COMPLETE ASYMPTOTIC EXPANSIONS OF THE SPECTRAL FUNCTION FOR SYMBOLIC PERTURBATIONS OF ALMOST PERIODIC SCHRÖDINGER OPERATORS IN DIMENSION ONE

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Abstract. In this article we consider asymptotics for the spectral function of Schrödinger operators on the real line. Let $P : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ have the form

$$P := -\tfrac{d^2}{dx^2} + W,$$

where W is a self-adjoint first order differential operator with certain modified almost periodic structure. We show that the kernel of the spectral projector, $\mathbb{1}_{(-\infty,\lambda^2]}(P)$ has a full asymptotic expansion in powers of λ. In particular, our class of potentials W is stable under perturbation by formally self-adjoint first order differential operators with smooth, compactly supported coefficients. Moreover, it includes certain potentials with *dense pure point spectrum*. The proof combines the gauge transform methods of Parnovski-Shterenberg and Sobolev with Melrose's scattering calculus.

1. Introduction

Let

$$P := D_x^2 + W_1 D_x + D_x W_1 + W_0 : L^2(\mathbb{R}) \to L^2(\mathbb{R}),$$

where $W_i \in C^\infty(\mathbb{R};\mathbb{R})$. We study the spectral projection for P, $\mathbb{1}_{(-\infty,\lambda^2]}(P)$, when W_i, $i = 1, 2$ satisfy certain almost periodic conditions. Denote by $e_\lambda(x, y)$ the kernel of $\mathbb{1}_{(-\infty,\lambda^2]}(P)$.

We assume that there is $\Theta \subset \mathbb{R}$ countable such that $-\Theta = \Theta$, $0 \in \Theta$, and

$$W_i(x) = \sum_{\theta\in\Theta} e^{i\theta x} w_\theta(x), \qquad |\partial_x^k w_\theta(x)| \le C_{k,N} \langle x\rangle^{-k} \langle\theta\rangle^{-N}. \tag{1.1}$$

Before stating the general conditions on w_θ (see §3), we give two consequences of our main theorem (Theorem 3.1). Let $\omega := (\omega_1, \dots \omega_d) \in \mathbb{R}^d$. We say ω satisfies the diophantine condition if there are $c, \mu > 0$ such that

$$|\mathbf{n} \cdot \omega| > c|\mathbf{n}|^{-\mu}, \qquad \mathbf{n} \in \mathbb{Z}^d \setminus \{0\}. \tag{1.2}$$

Theorem 1.1. *Suppose $\omega \in \mathbb{R}^d$ satisfies the diophantine condition* (1.2) *and W_i are as in* (1.1) *with $\Theta = \mathbb{Z}^d \cdot \omega$ and*

$$|\partial_x^k w_{\mathbf{n}\cdot\omega}(x)| \le C_{k,N} \langle x\rangle^{-k} \langle\mathbf{n}\rangle^{-N}, \qquad \mathbf{n} \in \mathbb{Z}^d$$

then for $|x - y| > c$,

$$e_\lambda(x, y) \sim \cos(\lambda(x - y)) \sum_j \lambda^{-j} a_j(x, y) + \sin(\lambda(x - y)) \sum_j \lambda^{-j} b_j(x, y), \qquad e_\lambda(x, x) \sim \sum_j \tilde{a}_j \lambda^{j+1} \tag{1.3}$$

where $a_0 = 0$ and $b_0 = \frac{2}{\pi(x-y)}$. Moreover, we have an oscillatory integral expression for $e_\lambda(x, y)$ valid uniformly for (x, y) in any compact subset of $\mathbb{R}^2$.

Remark 1.1. It is easy to see that the condition (1.2) is generic in the sense that it is satisfied for Lebesgue almost every $\omega \in [-1, 1]^d$.

Next, we state a theorem in the limit periodic case.

Theorem 1.2. *Let* $\{m_n\}_{n=1}^{\infty} \subset \mathbb{Z}_+$*, and* $\Theta = \Theta_+ \cup -\Theta_+ \cup \{0\}$ *where* $\Theta_+ = \{\theta_n\}_{n=1}^{\infty}$, $\theta_n := m_n/n$. *Suppose that* W_i *are as in* (1.1) *with*

$$|\partial_x^k w_{\theta_n}(x)| \leq C_{k,N}\langle x\rangle^{-k}\langle n\rangle^{-N}, \qquad n \geq 1$$

then (1.3) *holds.*

In both Theorems 1.1 and 1.2, one may add *any* formally self-adjoint first order differential operator $W_{sym} = a_1(x)D_x + b_1(x)$ whose coefficients satisfy $|\partial_x^k a_i(x)| \leq C_k\langle x\rangle^{-k}$ to W and $W + W_{sym}$ will satisfy the assumptions of the Theorem. In addition, Theorems 1.1 and 1.2 include examples with arbitrarily large embedded eigenvalues and Theorem 1.2 includes examples with dense pure point spectrum (See Appendix B).

While full asymptotic expansions are known in the the case that W is compactly supported [PS83, Vai84] and in the case that $W_\theta = \sum_\theta e^{i\theta x} v_\theta$ with $v_\theta \in \mathbb{C}$ and Θ satisfying the assumptions of Theorem 1.1 [PS16], to the author's knowledge Theorems 1.1 and 1.2 are the first to allow for both types of behavior. The work [PS16] followed the approach developed in [PS12, PS09] for the study of the integrated density of states a subject whicht, for periodic Schroödinger operators, has been the focus of a long line of articles (see e.g. [Sob05, Sob06, Kar00, HM98]).

1.1. **Discussion of the proof.** We choose not to state our general results until all of the necessary preliminaries have been introduced (see Theorem 3.1). Instead, we outline how our proof draws on and differs from the work of Parnovski–Shterenberg [PS09, PS12, PS16] and Morozov–Parnovski–Shterenberg [MPS14]. These papers handle the much more difficult higher dimensional case of the above problem when $W(x, D)$ is replaced by a potential $V(x) = \sum_{\theta\in\Theta} v_\theta e^{i\theta x}$ where $v_\theta \in \mathbb{C}$ and Θ is assumed to be discrete and satisfying certain diophantine conditions. The crucial technique used in those articles is the gauge transform (developed in [Sob05, Sob06, PS10]) i.e. conjugating the operator P by e^{iG} for some pseudodifferential G constructed so that the conjugated operator takes the form H_0+R where H_0 is a constant coefficient differential operator near frequencies $|\xi| \sim \lambda$ and away from certain resonant zones in the Fourier variable and where $R = O(\lambda^{-N})_{H^{-N}\to H^N}$. The authors are then able to make a sophisticated analysis of the operator H_0 acting on Besicovitch spaces. This analysis uses in a crucial way that H_0 acts nearly diagonally i.e. that the operator can be thought of as a direct sum of operators acting on resonant frequencies and is diagonal away from these frequencies. The authors write a more or less explicit, albeit complicated, integral formula for the spectral function and then directly analyze this integral.

In this article, we take a somewhat different approach to the second step of the above analysis. Namely, we start with our operator P and, after conjugation by e^{iG}, are able to reduce to the case of $H_0 + R$ where H_0 is a scattering pseudodifferential operator [Mel94] near the frequencies $|\xi| \sim \lambda$. However, because we have simplified our problem by working in one dimension, resonant zones do not occur. In particular, we will prove a limiting absorption principle for H_0 at high enough energies and show that the resulting resolvent operators $(H_0 - \lambda^2 \mp i0)^{-1}$ satisfy certain 'semiclassical outgoing/incoming' properties. These, roughly speaking, state that the resolvent transports singularities in only one direction along the Hamiltonian flow for the symbol of H_0 and that these singularities do not return from infinity. With this in hand, we are able understand the spectral projector for H_0 using the wave method of Levitan [Lev52], Avakumović [Ava56] and Hörmander [Hör68] and hence, using an elementary spectral theory argument, to understand the spectral function for P. The crucial fact allowing the proof of a limiting absorption principle is that H_0 may be chosen such that the 'non-scattering pseudodifferential' part is identically zero on frequencies near λ.

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2. General assumptions

2.1. **Pseudodifferential classes.** We work with pseudodifferential operators in Melrose's scattering calculus [Mel94]. Since we are working in the simple setting of $\mathbb{R}$, we will not review the construction of an invariant calculus. Instead, we say that $a \in C^\infty(\mathbb{R}^2)$ lies is $S^{m,n}$ if for all $\alpha, \beta \in \mathbb{N}$.

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta} \langle x\rangle^{n-\alpha}\langle \xi\rangle^{m-\beta}. \tag{2.1}$$

We define the seminorms on $S^{m,n}$ by

$$\|a\|_{\beta,\alpha}^{m,n} = \sum_{j=0}^{\alpha}\sum_{k=0}^{\beta} \sup |\partial_x^j \partial_\xi^\beta a(x,\xi)\langle x\rangle^{-n+j}\langle \xi\rangle^{-m+k}|$$

When it is convenient, we will say $\mathcal{N} = (m,n,\alpha,\beta) \subset \mathbb{N}^4$ is a choice of a seminorm on $S^{m,n}$.

It will also be convenient to have the standard symbol classes on $\mathbb{R}$. For this, we say $a \in C^\infty(\mathbb{R}^2)$ lies in S^m if

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta}\langle \xi\rangle^{m-\beta}.$$

Note that $S^{m,n} \subset S^m$. We also define the corresponding classes of pesudodifferential operators:

$$\Psi^{m,n} := \{a(x,hD) \mid a \in S^{m,n}\}, \qquad \Psi^m := \{a(x,hD) \mid a \in S^m\},$$

where for $a \in S^m$,

$$a(x,hD)u := \frac{1}{2\pi h}\int e^{\frac{i}{h}(x-y)\xi} a(x,\xi)u(y)dyd\xi.$$

We sometimes write $Op_h(a)$ for the operator $a(x,hD)$.

Our pseudodifferential operators will have polyhomogeneous symbols. That is, they will be given by $a \in S^{m,n}$, $b \in S^m$ such that there are $a_j \in S^{m-j,n-j}$, $b_j \in S^{m-j}$ satisfying

$$a(x,\xi) - \sum_{j=0}^{N-1} h^j a_j(x,\xi) \in h^N S^{m-N,n-N}, \qquad b(x,\xi) - \sum_{j=0}^{N-1} h^j b_j(x,\xi) \in h^N S^{m-N}.$$

We will abuse notation slightly from now and and write $a \in S^{m,n}$, $b \in S^m$ to mean that a and b have such expansions and $\Psi^{m,n}$, Ψ^m for the corresponding operators.

Note that both $\Psi^{m,n}$ and Ψ^m come with well behaved symbol maps, $\sigma_{m,n} : \Psi^{m,n} \to S^{m,n}$ and $\sigma_m : \Psi^m \to S^m$ respectively such that

$$0 \to hS^{m-1,n-1} \xrightarrow{a(x,hD)} \Psi^{m,n} \xrightarrow{\sigma_{m,n}} S^{m,n} \to 0, \qquad 0 \to hS^{m-1} \xrightarrow{a(x,hD)} \Psi^m \xrightarrow{\sigma_m} S^m \to 0.$$

are short exact sequences. Moreover,

$$\sigma_{m_1+m_2,n_1+n_2}(AB) = \sigma_{m_1,n_1}(A)\sigma_{m_2,n_2}(B), \qquad \sigma_m(AB) = \sigma_{m_1}(A)\sigma_{m_2}(B),$$

and

$$\begin{aligned}\sigma_{m_1+m_2-1,n_1+n_2-1}(ih^{-1}[A,B]) &= \{\sigma_{m_1,n_1}(A),\sigma_{m_2,n_2}(B)\},\\ \sigma_{m_1+m_2-1}(ih^{-1}[A,B]) &= \{\sigma_{m_1}(A),\sigma_{m_2}(B)\},\end{aligned} \qquad \{a,b\}:=\partial_\xi a\partial_x b-\partial_\xi b\partial_x a.$$

For future use, we define norms as follows,

$$\|u\|_{H_h^{s_1,s_2}}:=\|\langle x\rangle^{s_2}u\|_{H_h^{s_1}},\qquad \|u\|_{H_h^{s_1}}:=\|\langle -h^2\partial_x^2\rangle^{s_1/2}u\|_{L^2}.$$

We recall the following estimates for pseudodifferential operators

Lemma 2.1. *Let $a\in S^{m,n}$, $b\in S^m$. Then*

$$\|a(x,hD)u\|_{H_h^{s_1-m,s_1-n}}\le C_a\|u\|_{H_h^{s_1,s_2}},\qquad \|b(x,hD)u\|_{H_h^{s_1-m,s_2}}\le C_b\|u\|_{H_h^{s_1,s_2}},$$

The maps $S^{m,n}\overset{a(x,hD)}{\longrightarrow}\mathcal{L}(H_h^{s_1,s_2},H_h^{s_1-m,s_2-n})$ and $S^m\overset{b(x,hD)}{\longrightarrow}\mathcal{L}(H_h^{s_1},H_h^{s_1-m})$ are continuous.

In preparation for the gauge transform method, we prove two preliminary lemmas on exponentials of elements of Ψ^0.

Lemma 2.2. *Let $G\in\Psi^0$ self-adjoint. Then $e^{iG}\in\Psi^0$.*

Proof. Let $g\in S^0$ such that $G=Op_h(g)$ and $A_0(t):=Op_h(e^{itg})$. We compute

$$D_t(e^{-itG}A_0(t))=e^{-itG}(-GA_0+Op_h(ge^{itg}))=e^{-itG}hOp_h(r_1(t))$$

where $r_1\in S^{-1}$. Now, suppose that we have $B_j(t)$, $j=1,\dots,N-1$, $B_j\in\Psi^{-j}$ such that with $A_{N-1}(t):=A_0(t)+\sum_{j=1}^{N-1}h^jB_j(t)$,

$$D_t(e^{-itG}A_N(t))=e^{-itG}h^NOp_h(r_N(t))$$

with $r_N\in S^{-N}$. Then, putting $B_N(t)=Op_h(-i\int_0^t e^{i(t-s)g}r_N(s)ds)$ we have

$$\begin{aligned}D_t(e^{-itG}(A_N(t)+h^NB_N(t)) &= e^{-itG}h^N\big(Op_h(r_N(t))-GB_N(t)+D_tB_N(t)\big)\\ &= e^{-itG}h^{N+1}Op_h(r_{N+1}(t))\end{aligned}$$

for some $r_{N+1}\in S^{-N-1}$. Putting $A\sim A_0+\sum_j h^jB_j(t)$, we have

$$D_t(e^{-itG}A(t))=e^{-itG}O_t(h^\infty)_{\Psi^{-\infty}}.$$

In particular, integrating, we have

$$e^{itG}=A(t)+\int_0^t e^{i(t-s)G}R_\infty(s)ds,\qquad R_\infty(s)=O(h^\infty)_{\Psi^{-\infty}}$$

Therefore, since for all N, $A(t):H_h^{-N}\to H_h^{-N}$ and $R_\infty:H_h^{-N}\to H_h^N$ are bounded, the fact that $e^{itG}:L^2\to L^2$ is bounded implies that for $N\ge 0$, $e^{itG}:H_h^{-N}\to H_h^{-N}$. But then for $u,v\in C_c^\infty$,

$$|\langle e^{itG}u,v\rangle_{L^2}|=|\langle u,e^{-itG}v\rangle_{L^2}|\le\|u\|_{H_h^N}\|e^{-itG}v\|_{H_h^{-N}}\le C\|u\|_{H_h^N}\|v\|_{H_h^{-N}}.$$

In particular, by density, we have $e^{itG}:H_h^N\to H_h^N$ is bounded for all N and hence

$$e^{itG}=A(t)+O(h^\infty)_{\Psi^{-\infty}}.$$

From the construction, it is clear that since G is polyhomogeneous, so is e^{itG}. □

Lemma 2.3. *Let $G \in \Psi^0$ self adjoint, and $P \in \Psi^m$,*

$$e^{iG}Pe^{-iG} = \sum_{k=0}^{N-1} \frac{i^k \operatorname{ad}_G^k P}{k!} + O(h^N)_{H_h^s \to H_h^{s+N-m}}$$

where $\operatorname{ad}_A B = [A, B]$.

