using technology and that the intended use of technology in mathematics class must not necessarily evoke worries or anxiety for the teacher. This article aims at offering some answers to the question: What are Icelandic upper secondary school mathematics teachers' expectations of and experiences with using silent video tasks in their classes? Some preliminary results will be presented regarding three particular issues connected to teachers' anxiety when it comes to using new technology in class.

Silent video tasks

Silent video tasks were developed and tested in a Nordic and Baltic collaboration project in 2014. Silent videos are animated, short films without any text or sound that show mathematics dynamically, each video focusing on one mathematical concept. In a silent video task, students get the assignment to watch a silent video as often as they like, discuss it in groups of two, and add their commentary to it, i.e. to record a voice-over to the video.

This activity can be implemented as an introduction to a new mathematical concept or as a summary of a topic; the former serves as a tool for collecting preconceptions and initial ideas and the latter helps assessing pupils' understanding of previously studied concepts and to overview possible misconceptions and misunderstandings. In both cases, problems can arise from imprecise language use and it can be addressed to clarify concepts. Ideally, students should receive feedback on their solutions and in a follow-up lesson some student solutions can be chosen for viewing to initiate a group discussion.

Silent video tasks as formative assessment

Suurtamm et al. (2016) claim that the primary purpose of an assessment is to improve student learning of mathematics. One cognitive aspect designed to improve student learning is when students get the opportunity to explain to others and/or to receive explanations from their classmates. Silent video tasks not only offer students an opportunity to do so, but also that they become aware of the fact that once they are able to explain to others what they have learnt, they also improve their understanding. One of the students who completed the silent video task commented the following in an online survey: "You don't know the material well enough if you cannot explain it to others in a good [understandable] way."

Teacher feedback to silent video task solutions can be used to suggest what can be worked on further and not only to measure the students' current ability. In other words, silent video tasks can be used as formative assessment, supporting students' learning of mathematics.

THE STORY BEHIND SILENT VIDEO TASKS

The idea of silent video tasks emerged in the winter 2013-14 in a Nordic and Baltic GeoGebra Network teachers' and teacher educators' collaboration project. After developing the task, teachers in four countries participated in a teaching experiment and tried out silent video tasks with their pupils in grade 5-13. They divided their pupils into groups of 2-4 to work on and record their commentaries. In some exceptional cases only one pupil worked on a solution. In total, approximately 450 pupils and 21 teachers participated in the teaching experiment. Teachers answered an online survey with two open questions regarding their pupils' results in general and five Likert-scale questions each for every group of pupils regarding their reactions to the task. The teachers reported that 94% of their pupils' groups reacted positively to the task and their pupils' groups communicated more than in usual class.

As one of the teachers who participated in the teaching experiment, I felt sceptical before assigning the task. Listening to my pupils' solutions I had an "aha!-moment" realising the potential of the task to offer insight into my pupils' conceptual understanding. I wondered what expectations and experiences other teachers had and that is the key topic in my ongoing PhD study.

MOTIVATION

The Icelandic mathematics curriculum for upper secondary schools lists several goals regarding students' mathematical understanding and their ability to apply mathematical methods to tackle various cognitive demands. Still, according to a report written for the Ministry of Education, Science and Culture in Iceland, mathematics teaching practice in Icelandic upper secondary schools is mainly limited to standardized calculation methods and running down lists of things to cover (Jónsdóttir et al., 2014). This resonates with the international TIMSS study that revealed at least 80% of the lesson time in 8th grade mathematics class was used for solving problems and that on average in all countries but Japan most problems (between 63% and 77%) were problems of low procedural complexity (Hiebert, 2003). To reverse this trend, I was interested in seeing whether silent video tasks might be used to break up the common teaching practice routine and awaken teachers' interest in practices more related to idea of the *thinking classroom* described by Peter Liljedahl (2016). Namely, a *thinking classroom* is organised in a way that students are expected to think and given opportunities to think through activities through continuous discussions (Liljedahl, 2016).

As a teacher with only two years teaching experience in upper secondary school, I made small steps towards becoming the teacher that I wanted to be. Participating in the silent video teaching experiment reminded me that I wanted to change my teaching practice. When I started my PhD studies, three years later, this experience still stuck with me. I came to realize that the experience that I saw happening with my students, as they perceived knowledge in the process of communication resonated with Anna Sfard's (2010) idea of commognition. That idea, in fact, requires us to think of cognition as a process of communication (Sfard, 2010). In an earlier paper Sfard explains two equally important 'metaphors' for learning: individual, cognitively oriented knowledge acquisition conceptualisations, and social and participatory conceptualisations of learning (Sfard, 1998). The latter one being what I experienced in my classroom in 2014 with the silent video task.

Last but not least, what I experienced from the teaching experiment in 2014 was that even teachers who were not used to using technology in their classes seemed to find the silent video task to be a reasonably easy start when it came to trying out something new in their teaching.

METHODOLOGY

The research question initiated from my motivation: What are Icelandic upper secondary school mathematics teachers' expectations of and experiences with using silent video tasks in their classes?

In total, there are 30 upper secondary schools in Iceland that prepare pupils for studies at university. Ten of these upper secondary schools were randomly selected, and out of them six schools accepted the offer to participate. One teacher in each school assigned a silent video task to their 17-years-old pupils. The research design included two short online questionnaires for students and three semi-structured interviews with the teachers:

The first teacher interview aimed at collecting background information (e.g. teaching experience, working atmosphere) and expectations, discussing the task in advance. The second interview revolved around the experience so far, the student solutions and planning the follow-up lesson, and the third interview focused on the overall experience of working with a silent video task.

The silent video task chosen for this study

For many students in Iceland, the first new mathematical concept that they encounter in upper secondary schools is the unit circle and therefore it was chosen to be the theme of the silent video selected to use in this study [1]. The two minutes long video shows a circle getting parted into four quadrants before a coordinate system appears, defining the circle to be a unit circle. All the time, a dot moves around the circle in a positive direction. Line segments point to the coordinates of the point and at the same time can be interpreted to denote the sine and cosine values. The video was made with GeoGebra and was based on an idea from Alf Coles, a senior lecturer at the University of Bristol School of Education. According to Coles, he got the idea from Dick Tahta and Caleb Gattegno, and the original idea probably came from the Nicolet [2] films (Coles, 2008).

The silent video task was introduced to the participating teachers via phone calls, in email communication, and on a webpage before the first interview took place. Four teachers completed all three interviews, one only participated in the first interview and one in none of them. The teacher quitting after the first interview reported lack of both time and student interest. That, however, contradicts the fact that most participants found the task interesting. The other teacher quit because only six students signed up for the course. All interviews were made in meeting rooms or teacher offices in the schools. Principals and teachers received information about the purpose of research and signed informed consent regarding their participation.

Students received information about the research project and were asked for a permit to collect their silent video task solutions, which was accepted by all students. Links to short feedback questionnaires, each with five Likert-scale questions and one open commentary field, were sent by email to students before (86% answer rate) and after (70% answer rate) the follow-up lesson. Results from the student questionnaires were partly used as a reference in the second and third interview with each teacher.

After transcribing the interview data, I first viewed it through the theoretical lens that I used in the preparation phase. This leads into a cul-de-sac. Then I coded and analysed the data using a Grounded Theory approach (Charmaz, 2006). Based on the findings, I will be using a top-down approach in the next step of analysis.

PRELIMINARY RESULTS

In this section, I will give some preliminary results and I focus on the first three interesting issues related to technology that emerged during coding. Further results will be outlined in future publications.

The first issue that I would like to address is that teachers are not aware of the technological reality of their students. In Iceland, youth and young adults use social media such as Snapchat and Instagram to share video recordings from their daily lives with friends as well as strangers (Gallup, 2017). This is not a specific Icelandic phenomenon and the same trend is apparent for teens in the US and in numerous other countries. According to surveys by the Pew Research Center on American teens' social media use, 92% of American teens used the Internet daily in 2015, 71% used Facebook, 50% used Instagram, and 41% used Snapchat (Lenhart, 2015). As of 2017 social media use amongst teens and young adults in the US has increased to 79% using Snapchat, 76% using Facebook and 73% using Instagram (Edison Research, 2018). Most teens, therefore, are quite used to hearing their voice in a recorded form and are not shy of sharing such recordings with others. Despite this fact, three of the four teachers were very hesitant to play their students' silent video commentary recordings in the follow-up lessons.

Teacher G: […] If you say "And then I am going to play the recording for everybody" then I think that it immediately has a certain influence on how they solve the task so I decided not to tell them that I was going to play their recordings and afterwards I thought "Can I really play the recordings or not?" so I ended up not playing any recording […] one has to ask them for permit to play the recording […] it is a fragile age and even though you don't hear anything wrong with it yourself then somehow they spot something and that can be risky […] they can freak out.

In the example above, the teacher did not introduce the task to students such that they would expect some selected solutions to be shown in the follow-up lesson. Another teacher who had announced that some selected solutions would be shown in a follow-up lesson, backed out and ended up not showing any student results despite his announcement. The idea that most students somehow feel ashamed when their solutions are picked to discuss in the follow-up lesson is not necessarily always valid and must be cleared, i.e. the teachers need information on their students' reality.

A second issue is the fact that some teachers fear technology will fail in their classrooms and are therefore anxious about using it. Before assigning the task, technology other than calculators was not used by students in class on a regular basis. All five of the teachers that participated in the first interview expected their students to have difficulties with technology. This expectation was partly based on former experience with slow computers or failing software. Whilst one of them looked forward to seeing the students struggle with technological problems before succeeding, others were anxious and over-protective, taking full responsibility themselves for solving technological issues.

Teacher S: […] I was rather surprised […] I had expected that something like that [refers to problems with technology] would come up, but there was nothing.

Teachers assigned the silent video task in a manner depending on their own beliefs and ranging from giving very short verbal instructions asking the students to solve the technological part of the task themselves to handing out a long and detailed instruction sheet. Only one of the teachers experienced that students needed help with technology:

Teacher M: As it goes then it was quite diverse [refers to how it went to deal with technology] and maybe they described that in their — I asked them to explain it clearly — in the online survey […] I feel it is quite a hurdle for this task in order to let the energy — I wish that more — a larger part of the energy would go into thinking about the mathematics rather than fighting the technology.

This teacher was very positive towards trying out new methods in his teaching, but it was apparent from the interview data that he did not trust his students with understanding a task without getting detailed instructions. Since his students are not very different from the other participating teachers' students, it is quite likely that without the detailed instructions, the students would have had no problems in dealing with technology. Nonetheless, his reaction to whether he would change something next time was:

Teacher M: Yes, somehow because my instructions said […] and if I think about it in retrospect […] I should maybe rather add an extra step into [the instructions].

This brings us to the third issue that I would like to address; the transition from transmission-oriented teacher to an organizer or facilitator of learning. The teacher in the group that had most experience with using technology in class said:

Teacher L: Then there was one who said "As soon as I checked all the buttons in the browser I realized that there is one for recording" […] did you know that this was possible? [asking if the interviewer had realized this possibility].

This teacher was the only teacher who transferred the responsibility to solve possible technological issues completely to the students themselves.

Teacher L: They found it unbelievable to have figured out by themselves how to send me a sound recording from their phone […] it was so nice to hear how anxious they were (before starting the task) "Shall we figure out the technological issues ourselves?!" "Yes, you are so clever, you can do this" Then they just "But wait a minute, aren't you going to help us?" "No, you will figure this out," then they said "That will take us the whole time (referring to the length of the lesson) just to figure out the technological issues" […] and then when they finished (and they did so well within the lesson time) it was as if they had won the Olympics or world cup or something.

What this teacher experienced is that students can also teach teachers to use technology and that this is something that ought to be welcomed in the classroom. Also, underlying was this teacher's opinion that teachers in general should trust their students to get through the struggle and in doing so give them an opportunity to experience the good feeling of having succeeded.

DISCUSSION AND CONCLUSIONS

Teachers in this study reported having constantly limited time to devote to other things than preparation for the final test. They all (separately) agreed that the reason why they reacted positively to participating and trying out a silent video task in their class was that it only required a maximum two lessons and some preparation time. They noticed that the task helped to break up their normal teaching routines. Even if all but one teacher found it rather unlikely that they would use a silent video task in their class again, still, by participating, they were given insights into that most students are fully capable of solving possible technological issues themselves. Hopefully, this can encourage them to do more experiments utilizing technology in the classroom.

It was interesting to see how all but one of the teachers expected their students to be shy about their solutions being presented in class in the follow-up lessons. Possibly their memories of their own adolescent years affected their decision. However, two of the teachers did show examples of their students' solutions in the follow-up lesson, and the respective students showed no signs of timidity because of that.

Looking into the teaching practice of the teachers participating in this study, they all focused on preparing their students for a final test; more or less using transmission-oriented teaching methods. It might be interesting to ask teachers who use other more modern teaching methods for their view of the silent video task. At least, the teacher who was using less transmission-oriented methods than the others was the only one giving the impression to intend to use silent video task again in the classroom.

NOTES

1. The silent video task used in this study can be found here: https://ggbm.at/BfRqGSKq

2. Jean-Louis Nicolet was a Swiss mathematics teacher. He made some black and white animated silent mathematics films without any text called Animated Geometry in the 1930's that are still used in mathematics classrooms today (Tahta, 1981).

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Evaluation of Apps using the ACAT Framework

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Since the introduction of the first iPad in 2010, the overall number of apps and, consequently the number of apps intended to support mathematical learning, have increased exponentially. This observation results in the need for quality information about apps and also on the various possibilities for teachers to evaluate apps in an efficient and reliable way. Artifact-Centric Activity Theory (ACAT) is a model developed to capture complex situations that arise when digital technology is introduced in classroom situations. Furthermore, ACAT provides a framework to help teachers to evaluate apps. In this article we show how to use the ACAT framework for the evaluation of mathematics apps.

Keywords: Activity Theory. Apps. Primary Education. Geometry. Review.

TECHNOLOGY AND APPLICATIONS IN PRIMARY MATH EDUCATION

Digitization is both one of the most significant challenges, and also one of the most significant opportunities in today's world. This is also the case in educational contexts. When technology is used in a useful and goal-oriented way, it can help people to simplify everyday life as well as support various occupational routines. In the field of education, digitization can also help children to learn mathematics, a discipline that is often seen as difficult by many students (Larkin & Jorgensen, 2016). As in the broader societal context, the usefulness of technology in mathematics classrooms depends upon whether teachers are willing and able to take advantage of the potential of technology to support student learning.

The introduction and use of desktop computers in primary school has more or less failed for several reasons: one reason identified is that the hand-eye-coordination of young children is often insufficiently developed. That is why young children often have problems moving the mouse and coordinating the movements of their hands with their eyes and what happens on the screen (Ertmer, 1999; Kortenkamp & Dohrmann 2010; Ladel, 2017). This problem is exacerbated by the fact that the scale of the movements with the hand does not correspond to the scale of the appropriate movements on the screen (Ladel, 2017). With the introduction of the first iPad in 2010, these detrimental factors to student use of technology diminished or disappeared entirely. iPad technology allows the user to interact directly with the screen using their fingers and hence with the objects visually represented there. Thus, the obstacles of the insufficiently developed hand-eye-coordination of young children, as well as the different scales of mouse and screen, are no longer present. The haptic technology of the iPad (and similar tablets) is very suitable in supporting the learning of young children (See Alade et al., 2016; Sinclair & Bruce, 2015).

In addition to developments with iPad hardware, the software (Apps) has also developed rapidly in recent years. Whereas the software for personal computers was (and still is) very often restricted to drill-and-practice (especially in the field of arithmetic), many current iPad apps offer increased interactivity and a broader range of possibilities for discovery learning, particularly when used in authentic contexts with young children (Arnott, 2016). Although the rapid increase in the availability of apps is a potentially positive outcome for education, the negative aspect of this equation is that amongst the rapidly increasing number of apps, there is an overwhelming prevalence of inadequate or unsuitable mathematics apps (Larkin, 2016). As the need to identify good apps that support mathematical learning is therefore critically important; researchers, developers, as well as teachers require an efficient and reliable instrument to evaluate apps in an easy, yet thorough, way. One such instrument to do so, presented in this conference paper, is a review guide based on Artifact-Centric Activity Theory (ACAT) (Ladel & Kortenkamp, 2016).

THE ARTIFACT-CENTRIC ACTIVITY THEORY (ACAT)

Vygotsky and Leont'ev developed Activity Theory and this theory is based on the cultural-historical conception of human beings and their development. The interaction (activity) between human beings and the world (subject-object) is the primary focus of this theoretical framework and it is assumed that human beings are shaped in particular by their activities in the world (Nerdinger, Blickle & Schaper, 2014). In this way, activity is a process characterized by a constant transformation (Leont'ev, 1982) (see Fig.1).

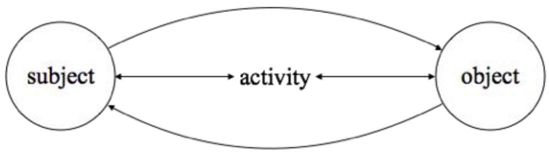

Figure 1. Ring structure of activity according to Leont'ev (In Nerdinger et al., 2014, p. 341)

It is understood that subjects (in this case children) have needs or objects (in this case learning about mathematics). In order to meet those needs they carry out activities and they achieve their object in an active way through object-oriented, changing and productive impacts. Thus, the activity is related to, and controlled by, the motive and is realized through object-oriented activities.

Vygotsky (1980) attributes an important role to mental tools in the interaction between subject and object. In the "so called" *instrumental act* (see Fig. 2), the mental tool mediates between the subject and the object.

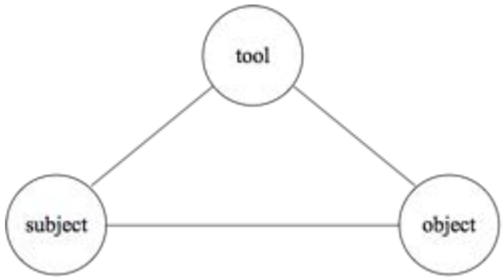

Figure 2. instrumental act

ACAT is a further development of Activity Theory and Activity Systems (where Community, Division of Labour and Rules are added to the initial Activity Theory triangle (Engeström, 1987) and is a methodological tool to understand the complex network of relationships in the interacting activity system of teaching and learning. The causal network of ACAT comprises five components (subject, artifact, object, rules, group), which interact with, and influence, each other in such a way that a change in any one of these components provokes a change in all other components. Central to ACAT is the artifact - a tool, an instrument or a mediating object - in this paper this artifact is the iPad or the iPad app. The main axis of subject-artifact-object includes the theoretical underpinnings of both Leont'ev's notion of activity as well as Vygotsky's instrumental act (see Fig. 3).

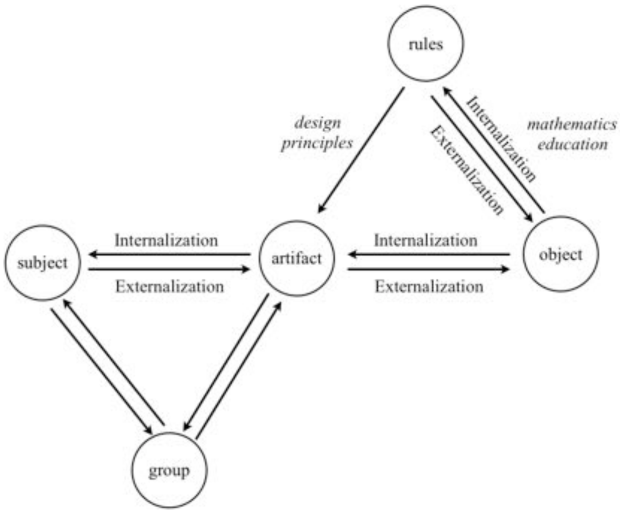

Figure 3. Artifact-Centric Activity Theory (Ladel & Kortenkamp, 2011, 2013).

The mediating artifact replaces the direct connection of subject and object with two connections, subject-artifact and artifact-object, as all interaction between subject and object is understood to be mediated by the artifact. This highlights the importance of the design and the analysis of the artifact for the activity (upper right triangle: artifact-object-rules).

The rules are primarily a result of the mathematical object itself, but they are also related to disciplines such as psychology, didactics of mathematics, or media didactics. The artifact represents the object itself, but simultaneously the object is encoded in the artifact and the properties and the aspects of the object limit the artifact.

The lower left triangle (artifact-subject-group) reflects the observation that the artifact does not only connect a human being with the object but also connects each individual with other human beings - the group. In this way the individual actions of the subject also include the experiences of other individuals in interacting with the artifact – in the case here this could include other students and also the teacher.

REVIEW OF APPS

ACAT is, therefore, a methodological tool to assist our understanding of the complex network of relationships in the interacting activity system of teaching and learning. Based on the assumption that ACAT could also help researchers and teachers to analyse and evaluate apps, we designed a review guide that should help them to easily evaluate apps and hence find the right ones to support their teaching and learning goals. The resulting review guide is organized into a sequence of five steps, following the main aspects of ACAT and principles of Activity Theory (Kaptelinin, 1996). This is similar to the approach of Kaptelinin et al. (1999), but differs in three ways: (1) The underlying model is specialised for analysing and designing artefacts and instruments; (2) The purpose of the guide is to help teachers without special training to judge whether a certain app can be useful for their pedagogical needs and thus includes specific instructions on how to answer the questions; (3) The guide is focused on education and not on Human Computer Interaction (HCI).

For each step, we formulated a key question that needs to be answered. In order to aid teachers in answering, we provide remarks and lead questions to consider, as well as possible data sources to find information regarding the five steps. The full development of the review guide is detailed in Larkin, Kortenkamp, Ladel and Etzold (2018, under review), whereas the review guide itself is available as an open educational resource at http://dlgs.uni-potsdam.de/oer.

Object orientation is a central principle of Activity Theory. All activities are directed towards an object. Hence it is important to know the object of the actions of students within an app precisely:

Step 1: What is the mathematical *object* of the app?	
Remarks	The reviewer identifies the mathematical object, i.e. the concept, content or mathematics process that is targeted by the app. Each app can address one or several mathematical objects.
Sources	Title and official description at iTunes Store; Additional material provided, e.g. downloadable worksheets; External references, e.g. recommendations by peers who have used the app; trials of app

In the process of designing an app, once the mathematical object has been established, the interaction design would be the next consideration. In the process of analysing an existing app, step 2 is used to examine the design of the user interaction:

Step 2: How do students *interact* with the mathematical object, mediated by the app?	
Remarks	What are the concepts that students have of the mathematical content? How do these concepts influence the use of the app? What possibilities and what limitations does the app have?
Sources	Own systematic testing of the app

Step 3 focuses on the *hierarchy* of activities, actions and operations, as well as conclusions about possible *developments* of students' interactions that influence their learning:

Step 3: How does the interaction *develop*?	
Remarks	What are the activities, the actions and the operations of the interactions?
Sources	Discussion of hypothetical scenarios; Empirical tests

The design of an app is guided by *rules* which in turn are guided by designer knowledge e.g. from mathematics education, HCI design, etc. Following those rules maximizes opportunities for the app to support learning of the targeted mathematical content:

Step 4: Is the app suitable for teaching and learning the mathematical object?	
Remarks	What insights do we have from mathematics specific pedagogy, the discipline of mathematics, and psychology? Do the previously analysed interactions support the desired or needed concepts, experiences and competencies?
Sources	Syntheses of the discussion above; Scientific background literature and references

Within ACAT, learning is never a purely individual activity of one single student. It must always be seen in a social and corporate context, in which learning content occurs by working together (group):

Step 5: How can the app be used in classroom instruction?	
Remarks	Is the app suitable for individual work, partner work or group work? What are possible impulses and tasks? What kind of differentiations and levels are possible? Is the goal of the app to train already understood content or is it intended to develop new concepts? Is the app based on an instructive or on a constructive paradigm? What are the competencies that the students need to work with the app?
Sources	Additional teacher's material; Trials with students; Imagination

In the appendix, we give an abridged example of an ACAT-based app review that was created by two teacher students following the above steps. Another in-depth example is available in Larkin et al. (2018, under review) and online.[1] In all these examples we can find conclusions that show how the reviewers come to deep conclusions based on their normal teaching skills. Also, we could see how the reviewers came up with suggestions for task design. They included suggestions for classroom integration of the app for students with special needs as well (in Step 5), which shows how the structure of the review guide helped them to come up with ideas for better teaching.

CONCLUSION AND FURTHER WORK

We propose an app review guideline that is based on the ACAT model. So far, this framework has been used for several reviews by German teacher students and proved useful for a theory-based approach without requiring in-depth training of the reviewers in ACAT. This is a welcome addition to more technical approaches that try to review apps based on non-pedagogic criteria such as number of features, configurability or technical soundness, or ad-hoc approaches that base the assessment on personal opinion or number of downloads. As an outcome of the project *Digitales Lernen Grundschule*, funded by Deutsche Telekom Stiftung, a German and English review guide and template were created and published as an Open Educational Resource.[2] Besides a platform to collect and publish ACAT based reviews, further work will include a systematic meta-evaluation of these reviews and the translation of the review guide and template to other languages.

APPENDIX: ACAT REVIEW FOR THE APP *SHAPES 3D – GEOMETRY LEARNING*

(Abridged translation of the original review Deßloch, L. & Hoffmann, L.-M., 2018)

App: Shapes 3D – Geometry Learning, *Version* 2.2.2. (Published 30th of June, 2017). At the App-Store intended users can find a lengthy description of the app, information regarding additional material to support its use, as well as rewards the app has won. Due to space limitations we have not included this information here and have only provided an abridged version of the teacher student review.

Step 1: What is the mathematical *object* of the app? The mathematical object of the app is to identify the spatial imagination in relation to geometric solids. The focus lies on the connection between the three-dimensional solid and the two-dimensional solid-net.

Step 2: How do students *interact* with the mathematical object, mediated by the app? Students choose one of the 27 solids offered by the app to explore them. They can move and scale them as well as rotate them using touch gestures. Using a swipe gesture or touches the solid can be unfolded dynamically or step-by-step into a net. Several nets for each solid are offered and more can be created by the student. Nets can be printed, with additional glue flaps added. Furthermore, the faces, vertices and edges of the nets can be colored.

[2] http://dlgs.uni-potsdam.de/oer

Step 3: How does the *interaction* develop? The students' actions manifest themselves as goal-directed, individual interactions on physical and virtual manipulatives. The combination of physical and virtual manipulatives leads to a (further) development of the spatial imagination, which results in the fact that the concrete actions gradually pass into operations and thus into internalized actions.

Step 4: Is the app suitable for teaching and learning the mathematical object? The consistent representation of the user interface is self-explanatory and intuitive for the learner as well as the symbolic representation. Dynamic elements for representing interactions also serve to intuitively interact with the touch device. In terms of mathematic didactics, it should also be emphasized that the principle of the spiral curriculum can be applied to the app, in particular due to the high number of geometric solids and the varying complexity. In this way, more and more geometric solids can be explored.

Step 5: How can the app be used in classroom instruction? For a meaningful use of the app in primary school, the students should have basic experience with and basic knowledge of geometric solids. Regarding knowledge transfer, the app supports the acquisition of conceptual knowledge related to the properties of geometric solids or solid groups as well as the nature of solid-nets. Regarding procedural knowledge, the app promotes knowledge about the composition of a solid's net, the relationship between two-dimensional solid-nets and the corresponding three-dimensional solids, and the relationships between the properties of a geometric solid.

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Minecraft in mathematics classrooms: A teacher´s perspective

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In this paper, one teacher´s experiences of using Minecraft in her mathematics classroom over several years is explored to determine the professional knowledge she drew on. The Technological Pedagogical Content Knowledge (TPACK) model is used to describe the different professional knowledges that the teacher used in bringing a digital game into her mathematics teaching. Insights from this teacher can inform other educators about the types of knowledge that need to be blended if digital games are to be used to support students' learning of mathematics.

Keywords: Minecraft, digital games, measurement, differentiated teaching.

INTRODUCTION

In this paper, we use the Technological Pedagogical Content Knowledge (TPACK) model (Mishra and Koehler, 2006) to unpack Ruzica´s expertise. Ruzica, the second author on this paper, is an experienced teacher, who has used Minecraft in her mathematics teaching over several years. We use TPACK to better understand the sorts of professional knowledges that Ruzica used for incorporating digital games into her mathematics teaching. We consider that this information is useful for teacher educators and professional development facilitators in determining how to support other teachers to incorporate digital games into their mathematics teaching.

In many countries, there have been suggestions to include digital games in mathematics teaching (Holden & Williams, 2014). Although mathematics education research has been slow to investigate them, digital games have become more prominent in the last decade, coinciding with the introduction of Minecraft into mathematics classrooms (Xolocotzin & Pretelín-Ricárdez, 2015). Minecraft (https://minecraft.net/nb-no/) is a sandbox game in which players create their own environment with a set of tools, allowing them to both create and solve their own problems (Williams, 2010; Seventko, Panorkou & Greenstein, 2017). Yet the newness of Minecraft means there is limited research into the kinds of professional knowledge that teachers need for incorporating sandbox games successfully into mathematics lessons.

What research there has been on Minecraft in mathematics classrooms has focused on geometrical understandings, such as moving from 2-D drawings to 3-D constructions, measurement concepts of area, volume and geometrical understandings to do with scale (Foerster, 2017). Measurement concepts have been shown to be difficult for children to grasp because of confusion about what attribute is being measured and the relationship between procedural formulae and conceptual understandings (Browning, Edson, Kimani, & Aslan-Tutak, 2014). Baturo and Nason (1996) noted that the preservice teachers in their study confused determining area with perimeter, which they related to a lack of understanding about the dynamic relationship between the two attributes. Foerster (2017) found that Grade 5 and 6 students improved their understandings by being front-loaded with information about these concepts in two lectures before playing with Minecraft. Similarly, Bos, Wilder, Cook and O´Donnell (2014) provided anecdotal evidence about third graders using Minecraft to develop perimeter and area understandings. They also suggested that Minecraft could be used in the teaching of other areas of mathematics but provided no research about this.

Although there seems to be agreement about the potential benefit of Minecraft in mathematics learning, concerns have been raised about the skills that teachers need for incorporating digital games, such as Minecraft, into their teaching, including their teaching of mathematics. Holden and Williams (2014) stated:

> Teaching mathematics with video games will require that teachers deftly consider trends, limitations, and the insights of case studies …; make professional judgments relevant to local context; and reflect upon the successes and challenges associated with teaching and student learning. (p. 5)

One issue is to do with the balance between teachers´ requirements for children´s learning and children´s expectations about playing a digital game (Mørch & Thomassen, 2016). Seventko et al. (2017) investigated how Minecraft projects gave children agency in their learning of mathematics. They found that children´s expectations about playing Minecraft at home interfered with the researchers need for them to explain to an adult what they were doing mathematically. At the same time, the children spontaneously interacted with their peers to share solution pathways.

Consequently, teachers may not have the necessary skills to implement digital games, such as Minecraft, from their teacher education (Nebel, Schneider & Rey, 2016). Based on overviews of previous research, Southgate, Budd and Smith (2017) designed a framework, with a series of questions that outlines what teachers should focus on when choosing digital games, including Minecraft. However, the lack of teacher input into such frameworks indicates that there remains a need to investigate teachers´ perspectives on the skills needed for successfully incorporating digital games.

TECHNOLOGICAL PEDAGOGICAL CONTENT KNOWLEDGE[1]

TPACK (see Figure 1) was chosen as the framework for this study as:

> It attempts to capture some of the essential qualities of teacher knowledge required for technology integration in teaching, while addressing the complex, multifaceted, and situated nature of this knowledge. We argue, briefly, that thoughtful pedagogical uses of technology require the development of a complex, situated form of knowledge that we call Technological Pedagogical Content Knowledge (TPCK). In doing so, we posit the complex roles of, and interplay among, three main components of learning environments: content, pedagogy, and technology. (Mishra & Koehler, 2006, p. 1017)

To consider how digital games, particular sandbox games such as Minecraft, can be used in mathematics classrooms, there is a need to determine which sets of knowledges, from the TPCK model, are combined and in what ways by experienced teachers, such as Ruzica. It is, therefore, useful to consider the ideas separately before considering how they were combined.

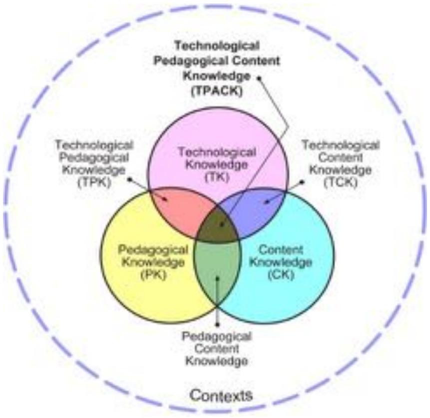

Figure 4: Model showing the components of Mishra and Koehler's (2006) Technological Pedagogical Content Knowledge (from: http://tpack.org/tpck/index.php?title=Main_Page)

The components of the TPACK model are:

Content knowledge (CK) is knowledge of mathematics and how different ideas are related, such as how perimeter, area and volume are related both at a conceptual level but also in regard to the procedures needed to calculate them (Browning et al., 2014).

Pedagogical knowledge (PK) is "deep knowledge about the processes and practices or methods of teaching and learning and how it encompasses, among other things, overall educational purposes, values, and aims" (Mishra & Koehler, 2006, p. 1026).

Technology knowledge (TK) is knowledge of different technologies, not just digital technologies, and what skills are required for using them.

Pedagogical content knowledge (PCK) describes the pedagogical knowledge needed for teaching a specific mathematical topic. It includes "knowledge of what makes concepts difficult or easy to learn, knowledge of students' prior knowledge, and theories of epistemology" (Mishra & Koehler, 2006, p. 1026).

Technological content knowledge (TCK) is knowledge about the possibilities or affordances that technology provides for illustrating specific mathematical concepts. Not all technologies will represent different mathematical ideas as well as others.

Technological pedagogical knowledge (TPK) is "knowledge of the existence, components, and capabilities of various technologies as they are used in teaching and learning settings, and conversely, knowing how teaching might change as the result of using particular technologies" (Mishra & Koehler, 2006, p. 1028).

Technological pedagogical content knowledge (TPCK) combines information about how the content and pedagogical affordances of technology can be used for the teaching and learning of specific mathematical ideas by different groups of students.

METHOD AND BACKGROUND

Having known each other since 2011, the two authors began talking about Ruzica´s teaching with Minecraft in 2016. Since then, we have collected teaching notes, photos and a recorded interview from one of our early discussions. Ruzica also collected interviews with the children and parents about Minecraft. In this paper, we analyse the 2016 interview using TPACK. The interview followed the format of life histories (Goodson, 2013), focussing on the three years that Ruzica had used Minecraft and the context in which her teaching was developing. As such, it was an unstructured interview with Tamsin´s contributions mainly being to prompt Ruzica´s reflections on her work. The interview was first analysed using the TPACK model by Tamsin. The results were then shared with Ruzica who provided feedback on the interpretation and elaborated on some of the points. The results are provided in the next section

Ruzica is a specialist mathematics and science teacher for grades 1-6 in Swedish compulsory schools. She began using Minecraft in her mathematics classes in the 2014-2015 academic year, following the interest shown by her students. Since then she has used it with different classes, from Grades 3 to 6, in the different schools, with different demographics of students. When she began using Minecraft, she did not know other teachers using it. Her initial instructions to the class were based on the work of another teacher, which she found on the internet. These instructions included:

> Each block is 1 cubic meter, i.e. one meter in each direction.
>
> • Your house´s perimeter should be between 48 and 60 m.
>
> • Your house should not be higher than 6 m.
>
> • You should make the house like a cube or straight block and build on a roof afterwards if you want to.

Once the students had built their houses, they then presented them to the rest of the class and discussed the perimeters, areas and volumes. The other students were expected to ask questions about the presentations.

RESULTS

Ruzica´s discussion about what she did and why illustrated the different kinds of knowledge that she drew upon. In this section, we provide examples from the interview of each of these.

CK: Ruzica´s choice of the content to focus on was based both on her knowledge of mathematics but also of her knowledge of the curriculum. As was the case in other work with Minecraft in mathematics classrooms (Foerster, 2017), her focus was primarily on the measurement concepts of perimeter, area and volume, which she connected to geometric knowledge about 2D and 3D shapes. In a third grade class in the following year, she included, “a focus also on multiplication because they didn´t know it at all. They have to know multiplication in third grade but they didn´t.”

PK: Ruzica described how she was influenced by a four stage model for teaching, common in Sweden, where the first stage is listening to the children to find out what they know about a topic and what they are interested in. This information provides the background for developing the teaching. From asking the children what they were interested, Ruzica decided that Minecraft would be a good context for teaching mathematics, “you can´t just go your own way, you have to listen, you have to know these children. That´s my key, listen to the children. That´s how I got to Minecraft”.

Another part of the model was about having children discuss and present their results. Ruzica adapted this approach to suit specific groups of students. For example, she considered that her students who had Swedish as their second language children should use their first language to understand what they were doing and then to work with another child to present their results in Swedish. As well as supporting their understanding and developing their mathematical languages, she felt it also allowed them to feel safe in the classroom:

> I had a Syrian girl and I had one other girl who was born here but she talked also Arabic. So I put them two together, just to make them feel safe.

TK: Ruzica was clear that she needed to know about different technologies but this did not mean that she had to have an in depth knowledge of Minecraft. When her students first told her about Minecraft, she had not seen it, nor played with it. She did not consider that this was uncommon, “I noticed that there was a big discrepancy between students' knowledge and game culture and the teachers' knowledge about gaming”. However, she did consider it was an issue when other teachers chose not to use gaming in their classroom because they were unfamiliar with it, “You know teachers, “Do something new with IT? No, no not my interest””.

Ruzica was also aware of potential problems with the technology. Initially the tablets that her class used were shared with other classes, resulting in a risk that her students´ work with Minecraft could be erased. This was a problem she could not fix. However, when she moved schools, she was able to ensure her class had their own set of tablets, which eliminated the problem.

PCK: Knowledge of mathematics and of pedagogy were combined in how Ruzica described her teaching. For example, she frontloaded geometric knowledge of 2D and 3D shapes for the students, by providing experiences with concrete materials, before beginning with Minecraft. As was the case with Foerster´s (2017) research, frontloading was helpful in supporting students to make sense of the problem that they were given to solve and using appropriate terms for discussing their ideas with others.

TCK: Ruzica recognised that the Minecraft blocks, as cubes, provided opportunities for children to explore differences between perimeter, area and volume. By giving the blocks specific dimensions, students could explore different possibilities for making their houses fit the requirements of the problem. With her third grade class, she also utilised the blocks to support children´s work with multiplication, “you can do work with a cube and they have to calculate something, everybody has a tablet in front of them".

TPK: Ruzica identified several points where she had combined the technological knowledge with pedagogical knowledge. One of these was to act as a novice who needed to ask questions as a way of supporting other students to ask questions and for the Minecraft experts to provide relevant information.

> When I initially had difficulty with the game, I noticed that the students, who had not played for a long time, were also having trouble. I asked the questions and the students could go through with the whole class how certain functions worked. Among other things, how to build a lift and how to go up and down with it. How one used, for example, redstone it provides electricity and you also learn about mechanics.

Different technologies also provided opportunities for instigating specific pedagogical approaches. For example, having a projector in the classroom, facilitated children presenting their findings to the whole class by showing their work on the tablets on a larger screen. The tablets also provided opportunities for two children to work together around a shared screen (see Figure 2). Ruzica considered that this contributed to them talking about what they were doing. However, it also meant that she had to monitor that no one child monopolised the tablet and make sure that they swapped control regularly.

Figure 2: A pair of children working on a task with the written plan beside them.

Ruzica also described how children from a Grade 1 class came to visit her Grade 3 class because their teacher had wanted to see how Ruzica worked. The Grade 3 students worked in small groups with the Grade 1 students. The outcome for the Grade 3 students was, "they felt that they knew something that the others couldn´t. And they showed them how to and they know the other details that didn´t work. It was great really, the co-operation is important." The affordances of the technology provided the Grade 3 students with an opportunity to act as knowledgeable peers and have an authentic audience to discuss what they knew.

TPACK: Identifying the components that make up TPACK illustrates the complexity of the points that Ruzica took into consideration when planning and implementing mathematics lessons based on Minecraft. Her pedagogical approach of listening to the children led her initially to investigating how she could use Minecraft. Checking on the internet provided her with a lesson plan that she could use for confronting her students´ potential confusion about perimeter, area and volume, part of the curriculum requirements that her students often had difficulties with. She saw her responsibility as a teacher as important in determining how best to support her students.

> Your goal here is perimeter, area and volume. It´s how the students are working with this question that you have provided for them and that´s why you as a teacher are really, really important. So you can do anything with Minecraft with mathematics. It´s up to you.

However, she did not considered that she needed to be an expert in Minecraft. Instead, she made use of expert students´ technological knowledge to ensure that everyone had the necessary Minecraft skills to complete the task. Use of the technology should not hinder some students from engaging with the mathematics. At the same time, she frontloaded mathematical language by engaging the students in tasks with concrete materials to show the differences between 2D and 3D shapes. She reinforced the need for the children to be able to talk about their ideas by having them present their houses and ideas about perimeter, area and volume to the rest of the class. She also combined her knowledge of the technology and the mathematics to support particular students. For example, she encouraged the use of Arabic by a recently arrived refugee student with another student to support their understandings about perimeter, area and volume. She also provided a word list with the prepositions in Swedish and asked the students to provide Arabic translations. However, by still requiring them to present their finding to the rest of the class, she supported them to use Swedish. In this way, she involved the children with three main technologies. Initially she had the children use chick peas and toothpicks to make models of 3D shapes. Then the children then used Minecraft on tablets to solve the problem and use the projector to present their finding to the class.

CONCLUSION

TPACK provides a useful way to unpack the different knowledges of an experienced teacher. The experiences described here illustrate how different knowledges are highlighted or combined when designing and implementing mathematics lessons based on Minecraft. Teacher educators need to recognise that in working with preservice teachers that they need to become aware of how their pedagogical approach will affect the choice of technologies and their willingness to allow students to be in control. Their main role will be to ensure that the mathematical learning goals remain transparent.

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Symbolic sketching: A challenge for online assessment in an interactive multiple-representation environment

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We explored the challenge of analyzing "symbolic sketching," using the Seeing the Entire Picture (STEP) online assessment platform. STEP enables students to submit examples of mathematical objects constructed as interactive diagrams. Students use various tools to draw functions and analyze specified properties. We describe submitted symbolic expressions that are (a) incorrect, although the graph expresses the main properties of the solution, which when analyzed as a sketch may meet the requirements of the task, and (b) correct sketches that are too complex to analyze symbolically, which must therefore be treated as a sketch.

Keywords: Task design, calculus, multiple representations, online assessment

RATIONALE & FRAMEWORK

Students' choices of representations and of transitions between these representations serve as an indication of students' proficiency or strategy preference (Duval, 2006). Tasks designed in multiple linked representation (MLR) environments have the potential to support problem-solving processes, to catalyze perceptions, and to serve as indications of students' perceptions and concept images. Naftaliev and Yerushalmy (2017) provided evidence that the multiplicity of representational and viewing options in interactive environments shapes students' mathematical activity and thinking, enables them to grasp the qualitative properties of objects and relationships, and highlights important aspects of their reasoning. Using the Seeing the Entire Picture (STEP) online assessment platform, in our design of technology-based rich assessment tasks, we provide means for generating examples in multiple representations, to support calculus problem-solving, and to mirror reasoning processes. STEP enables students to submit examples of mathematical objects constructed as interactive diagrams (based on Geogebra). One of our efforts focuses on the design of tasks that require freehand construction of the graph of a function in ways that facilitate automatic analysis of submissions. We found that a design that allows students to choose in which representation to submit the answer, either by freehand sketching or through a symbolic expression that is automatically graphed, offers a mathematically appropriate means of expression. Most often, such design encourages students to start by "sensing" the problem using freehand sketching, and provides informative indication of the mathematical knowledge of the student (Yerushalmy et al., 2017). Accordingly, calculus tasks in STEP[1] are designed to include several tools that provide different ways of expressing mathematical ideas. This design principle assumes that for many students, symbolic expressions are not the only choice of communication, that it is important to retain the natural communication of mathematical ideas through freehand drawing, and that such tasks increase the ability to make informed assessment decisions about students' work. The drawing tools support freehand marking of points (e.g., to indicate discontinuity) and lines (e.g., vertical and horizontal asymptotes and tangent lines). We call these inscriptions iconic. These iconic tools are not intended to replace the full scale of the symbol structures, symbol manipulations, and mathematical procedures that support validation of symbolically-represented examples, but rather to: (a) support personal choice of mathematical inscriptions; (b) support semantic understanding and informal sense making of expressions (along the lines indicated by Yerushalmy (1997), Gafni & Yerushalmy (1992), Arcavi (1994)); (c) enrich the range of submitted answers and the ability to devise informative and meaningful assessment; and (d) allow for more accurate and less interpretive automatic online assessment of freehand sketches.

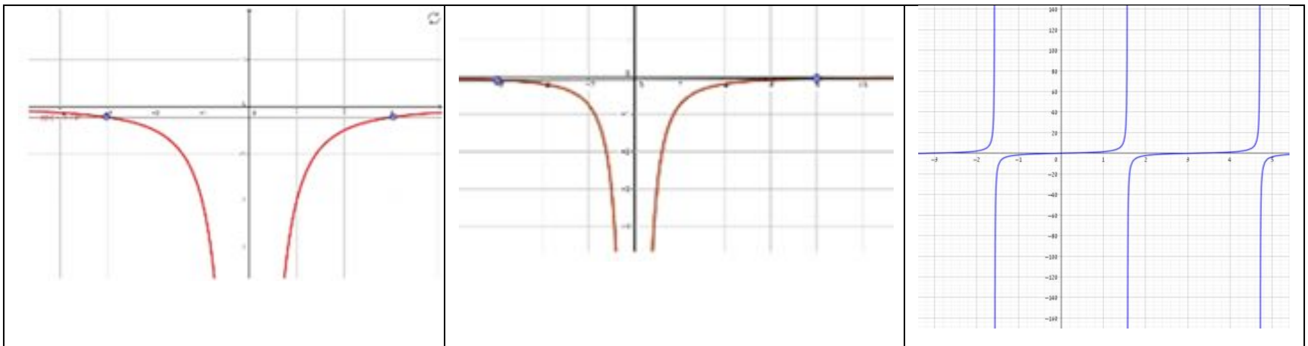
To demonstrate how these assumptions are reflected in the design of an interactive task, and what characteristics of an answer can be assessed automatically, we describe a hypothetical flaw in design. Consider a task that requires constructing and submitting examples either by freehand sketching or through a symbolic expression, of the graph of a function with a removable discontinuity point at x=1. We would expect several types of answers, for example: (a) "pure sketch," consisting of a free-hand sketch of a line and a hollow point mark (that looks like $f_1(x) = x$, with a hollow mark at (1,1)); (b) "pure symbol," consisting of a graph of a function represented symbolically as $h(x) = x \cdot \frac{x-1}{x-1}$ (the removable discontinuity point is embedded in the symbolic expression); (c) "symbolic sketch," consisting of a mix of a "neat" graph of the function $f_1(x) = x$, where a point of discontinuity is marked with the hollow point tool. This example demonstrates that characterizing students' submissions by the type of representation used is insufficient; we need to further analyze how it was used. All three submissions are acceptable correct answers: options (b) and (c) ($f_1(x) = x$ and $h(x) = x \cdot \frac{x-1}{x-1}$) describe similar graphs; option (b) offers effective symbolic validation; options (a) and (c) are sketches that represent the idea of removable discontinuity. Three such submissions may represent different processes of reasoning (Zaslavsky & Zodik, 2014), which we attempt to reveal through automatic analysis of the mathematics describing each submission. This study is part of our attempt to investigate the challenges of online analysis of rich problem solving. We argue that online assessment environments that support freehand drawing, expression input, and iconic tools for constructing and analyzing graphs of functions should be developed with awareness of the possible mixed technology-based inscriptions. Here we demonstrate an investigation of submissions that use a mix of communication channels, especially the notion of a "symbolic sketch," and the strategic choices that students make in this regard.

THE STUDY AND METHODOLOGY

The present study is part of a larger research project on the principles of innovative assessment designs in an MLR environment. Using a design-based research methodology (DBR), we conducted a two-cycle study focusing on one e-task concerning tangency to the graph of a function. Data for this report consist of submissions by high school students aged 16-17, who volunteered to anonymously solve the tasks. All students were enrolled in the most advanced high school calculus course. They all used the same curricular resources, but were taught by different teachers in different schools. They were all conversant with the basic graphing technology. Before the experiment, students participated in a preparatory session to familiarize themselves with the STEP environment. Eighty-six students participated in the experiment. The tasks require constructing examples of functions that have a single tangent line at two different points. In the instructions, students were asked to submit a freehand sketch or an expression of a function, to highlight the points of tangency, and to add the tangent line defined by these two points.[2] The STEP platform can identify and characterize automatically the method of construction, its mathematical characteristics, the tools used, and its correctness. For symbolic input, correctness is unequivocally determined by symbolic analysis. Freehand sketches are analyzed using a set of graphic filters that can determine whether the sketch represents a function, and if so, its characteristics (e.g., continuity, concavity, etc.). Using this analysis process, we encountered several instances in which it was not clear whether the type or representation of the input, in itself, was able to constructively inform the assessment.

ANALYSIS

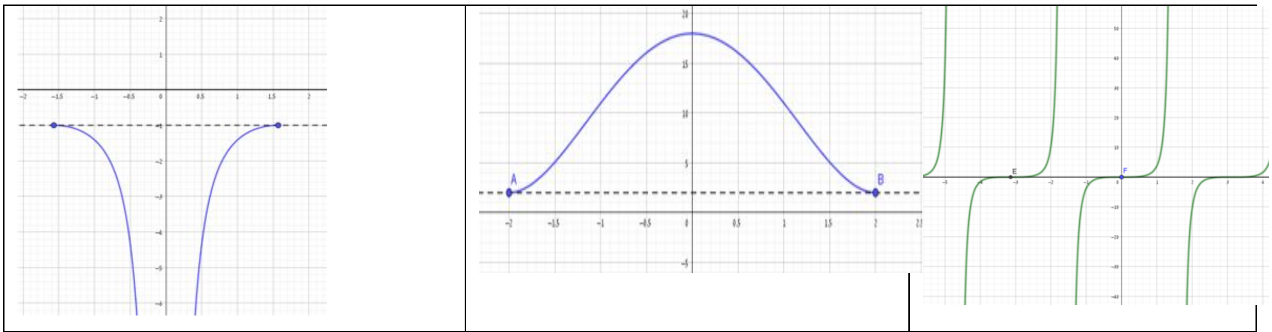
When asked to submit either a sketch or an expression of a function with a line that is tangential to the graph in two different points, some students submitted symbolic expression of functions such as $\mathrm{f}(\mathrm{x}) = \frac{-3}{\mathrm{x}^2}\mathrm{f}(\mathrm{x}) = \frac{-3}{\mathrm{x}^2}$, $\mathrm{g}(\mathrm{x}) = \frac{-2}{\mathrm{x}^2}$, and $\mathrm{h}(\mathrm{x}) = \tan(\mathrm{x})$ (Figure 1), and using the line tool, added a horizontal line to represent the tangent. These submissions do not meet the requirements of the task, because the derivative of each of the above functions is non-zero for every x in the domain: $\mathrm{f}'(\mathrm{x}) = \frac{6}{\mathrm{x}^3} \neq 0$ $\mathrm{g}'(\mathrm{x}) = \frac{4}{\mathrm{x}^3} \neq 0$; $\mathrm{h}'(\mathrm{x}) = \frac{1}{(\cos(\mathrm{x}))^2} > 0$.



GH15: $\boldsymbol{f(x)} = \frac{-3}{x^2}$	GH21: $\boldsymbol{g(x)} = \frac{-2}{x^2}$	P10: $\boldsymbol{h(x)} = \mathbf{tan}\ (\boldsymbol{x})$

Figure 1. Submissions of expressions of functions with horizontal asymptote, “posing” as a horizontal tangent line in multiple points (authentic submissions)

Students GH15, GH21 (Figure 1) may have visually or conceptually confused horizontal asymptote with horizontal tangent. This may have to do with the students’ views of actual or theoretical infinity, as follows: some functions defined on an interval are tangential to a horizontal line at the edges of their domain, as shown in Figure 2, but the submissions included in Figure 1 did not indicate the domain.



$k(x) = \frac{1}{(\sin(x))^2} for -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$	$t(x) = ((x-2)(x+2))^2 + 2$ $for -2 \leq x \leq 2$	$m(x) = \tan^3 x$

Figure 2. Symbolic expressions (in a certain domain) of functions that have a common horizontal tangent line at the edge of the defined domain.

It is plausible that some students may mistake the horizontal asymptote for a kind of "tangent at infinity." Some students submitted freehand sketches of graphs that are tangent to a horizontal line in two points (Figure 3): GYA10 submitted a sketch that resembles the incorrect f(x) (Figure 1) and the correct k(x) (Figure 2). This demonstrates the difficulty of assessing students' reasoning based on freehand sketches of asymptotic graphs (Yerushalmy, 1997). By contrast, the function $m(x) = \tan^3 x$ (Figure 2) is tangent to the x-axis at $x = \pi k$, because $m'(\pi k) = \frac{3\tan^2 \pi k}{\cos^2 \pi k} = 0$, $\forall k \in Z$. Student P10 (Figure 1), submitted the function $h(x) = \tan x$ (which "looks" similar to $m(x) = \tan^3 x$) and a line that coincides with the x-axis as a tangent line at the inflection points. In contrast to m(x), the derivative of the function h(x) is at least equal to 1 everywhere, and the slope of the tangent cannot be zero. At the inflection points, the derivative is equal to $h'(\pi k) = \frac{1}{\cos^2 \pi k} = 1$. The student, however, appears to have rescaled the x and y axes, achieving a Cartesian system that makes the slope appear close to zero (as $m(x) = \tan^3 x$), thus meeting the requirement of the task. This strongly suggests that P10 intended the neat graph to represent a mathematical idea that is independent of the expression he used to generate the graph. The student did not calculate the derivative of the function correctly (if at all) at the relevant points.

Freehand sketches are not meant to be accurate, therefore they naturally resemble two or more different functions with similar visual characteristics, some of which are valid examples, and some of which are not. In such cases, it is difficult to determine the students' intentions and to evaluate their answers. This raises a concern for assessment: students may submit an incorrect symbolic expression (Figure 1) intending to make use of its graph as a sketch. How should the student's reasoning be assessed in such cases? Students may submit an expression to describe an idea. Did they mean to use the expression to generate an example of a neat graph that can be verified analytically, or to produce an example that graphically describes an idea that can be verified only visually, without algebraic analysis (similar to the submissions in Figure 3)?

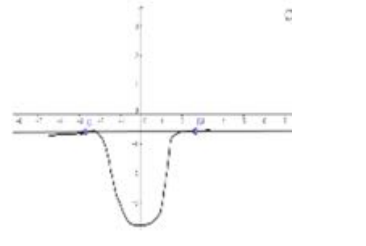
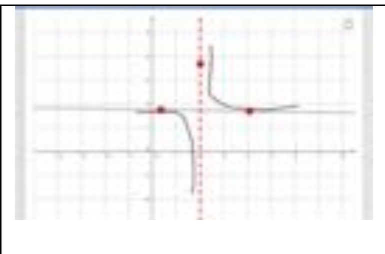
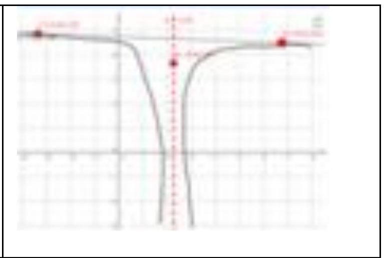
		
GH2	GH13	GYA10

Figure 3. Submissions of sketches of functions that are tangent to a horizontal line at two points at the boundary of their domain (Authentic submissions)

Other cases posed similar interpretation challenges. A freehand sketch is considered to be relatively intuitive, and can inform us about students' perceptions without mediation (see also Yerushalmy et al. (2017) on the characteristics of freehand sketching). To add the tangent line, students could have computed an expression for the tangent, or have used the "line between two points" tool, which was provided as part of the interactive environment, to approximate the tangent. In some cases no such line exists, even approximately, as demonstrated by the submissions in Figure 1. In other cases, the answer may not be unequivocal, or may be difficult to prove symbolically, and therefore it is not expected from high school students. Some students submitted highly complex expressions of functions, such as: $p(x)=\frac{x^2+2x+4}{\sqrt{x^2-4}}$ $p(x)=\frac{x^2+2x+4}{\sqrt{4+x^2}}$ (Figure 4 (a)), and $q(x)=\sqrt[3]{x}+x^5$ (Figure (a)), and added an iconic tangent line. It is difficult to validate this example by providing a symbolic representation of an appropriate tangent.

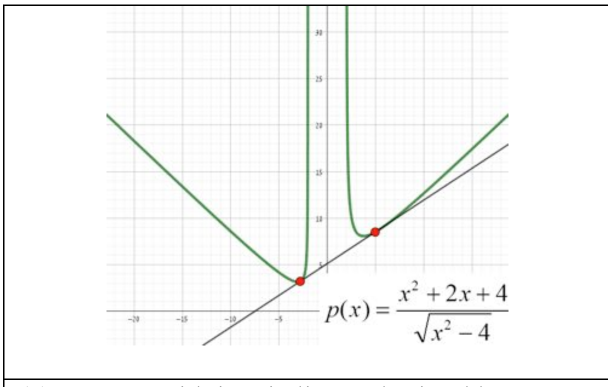
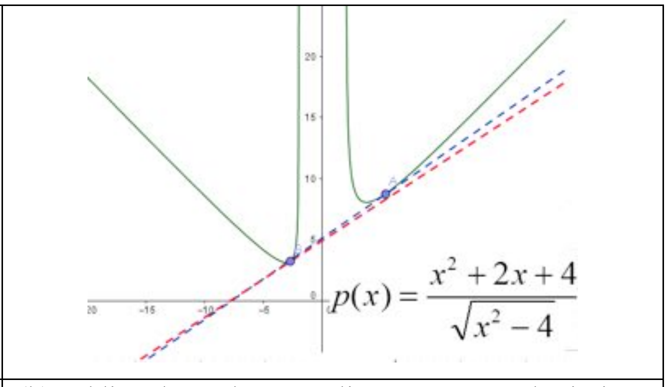
	
(a) Answer with iconic line, submitted by IRI2	(b) Adding the real tangent lines to IRI2's submissions

Figure 4: (a) the authentic submission for a multiple points of tangency task; (b) adding the real tangent lines to the submission for a multiple points of tangency task

If we add the real tangents at each of the points (Figure 4 (b), Figure 5 (b1, b2)), we see that they are very close to each other, but do not coincide. Dragging the points individually to cause the tangents to meet is challenging, but a different action could prove existence: constructing the tangent in one point, and dragging the point until the tangent no longer intersects the graph anywhere else, then dragging it until it does intersect. If the function is sufficiently smooth, there must be an intermediate case where a second point of tangency exists. Although this proof requires some delicate application of the intermediate value theorem, we propose that this thought experiment makes the existence of such a line perceptually salient.

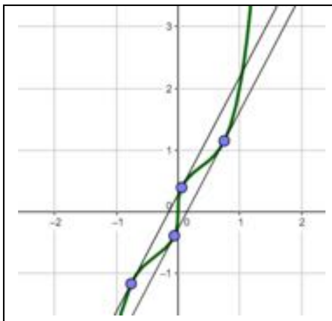
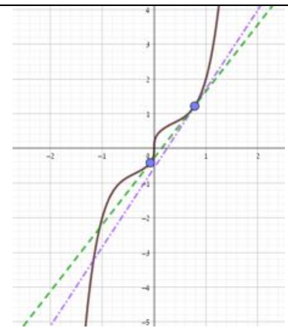
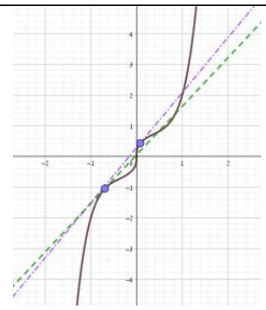
		
(a) Answer submitted by P10, with iconic lines: $q(x) = \sqrt[3]{x} + x^5$	(b_1) Adding the real tangent lines to P10's submission	(b_2) Adding the real tangent lines to P10's submissions

Figure 5: (a) The authentic submissions by P10 for the multiple points of tangency task; (b1, b2) adding the real tangent lines in P10's submission

DISCUSSION

If students are allowed to submit either a freehand sketch or an expression, in an environment of iconic representational tools, we can expect a broad range of submissions that must be assessed based on different types of criteria. At one end of the scale, we have purely symbolic solutions, with symbolic representations of function f(x) and line y=mx+n, where derivation is used to validate that the slopes of the line and of the function are equal in two common points. At the other end of the scale we have pure sketch, a freehand sketch of the graph and a line drawn between two points in such a way that the line looks as if it is tangential in these points. Between the two extremes there are multiple ways to solve the task, which may be considered to be symbolic sketching. In this type of submission, the function is graphed symbolically but the required analysis (the tangent line) is drawn as a sketch. In this study we explored some of the challenges in analyzing symbolic sketching, which arise in the presence of rich representational tools and sketching methods. We have shown that working in "sketch land," where it is possible to submit either a freehand sketch or an algebraic expression, both furnished with iconic annotations, students may choose to use expressions to represent a mathematical idea rather than seek precision.

We presented submissions of neat graphs that would have been considered incorrect if analyzed symbolically (Figure 1), but the students who submitted their work apparently did not look for analytic verifications, and analyzed the graph as a sketch (as in Figure 3), which may be interpreted as a correct example. When the submitted examples are elementary functions that involve simple calculations, we expect students to analytically verify them, and recognize when they are not valid examples. But if they intend their algebraic submission to be interpreted as a sketch, we expect them to add a comment to this effect. A different design may consider restricting the tools to fit the representation mode.

As demonstrated in Figures 4 and 5, occasionally the task required examples of functions that are algebraically known, but symbolic analysis for verification and proof of existence are beyond the scope of high school mathematics. Nevertheless, the rich example space of functions that were submitted in symbolic representation was intriguing; it invites exploration, raises interesting questions of existence and of mathematical substantiation, and has the potential to deepen the class discussion.

The principles that guided our design, together with rich representational resources provided to produce evidence of students' mathematical knowledge, create great challenges for automatic (as well as human) assessment aimed at gaining better information about compound strategies in students' work. These challenges include questions related to whether symbolically represented functions are intended to serve as conceptual sketches or as (possibly incorrect) symbolic examples. Regarding students' tacit argumentation, we are facing the challenge of interpreting the role of the iconic inscriptions. These challenges illustrate the need to develop class norms regarding what is considered to be a sketch, and in what sense sketches substantiate claims of existence, including discussions about the limitations of symbolic expressions that function as sketches. If such norms are adopted by students, they make possible a more reliable interpretation of their submissions. These norms, which are mathematical in nature, can themselves become an object for online assessment.

NOTES

1. Seeing the Entire Picture (STEP) is a formative assessment platform developed at the Mathematics Education Research and Innovation Center, at the University of Haifa. Details about this platform are available at www.visustep.com.

2. Several versions of this task were produced, but for the focus of the present paper the differences between them (such as adding constraints to the function and asking students to submit three different examples) are not relevant. For details, see Yerushalmy et al. (2017) and Nagari-Haddif & Yerushalmy (2018), where the exact text of the tasks and the available set of tools are described.

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Auditory material for specialised language support in mathematics education

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This paper focuses on the use of auditory material in form of radio features in mathematics education at primary school level, particularly for specialised language support. The aim of my current research is to investigate the effects of auditory material on the learning procedure and how supportive language examples can be given through auditory media, as a provided language model. This way, auditory material could be effectively implemented in teaching practice.

Keywords: Auditory learning, radio sources, design science, language support.

AUDITORY LEARNING

In order to research the effects and the use of auditory material, one has to understand how auditory learning is working. According to Baddeley (2007), in the working memory visual and auditory information are processed in different sensory channels. Auditory learning reduces sensory impressions on the visual channel, so that the auditory channel is required and therefore trained more. Acoustic information is processed through various memory processes. The echo memory saves mostly unprocessed sensory impressions for a short period of time. In a next step, the working memory with its limited capacity decodes and processes the acoustic information. It can also build temporal connections that are necessary for remembering the beginning of a sentence while hearing the end of it (Leuders, 2011). According to Leuders, an increase in efficiency of the working memory is a training effect. Since the processing of auditory information takes place in the working memory, one can also assume that the increase in auditory learning efficiency is a training effect. This is a fundamental trigger for my interest in research on auditory learning materials in mathematics education.

Another important aspect is the Cognitive Load Theory by Schweller (1994), more recently taken up by Rink (2014) in digital media research. This theory says that the capacities of the working memory are limited and should not be exhausted by extrinsic factors. One of those extrinsic factors is reading. Reading difficulties can exhaust the working memory and lead to not understanding mathematical content as well as not being able to solve mathematical tasks. Keeping mathematical concerns in the centre of learning processes while reducing extrinsic factors, is one important approach for auditory learning.

AUDITORY LEARNING MATERIAL

According to the model of orality and writtenness (Schreiber and Klose 2017, based on Koch & Österreicher 1985), auditory learning materials can foremost be categorized as being medial-oral. If they are designed for children, they are also more likely conceptual-oral because they utilize everyday language and explain in a child-orientated and situational manner. Still, they use mathematical terms and phrases of school register. In this way, they are also characterized through conceptual-textual elements and can lead children from spoken language to written language. At best, this guides listeners to use mathematical terminology both when speaking and writing. This makes the listeners act in a more conceptual-textual way.

In education, auditory material can help to clarify mathematical terms (Leuders, 2011). Auditory support can also aid children with reading difficulties to understand mathematical contents and tasks without the need of having to read coherently and extract the meaning. This, however, does not mean that reading should be replaced by hearing. Auditory support should only be provided if needed.

Another function of the use of auditory material is the development of active listening skills. Such competence is a primary requirement for education, but it is seldom supported or even trained (Pimm, 1987). In Germany, education standards for the subject of German language make it very clear: These require, not only the competences of reading and writing, but also speaking and listening. Speaking is required to be consciously organized, while terminology is to be trained and (the use of) language is to be examined. Regarding listening, children are to listen attentively and perceptively, while registering others' statements and constructively dealing with them. Of course, listening also takes place in the form of frontal teaching, but this kind of listening is difficult for most students. This is why supportive elements for auditory learning are necessary – both in frontal teaching and by means of group work.

EMBEDMENT INTO COOPERATION WITH THE RADIO STATION "HR2"

In 2015, the Institute for Mathematical Education at the University of Giessen started a project in cooperation with a regional radio station "hr2 – Hessen Radio for Culture". This radio station developed, *inter alia*, a series of radio broadcasts on mathematical topics for primary level, collected in the multimedia offering "Kinderfunkkolleg Mathematik" (www.kinderfunkkolleg-mathematik.de). Within this collaboration project, future mathematics teachers developed auditory material for use in mathematics education at the primary level. It was an important challenge to "write for listening" without a visual representation of the subject. For example, verbal explanations were required to counter the fugacity of spoken language through linear representation or reputation. They also needed to be very accurate, so that the students would be able to deal with each mathematical topic in a very deep way. Throughout the process, the future teachers also had to reflect upon the mathematical topic as well as their own explanations, which was an advancing learning experience.

In a second step, the participants planned teaching units for performance in schools, embedding auditory educational material as a central element. This could be realised through developing listening tasks for the whole class or for working phases in smaller group. The important thing was to interrupt longer listening phases and repeat the segments, while giving tasks in between. The auditory material could be used as preparation of a topic or as a base for discussions, as well as for explaining and deepening new contents or repeating already covered topics.

The units were realised in different schools and reflected on in the seminar, including feedback from peers, teachers and university teachers. After the reflection, the participants optimized their units and turned them in. After correcting and editing these units one last time, the units were then allocated for the download centre of the "Kinderfunkkolleg Mathematik," as accompanying material.

SPECIALISED LANGUAGE SUPPORT THROUGH AUDITORY MATERIAL

As already mentioned, the learning of a language can be supported by training listening competence. Active processing is important for meaningful processing and for memorization of what has been heard. Reasonable and profitable use of auditory material is needed for these processes, for example good embedment, listening tasks or segmenting principle (Rink, 2017). With this kept in mind, teaching concepts can be developed, in which radio features or other auditory material serves as impulse or stimulus in the sense of didactical reduction. As acoustic representations are volatilely, there is the need of adding opportunities to document the content of the heard and results of the related tasks within the teaching units. By these means, specialised language support can be ensured. Following these ideas, my research can be focussed on the evaluation of auditory educational material in various settings – particularly regarding possible learning effects. The main interest of this research is the use of radio in mathematical education for specialised language support at the primary level.

Prediger and Krägeloh (2016) are referring to a model of three registers relevant for mathematical learning (everyday register, school register and technical register). This model illustrates different levels of verbal representation and how they are connected or built onto each other. Particularly interesting for my research is the question how children can be led from everyday register to school or even technical register.

School register is an important and necessary factor for successful learning in mathematics. It is a shared language basis and helps with explaining, describing and justifying (Götze, 2015). However, children do not bring this type of language to school with them. It must be learned, like registering a new language. This applies not only to children with special needs in language development but to every other child. That is the reason why they need linguistic models to develop educational language and to fill terms with representations. These linguistic models are scaffoldings onto which children can lean (Gibson, 2002). Lexical storages, which only include words, are not sufficient, as new terms must be used in whole phrases and sentences. According to Götze (2015), language acquisition is, in practice, merely a continuous learning process. There can be setbacks and sometimes children express themselves better in written language than in spoken language. This is because everyday register is predominant in spoken language and, oftentimes, deictic expressions are used. This is valid, not only for children, but also for teachers – even if they do so unaware and unintentionally. At this point, auditory educational material could be a useful and profitable addition.

Research questions

The aim of my research is to find out in what way auditory learning material could be of use for language support in mathematics education. For this cause, I want to ask how auditory material, as a language model, can stimulate the development of the school register and how auditory material can support listening competence. In a second step, I want to research what a profitable use of such material could look like.

METHODOLOGICAL APPROACH

Regarding the data collection, I have decided to use the design research (Prediger et. al., 2012), based on Wittmann's Design Science (1995). Subject of the Design Science is the construction and research of teaching concepts, including accompanying theories. According to Wittmann (1995), this science is a practice-oriented core area for mathematical education, since it refers to the construction of artificial objects (teaching concepts, curricula etc.) and the research on possible effects in different educational settings. Innovations could find orientation in the concept of "reflective practitioners" (Schön 1987), which was developed for the training of engineers. According to this concept, practitioners are supposed to undergo a perpetual cycle of reflection, in which they repeatedly question and optimize their own actions – just as teachers have to in educational development.

Didactical Design research

Based on the Design Science, Prediger developed the model of design research (Prediger et al., 2012). The aim of this method is to effectively implement innovations for educational development in the teaching practice and empirical research, carried out under realistic conditions. In order to do this, one has to undergo a cycle as pictured in the following illustration.

The cycle starts at the upper left side of the illustration with the specification and structuring of the learning subject. In my case, this would be the auditory materials. Based on this, a design is to be developed for the specific learning topic – in my case, the teaching concept. In a third step, the developed design will be performed by means of a design-experiment. In this phase, I want to test the teaching concept in teaching practice, collect data and evaluate it. Based on the analysis of this data, local theories about the learning subject and the teaching concept can be developed in the last phase. The local theories are the starting point for a second round of the cycle (and after that, a few more), in which they can help to optimize the learning subject and concept. Thus, more experiments, data collections and evaluations are to follow, along with new local theories. In this way, after a few cycles, we will not have a perfect teaching concept or representative research results, but new and tested theories on the use of auditory media for specialized language support.

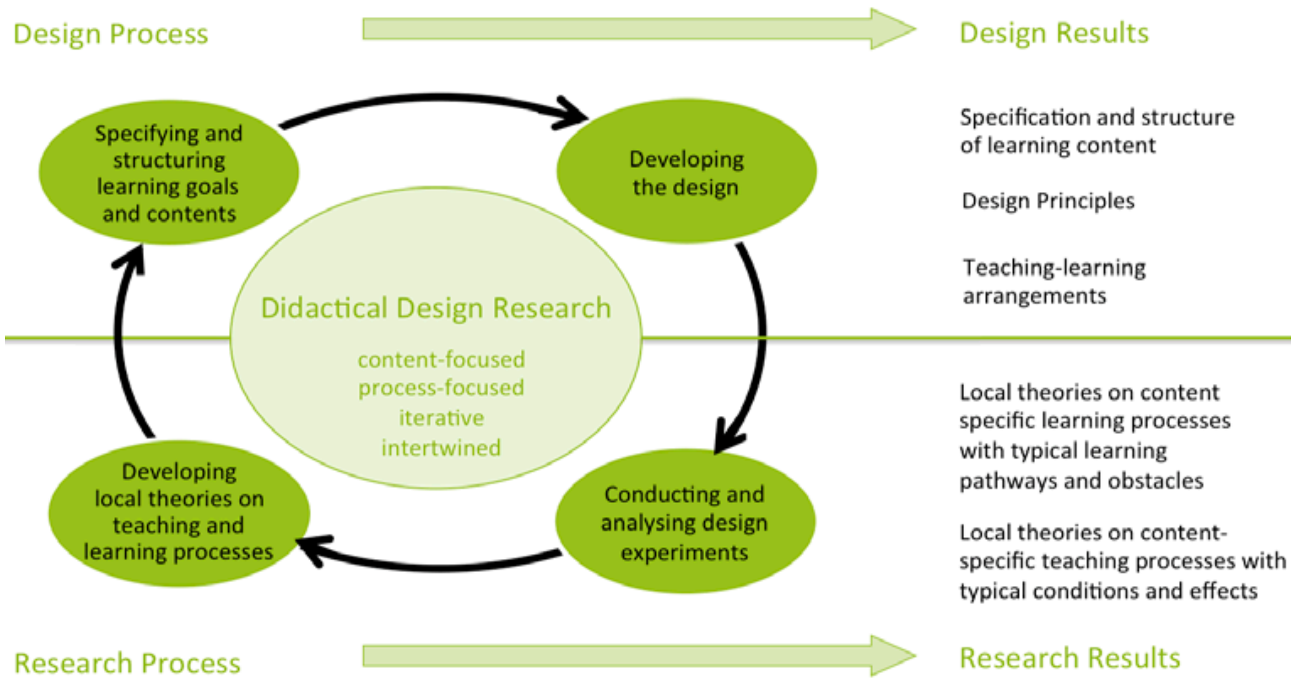

Figure 1: Model for didactical design research (Prediger & Krägeloh, 2016)

EVALUATION

To evaluate the data from my design experiments, I plan to use reconstructive social research methods (Bohnsack, 2010). This method aims to reproduce, by means of empiric data, how something is or was realised. I also want to refer to interpretative teaching research (e.g. Krummheuer & Naujok 1999), which examines a specific subject and interprets, not only the research itself, but also the specific situation of investigation – between the interaction partners and within their interaction.

Exemplary testing with mathematics teacher students started in February 2018 (Peters & Schreiber, in print). In summer 2018, I am going to run a pilot study, as well as prepare my main study. So, according to the plan, I will be able to present first results by the end of the year.

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Examining Elementary Teachers' Use of Digital Instructional Resources: A Cross-Cultural Study

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This research report examines teachers' purposes for working with digital resources in their mathematics instruction. We do so by undertaking an empirical, qualitative, cross-cultural analysis of interviews with 40 elementary school teachers from four educational contexts: Sweden, Finland, the US, and Belgium. We explore how teachers use digital resources for instructional and professional purposes and consider the possible opportunities for and challenges to transformation of teaching and learning. Despite wide variation in the types and quantity of digital resources used within and across contexts, we found some common purposes guiding teachers' selections and approaches. We also found that digital resources impacted multiple aspects of teachers' professional practices, including professional learning and interactions.

Keywords: Elementary Teachers, Elementary Mathematics, Digital Resources, Cross-cultural, Curriculum.

PURPOSE

The first decade and a half of the 21st Century have seen a rapid proliferation of digital instructional resources (DIRs)[i] available through publishers, online vendors, or open access initiatives. Many developers and educators see the potential for digital resources and tools to transform teachers' practices and students' learning opportunities, but research on teachers' interactions with them is scarce (Pepin, Choppin, Ruthven, & Sinclair, 2017). Research on DIRs has primarily been concerned with student interactions (Pepin, Gueudet, & Trouche, 2013). In light of the critical roles that teachers play in choosing and using DIRs and the demands that these resources place on teachers, scholars have argued for research that focuses on teachers' perspectives and practices (Healy & Lagrange, 2010; Remillard, 2016).