Proof. Note that

$$(D_t)^k e^{itG}Pe^{-itG} = e^{itG} \operatorname{ad}_G^k Pe^{-itG}$$

and in particular,

$$e^{itG}Pe^{-itG} = \sum_{k=0}^{N-1} \frac{t^k i^k}{k!} \operatorname{ad}_G^k P + \int_0^t \frac{(t-s)^{N-1} i^N}{(N-1)!} e^{isG} \operatorname{ad}_G^N Pe^{-isG} ds$$

Now, $\operatorname{ad}_G^N P \in h^N \Psi^{m-N}$ and hence, the lemma follows by putting $t = 1$ and recalling that $e^{isG} \in \Psi^0$. □

2.2. **Ellipticity.** Next, we recall the notion of the elliptic set for elements of Ψ^m and $\Psi^{m,n}$. To this end, we compactify $T^*\mathbb{R}$ in the fiber variables to $\overline{T^*\mathbb{R}} \cong \mathbb{R} \times [0,1]$ for Ψ^m and in both the fiber and position variables to ${}^{\mathrm{sc}}\overline{T^*\mathbb{R}} \cong [-1,1] \times [-1,1]$ for $\Psi^{m,n}$. In particular, the boundary defining functions on ${}^{\mathrm{sc}}\overline{T^*\mathbb{R}}$ are $\pm x^{-1}$ near $\pm x = \infty$ and $\pm\xi^{-1}$ near $\pm\xi = \infty$ and those for $\overline{T^*\mathbb{R}}$ are $\pm x^{-1}$. We can now define the elliptic set of $A \in \Psi^{m,n}/\Psi^m$, $\operatorname{ell}_{\mathrm{h}}^{\mathrm{sc}}(A) \subset^{\mathrm{sc}} \overline{T^*\mathbb{R}}$, and $\operatorname{ell}_{\mathrm{h}}(A) \subset \overline{T^*\mathbb{R}}$ respectively as follows. We say $\rho \in \operatorname{ell}_{\mathrm{h}}^{\mathrm{sc}}(A)$ if there is a neighborhood, $U \subset^{\mathrm{sc}} \overline{T^*\mathbb{R}}$ of ρ such that

$$\inf_{(x,\xi)\in U} \langle x\rangle^{-m}\langle \xi\rangle^{-n} |\sigma_{m,n}(A)(x,\xi)| > 0.$$

We say that $\rho \in \operatorname{ell}_{\mathrm{h}}(A)$ if there is a neighborhood, $U \subset \overline{T^*\mathbb{R}}$ of ρ such that

$$\inf_{(x,\xi)\in U} \langle \xi\rangle^{-n} |\sigma_{m,n}(A)(x,\xi)| > 0.$$

Next, we define the wavefront set for an element of Ψ^m, $\mathrm{WF}_{\mathrm{h}}(A) \subset \overline{T^*\mathbb{R}}$ and the scattering wavefront set of $A \in \Psi^{m,n}$, $\mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(A) \subset^{\mathrm{sc}} \overline{T^*\mathbb{R}}$. For $A \in \Psi^m$, we say $\rho \notin \mathrm{WF}_{\mathrm{h}}(A)$ if there is $B \in \Psi^0$ such that $\rho \in \operatorname{ell}_{\mathrm{h}}(B)$ and

$$\|BA\|_{H_h^{-N} \to H_h^N} \le C_N h^N.$$

For $A \in \Psi^{m,n}$, we say $\rho \notin \mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(A)$ if there is $B \in \Psi^{0,0}$ such that $\rho \in \operatorname{ell}_{\mathrm{h}}^{\mathrm{sc}}(B)$ and

$$\|BA\|_{H_h^{-N,-N} \to H_h^{N,N}} \le C_N h^N.$$

We can now state the standard elliptic estimates.

Lemma 2.4. *Suppose $P \in \Psi^{m,n}$, $A \in \Psi^{0,0}$, with $\mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(A) \subset \operatorname{ell}_{\mathrm{h}}^{\mathrm{sc}}(P)$. Then there is $C > 0$ such that for all N there is $C > 0$ such that*

$$\|Au\|_{H_h^{s,k}} \le C\|Pu\|_{H_h^{s-m,k-n}} + C_N h^N \|u\|_{H_h^{-N,-N}}.$$

If instead $P \in \Psi^m$, $A \in \Psi^0$, with $\mathrm{WF_h}(A) \subset \mathrm{ell_h}(P)$. Then there is $C > 0$ such that for all $N > 0$ there is $C_N > 0$ such that

$$\|Au\|_{H_h^s} \le C\|Pu\|_{H_h^{s-m}} + C_N h^N \|u\|_{H_h^{-N}}$$

2.3. **Propagation estimates.** We next recall some propagation estimates for scattering pseudodifferential operators. Since we will work with operators that are fiber classically elliptic, i.e. $\partial({}^{\mathrm{sc}}\overline{T^*\mathbb{R}})_\xi \subset \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(P)$, we do not need the full scattering calculus here, and will work with operators that are fiber compactly microlocalized. In particular, we say that $A \in \Psi^{m,n}$ is *fiber compactly microlocalized* and write $A \in \Psi^{\mathrm{comp},n}$ if there is $C > 0$ such that

$$\mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(A) \cap \{|\xi| > C\} = \emptyset.$$

For fiber compactly microlocalized operators, all propagation estimates from the standard calculus (see e.g. [DZ19a, Appendix E.4]) follow using the same proofs but interchanging the roles of x and ξ.

Throughout, we let $P \in \Psi^{m,n}$ self-adoint with $\sigma_{m,n}(P) = p$, and write

$$\varphi_t := \exp(t\langle\xi\rangle^{1-m}\langle x\rangle^{1-n}H_p) : {}^{\mathrm{sc}}\,\overline{T^*\mathbb{R}} \to {}^{\mathrm{sc}}\,\overline{T^*\mathbb{R}}$$

for the rescaled Hamiltonian flow. The following lemma follows as in [DZ19a, Theorem E.47]

Lemma 2.5. *Let $P \in \Psi^{m,n}$ self-adjiont and suppose that $A, B, B_1 \in \Psi^{\mathrm{comp},0}$. Furthermore, assume that for all $\rho \in \mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(A)$, there is $T \ge 0$ such that*

$$\varphi_{-T}(\rho) \in \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(B), \qquad \bigcup_{t\in[-T,0]} \varphi_t(\rho) \subset \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(B_1).$$

Then for all N there is $C > 0$ such that for $\varepsilon \ge 0$, $u \in \mathcal{S}'$ with $Bu \in H_h^{s,k}$, $B_1(P - i\varepsilon\langle x\rangle^n)u \in H_h^{s,k-n+1}$

$$\|Au\|_{H_h^{s,k}} \le C\|Bu\|_{H_h^{s,k}} + Ch^{-1}\|B_1(P - i\varepsilon\langle x\rangle^n)u\|_{H_h^{s,k-n+1}} + C_N h^N \|u\|_{H_h^{-N,-N}}.$$

We will also need the radial point estimates in the setting of fiber compactly microlocalized operators. The following two lemmas are a combination of [DZ19a, Theorem E.52, E.54] together with the arguments in [DZ19b, Section 3.1]

Lemma 2.6. *Let $P \in \Psi^{m,n}$ self adjoint with $n > 0$ and let*

$$L \Subset \{\langle x\rangle^{-n}p = 0\} \cap \partial({}^{\mathrm{sc}}\overline{T^*\mathbb{R}})_x$$

be a radial source for p. Let $k' > \frac{n-1}{2}$, fix $B_1 \in \Psi^{\mathrm{comp},0}$ such that $L \subset \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(B_1)$. Then there is $A \in \Psi^{\mathrm{comp},0}(M)$ such that $L \subset \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(A)$ and for all N, $k > k'$, $\varepsilon \ge 0$, and $u \in \mathcal{S}'$ such that $B_1 u \in H_h^{s,k'}$ and $B_1(P - i\varepsilon\langle x\rangle^n)u \in H_h^{s,k-n+1}$,

$$\|Au\|_{H_h^{s,k}} \le Ch^{-1}\|B_1(P - i\varepsilon\langle x\rangle^n)u\|_{H_h^{s,k-n+1}} + C_N h^N \|u\|_{H_h^{-N,-N}}.$$

Lemma 2.7. *Let $P \in \Psi^{m,n}$ as above with $n > 0$, let*

$$L \Subset \{\langle x\rangle^{-n}p = 0\} \cap \partial({}^{\mathrm{sc}}\overline{T^*\mathbb{R}})_x$$

be a radial sink for p Let $k < \frac{n-1}{2}$*, fix* $B_1 \in \Psi^{\mathrm{comp},0}$ *such that* $L \subset \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(B_1)$*. Then there are* $A, B \in \Psi^{\mathrm{comp},0}(M)$ *such that* $L \subset \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(A)$*,* $\mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(B) \subset \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(B_1) \setminus L$*, and for all* N*,* $\varepsilon \geq 0$*, and* $u \in \mathcal{S}'$ *such that* $Bu \in H_h^{s,k}$ *and* $B_1(P - i\varepsilon\langle x\rangle^n)u \in H_h^{s,k-n+1}$*,*

$$\|Au\|_{H_h^{s,k}} \leq C\|Bu\|_{H_h^{s,k}} + Ch^{-1}\|B_1(P - i\varepsilon\langle x\rangle^n)u\|_{H_h^{s,k-n+1}} + C_N h^N \|u\|_{H_h^{-N,-N}}.$$

3. Almost periodic potentials

3.1. **Assumptions on the potential.** We now introduce the objects necessary for our assumptions on the perturbation W. We say that $\Theta \subset \mathbb{R}$ is a *frequency set* if Θ is countable, $\Theta = -\Theta$ and $0 \in \Theta$. We write $\Theta^k := \Theta \times \underset{k-2}{\cdots} \times \Theta$ and $\Theta_k := \Theta + \underset{k-2}{\cdots} + \Theta$ and

For a frequency set Θ, and a seminorm $\mathcal{N}$, on $S^{m,n}$, we will need a family of maps $s_{k,\mathcal{N}} : \Theta^k \times (S^{m,n})^{\Theta} \to [0, \infty)$. We denote an element $(w_\theta)_{\theta\in\Theta} \in (S^{m,n})^{\Theta}$ by $\mathcal{W}$. Fix a seminorm, $\mathcal{N}$ and define

$$s_{0,\mathcal{N}}(\mathcal{W}) = 1, \qquad s_{1,K}(\theta, \mathcal{W}) = \begin{cases} \frac{\|w_\theta\|_{\mathcal{N}}}{|\theta|} & \theta \neq 0 \\ 0 & \theta = 0 \end{cases}.$$

Next, for $\alpha \in \mathbb{N}^j$ with $|\alpha| = k$, define $\beta_i(\alpha) = \sum_{\ell=1}^{i-1} \alpha_\ell$. Then, for $\theta \in \Theta^k$, we write $\theta_{\alpha,i} := (\theta_{\beta_i(\alpha)+1}, \dots \theta_{\beta_{i+1}(\alpha)}) \in \Theta^{\alpha_i}$. We can now define

$$s_{\alpha,\mathcal{N}}(\theta, \mathcal{W}) := \prod_{i=1}^{j} s_{\alpha_i,\mathcal{N}}(\theta_{\alpha,i}, \mathcal{W}),$$

$$s_{k,\mathcal{N}}(\theta, , \mathcal{W}) = \begin{cases} \frac{1}{|\sum_{i=1}^k \theta_i|} \sum_{p\in Sym(k)} \sum_{|\alpha|=k, \alpha_i \leq k/2} s_{\alpha,\mathcal{N}}(p(\theta)) & \sum_{i=1}^k \theta_i \neq 0 \\ 0 & \sum_i \theta_i = 0. \end{cases}$$

where $Sym(k)$ denotes the symmetric group on k elements.

The following two lemmas on the behavior of $s_{k,\mathcal{N}}$ will be useful below. Their proofs are elementary and we postpone them to Appendix A.

Lemma 3.1. *There are* $C_k, N_k > 0$ *such that for* $\theta \in \Theta^k$*,*

$$|s_{k,\mathcal{N}}(\theta, \mathcal{W})| \leq C_k \frac{\prod_{i=1}^k \|w_{\theta_i}\|_{\mathcal{N}}}{\inf\{|\omega|^{N_k} \mid \omega \in \{\theta_1, 0\} + \cdots + \{\theta_k, 0\} \setminus 0\}} \tag{3.1}$$

Lemma 3.2. *Suppose that* $\tilde{\mathcal{W}} \in (S^{m,n})^{\Theta_n}$ *with* $(\tilde{\mathcal{W}})_{\theta_1+\cdots+\theta_n} = \tilde{w}_{\theta_1\dots\theta_n}$ *such that for all* $\mathcal{N}$ *there is* $\mathcal{N}'$ *satisfying*

$$\|\tilde{w}_{\theta_1\dots\theta_n}\|_{\mathcal{N}} \leq \frac{\prod_{i=1}^n \|w_{\theta_i}\|_{\mathcal{N}'}}{|\theta_i|}.$$

Then for all $\mathcal{N}$*, there is* $\mathcal{N}'$ *such that*

$$s_{k,\mathcal{N}}(\theta_1 + \cdots + \theta_n, \tilde{\mathcal{W}}) \leq s_{nk,\mathcal{N}'}((\theta_1, \dots, \theta_n), \mathcal{W}).$$

We say that $W \in \Psi^1$ is *admissible* if

$$W = \sum_{\theta\in\Theta} e^{i\theta x} w_\theta(x, hD) \tag{3.2}$$

where $w_\theta \in S^{1,0}$ and for all $0 \leq k$, $\mathcal{N}$, and $N > 0$ we have

$$\sum_{\theta \in \Theta^k} s_{k,\mathcal{N}}(\theta, \mathcal{W}) \leq C_{k,\mathcal{N}}, \qquad \|w_\theta\|_{\mathcal{N}} < C_{N,\mathcal{N}} \langle \theta \rangle^{-N}, \tag{3.3}$$

where $\mathcal{W} = (w_\theta)_{\theta \in \Theta}$.

Remark 3.3. When W is smooth and periodic i.e. $\Theta = r\mathbb{Z}$, and $\|w_\theta\|_{\mathcal{N}} \leq C_{N,\mathcal{N}} \langle \theta \rangle^{-N}$, then W is admissible.

Remark 3.4. If W is an approximately almost periodic function of the form

$$W = \sum_{\mathbf{n} \in \mathbb{Z}^d} e^{i\mathbf{n}\cdot\omega x} w_{\mathbf{n}}(x, hD)$$

with $\|w_{\mathbf{n}}\|_{\mathcal{N}} \leq C_{N,\mathcal{N}} \langle \mathbf{n} \rangle^{-N}$ and if $\omega = (\omega_1, \omega_2, \dots \omega_d)$ satisfies the diophantine condition (1.2), then W is admissible. To see this, without loss of generality, we assume that $\omega \in B(0,1)$. Then if $\theta \in \Theta$, $\theta = \mathbf{n} \cdot \omega$ for some $\mathbf{n} \in \mathbb{Z}^d$. In particular, if

$$\theta_{\mathbf{n}_1}, \dots, \theta_{\mathbf{n}_k} \in \Theta, \qquad \sum_{i=1}^k \theta_{\mathbf{n}_i} = \sum_i \mathbf{n_i} \cdot \omega,$$

and hence, if $\sum_i \theta_{\mathbf{n_i}} \neq 0$, then $|\sum_{i=1}^k \theta_{\mathbf{n}_i}| \geq C |\sum_i \mathbf{n}_i|^{-\mu}$.