In this research report, we focus on teachers' purposes for working with digital resources in their mathematics instruction. We draw on a qualitative analysis of interviews with 40 elementary school teachers from four educational contexts: Sweden, Finland, the U.S., and Belgium. The study falls under the conference Theme 2: Mathematics curriculum development and task design in the digital age.

The study takes a cross-cultural perspective. Previous analysis of the design of print curriculum resources from different educational contexts (Hemmi, Krzywacki, & Koljonen, 2017; Remillard, Van Steenbrugge, & Bergqvist, 2016) has surfaced different assumptions about learning and teaching mathematics and the type of support teachers need, despite the identified commonalities across the official mathematics curriculum frameworks of these systems (Boesen et al., 2014). We believe that studying teachers' use of DIRs cross-culturally can shed light on the practices and norms within each context, and lead to wider understandings of how these resources can support the teaching and learning of mathematics. The following research questions guide our analysis:

1) How do teachers in different educational contexts describe the purposes and uses of digital resources in their mathematics instruction?
2) To what extent do these purposes portend opportunities for transformation of teaching and learning through the take up of digital resources?

Our analysis assumes that teachers are key mediators in the take-up and use of curriculum resources, and, as such, the potential for these resources to transform mathematics teaching and learning is contingent on how teachers use them.

BACKGROUND AND THEORETICAL FRAMING

Our research builds on existing frameworks for studying teachers' use of print curriculum resources (e.g., Gueudet & Trouche, 2009; Remillard, 2005) and a new framework oriented toward the design and affordances of digital curriculum resources (DCRs) (Pepin et al., 2017). Remillard (2005) conceptualizes teachers' curriculum use as a dynamic interplay between the teacher and the curriculum resource, viewing resource use as a participatory process, rather than one of passive implementation. In this process, teachers' beliefs, knowledge, and instructional purposes come into play, as do features of the resource. There is empirical evidence that particular designs can support teacher learning through their use (Collopy, 2003; Stein & Kaufman, 2010).

Pepin et al. (2017) offer a framework for examining aspects of the designs of DCRs. They conceptualize features of DCRs and discuss their potential for having transformative impacts on teaching and learning. Given the adaptable nature of digital media, these features can create new opportunities for learning and for the work of teaching. The four features that impact the learning space include: a) the presentation space (how material and topics are presented to students); b) the problem space (types of problems students encounter and the range of ways they might solve them); c) the work space (tools and resources available to students to solve problems); and d) the navigational space (possible paths for progressing through the content of the resource, including linear or non-linear routes). Despite the potential for transformation across these various features, Pepin et al. argue that much of the current design activity in mathematics education focuses on the presentation space and suggest that true transformation of learning through digital resource must include changes to the problem and work space, as well. Additionally, Pepin et al. discuss features that impact how teachers monitor and assess student learning, which can lead to shifts in teachers' practices related to formative assessment and related instruction.

DESIGN AND METHODS

The data come from the first of a four-year study of teachers' use of print and digital mathematics instructional resources in Belgium, Finland, Sweden, and the U.S. The selection of educational contexts represents the cultural backgrounds of the research team and allows us to leverage insider perspectives in our analysis. In Belgium, we focus on the Flanders context, the Dutch-speaking community, which is culturally, and educationally distinct from Wallonia (the French-speaking community).

To identify participants, we selected two elementary mathematics curriculum programs from each context, one highly used and the other with unique characteristics. We identified a convenience sample of 10 grade 1-6 teachers, 5 from each program, from schools in each context, without the intent of drawing a representative sample. They vary in years of experience and the type and size of school they teach in.

This paper presents findings from the first of two interviews of the 40 teachers. We conducted and audio recorded one-hour, semi-structured interviews, addressing teacher background and school characteristics; the print, digital, and concrete instructional resources used by the teacher; the teacher's views on these resources; and the teacher's general beliefs about teaching and learning mathematics.

Our analytical approach involved analysis based on conversations between insiders and outsiders of each context (c.f. Clarke, 2013; Hemmi & Ryve, 2015; Pepin & Haggarty, 2001; Stigler & Hiebert, 1999; Tobin, Hsueh, & Karasawa, 2009). Cultural insiders undertook initial coding, using a priori codes, in the original language of the interview, summarizing within context themes in a spreadsheet in English. The full team discussed selected themes and made comparisons across them. Through these discussions, several common purposes for choosing and using DIRs emerged. Cultural insiders then identified the purposes offered by all teachers in the interviews and summarized patterns in a matrix. The full team discussed similarities and differences, provided clarifications, and used the four features identified Pepin et al. (2017) to consider the potential for transformation of learning opportunities through the uses of DIRs across the data set. When needed, team members translated illustrative quotes into English, which led to further discussion and clarification of the five purposes.

FINDINGS

We identified three primary ways that teachers described using DIRs for mathematics instruction: a) enhancing whole-class instruction, b) structuring students' mathematics work, and c) professional participation and learning. Within these categories, teachers voiced different purposes for how they used DIRs. In this section, we outline commonly available digital tools and hardware in each context and then discuss the ways teachers used DIRs and the purposes behind their uses. We draw on Pepin et al. (2017) to consider the potential for transformation of learning spaces.

Available Resources across Contexts

We found a range of different digital resources and tools in classrooms in all four contexts. Almost all classrooms were equipped with interactive whiteboards and had access to computers or tablets on a regular basis. Several classrooms (in Sweden and U.S.) provided a laptop or tablet for each student throughout the day. All teachers had access to a range of resources available through the internet. In the three European contexts, teachers had access to online platforms, such as ViLLE or Bingel, which allowed teachers to manage students' work on various assignments.

Enhancing Whole-Class Instruction

Participating teachers described two primary ways that they incorporated digital resources into their lessons: supplementing the presentation of their print-based curriculum programs and increasing opportunities for students to interact or share their work. These uses of digital resources fall within Pepin et al.'s (2017) category of presentation space. In all four contexts, teachers reported using interactive whiteboards or computer projectors to display static or dynamic portions of the print textbook, either as an image or with copied text.

Eight of 10 U.S. teachers reported using ready-made presentations or videos sources from the primary program or found online to present concepts to students. The criteria most often identified for selecting these resources were ease of use, likelihood of engaging students, and alignment with the content in the core program. A grade 6 teacher using *Eureka Math* described the videos she found to introduce concepts as:

> They show visuals, they show models…. This guy's kind of making it entertaining for them. The strategies are the tools that we're teaching; they both [the two video publishers] use kind of the same strategies and tools as *Eureka*. (US8.EM6.Int1)

Only 1 of the 2 Swedish programs supplied digital presentations (*Favorit Matematik*) and 4 of the 5 teachers using this program regularly employed them during instruction. Many of the Finnish teachers also reported projecting dynamic or static components of the textbook during instruction, although they did so with varying frequency. Only 3 of 10 Belgian teachers used dynamic or static components of the textbook during the instructional phase of their lessons, but all projected an image of the student textbook page during the practice phase of the lesson.

A number of Swedish teachers also described using tools, such as document cameras or connected software, to incorporate student solutions into the lesson. This approach was particularly common among teachers trying the EPA instructional model, in which students first work on a task individually, then in pairs, and then discuss it with the whole class. As the following grade 5 teacher's explanation illustrates, teachers found digital tools valuable for integrating student work into whole-class discussion:

> I post some [of the] results that we received during the lesson and then we discuss these solutions by asking, for instance: How did this person think it through? ... I photograph some of the results every time. They [the students] love this. They all want to contribute with their solutions and to be posted on the board. (SW8.MD5.Int1)

Across contexts, we observed a tendency for teachers to use DIRs to change non-digital instructional presentations to digital and often dynamic ones. This practice primarily impacted the presentation space (Pepin, et al., 2017), but in somewhat modest ways. In some cases, through projecting student work, teachers found themselves engaging students differently in the presentation space of their lessons.

Structuring Students' Mathematics Work

Across all four contexts, teachers viewed DIRs as providing opportunities for students to work, individually or with others during math class or for homework. This type of use of DIRs can affect both the problem space (types of problems and solution paths) and work space (tools available to solve them), as defined by Pepin et al. (2017). Teachers described three purposes for this type of resource use: engagement, personalization, and monitoring student progress.

Making practice engaging. Most of the Swedish teachers described using digital programs to provide students with additional practice in computation. Six of the teachers emphasized that these tools were either "fun" or offered a break from the usual routine. Others emphasized the importance of skill building, but they saw fun as a possible motivating factor or bonus. Similarly, all but one U.S. teacher said they assigned students to use game-like programs for several hours each week. Most of these programs focused on developing computational fluency through repeated practice. Teachers indicated that students found these computer games fun and engaging, and the adaptive features made the practice targeted and efficient.

Finnish and Belgian teachers provided more measured responses to this type of use of DIRs, questioning their role or added value over textbook tasks. In Belgium, several teachers said that they allowed students to use digital environments to solve problems at school or at home, but, they saw computer-based work as supplemental to paper-based tasks and physical tools, as exemplified by this grade 6 teacher:

> The kids use that [DCR] instead of, when they have some time left or I don't know what. Instead of saying "Do [the usual additional exercises]", I think it's more interesting that they can do these [DCR] exercises. (FL4.KP6.Int1)

Personalized learning. Many teachers saw DIRs as providing a way to personalize learning experiences based on their needs and abilities. Pepin et al. (2017) argue that DCRs can allow for flexible and varied problem spaces, affording opportunities for students to work on different types of problems and to proceed at their own pace. In the two contexts (Finland and U.S.) where curriculum programs provide substantive guidance on adapting learning opportunities, many teachers described deploying digital features provided by their primary programs or other sources to differentiate practice. Teachers in Finland described using a course design platform to tailor assignments to specific students. These assignments and tasks were adapted from the primary textbook. A Finnish teacher explained: "You don't need to indicate the same [tasks] for everyone as there're plenty of them… low-performing students had some tasks that repeated really the basics" (FI1.TT3.Int1). The 9 U.S. teachers who used DIRs had their students use game-like platforms that either automatically adapted to students' abilities by providing hints, explanations, and videos, or which they could use to assign different tasks based on students' needs.

Use of DIRs for personalized learning was mentioned less frequently as a purpose by teachers from Belgium and Sweden, the two contexts where curriculum programs provide substantially less teacher guidance on adapting learning opportunities. For instance, only one Belgian teacher described differentiating instruction with the help of the platform Bingel, doing so based on student test scores.

Assessing and Monitoring. According to Pepin et al. (2017), DCRs have the potential to deepen and extend the role of formative assessment in mathematics instruction. We found that teachers used digital resources to enhance their approaches to assessing and monitoring student work in two ways: tailoring assessments to students and monitoring student progress through system-generated tracking and reports. Teachers in Finland and Sweden described using digital platforms, such as ViLLE or Bingel, to monitor student progress on individualized assignments. One Finnish teacher explained: “It's easy for a teacher to monitor and download new assignments weekly and then check who had completed them all” (FI1.TT3.Int1). The majority of U.S. teachers also indicated that they used data reports from game-based environments to monitor students' progress on assignments. In addition, several Finnish teachers used assessment problem banks to design tests, though they gave students printed versions.

When teachers spoke of using DIRs to assign students engaging or personalized tasks, the tasks typically involved computational practice, rather than other types of problem solving. In other words, although students are working in digital environments, they may not be encountering different types of problems than standard drill and practice.

Professional Participation and Learning

In addition to using DIRs to support their mathematics instruction, teachers in three of the contexts described enhancing their professional practice through online collaboration with other teachers, communicating with parents digitally, or engaging in online professional learning. Although these non-instructional aspects of teachers' work are infrequently discussed as affordances of DCRs, we believe they may underlie important instructional transformations.

File sharing with colleagues through online repositories was the most prominent way that teachers used digital resources to enhance their practice. All 10 U.S. teachers shared lesson plans, assessments, and other activities through cloud-based platforms maintained by teachers, schools, or districts. One Belgian teacher reported using Dropbox to share lessons and other resources with her grade colleague. U.S. and Swedish teachers also described using teacher websites to share lesson plans and find those developed by other teachers. This type of online collaboration was not mentioned by Finnish teachers. Because this particular purpose was emergent in our data, we did not ask about it explicitly in our interviews; it is possible that some teachers who shared lesson materials online did not report it.

One other professional purpose emerged primarily from the U.S. teacher interviews. Four teachers reported watching videos designed to educate teachers about the mathematics concepts and lesson progressions in the units they were teaching. Some videos were provided by the curriculum publishers, while others were found on YouTube. One Belgian teacher also reported changing her instructional approach for a lesson after viewing a YouTube video recommended by a colleague.

DISCUSSION AND SIGNIFICANCE

The preliminary findings discussed above suggest a wide range of DIRs being used by teachers across the four contexts and a set of common purposes behind their use. Within these purposes, however, we found substantial variation across, and sometimes within, contexts. In line with an observation made by Pepin et al. (2017), we found that the learning space most impacted by digital resources appears to be the presentation space. We also see DIRs impacting the problem space, but the types of problems students encounter remains mainly unchanged from those encountered in print textbooks. We found almost no impact on the work or navigational spaces. While participating teachers were selected due to their use of programs that have both print and digital components, the narrow impact of DIRs is worth noting and merits further exploration. On the other hand, the unanticipated theme of professional participation and learning offers a different place to look for the impact of digital tools on elementary teachers' work. We might understand these forms of professional collaboration and learning as one way that DIRs can support teachers to transform their practice.

NOTE

1. We use the term digital instructional resources (DIRs) to refer to all resources designed and appropriated to support instruction. The term digital curriculum resource (DCR) has been used to refer to such resources that are curricular in nature, in that they contain a scope and sequence and are designed to support instruction over time. DIR is a more general term. See Pepin et al. (2017) and Remillard (2016) for more on this distinction.

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Examining peer-interaction during individual work with a digital textbook in a primary mathematics classroom

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On the one hand, digital technology supports individual learning and on the other hand it enhances opportunities for collaborative learning. In this paper, we report on an exploratory study in a primary mathematics classroom (third grade), in which a digital mathematics textbook was used. While the digital textbook offers features in order to assist individual learning, it turned out to foster peer-interaction when used in the classroom setting. We analyse the peer-interaction that occurred according to the following research questions: 1) What characterizes peer-interaction and interactions between students and teachers while students are working with a digital mathematics textbook that does not particularly afford peer-collaboration?; 2) What triggers peer-interaction in these situations?; 3) What is the content of the interaction?; 4) How far can mathematical understanding be reconstructed?

Keywords: digital technology, e-textbook, peer-collaboration, interaction, mathematical understanding.

INTRODUCTION

Regarding research on the use of digital technology two opposed trends are observable: individualization vs. collaboration. On the one hand, digital technology has the potential to foster individual learning by "generic curricula" (Klep, 2002), which promote individual learning paths, to provide individual immediate feedback, and to enable adaptive assessment (e.g. Wu, Kuo, & Wang, 2017). On the other hand, one of the main affordances of new technologies is their potential for fostering communication and collaboration. Schrage (2001; as cited in Beatty and Geiger, 2010) argues related to new technologies that "the real value of a medium lies less in the information that it carries than in the communities it creates". Beatty and Geiger (2010) identify an increasing interest in research on the role of technology in collaborative mathematical practice. They survey research related to four types of digital technologies in order to identify their role in fostering communication and collaboration among students. These include technologies designed for (1) both learning mathematics and collaboration; (2) learning mathematics but not specifically for collaboration; (3) collaboration but not necessarily learning mathematics; (4) neither learning mathematics nor collaboration (Beatty & Geiger, 2010, p. 262). They conclude that

> All [studies] used rich open-ended tasks, and all specified the affordances of the particular kind of technology used for engendering collaborative communities of practice – whether based around aggregated dynamic representations, or archiving threads of discussion in student-managed discussion platforms. And in all studies, technology was viewed as a means of mediating social interaction (Beatty & Geiger, 2010, p. 278).

As Beatty and Geiger (2010) point out, affordances of the technology or the pedagogical setting are usually the starting point of research on peer-collaboration related to the use of digital technologies. In this paper, we report on an exploratory study in a German third grade classroom, in which neither the technology particularly affords communication and collaboration among peers nor did the pedagogical setting. The study aims at a better understanding of the effects of a newly introduced technology – in our case a tablet-based digital textbook – on classroom interaction and children learning. The used tablet-based mathematics textbook offers different kinds of feedback and further assistance (lexicon, tips, help) for students' individual learning, but does not foster peer-collaboration in a different way than the traditional textbook. Consequently, the digital textbook belongs to type 2 in the typology by Beatty and Geiger (2010). Nevertheless, we noticed that children were communicating and collaborating intensely – indeed much more than in a comparable classroom with traditional textbooks. On the one hand, this led us to question the role of the opportunities for assisting individual learning provided by the textbook and on the other hand, we want to better understand the quality of the peer-interaction during the use of a digital textbook, which does not afford peer-collaboration in a particular way. Therefore, we tackle the following research questions in the study:

RQ1) What characterizes peer-interaction and interactions between students and teachers while students are working with a digital mathematics textbook that does not particularly afford peer-collaboration?

RQ2) What triggers peer-interaction in these situations?

RQ3) What is the content of the interaction?

RQ4) How far can mathematical understanding be reconstructed?

Due to page limitation, we are only able to tackle these questions exemplarily in this paper.

THEORETICAL FRAMEWORK

We contextualize the use of the digital textbook in a mathematics class within the socio-didactical tetrahedron (Rezat & Sträßer, 2012). The digital textbook is the artefact that mediates students' classroom activity with mathematics. During this activity students interact with: 1) the digital textbook; 2) the teacher; 3) peers. Thus, we focus on the relationships between five relevant vertices within the socio-didactical tetrahedron (Fig. 1): student, teacher, peers, artefact, and mathematics.

The interaction between these vertices is conceptualized differently. While the interaction of the student, the artefact, and the mathematics is conceptualized in terms of the instrumental approach (Rabardel, 2002), the interactions between the student, peers / teacher, and mathematics are regarded in terms of collaboration.

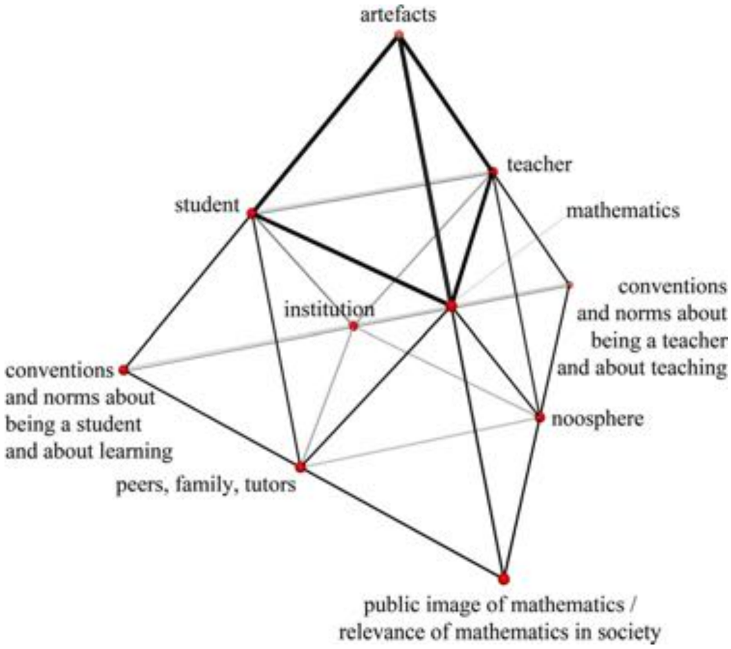

Figure 1: Socio-didactical-tetrahedron (Rezat & Sträßer, 2012)

Naujok (2000) defines collaboration as any kind of task related interaction among students. She distinguishes different collaborative actions according to the theme, the explicitness, the contribution to learning and autonomy, the (a)symmetry in the relationship, the duration, the intensity, and the openness of the collaboration. Based on these dimensions of collaborative actions, she distinguishes explaining, prompting, copying, comparing, inquiring, providing materials, meta-cooperating. We refer to these collaborative actions in order to characterize peer-interaction (RQ1).

In this paper we focus particularly on 'explanation' in order to analyse the trigger, the content and related understanding of peer interactions (RQ2 – RQ4). Dekker and Elshout-Mohr (1998) name explanation as one of four collaborative activities (to show, to explain, to justify, to reconstruct her/his own work) which have the potential to raise the level of mathematical understanding. Erath (2017, S. 1260) defines explaining "as multi-turn units which are interactively co-constructed, contextualized and serve to convey or construct knowledge". An explanation has also the essential function to make a content understandable. Thereby it must be differentiated between the considered epistemic aspects (concepts or procedures) and the modes (e.g. explicit formulation, exemplification) which are used for an explanation.

METHODS

The digital textbook was used in a whole class setting in a German third grade classroom. Each child was working with an individual tablet-computer. The observation focused on two groups of 4 children each that were video-recorded while working with the digital textbook for altogether 6 lessons. The video recordings were coded using MAXQDA in terms of a) assistance features from the digital textbook that students instrumentalized and b) different kinds of interaction among peers or students and the teacher. These kinds of interaction were coded in terms of collaborative actions according to Naujok (2000). Since Naujok (2000) only refers to peer-interaction in a special pedagogical setting, we developed further categories (in italics in table 1) in order to characterize interactions among peers or students and the teacher that were not covered by Naujok's categories. Additionally, the use of assistance-features (lexicon, tip, help) provided by the digital textbook was coded. From the coded video-recordings, we analysed the scenes coded as "explanation" in more depth, because these seemed to be the candidates for substantial mathematical collaboration. The in-depth analysis follows the rules of a systematical-extentional analysis (Beck & Maier, 1994) focussing on the epistemological learning processes (Steinbring, 2005).

RESULTS

Table 1 shows the distribution of different kinds of interactions with peers, the teacher and the assistance-features of the digital textbook that occur during the individual work with the digital textbook in the focus groups.

Kind of interaction	Number of instances		
	Peer-to-peer	**Student-teacher**	**Student-textbook**
inquiring	23	26	
providing materials	0	3	
explaining	18	10	
prompting	33	13	
offering assistance	6		
meta-cooperating	0		
comparing	3		
copying	15		
discussing	5		
feedback		9	
guiding		28	
giving a hint		16	
lexicon			1
tip			1
help			1

Table 1: Distribution of different kinds of interactions between peers, students and teacher and student and textbook

Besides the different kinds of interactions by Naujok (2000) we have observed additional kinds of interactions, which occur mainly between student and teacher, namely feedback, guiding, and giving a hint (in italics in table 1).

Among the different kinds of interaction inquiring, prompting, explaining, and guiding are the most prominent ones. Discussing, meta-cooperating, and comparing are relatively rare. Regarding that the students were neither explicitly asked to cooperate nor did the seating arrangement particularly foster cooperation during their work on the tasks the frequently observed explanations are remarkable.

Having in mind that the interaction occurs during the individual work with the digital textbook it is of particular interest what triggers peer interaction (RQ 2). What is the reason for the children to stop their individual work and start explaining? Does the explanation follow a question, a verbalized or observed problem or a mistake (Häsel-Weide, 2017)? How far can it be reconstructed, that the explaining person understands the mathematical content and in what extend is the explanation followed by a raised level of understanding of the recipient? (RQ 3 & RQ 4)

Following Beck and Maier (1998) we choose crucial episodes for a systematical-extentional analysis and focus on the epistemological learning processes.

Example: Thorsten´s explanation

Figure 2: Task from the digital textbook (www.denken-und-rechnen-interaktiv.de)

Sophie started solving the task (Fig. 2) and entered 5 into the empty field next to the 50 € bill. Probably she wants to express part of 175 € by five 50 € bills. However, the task asks the children to express the given amount of money using as few bills and coins as possible.

1	Sophie	How does it work?
2	Thorsten	Look, you have to express this number (*taps on 175*) using these bills (*taps on the bills*, *raises his head and seems to look at someone else*) I am explaining, (*looks at Sophie*) here you have to use (*taps on the input field in front of the 500 € bill*), use as few bills as possible. And the bills, the bills you do not need, for example this one (*taps on the input field in front of the 200 € bill, a digital numeric pad appears*) you do not need it. There you put a zero.
3	Sophie	Okay (*enters a 0 in the input field of the 1 € coin*)

The interaction between Sophie and Thorsten starts with a general open question. Sophie asks Thorsten how "it" works. She already tried to solve the task on her own, but did not succeed, so she asks for help. Thorsten explains – he himself uses the expression "explaining" to describe his action – by paraphrasing the task and using the given example. He stresses that bills that are not needed for a correct solution must be marked with a zero and demonstrates Sophie how to use the numeric pad. He also frames the task in his words and demonstrates the technical realization, but does not explain how to find the correct decomposition of the given amount of money.

Sophie continues solving the task and firstly considers the bills in the left column. She asks Thorsten at each step for confirmation.

5	Sophie	And here a zero, too (*taps on the input field next to the 500 € bill and enters 0)*?
6	Thorsten	Exactly
7	Sophie	And here a one (*taps on the input field next to the 50 € bill and enters 1)*
8	Thorsten	Yes, you need a one.
9	Sophie	And, and here zero (*taps on the input field next to the 5 € bill, the numeric pad appears)*
10	Torsten	No, a one, look five (*points with his finger at 175)*

The interaction between the children makes clear that Sophie does not have problems with the technical handling of the textbook, but with the content of the task. Thorsten's explanation at the beginning seems not to be helpful in a way that now she would be able to work on her own. As a result, the interaction might be characterized as a mix of confirmation, correction and prompting. The correction is accompanied by gestures, which allow Sophie to understand why she is wrong although Thorsten does not give a verbal explanation.

Since Dekker and Elshout-Mohr (1998) characterize showing as collaboration activity, which has the potential to raise the level of mathematical understanding, we further analyze if this potential is exploited within the interaction (RQ4).

The ongoing solution process shows that Sophie is not able to solve the task on her own, because she suggests to enter "1" into the empty fields next to the 200 € bill and the 20€ bill, respectively. The children continue the process of asking, correcting and prompting until the task is finished. At the end, Thorsten summarizes.

25	Thorsten	Do you see, how you have to, you have to put the bills? Do you see? As few bills as possible to express the number.

Thorsten uses the finished task for reflection. He probably wants to focus Sophie´s attention on the solution, which in his perception seems to function as an explanation for how to solve the task. However, like before he does not explain how one can find the decomposition of a given amount of money and decide which bills and coins are needed.

In summary, the analysis shows, that the explanations of the children consider both, the mathematical content of the tasks and the technical aspect of handling the digital textbook. Thorsten´s mix of explanation, promoting and correction lead to a correct solution, which shows that he has the competence to decompose given amounts of money in the required way. However, the collaboration of the children and the explanation seem not to raise Sophie´s level of understanding. Nevertheless, it allows her to solve the problem.

DISCUSSION

The results show that peer-interaction is characterized by the collaborative actions ‘inquiring’, ‘prompting’, and ‘explaining’. All these are asymmetrical interactions according to (Naujok, 2000), in which one of the partners is more knowledgeable than the other. Symmetrical interactions, such as ‘discussing’, ‘meta-cooperating’, and ‘comparing’ are relatively rare. In terms of ‘comparing’, the finding is not surprising, since the digital textbook provides knowledge of result feedback, so that comparing the results may be less attractive for the children.

Consequently, peer-interaction, in our case (working with a digital textbook that does not particularly afford peer-collaboration), is characterized by interactions, in which one students seems to approach another student in order to get assistance. Considering that the digital textbook offers support for individual learning this is an interesting finding. Simply comparing the number of instances where students use the assistance from the textbook with the number of instances where students interact with their peers or their teacher yields that students seem to approach their peers and their teacher in the first place rather than to use the assistance from the textbook. Consequently, the assistance of the textbook seems not play a significant role in the individual learning processes in class. If this finding is validated in larger studies, it has implications for the design of tasks in digital textbooks. For the in-class-use of digital textbooks the design of tasks that afford peer-collaboration seems to be more important than supporting individual learning.

The exemplarily analysis shows that a problem with solving the task triggers peer-interaction in this case. However, while the peer-interaction leads to a correct solution of the task at hand, it is not likely to contribute to the mathematical understanding of the child that looks for assistance.

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Teachers' perspectives on the use of technology to teach Functions at lower and upper secondary

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This study aims to understand the perceptions of lower and upper secondary age teachers of mathematics regarding the use of technology to teach functions. For that, a mixed methodology was adopted, and the perceptions of 129 teachers were collected through a questionnaire (quantitative section) and four teachers through an interview (qualitative section). The main conclusions point to similarities in teachers' perceptions, but also to some differences related to the level that they taught. Teachers show conviction about their knowledge on technology and about the potential of technology in what concerns their teaching and the students' learning. However, they are not so clear about the best way to articulate technology and paper-and-pencil methods, nor about the use of technology in assessment.

Keywords: Technology, Functions, Teachers of lower and upper secondary, Teachers' perspectives.

INTRODUCTION

The notion of function is one of the most important concepts in mathematics (Mesa, 2004) and technology can make an important contribution to its teaching. Teachers can rely on technology as a source of information to prepare their lessons or use it to get a deeper involvement of the students in the classroom, or even to enhance different forms of assessment. These are a few of the many options available to the teachers when considering the integration of technology with their practice; a practice that we know to be marked by the professional knowledge of the teachers, by their conceptions and by the teachers' teaching context.

In Portugal, the initial training program for lower and upper secondary teachers is the same. As so, studying these two groups of teachers can provide a deeper understanding over the impact of the teachers' professional experience on their practice. In this study, we analyze the perceptions that lower and upper secondary teachers have of the use of technology in the teaching of functions. Specifically, we seek knowledge over the teachers' perceptions regarding: (1) knowledge of technology, (2) technology use, (3) consequences of technology use, (4) technology versus paper and pencil, (5) skill development, and (6) technology and assessment.

In this study, we understand *perceptions* as the ways of thinking, or images expressed by the teachers, when talking about their professional practice. And we assume them as privileged windows into teachers' knowledge and conceptions.

FUNCTIONS AND TECHNOLOGY

According to the Portuguese programs (MEC, 2013), functions begin to be addressed at the 7th grade (age 12), in the first year of the lower secondary, and continue to be studied in each of the three years of this stage. After this, students continue to study functions at all the three years of the upper secondary school (if the students choose a course that includes mathematics, such as sciences) (MEC, 2014).

In the teaching and learning of functions, their different representations play an important role. In fact, they allow the students to understand in a different way what could not be understood in the initial representation and, as Kaput (1992) says, are fundamental to the understanding of the concept.

One of the potentialities of technology is to allow an easy and fast access to multiple representations (Rocha, 2016), which allows the students to establish or reinforce links in a way that otherwise would not be possible (Cavanagh & Mitchelmore, 2003), enhancing the development of a better understanding of functions, of the notion of variable and of problem solving (Burril, 2008). The connection between different representations creates a global vision, which is more than the joining of the knowledge relative to each of the representations. Additionally, the technology allows a full exploration of the numerical and graphic approaches in a way that until then was not possible, thus favoring an integrated approach of the different representations and consequently the development of a deeper understanding (Rocha, 2016). however, technology allows more than that. It makes possible the modeling of real situations, promoting the understanding of the potential of functions for the exploration and understanding of aspects of the real world.

TEACHERS' KNOWLEDGE

Deborah Ball and colleagues are inspired by Shulman's work to develop Mathematical Knowledge for Teaching (MKT). As part of Subject Matter Knowledge, Hill and Ball (2009) consider Common Content Knowledge (CCK), a knowledge identical to that used in other professions in which mathematical knowledge is involved; Specialized Content Knowledge (SCK), a specific knowledge of teachers; and Horizon Content Knowledge (HCK), a kind of comprehensive view of mathematics teaching. As part of Pedagogical Content Knowledge (PCK), they consider the Knowledge of Content and Students (KCS), which combines knowledge of students and of mathematics; the Knowledge of Content and Teaching (KCT), which articulates knowledge about mathematics and about teaching; and the Knowledge of Curriculum (KC).

Recognition of the importance of knowledge of technology leads Mishra and Koehler (2006) to argue that the articulation of this knowledge with the others is fundamental. They then propose a model that is inspired by previous works and which not only includes the three basic domains of knowledge (knowledge of Content, Pedagogy and Technology), but also attends to the connections, interactions and constraints that are established between them. They consider Technological Content Knowledge (TCK), Technological Pedagogical Knowledge (TPK) and PCK. These three areas of knowledge are the essence of this model, called TPACK, and what truly distinguishes it from others previously proposed.

METHODOLOGY

This study assumed a mixed methodology, collecting the perceptions of 129 teachers through a questionnaire (quantitative section) and of four teachers through interviews (qualitative section). The questionnaire included 19 items related to the use of technology to teach Functions. The response options were five and ranged from *I totally disagree* (coded by 1) to *I fully Agree* (coded by 5). The sample was defined by the convenience method (Hill & Hill, 2012), and the questionnaires were distributed in several schools by teachers known to the authors. The sample consisted of 129 teachers, 64 from lower secondary and 65 from upper secondary. The semi-structured interviews were carried out with two teachers from the lower secondary (T1 and T2) and two teachers from upper secondary (T3 and T4) with a professional experience between 23 and 28 years. The interviews were based on the questions of the questionnaire and include the following dimensions: knowledge of technology, technology use, consequences of technology use, technology versus paper and pencil, skill development, and technology and assessment. In the analysis of the questionnaire, we began by determining the mean values of the codifications in each item. Later, the T-Test for independent samples was applied in order to compare the means of the two groups defined (teachers of lower and upper secondary), emphasizing the items in which there were statistically significant differences between the groups. Statistical analysis was performed using the Statistical Package for the Social Sciences (SPSS), IBM SPSS Statistics 23 version for Windows, and in the decision about the existence of statistically significant differences, the significance level of 0.1 was considered appropriate for an exploratory study, such as the one reported here. In the analysis of the interviews (in the part presented here), we use as the main criteria the identification of answers that could offer justifications for the answers obtained in the questionnaire.

RESULTS

An analysis of the questionnaire allows characterizing teachers' perceptions regarding the dimensions considered. It also allows identifying statistically significant differences in five of the items considered (Table 1). The interviews with the teachers clarify some aspects that might promote a better understanding of the answers to the questionnaire.

		Lower Sec (n=64)		Upper Sec (n=65)		value
	Items	$\bar{x}$	s	$\bar{x}$	s	p
Knowledge of tec	I feel comfortable using a graphing calculator to teach functions.	3,56	1,332	4,55	0,685	0,000**
	I feel comfortable using specific software to teach functions.	3,47	1,154	3,74	0,889	
Technology use	Technology helps to do transformations on graphs of functions.	4,25	0,943	4,49	0,590	0,082*
	Technology is mainly useful for drawing graphs of functions.	3,06	1,125	3,29	1,057	

	Technology allows establishing relations between the different representations of functions (algebraic-graphical-tabular).	4,36	0,698	4,28	0,696	
	Technology should be used mainly to introduce concepts of functions.	3,25	0,891	3,15	1,049	
Consequences of technology use	The use of technology favors a teaching of Functions less expositive and more participative.	4,25	0,667	4,18	0,808	
	The use of technology makes the teacher look for tasks about Functions in other sources besides the textbook.	3,72	1,031	3,43	1,131	
	The use of technology frees the teacher and the students from routine activities.	3,55	1,022	3,40	1,028	
Technology versus paper and pencil	The resolution of tasks on Functions must be done with paper and pencil and checked with technology.	3,89	0,799	3,49	1,033	0,016**
	The resolution of tasks on Functions must be done with technology and checked with paper and pencil.	3,37	1,120	3,54	0,937	
	When solving tasks on Functions one must resort to technology when it is impossible to solve using analytical processes.	4,03	1,038	4,25	0,867	
Skill development	The use of technology develops the visualization skill.	4,33	0,736	3,97	0,728	0,006**
	The use of technology challenges the student to think.	4,13	0,917	3,74	0,691	0,008**
	The use of technology develops skills relevant for symbolic manipulation.	3,44	0,957	3,28	0,976	
Technology and assessment	Students should be allowed to use the technology they used in class during assessment.	3,83	0,952	3,88	0,960	
	The use of technology leads the teacher to change assessment.	3,39	1,107	3,34	1,020	
	The use of technology in assessment leads the teacher to ask more questions of understanding and problem solving than memorization.	3,89	0,961	3,77	0,897	

Note: statistically significant differences for $^{*}p<0$ $^{**}p<0,05$.

Table 1: Synthesis of teachers' answers to the questionnaire

With regard to knowledge of technology, teachers express some confidence in their ability to use it. However, statistically significant differences were found between the two groups of teachers, with upper secondary teachers showing much more confidence in the use of the graphing calculator. One difference that the teachers interviewed recognize and assume as predictable.

When the use of the graphing calculator became compulsory in the upper secondary, the teachers were somehow forced to learn how to use it. Some teachers will be more at ease than others, I mean, there are those who know how to program, but the basics everyone knows. In the lower secondary it isn't like that. (US, T3)

Regarding the use of technology in teaching, teachers consider it very appropriate to establish relationships between different representations and to perform transformations of function graphs. In the latter case, it is even possible to identify a statistically significant difference, and upper secondary teachers are the ones who most consider the use of technology as useful for performing function graphs transformations. Although less convincing, teachers still recognize the potential of technology to draw graphs and to introduce concepts.

Technology makes it easy for them to perceive the relationship between the graph and the expression, or between the graph and the table. It helps a lot. (...) It is not possible to deny it. Even those who don't like technology have to recognize it. (LS, T1)

Well, the transformations of graphs of functions are usually studied in the upper secondary, so I think it's to be expected that the teachers of the upper secondary are the ones who most refer to it. And technology is great for that... with paper and pencil it was basically us telling, with technology they can see it. It's completely different. (US, T4)

The active involvement of students in learning is what teachers, regardless of their teaching level, recognize as a consequence of the use of technology. Besides that, they consider that the integration of technology leads the teacher to look for tasks in places different than the textbook and they also consider that technology eases the teaching and learning of routine tasks.

When you use technology the tasks get a little bit different, it is not just exercises... and sometimes it isn't easy to find those kind of tasks in a textbook... but I think this is changing. More and more technology is a reality in the classroom and the textbooks are beginning to take this into account. In the upper secondary all the textbooks have tasks where the use of the graphing calculator is required. (LS, T1)

Teachers in both groups, lower and upper secondary, agree with the use of technology when a resolution is not possible by analytical procedures. However, agreement is no longer so strong as to whether technology should be used before or after using paper and pencil methods, although teachers in both groups take a favorable view in both cases. Still, it is the lower secondary teachers who are more in favor of a paper and pencil approach and the use of technology only after this, and this is a statistically significant difference. In what concerns a use of technology followed by a paper and pencil approach, it is the upper secondary teachers who are most in favor of it.