Using this, observe that by (3.1) there are C_k, N_k such that

$$s_{k,\mathcal{N}}(\theta_1, \dots, \theta_k) \leq C_k (\sum_i |\mathbf{n}_i|)^{\mu N_k} \prod_{i=1}^k C_N \langle \mathbf{n}_i \rangle^{-N} \leq C_k \prod_{i=1}^k C_N \langle \mathbf{n}_i \rangle^{-N + N_k \mu}$$

We thus obtain the desired estimate by taking $N > N_k \mu + d$ and summing over $\mathbf{n}_i$, $i = 1, \dots k$.

Remark 3.5. Next, we verify that certain approximately limit periodic functions are admissible. Suppose that $\{m_n\}_{n=1}^\infty \subset \mathbb{Z}$ contains 0 and satisfies $\{m_n\}_{n=1}^\infty = \{-m_n\}_{n=1}^\infty$. Suppose

$$W = \sum_n e^{im_n x/n} w_n(x, hD)$$

and $\|w_n\|_K \leq C_{N,K} \langle \max(n, |m_n|/n) \rangle^{-N}$, then w_n satisfies our conditions with $\mu_M \equiv 0$. Indeed, in this case, $\Theta = \{m_n/n\}_{n=1}^\infty$. Now, note that for $\theta_i \in \Theta$, $\theta_i = m_{n_i}/n_i$

$$\sum_{i=1}^k \theta_i \neq 0 \qquad \Rightarrow \qquad \Big| \sum_i \theta_i \Big| \geq \frac{1}{n_1 n_2 \cdots n_k}$$

Using this, observe that by (3.1) there are C_k, N_k such that

$$s_{k,\mathcal{N}}(\theta_1, \dots, \theta_k) \leq C_k (n_1 n_2 \cdots n_k)^{N_k} \|w_{\theta_1}\|_{\mathcal{N}} \cdots \|w_{\theta_k}\|_{\mathcal{N}}$$

In particular, for $N > N_k$

$$s_{k,\mathcal{N}}(\theta_1, \dots, \theta_k) \leq C_k \prod_{i=1}^k C_N^k n_i^{N_k} \langle \max(n_i, m_{n_i}/n_i) \rangle^{-N} \leq C_{N,k} \prod_{i=1}^k \langle n_i \rangle^{N_k - N}.$$

We thus obtain the desired estimate by taking $N > N_k + 1$ and summing over n_i, $i = 1, \dots k$.

Theorem 3.1. *Suppose that $W(x,hD)\in\Psi^1$ is self-adjoint and admissible (i.e.* (3.2) *and* (3.3) *hold). Let $0<\delta<1$,*

$$P:=-h^2\Delta+hW(x,hD).$$

Then there are $a_j\in C_c^\infty(\mathbb{R}^3)$ such that for all $R>0$ there is $T>0$ satisfying for all $E\in[1-\delta,1+\delta]$, $\hat{\rho}\in C_c^\infty(\mathbb{R};[0,1])$ with $\hat{\rho}\equiv 1$ on $[-T,T]$, and all $x,y\in B(0,R)$ the spectral projector $1_{(-\infty,E]}(P)$ satisfies

$$1_{(-\infty,E]}(P)(x,y)=h^{-2}\int_{-\infty}^{E}\int \hat{\rho}(t)e^{it(\mu-|\xi|^2)+(x-y)\xi)/h}a(x,y,\xi;h)d\xi dt d\mu+O(h^\infty)_{C^\infty},$$

where $a\sim\sum_j h^j a_j$.

After putting $h=\lambda^{-1}$, $W(x,hD)=h(W_1(x)hD_x+hD_xW_1(x))+h^2W_0(x)$, an application of the method of stationary phase, the analysis in Remarks 3.4 and 3.5, and an application Theorem 3.1 proves Theorems 1.1 and 1.2. (See [Ivr18] for a related problem.)

4. Gauge transforms

Before gauge transforming our operator, we need the following symbolic lemma which allows us to solve away errors.

Lemma 4.1. *Suppose that $a\in S^{k,0}$. Then, there is $b\in S^{k,0}$ such that $(D_x+\theta)b-a=r\in S^{k,-\infty}$ and*

$$\|b\|^{k,0}_{\beta,\alpha}\leq C_{\alpha\beta k}|\theta|^{-1}\|a\|^{k,0}_{\beta,\alpha+2},\qquad \|r\|^{k,-N}_{\beta,\alpha}\leq C_{\alpha\beta N}|\theta|^{-1}\|a\|^{k,0}_{\beta,\alpha+N+2}f$$

Proof. **Case 1:** $|\theta|\geq 1$. Let $\chi\in C_c^\infty(\mathbb{R})$ with $\chi\equiv 1$ on $[-1/3,1/3]$ and $\operatorname{supp}\chi\subset(-1,1)$. Then define

$$b(x,\xi):=\frac{1}{2\pi}\int e^{i(x-y)\eta}\frac{1-\chi(\theta+\eta)}{\eta+\theta}a(y,\xi)dyd\eta.$$

where the integral in y interpreted as the Fourier transform. Then, $(D_y+\theta)b-a=r$ where

$$r(x,\xi):=-\frac{1}{2\pi}\int e^{i(x-y)\eta}\chi(\theta+\eta)a(y,\xi)dyd\eta=-\frac{1}{2\pi}\int e^{i(x-y)\eta}\chi(\theta+\eta)|\eta|^{-N}D_y^Na(y,\xi)dyd\eta$$

Then, since $\tilde{\chi}_\theta:=\chi(\theta)|\eta-\theta|^{-N}$ is smooth and compactly supported with seminorms bounded uniformly in $|\theta|\geq 1$,

$$\begin{aligned}|D_x^\alpha D_\xi^\beta r(x,\xi)|&=\Big|-\frac{1}{2\pi}\int e^{i\theta(x-y)}\hat{\tilde{\chi}}_\theta(y-x)D_y^{\alpha+N}D_\xi^\beta a(y,\xi)dy\Big|\\&\leq C_{N,M}\int|\theta|^{-1}\langle x-y\rangle^{-M}\langle y\rangle^{-\alpha-N}\langle\xi\rangle^{k-\beta}\|a\|^{k,0}_{\beta,\alpha+N+1}dy\\&\leq C|\theta|^{-1}\langle x\rangle^{-\alpha-N}\langle\xi\rangle^{k-\beta}\|a\|^{k,0}_{\beta,\alpha+N+1},\end{aligned}$$

and

$$\begin{aligned}|D_x^\alpha D_\xi^\beta b(x,\xi)| &= \Big|\frac{1}{2\pi}\int e^{i(x-y)\eta}\Big(\frac{1-(x-y)D_\eta}{1+|x-y|^2}\Big)^N\frac{1-\chi(\theta+\eta)}{\eta+\theta}\Big(\frac{1+\eta D_y}{1+|\eta|^2}\Big)^2 D_y^\alpha D_\xi^\beta a(y,\xi)dyd\eta\Big|\\ &\le C_N\Big|\int\langle x-y\rangle^{-N}\langle\eta\rangle^{-2}\langle\eta+\theta\rangle^{-1}\langle\xi\rangle^{k-\beta}\langle y\rangle^{-\alpha}\|a\|^{k,0}_{\beta,\alpha+2}dyd\eta\Big|\\ &\le C|\theta|^{-1}\langle\xi\rangle^{k-\beta}\langle x\rangle^{-\alpha}\|a\|^{k,0}_{\beta,\alpha+2}.\end{aligned}$$

Case 2: $|\theta|\le 1$. Define $L: S^{k,\ell}\to C^\infty(\mathbb{R}^2)$ by

$$L\tilde{a} := i\int_0^x e^{i\theta(s-x)}\tilde{a}(s,\xi)ds.$$

Then, $(D_x+\theta)L\tilde{a}=\tilde{a}$.

Moreover, if $\tilde{a}$ vanishes at $x=0$, then

$$\begin{aligned}L\tilde{a} &= i\int_0^x[\frac{D_s}{\theta}e^{i\theta(s-x)}]\tilde{a}(s,\xi)ds = -i\theta^{-1}\int_0^x e^{i\theta(s-x)}D_s\tilde{a}(s,\xi)ds+\frac{1}{\theta}\tilde{a}(x,\xi)\\ &= -\theta^{-1}LD_x\tilde{a}+\theta^{-1}\tilde{a}\end{aligned}$$

In particular,

$$D_xL\tilde{a}=\tilde{a}-\theta L\tilde{a}=LD_x\tilde{a}$$

Now, suppose that $\tilde{A}\in S^{k,0}$ and $\tilde{a}$ vanishes to infinite order at $x=0$. Then, for $xr\ge 0$ with $|x|\le|r|$

$$|L\tilde{a}(x,\xi)|\le|r|\|\tilde{a}\|^{k,0}_{0,0}\langle\xi\rangle^k$$

For $|x|\ge|r|$,

$$\begin{aligned}\Big|L\tilde{a}(x,\xi)\Big| &\le\Big|\int_0^r e^{i\theta s}\tilde{a}(s,\xi)ds\Big|+|\theta|^{-1}\Big(\Big|\int_r^x e^{i\theta s}D_s\tilde{a}(s,\xi)ds\Big|+|\tilde{a}(x,\xi)|+|\tilde{a}(r,\xi)|\Big)\\ &\le(|r|+2|\theta|^{-1})\|\tilde{a}\|^{k,0}_{0,0}\langle\xi\rangle^k+|\theta|^{-2}\Big(\Big|\int_r^x e^{i\theta s}D_s^2\tilde{a}(s,\xi)ds\Big|+|D_x\tilde{a}(x,\xi)|+|D_x\tilde{a}(r,\xi)|\Big)\\ &\le\langle\xi\rangle^k\big((|r|+2|\theta|^{-1})\|\tilde{a}\|^{k,0}_{0,0}+|\theta|^{-2}\langle\xi\rangle^k(C\|D_x^2\tilde{a}\|^{k,-2}_{0,0}\langle r\rangle^{-1}+2\|D_x\tilde{a}\|^{k,-1}_{0,0}\langle r\rangle^{-1})\big)\end{aligned}$$

Optimizing in r, we obtain $|r|=|\theta|^{-1}$ and in particular,

$$\|L\tilde{a}\|^{k,0}_{0,0}\le C|\theta|^{-1}\|\tilde{a}\|^{k,0}_{0,2}$$

Therefore, since D_ξ commutes with L, if $b\in S^{k,0}$ vanishes to infinite order at $x=0$, we have

$$\|L\tilde{a}\|^{k,0}_{\beta,0}\le C|\theta|^{-1}\|\tilde{a}\|^{k,0}_{\beta,2}$$

Now, consider

$$D_xL\tilde{a}=\theta^{-1}(-LD_x^2\tilde{a}+D_x\tilde{a})$$

and define

$$\tilde{a}_\pm(\xi):=i\int_0^{\pm\infty}e^{i\theta s}D_s^2\tilde{a}(s)ds.$$

Arguing as above, we can see that

$$|\partial_\xi^\beta \tilde{a}_\pm(\xi)| \leq C|\theta| \|\tilde{a}\|_{\beta,2}^{k,0}\langle\xi\rangle^{k-\beta}.$$

Fix $c_\pm(x) \in C_c^\infty$ such that $\int c_\pm dx = 1$, $\operatorname{supp} c_\pm \subset \{\pm x > 0\}$. Then,

$$\int e^{i\theta s} D_s^2 c_\pm(s) ds \geq c|\theta|^2,$$

and putting $\tilde{a}_{mod}(x,\xi) = \tilde{a}(x,\xi) - \frac{c_+(x)\tilde{a}_+(\xi)}{\int e^{i\theta s}D_s^2 c_+(s)ds} - \frac{c_-(x)\tilde{a}_-(\xi)}{\int e^{i\theta s}D_s^2 c_-(s)ds}$, we have

$$\int_0^\infty e^{i\theta s} D_s^2 D_\xi^\beta \tilde{a}_{mod}(x,\xi) ds = \int_0^{-\infty} e^{i\theta s} D_s^2 D_\xi^\beta \tilde{a}_{mod}(s,\xi) ds = 0.$$

Moreover, since $\tilde{a}_{mod}$ vanishes to infinite order at 0, we can integrate by parts to see that

$$\int_0^\infty e^{i\theta s} D_s^k D_\xi^\beta \tilde{a}_{mod} ds = \int_0^{-\infty} e^{i\theta s} D_s^k D_\xi^\beta \tilde{a}_{mod} ds = 0, \qquad k \geq 2.$$

Finally, note that for $\alpha \geq 1$,

$$D_x^\alpha L\tilde{a}_{mod} = \theta^{-1}(-LD_x^{\alpha+1}\tilde{a}_{mod} + D_x^\alpha \tilde{a}_{mod})$$

and we have

$$\Big|\int_0^x e^{is\theta} D_s^{\alpha+1} D_\xi^\beta \tilde{a}_{mod} ds\Big| = \Big|\int_x^{\operatorname{sgn} x\infty} e^{is\theta} D_s^{\alpha+1} D_\xi^\beta \tilde{a}_{mod} ds\Big| \leq C_N \|\tilde{a}\|_{\beta,\alpha+1}^{k.0}\langle x\rangle^{-\alpha}\langle\xi\rangle^{k-\beta}.$$

To complete the proof we let $\chi \in C_c^\infty(\mathbb{R})$ with $\chi \equiv 1$ near 0 and put $\tilde{a} = (1-\chi(x))a(x,\xi)$, $b = L\tilde{a}_{mod}$.

□

Next, we need a lemma which controls scattering symbols after conjugation by $e^{i\theta x}$.

Lemma 4.2. *Suppose that $B \in \Psi^{n,m}$ and $\theta \in \mathbb{R}$, $|\theta| \leq Ch^{-1}$. Then, there is $B_\theta \in \Psi^{n,m}$ such that*

$$e^{i\theta x} B e^{-i\theta x} = B_\theta.$$

and $\mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(B_\theta) = \mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(B)$. Moreover, if $B = b(x,hD)$, then $B_\theta = b_\theta(x,hD)$ where

$$b_\theta(x,\xi) = b(x,\xi - h\theta) \sim \sum_{j=0}^\infty \frac{h^j(-1)^j}{j!}\langle\theta,\partial_\xi\rangle^j b.$$

In particular,

$$\|b - b_\theta\|_{\alpha,\beta}^{n-1,m} \leq \|b\|_{\alpha,\beta+1}^{n,m} h|\theta|\langle h|\theta|\rangle^{n-|\beta|-1}.$$

Proof. Write $B = b(x,hD) + O(h^\infty)_{\Psi^{-\infty,-\infty}}$. Then,

$$e^{i\theta x} b(x,hD) e^{-i\theta x} = b_\theta(x,hD), \qquad b_\theta(x,\xi) = b(x,\xi - h\theta).$$

Now,

$$\begin{aligned}|\partial_x^\alpha \partial_\xi^\beta b_\theta(x,\xi)| &= \left|\partial_y^\alpha \partial_\eta^\beta b(y,\eta)_{y=x,\eta=\xi+h\theta}\right| \leq C_{\alpha\beta}\langle x\rangle^{m-|\alpha|}\langle\xi - h\theta\rangle^{n-|\beta|} \\ &\leq C_{\alpha\beta}\langle h\theta\rangle^{n-|\beta|}\langle x\rangle^{m-|\alpha|}\langle\xi\rangle^{n-|\beta|} \leq \tilde{C}_{\alpha\beta}\langle x\rangle^{m-|\alpha|}\langle\xi\rangle^{n-|\beta|}\end{aligned}$$

and the first part of the lemma follows from Taylor's theorem.

Note also that

$$\begin{aligned}\partial_x^\alpha\partial_\xi^\beta(b(x,\xi)-b_\theta(x,\xi)) &= h\int_0^1 -\langle\partial_x^\alpha\partial_\xi^{\beta+1}b(x,\xi-th\theta),\theta\rangle dt\\ &\leq \|b\|^{n,m}_{\alpha,\beta+1}h|\theta|\langle\xi\rangle^{n-|\beta|-1}\langle x\rangle^{m-|\alpha|}\langle h\theta\rangle^{n-|\beta|-1}\end{aligned}$$

□

Lemma 4.3. *Suppose that $\theta_1,\theta_2\in B(0,Dh^{-1})$ and that $a\in S^{m_1,n_1}$ and $b\in S^{m_2,n_2}$. Then,*

$$h^{-1}[e^{i\theta_1x}Op_h(a),e^{i\theta_2x}Op_h(b)]=e^{i(\theta_1+\theta_2)x}(h^{-1}[Op_h(a),Op_h(b)]+|\theta|c_2(x,hD))$$

where the map $L:S^{m_1,n_1}\times S^{m_2,n_2}\to S^{m_1+m_2-1,n_1+n_2}$, $(a,b)\mapsto c_2$ is bounded uniformly in h with bound depending only on the constant D.

Proof. Note that

$$\begin{aligned}[e^{i\theta_1x}Op_h(a),e^{i\theta_2x}Op_h(b)] &= e^{i(\theta_1+\theta_2)x}(Op_h(a_{-\theta_2})Op_h(b)-Op_h(b_{-\theta_1})Op_h(a))\\ &= e^{i(\theta_1+\theta_2)x}([Op_h(a),Op_h(b)]+(Op_h(a_{-\theta_2}-a))Op_h(b)\\ &\quad -(Op_h(b_{-\theta_1}-b))Op_h(a))).\end{aligned}$$

We now apply Lemma 4.2 to finish the proof. □

Using Lemma 4.3, we can see that if $\Theta_1,\Theta_2\subset B(0,Dh^{-1})$

$$G=\sum_{\theta\in\Theta_1}e^{i\theta x}g_\theta(x,hD),\quad g_\theta\in S^{m_1,n_1}\qquad B=\sum_{\theta\in\Theta_2}e^{i\theta x}b_\theta(x,hD),\quad b_\theta\in S^{m_2,n_2}\tag{4.1}$$

then,

$$h^{-1}[G,B]=\sum_{\theta_i\in\Theta_1,\theta_j\in\Theta_2}e^{i(\theta_1+\theta_2)x}\tilde{g}_{\theta_1,\theta_2}(x,hD)$$

where, for all m_i,n_i, $i=1,2$ and $\alpha,\beta\in\mathbb{N}$, there are $K,C>0$ such that

$$\|\tilde{g}_{\theta_1,\theta_2}\|^{m_1+m_2-1,n_1+n_2}_{\alpha\beta}\leq C(1+\max(|\theta_1|,|\theta_2|)\|g_{\theta_1}\|^{m_1,n_1}_{\beta+K,\alpha+K}\|b_{\theta_2}\|^{m_2,n_2}_{\beta+K,\alpha+K}$$

Thus, applying Lemma 2.3, we have the following lemma:

Lemma 4.4. *Let $G\in\Psi^{-\infty}$ self-adjoint and B are as in (4.1) with $m_1=m_2=-\infty$ and $n_1=n_2=0$. Then,*

$$e^{iG}Be^{-iG}=B+\sum_{j=1}^{k-1}\sum_{\substack{\mathbf{\Phi}\in\Theta_1^j\\ \theta\in\Theta_2}}h^je^{i(\sum_{i=1}^j\mathbf{\Phi_i}+\theta)x}\tilde{g}_{\mathbf{\Phi},\theta}+O(h^k)_{H_h^{-N}\to H_h^N}$$

where for any $\sum_{i=0}^jN_i=N$, α,β there are K and $C_{N\alpha\beta j}$ such that

$$\|\tilde{g}_{\mathbf{\Phi},\theta}\|^{-N,0}_{\beta,\alpha}\leq C_{j\alpha\beta}(1+|\theta|)\|b_\theta\|^{-N_0,0}_{\beta+K,\alpha+K}\prod_{i=1}^j(1+|\mathbf{\Phi}_i|)\|g_{\mathbf{\Phi}_i}\|^{-N_i,0}_{\beta+K,\alpha+K}.$$

4.1. **The gauge transform.** We are now in a position to prove the inductive lemma used for gauge transformation.

Lemma 4.5. *Suppose that* $0 < a < b$ *and*

$$\mathrm{WF_h}\left(\tilde{P} - (P_0 + hQ_k + h^{1+k}W_k + h^N R_k\right) \cap \{|\xi| \in [a,b]\} = \emptyset$$

where $Q_k \in \Psi^{-\infty,0}$, $R_k \in \Psi^{-\infty}$,

$$W_k = \sum_{\theta\in\Theta\setminus 0} e^{i\theta x} w_{\theta,k}(x,hD)$$

with $\{w_{\theta,k}\}_{\theta\in\Theta}$ *satisfying* (3.3) *and* W_k, Q_k *self adjoint. Then there is* $G \in h^{-\delta+k(1-\delta)}S^{-\infty,0}$ *self adjoint such that*

$$\mathrm{WF_h}\left(\tilde{P}_G - (P_0 + hQ_{k+1} + h^{1+k+1}W_{k+1} + h^N R_{k+1})\right) \cap \{|\xi| \in [a,b]\} = \emptyset$$

where $R_{k+1} \in \Psi^{-\infty}$,

$$Q_{k+1} = Q_k + h^{k+1}\tilde{Q}_k \in \Psi^{-\infty,0}$$

with $\tilde{Q}_k$ *self adjoint and* W_{k+1} *is self adjoint with*

$$W_{k+1} = \sum_{\theta\in\Theta_{\lceil \frac{N}{k+1}-1\rceil}\setminus 0} e^{i\theta x} w_{\theta,k+1}(x,hD)$$

satisfies (3.3) *with* Θ, *replaced by*

$$\Theta_{\lceil \frac{N}{k+1}-1\rceil}.$$

Proof. Let $\chi \in C_c^\infty(0,\infty)$ such that $\chi \equiv 1$ near $[1/2,2]$ and

$$\tilde{P}\chi(|hD|) = (P_0 + hQ_k + h^k W_k - h^N R_k)\chi(|hD|) + O(h^\infty)_{\Psi^{-\infty}}$$

We aim to use the fact that P_0 dominates $\tilde{P}$ to conjugate away W_k. Therefore, we look for G such that, modulo lower order terms,

$$ih^{-1-k}[P_0, G] = W_k.$$

To do this, we solve

$$2\xi\partial_x g = \sigma_{-\infty}(W_k\chi(|hD|)).$$

Now,

$$W_k\chi(|hD|) = \sum_{\theta\in\Theta\setminus\{0\}} e^{i\theta x}(w_\theta\chi(|\xi|))(x,hD)$$

where $w_\theta \in S^{-\infty,0}$ satisfy (3.3). Let $\chi_i \in C_c^\infty(0,\infty)$, $i = 1,2$, such that $\chi_1, \chi_2 \equiv 1$ near $[a,b]$ and $\operatorname{supp}\chi_2 \subset \operatorname{supp}\chi_1 \subset \operatorname{supp}\chi$. By, Lemma 4.1, there is $g_\theta \in S^{-\infty,0}$ such that

$$(D_x + \theta)g_\theta(x,\xi) - iw_{\theta,k}\chi_1(|\xi|)/2\xi \in S^{-\infty,-\infty}, \qquad \|g_\theta\|_{\beta,\alpha}^{-N,0} \le C_{\alpha\beta N}|\theta|^{-1}\|w_\theta\chi_1\|_{\beta,\alpha+2}^{-N,0}.$$

Modifying lower order terms in g_θ to make $e^{i\theta x}g_\theta + e^{-i\theta x}g_{-\theta}$ self adjoint, we put

$$G := h^k \sum_{\theta\in\Theta\setminus\{0\}} e^{i\theta x} g_\theta(x,hD).$$

Then, $G\in h^kS^{-\infty}$, and, letting $\tilde{k}=(k+1)$, by Lemma 4.4, for any N_1

$$\chi_2(|hD|)\tilde{P}_G=\chi_2(|hD|)(P_0+hQ_k)+\sum_{j=2}^{N_1-1}\sum_{\mathbf{\Phi}\in\Theta^j}e^{i(\sum_{i=1}^j\mathbf{\Phi_i})x}h^{j\tilde{k}}\tilde{g}^1_{\mathbf{\Phi}}(x,hD)$$
$$+\sum_{j=1}^{N_1-1}\sum_{\mathbf{\Phi}\in\Theta^j}h^{j\tilde{k}+1}e^{i(\sum_{i=1}^j\mathbf{\Phi_i})x}\tilde{g}^2_{\mathbf{\Phi}}(x,hD)+O(h^{N_1\tilde{k}})_{H_h^{-N}\to H_h^N}+O(h^N)_{H_h^{-N}\to H_h^N}$$

where for $\mathbf{\Phi}\in\Theta^n$,

$$\|\tilde{g}^\ell_{\mathbf{\Phi}}\|^{-N,0}_{\alpha\beta}\leq C_{j\alpha\beta N}(1+\|Q_k\|^{-N,0}_{\alpha+K,\beta+K})\prod_{i=1}^n(1+|\mathbf{\Phi}_i|)|\mathbf{\Phi}_i|^{-1}\|w_{\mathbf{\Phi}_i}\chi_1\|^{-N,0}_{\beta+K,\alpha+K+2}.$$

In particular, putting $N_1=\lceil\frac{N}{k+1}\rceil$, and

$$W_{k+1}=\sum_{j=2}^{N_1-1}\sum_{\substack{\mathbf{\Phi}\in\Theta^j\\ \sum\mathbf{\Phi}_i\neq0}}e^{i(\sum_{i=1}^j\mathbf{\Phi_i})x}h^{j\tilde{k}}\tilde{g}^1_{\mathbf{\Phi}}(x,hD)+\sum_{j=1}^{N_1-1}\sum_{\substack{\mathbf{\Phi}\in\Theta^j\\ \sum\mathbf{\Phi_i}\neq0}}h^{j\tilde{k}+1}e^{i(\sum_{i=1}^j\mathbf{\Phi_i})x}\tilde{g}^2_{\mathbf{\Phi}}(x,hD)$$

and

$$Q_{k+1}=Q_k+\sum_{j=2}^{N_1-1}\sum_{\substack{\mathbf{\Phi}\in\Theta^j\\ \sum\mathbf{\Phi}_i=0}}h^{j\tilde{k}}\tilde{g}^1_{\mathbf{\Phi}}(x,hD)+\sum_{j=1}^{N_1-1}\sum_{\substack{\mathbf{\Phi}\in\Theta^j\\ \sum\mathbf{\Phi_i}=0}}h^{j\tilde{k}+1}\tilde{g}^2_{\mathbf{\Phi}}(x,hD)$$

we have by Lemma 3.2 that W_{k+1} satisfies (3.3) with Θ replaced by $\Theta=\Theta_{\lceil N/(k+1)-1\rceil}$. □

The following is now an immediate corollary of the previous lemma

Corollary 4.6. *Let $P=-h^2\Delta+hW$ where W is admissible and $0<a<b$. Then for all N there is $G\in\Psi^0$ self-adjoint such that*

$$e^{iG}Pe^{-iG}=-h^2\Delta+hQ+(1-\chi(h^2\Delta-1))h\tilde{W}(1-\chi(h^2\Delta-1))+O(h^N)_{\Psi^{-\infty}}$$

where $Q\in\Psi^{-\infty,0}$, $\tilde{W}\in\Psi^1$, are self adjoint, and $\chi\in C_c^\infty$ with $\chi\equiv1$ on $[a,b]$.

5. Limiting absorption for the gauge transformed operator

Throughout this section, we work with an operator

$$\begin{gathered}P=P_0+h(1-\chi(-h^2\Delta-1))W(x,hD)(1-\chi(-h^2\Delta-1))),\\ P_0\in S^{2,0},\ \sigma_{2,0}(P_0)=|\xi|^2,\qquad\sigma_{1,-1}(h^{-1}\operatorname{Im}P_0)=0.\end{gathered}\tag{5.1}$$

where $\chi\in C_c^\infty(\mathbb{R})$ with $\chi\equiv1$ in a neighborhood of $[-\delta,\delta]$. and $W\in\Psi^1$. We will show that for $E\in[1-\delta,1+\delta]$, $R_\pm(E):=(P-E\mp i0)^{-1}$ exist as limiting absorption type limits. Moreover, we will show that $R_\pm(E)$ satisfy certain outgoing/incoming properties.

Throughout this section, we let $\chi_i\in C_c^\infty(\mathbb{R})$ $i=1,2,3$ with

$$\begin{gathered}\chi_i\equiv1\text{ near }[-\delta,\delta],\qquad\operatorname{supp}\chi_i\subset\{\chi_{i-1}\equiv1\},\ i=2,3,\qquad\operatorname{supp}\chi_1\subset\{\chi\equiv1\},\\ \psi_i:=(1-\chi_i((-h^2\Delta-1))),\qquad X_i:=\chi_i((-h^2\Delta-1))\end{gathered}\tag{5.2}$$

5.1. **Elliptic Estimates.** We first obtain estimates in the elliptic region where the perturbation of P_0 is supported.

Lemma 5.1. *With ψ_i as in* (5.2),

$$c\|\psi_2 u\|^2_{H_h^{s+2,k}} \leq \|\psi_3(P-E\pm i\varepsilon)u\|_{H_h^{s,k}} + Ch^N\|u\|^2_{H_h^{-N,-N}}. \tag{5.3}$$

Proof. Observe that

$$\psi_i(P-E) = \psi_i(P_0-E) + h(1-\chi(-h^2\Delta-1))W(x,hD)(1-\chi(-h^2\Delta-1)))$$

since $\psi_i(1-\chi(-h^2\Delta-1))) = (1-\chi(-h^2\Delta-1))$. Note that $\mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(\psi_2)\subset \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(\psi_3(P_0-E))$, and hence by Lemma 2.4, for $\varepsilon>0$,

$$\begin{aligned}\|\psi_3(P-E\pm i\varepsilon)u\|_{H_h^{s,k}} &\geq \|\psi_3(P_0-E\pm i\varepsilon)u\|^2_{H_h^{s,k}} - Ch\|(1-\chi)u\|_{H_h^{s+1,k}}\\ &\geq c\|\psi_2 u\|_{H_h^{s+2,k}} - Ch\|(1-\chi)u\|_{H_h^{s+1,k}} - Ch^N\|u\|_{H_h^{-N,-N}}\\ &\geq c\|\psi_2 u\|^2_{H_h^{s+2,k}} - Ch^N\|u\|^2_{H_h^{-N,-N}}.\end{aligned}$$

Here, in the last line we have used that $(1-\chi) = (1-\chi)\psi_2$. □

5.2. **Propagation estimates.** Consider $\tilde{P}_E := \langle x\rangle^{1/2}(P_0-E)\langle x\rangle^{1/2}$ so that $\tilde{P}_E\in\Psi^{-\infty,1}$ is self-adioint and

$$\sigma_{2,1}(\tilde{P}_E) = \langle x\rangle(\xi^2-E) =: \tilde{p}.$$

Note that

$$H_{\tilde{p}} = 2\xi\langle x\rangle\partial_x - (\xi^2-E)x\langle x\rangle^{-1},$$

and therefore, letting,

$$\begin{aligned} L_+ &= \bigcup_{\pm} L_{+,\pm}, \qquad L_{+,\pm} := \{\xi = \pm\sqrt{E}, x = \pm\infty\},\\ L_- &= \bigcup_{\pm} L_{-,\pm}, \qquad L_{-,\pm} := \{\xi = \mp\sqrt{E}, x = \pm\infty\},\end{aligned}$$

we have that $L_{+,\pm}$ are radial sinks for $\tilde{p}$ and $L_{-,\pm}$ are radial sources (see [DZ19a, Definition E.50]).