I think this is the big problem... and it will get worse with the new generations that are increasingly technological generations. What can you do with technology and what do you have to do without it? Tradition has a very strong impact in school. And teachers are still from a generation where knowledge is what you do by yourself. And the curriculum is like this, isn't it? It was developed by people of our generation. (LS, T2)

I see technology as something that helps you understand and so you use technology to improve understanding and to learn how to do it without technology. If you use technology just to do it, then afterwards how do you convince students to do it without technology? And if you already know how to do it without technology, what is the point in using technology? Only if you intend to be faster. (US, T4)

Technology is viewed by teachers as including the potential for developing visualization skills and encouraging students to think, and it is the teachers of the lower secondary who most recognize these potentialities, doing so in a statistically significant way. Regarding the potential of technology to develop the capacity for symbolic manipulation, teachers assume a more neutral, yet positive, position.

It always depends on how you use it, but I think technology always allows you to focus on understanding and not so much on mechanization. And I think this is important at all levels. In the upper secondary because you begin to have some more elaborate concepts and it can help to achieve a deeper understanding. In the lower secondary the concepts are not so complex, but the students are also younger and it can equally help them to realize meaning. (US, T3)

For assessment, teachers believe that students should be able to use the technology just as in class. This option, however, has an impact on the questions posed to the students, and it is teachers' opinion that this leads them to formulate more questions focused on understanding and problem solving than on memorization. There is also the idea that the use of technology may lead to changes in the forms of assessment.

In my opinion, the use of technology in exams allows posing different types of questions and allows reducing the weight of calculations, since it becomes possible, for example, to adopt a graphical resolution. But in fact I don’t think we have had these big changes. There was a lot of talk about it when the graphing calculators came along, but it didn’t change that much. (US, T4)

CONCLUSION

Knowledge of technology does not appear to be in any way identified by teachers as an obstacle to their use of technology. Even so, upper secondary teachers feel more comfortable with the graphing calculator, a situation that can result from the fact that the curriculum has made its use mandatory for about 20 years.

Teachers understand that technology is particularly suited to work with function graph transformations and to establish relationships between representations. This is a circumstance that expresses the teachers’ knowledge about how technology can bring new approaches to mathematics, but also about the contribution that work with different representations can bring to the understanding of Mathematics, in line with Burril’s (2008) ideas. And the potential of technology for the study of transformations of graphs of functions is significantly more valued by upper secondary teachers. This is an aspect that can be expected, even for the teachers themselves, if we take into account that it is at this level of education that this type of content is most worked.

Teachers in both lower and upper secondary agree that the use of technology favors a less expositive and a more interactive teaching approach - something that reflects the teachers' knowledge regarding the way technology allows the adoption of new forms of teaching (Niess et al., 2009). The place where the teachers look for tasks for their students is however a little different in the two levels of teaching. The teachers of lower secondary look for tasks in sources other than the textbook, which is not the case of upper secondary teacher. This choice might be related to the mandatory use of graphing calculators at upper secondary, a situation that might have caused an increase on the quantity of tasks on textbooks requiring this technology.

The teachers agree to the use of technology when analytical approaches are not available. In the other cases, they agree with its use to check paper and pencil approaches (this trend being significantly stronger for lower secondary teachers). However, the lower secondary teachers are less supportive of using a paper and pencil approach to check a solution achieved using technology. This articulation between technology and paper and pencil seems to be a delicate aspect of technology integration, with the teachers interviewed hesitant to take a position. This suggests that the knowledge of how to teach with technology still lacks some development in order to achieve a full integration of technology, as advanced by Niess et al. (2009).

The use of technology by the students promotes the development of visualization skills and challenges the students to think (Rocha, 2016). This is a significantly stronger perspective among the teachers of the lower secondary. This may be related to the less formal reasoning of the younger students and to the associated intention of the teachers to present some more intuitive approaches. It is still another indication of the teachers' knowledge regarding the contributions that technology can bring to student learning.

The teachers agree with the students' use of technology in the tests, since this allows teachers to put more questions requiring understanding instead of memorization. This focus on conceptual rather on procedural understanding is a consequence of technology use on assessment, as pointed by Niess et al. (2009). However, the teachers interviewed consider that in practice the changes are smaller than what may be expected. The teachers' perspectives about technology use in assessment are however less convincing, suggesting that this is another field where technology integration has not yet been fully achieved.

Overall, teachers seem to have professional knowledge regarding the integration of technology in mathematics teaching. The knowledge of technology seems to be considered adequate by the teachers - that is, the teachers do not seem to feel gaps at the level of their TK, according to the TPACK model. References to the potential of technology related to Mathematics (eg work with different representations) and its teaching (eg adopting methodologies where the students take a more active role) are identified - ie TCK and TPK, according to the TPACK model. But circumstances are also identifiable where the integration of technology is not complete (for example, with regard to assessment and to articulation between paper and pencil and technological approaches), suggesting that teachers' knowledge could still be deepened - or that their

TPACK could be developed. According to Niess et al. (2009) development TPACK model, teachers *recognize* and *accept* the use of technology in assessment and in the teaching and learning process but they are still *adapting* to it. Some development is needed in order to achieve the *exploring* and *advancing* levels, where the teachers engage students at high-level thinking activities, exploring various instructional strategies and assuming technology fully as a teaching and learning tool.

It would be interesting to deepen research into precisely the reasons why teachers who seem to have a good knowledge of technology and its potential for learning mathematics find it difficult to take a position on the articulation between the use of paper-and-pencil and technology and also relatively to the impact of technology on assessment.

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Feedback models in task and digital environment design: Experiences from modelling and algebraic transformation activities

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This article shows relevant aspects of feedback systems incorporated to two different digital learning environments, in the light of the experience of having tested them in experimental studies with tertiary education students. The outcomes concerning pupils learning processes have already been reported in previous publications, and this paper focuses on the design fundamentals of the feedback models as well as on the features that played a crucial role so that the students could progress in performance of modelling activities and in tasks involving algebra structure sense.

Keywords: feedback, digital learning environment design, intelligent support, parameterized modelling, algebra structure sense.

INTRODUCTION

Many studies have recorded the existence of critical moments when undertaking tasks, during which specific and timely feedback is crucial for students to be able to overcome the obstacles before them and to move forward with a mathematics activity. The situation is heightened when the tasks are carried out in digital learning environments given that they tend to promote student exploration and, consequently, these crucial moments may arise throughout very different trajectories and at different times. In the research area of algebraic thinking, one of the major difficulties reported is the move from recognizing and analysing a pattern to express it algebraically. That step is an example of one of those critical moments that require focused feedback. The MiGen project from UCL-Knowledge-Lab (http://www.ucl.ac.uk/ioe/departments-centres/centres/ucl-knowledge-lab) dealt with this problem by developing an intelligent support system for the *eXpresser* microworld. This microworld was designed to help students with rule generalization based on the structure of figural patterns made up of square tiles and in which the intelligent system prompts students while they perform the task, particularly at times when they are formulating and symbolising the rule. The MiGen project work inspired researchers at the Centre for Research and Advanced Studies (Cinvestav) and the National University in Mexico to develop feedback systems within two different digital learning environments [1] [2], which design is based on the results of previous mathematics education research, specifically in the areas of modelling and algebra structure sense. This article shows relevant aspects of such feedback models, in the light of the experience of having tested them in experimental studies. The outcomes concerning participant learning processes in those studies have already been reported in previous publications (Rojano & García-Campos, 20017; Muñoz & Rojano, 2017), and this paper focuses on the design fundamentals of the feedback models in both environments, as well as on the features that played a crucial role so that the students could progress in performance of the modelling activities and in the tasks involving structure sense.

ON FEEDBACK AND DIGITAL ENVIRONMENTS

The term feedback is most broadly defined by Hattie & Timperley (2007, pg. 81): "[...] information provided by an agent (e.g., teacher, peer, book, parent, self, experience) regarding aspects of one's performance or understanding". Thus feedback is the result of performance and according to the authors cited, it may be of a corrective nature or offer alternative strategies, to encourage new actions or to provide the correct answer and hence validate one's response. The 'feedback agent' in digital learning environments is the computer program per se, which responds to the user's interactions with it. In the specific case of digital environments for mathematics learning, users can interact with multiple dynamic representations of mathematics objects and concepts (that are hot-linked to each other), such as graphs, formulae, geometric figures, number tables and natural language, *inter alia*. This characteristic allows for a wide range of feedback models. Two of those models are described below. Their design was based on findings reported in the specialized research literature on modelling and on development of algebra structure sense. The feedback model in the modelling environment incorporates an <u>intelligent support system, in natural language</u>; while in the structure sense environment, <u>a level-up approach</u> is used to guide user actions, and in which there is practically no presence of natural language whatsoever. With the description and analysis of the basic design principles in both environments, the intent in this article is to show the feedback potential of the vehicle used. On the one hand, the decision to incorporate natural language in the first case is supported by the hypothesis proposed by Litman (2009), which purports that an increase in rich talk in the activities is correlated with greater learning achievement. While on the other hand, opting in the second case for a design devoid of natural language, delegating feedback to the explicit hierarchy of the tasks, is based on the hypothesis that allowing a user to choose the level of complexity of the tasks she is to solve triggers reflection on and raises awareness (Bødker, 1995) of the user's own level of development of her structure sense.

Intelligent dialogues in parameterized modelling activities

In the study "Intelligent dialogues with tertiary education and university students" [1] natural language support is incorporated to provide feedback to user ideas and actions when he/she is working on parameterized modelling activities. To develop that system, the *Dialogues with Theo* (DiT) (http://recursostic.education.es/descartes) software was used. Using that software it is possible to simultaneously display a dialogue window and a microworld window on the computer screen, both of which are dynamically hot-linked, as can be seen in Figures 1 and 2. In that environment students can work on specific tasks in the microworld and have dialogues with the system in natural language. As such students are given synchronized feedback both within the microworld and through the dialogue window.

Figure 1. Screen with simulation of Molecular diffusion in a cell.

Figure 2. Screen with the dialogue (on the left) and the microworld (on the right) windows

The DiT design and technology development principles were presented at the Conference *Constructionism 2012* (Rojano & Abreu, 2012). The didactic design of the integrated system (microworld-intelligent support) is based on findings of the Anglo-Mexican project "The role of spreadsheets within school-based mathematical practices in sciences" (Molyneux, Rojano et al, 1999). That study identified moments considered to be critical for all participating students when undertaking modelling activities. The moments identified were as follows: prediction of phenomenon behaviour, verification of the prediction and generalization of the model. These findings together with the DiT development served as the basis for subsequent design and development of the integrated environment *Intelligent dialogues in parameterized modelling activities.*

In order to delve deeper into the role of the intelligent support feedback, an experimental study was undertaken with a group of six 16-17-year-old students, using the tool in modelling activities involving phenomena from the physical world, in a spreadsheet microworld. Figure 1 shows a scene of the activity in DiT *Molecular diffusion in a cell*, in which the screen displays a simulation of the phenomenon that consists of considering a simplified cell (in two dimensions) with six compartments. The outer walls are impermeable, although the internal membranes between each two compartments do allow molecules to move from one compartment to another. At time t=0, there are 1,200 molecules in the first compartment. In each time unit the molecules move in the four directions with the same probability. The students are asked to build a spreadsheet model that represents the molecules spreading into the different compartments over time. The text in Figure 1 describes the behavior of the phenomenon at time t=0 and t=1; Figure 2 shows the questions asked by the system and student answers (dialogue window), as well as the calculations performed by the student (window on the right). The microworld offers students the option of working numerically or with formulae on a spreadsheet.

The goal of the experimental work was to analyse the role of feedback in the integrated system at critical modelling moments. In designing the modelling activities, only the following stages were considered: *understanding the phenomenon*, *building the model*, and *predicting the phenomenon behaviour in the long term.* The critical moments were anticipated at each of the stages in order to design the feedback in the dialogue window. Figure 3 shows how to keep them from making false generalizations, students are presented with the option of a correct and an incorrect prediction response of the long-term behaviour of the molecular diffusion. To help students move through the varying phases of the modelling task, open questions, multiple-choice items, suggestions and prompts were all used within the dialogue window; whereas in the microworld window, different mathematical representation systems were used.

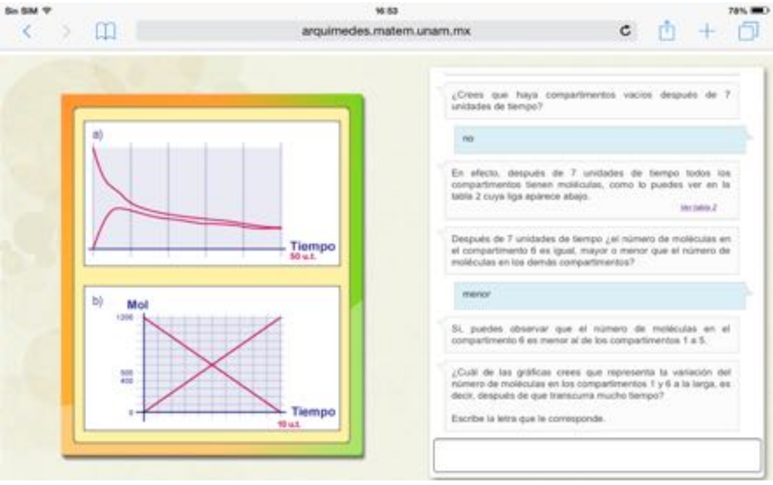

Figure 3. Dialogue and micro-world windows. Choosing graphs representing the number of molecules in compartments 1 and 6, in the long term. Option a) is the correct prediction. Pupils choosing option b) rectify their prediction after entering in dialogue with the intelligent

The description of the activities designed and used, along with the details of the methodology and data analysis of the experimental study can be found in Rojano & García-Campos (2017). The analyses of the three cases (three student pairs) reported in that publication show that the integrated system responds differently to different learner approaches (spreadsheet-numerical or spreadsheet formulae (algebraic) approach) and that feedback facilitates the flow of performance and completion of the modelling activity. It is worth noting that in cases where students showed that they were experiencing great difficulty from the initial stage of the modelling, substantive intervention from the teacher-researcher was necessary. As for the role of natural language, the study with intelligent dialogues confirms the hypothesis raised by authors positing that the use of interaction with natural language contributes to improving learning achievements, specifically of learning with computer systems. The foregoing hypothesis is based, on the one hand, on the fact that a higher percentage of content-rich talk is correlated with higher learning gain (Litman, et al, 2009) and, on the other, on evidence that shows that achieving self-explanation among students significantly enhances learning (Chi, et al, 1994).

Algebra structure sense in a web environment

According to Hoch and Dreyfus, one has algebra structure sense if a) a familiar structure in its most simple form is recognized; b) a compound term is treated as an entity and, after making the appropriate substitutions, a familiar form is recognized within a more complex form; and, c) appropriate manipulations are chosen to make better use of the structure (Hoch & Dreyfus, 2007, pg. 436). Despite the fact that algebra structure sense is recognized as important, and that its teaching is acknowledged as an enormous challenge, not much research has been done on the topic. In the project "Algebra structure sense in a web environment" [2], the *expression machine* (EM) environment was built in order to help users to develop that ability. Based on the Hoch and Dreyfus definition, EM was developed as a web application so that by experimenting and practicing students learn the rules that the program uses to generate the tasks, at the same time as they acquire structure sense in terms of actions a)-c) described above.

Design of EM emphasizes the role of affordances (preconditions for action), level-up (task design by levels of complexity) and feedback in developing the algebra structure sense of tertiary education students. Algebraic substitution is the main procedure involved in the way the software works. Its design and testing methodology are based on the Human - Computer Interaction aspect of Activity Theory (see Muñoz & Rojano, 2017, for a detailed description of these EM features).

The interactive sequence was designed based on a scheme where the elements of the machine are the *input* (two expressions IE1 and IE2), *process* (a generating expression GE), and *output* (a resulting expression OE after substituting IE1 and IE2 in GE). The expression machine generates an output expression upon substituting the input expressions in the generating expression. For instance, if the input expressions are $6x$ and $2y,$ and if the generating expression is $6(a+b)$, then the machine will output 6(6x+2y) as it substitutes a with $6x$ and b with $2y$ (Figure 4). The activities proposed with this machine are of three types (conjecture input expressions; predict the output expression; and conjecture generating expression). In technological terms, EM design was inspired by *serious games* and *touch* applications, with almost no training time at the use of the artefact level.

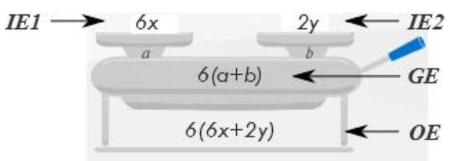

Figure 4. Components of the expression machine: IE1 and IE2, input expressions; GE, generating expression; OE, output expression

Figure 5. A clean or crossed equality sign marks a correct or incorrect response

EM was tested with a group of 35 tertiary education students of a public school in Mexico City. Pre-post questionnaires were applied to assess participants' mastery of symbolic manipulation as well as their structure sense level. Participants had a period of on-line interaction (on average one and a half hours) between tests and worked through different levels of complexity with regards to the algebraic expressions involved in the activities (level-up approach). The 'doing' of the students included paper and pencil algebraic manipulations to introduce the expressions required into the three types of activities. Their actions were recorded during the experimental period, and data were analysed with the method proposed by Bødker (1995) to detect focus shifts and breakdowns [3]. It should be pointed out that although the tasks are randomly proposed to students, they were able to choose the level of complexity of the tasks.

Results from this study suggest that as the EM is essentially an algebraic substitution machine, it favours students adopting this technique. However, the tasks proposed also require actions such as structure recognition, switching between different forms of an expression, and application of known algebraic identities. In this sense, and considering that the group of participants showed improvement in their structure sense, it can be said that interaction with EM fosters development of this ability. By the same token, that experimentation lead to the following reformulation of 'structure sense', which supplements the definition of Hoch and Dreyfus: 'one has a sense of the structure if one is able to perform a combination of the following actions: a) Recognize familiar structures (for instance, of notable products); b) See-how, that is switching between various forms of an expression, to take advantage of the structures (learn to see sub-expressions as an object or entity); c) Substitution (whether internal or explicit); and d) Timely application of known algebraic identities'.

The minimalist design of EM based on *affordances, feedback* and *level-up* (task design by levels of complexity) had a positive usability effect on students. This was confirmed by the nearly null training period required to become familiar with the artefact. In addition, students were able to continue exploring and solving the exercises without intervention from the teacher or researcher. With respect to the EM level-up feature, the results suggest that in a next version, EM should include tasks with greater levels of complexity in order to trigger shifts and breakdowns and, thus, expand the students' learning experience. In addition to the feedback that consists of using a clean or crossed equality to mark the correct or incorrect response (Figure 5), the level-up design played an important role in feedback and encouragement because users were able to see their level of mastery of symbolic manipulation in each activity they undertook.

FINAL REMARKS

The two learning environments described in the section above include different feedback models. The model in the Intelligent Dialogues environment is a continuum that weaves in instruction and feedback, and in which natural language is the vehicle of communication, of explicit dialogue, between users and the system. The results of the experimental work with this environment confirm hypothesis concerning the relevant roles played by the use of natural language in learning achievements. On the other hand, the Expression Machine model uses the resource of mathematics language (equal sign) to mark the correct and incorrect answers. Moreover and primarily, however, it incorporates the resource of task design by levels of complexity (level-up), which enables users to know operatively what their level of competence is in manipulative algebra. The latter depicts the environment as a potentially autonomous learning system, in which users can progressively expand their sense of structure. The foregoing can be consolidated in a subsequent version, by programming the Expression Machine as an adaptive system and by including tasks that involve expressions with more complex structures. In both cases the developers of the learning environments and their respective feedback models resorted in their design to previous research findings. Hence knowledge of critical moments of parameterized modelling of the Anglo-Mexican project was essential in designing the activities in the modelling microworld 'by phases', as was their knowledge of the system intervention scheme by way of the dialogue window. The initial definition of structure sense and the action of algebraic substitution were essential elements in designing the Expression Machine. Moreover, outcomes from the experimental work with this environment led to a supplementary definition of structure sense. In other words, these two cases show how designing tasks in digital environments can go hand in hand with basic research in mathematics education, and how that link favours quality in the design of the environments *per se*.

NOTES

1. *Intelligent Dialogues with tertiary and university education students* is an on-going 3-year research project, funded by the National Council of Science and Technology (Conacyt) in Mexico, Reference No. 168620

2. The web environment *Expression Machine* was designed and developed at Centre for Research and Advanced Studies (Cinvestav) and funded by the National Council of Science and Technology (Conacyt) in Mexico.

3. Bødker (1966, pg.6) uses the term *breakdown* when the learning activity is interrupted because something did not happen as it was expected to (for example, if a button is pressed but nothing happens). This author uses the term *focus shift* when interruption of the activity is more deliberate and does not necessarily happen due to a system failure. A *breakdown* causes a *focus shift* from the object of the activity mediated by the artefact to the artefact itself.

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An inferentialist perspective on the understanding of students' uses of digital textbooks in mathematics

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Inferentialism is a philosophical theory of meaning. It is used in this paper as an epistemological framework to understand student argumentation processes in mathematics. It also serves as a methodological tool in order to precisely describe such processes. The empirical examples of a study conducted in a geometry context in grade 5 show in which way the students' argumentation changes in the light of the given immediate feedback by the digital textbook. The findings reveal that the students reconstruct the argumentative structure behind the given feedback and relate it to their own argumentation. The argumentation process itself becomes an object of justification by using the digital textbook.

Keywords: inferentialism, digital textbook, argumentation, area.

MOTIVATION

Digital textbooks are an important topic in the development and research in mathematics education. Yet, little attention is payed to the individual conceptual processes and student argumentation when working with such textbooks, while this is necessary since the use of such tools highly influence argumentation processes. A theory that helps to understand student argumentation both must conceptualize the phenomenon of argumentation in the light of the complex conceptual (mathematical) processes and give respect to the characteristics of the use of such textbooks. This theoretical coherence on the one hand has to fit to the methodological considerations to offer a precise language to reconstruct such processes on the other hand. Whereas general terminologies such as the Toulmin scheme offer transdisciplinary tools to analyse such processes, they seem to be limited both in giving respect to the use of digital tools and regarding complex conceptual situations. This paper introduces an inferentialist perspective on student argumentation when using digital tools. Inferentialism is a relatively new semantic theory from philosophy (Brandom, 1994) and it has gained recent attention in mathematics education (e.g. Bakker & Hußmann, 2017; Schacht, 2012). In this approach, individual processes are theoretically conceptualized in respect to argumentation in the game of giving and asking for reasons (GOGAR, c.f. Brandom, 1994).

DIGITAL TEXTBOOKS IN THE MATHEMATICS CLASSROOM

Textbooks play an important role in the mathematics classroom (Fan, Zhu, & Miao, 2013), especially in their function of guidance for teachers in preparing and organizing the lessons and in the way the mathematical content is presented to the students (Chazan & Yerushalmy, 2014). Recently, much effort is spent on the development of digital textbooks. Such digital textbooks seem to have enormous potential for the (mathematics) classroom. This is especially the case in respect of the role of the teacher since digital textbooks can change the relation between textbooks, the mathematical subject matter and the teacher in a way that they can (re-)organize the content individually and work on such organization processes collectively (Chazan & Yerushalmy, 2014, p. 69). Hence, a current research focus is on the structure and the teachers' use of digital textbooks. Pohl & Schacht (2017) studied the structural elements of German digital textbooks. The analysis shows that many versions of digital textbooks are a pdf version of the non-digital book. On the other hand, some digital textbooks offer structural elements that reflect their digital nature. For example, structural elements like drag-and-drop exercises or the automatic and dynamic feedback for individual solutions offer new possibilities of textbook usages.

In this paper, the focus will be on possibilities of immediate feedback that such textbooks offer. Kehrer et al. (2013) have pointed out for homework situations with electronic devices that "*immediate feedback* does improve learning" (ibid, p. 544). The role of such kind of feedback has also been discussed in the context of formative assessment (c.f. Drijvers et al., 2016; Ruchniewicz, 2016). Although these results seem promising regarding the effects of immediate feedback there is a need to further study the role of such feedback in the context of digital textbooks in regular mathematics classroom situations because both homework situations (with the goal to "prepare the student for the next lesson" (Kehrer, 2013, p. 544)) and situations of formative assessment (with the goal of adapting "learning and teaching lessons to fit students' learning" (Drijvers et al., 2016, p. 16)) are structurally different to regular classroom situations in which digital textbooks are used. Although there are several studies that report on student uses of non-digital textbooks (c.f. Rezat, 2009), yet little is known about the way students actually use digital textbooks in the light of such feedback and about the way how such usages influence individual conceptual processes. This is especially relevant in the light of the reconstructed structural elements reflecting the digital nature of the textbook, since such elements possibly influence argumentation processes. The next section introduces a philosophical framework that helps to understand (theoretical level) and to precisely reconstruct (methodological level) individual argumentation processes.

INFERNTIAL CONSIDERATIONS

The study of argumentation plays a major role in mathematics education. Many authors such as Boero et al. (2010) or Pedemonte (2007) refer to Toulmin's model (2008) in order to study argumentation and proof by reconstructing inferential steps in argumentation. This paper uses Inferentialism as a philosophical theory of meaning, developed by the philosopher Robert B. Brandom (1994), in order to study argumentation processes. This framework especially allows the reconstruction of meaning making in discursive situations by reconstructing the processes of attributing and acknowledging reasons. Due to space limitations, this paper will not make the attempt to elaborate the theory in detail. Main aspects of the theory will be outlined though to the extent to which it can be useful to better understand individual student processes when using digital textbooks.

In an inferentialist perspective, *conceptual understanding* "can be understood, not as the turning on of a Cartesian light, but as practical mastery of a certain kind of inferentially articulated doing: responding differentially according to the circumstances of proper application of a concept and distinguishing the proper inferentialist consequences of such application." (Brandom, 1994, p. 120) As a pragmatic theory, meaning is understood in terms of doing rather than in terms of representation: to grasp a concept is understood as to know how to use it in discourse (*practical mastery*) rather than to have a correct mental representation of it (*Cartesian light*). Understanding a mathematical concept – such as the concept of area – means to practically master its inferential relations, e.g. that the area itself is different to its size, that areas can be ordered according to their size or that the size itself is always – independent of the dimension of the volume – one-dimensional. As Brandom (1994) puts it: "grasping the semantic content expressed by the assertional utterance of a sentence requires being able to determine both what follows from the claim, given the further commitments the scorekeeper attributes to the assertor, and what follows from the claim, given the further commitments the scorekeeper undertakes." (pp. 591/592) Hence, inferentialism conceptualizes understanding in terms of inference (What is the reason for a certain assertion and what follows from it?) rather than in terms of reference (Does the student have a correct mental representation of the concept?). In the light of Brandom's conceptualization of the game of giving and asking for reasons (GOGAR), Schacht (2012) developed a theoretical framework to better understand individual concept formation by reconstructing individual commitments that students make explicit. Individual commitments are student assertions that can serve as reasons in the GOGAR. Every social classroom situation can be such a GOGAR, especially when students work together in small groups exploring a certain mathematical concept or in periods of student-teacher interaction. In such discursive situations, students can make their individual commitments and the inferential relation between them explicit. By reconstructing commitments and their inferential relation within processes of reasoning, webs of reasons can be reconstructed and by doing so over a period of time, it is possible to reconstruct processes of individual concept formation. A second central element of this theory is closely related to the nature of mathematics itself: One of the reasons that mathematics educators are attracted to this theory is because mathematics itself is about reasoning and inferences. Hence, inferentialism can be used to precisely describe a mathematical subject matter in terms of commitments and inferences and webs of reasons. Mathematical theorems for example can make conditional relations explicit (What follows from the premise?), which have to be proven (What is the reason for the conclusion?). In this sense, Schacht (2012) uses inferentialism to both describe the mathematical subject matter and the individual processes of concept formation, which then can be understood by comparing the individual webs of reasons with the mathematical webs.

Although inferentialism has not yet been used to explore individual conceptual processes with a focus on the use of digital tools and especially of digital textbooks, the theory seems promising for an application in research in this context. As Derry (2008) points out the inferentialist priorization of inference over reference has fundamental consequences in terms of pedagogy because "the grasping of a concept (knowing) requires committing to the inferences implicit in its use in a social practice (...). Effective teaching involves providing the opportunity for learners to operate with a concept in the space of reasons within which it falls and by which its meaning is constituted" (p. 58). Such a pragmatic view on concept use and understanding is in line with an understanding of using tools in the mathematics classroom, in which the learner operates with a digital tool (e.g. as a black box) in order to explore concepts and conceptual relations *within the space of reasons*. Furthermore, inferentialism can be related theoretically to central epistemologies in mathematics education. As Schacht & Hußmann (2015) point out inferentialism can help to bridge the gap between social and individual perspectives on student processes and can – as a background theory - be connected to local theories with compatible theoretical roots. Schacht (2012) connected inferentialism to the theory of conceptual fields (Vergnaud, 1996), Derry (2008) to a theoretical perspective following Vygotsky. Although not a focus of this paper, it will be a central task to connect inferentialism to foundational theories concerning the use of tools in mathematics education. In the light of the theoretical foundations discussed above, this paper explores the individual student use of digital textbooks by using an inferentialist perspective (Bakker et al., 2017; Brandom, 1994; Schacht, 2012; Schacht & Hußmann, 2015). Using this perspective, the following research question will be posed: *How does the use of digital textbooks influence individual argumentation processes in the light of immediate feedback by the digital textbook?*

METHODS & DESIGN

To answer this question, a qualitative interview study with three German students in grade five was conducted (Pohl & Schacht, 2017).

Assign the size estimations. Compare the sizes with each other before you do so.

Soccer field [very big✔] | your room | lawn in front of your house [big✔] | a book [small x]

garage for a car | stamp | room door

chessboard | your desk | area of the roof of your house

cent coin | pinhead [very small✔] | gymnasium

very big | medium size big | small
big | smaller than medium size big | very small

3 of 16 size estimations are right.

source: Hornisch et al., 2017

Figure 1: Screenshot and translation of a task bringing size estimations into an order

The geometrical mathematical content deals with the concept of area. The students worked with a digital textbook (Hornisch et al., 2017) in a small group with phases of group activities and working alone. The selection of the digital textbook followed the decision to work with an elaborated version of a digital textbook rather than a pdf version of a regular textbook. The digital textbook in use offers both a combination of static and dynamic as well as interactive tasks. The transcript presented in this paper deals with a task in which the students have to compare different areas regarding their size and dynamically drag and drop the given areas into the right order. When the students had finished the task, the textbook gave an immediate answer if the order was correct or not (fig. 1). The use of the textbook was recorded both with screen capture recordings as well as video recordings of the students' actions.

RESULTS AND DISCUSSION

The students have worked on the task discussed above and brought the sizes into an order. The two male students S1 and S2 worked together, the female student S3 worked alone. The following examples show argumentation processes before and after the feedback of the digital textbook. The inferentialist analysis will show the way in which the argumentation process changes after the feedback was given.

Argumentation ***before*** *the feedback by the digital textbook*

The empirical reconstruction of argumentation processes starts by analysing the individual commitments and the inferential relations between them within the student discussion. In the first example discussed here, the students S1 and S2 (who worked as a pair) compare their solution with S3. S3 has assigned a book to be *smaller than medium size* and a room as *medium size* (size choices given in fig. 1). S1 and S2 disagree using the following argumentation (reconstructed commitments with their inferential relations (→)):

Example 1) S1 & S2: *The book is definitely much much much smaller than your room.*
→ *Its area has to be more than one size smaller*
→ *The solution of S3 has to be wrong.*

In this example the students S1 and S2 first compare their own solution with S3's solution and within their argumentation they refer to the real objects (book and room) in order to conclude within the GOGAR that S3's order cannot be correct. In respect to the concept of area, the students compare two given areas in this situation.

Student S3 is not convinced though and she makes her own reasons explicit. Instead of acknowledging the commitments, S3 justifies her decision why a book is medium size and a cent coin is small by giving reasons for her choice.

Ex. 2) S3: *A pinhead is very small.* → *A cent coin is small*
→ *If I cover a book with cent coins, I need a lot of cent coins.*
→ *The book has a medium size and is not smaller than medium size.*

In this reconstruction, the student's inferential relation between her commitments reveals her way to compare the different size of the areas. She uses an indirect comparison (by covering a book with cent coins) in order to conclude that a book has to be two sizes bigger than the coin. In this case she refers to the physical objects.

Both examples show that the students challenge their individual solutions by making inferential relations explicit and in order to justify their decisions. The examples also show that the students compare their argumentation by referring to physical objects within the GOGAR.

Argumentation ***after*** *the feedback by the digital textbook*

After the feedback had been given by the digital textbook (an example is shown in fig. 1) the argumentation changes severely. The feedback shows that S1 and S2 have three (out of 13) right answers whereas S3 has six (out of 13) right answers. Within the discussion (GOGAR) the following commitments and inferential relations were reconstructed:

Ex. 3) S3: *I have six right answers and S1 & S2 have two right answers. → My logic was better.*

Ex. 4) S1 & S2: *We have two right answers and S3 has six. → The computer was wrong.*

Both examples show that the students justify their results differently within the GOGAR. S3 first compares the different arguments and concludes that her "logic was better". Here the students S1 and S2 acknowledge the commitment that S3 thinks that their solution is not as good as hers. Hence, they conclude that the "computer was wrong". Both commitments reveal that the students follow the rules of the GOGAR and directly compare the different reasons and their inferential relations in the light of the given feedback quite similarly (although with different conclusions). S1 finally even gives an argument referring to the textbook author:

Ex. 5) S1: *The textbook author could mean a different coin. → The computer solution would be plausible.*

In this situation, S1 judges reasons for the solution of the digital textbook. S2 later makes a detailed inferential relation explicit between different commitments in order to show that – from his point of view – the solution of the digital textbook has to be wrong.

Ex. 6) S2: *The cent coin is small. → The stamp is bigger. → The stamp has to be smaller than medium size. → The digital textbook is wrong because the book is already smaller than medium size.*

In both examples 5 and 6 the students attribute commitments to the digital textbook and make a possible logic explicit. Especially example 6 reveals that S2 makes explicit several commitments that are inferentially related. In the final step of this argumentation process he concludes that the feedback of the digital textbook must be wrong in the light of his argumentation. Hence, he relates his own inferential argumentation structure to the results being given by the digital textbook. Hence, the immediate feedback by the digital textbook evokes the giving of reasons by the students.

This short inferentialist analysis shows that the use of the digital textbook influences the individual argumentation process. *First*, the digital textbook almost becomes a rational actor within the discussion. The students acknowledge and attribute inferentially-structured commitments to the digital textbook. *Second*, the students try to reconstruct the inferential relations behind the given feedback from the digital textbook by investigating the inferential webs of reasons of the solution. The analysis also shows different argumentation processes in the light of the given feedback: First, the students use direct and indirect comparison strategies in order to determine the different sizes. After the feedback by the digital textbook, they reconstruct their own and the digital webs of reasons. *Third*, the argumentation itself becomes object of justification. The students compare the different webs of reasons regarding their power within the process of argumentation (ex. 3 "my logic was better"), they reconstruct the webs of reasons behind the computer solution (ex. 5 "they could mean a different cent coin") and they make detailed inferential relations explicit among the different commitments behind the computer solution in order to show that the "laptop was wrong" (ex. 6).

OUTLOOK

The inferentialist analysis of individual argumentation processes using digital textbooks reveals that the given feedback of the digital textbook influences students' argumentation processes. In this GOGAR, the textbook is treated as a rational actor with its own webs of reasons that the students reconstruct and interact with. While using direct and indirect comparing-strategies before the feedback was given, the students reflect on the inferential structure itself after the feedback has been given. Such reflective processes reveal potential for the use of digital textbooks in the mathematics classroom.

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Implications of Digital Media and New Cognitive Theories for Research and Practice in the Mathematics Classroom

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Contemporary mathematical education in many parts of the world revolves primarily around the lecture-based classroom, which in turn rests on two pillars: (i) text as dominant medium and (ii) a mind-as-processor paradigm of cognition. Both these pillars are under attack, by digital media and contemporary theories of cognition, respectively. Importantly, research suggests that the relationship between our minds and the dominant medium of our times is extremely deep, as well as that our bodies and environment are integral to our cognitive abilities. This has significant implications for mathematics learning in the digital age. Research in mathematics education needs to study these implications by using relatively novel methods such as design-based research, which combine analysis with creativity to enable study of learning processes that might not yet exist.

Keywords: digital media, embodied cognition, distributed cognition, design-based research.

INTRODUCTION

Education systems have been traditionally structured around the pedagogical institution of the classroom, typically characterised by an 'expert' lecture session that is supplemented with additional practice tasks for learners. While contemporary mathematics education has moved away from this format in some countries, many math education systems around the world continue to be dominated by the lecture as the primary pedagogical tool. In this article we are interested in two of the fundamental pillars on which this traditionally dominant pedagogical institution must rest.

First, such institutions are deeply dependent on the dominant media of communication and knowledge-sharing. Second, they must hold, even if implicitly, some notions regarding the nature of the mind and how it learns. The lecture-based mathematics classroom is an institution built around the dominant medium of text, and one that sees the mind primarily as a computational processor of abstract symbols. Recent developments have now raised challenges to both these pillars.

THE FIRST CHALLENGE: DIGITAL MEDIA

The first challenge comes from digital technology transforming the media available to us, not just for communication, but also for storing, representing and interacting with knowledge. Given that digital media is a revolution comparable to the invention of writing and the printing press, its potential effects can be difficult to anticipate. This is because the dominant media of a culture has a huge influence on its cognitive and social structures, as has already been shown for spoken language (Vygotsky, 1980) and written text (Ong, 2013).

Vygotsky (1980) extensively studied children just learning to speak, and found a deep relationship between their linguistic development and the development of cognitive functions such as attention, perception, memory, etc. Vygotsky found that children in his studies were unable to solve certain practical problems if they were not allowed to speak, at least until they gradually internalized this speech into thought. His work provides substantial support for the idea that spoken language is not just a medium of communication, but fundamentally alters our cognitive structures and is indeed crucial for the development of our uniquely human intellect.