Lemma 5.2. *Let $B_+, B_-\in\Psi^{\mathrm{comp},0}$,*

$$L_\pm\subset \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(B_\pm), \qquad \mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(B_\pm)\cap L_\mp = \emptyset, \qquad \{p=E\}\subset(\mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(B_-)\cup \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(B_+)) \tag{5.4}$$

and $B'_\pm\in\Psi^{\mathrm{comp},0}$ with the same property, and $\mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(B'_\pm)\subset \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(B_\pm)$. Then, for all $k_+<-\frac12$ and $k_->k'_->-\frac12$, and N there is $C>0$ and $\delta>0$ such that for $\varepsilon\geq 0$, $E\in[1-\delta,1+\delta]$, and $u\in\mathcal{S}'(\mathbb{R})$ with $B_\pm(P_0-E-i\varepsilon)\in H_h^{0,k_\pm}$, and $B_-u\in H_h^{0,k'_-}$,

$$\begin{aligned}&\|B'_+u\|_{H_h^{0,k_+}} + \|B'_-u\|_{H_h^{0,k_-}}\\ &\qquad\leq Ch^{-1}(\|B_+(P_0-E-i\varepsilon)\|_{H_h^{0,k_++1}} + \|B_-(P_0-E-i\varepsilon)u\|_{H_h^{0,k_-+1}}) + Ch^N\|u\|_{H_h^{-N,-N}}.\end{aligned}$$

Similarly, for all $\tilde{k}_+ > \tilde{k}_+' > -\frac{1}{2}$ *and* $\tilde{k}_- < -\frac{1}{2}$*, and* N *there are* $C > 0$ *and* $\delta > 0$ *such that for* $\varepsilon \geq 0$*,* $E \in [1-\delta, 1+\delta]$*, and* $u \in \mathcal{S}'(\mathbb{R})$ *with* $B_\pm(P_0 - E + i\varepsilon) \in H_h^{0,\tilde{k}_\pm}$*, and* $B_+ u \in H_h^{0,\tilde{k}_+'}$*,*

$$\begin{aligned}&\|B_+' u\|_{H_h^{0,\tilde{k}_+}} + \|B_-' u\|_{H_h^{0,\tilde{k}_-}} \\ &\qquad \leq Ch^{-1}(\|B_+(P_0 - E + i\varepsilon)\|_{H_h^{0,\tilde{k}_+ + 1}} + \|B_-(P_0 - E + i\varepsilon)u\|_{H_h^{0,\tilde{k}_- + 1}}) + Ch^N \|u\|_{H_h^{-N,-N}}.\end{aligned}$$

Proof. Let $\tilde{B}_- \in \Psi^{\mathrm{comp},0}$ such that $L_- \subset \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(\tilde{B}_-)$, $\mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(\tilde{B}_-) \subset \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(B_-)$. Then, by Lemma 2.6 there is $A_- \in \Psi^{\mathrm{comp},0}$ such that $L_- \subset \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(A_-)$ and for all $\tilde{k}_- > \tilde{k}_-' > 0$, $\varepsilon \geq 0$ and $v \in \mathcal{S}'(\mathbb{R})$ with $\tilde{B}_- v \in H_h^{0,\tilde{k}_-'}$, $\tilde{B}_-(\tilde{P}_E - i\varepsilon\langle x\rangle)v \in H_h^{0,\tilde{k}_-}$,

$$\|A_- v\|_{H_h^{0,\tilde{k}_-}} \leq Ch^{-1}\|\tilde{B}_-(\tilde{P}_E - i\varepsilon\langle x\rangle)v\|_{H_h^{0,\tilde{k}_-}} + C_N h^N \|v\|_{H_h^{-N,-N}}. \tag{5.5}$$

Next, let $\tilde{B}_+ \in \Psi^{\mathrm{comp},0}$ such that $L_+ \subset \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(\tilde{B}_+)$, $\mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(\tilde{B}_+) \subset \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(B_+)$. Then, by Lemma 2.7 there are $A_+, B \in \Psi^{\mathrm{comp},0}$ such that $L_+ \subset \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(A_+)$, $\mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(B) \subset \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(\tilde{B}_+) \setminus L_+$, and for all $\tilde{k}_+ < 0$, $\varepsilon \geq 0$ and $v \in \mathcal{S}'(\mathbb{R})$ with $Bv \in H_h^{0,\tilde{k}_+}$, $\tilde{B}_+(\tilde{P}_E - i\varepsilon\langle x\rangle)v \in H_h^{0,\tilde{k}_+}$,

$$\|A_+ v\|_{H_h^{0,\tilde{k}_+}} \leq C\|Bv\|_{H_h^{0,\tilde{k}_+}} + Ch^{-1}\|\tilde{B}_+(\tilde{P}_E - i\varepsilon\langle x\rangle)v\|_{H_h^{0,\tilde{k}_+}} + C_N h^N \|v\|_{H_h^{-N,-N}}. \tag{5.6}$$

Finally, let $B_0 \in \Psi^{\mathrm{comp},0}$ with $\mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(B_0) \subset \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(B_+)$,

$$\{\tilde{p} = 0\} \subset \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(B_0) \cup \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(B_-').$$

Then, there is $A_0 \in \Psi^{\mathrm{comp},0}$ such that $\mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(A_0) \cap (L_+ \cup L_-) = \emptyset$ and there is $T > 0$ with

$$\mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(A_0) \subset \bigcup_{0 \leq t \leq T} \varphi_t(\mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(A_-)) \cap \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(B_0), \tag{5.7}$$

$$\{\langle x\rangle^{-1}\tilde{p} = 0\} \subset \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(A_0) \cup \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(A_-) \cup \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(A_+). \tag{5.8}$$

Now, by (5.7) and Lemma 2.5 for all $\varepsilon \geq 0$, and $u \in \mathcal{S}'(\mathbb{R})$ such that $A_- v \in H_h^{0,\tilde{k}_-}$, $B_0(\tilde{P}_E - i\varepsilon\langle x\rangle)v \in H_h^{0,\tilde{k}_-}$,

$$\|A_0 v\|_{H_h^{0,\tilde{k}_-}} \leq C\|A_- u\|_{H_h^{0,\tilde{k}_-}} + Ch^{-1}\|B_0(\tilde{P}_E - i\varepsilon\langle x\rangle)v\|_{H_h^{0,\tilde{k}_-}} + C_N h^N \|v\|_{H_h^{-N,-N}}. \tag{5.9}$$

Next, observe that if $B_i \in \Psi^{\mathrm{comp},0}$ with $\mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(B_1) \subset \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(B_2)$, then there is $C_{k,s} > 0$ such that for all $w \in \mathcal{S}'(\mathbb{R})$ with $B_2 w \in H_h^{0,k+s}$,

$$\|B_1\langle x\rangle^s w\|_{H_h^{0,k}} \leq C\|B_2 w\|_{H_h^{0,k+s}},$$

Combining (5.5), (5.6), (5.9), and using (5.8) and Lemma 2.4 finishes the proof of the first inequality after putting $v = \langle x\rangle^{-1/2}u$ and letting $\tilde{k}_+ = k_+ + \frac{1}{2} =$, $\tilde{k}_- = k_- + \frac{1}{2}$.

The second inequality follows by replacing $\tilde{P}$ by $-\tilde{P}$. ∎

Now, for each $\Gamma \subset^{\mathrm{sc}} \overline{T^*\mathbb{R}}$, let $B_\Gamma \in \Psi^{0,0}$ such that $\mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(B_\Gamma) \subset \overline{\Gamma}$, $\mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(B_\Gamma) = \Gamma^o$. Then, for $k_\Gamma \geq k$, $s \in \mathbb{R}$ define the norm,

$$\|u\|_{\mathcal{X}_\Gamma^{s,k_\Gamma,k}} := \|B_\Gamma u\|_{H_h^{s,k_\Gamma}} + \|u\|_{H_h^{s,k}}.$$

Lemma 5.3. *For $k_- > -\frac{1}{2}$, $k_+ < -\frac{1}{2}$, $\Gamma_-, \Gamma'_- \subset^{\mathrm{sc}} \overline{T^*\mathbb{R}}$ open with $L_- \subset \Gamma \Subset \Gamma' \Subset \{\chi_3(|\xi|^2 - 1) \equiv 1\} \setminus L_+$, there is $h_0 > 0$ such that for all $u \in \mathcal{X}_{\Gamma,+}^{s,k_-,k_+}$, $\varepsilon > 0$, and $0 < h < h_0$*

$$\|u\|_{\mathcal{X}_{\Gamma_-}^{s,k_-,k_+}} \leq Ch^{-1}\|(P - E - i\varepsilon)u\|_{\mathcal{X}_{\Gamma'_-}^{s-2,k_-+1,k_++1}}.$$

For $k_- < -\frac{1}{2}$, $k_+ > -\frac{1}{2}$ and $\Gamma_+, \Gamma'_+ \subset^{\mathrm{sc}} \overline{T^*\mathbb{R}}$ open with $L_+ \subset \Gamma_+ \Subset \Gamma'_+ \Subset \{\chi_3(|\xi|^2 - 1) \equiv 1\} \setminus L_-$, there is $h_0 > 0$ such that for all $u \in \mathcal{X}_{\Gamma,-}^{s,k_-,k_+}$, $\varepsilon > 0$, and $0 < h < h_0$

$$\|u\|_{\mathcal{X}_{\Gamma_+}^{s,k_+,k_-}} \leq Ch^{-1}\|(P - E - i\varepsilon)u\|_{\mathcal{X}_{\Gamma'_+}^{s-2,k_++1,k_-+1}}.$$

Proof. Put $f_\varepsilon = (P - E - i\varepsilon)u$. Let $\Gamma_- \Subset \Gamma_1 \Subset \Gamma_2 \Subset \Gamma'_-$ and A_{Γ_1}, $A_{\Gamma_2} \in \Psi^{\mathrm{comp},0}$ such that

$$\Gamma_- \Subset \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(A_{\Gamma_1}) \subset \mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(A_{\Gamma_1}) \subset \Gamma_1 \subset \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(A_{\Gamma_2}) \subset \mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(A_{\Gamma_2}) \subset \Gamma_2,$$
$$\mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(\mathrm{Id} - A_{\Gamma_i}) \cap L_- = \emptyset, \qquad \mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(\mathrm{Id} - A_{\Gamma_1}) \subset \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(\mathrm{Id} - A_{\Gamma_2}).$$

Next, define

$$B_+ := (\mathrm{Id} - A_{\Gamma_1})X_1, \qquad B_- := A_{\Gamma_2}X_1, \qquad B'_+ := (\mathrm{Id} - A_{\Gamma_2})X_2, \qquad B'_- := A_{\Gamma_1}X_2. \tag{5.10}$$

Then, (5.4) is satisfied and by Lemma 5.2 together with the fact that $X_2P = X_2P_0$,

$$\begin{aligned}\|B'_-u\|_{H_h^{0,k_-}} + \|B'_+u\|_{H_h^{0,k_+}} &\leq Ch^{-1}(\|B_+f_\varepsilon\|_{H_h^{0,k_++1}} + \|B_-f_\varepsilon\|_{H_h^{0,k_-+1}}) + Ch^N\|u\|_{H_h^{-N,-N}} \\ &\leq Ch^{-1}(\|f_\varepsilon\|_{H_h^{0,k_++1}} + \|B_-f_\varepsilon\|_{H_h^{0,k_-+1}}) + Ch^N\|u\|_{H_h^{-N,-N}}\end{aligned} \tag{5.11}$$

Now, since $\mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(A_{\Gamma_i}) \subset \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(B_{\Gamma'_-})$, we have by Lemma 2.4

$$\begin{aligned}\|B_-f_\varepsilon\|_{H_h^{s,k_-+1}} + \|A_{\Gamma_1}f_\varepsilon\|_{H_h^{s,k_-+1}} &\leq C\|B_{\Gamma'_-}f_\varepsilon\|_{H_h^{s,k_-+1}} + C_Nh^N\|f_\varepsilon\|_{H_h^{-N,-N}} \\ &\leq C\|B_{\Gamma'_-}f_\varepsilon\|_{H_h^{s,k_-+1}} + C_Nh^N\|u\|_{H_h^{-N,-N}}.\end{aligned} \tag{5.12}$$

Next, since $\mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(A_{\Gamma_2}) \cap \mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(\mathrm{Id} - X_3) = \emptyset$, and $\mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(\mathrm{Id} - X_2) \subset \mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(\mathrm{Id} - X_3)$, we have by (5.10), (5.11) and (5.12) that

$$\begin{aligned}\|A_{\Gamma_1}u\|_{H_h^{s,k_-}} &\leq C_s\|A_{\Gamma_1}X_2u\|_{H_h^{0,k_-}} + \|A_{\Gamma_1}(\mathrm{Id} - X_2)u\|_{H_h^{s,k_-}} \\ &\leq Ch^{-1}\|B_{\Gamma'_-}f_\varepsilon\|_{H_h^{0,k_-+1}} + \|f_\varepsilon\|_{H_h^{0,k_++1}} + Ch^N\|u\|_{H_h^{-N,-N}} \\ &\leq Ch^{-1}\|f_\varepsilon\|_{\mathcal{X}_{\Gamma'_-}^{s-2,k_-+1,k_++1}} + Ch^N\|u\|_{H_h^{-N,-N}}.\end{aligned} \tag{5.13}$$

Now, since $\mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(X_2) \subset \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(X_1)$, and $\{p = E\} \subset \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(\mathrm{Id} - A_{\Gamma_2}) \cup \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(A_{\Gamma_1})$,

$$\mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(X_2) \setminus (\mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(\mathrm{Id} - A_{\Gamma_2}) \cup \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(A_{\Gamma_1})) \subset \mathrm{ell}_{\mathrm{h}}(X_1(P_0 - E - i\varepsilon)),$$

with uniform bounds in $\varepsilon \geq 0$. Therefore, using (5.11), (5.12) together with the the elliptic estimate from Lemma 2.4, we have

$$\begin{aligned}\|X_2u\|_{H_h^{0,k_+}} &\leq Ch^{-1}\|f_\varepsilon\|_{\mathcal{X}_{\Gamma'_-}^{s-2,k_-+1,k_++1}} + C\|X_1f_\varepsilon\|_{H_h^{0,k_+}} + Ch^N\|u\|_{H_h^{-N,-N}}\\ &\leq Ch^{-1}\|f_\varepsilon\|_{\mathcal{X}_{\Gamma'_-}^{s-2,k_-+1,k_++1}} + Ch^N\|u\|_{H_h^{-N,-N}}.\end{aligned}$$

So, using (5.3),

$$\|u\|_{H_h^{s,k_+}} \leq Ch^{-1}\|f_\varepsilon\|_{\mathcal{X}_{\Gamma'_-}^{s-2,k_-+1,k_++1}} + Ch^N\|u\|_{H_h^{-N,-N}}. \tag{5.14}$$

For h small enough, the first part of the lemma follows from (5.13) and (5.14), the fact that $\mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(B_{\Gamma_-}) \subset \mathrm{ell}_{\mathrm{h}}^{\mathrm{sc}}(A_{\Gamma_1})$, and the elliptic estimate (Lemma 2.4). The second claim follows from an identical argument. □

5.3. **The limiting absorption principle and the outgoing property.** We are now in a position to prove the limiting absorption principle. For this, we define $R(\lambda) := (P-\lambda)^{-1} : H_h^{s,k} \to H_h^{s+2,k}$ for $\operatorname{Im}\lambda \neq 0$.