Ong (2013) brings together a diverse range of studies, primarily comparing pre-literate and literate societies, to conclude that writing had similarly significant effects on human consciousness. An obvious one was to free the mind for analytical, innovative thought by taking on the task of conserving knowledge, which earlier had to be held in social memory through repetition. More subtle and significant, however, is the way in which the 'interiorization' of a vowel-based alphabet by the Greek psyche "made a permanent part of its noetic resources the kind of thinking that alphabetic writing made possible" (p. 51) and enabled its "later abstract intellectual achievements" (p. 28), such as the invention of formal logic.

Both mathematical practice and education have been, and remain, centred on the textual medium, which has defined them in important ways. A large part of scientific endeavour could be seen as an effort to capture and explain dynamic phenomena through shareable representations, representations that have been constrained to static objects (e.g. graphs, equations, formulae) because of the dominance of the textual medium. Mathematics is largely responsible for constructing and advancing this world of textual representations, and mathematics education for inducting learners into it.

With digital media, however, not only do dynamic representations become possible, but also the transmission of prosody and gesture that text had robbed speech of (Rotman, 2008). Gesture, as Chatelet (cited in de Freitas and Sinclair, 2014) points out, is essential to mathematics, since both exist where the physical and the virtual meet. As with speech and writing, the internalization of this new media could have significant implications for the way we think, among other things about mathematics.

THE SECOND CHALLENGE: EMERGING NARRATIVES OF COGNITION

The second challenge to the pillars of the lecture-based classroom comes from emerging narratives in cognitive science, which are making untenable the classical cognitive paradigm of the mind-as-processor. The lecture-based classroom still seems to rely on the idea of the mind as an abstract entity that takes in and processes symbolic inputs into symbolic outputs, a class of intellectual activity completely separable from the body's physical interaction with its environment. This is especially visible in mathematics, which as a discipline is considered the epitome of abstract thought – material practices are simply seen as ways of expressing or communicating its abstract symbols and operations.

More contemporary narratives suggest, however, that cognition is (i) more embodied and (ii) more distributed across socio-material elements in the environment than the classical paradigm allows for. To illustrate the former, a study by Domahs, Moeller, Huber, Willmes, and Nuerk (2010) shows how cultural differences in finger counting during childhood impacts arithmetic abilities in adults. These differences cannot be explained if the brain is dealing in purely abstract representations of numbers – rather, the idea of a number retains its roots in the embodied experience of finger counting, even though the physical use of fingers is suppressed in adulthood.

The idea of distributed cognition (DC) is essentially that cognition does not just occur within individual human bodies. There are two conceptualizations of this that are relevant here. The first is the idea that the mind can 'extend' its body schema and mental identity to include parts of the environment that are 'external' to the body, such as tools. Philosophical arguments have been made earlier in this regard (see e.g., Hull, 2013), and now experiments in cognitive science are supporting such narratives: at a neuronal level in monkeys (Iriki, Tanaka and Iwamura, 1996) and with less invasive techniques in human beings (Chandrasekharan, 2016).

A second DC conceptualization is of cognition as an emergent phenomenon that can be seen not just within individual minds but also across socio-technical systems (e.g. an aircraft cockpit), which have cognitive capacities irreducible to those of their human elements alone (Hutchins, 1995). These two conceptualisations can interact, for example in the case of a computer model developer in a biochemistry research group, whose external model extended his imagination, as well as formed a distributed cognitive system with his mind that had emergent cognitive properties irreducible to the cognition of either entity alone (Chandrasekharan and Nersessian, 2015).

The typical lecture-based mathematics classroom reveals its implicit assumptions of the mind as an individual, disembodied processor of abstract symbols in several ways: it excludes the body from classroom activities, except to interact with symbolic media; it focuses on abstract, decontextualized knowledge that it assumes can be transmitted through this symbolic media from 'expert' to 'novice'; it brings together learners only to forbid them to interact with each other, and evaluates them individually.

IMPLICATIONS FOR LEARNING PROCESSES

It is clear that our current assumptions about learning, and the institutions based on them, need revision. As Bruner (1985, p. 8) suggested, one "cannot improve the state of education without a model of the learner." We will thus begin by constructing a possible model of learning, keeping in mind Bruner's warning that "the model of the learner is not fixed but various," and that "it was the vanity of a preceding generation to think that the battle over learning theories would eventuate in one winning over all the others" (p. 8). The aim is not, therefore, to prescribe a single, optimal, rigid model of learning but to suggest a learning pathway based on contemporary cognitive theories that could enable us to anticipate, analyse or even design new learning processes.

A Distributed and Embodied Account of Learning

Based on the theories of embodied and distributed cognition outlined above, one can imagine an initial stage of learning where the learner(s) forms distributed cognitive systems with their socio-material environment, systems that can complete tasks (e.g. solve math problems) beyond any individual learner's cognitive capacity. As the learner actively constructs learning within such a system, his or her mind could then extend itself to incorporate and assimilate the other parts. Increasing interactions lead to internalization of the embodied practices, allowing for a move to abstract stages of knowledge as well as for a shift in the learner's individual cognitive abilities.

Interestingly, this narrative provides us with a possible account of how learners might expand their cognitive abilities through a 'zone of proximal development' as Vygotsky (as cited in Gray, 2010) suggests, this zone being the area where distributed and extended cognition enable them to perform tasks beyond their individual abilities.

Importantly, however, contemporary learning is not simply about internalizing knowledge or mental skills. Learning processes should ensure that learners are not just gaining intuitive knowledge by internalizing media, but are also able to externalize that knowledge back through the various media at their disposal, as well as transfer it to various situations. Learners should also be made critically aware of the nature, sources and limitations of their knowledge.

Potential Role of Digital Media

Digital media provides us both new opportunities and constraints in this task of revising our mathematics classrooms – opportunities because, as we have seen above, it has certain features that are lacking in both orality and text, and constraints because its pervasiveness in society is likely to affect individual cognitive structures in ways that learning processes will need to acknowledge.

Unfortunately, "most teachers do not make use of the potential of ICT to contribute to the power of learning environments. Thus, computers are used mainly to complement rather than change existing pedagogical practice" (Smeets, 2005, p. 353). Most popular applications of digital media in education today replicate text-based learning processes in digital form, for example through e-books, MOOCs or drill-and-practice software applications. However, given the close relationship between mind and media, it is of both theoretical and practical interest to explore how the unique features of digital media enables novel forms of learning that break away from textual norms.

One example that is being increasingly explored in learning today is the use of digital simulations, which utilise the dynamic and manipulable nature of digital representations. Wishart (2013) outlines some of the theoretical potential of simulations for learning, along with detailed case studies. However, analysis along the lines of distributed and embodied cognition has not yet been done.

From an embodied cognition perspective, of great potential interest to mathematics education is the new range of embodied interactions with media that digital technology enables. Touchscreens have quickly become the new norm for digital devices, but newer 'gesturo-haptic' technologies are also on the rise (Rotman, 2008), such as the full-body motion sensors in the newest generation of video game consoles. de Freitas and Sinclair (2014) explore how a touchscreen-based counting application that they develop could lead to a different set of embodied representations, and thus potentially different conceptualisations of basic numbers in arithmetic learners.

If the formation of abstract concepts is seen as the internalisation of interactions with learning media, then digital media can indeed play a role in ending what Papert (1993) calls the 'brutalisation' of learners' movement from concrete to abstract stages of learning. Digital media can provide a wider range of interactions with a broader set of representations, while also co-presenting multiple representations. This can help link the concrete and the abstract in ways impossible with textual media.

Distributed cognition theories could posit potential roles for digital media even beyond the ways in which learners interact with the media itself. For example, the flipped classroom is a pedagogical arrangement wherein lectures are recorded and made available to students at home, thus freeing up classroom time for several collaborative and interactive activities. This arrangement significantly alters the scope for distributed cognition in the classroom by enhancing the space of possible interactions compared to the conventional lecture (Shah & Chandrasekharan, 2015).

IMPLICATIONS FOR EDUCATIONAL RESEARCH

Ideally, research in mathematics education should lead the way in suggesting ways to transition our learning institutions into the digital age. This requires combining theory and innovation through research methods that are not just analytical, but also creative.

Most contemporary research in educational technology does not seem to meet these needs, rather using "inherently flawed research methodologies" such as media comparison studies, which "varies the media and holds constant the instructional method," such that "the different capabilities or affordances of the media are often removed from experimental evaluation" (Hokanson & Hooper, 2000, pp. 542-543). Several other authors have also critiqued evaluative research, especially comparative studies, in educational technology (e.g. Kennewell, 2001; Kirkwood & Price, 2013).

The limitations of comparative studies have led to some limited and conservative views on the scope for digital technology in teaching and learning. For example, Clark (as cited in Kirkwood & Price, 2013, p. 4), after reviewing several such studies, suggests that the new forms of media offered by ICTs "are mere vehicles that deliver instruction but do not influence student achievement any more than the truck that delivers our groceries causes changes in our nutrition."

Clark's conclusion is understandable, given the contemporary uses of digital media in education. If we use digital media only to replicate text-based practices, it will remain a vehicle. "If there is no relationship between media and learning it may be because we have not yet made one" (Kozma, 1994, p. 7). There is a need for educational technology research to creatively explore the possibilities for creating these relationships, based on a contemporary scientific understanding of the mind and how it learns.

Design-Based Research as A Way Forward

Design-based research (DBR) is one set of methods that could help overcome this gap in mathematics education research. DBR involves designing novel learning environments or interventions and implementing them in naturalistic settings such as classrooms (see Anderson & Shattuck, 2012, for a review). The aim is to make theoretical contributions to the science of learning by analysing how learning happens in these designed settings.

The primary advantage of DBR is that it permits the study of learning environments and processes that *might not yet exist*. This seems essential given our contemplation so far of the deep effects that digital media might have on our learning processes and institutions. By enhancing theory and innovation through iterative cycles, DBR can help understand how theoretical accounts of cognition and learning might play out in mathematics classrooms when using digital media in novel ways.

Another advantage of DBR is that its primary focus is on creating qualitative design narratives rather than on the statistical evaluation of interventions. Design narratives are deeply contextual, and while lacking the generalisability of controlled trials, also free the researcher from the many constraints of comparative methodologies, which can curb the freedom to innovate with the learning process.

DESCRIPTION OF AN ONGOING DBR STUDY

Our research group has been interested in the cognition-media interface for several years, and has recently begun to move out of the lab to conduct DBR studies in classrooms. One such study focused on the chapter of 3D mensuration taught to a class of about forty eighth-grade students.

As part of the study, we designed a smartphone application that allowed the learners to interact with the 3D figures that were part of their curriculum. Interactions included free rotation, change of dimensions, and unfolding the shapes into their 2D surfaces (e.g. rectangle and circles for a cylinder). The corresponding equation for volume or surface area accompanied each figure.

Three sets of 'lessons' were created, based on discussions with the teacher about her lesson plans, and were shared with the students before the corresponding classroom session. The lessons were expected to take no more than ten minutes of the students' time. The application was not accessible in the classroom, since the aim was to use the classroom time for collaboration and discussion.

Some of the unique features of digital media that we tried to use in our intervention design include: (i) Allowing learners to interact with dynamic, manipulable 3D objects and equations rather than just static drawings and formulae; (ii) co-presenting the dynamic figure and dynamic equation to allow learners to correlate the two; (iii) Providing a relatively easy (compared to text) initial exposition to the topic so that learners can participate more actively in the classroom discussion.

Data collected included classroom observations, video recordings of the classroom, worksheet responses, digital logs of the application's usage and a student survey. Analysis is still ongoing, but some of the anecdotal evidence has proved interesting. For example, in one instance the digital application, despite being absent in the classroom, played an important role as a cognitive artefact in the distributed cognition taking place between teacher, students and the virtual shapes. By referring to the application and what they saw on it, the students were able to reconstruct an understanding of the surface area equation of one of the figures.

CONCLUSION

Different models of learning and theoretical interests could lead to very different types of DBR studies. Extensive and innovative research is required if mathematics classrooms are to successfully bridge the application of digital media in education with emerging narratives of cognition. Otherwise we risk leaving the future of mathematics learning in the hands of the fast-growing educational technology market, which will push what solutions it can create rather than focus on deeper theoretical issues. Worse, conservative text-based norms might continue to be rigidly implemented from a top-down hierarchy, creating the kind of immune system response that Papert (1993) describes by ejecting technology from schools altogether, possibly alienating entire generations of digital natives from mathematics classrooms.

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Teacher Knowledge for Teaching Geometric Similarity with Technology: A Review of Literature

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Teacher knowledge for the teaching of key topics in secondary mathematics (in particular geometric similarity) is of prime interest to the mathematics education community. This theoretical paper, which forms part of a literature review conducted for a doctoral study, aims to discuss the few existing research studies addressing teacher knowledge of this topic and to highlight how the integration of digital tools requires the nature of this knowledge to be reconsidered. The implications of these findings for the future research agenda are outlined.

Keywords: Teacher Knowledge, Geometric Similarity, Technology.

INTRODUCTION

Teacher knowledge is a key factor for successful teaching of mathematics. A variety of theoretical frameworks provide useful conceptual lenses for analysing teacher mathematical knowledge (e.g., Ball, Thames and Phelps, 2008; Mishra & Koehler, 2006; Shulman, 1987). Shulman's (1987) two content-related categories of knowledge, content knowledge (CK) and pedagogical content knowledge (PCK), have become foundational ideas for many subsequent researchers.

Geometric similarity (GS), an aspect of most school geometry curricula, is a mathematical concept for which researchers have sought to deeply examine teacher knowledge for teaching. GS is considered a fundamental integrative concept in both secondary and higher-level school mathematics (Cox & Lo, 2014; Watson, Jones & Pratt, 2013) as it connects the two core concepts of proportionality and geometric transformations, linking both numeric and geometric reasoning (Cox, 2013). However, as a result, students are known to encounter difficulties in making sense of GS (Chazan, 1988; Edwards & Cox, 2011, Noss & Hoyles, 1996). For example, students tend to fail to apply multiplicative strategies and proportional reasoning in the context of solving problems relating to GS due to an apparent over-reliance on additive strategies. Given both the significance of GS for school mathematics and the difficulties that it presents to students, identifying and describing teacher mathematical knowledge for teaching GS appears to be valuable and important endeavour to elucidate what is necessary for teachers to know in order to teach it efficiently. It is crucial for teachers to have a wide and deep understanding of GS, which involves appreciating the different approaches to its definition and how this relates to the properties of similar shapes.

As part of an ongoing doctoral study, the first author of this paper performed the literature search for articles, conference proceedings, and dissertations published in English using the following data-bases: British Education Index (BEI) (EBSCO), ERIC (EBSCO), Google Scholar, UCL Discovery, and EThOS (the British Library's Electronic Theses Online Service). The key search terms used were "similarity", "geometric similarity", "proportionality", "teacher", "classroom practice", and "technology". This method revealed surprisingly few research studies on teachers' mathematical knowledge for teaching with regard to GS (both with and without reference to any technology), evidence of a clear gap in the literature. This paper aims to synthesise key findings from the identified studies and, based on this review, to provide recommendations for future research.

RESEARCH ON TEACHERS' KNOWLEDGE OF GEOMETRIC SIMILARITY

Identified research studies have tended to examine GS as a sub-concept of proportionality (treated numerically), despite its fundamental role within aspects of geometric reasoning (Cox, 2013). The research has tended to focus on students and be centred on the diagnostic, aiming to identify the kinds of difficulties students encounter (and the strategies they use) to solve particular sets of problems (e.g. Cox, 2013; Cox & Lo, 2014; Friedlander, Lappan & Fitzgerald, 1985).

By comparison, there has been little research undertaken that focuses on teacher knowledge of GS (e.g, Clark-Wilson & Hoyles, 2017; Cunningham & Rappa, 2016; Seago, Jacobs, Heck, Nelson & Malzahn, 2014; Son, 2013). In these studies, the broad aim was either to examine the types of teacher mathematical knowledge of GS (such as CK or PCK) or to explore the growth of teachers' mathematical knowledge for teaching.

For example, Son (2013) examined 57 primary and secondary pre-service teachers' (PSTs') CK and PCK in relation to GS, paying particularly close attention to their additive reasoning in 'missing value' tasks. She argued that these tasks lead students to understand proportionality and GS deeply, from both conceptual and procedural aspects.

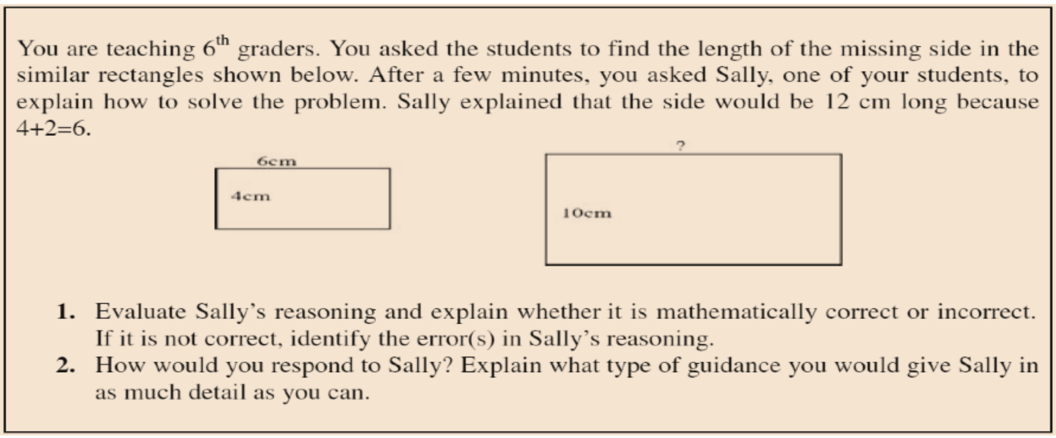

Figure 1: Pedagogical missing value problem: "What is the length of the missing length in similar rectangles?"

In Son's study, the PSTs were first asked to produce an answer to a particular missing value problem (as in Figure 1). They were then invited to interpret and respond to the student's error(s) through a teaching scenario task (Figure 1), which stems from the incorrect use of an additive strategy, and to suggest strategies to help Sally to make sense of here error.

According to Son, the successful steps to solve missing value problems featuring similar figures are: (1) understand the concept of GS; (2) be able to recognise the proportionality embedded in similar shapes by comparing lengths and widths between figures (between ratio) or by comparing the length to width within a rectangle (within ratio) or determining a scale factor; (3) explain the relationship between two similar figures using a ratio, a proportion, or a scale factor; and (4) carry out the calculation correctly. In relation to these four steps, Son noted that while the first two are related to conceptual aspects of GS, the others are associated with procedural aspects.

Son's data analysis categorised three approaches by the PSTs in the ways that they identified and interpreted Sally's misconception in terms of a procedural and conceptual approach.

- *Concept-based approach*: the PSTs paid attention to the meaning of GS in rectangles in that "two figures are similar if (1) the lengths of their corresponding sides increase (or decrease) by the same factor, called the scale factor, while their corresponding angles are equal, and (2) the perimeter from one rectangle to another rectangle also increases by the same scale factor" (p. 59).
- *Procedure-based approach*, concerning finding the value of missing side in similar figures. The PSTs underscored that in the procedure-based approach, one needs to calculate a ratio, a proportion, or a scale factor to find a missing length without necessarily understanding the meaning of GS. i.e. this approach relates to building a numerical expression indicating an equivalence between the two rectangles.
- *Misidentification* of the error(s) in terms of additive reasoning or improper focus.

One of the most notable findings of Son's study is that, despite the fact that Sally's misconception related to her use of an additive strategy might be due to a limited understanding of the concept of similarity, most of the PSTs considered that the error was due to her procedural misunderstanding.

A second study, conducted by Seago et al. (2014), aimed to promote secondary mathematics teachers' MKT in relation to a transformations-based approach to the definition of GS. It is noteworthy to state here that in the literature, GS can be conceptualised in three related but distinctive ways. The first conceptualises GS as "the same shape, but not necessarily the same size". The second, which is named as a static-based approach, conceptualises GS on the basis of a numeric relationship between measures of lengths of figures and their sizes of angles. This implies that if two figures are similar, the measures of their corresponding lengths are proportional, and the sizes of their corresponding angles are equal. The third is a transformations-based approach, whereby GS is conceptualised in terms of translations, reflections, rotations and dilations.

Seago et al. (2014) define GS based on a transformations-based approach as follows: in order for two figures to be similar, it is required that the second figure can be acquired from the first one by applying a sequence of translations, reflections, rotations, and dilations. This pedagogical approach encourages students to solve GS problems by reasoning and applying geometric transformations, rather than by merely applying numeric strategies. The researchers hypothesised that incorporating a transformations-based approach into the process of teaching GS enables students to develop a deeper understanding. For this reason, they aimed to support teachers to "gain a robust conception of similar figures as part of an infinite family that can be formed by applying one or more geometric transformations" (p. 632).

Their study involved a professional development programme (PD) in which a sequence of video cases was used to present the mathematical ideas to teachers so as to address the challenges teachers may face when adopting a transformations-based approach. The research findings indicated that through the PD provided, the teachers improved their understanding of GS for teaching, in particular concerning their mathematical knowledge regarding definitions of GS relating to congruence and dilation.

Further to the work of Son (2013) and Seago et al. (2014), Cunningham and Rappa (2016) also investigated mathematics teachers' ability to solve GS problems. The researchers surmise that, like Seago et al., when teachers introduce a transformations-based approach together with a static-based approach when teaching GS, students are likely to understand the underlying ideas of GS more deeply. Therefore, they asserted that it is important to investigate teachers' mathematical knowledge of GS from both perspectives because the teachers' mathematical knowledge could play a key role in the development of students' understanding.

In their small-scale study, Cunningham and Rappa asked 15 secondary mathematics teachers to solve seven problems related to GS, in which either a static-based approach or a transformations-based approach was the stipulated method.

The concluded that, while the problems requiring a static perspective were successfully solved by all of the teachers, only eight teachers were able to successfully solve the problems requiring a transformational perspective. Cunningham and Rapp conclude that the latter group of teachers perceived GS more procedurally, which led them to rely only on the numerical relationship embedded in the similar figures. This result resonates with Son's (2013) finding that teachers may favour using procedure-based method to solve problems related to GS.

The aforementioned studies were not designed to research teachers' specific knowledge and practice to use dynamic technology in the teaching of GS, although the study by Seago et al. (2014) did employ technology within the PD.

This aspect has been partially addressed in research conducted by Clark-Wilson and Hoyles (2017), which explored the impact of 40 secondary mathematics teachers' engagement with PD and classroom teaching on their mathematical knowledge for teaching GS. Their study explored the teachers' starting points using data collected through survey-items, PD tasks and lesson plans. Key to the design of the PD were a number of tasks for teachers that required them to closely analyse hypothesised student responses whilst engaging with a particular dynamic mathematical technology (DMT), 'Cornerstone Maths' (CM) (For an example, see Figure 2).

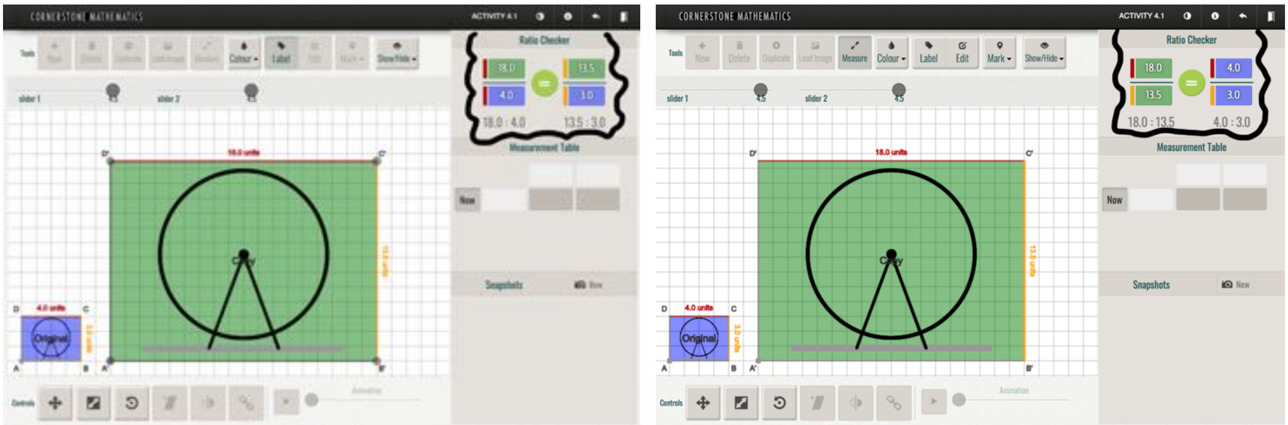

Figure 2: Students' work created in the DMT used as a context for teachers to identify the meaning of invariant ratios revealed by the two statements in the ratio checker.

Clark-Wilson and Hoyles gained further insight into the teachers' enacted knowledge by observing a common lesson from the CM teaching sequence. 7 of the 40 participating teachers were observed and subsequently interviewed about their lesson.

Clark-Wilson and Hoyles found that the combination of PD activity focused on GS and classroom teaching involving DMT led to notable improvements in teachers MKT in relation to GS that "concerned more robust definitions of [GS] for a broader range of polygons and the appreciation of the invariant ratio property for pairs of corresponding sides within similar polygons" (p. 18). According to the researchers, the use of students' work created in the DMT environment (as in Figure 2) encouraged the teachers to think deeply about the "within ratio" invariant property. Having engaged with the task in the DMT environment, they were able to successfully articulate the underlying mathematical ideas related to the property that, for similar shapes, the ratios of the side lengths for any pair of corresponding sides within the shape is invariant.

CONCLUDING COMMENTS AND FUTURE DIRECTIONS

The aforementioned studies sought to elicidate and/or improve teacher knowledge in relation to GS and they provide multiple insights into both what teachers understand concerning GS and how they develop related MKT. For example, Son's (2013) revealed a lack of teachers GS-related PCK as evidenced by the misidentification and/or misinterpretation of Sally's error. The studies by Seago et al. (2014) and Cunningham and Rappa (2016), reveal that definitions of GS from the perspective of geometric transformations is novel to teachers and the key role that PD plays in the development of teachers' understanding of the underlying curriculum links. Likewise, Clark-Wilson and Hoyles' (2017) study suggests that although gaps in teachers' knowledge for teaching GS are apparent, PD programmes in which teachers use dynamic technology to engage with mathematical activities away from and in the classroom do stimulate notable improvements in their knowledge.

Furthermore, research has consistently highlighted that GS is a mathematical topic with which both students and teachers encounter difficulties. Simultaneously, some studies do suggest that carefully designed DMTs, might help students (and teachers) to overcome difficulties and misconceptions with regard to GS as the dynamic and visual nature of digital technology offers opportunities (e.g., dragging, visualisation, measurement) to explore the underlying concepts and discover the embedded variant and invariant relationships (Chazan, 1988; Denton, 2017; Edwards & Cox, 2011). Such opportunities might enable teachers and students to experience and examine the dynamic nature of GS in more tangible ways. For example, teachers can exploit the affordances of digital technology to help students build connections between geometric transformations and GS so that students understand how to use translations, reflections, rotations and dilations to determine if two figures are similar. Additionally, making use of technology in a dynamic environment where students can formulate, test, and verify mathematical conjectures, teachers can support students to surmount their misconceptions about the ideas of GS, particularly those who make the incorrect use of an additive strategy as the student in Son's (2013) study. How digital technology can support students in addressing their misconceptions related to non-multiplicative strategies has been illustrated by Edwards and Cox (2011).

We conclude that carefully designed DMT can be a useful didactical tool that can provide teachers with both a context and opportunities to develop their students' understanding of the ideas of GS. A focused and longitudinal investigation into teacher knowledge for teaching GS using digital technology could identify and articulate teachers' relevant mathematical knowledge, an aspect that none of the aforementioned research studies have *specifically* explored. Consequently, such a study would focus the research lens on characteristics of teacher knowledge (both espoused and enacted in the classroom) in relation to using digital technology to teach GS.

Moreover, researchers underline that one of the key factors for the success of the integration of digital technology into classroom practice is the teacher, and the interactive and dynamic nature of their knowledge plays a central role in underpinning the practice (Ruthven, 2014). Nonetheless, it is widely acknowledged that that the integration of new digital technologies into ordinary classroom practices poses a significant challenge for mathematics teachers.

In terms of the concept of GS, Clark-Wilson and Hoyles' (2017) research project is the only study in the literature that focuses on selected teachers' classroom practices with DMT in relation to GS. However, as their research probed only one lesson of each teacher, their data provides useful but limited insights into the development of their mathematical knowledge and associated classroom practices on their teaching of GS with dynamic technology. Hence, very little is currently known as to *how and why* teachers exploit the opportunities that dynamic digital technology offers when teaching GS in the mainstream classroom and how their associated knowledge shapes and is shaped by their thoughts and actions, and thereby their practices. There is a need to identify the important aspects of teacher knowledge and classroom practices that promote students' robust understanding of GS within technology enhanced classroom environments through conducting more systematic investigation. This necessity, therefore, calls for more research aiming to ascertain what mathematical knowledge for teaching and mathematical pedagogic practices are required for teachers to productively make use of dynamic digital technologies in their classroom teaching of GS.

To address the identified gap, the first author's doctoral study is researching the actual classroom practices of three English secondary mathematics teachers using a particular DMT to teach GS explored through classroom observation, teacher interview, and the scrutiny of lesson plans and resources. The combination of the Structuring Features of Classroom Practices (Ruthven, 2014) and Instrumental Orchestration (Drijvers et al., 2010) frameworks guide both the data collection and analysis. The research aims, in particular, to develop a more comprehensive understanding of the development of teachers' classroom practices when teaching GS with dynamic technology along with the nature and content of their associated MKT and, in general, to add to the growing body of knowledge on teachers' integration of DMT in mathematics classrooms.

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Studying the use of digital resources in mathematics classrooms: A deeper focus on the reasons underlying teachers' choices

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In this paper, we report research on one mathematics teacher's use of digital resources in the context of a professional development project targeting inquiry in statistics. Since the teacher was an accomplished user of digital resources, it was important for us to study if and how she aligned her teaching with the project aims, the conflicts that emerged regarding mathematical activities and communication, and the reasons behind her decisions. The analysis reveals that the teacher's choices were rooted mainly in her conventional practice, which did not include analysing and connecting the affordances of digital resources with mathematical content. This led her to block students' communication and engagement in inquiry processes.

Keywords: Use of digital resources, classroom teaching practice, documentational approach, teachers' decision making.

INTRODUCTION

Digital resources (DR) have been increasingly available for mathematics teaching overt the last years primarily with the aim to promote development of inquiry processes, and support cultures of richer mathematical activity and communication in classrooms. Given the teachers' reported difficulties in achieving that goal, this paper aims to contribute in getting a deeper insight into teachers' underlying reasons for how they integrate DR in their daily practice. Research shows that integration of DR in mathematics teaching appears to be a highly complex task for teachers. This complexity relates to the fact that the introduction of DR involves the establishment of new cultural practices that requires comprehensive changes in pedagogy, if it is to facilitate student learning as envisaged (Hoyles & Lagrange, 2010). In addition to identify the potential of different DR for mathematical understanding, teachers face a number of critical issues underlying the teaching and learning process with DR. These include the nature of the designed tasks, the appropriation of the available (digital and non-digital) resources into classroom teaching as well as the need to transform dominant modes of classroom communication in more participatory directions where new forms of exploration with digital and dynamic mathematical representations are present (Ruthven & Hennessy, 2002). However, research points out that little is known about the impact of DR's actual use on teachers' work and the interrelationship between factors that influence teachers' decision making in DR settings and established norms, teaching conventions and personal background (Goos & Soury-Lavergne, 2010). To address this issue, we study how a primary teacher (Sofia) regarded as an accomplished user of DR, use these resources in her classroom teaching in the context of her participation in a large Teacher Professional Development (TPD) program aiming to engage students in inquiry processes by use of DR. It was critical for us to study if and how she aligned her teaching with the project aims, how she drew on DR, and what conflicts emerged in her teaching with regard to mathematical activities and communication.

THEORETICAL FRAMEWORK

The *documentational approach to didactics* acknowledges the crucial role of resources for teachers' work and professional growth. Gueudet and Trouche (2011) retain a broad meaning for resources comprising material and non-material elements (such as a textbook, a discussion with colleagues etc.) while they consider that the teacher's work with and on resources constitutes a dialectic process where design and enactment are intertwined. Teachers use different kinds of resources that shape not only the mathematical content and the ways it is (re)presented, but also students' mathematical learning. Teachers adapt their appropriation and use of resources to their needs and customs. An implication of this approach is that curriculum material is not conceived as a static body of resources that guides instruction, but rather as a set of objects amenable to changes and modifications depending on the teacher's didactical design and actual use. This dynamic process of (re)-design and interpretation continues ('in use') during enactment of the resources (e.g., Pepin et al., 2013) leading to teachers' creation of *documents*. A document incorporates "practice (how to use these resources for teaching a given subject) and knowledge (on mathematics, on mathematics teaching, on students, on technology)" (Gueudet & Trouche, 2011, p. 401). On the one hand, practice entails observable parts of teachers' stable behaviour for a given class of situations called *usages*. On the other hand, teacher's knowledge about the use of the resources is often implicit, but can be inferred from the usages, i.e. the 'reasons' piloting the usages called *operational invariants*. Creation of documents is considered as unfolding through a dual process of instrumentation (the resources influence teachers' practice and knowledge) and instrumentalisation (teachers act upon these resources as they appropriate them).

Another strand of existing research indicates that integration of DR depends highly on teachers adapting and developing appropriate craft knowledge to underpin their classroom work (Ruthven, 2009). Aiming to make visible and analysable the constituent elements of this kind of practical knowledge in bearing technology integration in mathematics teaching, Ruthven developed the conceptual framework *Structuring Features of Classroom Practice (SFCP)*. It brings to the foreground five structuring features of classroom practices which shape the ways in which teachers integrate (or fall short of integrating) new technologies in their teaching: *Working environment* refers to where the lessons take place (room location, class organization etc.); *Resource system* includes the digital and material resources in use towards student activity and learning goals; *Activity structure* describes activity formats that frame the action and interaction of teacher and students during particular styles of lesson; *Curriculum script* is a loosely ordered mental model of goals, resources and actions for teaching a mathematical topic that guides lesson enactment (e.g., potential emergent issues, alternative paths of action); *Time economy* refers to the teacher's management of the time available for class activity so as to convert it into "didactic time" for student learning.

In our study, the documentational approach offers an opportune window into Sofia's classroom work to explore if and how the project innovation was taken into account (e.g., task redesign, aligning her lesson plans with the project aims). It also allows us to address implicit and explicit knowledge, and conventions, norms, views and values that drive her choices at the level of design and enactment. SFPC allowed us to access key aspects of Sofia's classroom practices so as providing a more comprehensive interpretation of the emergent current conflicts by highlighting their connections to Sofia's teaching history in terms of long term activity structures.

METHODOLOGY

Even though the focus of the TPD project was not on teachers' classroom practices in particular, it turned out to be a useful context to study Sofia's teaching in terms of identifying conventions, norms, views and values that underlay her integration of DR. The context was especially useful, when Sofia attempted to adapt her practices in line with the project, but where it became difficult for the students to engage meaningfully in the statistical activities and communication. These instances of practice provided insights into conventional practices that seemed to contribute to complicate the intended inquiry processes. We termed such instances as critical instances of practice, much in line with Skott (2001), and used them as an analytical strategy to make visible the reasons behind her daily integration of DR.

We observed and recorded 31 classroom lessons (16 from the project, 15 before or after it) and conducted five semi-structured interviews (three during the project and two five month later). Apart from the last one, which was a focus group interview evaluating the entire project, the others were conducted face-to-face and focused on Sofia's perception and use of DR in her daily teaching as compared to her teaching carried out in the context of the project. Based on the interviews that were audio-recorded and transcribed, we constructed a narrative of Sofia focusing primarily on how she became a mathematics teacher preoccupied by DR and how she perceived mathematics teaching with/without DR. We identified critical instances of practice in the videos of the project lessons, which we analysed in terms of (a) the ways Sofia conceived and used DR, and (b) the SFCP framework. With the aim of understanding Sofia's practice and reasons behind her decisions, we compared and synthesized (triangulation) evidence from interviews and observations. For example, a verbatim transcript of the lessons (e.g., *"Write it all down. You just have to make a list [of your everyday use of statistics]"*) and an interview quote (e.g., *"I think I only teach basic skills … we never have time to do something [inquiry oriented]."*) were used to establish the presence of a specific operational invariant or activity format (e.g., Sofia tends to interact in procedural ways with students).

THE CASE OF SOFIA

Sofia has taught mathematics in grade 4-6 at the same small municipality school in a Danish village, since she graduated in 2000. She describes the mathematical part of her four years of education as an amplifier of her high school mathematics, dominated by presentations of theorems and proofs. Didactical aspects, such as how to use DR and communicate in mathematics classrooms, were only insignificant parts of her education. Inspired by her own everyday use of DR, she started early on to use them in teaching mathematics. During the last 15 years, she participated in various TPD on DR such as national projects, publisher arrangements and municipal initiatives. On the one hand, she emphasises such opportunities as important for keeping her spirits up on teaching. On the other hand, she expresses lack of real development opportunities as she experiences to be the most knowledgeable about DR: *"Each time I have that naive belief that someone can inspire me and tell me what they do and how… but no-one does"*. Hence, Sofia appears very secure and confident about her teaching and reasons for integrating DR. Her colleagues and the school management regard her as an accomplished user of DR and recently the management appointed her as DR-coach in the municipality. In her daily teaching, Sofia uses various DR such as an electronic training-database, applets (e.g., Book Creator, Explain Everything and Showbie), Microsoft Office package and dynamic geometry system (seldom). She uses such resources primarily to engage students in skill practice; check their understandings and fluency; and adapt the content to students' individual needs.

In 2013-2014, Sofia participated in the mathematical part of the national TPD project with the overall aim to support teachers in developing new teaching practices that take DR into account and are in line with the reform orientation of mathematics teaching (NCTM, 2000). A unit, called *Youngsters and ICTs,* covering 15 lessons of statistics teaching in grade 6 was its focal point. The aim of the unit was to support teachers in using DR to initiate, negotiate and establish two classroom mathematical practices: (1) to be critical towards the use of statistics, (2) to investigate and reason about patterns in data sets. To realise these aims, the unit framed and suggested ways for teachers to engage students in statistics inquiry by use of electronic surveys to aid them generate their own data, spreadsheets to support their data analysis and reasoning, and Explain Everything to support their reasoning and interpretation.