Lemma 5.4. *Let Γ_- be a neighborhood of L_- satisfying the assumptions of Lemma 5.3, $k_- > \frac{1}{2}$, $k_+ < -\frac{1}{2}$, $s \in \mathbb{R}$, and $E \in [1-\delta, 1+\delta]$, the strong limit $R(E+i0) : H_h^{s,k_-} \to \mathcal{X}_{\Gamma_-}^{s+2,k_--1,k_+}$ exists and satisfies the bound*

$$\|R(E+i0)f\|_{\mathcal{X}_{\Gamma_-}^{s+2,k_--1,k_+}} \leq Ch^{-1}\|f\|_{H_h^{s,k_-}}.$$

Similarly, for Γ_+ a neighborhood of L_+ satisfying the assumptions of Lemma 5.3, $k_- < -\frac{1}{2}$, $k_+ > \frac{1}{2}$, $s \in \mathbb{R}$, and $E \in [1-\delta, 1+\delta]$, the strong limit $R(E-i0) : H_h^{s,k_+} \to \mathcal{X}_{\Gamma_+}^{s+2,k_+-1,k_-}$ exists and satisfies the bound

$$\|R(E-i0)f\|_{\mathcal{X}_{\Gamma_+}^{s+2,k_+-1,k_-}} \leq Ch^{-1}\|f\|_{H_h^{s,k_+}}.$$

Proof. We start by showing that for $k_- > \frac{1}{2}$, $k_+ < -\frac{1}{2}$, $R(E+i\varepsilon) := (P-E-i\varepsilon)^{-1} : H_h^{s,k_-} \to H_h^{s+2,k_+}$ converges as $\varepsilon \to 0^+$. First, note that for each fixed $\varepsilon > 0$, $R(E+i\varepsilon) : H_h^{s,k} \to H_h^{s+2,k}$ is well defined. Let Γ'_- be a neighborhood of L_- with $\Gamma_- \Subset \Gamma'_-$.

Suppose there is $f \in H_h^{s,k_-}$ such that $R(E+i\varepsilon)f$ is not bounded in H_h^{s+2,k_+}. Then, there are $\varepsilon_n \to 0^+$ such that, defining $u_n := R(E+i\varepsilon_n)f \in H_h^{s+2,k_-}$, we have $\|u_n\|_{H_h^{s+2,k_+}} \to \infty$. Putting $v_n = u_n/\|u_n\|_{H_h^{s+2,k_+}}$, we have that $\|v_n\|_{H_h^{s+2,k_+}} = 1$ is bounded and $(P-E-i\varepsilon_n)v_n = f/\|u_k\|_{H_h^{s,k_+}} \to 0$ in H_h^{s,k_-}.

Since $f \in H_h^{s,k_-}$, for all $\varepsilon > 0$, $R(E+i\varepsilon) : H_h^{s,k} \to H_h^{s+2,k}$, and $k_- - 1 > k_+$, we have $v_n \in \mathcal{X}_{\Gamma_-}^{s+2,k_--1,k_+}$ and $f \in \mathcal{X}_{\Gamma'_-}^{s,k_-,k_++1}$. Therefore, by Lemma 5.3 all n,

$$\begin{aligned}\|v_n\|_{H_h^{s+2,k_+}} &\le C\|v_n\|_{\mathcal{X}_{\Gamma_-}^{s+2,k_--1,k_+}} \\ &\le Ch^{-1}\|f\|_{\mathcal{X}_{\Gamma'_-}^{s+2,k_-,k_++1}}/\|u_k\|_{H_h^{s+2,k_+}} \le Ch^{-1}\|f\|_{H_h^{s,k_-}}/\|u_k\|_{H_h^{s+2,k_+}} \to 0\end{aligned}$$

which contradicts the fact that $\|v_n\|_{H_h^{s+2,k_+}} = 1$. In particular, $u = R(E+i\varepsilon)f$ is uniformly bounded in H_h^{s+2,k_+}, and, arguing as above

$$\|u\|_{\mathcal{X}_{\Gamma_-}^{s+2,k_--1,k_+}} \le Ch^{-1}\|f\|_{H_h^{s,k_-}}.$$

Now, we show that $R(E+i\varepsilon)f$ converges as $\varepsilon \to 0^+$. To see this, first take any sequence $\varepsilon_n \to 0^+$. Then, $R(E+i\varepsilon_n)f$ is bounded in $X_{\Gamma_-}^{s+2,k_--1,k_+}$ and hence, for any $s' < s$, $\frac{1}{2} < k' < k_-$, and $k'_+ < k_+$, we may extract a subsequence and assume that $u_n = R(E+i\varepsilon_n)f \to u$ in $X_{\Gamma_-}^{s'+2,k'_--1,k'_+}$, $(P-E)u = f$, and $u_n \rightharpoonup u$ in $X_{\Gamma_-}^{s+2,k_--1,k_+}$.

Suppose that there is another sequence which converges to $u' \in X_{\Gamma_-}^{s'+2,k'_--1,k'_+}$ and satisfies $(P-E)u' = f$. But then we have

$$\|u-u'\|_{H_h^{s'+2,k'_+}} \le C\|u-u'\|_{\mathcal{X}_{\Gamma_-}^{s'+2,k'_--1,k'_+}} \le Ch^{-1}\|(P-E)(u-u')\|_{\mathcal{X}_{\Gamma'_-}^{s'+2,k'_-+1,k'_++1}} = 0,$$

so $u = u'$. Now, suppose that there is a sequence $\varepsilon_m \to 0^+$ such that $u''_m := R(E+i\varepsilon_m)f$ does not converge to u in $X_{\Gamma_-}^{s+2,k_--1,k_+}$. Then, extracting a subsequence we may assume that $u''_m \to u'' \in X_{\Gamma_-}^{s'+2,k'_--1,k'_+}$ and hence $u = u''$, which is a contradiction. In particular, $R(E+i\varepsilon)f \to u$ in $\mathcal{X}_{\Gamma_-}^{s+2,k_--1,k_+}$ as $\varepsilon \to 0^+$. Boundedness of the operator follows from the above estimates. Moreover, we see that if $f \in H_h^{s,k_-}$, for some $k_- > \frac{1}{2}$, then $R(E+i0)f \in \mathcal{X}_{\Gamma_-}^{s+2,k_--1,k_+}$, for any $k_+ < -\frac{1}{2}$.

The case of $R(E-i\varepsilon)$ follows by an identical argument. □

Finally, we are in a position to prove that the limiting absorption resolvent satisfies the outgoing/incoming property.

Lemma 5.5. *For $f \in \mathcal{E}'(\mathbb{R})$,*

$$\mathrm{WF_h}(R(E \pm i0))f \subset \mathrm{WF_h}(f) \cup \bigcup_{\pm t \ge 0} \exp(tH_{|\xi|^2})\big(\mathrm{WF_h}(f) \cap \{|\xi|^2 = E\}\big).$$

Proof. First, note that for $A \in \Psi^{0,\mathrm{comp}}$ with $\mathrm{WF_h}(A) \subset \mathrm{ell_h}(B) \cap \mathrm{ell_h}(P)$, and

$$\|Au\|_{H_h^{k,s}} \le C\|BPu\|_{H_h^{k-2,s}} + Ch^N\|u\|_{H_h^{-N,-N}}.$$

Therefore, letting $v_\pm = R(E \pm i0)f$, we have

$$\mathrm{WF_h}(v_\pm) \cap \{|\xi|^2 \ne E\} \subset \mathrm{WF_h}(f).$$

Next, note that since $f \in H_h^{-N,\infty}$, for some N, by Lemma 5.4 we have $v_+ \in \mathcal{X}_{\Gamma_-}^{-N+2,\infty,k}$, and $v_- \in \mathcal{X}_{\Gamma_+}^{-N+2,\infty,k}$, for any $k < -\frac{1}{2}$ and any $\Gamma_\pm$ open neighborhoods of $L_\pm$ such that $\Gamma_\pm \cap L_\mp = \emptyset$. In particular, by Lemmas 2.5, 2.6, and 2.7, together with the fact that $X_3 P \in \Psi^{2,0}$,

$$\mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(R(E \pm i0)f) \cap \{|\xi|^2 = E\} \subset \bigcup_{\pm t \geq 0} \exp(tH_p)\Big(\mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(f) \cap \{|\xi|^2 = E\}\Big) \cup L_\pm.$$

Next, since $f \in \mathcal{E}'$, $\mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(f) \subset \{|x| \leq C\}$ and in particular, $\mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(f) = \mathrm{WF}_{\mathrm{h}}(f)$. Therefore, the claim follows. □

Using the outgoing property, we can write an effective expression for the incoming/outgoing resolvent (see also [DZ19a, Lemma 3.60])

Lemma 5.6. *Let $R > 0$. Then there is $T > 0$ such that for all $f \in \mathcal{E}'$ supported in $B(0,R)$ and $B \in \Psi^{\mathrm{comp},-\infty}$ with $\mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(B) \subset T^*B(0,R) \cap \{1/2 \leq |\xi| \leq 2\}$, and $\chi \in C_c^\infty(B(0,R))$,*

$$\chi R(E \pm i0)B = \frac{i}{h}\int_0^{\pm T} \chi e^{-it(P-E)/h} Bf + O(h^\infty)_{\mathcal{D}' \to C_c^\infty}.$$

Proof. Let $\psi \in C_c^\infty(\mathbb{R})$ such that $\psi \equiv 1$ on $B(0, R + 10T)$. Let

$$v = R(E + i0)Bf - ih^{-1}\int_0^T \psi e^{-it(P-E)/h} Bf.$$

Then,

$$\begin{aligned}(P - E)v &= Bf - ih^{-1}\int_0^T (hD_t\psi + [P, \psi])e^{-it(P-E)/h}Bf dt \\ &= \psi e^{-iT(P-E)/h}Bf - ih^{-1}\int_0^T [P, \psi]e^{-it(P-E)/h}Bf dt + O(h^\infty)_{\Psi^{-\infty,-\infty}} f.\end{aligned}$$

Now,

$$\mathrm{WF}_{\mathrm{h}}(e^{-it(P-E)/h}B) \subset \{(x + 2t\xi, \xi) \mid |x| \leq R, |\xi| \in [1/2, 2]\}.$$

In particular, for $t \in [0, T]$,

$$[P, \psi]e^{-it(P-E)/h}Bf = O(h^\infty)_{C_c^\infty}$$

and we have

$$(P - E)v = \psi e^{-iT(P-E)/h}Bf + O(h^\infty)_{C_c^\infty}.$$

Since $v - R(E + i0)Bf \in C_c^\infty$, for any Γ_- a neighborhood of L_- satisfying the assumptions of Lemma 5.3, $v \in \mathcal{X}_{\Gamma_-}^{s,k_-,k_+}$ for all s, k_- and $k_+ < -\frac{1}{2}$ and hence

$$v = R(E + i0)(\psi e^{-iT(P-E)/h}Bf + O(h^\infty)_{C_c^\infty}).$$

But then,

$$\begin{aligned}\mathrm{WF}_{\mathrm{h}}(v) &\subset \{(x + 2t\xi, \xi) \mid t \geq 0,\, (x, \xi) \in \mathrm{WF}_{\mathrm{h}}(\psi e^{iT(P-E)/h}B)\} \\ &\subset \{(x + 2(t + T)\xi, \xi) \mid |x| \leq R, \xi \in [1/2, 2]\} \\ &\subset \{(x, \xi) \mid B(0, R + 4T) \setminus B(0, T - 2R),\, \xi \in [1/2, 2]\}.\end{aligned}$$

In particular, for $T > 3R$, $\chi v = O(h^\infty)_{C_c^\infty}$ and hence

$$\chi R(E+i0)Bf = \frac{i}{h}\int_0^T \chi e^{-it(P-E)/h}Bf dt + O(h^\infty)_{C_c^\infty}$$

as claimed. The proof for $R(E-i0)$ is identical.

□

6. Completion of the proof of Theorem 3.1

We now complete the proof of the main theorem. Let $P = -h^2\Delta + hW$ where W is admissible (i.e. satisfies (3.2) and (3.3)). Let $0<\delta<\delta'<1$. Then by Corollary 4.6, for any $N>0$, there is $G\in\Psi^0$ self adjoint such that

$$P_G := e^{iG}Pe^{-iG} = -h^2\Delta + hQ + (1-\chi(h^2\Delta-1))h\tilde{W}(1-\chi(h^2\Delta-1)) + R_N$$

where $Q\in\Psi^{-\infty,0}$, $\tilde{W}\in\Psi^1$, are self adjoint, $R_N = O(h^{3N})_{\Psi^{-\infty}}$, and $\chi\in C_c^\infty$ with $\chi\equiv 1$ on $[-\delta',\delta']$. In particular, $\tilde{P}_G := P_G - R_N$ takes the form (5.1).

Next, note that

$$\mathbb{1}_{(-\infty,E]}(P)(x,y) = \langle \mathbb{1}_{(-\infty,E]}(P)\delta_x, 1_{(-\infty,E]}(P)\delta_y\rangle_{L^2} = \langle \mathbb{1}_{(-\infty,E]}(P_G)e^{iG}\delta_x, 1_{(-\infty,E]}(P_G)e^{iG}\delta_y\rangle_{L^2}.$$

Now, by [PS16, Lemma 4.2],

$$\begin{aligned}\|(\mathbb{1}_{(-\infty,E)}(\tilde{P}_G) - &1_{(\infty,E]}(P_G))f\|_{L^2} \\ &\leq 2\|\mathbb{1}_{[E-\mu,E+\mu]}(\tilde{P}_G)f\|_{L^2} + Ch^{3N}\mu^{-1}(\|\mathbb{1}_{(-\infty,E]}(\tilde{P}_G)f\|_{L^2} + \|(\tilde{P}_G+1)^{-s}f\|_{L^2})\end{aligned}$$

Let $f\in H^{-\ell}$ and $\mu = h^N$. Then for $N,s>\ell$, the last two terms above are bounded by h^N. Therefore, we need only understand $\mathbb{1}_{(-\infty,E]}(\tilde{P}_G)e^{iG}\delta_x$ and $\|\mathbb{1}_{[E-h^N,E+h^N]}(\tilde{P}_G)e^{iG}\delta_x\|_{L^2}$.