Sofia participated in three workshops and taught the unit two times (spring 2013 and winter 2014). Generally, she considers the unit as not sufficiently innovative as regards DR, while she appreciates its different teaching approach: *"I do not think there has been enough DR ... but, mathematically, it's another way of teaching than I have taught in the past"*.

A CRITICAL INSTANCE OF STATISTICS TEACHING

We present one critical incident from the second lesson of Sofia's teaching in 6th grade in 2014 where the pivot was a newspaper article using statistical methods and reasoning in inappropriate and opaque ways. By engaging students in a first critical examination of such methods with the aid of a spreadsheet, the purpose of this lesson was to set the stage for the rest of the unit. The lesson started with Sofia introducing the article to the students and asking them to answer questions she had written in Showbie. At this phase of the unit, Sofia had adapted her activity format and curriculum script on several occasions as when she engaged students in exploring the questions through a combination of group work and whole class discussion. However, these promising adaptations concerned only parts of her teaching that did not involve DR. In comparison, the selected episode involves use of DR and it brings to the fore how Sofia's conventional practices make it hard for the students to engage meaningfully in statistical inquiry and communication.

The incident occurs when Sofia encourages a group of four engaged girls to examine critically the article's calculation of the statistical mean of children's use of ICTs given four pairs of hours and minutes for different age groups. Sofia initiates, *"Do you have a calculator, where you can try to do it?"* She carries on instructing the girls on how to convert from hours and minutes to minutes, before they realise this as a necessary step. The girls manage on their own to convert the numbers with the use of the calculator, add them and ask Sofia, *"Do we have to divide by four?"* In a long interaction (more than 2 min.), Sofia initially encourages the girls to think for themselves, but as they repeatedly type the numbers wrong, gaining odd results, the mutual frustration grows. When a girl asks, *"How can you interpret the mean?* Sofia takes over on the calculator, types the four numbers and divide them by 60, while she says, *"You have to find out. You have to make a guess"*. Sofia does not succeed in finding the mean, and together they try more or less randomly, before they realise that they need to divide by four. One girl types the addition of the four numbers, divide by 60 and then by four. When the calculator returns 7.7333, she says, *"That was close [the mean in the article is 7 hours and 48 min.] but clearly not right"*. Sofia asks, *"What does the decimal 7333 mean? How do you convert into minutes?"* As no one reacts, Sofia moves on *"We agree that it is 7 hours? We remove the 7 hours and 0.7333 remains. If you multiply by 60 then you get that part of an hour ... You just have to find 73/100 out of 60"*. Clearly guessing, the girls suggest to subtract 73/100 and 60 and then to divide the two numbers. Sofia repeats her explanation and one girl says, *"You confuse me more than I was before"*. Sofia instructs the girls what to do and says, *"But you have not understood it. You just know what to do"*.

REASONS UNDERLYING SOFIA'S CHOICES

In the instance, Sofia enacts the unit but she decides on behalf of the girls to use a calculator as a DR, not a spreadsheet as suggested by the project. However, the calculator does not require the girls to develop a model or sheet that could provide them an overview of the situation and help them structure their solution. On the contrary, they have to sequence the calculations in separate parts without being able to log their calculation history or model their common thinking. They could have used paper to keep track of their calculations and provide an overview, but only using the calculator seems to complicate the activity. This example is evidence of a practice that we see generally in Sofia's teaching: she chooses to use DR without considering how to support students' learning through their use.

In terms of the documentational approach, Sofia initiated a usage for the calculator focusing on computing the statistical mean through automatization skills. At the same time, she failed to integrate the resource use in an inquiry way as targeted by the project. We seek to make visible the underlying reasons driving Sofia's choice and use of the selected resource as well as the features of her practice that pilot her enactment of the project's unit in the classroom. Based on the narrative we deduced four reasons behind her choice and use of DR. First, Sofia has established a norm where she learns a new DR together with the students. She copes with the fact that they often learn it faster than her as a sound exchange of roles *"It is fine, that they can teach me something. It is sound for them to see that I am not able to do everything"*. Sofia developed this practice in response to the often-heard need of Danish teachers to be in full control of the resources they use in their classrooms and not dare to use an unknown or less familiar one. Second, Sofia's choice and use of DR also connect closely to her balancing her working hours. She reports seldom to have time to do a designed task with DR before teaching. She considers this unpreparedness as a necessary condition of her job: *"I haven't done any of the tasks [with a DR]. Of course, then you sometimes fail as a teacher, because you have not prepared yourself properly. That is part of being a teacher"*. This means, that Sofia is neither conscious nor prepared to handle the mathematical or technical difficulties students might encounter when they use a new DR or solve a new task. Third, she constantly balances between the time students require to learn a new DR and their expected learning outcome, i.e. the time economy. So far, this balance has tilted to the side of not introducing spreadsheets, as she considers it too complex for this class. Therefore, the range of DR that she can draw on in the activity is limited. Fourth and most crucial, when choosing to use DR Sofia predominantly steers by general pedagogical concerns such as to motivate students by variating activities during a lesson and to meet their different needs by individualizing their work. Her decisions do not rest on analysis nor reflections on the mathematical affordances of DR in relation to students' learning through their use. We deduce two operational invariants that seem to pilot her daily usages regarding DR. One is that "The use of DR generally contribute to students' learning", and the other that "The use of DR does not need to be addressed from a mathematical point of view".

Taking an overall view of the episode in terms of SFCP, we see that at the level of the resource system Sofia only uses DR familiar to students (e.g., calculator, Showbie) and at this time she chooses not to use the ones suggested by the project (e.g., spreadsheet) that facilitated a critical stance towards use of statistics. In contrast to the projects' focus on selecting resources according to their potential for supporting students' statistical learning, Sofia's underlying reasons are related to her working conditions (limited time for introducing new DR), and to handle students' low and different ability level in mathematics from general pedagogical perspectives.

In planning to teach the unit on statistics, Sofia draws primarily on her existing curriculum script built over years of experience. In particular, as she mentioned in interviews she views DR as an efficient tool to provide new, different and dynamic representations. Nevertheless, she appears to underestimate the refinements that curriculum scripts need in order to integrate a new DR. For that reason, the inquiry processes targeted by the project conflict Sofia's views on the aim of mathematics teaching, which is to provide students *"with a tool box ... The students have to be able to calculate area and circumference, to add, subtract, multiply and divide and solve an equation"*. In the episode, we see Sofia concentrating on the standard techniques (e.g., divisions, linking decimals to time units) without having responded to ways in which a DR (i.e. calculator) may help/hinder specific processes and objectives involved in learning the targeted topic (i.e. statistical mean). Here it seemed to hinder the students' engagement in the activity and their communication.

At the level of activity format, we see that in the beginning of the lesson Sofia seems to adapt her normal activity structure to include the suggested whole-class discussions and classroom activity format, when DR is not involved. However, when DR is part of the activity, Sofia interacts with students in her conventional procedural way by instructing them on what to do and asking questions that fit with her own interpretations. Based on the narrative, her underlying reasons for such procedural interactions relate partly to her view of the aim of teaching mathematics targeting basic skills as a prerequisite to processes of inquiry, through *"a lot of practice. It is important to make a lot of the same type of tasks because then you get good at it"*, and partly to her view of mathematics communication. On the one hand, she stresses communication as one of her teaching values. On the other hand, she does not consider it as one of her strengths as a teacher: *"I have never been a mathematics teacher that can talk for twenty minutes"*. Hence, her activity format encompasses short initial lesson instructions to students on what to do and interactions with one or a few students working primarily on closed tasks using DR. Only rarely does she assemble the class for joint classroom discussions. Sofia considers her role as, *"... the role of a supervisor. I think I often supervise more than I teach"*. Sofia's integration of DR have enforced both her role as supervisor and her procedural interaction format as she has to technically support and structure students' individual work on mainly closed tasks in relation to more resources.

CONCLUSION

The TDP project targeted statistical inquiry with goals, resources and actions emphasising a critical approach to statistical thinking. This required a resource system exploiting the affordances of the different resources (digital and non-digital) to support students' conceptual understandings. Although Sofia found the unit challenging from a mathematical point of view, she appears to draw mainly on resources familiar to her and her students as well as to follow her conventional ways of interacting and communicating with the students when using DR. Without recognising the need to analyse and connect the affordances of a DR with mathematical content, she finds herself in a position that blocks students' communication and engagement in inquiry processes. A potential suggestion for teacher education targeted integration of DR in mathematics teaching can be a deeper focus on analysing potentialities of DR in relation to students' learning.

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Educating teachers to use e-textbooks as a means to prompt argumentation and creative reasoning

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This study aims at examining the effectiveness of a professional development course to educate teachers to use e-textbooks in a way that prompts creative reasoning and argumentation processes. Seven middle school teachers participated in this study, which was guided by the framework for creative reasoning and the inquiry community approach. Data were analyzed using the creative and imitative reasoning framework. Findings show that teachers' knowledge developed in a way that enabled them to use e-textbook features to prompt creative reasoning and argumentation processes.

INTRODUCTION

Although much is known about types of reasoning that have evolved following the use of traditional textbooks, less is known about types of reasoning that might evolve when students use electronic textbooks. This issue has posed an educational challenge that must be solved and understood better in order to shed light on how the use of e-textbooks may change traditional types of reasoning.

We assume that e-textbooks, insofar as they merge dynamic features of scientific concepts with those of its abstract entity, may boost students' creative reasoning (CR) and argumentation processes (AR). In view of the importance of developing these types of reasoning and of the role that may be played by teachers in using e-textbooks to boost the emergence of new styles of reasoning, we seek to identify teaching-learning processes and potential learning benefits associated with the use of e-textbooks in mathematics education. We also seek to explain how these e-textbooks provide new opportunities for learning mathematics.

In Israel, the Ministry of Education has begun to require textbooks publishers, in parallel to publishing textbooks, also to publish e-textbook versions, which can be made available on the Internet. Some publishers fulfil this requirement minimally by merely scanning existing textbooks and preparing a PDF version of them. Others have published e-textbooks, taking into account the resources of technological tools, producing thus a work qualitatively different from ordinary paper textbooks. Since these e-textbooks allow several modes of interaction, we assumed that teaching-learning with them might stimulate the emergence of new types of reasoning that were not common when students had used regular textbooks (Sciortino, 2016).

Nevertheless, teachers in Israel typically use e-textbooks in ways similar to those used with ordinary textbooks. Therefore, we need to understand better, how teachers and students might learn to use e-textbooks to stimulate the emergence of new types of reasoning. One way to approach this is by way of professional development courses (PDC) to educate teachers in using e-textbooks. In this paper, we report on a PDC for middle school teachers that aims at educating them in the use of e-textbooks to stimulate AR and CR. In particular, this study aims at examining how the PDC may affect the ways teachers use e-textbooks to prompt AR and CR.

THEORETICAL FRAMEWORK

This section is divided into three parts: (a) AR and CR concepts; (b) the concept of e-textbooks, their affordances and types; and c) the community of inquiry as a framework around which we constructed the PDC.

Creative reasoning and argumentation

Although a consensus exists among scholars that CR is essential for educating students to be able to face the challenges of the 21st century, the scientific literature still has several definitions of CR (Magnani & Nersessian, 2002). Here, we adopted the definition of CR introduced by Lithner (2008). For Lithner, CR is opposed to what he calls imitative learning, which he sees at the heart of "rote learning" and "algorithmic learning". He defined CR as a reasoning style that fulfils the following criteria: (a) Novelty. A new (to the reasoner) reasoning sequence is created, or a forgotten one is re-created; (b) Plausibility. There are arguments supporting the strategy choice and/or strategy implementation motivating why the conclusions are true or plausible; and (c) Mathematical foundation. The arguments are anchored in intrinsic mathematical properties of the components involved in the reasoning (Lithner, 2008, p. 266).

Like creative reasoning, AR is also important and fundamental in teaching-learning mathematics within schools and beyond the them as well. Mercier (2011) argues that the development of students' argumentation abilities may prepare them for life after school. In the context of teaching-learning mathematics, reinforcing argumentation skills may help students move from inductive reasoning to deductive reasoning, which is characteristic of mathematical thinking (Arzarello & Sabena, 2011). Lithner tied AR with CR. He argued that two characteristics of CR include a high level of AR. For this reason, Lithner referred to Schoenfeld's six phases of problem solving to connect between CR, problem solving and AR: reading the task; analysis; exploration; planning; implementation; and verification. To link problem solving with AR, he introduced the concepts of predictive argumentation (Why will the strategy solve the task?) and verificative argumentation (Why did the strategy solve the task?). He claimed that predictive argumentation is often linked with the first three phases, while verification argumentation is often linked with the last phases.

Nordmark and Milrad (2015) claimed that e-textbook features, especially interactive content, have changed the way that students learn and think (Nordmark & Milrad, 2015). Along the same line of thought, we assumed that e-textbook affordances may prompt the emergence of CR, AR and problem solving. To examine this assumption, we found Lithner's framework useful for examining the CR that might emerge when students learn with e-textbooks, as well serve as a tool for educating teachers to use e-textbooks to prompt the emergence of CR.

E-textbooks: definition, types and affordances

Pepin et al. (2016) defined e-textbooks as "an evolving structured set of digital resources, dedicated to teaching, initially designed by different types of authors, but open for re-design by teachers, both individually and collectively" (p. 644). Pepin et al. (2016) distinguished between three types of e-textbooks:

- *Integrative e-textbook*: referring to an 'adds-on' type of model where the digital version of a (traditional) textbook is connected to other learning objects
- *Interactive e-textbook*: referring to a 'toolkit' model where the e-textbook is based on a set of learning objects, tasks and interactives that can be linked
- *Evolving e-textbook*: referring to a developing type model, where a core community has authored a digital textbook that is developing permanently due to the input of other practicing members/teachers

Comparing e-textbook affordances with those of ordinary textbooks, Holzinger (2013) argued that unlike ordinary textbooks, which contain content that cannot be easily updated and changed, e-textbooks enable rich and dynamic media access, and provide learners with interactive experiences. Rockinson et al. (2013) listed the main features of e-textbooks, characterizing them as books that contain information represented by diverse media (text, image, video), an interactive environment enabling feedback and evaluation.

Since teachers' use of e-textbooks, at least in Israel, is not widespread, to explore how teachers use e-textbooks to prompt CR, we proposed a PDC to educate them in how to use e-textbooks and to share with them e-textbook the affordances. In doing so, we hoped to build a community of inquiry (Jaworski, 2006) in which we share our theoretical knowledge with teachers and they share with us ideas regarding practical ways of using e-textbooks in the classroom.

Community of inquiry

Jaworski (2006) proposed a model for engaging several communities in inquiry processes that addresses three forms of inquiry practice:

- *Inquiry in mathematics*: students in schools learn mathematics by exploring tasks and problems in the classroom; teachers use inquiry as a tool to promote students' learning of mathematics
- *Inquiry in teaching mathematics*: teachers use inquiry to explore the design and implementation of tasks, problems and activities in the classroom; educators use inquiry as a tool to enable teachers to develop teaching
- *Inquiry in research that results in developing the teaching of mathematics*: teachers and educators research the processes of using inquiry in mathematics and in the teaching of mathematics

This paper focuses on the teachers' evolutionary processes through the PDC. To examine these processes, we built a community of inquiry that included teachers and researchers according to the third inquiry form mentioned above.

PROFESSIONAL DEVELOPMENT COURSE

The PDC comprises five sessions and aims at educating teachers in how to use e-textbooks to prompt CR and AR through problem solving. In the first session, the theoretical background of CR, AR and problem solving was discussed and shared. The second session was devoted to the theoretical background about e-textbooks, their types and affordances. An analysis of items from e-textbooks according to Lithner's frameworks was shared and discussed. The third session was devoted to planning a lesson scenario whose educational goal was to prompt students in using AR and CR. Each teacher chose a geometry topic from an e-textbook and planned a lesson scenario.

Planning the scenarios was a collective activity in which both the teachers and the researchers worked together and discussed the process of planning the scenario. The scenarios were written using the Comics software, which simulates a classroom situation. Teachers were requested to improve and elaborate on their scenarios according to the feedback they received from the community members. In the fourth session, the teachers discussed the elaborated scenario. The scenario presented was criticized according to two principles: a) whether and how the presenting teacher made use of e-textbook affordances b) whether and how the scenario prompted CR and AR. In the last session, each teacher played out their scenarios while the other teachers served as students. This time as well, each scenario was criticized and the teachers were requested to modify them in order to apply them in their real classrooms.

METHODOLOGY

The e-textbook used in this study

For, this study we chose Kotar e-textbook (CET, 2018) that utilizes the GeoGebra interface. Kotar contains diverse topics in the Israeli mathematics curriculum. This e-textbook presents content in an interactive and dynamic manner, and the questions that appear in Kotar are written in such a way as to prompt students to explore and explain their answers. For each item, Kotar devotes a space in which students can drag mathematical objects and explore them. It also provides various tools, as in GeoGebra, which might help students to construct mathematical objects and solve the problem. In some cases, Kotar provides learners with all the tools that are available in GeoGebra'; in other cases, it limits the tools to be used by the learners.

Figure 1 illustrates an example of an item from Kotar for which the students are asked to examine the validity of a particular argument. To do so, the students are required to use both construction tools and dragging tools (Fig. 1). The students are able to drag and move the drawing from any point (A, B or C) they choose. To validate or refute the argument, the students can use specific tools or several tools. For example, the student can measure length, construct a midpoint, construct a perpendicular that passes over the mid-segment, or construct a circle and find the intersection points.

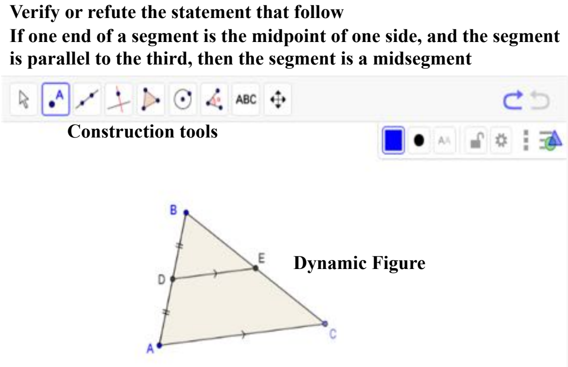

Figure 1: Example of an item from Kotar

Study design, data collection and analysis

Seven middle school teachers volunteered to participate in five after-school sessions that aimed at educating them in how to conduct instruction using e-textbooks that emphasize CR and AR. The teachers who participated in this study are novice teachers and they were not familiar with using e-textbooks to teach mathematics.

The data were collected in several ways. First, the discussions held between the participants during the sessions were video-recorded. Second, the teachers were requested to complete a journal regarding their reflections about the sessions. These journals were collected and the data analyzed. Third, the lesson plans that the teachers had prepared were also collected and analyzed later.

We took three dimensions into consideration in analyzing the data: argumentation; creative reasoning; and affordances of the e-textbook. We coded the participants' statements, distinguishing between the different elements of the three dimensions. We paid special attention to situations in which the participants shifted from one dimension to the other.

FINDINGS

In this section, we present the results of the evolutionary process of one of the teachers, Faisal, who had participated in this research.

Faisal's knowledge about reasoning styles

In the first session, Faisal demonstrated a lack of knowledge regarding the meaning of the creative reasoning and argumentation. Follow Osama's question to the teachers regarding their knowledge about argumentation, Faisal said:

Faisal: I have never heard of this concept before. Maybe we use it but I do not know what you mean by it.

After discussing the theoretical definitions of AR and CR with the teachers and enriching the discussion with examples, Faisal's knowledge about them changed. He said:

Faisal: Now I am beginning to rethink things that had seemed obvious to me, such as thinking process and creative reasoning. I can now say that I know what argumentation and creative reasoning is. I will also try to apply these reasoning styles in the class with my students, especially that now I am teaching the subject of word problems.

Faisal stated that he recognized the concepts of AR and CR. For him, these two concepts are obvious and he was regularly using them automatically, without thinking deeply about their meaning. Faisal was not satisfied with just recognizing the meaning of AR and CR; he also intended to apply the concepts in his future instruction. More specifically, he determined the mathematical topic for which these kinds of reasoning might be suitable.

Faisal's knowledge about e-textbooks

In the second session, Faisal demonstrated a lack of knowledge regarding e-textbooks. This lack of knowledge was expected since the teachers rarely used a variety of e-textbooks for teaching-learning mathematics.

Hanan: What does e-textbook mean to you?

Faisal: It is the same book but scanned and uploaded in PDF format to the Internet.

This transcript illustrates Faisal's limited access to a variety of e-textbooks. He was familiar with the integrated e-textbook. At the end of the second session, Faisal had broadened his knowledge about e-textbooks. At this time, he started to take into account e-textbook features and how these features might boost CR and AR among students.

Hanan: How could you use e-textbook to boost creative reasoning and argumentation processes among your students?

Faisal: By correctly using the various features of e-textbooks and choosing the right questions. For example, prompting the students to solve the same question each time using a different tool might foster the creation of novel solution. I also think that dragging a figure and observing the changes that happen to it might prompt argumentation processes.

Faisal referred to e-textbook features as being potential tools for prompting CR and AR. He argued that construction tools may play a central role in prompting CR. Focusing students' attention on a specific tool each time may lead to multiple solutions for the same question. If this happens, it is likely to create novel solutions for the reasoners. Indeed, the novelty of the solution is one aspect of creative reasoning. In this transcript, Faisal did not mention the other two elements of creative reasoning.

Examining the other two dimensions, namely, argumentation and e-textbook features, Faisal considered the dragging tools as a device for boosting argumentation processes. He believes that dragging tools could foster planning and implementation activities during the process of solving problems. If so, engaging students in these two activities may boost their verificative argumentation. Furthermore, he took into account the questions' formulation. He divided the questions in the e-textbook into questions that prompt CR and AR and those that do not.

Indications of AR and CR in the prepared scenario

Although in the first two sessions, Faisal acknowledged the contribution of e-textbook affordances to boosting AR and CR, he placed greater emphasis on the types of question formulations when preparing scenarios for a lesson. He completely ignored the role of e-textbook features in prompting CR and AR.

Faisal: A scenario should be prepared in a way that includes questions that stimulate students' thinking and allow discussion in the classroom, which could encourage CR and AR. The process of choosing such questions is a difficult process.

Osama: I see that you have ignored e-textbook features.

Faisal: Yes, maybe, we need to explore the various possibilities and tools that the e-textbook provides, and then select the tool that could help in answering the question.

This transcript illustrates Faisal's methods of fostering AR and CR types. Faisal referred to types of question formulations in e-textbooks as an important resource for boosting AR and CR. Even after Osama's question, Faisal did not refer to a specific e-textbook feature and how this feature was reflected in the AR and CR. Rather he referred to general e-textbook features – various possibilities and tools – that could boost AR and CR.

Playing the scenario in front of the teachers

When Faisal played the scenario in front of his colleagues, he shifted flexibly between the three dimensions. He afforded special attention to e-textbook features in an attempt to boost AR and CR.

Faisal: Try to solve the question now in an unusual way. You can use the tools and resources that are available to this end.

Colleague 1: I think the question could be solved by drawing a circle.

Faisal: That's a very good idea. However, please tell me why you chose to use this specific tool? And why do you think that this is an appropriate method for answering the question? And finally, why do you think that you reached the right solution using this method?

Faisal drew his colleagues' attention to the constructing tools available in the e-textbook (first dimension) by requesting them to answer the question in an unusual way using those tools. His request to answer the question in an unusual way suggests that he was trying to boost CR. Furthermore, his request to explain the reasons for choosing the tools to answer the question suggests his shift toward boosting teachers' argumentation processes.

CONCLUDING REMARKS

This paper reports on the initial part of an ongoing study that aims at examining the ways teacher use e-textbooks to prompt argumentation and creative reasoning. The results show that the PDC had some effect in guiding teachers to become aware of e-textbook affordances and in using them in a way that may develop creative reasoning and argumentation processes. Indeed, the analysis presented in this paper related to only one teacher, but we found similar professional growth among other teachers who had participated in this study.

Planning, writing and playing lesson scenarios and criticizing them was found to be a useful strategy for educating teachers to use e-textbooks in ways that may prompt creative reasoning and foster argumentation. Engaging teachers in planning and preparing lesson scenarios allows them to reflect on the theoretical knowledge shared in the course, to practice this theoretical knowledge virtually in advance, and to be well prepared for applying the lesson in a real-life setting. Indeed, we do not yet have any data from real-life settings when teachers actually teach the scenarios in their classrooms but based on the lessons they taught in the course, it is reasonable to assume that the teachers will refer to the scenarios they had prepared.

To better understand how the professional development course concerning e-textbooks will translate into classroom practice, a new research study is needed. This study should focus on the interactions between teachers, students and e-textbooks. Understanding these interactions may shed light on the ways teachers use e-textbook affordances to nurture creative reasoning and boost argumentation processes.

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ICT standards for teachers: Toward a frame defining mathematics teachers' digital knowledge

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A literature review on teacher education toward the use of technology in mathematics classes we have conducted recently points out a disappointment with the outcomes of most of the teacher education initiatives. A gap between the teachers' needs and the teacher education programs appears as the main reason. This consideration led us to search for international and national standards specifying knowledge and skills mathematics teachers need for an efficient use of technology. Most of the standards that we analyzed are neither subject matter, nor school level specific. In this contribution, we propose a framework that we have elaborated aiming at defining this knowledge and skills. We report the process of elaboration of this framework, which stands for its rationale.

Keywords: ICT standards; Mathematics teachers' digital knowledge; Mathematical knowledge for teaching; Double instrumental genesis

INTRODUCTION

Whether to use or not information and communication technology (ICT) in education is not an issue anymore. Nowadays, the question rather shifted to how to use technology in classrooms more efficiently and take the best profit of its affordances. Teacher education appears as one of the key elements of response to this question. Indeed, in a literature review on "*barriers to the uptake of ICT by teachers*", Jones (2004) highlights that "*there is a great deal of literature evidence to suggest that effective training is crucial if teachers are to implement ICT effectively in their teaching*" (p. 8). This clearly addresses the issue of pre-service teacher education (TE) and (in-service) teacher professional development (TPD) toward the use of digital technology.

A number of TE-TPD initiatives are discussed in the literature. Hegedus et al. (2017) pointed out that in a number of cases, disappointment with the outcomes of these initiatives are reported. The gap between teachers' needs and the TE-TPD contents was identified as one of the main reasons (Emprin, 2010). This calls attention to the necessity for teacher educators to better understand what teachers need to know in order to use ICT effectively, thus raising the issue of ICT competency standards.

In our previous work (Hegedus et al., 2017), we started searching for an institutional framework regarding teachers' knowledge for teaching mathematics with technology. We were surprised about how few such standards for mathematics teachers or even for teachers in general, at the national and international levels we found. Most of them are neither subject matter nor school level specific. Therefore, we recommended that "*[e]laboration of ICT standards for mathematics teacher education might become one of the goals of the mathematics education international community*" (ibid, p. 30). We continued exploring other documents, which helped us to deepen our understanding of technology-specific knowledge mathematics teachers need to develop in order to use efficiently ICT in their classes. In this paper, we report about our journey through various national and international educational contexts, aiming at proposing a framework enabling to define such knowledge. Each stage of the journey draws on specific theoretical frameworks that we briefly expose, before outlining main outcomes.

OUR JOURNEY TOWARDS A FRAMEWORK OF DIGITAL COMPETENCY FOR TEACHING MATHEMATICS WITH TECHNOLOGY

Stage 1: Technological pedagogical content knowledge (TPACK) and double instrumental genesis

At the beginning of our search for existing ICT standards for teachers, quite naturally we chose the TPACK model (Mishra and Koehler, 2006) that

> attempts to identify the nature of knowledge required by teachers for technology integration in their teaching, while addressing the complex, multifaceted and situated nature of teacher knowledge (Koehler, 2012).

The main reason for this choice was the fact that the framework was specifically conceived for addressing technology integration. Indeed, in theorizing about the unique knowledge needed for teaching with digital technology, Mishra and Koehler (2006) introduced the concept of TPCK (or TPACK): the knowledge teachers need to meaningfully integrate technology into instruction in specific content areas. The authors suggested an additional body of knowledge to the PCK model suggested by Shulman (1986), namely *technological knowledge* (TK), which partially overlaps content knowledge (CK) and pedagogical knowledge (PK). Figure 1 depicts the resulting teachers' knowledge, which includes seven knowledge bodies.

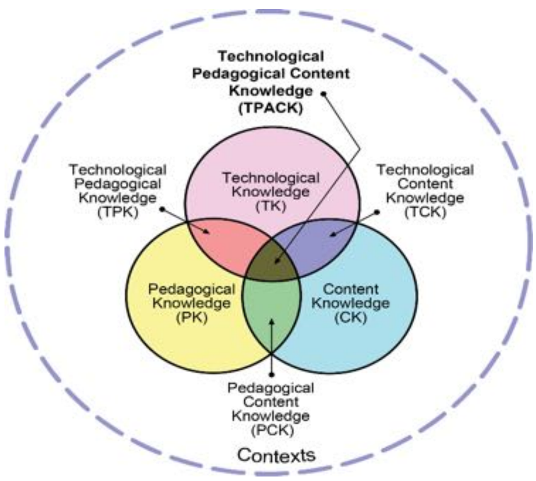

Figure 1: TPACK framework (TPACK.org)

The TPACK framework offers a theoretical lens that enables researchers to analyze teachers' professional knowledge at a general level. It is used by many researchers and several different interpretations are currently accepted (Voogt et al., 2012): T(PCK) as extended PCK; TPCK as a unique and distinct body of knowledge; and TP(A)CK as the interplay between three domains of knowledge and their intersections.

Besides the teacher knowledge, we needed a theoretical frame enabling us to capture teachers' skills required for technology enactment in classroom. The construct of *double instrumental genesis* (Haspekian, 2011) provides such a lens. Developed in line with the *instrumental approach* (Rabardel, 2002), it encompasses both personal and professional instrumental geneses in teachers while using ICT. Whereas the *personal instrumental genesis* is related to the development of a teacher's personal instrument for a mathematical activity from a given artefact, the *professional instrumental genesis* yields a professional instrument for the teacher's didactical activity. These two processes mobilize knowledge of the artefact and the abilities to solve problems using it in the personal genesis, and the abilities to teach mathematics with ICT and orchestrate ICT-supported learning situations in the professional one.

An analysis of existing national and international standards with these theoretical lenses (Hegedus et al., 2017) allowed to highlight that while in some documents, the technological knowledge (TK) tends to be over-emphasized (e.g., Israel), other suppose that teachers enter the profession with a basic mastery of digital technology and their personal instrumental genesis is thus taken as a starting point on which the professional genesis can build (e.g., ISTE standards for teachers (ISTE, 2008)). Most of the standards we analyzed are not subject matter specific, thus although all categories of TPACK knowledge are addressed, their description remains at a very general level, as we can see in the following excerpt from the UNESCO ICT competency framework for teachers (UNESCO, 2011, p. 21): "Incorporate appropriate ICT activities into lesson plans so as to support students' acquisition of school subject matter knowledge".

While carrying out the analyses of the standards, we felt a need for a frame that would allow us to address specifically mathematics teachers' ICT knowledge. Moreover, we realized that the theoretical framework we chose does not allow capturing but the cognitive aspects. Like Lynch et al. (2009), we believe that one cannot study teachers' work and practices by addressing the cognitive component alone. This consideration led us to the choice of the pedagogical technology knowledge framework.

Stage 2: Pedagogical technology knowledge and mathematical knowledge for teaching

Thomas and Hong (2005) use the term *Pedagogical Technology Knowledge* (PTK) which is "knowing how to teach mathematics with the technology" (p. 256). This was further developed by Thomas and Palmer (2014) to interweave a number of teacher intrinsic factors to produce PTK, including: teachers' instrumental genesis; mathematical knowledge for teaching (Ball et al., 2008); teacher orientations and goals (Schoenfeld, 2011), especially beliefs about the value of technology and the nature of learning mathematics, and other affective aspects, such as confidence in using technology (see Figure 2).

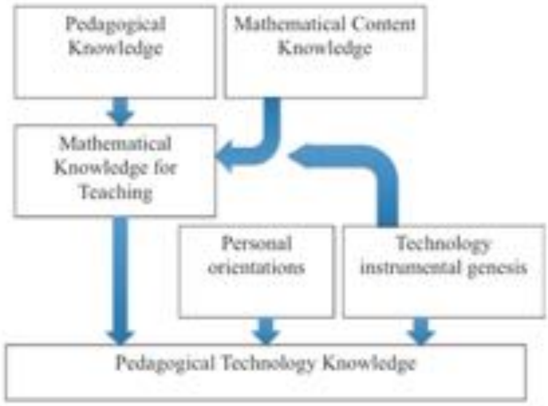

Figure 2: A model for the framework for PTK (Thomas and Palmer, 2014)

As we are focusing on mathematics teachers, we felt a need to refine the *mathematical knowledge for teaching* (MKT) component. Ball et al. (2008) suggest several categories of this knowledge that refine both content (CK) and pedagogical content knowledge (PCK), in reference to Shulman's PCK. Content knowledge consists of common, horizon and specialized content knowledge, while pedagogical content knowledge comprises knowledge of content and students, knowledge of content and teaching and knowledge of content and curriculum. However, since MKT does not address technology, we found it necessary to adapt four out of its six knowledge areas to technology. In the modified framework, which we call *Mathematics Digital Knowledge for Teaching* (MDKT – Fig. 3), these four knowledge areas are:

- Specialized Digital Content Knowledge (SDCK) of the teachers with respect to the mathematics to be taught;
- Knowledge of Digital Content and Students (KDCS), which includes additional aspects in a technological environment;
- Knowledge of Digital Content and Teaching (KDCR) that in a technological environment may vary due to digital resources;
- Knowledge of Digital Content and Curriculum (KDCC) e.g., knowledge of prescribed use of ICT.

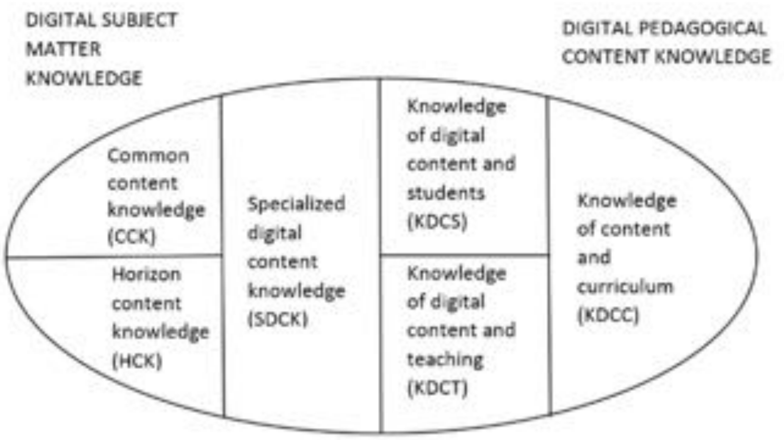

Figure 3. Mathematical Digital Knowledge for Teaching, adapted from Ball et al.'s Mathematical Knowledge for Teaching to technological environment

SDCK is closely linked with the personal instrumental genesis, that of the teacher for her/himself, while the other three are linked to the professional instrumental genesis, that of supporting students' instrumental geneses and mathematics learning with digital technology.

Analyzing institutional documents related to ICT standards at the international level (UNESCO, ISTE[1]), and the national level (US, NCTM[2], P21.org[3], Australia, Israel and France) with this new theoretical lens led us to three main conclusions.

Professional instrumental genesis emphasized but the personal one neglected

Most of the standards highlight only knowledge and skills related to the ICT use for teaching. In our view, ignoring the need for teachers' personal instrumental genesis is a mistake for two reasons: (1) the current teacher population varies considerably in their technological skills. Hence developing their personal use of digital mathematical software is a prerequisite for developing professional genesis for using these tools as instruments for mathematics teaching, and (2) digital tools are rapidly evolving, hence teachers' personal genesis in relation with new tools needs to be taken into consideration.

Neglecting teachers' personal orientation toward ICT

Teachers' personal orientation towards integrating technology is not considered in all documents. The UNESCO framework and the Australian documents do not mention any affective aspects. This is surprising, yet in line with Thomas and Palmer's claim:

> We believe that this latter aspect of teacher orientations and their effect on confidence in using technology has been given less attention in research and development than it deserves (2014, p. 76).

Two international documents, elaborated by P21 organization and by Neiss et al. (2009) do relate to orientation. P21 explicitly relates to the importance of teachers' positive attitudes in each of its three domains of practice: pedagogy, content and technology. At the national level, the French ICT standards[4] also refer to positive orientation as a necessary aspect. We claim that the importance of personal orientation is perhaps under-estimated as a major factor in teachers' competencies.

Categories of knowledge

ICT standards that we analyzed relate to different components of the MDKT frame. While only the UNESCO (2011) framework relates to the four knowledge domains, the other documents relate to knowledge of content and teaching in a digital environment (KDCT) and to knowledge of digital content and curriculum (KDCC). The first of the two is understandable, as teaching competencies and skills are at the heart of these frames. Likewise, the content is a major focus of the frames.

Stage 3: Toward a definition of mathematics teachers' digital competencies

As mentioned at the outset, our aim in this paper is to propose a framework that will enable us to define and to study knowledge needed for mathematics teachers who implement ICT in their practice. We detailed above two stages of our journey that started with general cognitive frame (TPACK), refined by a specific mathematical frame (MDKT) adapted to technology. The instrumental genesis frame was also present in our conceptualization along the cognitive one. Finally, the necessity of an affective lens led us to follow Thomas and Palmer (2014) and adopt personal orientation as a third lens.

We feel that networking (Prediger, Bikner-Ahsbahs and Arzarello, 2008) these three frames is necessary to yield digital competencies for teaching mathematics with technology framework (Figure 4).

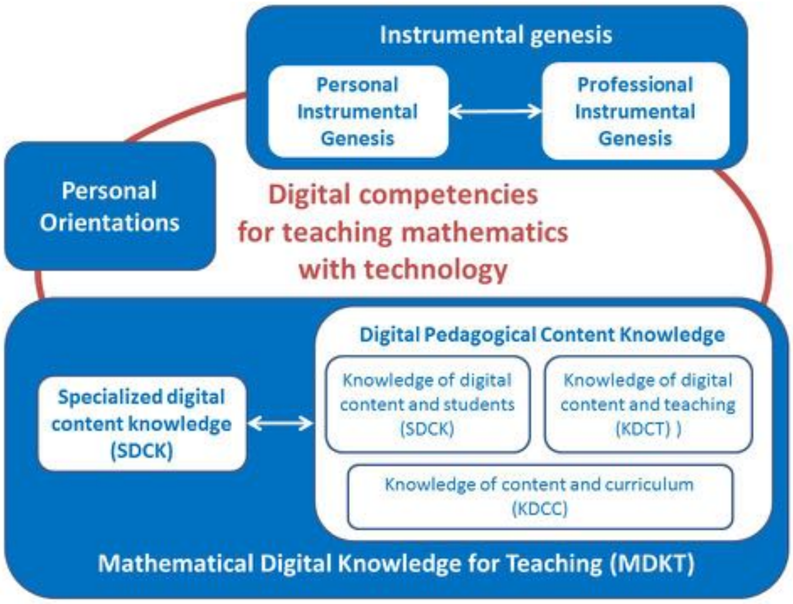

Figure 4. Digital competencies for teaching mathematics with technology

CONCLUDING COMMENTS

While reading through and analyzing institutional documents, the issue of vocabulary should be raised. Specifically, there is a need to define three major terms, namely knowledge, skills and competencies. Our analysis showed that there are diverse ways of relating to these three terms, some of them inconsistent, as in the following example cross-referencing the term competency: “The framework consists of seven competence areas, which contain descriptions of knowledge, skills and competence” (Kelentrić et al., 2017, p. 3). It seems that both skill and competency have to do with acting in a technological environment. Yet skill seems to be “less” than a competence, which seems to be acknowledged also by OECD:

> A competency is more than just knowledge or skills. It involves the ability to meet complex demands, by drawing on and mobilising psychosocial resources (including skills and attitudes) in a particular context. (OECD, 2003, p. 4)

Knowledge is a basis for both skill and competency, but these connections are not specified. We would like to call for clearer definitions of the basic terms in each frame.

NOTES

1. International Society for Technology in Education, https://www.iste.org/.

2. National Council of Teachers of Mathematics, https://www.nctm.org/.

3. The Partnership for 21st Century Learning, http://www.p21.org.

4. Compétences numériques (Digital skills), https://c2i.enseignementsup-recherche.gouv.fr/enseignant/quelles-competences-pour-le-c2i2e

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Integrating Mathematics Teaching with Digital Resources: Where to Begin?

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In a graduate course focused on technology for teaching mathematics, eleven teachers of mathematics planned a pair of mathematics lessons to include technology integrations. In one lesson, teachers began with existing mathematics curriculum and planned for digital instructional materials to enhance the lesson; and, in the second lesson, they chose digital mathematical resources and planned for what mathematics content could be taught using those resources. Analysis of teachers' reflections and lesson plans indicate commonalities between the curriculum-first and technology-first lessons, but more teachers preferred curriculum-first planning and more often focused those lessons on developing students' procedural fluency. Implications for mathematics teacher education and professional development are discussed.

Keywords: mathematics teacher education, digital instructional materials, mathematics lesson planning, technology integration, mathematics curriculum.

INTRODUCTION

Technology has the potential to transform the teaching and learning of mathematics. Historically, this potential has been demonstrated with regards to technologies such as calculators (e.g., Reys & Smith, 1994), computer programming (e.g., Clements & Meredith, 1993), dynamic geometry (e.g., Laborde, 2002), and computer algebra systems (e.g., Ruthven, 2002). In the digital age, teachers and students often have access to all of these types of resources and many more via connected, mobile devices. The widespread availability of technology resources introduces new challenges for teachers of mathematics, namely the selection, evaluation, and implementation of worthwhile resources among a myriad of options.

THEORETICAL PERSPECTIVE AND BACKGROUND

This study is theoretically grounded in the documentational approach (Gueudet & Trouche, 2009) and the interaction of teacher knowledge with documentational work (Gueudet, Bueno-Ravel, & Poisand, 2014; Clark-Wilson, 2010). Technology integration in mathematics teaching occurs within complex curricular systems that require coordination among curricular objectives, instructional materials, teachers' intentions, and enactment of mathematics curriculum in the classroom (Remillard & Heck, 2014). As technology resources such as digital instructional materials (DIMs) (Thomas & Edson, 2018) become increasingly available, teachers face new opportunities and challenges to select instructional materials and plan for their enactment in the classroom. Gueudet and Trouche (2009) describe the selection, planning, and enactment of resources as a process of *documentational genesis* and argue that "this documentation work is at the core of teachers' professional activity and professional development" (p. 199).

Documentational genesis also relates to interactions between documentational work and teacher knowledge (Gueudet et al., 2014). In this study, teacher knowledge was conceptualized through the lens of the Technological, Pedagogical, and Content Knowledge (TPACK) framework (Mishra & Koehler, 2006). In the documentational approach, it is theorized that teachers' use of resources such as DIMs impacts their knowledge (TPACK) (instrumentation); teachers' knowledge (TPACK) impacts their use of DIMs (instrumentalization) (Gueudet et al., 2013).

Technology integration can be a tool for innovation in teaching and learning, yet teachers' effective use of classroom technologies is complex and many barriers exist (e.g., Groff & Mouza, 2008; Zhao, Pugh, Sheldon, & Byers, 2002). Approaches that emphasize instructional planning with technology may impact teaching and TPACK more effectively than technocratic efforts that foreground technology (Harris, Mishra, & Kohler, 2009). Mathematics teacher education and professional development must support teachers in selecting and planning for enactment of technologies that can support effective mathematics teaching practices and student learning of rich mathematics content (Edson & Thomas, 2016; NCTM, 2014). That is, rather than integrating technology for technology's sake, technology integration must serve to advance mathematics teaching and learning. This distinction is particularly important in contexts where school administrators and policy-makers view educational technology as a change agent and teachers are expected to use technology in their teaching within and across disciplines.

METHODS

This empirical study took place in a graduate course for teachers that sought to simultaneously develop teachers' TPACK and enhance their planning for effective use of DIMs in the classroom. The positive impact of the course's approach on teachers' TPACK has been reported elsewhere (e.g., Thomas, Edson, & Abebe, 2018). The purpose of this study was to examine teachers' planning for technology integration in mathematics lessons, particularly focusing on how teachers integrated digital instructional materials (DIMs) within their existing mathematics curriculum. Here, we focus on the following questions: (1) To what extent do teacher-identified technology integration levels, mathematics teaching practices, and orchestration types differ between technology-integrated math lessons that begin with curriculum materials (*curriculum-first lesson planning)* and those that begin with selecting technology resources (*technology-first lesson planning)*? and, (2) What preferences do teachers express regarding *curriculum-first* and *technology-first* lesson plans?

Participants included 11 practicing US teachers who enrolled in a summer graduate-level course focused on technology for teaching mathematics. Nine taught elementary-grades mathematics, and two taught middle-grades mathematics. Participants' teaching experience ranged from two to twelve years; all were female.

During the course, teachers surveyed, discussed, and evaluated a variety of DIMs including mathematics-focused tools (e.g., virtual manipulatives, math-focused websites, mathematics games and applets, dynamic geometry environments) and general tools with possible applications in mathematics classrooms (e.g., screencasting and interactive whiteboards). Course readings and discussions familiarized teachers with the Replacement–Amplification–Transformation (RAT) framework (Hughes, Thomas & Scharber, 2006), effective mathematics teaching practices (NCTM, 2014), and technology orchestration types (Drijvers et al., 2010), providing foundational knowledge for their final project.

At the end of the course, teachers developed two lesson plans and reflected on them. We refer to the first technology-integrated mathematics lesson plan as *curriculum-first* (teachers identified a mathematics lesson in their existing curriculum and enhance it through technology integration), and the second as *technology-first* (teachers identified one or more DIMs with the potential to enhance mathematics teaching/learning and fit the technology into a lesson from their curriculum). Teachers explicitly identified three aspects of their lessons: technology integration levels, effective mathematics teaching practices, and orchestration types. Technology integration levels referred to the RAT framework (Hughes et al., 2006). Teachers considered these levels in relation to eight research-based, effective mathematics teaching practices (e.g., Amplifying the Posing of purposeful questions) (NCTM, 2014; Thomas & Edson, 2017). Teachers also identified the types of orchestration types they intended to implement in the lessons, ranging from teacher-centered orchestrations (*technical-demo, explain-the-screen, link-screen-board*) to student-centered orchestrations (*discuss-the-screen, spot-and-show, Sherpa-at-work)* (Drijvers, Doorman, Boon, Reed, & Gravemeijer, 2010). Teachers' reflections also compared their experiences with *curriculum-first* and *technology-first* planning.

Data sources for this study included teachers' lesson plans and reflections, submitted for the course. Qualitative methods were employed to analyse these written artefacts. Data were selectively coded for technology integrations, mathematical teaching practices, and orchestration types. Teachers' reflections about comparing *curriculum-first* and *technology-first* planning were openly coded, followed by axial coding resulting in four themes: ease of planning, challenges of technology-first planning, concerns about fidelity of implementation, and preferences for types of planning.

FINDINGS

Technology-integration Levels, Teaching Practices, and Orchestration Types

Findings from selective coding are summarized in Table 1. Teachers planned for 21 opportunities to replace, amplify, or transform instruction with technology in curriculum-first lessons, versus 14 in technology-first lessons. Curriculum-first lessons included a larger proportion of replacement integrations, whereas technology-first lesson plans planned for 85.7% of integrations at the amplification (35.7%) or transformation levels (50%). Teachers identified fewer mathematics teaching practices in the technology-first lesson plans, compared to curriculum-first lesson plans (33 and 37, respectively). Notably, teachers more frequently identified *Build procedural frequency from conceptual understanding* in curriculum-first plans, and *Implement tasks that promote reasoning and sense making* in technology-first plans. Consistent with technology integration levels and teaching practices, teachers more frequently identified orchestration types in curriculum-first lesson plans compared with technology-first lesson plans. The proportion of teacher-centered versus student-centered orchestration types was consistent across the two types of lesson plans.

Table 1: Number of teacher-identified integration types, teaching practices, and orchestration types in lesson plans

	Curriculum-first Lesson Plan	Technology-first Lesson Plan
Technology Integrations (Hughes, Thomas, & Scharber, 2006)		
Replacement	6	2
Amplification	9	5
Transformation	6	7
Not articulated	1	3
Effective Mathematics Teaching Practices (NCTM, 2014)		
Establish mathematics goals to focus learning	1	1
Implement tasks that promote reasoning and problem solving	1	5
Use and connect mathematical representations	6	7
Facilitate meaningful mathematical discourse	9	7
Pose purposeful questions	4	4
Build procedural fluency from conceptual understanding	8	2
Support productive struggle in learning mathematics	3	4
Elicit and use evidence of student thinking	5	3
Technology Orchestration Types (Drijvers et al., 2010)		
Technical-demo	6	3
Explain-the-screen	3	6
Link-screen-board	5	3

Discuss-the-screen	5	7
Spot-and-show	7	4
Sherpa-at-work	6	5

Teachers' Preferences

Only one of the eleven teachers identified a preference for technology-first lesson planning. Five preferred curriculum-first planning, and five indicated no preference. Analysis of reflections revealed that teachers tended to find curriculum-first planning easier and articulated more difficulty with technology-first planning. The teacher who preferred technology-first planning explained, "I felt like it was much more 'exciting' to think of the technology first, and then see how I could fit it into a variety of math lessons." This teacher also expressed concerns about fidelity to the curriculum, a concern also shared by teachers who preferred curriculum-first lesson planning. Table 2 summarizes the themes from teachers' reflections.

Table 2: Teacher preferences regarding curriculum-first and technology-first planning

Theme	*Excerpts from teacher reflections*
Ease of planning	*In my opinion, I think the better way to integrate technology into lessons is to first look at the curriculum and where certain technologies would fit.* *I think that curriculum first technology integration was easier than the technology first integration. I can look at any lesson and pick out technology that may improve student learning but fitting a piece of technology into the curriculum was harder when I had to find a lesson to put with it.* *For me, it was easier to approach the planning for technology with the curriculum-first because I was able to use the lesson provided and adapt the technology I will use to fit the needs of my students.* *When I compare and contrast the two lessons planned seems to be less to take on than the technology first lesson. With the curriculum first lesson I already have all the materials and will be seamlessly working in the technology to help enhance the lesson…I can conclude that it will be easier to plan and carry out a curriculum first lesson than a technology first lesson.*
Challenges of technology-first integration	*Unless the technology is made for the lesson, it is hard to integrate it. However, just having a neat piece of technology doesn't help if I have nowhere to put it.* *A conclusion I have come to is that it does take a little front-end planning to use technology in a meaningful way; it's not something you can just slap into a lesson and hope it works.* *With the technology first lesson I am completely changing the format of the lesson…There will be a lot more planning and preparation for the technology first lesson.*

Concerns about fidelity of curricular implementation	*I feel like if I consider the curriculum first and try to find technology to fit within it, I might be losing fidelity to the curriculum in order to integrate technology.* *It may be easier to use the technology-first planning approach if the curriculum was more lax in its' pacing or requirements.* *I am, of course, keeping the curriculum the same and still using the teacher's manual lesson, however, I am restructuring it to better fit my students' needs.*
Lack of preference	*Overall, I think both lessons used technology to further student learning, but a different beginning viewpoint was used to reach the end goals of the two lessons.* *What matters is HOW the technology is used. I don't think it makes a difference if you're sitting there looking at a lesson and decide to try to find a DIM to enhance a lesson, or if you find an intriguing DIM and try to find a lesson that it would work well in. Either way, you need to evaluate the tool and use it in a way that improves your instruction and students' learning.* *Overall, I think the teacher absolutely must be intentional and take time to plan the lesson using any technology. Technology can be fun and "frilly" and wow the students but not impact the learning.*

DISCUSSION

This study reports on two ways in which teachers of mathematics approached technology-integration in their documentational work during a graduate course: (a) curriculum-first technology-integration planning and (b) technology-first technology-integration lesson planning. In designing and developing activities for mathematics teacher learning, attending to technology integration is increasingly important (Niess, 2005) and often operationalized in two ways: technology-focused and content-embedded professional development (Lawless & Pelligrino, 2007). The findings reported in this paper provide some evidence to suggest that a content-embedded approach to professional development may be well-received by teachers. Because documentational work and teacher knowledge interact during documentational genesis (Gueudet et al., 2014, Clark-Wilson, 2010), developing impactful technology integration in mathematics teaching must attend to both. Our related research has documented growth in TPACK of teachers as they engaged in documentational work during this study (e.g., Thomas et al, 2018). Further empirical research is needed to examine instrumentalization–in this case, the impact of TPACK on teachers' documentational work. Findings in our study raise new questions with respect to teachers' approaches to integrating technology. What teacher knowledge domains in TPACK do teachers draw upon with respect to their preferred approaches for integration technology? How do preferred approaches to technology integration in lesson planning translate during classroom practice? Limitations of this study include a limited sample and the reliance on teachers' own interpretations of technology integration levels, effective teaching practices, and orchestration types. Although teachers in this study had opportunities to develop understanding of these aspects of consideration, further analysis is needed to examine the extent to which teachers' lesson plans fully supported these aspects of teaching and technology integration. Ongoing work related to this project focuses on other aspects of their documentational work, extending from the study of lesson planning to also examine enactment of technology-integrated mathematics lessons. This will provide further opportunity to examine the extent to which teachers' intentions during the lesson planning phase are realized during the classroom enactment of the lesson.

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Introducing teachers to a technology-supported flipped mathematics classroom teaching approach

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In recent years, Mathematics education research has been focusing on technology-enhanced forms of learning and modern, student-centred education methods. An example of this is the Flipped Classroom Approach (FCA). The project presented here aims to develop a framework for Mathematics teacher training for these forms of learning in Austria. In a first step, expert interviews were conducted and evaluated according to a Grounded Theory Approach. What was especially crucial for the interview data were the factors such as being active and collaborating with other teachers, as well as using and applying concrete examples of the new teaching method. Consequently, the temporal structure and the methodology of teacher training in Austria should be modified.

Keywords: Mathematics education, Flipped Classroom, Inverted Classroom, Technology-enhanced learning environment, Teacher Training.

INTRODUCTION

For successful Mathematics learning in our digital era, it is important to ensure that modern technologies and student-centred methods are used where specifically needed. The Flipped Classroom Approach (FCA) is a possible way to accomplish this synthesis, because on the one hand technologies form the framework of this educational approach and on the other hand it is problem-based and hands-on learning in the classroom, which is the core of a Flipped Classroom. In this paper, the terms new and modern technologies include electronic media for communication, provision of learning materials and presentation of learning products (e.g. Mahara). The terms also include software products for visualization, investigation and solution of mathematical problems (e.g. GeoGebra). In order to benefit from the use of the FCA, well-trained teachers are needed who both have a sound knowledge of educational technologies and can offer a mix of teaching approaches. Appropriate training supports teachers to utilise the interplay of communication technologies and education technologies that the FCA has to offer. It also helps teachers to provide student-centred forms of education to meet students' needs. In Austria, as of the school year 2018/19, the implementation of the content of *Digitale Grundbildung* [1] in lower secondary education will be mandatory, which in turn requires such training of teachers.

Methods of qualitative social research were chosen to develop a framework for Mathematics teacher training so that teachers become able to experiment with a technology-enhanced FCA in real school situations. This was done to facilitate the acquisition of new knowledge in the field of Mathematics teacher training for modern forms of teaching supported by technologies like the FCA. Both interviews with experts and in-service teachers as well as design experiments with physical and digital learning environments were chosen to glean information for the research project. By using various survey instruments (e.g. interviews, observations, analysis of lesson reports) and multiple methods (Grounded Theory Approach, Action Research, Design-Based Research) an attempt was made to facilitate triangulation and thus increase the quality of the results. This paper presents the initial findings of the first expert interviews, needed to develop the research project at hand further.

THE FLIPPED APPROACH IN MATHEMATICS CLASSROOMS

Although learning and teaching using the FCA has been becoming increasingly popular, there is still no clear definition and description of this type of education. The largest common divider of the definitions is that in a Flipped Class, the acquisition of basic knowledge and achievement of lower cognitive goals should happen before the lesson, so that the time in the classroom can be used for student-centred activities and the pursuit of higher cognitive-level goals (Galway et al., 2014; Wasserman et al., 2015). Technologies play a central role in Flipped Classroom Education. At the beginning of the modern Flipped Classroom era, technologies were mainly used for exporting direct instruction from the classroom (Esperanza et al., 2016; Fischer & Spannagel, 2012). However, in a more advanced modern Flipped Classroom, this is only a necessary condition. Thus, the purposeful use of technologies in the classrooms is much more important, nevertheless technologies must never become an end in themselves, they must always serve the learning process (Galway et al., 2014; Tague & Czocher, 2016).

TEACHER TRAINING FOR FLIPPED MATHEMATICS EDUCATION

Teaching Mathematics is a complex task that requires continuous and specific training, especially if Mathematics lessons deal with real-world problems and are supported by new technologies (Gainsburg, 2008; White, 2002). Also, the last TALIS study (OECD, 2014) indicates that in particular when it comes to the use of technologies in education, teachers identify a great need to develop their skills. In the category *ICT skills for teaching*, 19% of teachers feel that there is a high demand for professional development and in the category of *New technologies in the workplace*, 18% of teachers surveyed stated this to be so. Similar results are to be found in relation to teaching according to the FCA. The potential of this education strategy can only be exploited to the full by well-trained and skilled teachers. They utilizing an FCA have to be able to take on different roles and need meaningful content, didactic-pedagogical as well as technological knowledge and competences that need to be continuously brushed up (Herreid & Schiller, 2013; Nagel, 2013).

The learning process in professional teacher development takes place in the educational triangle, which is formed by the subject matter of teacher training, the workplace of the teachers, and teachers' attitudes and beliefs (Maxwell, 2010). Since teaching according to the FCA has an impact on all three corners of this educational triangle, the teaching and learning process in teacher training for the FCA must also be adapted.

By modifying the subject matter, the settings in the classroom, and the role of the teacher in education according to the FCA, teachers may already feel uncomfortable during professional development or the initial application of the new approach. Therefore, it is recommended that teacher training aiming at a fundamental changes in education should be carried out in small steps (Breckwoldt et al., 2014; Chapman & An, 2017). According to Kuntze (2006), a slow and supported approach to teacher training is also recommended because the first applications of new methods and/or technologies in the classroom usually lead to a reduction in the quality of education. Only when teachers have attained a certain level of competence with the new form of education, an increase in the quality of teaching and learning will become apparent. Therefore, teachers need special support to master this vulnerable phase of the first application of a new teaching approach and for the new approach to be sustainably implemented in the classroom.

A general change in the educational paradigm at the beginning of the 21st century has occurred, namely that the focus of education has shifted from the teaching process to the learning process – at least at the academic level (Midoro, 2001). This change particularly concerns the FCA. According to Selter et al. (2015), in-service teacher training is required above all, so that such a change can soon also be recognised in the classrooms.

METHODS

First semi-structured interviews were conducted to approach the research objective in the winter semester 2017/18. Experts on this topic include teacher trainers, who offer courses for the FCA and/or technology-enhanced Mathematics education. Eight experts were interviewed, six men and two women. Two experts work at universities, three experts work at university colleges of teacher education, one expert is a secondary school and university college teacher, and two experts are secondary school and teacher training teachers. Regarding expertise in teacher training, the following classification of the experts can be made: Two experts have experience with FCA teacher training, four have experience with professional teacher development for technologies in Mathematics education and two experts have experience in both areas. The interview guide focuses on the experts' opinions on the following topics: 1) teachers' motivation for professional development participation, 2) teachers' needs and wishes in teacher trainings, 3) specifics of further education courses that focuses on educational technologies, and 4) viable structures in further education courses on the use of technologies and modern didactics. The interviews were conducted in German. The quotations given in Preliminary Results and Categories (see below) were translated by the first author. The average duration of the interviews was approximately 30 minutes and the data were evaluated based on to the Grounded Theory Approach according to Glaser and Strauss (1999) and Charmaz (2006).

The open coding of the eight interviews led to 134 codes and sub-codes. By comparing the definitions and text excerpts of the transcripts of the individual codes, these could be reduced to 91 codes. These 91 codes were then re-applied to the transcripts of the eight interviews. After this, groups of codes were formed and raised to a higher level of abstraction by a general description of these groups. This resulted in a reduction of the codes to 22, 13 of which are considered central. The following preliminary results and categories have been derived from these 13 central codes.

PRELIMINARY RESULTS AND CATEGORIES

Learn the same way you want your students to learn afterwards. Once they have got an insight into the role of learners, teachers can gain confidence using new approach and/or technologies. They will realize by learning themselves according to a technology-enhanced FCA, that this form of education makes a learning process possible. Furthermore, teachers also recognize the strengths and weaknesses of the approach and technologies so that they can develop various solution strategies. Learning in accordance with the FCA can also reduce teachers' fear of the new approach, which proves to be a major obstacle in the introduction of new technologies and modern education strategies. This focus on technologies and the new method in teacher training was particularly evident among professional development teachers, who also teach at secondary level:

> Expert J: They [learning teachers] learn in teacher training according to the Flipped Classroom principle. They have to watch videos and complete work assignments in advance which will then be discussed in the course.

> Expert K: You have to set activities so that people really have to apply it [FCA] themselves. Thus, they first learn in a flipped manner so that they themselves can then also offer a real flipped lesson.

The following expert working exclusively in pre- and in-service teacher training had an even more general opinion on this aspect:

> Expert E: […] as the teachers learn, so do they teach. This applies to one's own learning at school and university, and even more so to teacher training.

Statements such as the above could be used to conclude that a person's general learning history has a strong impact on their actions and attitudes as a teacher.

Spreading time for teacher training to different periods. This is closely related to the first topic. Teaching according to the FCA can only be learned by a temporal allocation of active training, application in school and feedback units. Above all, the criticism of the current time structure of teacher training in Austria was clearly apparent among most experts.

> Expert E: I call our afternoon training courses oral vaccinations. Although, they are not even oral vaccinations. If you are trying to expand your knowledge of something, you should spend a semester on it. However, if you learn something entirely new, this will take even longer.

This statement makes it blatantly clear how low the trust of some professional development teachers is in the temporal structure of teacher training in Austria.

Added value and advantages of the FCA and technologies: Teachers are to be made aware of the advantages of teaching according to a technology-enhanced FCA. Highlighting the advantages of a technology-enhanced FCA can increase the motivation to actually use this type of education. Thus, learning yourself using the new method can be used for this purpose and best practice examples can be presented to peer teachers. Other teachers, who are teaching according to the FCA, can share their experiences and successes with the new method (via video, online, or personally in the course).

Attention to heterogeneity of teachers: When it comes to new teaching approaches and new educational technologies, different teachers have particularly divergent backgrounds. Many professional development teachers complain about this especially when it comes to technologies:

Expert M: Usually everything is always there – from A to Z. Some colleagues have problems using a mouse or right-clicking and others are more familiar with certain programs than I am […] most colleagues do not dare to say that – that makes it very difficult.

For the head of a teacher training this means that a wide range of learning opportunities must be made available to teachers. This should prevent teachers from overstraining or getting bored which both may lead to demotivation. All experts had developed their own strategies for this, but the principle "from easy to challenging" (see Expert K) can be recognized more often.

Expert K: At the beginning I start with very simple exercises and then it gets more and more challenging […] because when technologies are involved, the teachers' skills are extremely diverse – and you should not demotivate people right at the beginning [pause] so that they say "I will never manage to learn that"

Examples and applicability of the FCA and technologies: In most cases, teachers expect to be able to integrate the content of teacher training quickly into their lessons. This expectation is a decisive factor in determining whether teachers decide to participate in a teacher education programme or not – i. e. one must already take these aspects into account when advertising the teacher education programme. In teacher training, examples provided are to be further developed and then also applied in practice. However, teacher training must not degenerate into a mere passing on of lesson plans.

Collaboration: Collaboration should take place both among the teachers and between the teacher trainer and teachers. The aim is to jointly construct knowledge. This collaboration should continue after the training unit. This should lead to an extension of the learning process and internalising of contents as well as identifying new technologies and the new teaching-learning approach. To enable this, collaboration must be an integral part of the teacher education unit:

Expert F: […] introduce teachers from different schools who have similar problems or needs and motivate them to work together in the course. This makes it easier for them to solve the problems. […] they stay in touch from time to time after the course […]

Ensuring teachers' learning performance. At the end of the training course, teachers should have a product that reflects their learning process.

Expert E: In the end, the participants must have a product [...] something concrete that they can then also use in class.

Four experts recommend that the learning performance be ensured quickly and that a learning product be created by the learning teachers, especially if the technologies are the focus of the training:

Expert A: […] the technology is introduced and then people must work with it right away. They must fulfil work orders […] it is important that they quickly hold their own products in their hands.

This learning product can consist of their own examples and plans as well as a script or other documents from the head of the teacher education. This should also facilitate the teachers' application of the new technologies and approach in their own lessons.

DISCUSSION AND CONCLUSIONS

Learning Mathematics by means of a technology-enhanced FCA, means changing learning from a mere repetition of content to becoming an application of competencies and an extension of possibilities to tackle real-world issues. Modern technologies and a sensible and appropriate use of them are an integral part of contemporary and future education. Teachers must get prepared to provide a learning environment that supports and reinforces these key competences, so that the aforementioned approach to education can be offered to many students in the near future. Educational technologies and modern didactics that should therefore be at the heart of teacher training.

The evaluation of the first expert interviews demonstrate that it is important for teachers to be active in professional development courses and to get better at working in groups. This importance of collaboration in teacher training has also been demonstrated in Mathematics (Fried & Amit, 2005), online Mathematics asynchronous teacher training (Silverman & Clay, 2009) and in other subjects (Coenders & Terlouw, 2015). But expanding existing research, collaboration should not only be a joint discussion, but above all a joint task – in the development classroom as well as asynchronously and synchronously online. It can be concluded from this that teacher activities are central in teacher training. Teachers' actions are accompanied by the creation of learning products. On the one hand, the learning products can be made available as raw products and thus trigger a learning process. On the other hand, completely new learning products may be created in professional development courses. In both cases, the learning process should end with a specific product. Amanatidis (2014) has already been able to investigate this, but an interplay between teacher training and school is necessary to ensure that the learning product is of high quality and that the added value of the newly learned – in this project a technology-supported

FCA – as compared to classical teacher-centred Mathematics teaching is easily recognizable. However, this interplay requires a modified temporal structure of teacher training. The learning time is to be broken down. This means that there will be a shift away from the one-off afternoon teacher training courses which are currently the standard for professional teacher development in Austria. Rather, there should be at least tripartite training courses so that teachers can experiment with the new skills in practice and have opportunities for support. Also, the time extension of teacher training should lead to an increase in collaboration among the participants in teacher training.

NOTES

1. Ministry guidelines and curriculum for *Digitale Grundbildung* (digital basic education) in Austria: https://bildung.bmbwf.gv.at/schulen/schule40/dgb/index.html (24.02.2018)

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Posters

Design based approach to ICT enabled proportional reasoning module: An analysis

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WHY PROPORTIONAL REASONING AND WHY ICT?

Proportional reasoning (PR) is a critical concept in school mathematics curriculum arguably for its wide applicability in many other domains of school mathematics and science curriculum as an underlying foundational concept. Therefore, strengthening of PR is seen as instrumental for furthering into concepts such as percentage, profit and loss, simple and compound interest, discount, work and time, speed, distance and time, linear equations, mensuration, trigonometry, concept of similarity, coordinate geometry, probability and so on, as well as other topic areas namely, torque in physics volumetric computations in chemistry, etc. However, there appears a gap in most middle school and high school curricula across the globe, in the way the concept of PR is developed (Lamon, 2007). Development of PR among students requires them to follow a smooth trajectory from additive to multiplicative thinking but, often in textbooks, there is a jump from additive to multiplicative reasoning. It remains unclear to students as to how different mathematical concepts (listed above) use PR differently (different cases of multiplicative reasoning). For instance, although both additive and multiplicative reasoning together build PR, students often struggle to think multiplicatively about the given task situations. This shift from additive to multiplicative reasoning remains a roadblock for students (Rahaman, Subramaniam & Chandrasekharan, 2012). Affordances created by technology are widely acknowledged and used, but in what ways do these digital avatars of the content address specific pedagogical issues needs a continuous monitoring (Drijvers, 2015)? The approach of this study, a blend of both digital tool and discursive practice led students to see and justify their answers while they kept moving between additive and multiplicative thinking. In particular, this poster will highlight design principles and analysis of digital resources and tasks used in developing a blended module on PR aimed at enabling students to identify and understand multiplicative relationships in contexts involving comparisons, sharing, and scaling, leading to conceptual applications both within and across subject domains.

Connected Learning Initiative (CLIx)

CLIx is a collaborative initiative between MIT (Cambridge, USA), TISS (Mumbai, India) and Tata Trusts to harness technology at a scale for constructivist learning by integrating ICT into design and development of innovative pedagogy. The goal is to get students to think, solve problems, explore, and collaborate with co-learners and to involve teachers in developing pedagogy for active learning. This poster discusses the design trajectory and analyses the approach to ICT enabled CLIx PR module.

Design trajectory

The main design principles for the CLIx module on PR is to develop a learning trajectory for students to move from additive to multiplicative thinking. A design based approach is adopted in its four Units, for design is seen as an effective player for its major role in addressing pedagogical gaps (Drijvers, 2015). The first Unit moves from additive to multiplicative thinking using the sharing notion through a digital story of a ninth grader. The emphasis is on understanding that the number of food packets and people has to increase in the same ratio and decisions about increasing or decreasing a quantity is based on the understanding of mathematical relationships. An affordance is created to experience constant difference strategy in contrast to scaling through a digital prototype experience. Students figure out whether the distributions between the two groups is equal or not and redistribute to make distribution equal giving reasons for their answer. The second Unit strengthens multiplicative thinking by moving from discrete to continuous quantities. Digital prototypes on pattern scaling and hands-on activities derived from Duckworth's (2012) similar task on Coffee and Milk using pellet counting and GeoGebra based image stretching and compressing tasks lead to developing thinking keeping proportions in mind. The next Unit on ratio and proportion sequels the previous image contraction task and introduces ratio notation and connections with regular usage in textbooks. At this juncture, the notion of scale used in maps and calculating distances is introduced using a hand-on activity which is then led to a discussion on inverse and direct variations using a digital game of ice-cubes dropped in lemonade affecting its volume by varying ice-sizes and a number of them used. The module is appended by the last Unit on applications which presents proportions in linear equations and probability, and discusses compound ratio through worksheets. This poster will report how this module was used in schools with episodes from the field and how those fed into the content and tools revision as a design based research.

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Features of 'Authentic' Programming-based Mathematical Tasks

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Policy makers, teachers, and scholars have expressed widespread interest in the integration of programming into school curricula, or more broadly "computational thinking", now considered a 21st century skill. In this poster we analyse a sequence of programming-based mathematics tasks found in an undergraduate course series taught since 2001. We identify 4 features of 'authentic' programming-based tasks, i.e., of tasks that aim at empowering students to engage in programming-based mathematical work 'as mathematicians would do'.

Keywords: programming, task design, computational thinking.

INTRODUCTION

There is a resurgence of interest in integrating programming—more broadly, computational thinking (CT)—in education (e.g. in the UK: Benton et al., 2016), which many argue reflects the number of scientific fields that have developed a computational counterpart (Weintrop et al., 2016), and the rise of a 21st century skill and need for proficiency in computational practices. Our interest is in CT curriculum development and task design that would equip students with skills and competencies to address this need, particularly in relation to programming for math learning. Weintrop et al. (2016) argue for 'authentic' computational tasks in math and science classrooms and provide a taxonomy of computational practices that students would use in such tasks.

This poster discusses design features of tasks, implemented annually for more than 15 years, in a sequence of three undergraduate programming-based mathematics courses named *Mathematics Integrated with Computers and Applications* (MICA) offered at Brock University (Canada). Based on a case study examining a student's learning experience through her 14 project tasks during the sequence of the three MICA courses (Buteau et al., 2016), we argue that such courses develop students' proficiency in CT engagement for mathematics. Furthermore, these 14 tasks (accounting for 70-80% of students' final grades) afford students to engage in 'authentic' computational practices for mathematics (Broley et al., 2017).

CONCEPTUAL FRAMEWORK & METHODOLOGY

The conceptual framework draws from various interrelated concepts reflected in the literature on CT, CT in mathematics, and CT in mathematics education (Buteau et al., 2018). Wing (2014) defines CT as "the thought processes involved in formulating a problem and expressing its solution(s) in such a way that a computer —human or machine—can effectively carry out" (p. 5), whereby computer programming is an underlying activity. In the field of mathematics education, CT has a 45-year legacy that started with LOGO programming language (Papert, 1980). In our work we view students' learning of mathematics through CT-based mathematical activities with Lave and Wenger's (1991) concept of "legitimate peripheral participation," which describes how learners enter into a community of practice and gradually take up its practices. The focus of this proposal is on the features of 'authentic' tasks in which students (newcomers) engage peripherally in CT for mathematics practices - as mathematicians (elders) would do. We use affordances of CT for mathematics learning (Gadanidis et al., 2017) to guide our analysis by exploring the relevance of such affordances in a mathematician's work, and then examine the 14 programming-based mathematics tasks from the MICA courses used in Buteau et al.'s (2016) study in order to identify common task features as 'authentic' CT-based mathematics tasks.

FOUR 'AUTHENTIC' PROGRAMMING-BASED MATH TASK FEATURES

The resulting 4 task features identified in our analysis are: i) involves **mathematics that cannot be done by hand**; ii) involves **(dynamic) visualization**; iii) **should lead to conjecturing/exploring of unknown mathematics** (to the student) or to interpreting mathematics applications; and iv) should be **meaningful to students**.

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Vetting digital resources that support teachers' promotion of mathematical habits of mind during instruction and assessments

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In this poster, four U.S. secondary school mathematics teachers and one university mathematics teacher educator present a rubric designed to evaluate the usefulness and relevance of digital resources in helping teachers to develop practical conceptions of mathematical habits of mind as described by the Common Core's Standards for Mathematical Practice. Descriptions of the rubric's development, its component parts, and an example of its use are expounded in the poster.

Keywords: mathematical habits of mind, Standards for Mathematical Practice, mathematical processes.

As U.S. schools and teachers focus on improved and sustained implementation of college and career ready K-12 mathematics standards, the number of digital resources designed to guide, and support enactment continues to expand. Even with such increased support, many teachers realize that faithful implementation and assessment of the Common Core State Standards for Mathematics (CCSSM) (NGA Center & CCSSO, 2010) or aligned standards can be formidable endeavours. Along with changes in mathematics content standards and their progressions, such standards place increased emphasis on powerful mathematical habits of mind—ways of thinking, reasoning, modelling, and communicating. The poster presents research deriving from attempts to address the research questions: How do mathematics teachers determine which digital resources (1) have the greatest potential to provide their students with opportunities to engage in and exhibit engagement in the mathematical practices? (2) have the greatest potential to develop their own conceptions of the mathematical practices? Four U.S. grades 9-12 mathematics teachers and one university mathematics teacher educator (research team) explored these questions, resulting in the 'Mathematical Practices Digital Resources Rubric', a tool designed to help teachers promote mathematical habits of mind, as described by the Common Core's Standards for Mathematical Practice, during instruction and assessments.

FRAMEWORK

A focus on the Standards of Mathematical Practice was three-fold: (1) CCSSM are a set of research-based learning standards (e.g., Mullis, Martin, & Foy, 2007; Schmidt, Houang, & Cogan, 2002), (2) the mathematical practices rest on longstanding and well-researched processes (NCTM, 2000) and proficiencies (National Research Council, 2001), and (3) CCSSM impact over 41 million students. Although this number comprises only 2.5% of the world's school age children (ages 5-18), the work presented here has the potential to influence other frameworks that concern mathematical habits of mind (e.g., Cuoco, Goldenberg, & Mark, 1996; Leikin, 2007).

METHODS

The rubric's development involved several stages, including: (1) generating a list of existing digital resources related to CCSSM or aligned standards; (2) division of sub-lists of resources into 'Keep', 'Discard', and 'Maybe' categories by team members; (3) exchange of resources from respective 'Maybe' categories to those that should be transferred to 'Keep' by team members; and (4) determining where each resource fit in the team's situational categorization-scheme.

RESULTS

The resulting rubric is illustrated in Figure 1 below. An accompanying scoring scheme (not shown), will be provided on the poster with an example for how the rubric can be used to evaluate digital resources.