Before, we examine $1_{[a,b]}(\tilde{P}_G)$, we consider the distribution $e^{iG}\delta_x$. By Lemma 2.2, $e^{iG}\in S^0$, and hence for any $y\in\mathbb{R}$ fixed,

$$(e^{iG}\delta_y)(x) = \frac{1}{2\pi h}\int e^{\frac{i}{h}(x-y)\xi}b(x,\xi)d\xi \tag{6.1}$$

where $b\in S^0$ with $b\sim\sum_j h^j b_j$, $b_j\in S^{-j}$. In particular, for $|x-y|\geq 1$, and $N>k+1$,

$$((hD_x)^k e^{iG}\delta_y)(x) = \frac{1}{2\pi h}\int e^{\frac{i}{h}(x-y)\xi}\frac{(-hD_\xi)^N}{|x-y|^N}\xi^k b(x,\xi)d\xi = O(|x-y|^{-N}h^{N-1}).$$

and hence for $\chi\in C_c^\infty$ with $\chi(x)\equiv 1$ on $|x|<R$ and all $|y|<R-1$

$$(1-\chi)(e^{iG}\delta_y) = O(h^\infty)_{\mathcal{S}}.$$

Therefore,

$$\mathbb{1}_{[a,b]}(\tilde{P}_G)e^{iG}\delta_x = \mathbb{1}_{[a,b]}(\tilde{P}_G)\chi e^{iG}\delta_x + O(h^\infty)_{C^\infty}.$$

Next, we consider $\chi\mathbb{1}_{[a,b]}(\tilde{P}_G)\chi$. Let dE_h be the spectral measure for $\tilde{P}_G$.

Lemma 6.1. *Let $\chi_1 \in C_c^\infty$ and $\psi \in C_c^\infty$ with $\psi \equiv 1$ on $[-1,1]$. Then, there is $T>0$ such that for $E \in [1-\delta', 1+\delta']$, and h small enough,*

$$\chi_1 dE_h \chi_1 = \frac{1}{2\pi h}\int_{-T}^{T} \chi_1 e^{-it(\tilde{P}_G - E)/h}\psi(hD)\chi_1 dt + O(h^\infty)_{\mathcal{D}'\to C_c^\infty}.$$

In particular,

$$\chi_1 dE_h(E)\chi_1 = \frac{1}{(2\pi h)^2}\int_{-T}^{T}\int e^{\frac{i}{h}(-t(|\xi|^2-E)-\langle x-y,\xi\rangle)}a_E(t,x,y,\xi)d\xi dt + O(h^\infty)_{\mathcal{D}'\to C_c^\infty}$$

where $a_E \sim \sum_j h^j a_{j,E}$ with $a_{j,E} \in C_c^\infty$.

Proof. We will use Lemma 5.6. In particular, by Stone's formula

$$\mathbb{1}_{[a,b]}(\tilde{P}_G) = \frac{1}{2\pi i}\int_a^b (R(E+i0) - R(E-i0))dE,$$

so we need to understand $dE_h := (2\pi i)^{-1}(R(E+i0)-R(E-i0))$. For this, let $\chi_2 \in C_c^\infty(B(0,R))$ with $\chi_2 \equiv 1$ on $\operatorname{supp}\chi_1$. Then consider

$$\begin{aligned}\chi dE_h \chi &= \frac{1}{2\pi i}\chi_2(R(E+i0)-R(E-i0))(\psi(hD)+1-\psi(hD))\chi_2\\ &= \frac{1}{2\pi h}\int_{-T}^{T}\chi_2 e^{-it(P-E)/h}\psi(hD)\chi + \frac{1}{2\pi i}\chi_2(R(E+i0)-R(E-i0))(1-\psi(hD))\chi_2\end{aligned}$$

Let $v_\pm = R(E\pm i0)(1-\psi(hD))\chi_2 f$. Then, since $(1-\psi(hD))\chi_2 f$ is rapidly decaying, $v_\pm$ is semiclassically outgoing/incoming and

$$(\tilde{P}_G - E)v_\pm = (1-\psi(hD))\chi_2 f.$$

In particular, since $\mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}((1-\psi(hD))\chi_2 f)\cap\{p=E\}=\emptyset$, we have $\mathrm{WF}_{\mathrm{h}}^{\mathrm{sc}}(v_\pm)\cap\{p=E\}=\emptyset$.

Now,

$$(\tilde{P}_G - E)(v_+ - v_-) = 0 \qquad \Rightarrow \qquad \mathrm{WF}_{\mathrm{h}}(v_+ - v_-)\setminus\{p=E\}=\emptyset.$$

In particular, since, a priori both terms have $\mathrm{WF}_{\mathrm{h}}(v_\pm)\cap\{p=E\}=\emptyset$, we obtain

$$\mathrm{WF}_{\mathrm{h}}(v_+ - v_-)=\emptyset$$

and hence

$$\frac{1}{2\pi i}\chi_2(R(E+i0)-R(E-i0))(1-\psi(hD))\chi_2 f = O(h^\infty)_{C_c^\infty}.$$

Therefore,

$$\chi_2 i dE_h \chi_2 = \frac{1}{2\pi h}\int_{-T}^{T}\chi_2 e^{-it(P-E)/h}\psi(hD)\chi_2 dt + O(h^\infty)_{\mathcal{D}'\to C_c^\infty}.$$

The lemma follows from the oscillatory integral formula for $e^{it(P-E)/h}$ ([Zwo12, Theorem 1.4]). ☐

As a corollary of Lemma 6.1, we obtain for $t,s \in [1-\delta, 1+\delta]$,

$$|(hD_x)^\alpha (hD_y)^\beta \chi_1 \mathbb{1}_{(s,t]}(\tilde{P}_G)\chi_1(x,y)| \le C_{\alpha\beta}h^{-2}|t-s|.$$

In particular, this implies

$$\|\mathbb{1}_{[E-h^N, E+h^N]}(\tilde{P}_G)e^{iG}\delta_x\|_{L^2} \le Ch^{N-\ell}$$

for some $\ell > 0$ and hence it only remains to have an asymptotic formula for $\chi_1 \mathbb{1}_{(-\infty,E]}(\tilde{P}_G)\chi_1$.

Let $\hat{\rho} \in C_c^\infty((-2T, 2T))$ with $\hat{\rho} \equiv 1$ on $[-T, T]$ and put $\rho_{h,k}(t) = h^{-k}\rho(th^{-k})$. Define

$$R_1(E, x, y) := \chi_1(\rho_{h,k} * 1_{(-\infty,\cdot]}(\tilde{P}_G) - 1_{(-\infty,E]}(\tilde{P}_G))\chi_1(x, y) \tag{6.2}$$

$$R_2(E, x, y) := \chi_1(\rho_{h,k} - \rho_{h,1}) * 1_{(-\infty,\cdot]}(\tilde{P}_G)\chi_1(E, x, y). \tag{6.3}$$

Then, we will show for $E \in [1 - \delta/2, 1 + \delta/2]$

$$|(hD_x)^\alpha(hD_y)^\beta R_1(E, x, y)| = O_{\alpha\beta}(h^{k-2}), \qquad |(hD_x)^\alpha(hD_y)^\beta R_2(E, x, y)| = O_{\alpha\beta}(h^{k-2}) \tag{6.4}$$

In order to show the first inequality in (6.4) we recall that standard estimates also show that there is $M > 0$ such that for $t \in \mathbb{R}$

$$|(hD_x)^\alpha(hD_y)^\beta \chi_1 \mathbb{1}_{(-\infty,t]}(\tilde{P}_G)\chi_1(x, y)| \le C_{\alpha\beta} h^{-M}\langle t\rangle^M.$$

Then, for $E \in [1 - \delta/2, 1 + \delta/2]$

$$\begin{aligned}
&|(hD_x)^\alpha(hD_y)^\beta R_1(E, x, y)| \\
&= \Big| \int h^{-k}\rho(sh^{-k})(hD_x)^\alpha(hD_y)^\beta \chi_1(\mathbb{1}_{(E-s,E]}(\tilde{P}_G))\chi_1 ds\Big| \\
&\le \Big| \int_{|s|\le\delta/2} h^{-k}\langle sh^{-k}\rangle^{-N} C_\alpha \beta h^{-2}|s|ds\Big| + \Big| \int_{|s|\ge\delta/2} h^{-k}\langle sh^{-k}\rangle^{-N} C_\alpha \beta h^{-M}|s|^M ds\Big|
\end{aligned}$$

Choosing N large enough, the first inequality in (6.4) follows.

To obtain the second inequality, we observe that, since $\tilde{P}_G$ is bounded below,

$$\begin{aligned}
R_2(E) &= \chi_1 \int_{-\infty}^{E} (h^{-k}(\rho((s - \tilde{P}_G)/h^k) - h^{-1}(\rho((s - \tilde{P}_G)/h))\chi_1 \\
&= \frac{1}{2\pi i}\int t^{-1}\hat{\rho}(th^{k-1})(1 - \hat{\rho}(t))\chi_1 e^{it(E-\tilde{P}_G)/h}\chi_1 dt = \chi_1 f_h\Big(\frac{E - \tilde{P}_G}{h}\Big)\chi_1
\end{aligned}$$

where

$$f_h(\lambda) = \frac{1}{2\pi i}\int t^{-1}\hat{\rho}(th^{k-1})(1 - \hat{\rho}(t))e^{it\lambda}dt$$

In particular, note that $|f_h(\lambda)| \leq C_N\langle\lambda\rangle^{-N}$. Now, let $\psi \in C_c^\infty(-\delta,\delta)$ with $\psi \equiv 1$ near 0. Then,

$$\begin{aligned}
\chi_1 f_h\Big(\frac{E-\tilde{P}_G}{h}\Big)\chi_1 &= \int f_h\Big(\frac{E-s}{h}\Big)\chi_1 dE_h(s)\chi_1 \\
&= \int \psi(E-s) f_h\Big(\frac{E-s}{h}\Big)\chi_1 dE_h(s)\chi_1 + \int (1-\psi(E-s)) f_h\Big(\frac{E-s}{h}\Big)\chi_1 dE_h \chi_1(s) \\
&= \int f_h\Big(\frac{E-s}{h}\Big)\psi(E-s)\chi_1 dE_h(s)\chi_1 + O(h^\infty)_{\mathcal{D}'\to C_c^\infty} \\
&= -\frac{1}{2\pi}\int_{-T}^{T}\int f_h(w)\chi_1 e^{-it(\tilde{P}_G-E+hw)/h}\psi(hD)\chi_1 dw dt + O(h^\infty)_{\mathcal{D}'\to C_c^\infty} \\
&= \frac{1}{2\pi}\int_{-T}^{T}\int it^{-1}\hat{\rho}(th^{k-1})(1-\hat{\rho}(t))\chi_1 e^{-it(\tilde{P}_G-E)/h}\psi(hD)\chi_1 dt + O(h^\infty)_{\mathcal{D}'\to C_c^\infty} \\
&= O(h^\infty)_{\mathcal{D}'\to C_c^\infty}.
\end{aligned}$$

Therefore, the second inequality in (6.4) holds.

Together, the inequalities in (6.4) imply that

$$\chi_1(\mathbb{1}_{(-\infty,E]}(\tilde{P}_G) - \rho_{h,1} * \mathbb{1}_{(-\infty,\cdot]}(\tilde{P}_G)(E))\chi_1 = O(h^\infty)_{\mathcal{D}'\to C_c^\infty}$$

and we finish the proof of the main theorem by observing that

$$\begin{aligned}
\chi_1\rho_{h,1} * \mathbb{1}_{(-\infty,\cdot]}(\tilde{P}_G)(E)\chi_1 &= \frac{1}{2\pi h}\int_{-\infty}^{E}\int \hat{\rho}(t)\chi_1 e^{it(\mu-\tilde{P}_G)/h}\chi_1 dt d\mu \\
&= \frac{1}{(2\pi h)^2}\int_{-\infty}^{E}\int \hat{\rho}(t)\chi_1(x) e^{i(t(\mu-|\xi|^2)+(x-y)\xi)/h} a(x,y,\xi)\chi_1(y) d\xi dt d\mu
\end{aligned} \tag{6.5}$$

where $a \sim \sum_j a_j h^j$ and $a_j \in C_c^\infty$. Conjugating by e^{iG} using the formula from (6.1) completes the proof.

Appendix A. Properties of $s_{k,\mathcal{N}}$

In this appendix, we collect the proofs of the required properties of $s_{k,\mathcal{N}}$.

Proof of lemma 3.1. The case $k = 1, 0$ are clear with $N_0 = 0$, $N_1 = 1$. Suppose (3.1) holds for $k = n-1$. Then,

$$s_{n,\mathcal{N}}(\theta,,\mathcal{W}) = \begin{cases} \frac{1}{|\sum_{i=1}^k \theta_i|}\sum_{p\in Sym(k)}\sum_{|\alpha|=k,\alpha_i\leq k/2} s_{\alpha,\mathcal{N}}(p(\theta)) & \sum_{i=1}^k \theta_i \neq 0 \\ 0 & \sum_i \theta_i = 0. \end{cases}$$

The statement is trivial when $\sum_i \theta_i = 0$. Therefore, we assume the opposite. In that case

$$s_{n,\mathcal{N}}(\theta, \mathcal{W})$$

$$\leq \frac{1}{|\sum_{i=1}^n \theta_i|} \sum_{p \in Sym(n)} \sum_{|\alpha|=n, \alpha_i \leq n/2} \prod_{i=1}^{j} C_{|\alpha_i|} \frac{\prod_{\ell=1}^{|\alpha_i|} \|w_{p(\theta)_{\beta_i(\alpha)+\ell}}\|_{\mathcal{N}}}{\inf\{|\omega|^{N_{|\alpha_i|}} \mid \omega \in \{p(\theta)_{\beta_i(\alpha)+1}, 0\} + \cdots + \{p(\theta)_{\beta_{i+1}(\alpha)}, 0\} \setminus 0\}}$$

$$\leq \frac{1}{|\sum_{i=1}^n \theta_i|} \sum_{p \in Sym(k)} \sum_{|\alpha|=n, \alpha_i \leq n/2} \prod_{\ell=1}^{n} \|w_{\theta_\ell}\|_{\mathcal{N}} \prod_{i=1}^{j} \frac{C_{|\alpha_i|}}{\inf\{|\omega|^{N_{|\alpha_i|}} \mid \omega \in \{\theta_1, 0\} + \cdots + \{\theta_n, 0\} \setminus 0\}}$$

Then, defining $N_0 = 0$, $N_1 = 1$ and

$$N_k := \sup\{1 + \sum_i N_{|\alpha_i|} \mid |\alpha| = n, \, |\alpha_i| \leq \frac{n}{2}\},$$

we have

$$s_{n,\mathcal{N}}(\theta, \mathcal{W}) \leq \frac{\prod_{\ell=1}^n \|w_{\theta_\ell}\|_K}{\inf\{|\omega|^{N_k} \mid \omega \in \{\theta_1, 0\} + \cdots + \{\theta_n, 0\} \setminus 0\}} \sum_{p \in Sym(k)} \sum_{|\alpha|=n, \alpha_i \leq n/2} \prod_{i=1}^{j} C_{|\alpha_i|},$$

and hence the lemma follows by induction. □

Proof of Lemma 3.2. For $k = 0$ the claim is clear. For $k = 1$, observe that

$$s_{1,\mathcal{N}}(\theta_1 + \cdots + \theta_n, \tilde{\mathcal{W}}) = \begin{cases} \frac{\|\tilde{w}_{\theta_1 \ldots \theta_n}\|_{\mathcal{N}}}{|\sum_{i=1}^n \theta_i|} & \sum_i \theta_i \neq 0 \\ 0 & \sum_i \theta_i = 0 \end{cases}.$$

Note that

$$\frac{\|\tilde{w}_{\theta_1 \ldots \theta_n}\|_{\mathcal{N}}}{|\sum_{i=1}^n \theta_i|} \leq \frac{1}{|\sum_{i=1}^n \theta_i|} \prod_{i=1}^{n} \frac{\|w_{\theta_i}\|_{\mathcal{N}'}}{|\theta_i|} \leq s_{n,\mathcal{N}'}((\theta_1, \ldots, \theta_n), \mathcal{W}).$$

Suppose that the claim holds for $k - 1 \geq 1$. Then, when $\sum_i \sum_{j=1}^k (\theta_i)_j \neq 0$

$$s_{k,\mathcal{N}}(\theta_1 + \cdots + \theta_n, \tilde{\mathcal{W}})$$

$$= \frac{1}{|\sum_{i,j} (\theta_i)_j|} \sum_{p \in Sym(k)} \sum_{|\alpha|=k, \alpha_i \leq k/2} \prod_{i=1}^{j} s_{\alpha_i,\mathcal{N}}((p(\theta_1 + \cdots + \theta_n))_{\alpha,i}, \tilde{\mathcal{W}})$$

$$\leq \frac{1}{|\sum_{i,j} (\theta_i)_j|} \sum_{p \in Sym(k)} \sum_{|\alpha|=k, \alpha_i \leq k/2} \prod_{i=1}^{j} s_{n\alpha_i,\mathcal{N}}((p(\theta_1))_{\alpha,i}, \ldots (p(\theta_n))_{\alpha,i}), \mathcal{W})$$

$$\leq \frac{1}{|\sum_{i,j} (\theta_i)_j|} \sum_{p \in Sym(nk)} \sum_{|\alpha|=nk, \alpha_i \leq nk/2} \prod_{i=1}^{j} s_{\alpha_i,\mathcal{N}}((p(\theta_1, \ldots \theta_n))_{\alpha,i}), \mathcal{W})$$

$$= s_{nk,\mathcal{N}'}(\theta_1, \ldots, \theta_n, \mathcal{W})$$

□

Appendix B. Examples with infinitely many embedded eigenvalues

We now construct some examples to which our main theorem applies that, nevertheless, have arbitrarily large eigenvalues.