Usefulness and Applications of Online Resource	Rating (1, 2, 3, or 4) 1 - not satisfying / lacking a component 4 - satisfying / fulfilling all aspects of a component
Resource:	
Website (URL):	
Grade Level(s) Targeted:	
Technical Aspects of Resource (rate each component: 1, 2, 3, or 4)	Sub-total:
1. Ease of access and navigation	
2. Regular updates	
Support from Resource (rate each component: 1, 2, 3, or 4)	Sub-total (×2):
3. Operationalization of Mathematical Practices	
4. Student engagement	
5. Teacher engagement	
Resource Usefulness (higher sums, or total, indicate higher levels of effectiveness)	Total:
Resource Application(s) (rate each component: 1, 2, 3, or 4)	
6. Can be used as a mathematical practice general resource	
7. Can be used for curriculum development	
8. Can be used for lesson or unit planning and implementation	
9. Can be used for conferences with parents, guardians, or administrators	
Situational Use(s):	

Figure 1: Mathematical Practices Digital Resources Rubric

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A new defragmenting teaching format for teacher education using mathematical maps

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A new defragmenting teaching format for teacher education was developed and first tested at the University of Passau in the winter semester 2017. The aim of the teaching format is to show the interrelation of geometry learned at school and geometry taught at university. Therefore we used mathematical maps which show interrelation between topics in the horizontal dimension and the development of a subject matter in time in the vertical dimension. We also used an e-learning platform to support the defragmentation.

Keywords: Defragmentation, teacher education, mathematical maps, e-learning, geometry.

TRANSITION AND FRAGMENTATION

Tall (2008) presented a theoretical model which he called the three worlds of mathematics to describe the transition problem from school to university in mathematics. This model helps to account for some difficulties of many students' transition to university in mathematics, resulting in missing common threads and not knowing connections. Consequently, one of the ideas of our visualization project is the attempt of following "known" conceptual-embodied and proceptual-symbolic truths – where possible through development in time – into the axiomatic-formal world in order to see their genetic connection, the desired and meaningful so-called "golden thread".

The concept of "mathematical maps" was introduced by Brandl (2008) as a didactical tool in the form of a virtual tree or net, which shows interrelation between topics (horizontal dimension) as well as the development of a subject matter – starting from an initial problem – in time (vertical dimension). This concept offers several opportunities to foster joined-up thinking and will allow the student to follow the development of an initial problem in time. For example, the visualization in three dimensions allows for an ideal transparency of the interdependencies or the connection of single nodes which additionally offer contents from other platforms by link (Brandl, 2008, pp. 106–109).

TEACHING FORMAT FOR GEOMETRY

The new teaching format which covers geometry was first tested in the winter semester 2017/18 in mathematics teacher education for higher secondary schools. We discussed the development of different aspects of geometry – like for example the axiomatic construction – over time. We also compared the view of school mathematics and university mathematics on different topics of geometry. Here we used the interactive mathematical map for geometry, which is still being improved. So far we have a map that shows the interrelation between topics (figure 1) and another map that shows the development of different topics of geometry over time (figure 2)

To enhance the defragmentation process, we used e-learning – particularly blended learning – in connection with these interactive mathematical maps. The nodes of the mathematical map are linked to external content using the e-learning platform ILIAS. There we have created courses covering the most important aspects of geometry which we used for most of the sessions of the teaching format. For detailed information about the ILIAS course see Datzmann (2017).

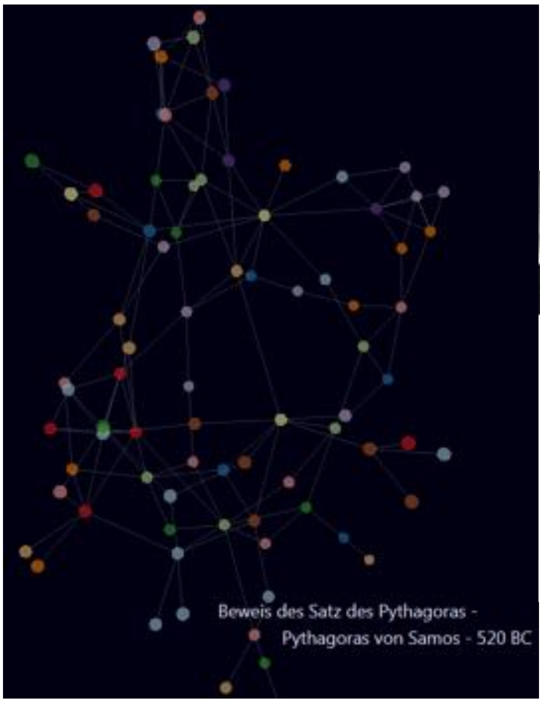

Figure 1: Interrelation between topics

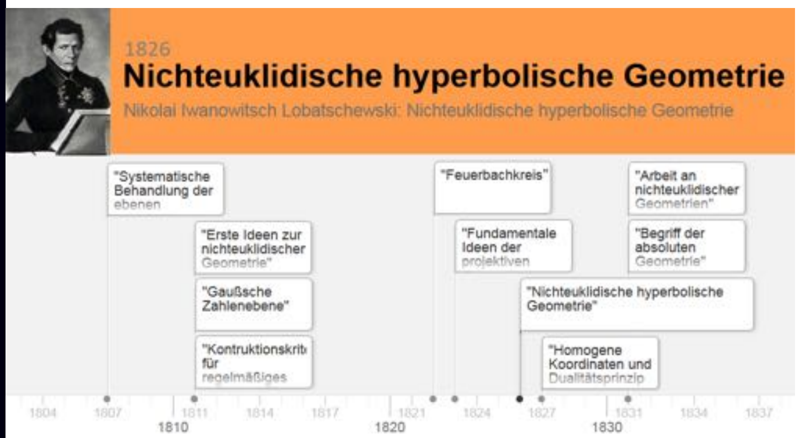

Figure 2: Development of topic over time

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Online platforms: new connections and new possibilities for teacher documentation work?

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In this poster we present a national database recently developed in France. We analyse it in terms of connectivity, distinguishing between macro-level connectivity (potential connections with resources outside of the database) and micro-level connectivity (potential connections inside the database), for the subject of functions. This is linked with new possibilities for teacher documentation work.

Keywords: Connectivity, Documentation work, Online resources

INTRODUCTION AND CONTEXT

The study presented here is situated in the context of the implementation in France of a new curriculum for lower secondary school (grades 6 to 9, the new curriculum started in September 2016). The education ministry financed the development, by private editors, of an online "digital resources for school database" (DRSD[3]), which opened in 2017. Our aim here is to study the potential of this database, in terms of teacher design, with the perspective of the documentational approach.

THEORETICAL FRAMEWORK AND RESEARCH QUESTION

We refer here to the documentational approach to didactics (Gueudet & Trouche 2009). This approach invites to view teachers as designers of their teaching: they search for resources, associate them, modify them. Along this work (called *documentation work*) they develop *resources systems:* structured sets of resources. Digital resources in particular open new possibilities for the design and sharing of resources by teachers; we have in previous works proposed the concept of *connectivity* (Gueudet, Pepin, Restrepo, Sabra & Trouche 2018) to study the potential of e-textbooks, in a documentational approach perspective.

We distinguish between *macro-level connectivity*, which "refers to the potential of linking to and between subjects/users and resources/tools outside the textbook" (ibid. p.545), and *micro-level connectivity*, which are made inside the e-textbook for a given mathematical content. We consider that the DRSD is an e-textbook and the research questions we study are:

What it the connectivity of the DRSD, at macro and micro level? Which possibilities does it open for teachers' documentation work?

[4] In French, « Banque de ressources numériques pour l'école » (BRNE), http://www.barem-hatier.fr/

SHORT PRESENTATION OF THE DATABASE

In the poster we will shortly describe the database content and structure, using appropriate figures. The DRSD offers many different kinds of resources: single "bricks" like ordinary texts in pdf, interactive exercises or videos; complete scenarios for lessons; bricks already associated in a "path". It also offers a space for building lessons with the DRSD resources and other resources, and a space for sharing with colleagues. Through the DRSD, the teachers can offer content to their students and can follow their work.

ANALYSIS OF THE DATABASE CONNECTIVITY

Macro-level connectivity

The macro-level connectivity of the DRSD is very significant, in terms of possibility to integrate resources from another origin, and of sharing resources with other colleagues and with students. Moreover all the resources offered can be freely downloaded by the users. So the macro-level connectivity is important. Nevertheless, there is no possibility to contact the authors in order to suggest modifications. The resources from the DRSD will not evolve – the only evolving part is the space of resources shared by teachers.

Micro-level connectivity, the case of functions

Functions is a subject for which many connections can be made (Akkoç & Tall, 2005), in particular between representations (tables, graphs etc.) or with other subjects (economy, physics etc.). The DRSD offers 253 different resources attached to the keyword "Functions". The connection between representations is present in several resources, and especially emphasized in the "interactive mind map". The connection with other domains is also present in many resources. Nevertheless, there is no clear "learning trajectory": organized connection between concepts.

The database offers many possibilities for new forms of teachers' documentation work. In a further research we will investigate its actual use.

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Research Based Design of Mathematics Teaching with Dynamic Geometry

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The poster describes a project investigating the potentials of dynamic geometry environments (DGE) in teaching that aims at developing students' mathematical reasoning competency. The ambition is to produce knowledge regarding guidelines for design of teaching, which utilises the potentials of DGE to support students′ development of reasoning competency. Furthermore, the current usage of DGE in Danish lower secondary school is investigated. Theoretically, the project is rooted in instrumental genesis and design-based research.

Keywords: Dynamic geometry environment, reasoning competency, design-based research.

Research on DGE in mathematics teaching has shown potentials for supporting the development of students' mathematical reasoning (e.g. Leung, 2015). These are welcomed potentials since students' inadequate reasoning abilities is a widespread problem, both in Denmark and internationally (e.g. Hoyles & Healy, 2007). Substantial resources have been used in Denmark to boost usage and accessibility of information and computer technology (ICT), which has led to DGE being an apparent part of mathematics teaching, particularly in primary and lower secondary school. However, students' accessibility to ICT such as DGE, does not guarantee a greater learning outcome (OECD, 2015). The manner in which the technology is used, and the way students appropriate it, is essential. Therefore, more knowledge is needed on how DGE teaching can be designed so that the potentials may be utilised. The project seeks to contribute in this regard by investigating and formulating guidelines for design of teaching with DGE, which can support the development of students′ reasoning competency (Niss & Jensen, 2011). This includes focus on task design, on the role of the teacher and on students′ interaction with DGE (e.g. Leung & Baccaglini-Frank, 2017; Leung, 2015).

In order to reach the projects ambition a tripartite process is undertaken: 1. Mapping of DGE potentials specific to reasoning competency by reviewing research on the topic and using the results to formulate a-priori guidelines. 2. Investigating, by means of a survey, how widespread the usage of DGE is in Danish lower secondary school and, in particular, how it is used. 3. Using the guidelines to design a DGE teaching lesson to be implemented in lower secondary school. Design-based research methodology is used in the teaching lesson design process (e.g. Gravemeijer et al., 2000) and two design iterations are to be carried out with the ambition of refining the particular design as well as the a-priori guidelines. The framework of mathematical

competencies (Niss & Jensen, 2011), especially the description of reasoning competency, plays an important role in the project, as it frames the mathematical goal of the DGE usage, and therefore acts as an optic in the review process, in the design of the guidelines and in the analysis of the DGE teaching lesson. Additionally, the focus on students' appropriation of DGE prompts a complementary theoretical approach embedded in the instrumental approach (Guin & Trouche, 1999). The novelty of the project lies in its aim to contribute to the research on DGE in the context of reasoning competency, and to suggest how DGE can be used in practice when the aim is to support students´ development of reasoning competency.

At the time of the conference, results from the mapping of DGE potentials and initial a-priori guidelines will be presented. In addition, an example of task design, the intended student interaction with the task and the role of the teacher will be shown using the guidelines.

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Adventures on the beach

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The mathematical basis of the software is the lesser known uniform metric definition of conic sections: given are a point (F) and a line (d) and a positive real number (e), the locus of those points P in their plane for which. dist(P,F)=e × dist(P,d) is a hyperbola if e < 1; a parabola if e = 1; an ellipse if e > 1.

Keywords: Mathematical software, Conic sections, Experimental mathematics.

The oceanwaves software had been developed as a result of the cooperation of the authors: a creative programmer and a high school teacher of mathematics. It can be downloaded from cie.web.elte.hu/oceanwaves.

The software implements this definition in the illuminating context of the "beach model": the respective curves are formed as the interference of two wave-fronts: a longitudinal and a circular one. With the help of additional, user friendly options this simplest initial setup can be altered to inspire the interested student and the competent teacher to devise their own laboratory, put forward their own questions or simply share their discoveries. This is particularly so in the *Advanced Arrangement* (see below).

Apart from the pure attraction of visualizing this definition in the most natural way the high expertise and creativity of the first author had also been of paramount importance in the creation of the software. Last but not least, the initiative is deeply embedded in the rich Hungarian math teaching tradition.

THE FEATURES OF THE SOFTWARE OCEANWAVES

There are basically two settings which can also be used in a mixed manner.

i) *Simple setting* A single longitudinal wave is emitted along the horizontal "seashore" and, simultaneously, a pointlike source -- the ship -- emits a single shot of circular wave on the sea. Depending on the ratio of the wave-fronts' respective velocities the interference of the moving wave-fronts is one of the conic sections above. The simpler component of the software displays this phenomenon while the user can experiment by changing the value of *e*. It is also possible to "zoom out" to see the evolving curves on a larger scale. Additional scenarios: the timing of the start of any one of the wave-fronts can be delayed with respect to the other one and additional wave-fronts can be launched manually.

Challenge questions: (1) How does this simple model correspond to the classical definition of conic sections as planar slices of a circular cone? (2) If $e \neq 1$ then the arising interference curves possess a horizontal mirror symmetry. (Their vertical mirror symmetry is obvious.) Give a synthetic reason for this phenomenon. (3) What kind of curves may evolve in this model? Devise your own experiments, prepare and share the corresponding lab-reports.

ii) *Advanced setting* Switch off the beach and disembark to the open sea! Here the number of ships is arbitrary and each one's timing and their respective velocities can be set individually. Playing around you can observe rich fauna: one can generate the familiar circles of Apollony, ellipses again without directrices, exotic curves and their families, and even the shining radiation of the Big bang.

Challenge question: Characterize the curves you find "out the open sea."

The software has been developed with the Processing toolkit: a programming language and environment made for teaching programming through creating interactive visual and artistic programs [https://processing.org/overview/]. It eliminates many of the entry barriers for beginners, and serves as a productive tool even for experienced programmers.

In the future, we plan to make a web-based version, and a mobile/tablet version. We have plans to extend the software with new features, including time reversal of the animation, and a spherical mode, where the setting is on a spherical planet covered with water.

The software had been successfully tested in the math-club of the Lauder Javne High School of Budapest where the second author is an ordinary teacher. Apart from enthusiastic personal reactions from college teachers and professional friend mathematicians in Hungary and abroad we have no experimental evidence of its educational value. It certainly cannot be inserted in curricula and it may well remain as a kind of leisure pastime for the happy few. However, motivated students in math clubs with no particular background can just "play around" with it. The software is also recommended to advanced-math classes, to extend their knowledge and improve their visual imagination. Last but not least, ambitious teachers may find it challenging to test it in their courses and the authors would be glad to get feedback, positive as well as negative.

Finally, the project as a whole can be considered as a partial and limited answer to the challenge: what are the perspectives of mathematical education in the digital age.

Touchy Feely Vectors: exploring how embodied interactions based on new computational media can help learn complex math concepts

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Students find abstract concepts such as vectors difficult to learn. New computational media is increasingly being used to address this problem. Such media allow novel interactive and collaborative ways of learning and teaching, but these features are under-explored. We present a design (Touchy-Feely Vectors, TFV) that allows students to learn vectors through embodied interactions, and discuss how the design extends existing textbook representations, to develop an interactive and collaborative approach to learning and teaching mathematics. The design allows vectors to be constituted differently in students' minds, compared to current mental models based on individual interactions with print representations.

Keywords: abstract concepts, vectors, technology, embodied interactions

Students face difficulties in understanding and applying vectors, and they usually lack a geometric understanding of vectors. They memorise formulae, or rote learn procedures and algorithms (Aguirre & Erickson, 1984; Flores et al., 2004; Knight, 1995), and falter with directions, finding *ijk* components (algebraic) easier to use. These difficulties could be partly due to the static nature of vector representations in textbooks, which are the dominant media used for teaching-learning vectors around the world. Digital media can help extend traditional print media, and thus overcome its static constraints, to allow new kinds of classroom interactions and learning. We outline the design principles of two versions of a system to learn vectors, illustrating a possible approach to develop digital interventions that smoothly extend textbooks around the world. TFV-1 (bit.ly/tfv-1), a mouse-based version, allows creating and manipulating vectors as geometric entities linked with algebraic representations in real time. It supports addition using triangle law, and resolution (rectangular components). Touch-based TFV-2 (bit.ly/tfv-2) supports the same conceptual features, along with the parallelogram law of addition.

KEY DESIGN FEATURES

Dynamicity: Math education researchers (Balacheff & Kaput, 1996) appreciate computational media's ability to make static representations dynamic, particularly for complex concepts like vectors (Donevska-Todorova, 2018). Both TFV versions support manipulation of vectors, linking geometric and algebraic modes. Short animations unpack underlying mechanisms. For instance, for resolution, sides (as arrows) of right triangle move to the axes, as equations change to rcosθ rsinθ form.

Integration: The scaffold of a circle around the vector (in both versions) was found very effective in integrating both concepts (geometry of triangles and circle, and trigonometry) and learner interactions (especially when manipulating the vector).

Embodiment: Sensorimotor interactions play a key role in math learning (Martin & Schwartz, 2005; Rahaman et al., 2017). Designs like Touch-Counts (Sinclair & de Freitas, 2014), GM (Ottmar et al., 2015), MIT-P (Abrahamson & Sánchez-García, 2016) extend this finding, and help study how body-media interactions support math learning. TFV-2 improved the mouse-based interactions in TFV1 to conceptually consistent embodied touch-gestures. For instance, double-consecutive taps (with 2 fingers) for creating, adding and reversing resolution of vectors; pinch-away for resolution, and addition with rectangular components were used.

Enactive textbooks: To scaffold current textbook based teaching-learning practices, we connected TFV to textbook figures using QR-codes (bit.ly/tfv-qr). The codes linked to TFV tasks, designed in collaboration with teachers.

TFV builds on existing systems (like CAS, DGE) which make advanced math concepts accessible, but focuses on embodied interactions, conceptual integration, and a smooth extension of textbooks. Our studies show that these features together make vectors easier for learners to understand, and allow teachers to extend their classroom practices, towards enactive and collaborative learning (bit.ly/tfv-cl).

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Improving an online diagnostic test via item analysis

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This poster reports on a project to apply the theoretical framework of psychometric testing to evaluate the performance of an online diagnostic test in mathematics, and to design and implement consequent improvements. The steps of the evaluation methodology are described, with a view to the techniques being more routinely used in the re-design of digital tasks. A key finding is that for the diagnostic test, an exploratory factor analysis gave results consistent with an entirely separate classification according to an established taxonomy.

Keywords: diagnostic testing, item response theory, MATH taxonomy.

INTRODUCTION

Diagnostic tests are widely used in university mathematics (Lawson, 2003; Heck & van Gastel, 2006) to inform staff about the level of students' basic mathematical skills, and to inform individual students of gaps in the level of assumed mathematical knowledge. The mathematics diagnostic test used by the University of Edinburgh is completed online by around 1000 students each year via the STACK computer-aided assessment system (Sangwin, 2013). This poster will describe the methods used to evaluate the test performance, and to design and implement improvements based on the findings. The evaluation methodology includes the use of psychometric item analysis (Lord, 2008) alongside the domain-specific MATH assessment taxonomy (Smith *et al*., 1996). The novel structured application of these different theories provides a model for the cyclic improvement of mathematics assessments more generally; the approach is particularly applicable to online assessments which gather much of the relevant data automatically.

METHODOLOGY AND RESULTS

The poster details the four key steps in the evaluation methodology, illustrated with the findings for the diagnostic test.

Mathematical content: the MATH taxonomy

The MATH taxonomy was designed to help exam authors to construct exams which test a balanced range of mathematical skills. The poster will describe and illustrate the taxonomy, and present the classification of the items on the test which shows that it is predominantly based on "routine procedures".

Factor analysis

Factor analysis is a statistical technique which essentially seeks to break the test up into groups of questions where the scores in each group correlate well but the correlation between groups is lower. This is driven by student response data, but the aim is to find groupings which can be given a meaningful description, e.g. "algebra" versus “calculus”. For the diagnostic test, two dominant factors were identified. Strikingly, the questions which had been classified as MATH Group B were loaded heavily onto Factor 1, while the Group A questions were loaded onto Factor 2.

Item response theory

A psychometric item analysis was carried out, providing a model of how students of various abilities would perform on each question. This gives a visual representation which shows the spread of question difficulties, and also how well each question discriminates between the most and least able students; how to interpret these properties from the visual representation will be explained on the poster. For the diagnostic test, three of the least discriminating items (all MATH Group A) were identified as candidates to be replaced.

Predictive validity

Students’ test scores were compared with their performance in various subsequent mathematics courses. Since the test scores are used in practice to help inform course choice, it is important to understand this relationship. The moderate correlations of the test with most subsequent courses will be detailed graphically on the poster.

REVISING THE TEST

Based on the findings, three questions on the test were replaced (guided by the MATH taxonomy, three Group A questions replaced by two Group B and one Group C question). The revised version of the test will be re-evaluated during summer 2018 and the poster will display preliminary results from this analysis.

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First year engineering students' selection and use of resources to learn mathematics

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In this study we have investigated engineering students' selection and use of resources in their first year university mathematics courses (Calculus; Linear Algebra). We report on the results of a survey in a technical university in the Netherlands, in which students indicated the most important resources they used for their learning of mathematics. Preliminary results indicate that traditional resources (textbook, teacher explanations) were important at secondary school and at university, and that selected digital resources and peer conversation gained importance at university. Moreover, the resources differed in terms of importance for the mathematics courses.

Keywords: Student use of resources, Calculus and Linear Algebra, Survey.

First year university students have access to a large variety of resources to learn mathematics. These include digital (curriculum) resources (e.g. interactive practice materials, YouTube videos); traditional curriculum materials (e.g. textbooks); and human resources (e.g. communication and cooperation with peers). This study aims at understanding the resources students choose to learn mathematics, and for which purpose they use them (e.g. initial content learning; problem solving; seeking help). Thus, it may contribute to the discussion on the design and intended use of resources for technology-mediated mathematics learning at the start of tertiary education. We focus on resources used by first year engineering students at a technical university in the Netherlands. The research question is: which kinds of resources are selected, and for what purpose, by engineering students to learn mathematics, in secondary school and in university Calculus (CS) and Linear Algebra (LA) courses?

THEORETICAL FRAMEWORK: THE LENS OF RESOURCES

We assume that student learning of mathematics is influenced by their use of resources. Following Pepin and Gueudet (2014) we distinguish between (1) human resources, and (2) (digital) material resources, which we further classify as (a) curriculum resources (developed by the teacher, used by students, and aligned with a particular course), and (b) general resources (e.g. web resources identified by students). Particular resources have been designed to allow visualization or manipulation of digital objects, and in that way facilitate subject specific modes of learning, such as proving and generalizing in LA (Donevska-Todorova, 2017), or understanding the concept of continuity in CS (Bressoud et al., 2016). Human resources refer to human interactions, for example conversations with peers or tutors. Selected studies on university students' use of resources focus on a limited range of resources (Inglis et al., 2011), or on second year engineering students (Anastasakis et al., 2017). Anastasakis et al. (2017) found the most popular resources to be curriculum resources and students' notes; and their use was motivated by obtaining high exam grades.

METHOD

A 27 item questionnaire was designed, partly amended from the Transmath research programme in the UK, (see www.transmath.org), and administered in the fall of 2017 to first year engineering students in the Bachelor College of a technical university in the Netherlands (15 Bachelor programmes; over 2000 first year students; N=430). The students had followed a first semester CS course, differentiated at three levels depending on the students' majors and preferences. Students in the Applied Mathematics/Computer Sciences programmes had also followed an LA course. The questionnaire was administered on paper to two CS groups and one LA group (N=278), and electronically to the remaining CS students (N=152). Here, we report on the parts of the questionnaire regarding the selection and use of resources.

PRELIMINARY RESULTS

Preliminary results indicate that traditional resources (textbook, teacher explanations) are important at both secondary school and university. However, at university, general digital resources, peer conversations, and discussions with university tutors gained importance. The results also indicate differences between the CS and the LA courses, e.g., regarding the importance of students' own notes, lecturers' explanations, collaboration among students, and tutor discussions. To understand these differences, factors such as course organization have to be taken into account. The poster presents more detailed results.

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Geogebra: the Fermat case

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As an alternative to presentation-like Geogebra applications I propose the option of utilizing the Geogebra environment as a mathematical laboratory. Experience shows that homemade simple applications prepared by students and teacher promote problem solving. The evolution of relevant and sometimes profound mathematical concepts is based on mathematical experiments in this laboratory and their thorough discussion. A project launched in a class aimed at solving Fermat's classical geometry problem. The poster is a brief and inevitably sketchy report on the progress. The proof is still ahead but having made good use of the opportunity for genuine long term research the students have gained experiences in many senses.

Keywords: Geogebra, problem solving, mathematical experiments, project work, Fermat point, level curve, isogonic center, particular case.

THE POSTER AND ITS ENVIRONMENT

The mathematical contents of too many Geogebra applications is either weightless or out of date or both. There are, of course, several masterpieces in the exponentially growing treasury of geogebra.org. Paradox as it is but their high quality renders their use in the classroom somewhat limited. Presentations in the best sense they are replacing or even outdoing the teacher: the chalk on the blackboard is handicapped. Working on the blackboard, however, diagrams and, more importantly, the underlying ideas are unfolding in a leisure, natural pace. Apart from that, fancy applications are like overbred hounds: their maintenance is tedious if not hopeless.

I propose a different approach: to work on a complex problem in the framework of a long-term project. The activity is the conduction of mathematical experiments and preparation of lab reports. To formulate relevant concepts, like the notion of level curves in our case and **use them** immediately in homemade applications. The goal should be clear all over the time: we want to find the answer. In my experiment of teaching we spent one lesson a week on the project where I set the following classical problem this February to my 9th grade class.

The problem Find the **Fermat point** of a triangle, a point such that the total distance from the three vertices of the triangle to the point is the minimum possible. With more and more powerful but still homemade apps both the location of this point and the minimal sum can be found with the highest accuracy provided by Geogebra. The relevant notion at this experimental phase was that of *level curves*. This is a curve in the plane and it is assigned to a quantity varying pointwise. A familiar example is the system of level curves on a map: the varying quantity is the height above sea level. Points at the same height are on the same curve. An experienced tourist is able to picture the whole terrain from this two dimensional representation. Familiar curves from the mathematical zoo can be produced this way: if the varying quantity -- a two variable function -- is the *distance* of the varying point from a fixed one then we get circles, all about the fixed point: the *pencil* of concentric circles. If, as in the case of the circle -- or the ellipse, by the way -- this varying quantity is defined in terms of distance, then the distance formula and the ImplicitFunction command of Geogebra yield the tool to generate the level curves of Fermat's problem. Stunning examples are shown on the videos linked to the poster. At this phase a relevant distinction has been formulated: while the experimental approach was perfectly appropriate for the surveyor or the engineer, the mathematician or geometer, as this vocation had been named in the past, wants to construct the Fermat point with ruler and compass and, first of all, wants proofs; either by experimenting or by insight. Torricelli's insight has remained concealed, and with good reason; so what sort of experiments are available for the geometer? Well, he can switch from the general to the particular, he can investigate particular cases of the problem. According to Georg Polya's advice:

> If you cannot solve the proposed problem, try to solve first some simpler, related problem.

On the poster there is a list of such simpler particular cases of Fermat's problem. Even some principles can be discerned about such simplifications. A special particular case is that of the equilateral triangle. The students unanimously put their word for the only reasonable candidate: the center. Then it has been justified as a kind of routine with the help of the home developed level curve toolkit. In spite of the deceiving simplicity of this reduced problem, the proof was still too hard and the question is unsettled up to this day. It is not an easy one, especially because there is a plausible hypothesis. We still had a worthwhile discussion about the truth versus proof issue. The students could also give a thought to the difference between the intelligent guess of the layman, the empirically justified hypothesis of the scientist and the proof of the geometer. To make the story more interesting, there is a profound argument kept to myself as yet, showing that **assuming** the truth of the empirically justified claim about the particular case of the equilateral triangle, the geometer can actually **find** and **prove** the truth about the general case. The solution to Fermat's problem -- at least for those triangles whose largest angle is less than 120° -- is the so called *isogonic center*: a point at which the sides of the triangle subtend equal angles, namely 120°.

Conclusion The project at the present, end of year state of affairs is at the threshold of this conditional proof: a considerable growth as far as their mathematical maturity is concerned. They could try themselves as engineers, scientists, laymen from time to time and yes: they could even act the geometer, sometimes. Apart from these role games my students have gained experience as software developers and they could apply their "products" in real time. As for the sources and objectives of this presentation: there is no particular reference but my more than 40 years of teaching experience, my luck to have the chance of working with really bright students sometimes, good colleagues and my conviction that I can and should show to my ordinary students that they are able to play the game of mathematics.

Transforming textbook geometry using new technology

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For most school students mathematics simply appears as a discipline consisting of calculations, use of formulae and strict rules. This feature becomes further visible in topics like geometry. Our school curriculum presents geometry in a very restricted form, where students didn't get the chance to explore and construct their own knowledge. Although, there are a lot of technological tools available which could provide the opportunity for exploration and construction, but they don't directly map with the existing curricular demands. Thus, in the present poster, we will report how the technologically transformed curricular geometry played out in the field.

Keyword: Geometry, Tools, Logo, GeoGebra

NEED FOR TRANSFORMING TEXTBOOK

Students' poor performance in geometry is well reported in literature (Battista, 2007). Our school curriculum and textbook play a major role in this poor performance, where the only available tools for geometry learning are paper, pencil and the geometry box consisting of the physical tools: ruler, protractor, compass, divider. Conventionally, in a classroom, when students are asked to construct a shape, e.g., a square, they are expected to do it on a notebook using pencil, so that they can erase mistakes. They use tools like ruler and compass to ensure precise construction. Subsequently, students are expected to make a single perfect square, devoid of opportunities to appreciate specific mathematical properties or principles involved in the construction. Most of the effort goes in doing precise construction rather than appreciating the underlying principles of the construction, which are left hidden and hardly uncovered in the classroom. Lesson-end exercises given in textbooks are often closed-ended with little to no possibilities for students to come up with different ways to solve them.

The possibility of Logo and dynamic geometry software in creating authentic geometry learning experiences is well reported in literature (Clements, 1985; Sinclair & Bruce, 2015). In order to make the existing school geometry content interesting and accessible to students with open-ended possibilities, technology could be used to provide opportunity to freely explore and construct different mathematical structures (Sinclair & Bruce, 2015). Although there are several open sourced software and tools available, they don't address curricular demands since they are not directly linked to school content. This poster discusses technology leverage provided by some software or tools which gives students the flexibility to creatively engage with geometry. We examine activities on Logo and Geogebra which provide different affordances to students in order to transform and impact their geometry learning by facilitating exploration of different variations unlike the construction in classroom.

About the study

This study is part of a project called Connected Learning Initiative (CLIx) aimed at improving the quality of education by using innovative technologies. The partner collaborators for this project are MIT, Cambridge, USA, TISS, Mumbai, India, Tata Trusts along with the state government from the four states of India. The project is aimed towards closely working with government schools in India, which cater to mostly the underprivileged section of the society. The data collection methodology was based on analysing the observations of regular and technology supported classrooms, and interacting with students and teachers.

The focus of the poster is to discuss the principles of technology supported tasks that emerge out of classroom observations in Mizoram (a north-eastern state of India). We will report instances from classroom where students could explore and construct new geometric knowledge using Logo and Geogebra.

Design of the task

Successful transformation of the textbook content with the given technologies needs well thought design considerations. The questions raised by Drijvers (2015) regarding the impact of ICT in classroom have guided our investigation on how these promising technological tools can be made use of in the classroom. In the present poster we will elicit how our classroom observation informed the design principles for transforming the textbook content to tasks mapped on technological tool. For example, adding an expect of imagination for students to guess what will be the effect of adding a turn right 45° command on their initial set of commands used to draw a square will allow students to connect more with the tool rather incessantly using the commands without thinking. Further, in one classroom two students used 270° and 450° to make a square, but couldn't explain why it gives a square or how is it related to the angles of a square. Again, with Geogebra students tend to make house without any instruction. Thus, mapping a curricular content on technology requires constant feedback from real classroom for its actual development.

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Programming and mathematics in lower secondary school

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The proposed poster will present the results of a design study where we try out an educational design combining mathematics and programming with pupils in Year 8. Our aim is to ensure that the learning of mathematics can be an outcome of programming tasks.

Keywords: computer programming, mathematics curriculum, secondary school, design research, Scratch.

INTRODUCTION

The national curriculum in Norway is under revision, and the government wants to include computer programming as a compulsory part of the mathematics curriculum for all pupils in primary and secondary school. There is an ongoing debate about this now, and many mathematics teachers and teacher educators in Norway argue against computer programming as part of the compulsory mathematics curriculum. Typical arguments are that it is already too many learning objectives in the plans, coding takes focus away from the mathematics, and that there is a weak connection between programming and learning mathematics.

We want to gain more experience with programming and mathematic learning in a lower secondary school context. The proposed poster will tell about some first attempts to introduce programming in a Year 8 mathematics class during the spring term 2018. How can we do this within the frames of the existing curriculum keeping the mathematics in focus?

CONTENT OF THE POSTER

The Norwegian University of Science and Technology (NTNU) has a special collaboration with some local schools in Trondheim. These university schools are ordinary public schools, which offer us access to real classrooms. The university provides resources and teacher training for the teachers at the university schools. Our research project takes place at one of the university schools where one of the teachers will implement programming as a part of his mathematics teaching during spring 2018.

The challenges reported by Misfeldt & Ejsing-Duun (2015) and Ainley, Pratt, & Hansen (2006) consist of designing an education that can prove both meaningful for the students and still ensure that mathematics is learned. There is a tension between enjoyment and motivation, on one hand, and the requirements from the mathematics curriculum at the other. Ainley, Pratt, & Hansen (2006) advocates that the tasks given to the students should both have a purpose and be a utility for the learners. Our over main concern is how to integrate programming as a useful tool for the learning of mathematics.

In collaboration with a mathematics teacher, we designed a teaching experiment involving programming in Scratch where the student task was to program an "Area Calculator". The poster describes the part of the experiment where the pupils in Year 8 tried to make a program which calculated the area of a triangle given the length of the baseline and the height. Before performing the programming task, the pupils worked in pairs, first with geoboards, paper and pencil, and then went on to program on their computers. Their computer screens were captured, and conversations were recorded. We collected the worksheets from the lesson with the pupils' writings. An analysis from this data will be a part of the poster. In focus is both how to calculate area of a triangle and to generalize the variables needed.

The methodological framework will be a design research (Collins, Joseph, & Bielaczyc, 2004) consisting of making a local instruction theory (Gravemeijer, 2004) for implementing some programming tasks. Following Misfeldt & Ejsing-Duun (2015) and Ainley, Pratt, & Hansen (2006) our design will try to ensure both the learning of mathematics and the benefits of introducing some enjoyable new tools.

The poster will present the process of the educational design, the outcome of the teaching, and an evaluation.

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Designing a Digital Game as a Response to the Challenges of Learning Geometry in High Schools in India

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Students often have difficulties when introduced to formal deduction and proofs in high school geometry. While teachers and students in privileged schools now have the means and access to multiple digital technologies to address this issue, in a large number of government high schools in India, there are multiple constraints that limit the use of these technologies. This poster argues for an alternative digital resource – a learning game, designed to provide a more customized response to the challenges of learning formal geometry in the context and scale of government schools in India.

Keywords: digital game, high school geometry, van Hiele levels

BACKGROUND AND CONCEPTUAL FRAMEWORK

The van Hiele theory has been used over the years by several researchers to trace the development of a learner's geometric thinking (Battista, 2007). In India, the early years of the high school curriculum focuses on formal Euclidean geometry. The curriculum demands that students operate at the higher levels of geometric thinking and write deductive proofs, having advanced through the lower levels in primary and upper primary grades. In the context of government schools, this curricular expectation is against a backdrop of a complex web of issues and challenges – high pupil-teacher ratios, poor infrastructure, absence of engaging material and low motivation to name just a few. Consequently, many students in high schools are often at the lowest level (Srinivas, Khanna, Rahaman, & Kumar, 2016) often unable to recognize even basic shapes in an orientation different from the visual prototype they identify it with. This calls for a substantive amount of what researchers have called the 'Spadework Prior to deduction' (Shaughnessy & Burger, 1985).

RATIONALE

While it might be possible to do this 'spadework' through regular classroom work, the use of digital resources is possibly more appropriate as a response to the high Pupil-Teacher ratio situation, for scalability, and also to address the specific pedagogic challenges. Also, in the given context, open resources and software which have high affordances but call for a high level of teacher involvement and expertise have, at best, limited feasibility. This was the premise for the design of a new digital resource - a learning game called Police Quad. This poster discusses how the design of this game responds to the specific challenges on the ground, and in fact, converts some of the challenges into opportunities.

KEY DESIGN PRINCIPLES

Drivjers (2015) proposed that three factors are crucial in determining whether 'digital technology in mathematics education works or does not': design of the digital technology and of the tasks within, role of the teacher, and educational context of the learner. These form the underpinnings of the design of the game, with the educational context of the learner being the driving factor. This section will discuss exactly how each of the three factors informs game design to expressly respond to (and in some cases, build upon) the challenges and constraints that exist at the ground level in the government high school scenario.

TASK DISCUSSION

The digital game is designed to create a learning environment that motivates students to engage in tasks and peer discussions that help them to understand foundational geometric concepts, develop mathematical vocabulary and advance to higher levels of reasoning. Also, the tasks embedded within the game work in tandem with teacher-facilitated discourse. The game progresses through different gameplays of which one will be showcased and discussed in some detail in this section, while also elaborating upon some of the discussions in the previous section.

INSIGHTS FROM IMPLEMENTATION

In this section, some preliminary data including sample student responses will be presented from the initial stages of implementation on the field. This will be linked to proposed future research at the full implementation stage.

The overall objective of the poster is to share with the audience the experience of designing a digital resource to address specific ground level challenges. It is also meant to invite discussion or comment on the design of the game, and discuss its applicability to other audiences and contexts.

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