Theorem B.1. *Let $\omega \in \mathbb{R}^d$ satisfy the diophantine condition* (1.2) *and $\Theta = \mathbb{Z}^d \cdot \omega$. Then there is $W \in C^\infty(\mathbb{R};\mathbb{R})$ satisfying the assumptions of Theorem 1.1 and such that $\{\frac{\theta^2}{4} \mid \theta \in \Theta \setminus \{0\}\}$ is contained in the point spectrum of $-\Delta + W$.*

Theorem B.2. *Let $\{m_n\}_{n=1}^\infty \subset \mathbb{Z}_+$ and Θ as in Theorem 1.2. Then there is $W \in C^\infty(\mathbb{R};\mathbb{R})$ satisfying the assumptions of Theorem 1.2 and such that for all n, $\frac{m_n^2}{4n^2}$ is contained in the point spectrum of $-\Delta + W$. In particular, if $\mathbb{Q} \cap \mathbb{R}_+ = \{\frac{m_n}{n}\}_{n=1}^\infty$, then this operator has dense pure point spectrum.*

Theorems B.1 and B.2 follow easily from the following theorem.

Theorem B.3. *Let $\{\kappa_n\}_{n=1}^\infty$ be an arbitrary sequence of positive real numbers. Then there is $W \in C^\infty(\mathbb{R};\mathbb{R})$ such that κ_n^2 is an eigenvalue of $-\Delta + W$. Moreover, we can find W such that*

$$W = \sum_n e^{2i\kappa_n x} w_{2\kappa_n}(x) + \sum_n e^{-2i\kappa_n x} w_{-2\kappa_n}(x) + w_0(x)$$

where $w_0 \in C_c^\infty$ and for any N,

$$|\partial_x^k w_{\pm 2\kappa_n}(x)| \leq C_N \langle n\rangle^{-N} \langle \kappa_n\rangle^{-N} \langle x\rangle^{-k}. \tag{B.1}$$

We follow the construction in [Sim97] with a few modifications to guarantee smoothness. First, we need to replace [Sim97, Theorem 5] to allow for smoothness in V.

Recall that the Prüfer angles , $\phi(x)$, are defined by

$$u'(x) = kA(x)\cos(\phi(x)), \qquad u(x) = A(x)\sin(\phi(x))$$

where $-u'' + V(x)u = k^2 u$. Then, $\phi(x)$ satisfies

$$\phi'(x) = k - k^{-1} V(x) \sin^2(\phi(x)). \tag{B.2}$$

For any $N \geq 0$, $a < b \in \mathbb{R}$. let $F : C^N([a,b]) \times \mathbb{R}^n \times \mathbb{T}^n \to \mathbb{T}^n$ to be the generalized Prüfer angles with potential V, $\phi_i(x; V, k, \theta)|_{x=b}$, where $\phi_i(0; V, k, \theta) = \theta_i$ and we put $k = k_i$ in (B.2).

Lemma B.1. *Fix $[a,b] \subset (0,\infty)$, $U \Subset (a,b)$ open, $N > 0$, $k_1, \dots k_n > 0$ distinct, $\theta^{(0)} \in \mathbb{T}^n$, and $\varepsilon > 0$. Then there is $\delta > 0$ such that for all angles $\theta^{(1)} \in \mathbb{T}^n$ satisfying*

$$|\theta^{(1)} - kb - \theta^{(0)}| < \delta,$$

there is $V \in C_c^\infty(U)$ with $\|V\|_{C^N} < \varepsilon$ and $F(V, k, \theta^{(0)}) = \theta^{(1)}$.

Proof. Note that $F(0, k, \theta^0) = (\theta_1^{(0)} + k_1 b, \dots \theta_n^{(0)} + k_n b)$ and $\phi_i(x; V = 0) = \theta_i^{(0)} + k_i x$. Therefore, we need only show that the differential (in V) is surjective when restricted to functions in $C_c^\infty(U)$. For this, let $\chi \in C_c^\infty(U)$ with $\chi \equiv 1$ on a nonempty open interval I. Note that if $V_\varepsilon = \varepsilon \chi V(x)$,

$$\partial_\varepsilon \phi_i'(x; V_\varepsilon)|_{\varepsilon=0} = -k_i^{-1} \chi(x) V(x) \sin^2(k_i x + \theta_i^{(0)}), \qquad \partial_\varepsilon \phi_i(0; V_\varepsilon)|_{\varepsilon=0} = 0.$$

Hence,

$$\partial_\varepsilon F_i(V_\varepsilon)|_{\varepsilon=0} = -k_i^{-1}\int \chi(x)V(x)\sin^2(k_ix+\theta_i^{(0)})dx.$$

We claim that $u_i(x) := \chi(x)\sin^2(k_ix+\theta_i^{(0)})$ are linearly independent in L^2. Indeed, suppose $0<k_1<\dots k_n$ and $\sum_{i=1}^K \alpha_i u_i(x)=0$ a.e. with $\alpha_K\neq 0$ (and hence, by continuity for all x). Differentiating enough times, we see that $\alpha_K\equiv 0$, a contradiction.

Thus, there are $V_1,\dots V_n\in C^\infty$ such that $(\partial_\varepsilon F(\varepsilon\chi V_i))_{i=1}^n$ is a basis for $\mathbb{R}^n$ and the implicit function theorem finishes the proof. □

Proof of Theorem B.3. We work on the half line and find $W(x)$ vanishing to infinite order at 0 such that there are L^2 solutions, u_n of

$$-u_n''(x)+W(x)u_n(x)=\kappa_n^2u_n(x),\quad x\in[0,\infty)\qquad u_n(0)=0.$$

The case of the line then follows by extending W to an even function and u_n to an odd function.

Let $\chi\in C^\infty(\mathbb{R})$ with $\chi\equiv 1$ on $[2,\infty)$, $\operatorname{supp}\chi\subset(1,\infty)$ and define $\chi_n(x):=\chi(R_n^{-1}x)$ where $R_n\to\infty$, $R_n\geq 1$ are to be chosen later. We put

$$(\Delta L_n)(x):=4\kappa_n\frac{\chi_n(x)}{x}\sin(2\kappa_nx+\varphi_n)$$

where φ_n is also to be chosen. We will also find ΔS_n to be smooth function supported on $(2^{-n},2^{-n+1})$ with $\|\Delta S_n\|_{C^n}\leq\frac{1}{2^n}$ and put

$$W_m(x)=\sum_{n=1}^m(\Delta L_n+\Delta S_n)(x),\qquad W(x):=\lim_{m\to\infty}W_m(x),\qquad \tilde{W}_m:=W_m-\Delta S_m.$$

Note that by construction $\sum_n\Delta S_n\in C^\infty([0,1))$, $\sum_n\Delta S_n$ vanishes to infinite order at 0, and

$$\Delta L_n(x)=-e^{2i\kappa_nx}2i\kappa_ne^{i\varphi_n}\chi_n(x)x^{-1}+e^{-2i\kappa_nx}2i\kappa_ne^{-i\varphi_n}\chi_n(x)x^{-1}.$$

In particular,

$$\Delta L_n(x)=e^{2i\kappa_nx}w_{2\kappa_n}(x)+e^{-2i\kappa_nx}w_{-2\kappa_n}(x)$$

with $w_{\pm2\kappa_n}=\mp2i\kappa_ne^{\pm i\varphi_n}\chi_n(x)x^{-1}$. In particular,

$$|\partial_x^kw_{\pm2\kappa_n}|\leq C_k\kappa_nR_n^{-1}\langle x\rangle^{-k}. \tag{B.3}$$

In order to obtain the estimate (B.1), we fix a positive Schwartz function f and choose R_n $R_n\geq\frac{1}{f(\langle\kappa_n\rangle\langle n\rangle)}$. The estimate (B.3) then guarantees that $\sum_n\Delta L_n$ is bounded with all derivatives. The fact that $\Delta S_n\in C_c^\infty(2^{-n},2^{-n+1})$ and $\|\Delta S_n\|_{C^n}\leq\frac{1}{2^n}$ guarantees that $w_0=\sum_n\Delta S_n\in C^\infty([0,1))$ and w_0 vanishes to infinite order at 0.

Now, note that $\tilde{\chi}_n(\xi):=\mathcal{F}((\cdot)^{-1}\chi_n(\cdot))(\xi)$ is smooth away from $\xi=0$. Therefore, for each $m\neq n$, we can find $\psi_{n,m}\in C_c^\infty(0,1)$ such that

$$\mathcal{F}(\psi_{n.m})(0)=-2i\kappa_n(-\tilde{\chi}(2\kappa_n)e^{i\varphi_n}-\tilde{\chi}(2\kappa_n)e^{-i\varphi_n})$$

$$\mathcal{F}(\psi_{n,m})(\pm2\kappa_m)=-2i\kappa_n(\tilde{\chi}(2(\pm\kappa_m-\kappa_n))e^{i\varphi_n}-\tilde{\chi}(2(\kappa_n\pm\kappa_m))e^{-i\varphi_n}).$$

Then, letting $\psi_{n,n}=0$ and defining $\tilde{L}_{n,m}:=\Delta L_n-\psi_{n,m}$, there are $A_{n,m}$, $A^{\pm}_{n,m}$ such that

$$\begin{gathered}|\tilde{L}_{n,m}|\leq C|x|^{-1},\qquad \tilde{L}_{n,m}=A'_{n.m},\quad |A_{n,m}|\leq C|x|^{-1},\\ e^{\pm 2i\kappa_m x}\tilde{L}_{n,m}=(A^{\pm}_{n,m})',\quad |A^{\pm}_{n,m}(x)|\leq C|x|^{-1}.\end{gathered}\tag{B.4}$$

By the conditions (B.4) and [Sim97, Theorem 3], there is a unique function $u_n^{(m)}(x)$ satisfying

$$-(u_n^{(m)})''+W_m(x)u_n^{(m)}=\kappa_n^2u_n^{(m)},\qquad \left|\!\left|\!\left|u_n^{(m)}-\sin((\kappa_n+\tfrac{1}{2}\varphi_n)\cdot)(1+|\cdot|)^{-1}\right|\!\right|\!\right|<\infty.\tag{B.5}$$

where $|||u|||=\|(1+x^2)u\|_\infty+\|(1+x^2)u'\|_\infty$. Similarly, there is a unique function $\tilde{u}_n^{(m)}(x)$ satisfying

$$-(\tilde{u}_n^{(m)})''+\tilde{W}_m(x)\tilde{u}_n^{(m)}=\kappa_n^2\tilde{u}_n^{(m)},\qquad \left|\!\left|\!\left|\tilde{u}_n^{(m)}-\sin((\kappa_n+\tfrac{1}{2}\varphi_n)\cdot)(1+|\cdot|)^{-1}\right|\!\right|\!\right|<\infty.\tag{B.6}$$

Now, we construct ΔL_n, ΔS_n such that

$$\left|\!\left|\!\left|u_n^{(m)}-u_n^{(m-1)}\right|\!\right|\!\right|\leq 2^{-m},\quad n=1,2,\dots,m-1,\qquad u_n^{(m)}(0)=0,\quad n=1,\dots,m.\tag{B.7}$$

Once we have done this, we can let $u_n=\lim_m u_n^{(m)}$ (in the $|||\cdot|||$ norm) to obtain L^2 eigenfunctions with eigenvalue κ_n.

Let $m\geq 1$ and suppose we have chosen $\{(R_n,\varphi_n)\}_{n=1}^{m-1}$, and $\Delta S_1,\dots\Delta S_{m-1}\in C_c^\infty$ with $\operatorname{supp}\Delta S_n\subset(2^{-n},2^{-n+1})$ and $\|\Delta S_n\|_{C^n}\leq\frac{1}{2^n}$ such that (B.7) holds and $R_n\geq 1/f(\langle n\rangle\langle\kappa_n\rangle)$.

By [Sim97, Theorem 3], there are ε_m $\tilde{R}_m$ such that for all $R_m\geq\tilde{R}_m$, and $\varphi_m\in[0,2\pi/(2\kappa_m)]$, if $\|\Delta S_m\|_{C^0}\leq\varepsilon_m$, then

$$|||u_i^m-\tilde{u}_i^m|||\leq 2^{-m-1}.$$

Observe that by Lemma B.1, there is $\delta_m>0$ small enough such that if $|\theta_i^{(1)}-\kappa_i 2^{-m+1}|<\delta_m$ and $\theta_i^{(1)}$ are the Prüfer angles of the solutions $\tilde{u}_i^m$, $i=1,\dots,m$ at 2^{-m+1}, then there is $\Delta S_m\in C_c^\infty(2^{-m},2^{-m+1})$ with $\|\Delta S_m\|_{C^m}\leq\min(2^{-m},\varepsilon_m)$ and such that $u_i^{(m)}(0)=0$. Therefore, if we can find $R_m\geq\tilde{R}_m$ and φ_m such that $|\theta_i^{(1)}-\kappa_i 2^{-m+1}|<\delta_m$, and

$$\left|\!\left|\!\left|u_i^{m-1}-\tilde{u}_i^m\right|\!\right|\!\right|\leq 2^{-m-1},$$

the proof will be complete.

Once again by [Sim97, Theorem 3], for R_m large enough, we have (uniformly in $\varphi_m\in[0,2\pi/(2\kappa_m)]$), $\left|\!\left|\!\left|u_i^{(m-1)}-\tilde{u}_i^{(m)}\right|\!\right|\!\right|<2^{-m-1}$ for $i=1,\dots,m-1$ and the Prüfer angles for $\tilde{u}_i^{(m)}$ at 2^{-m+1} satisfy $|\theta_i^{(1)}-\kappa_i b_i|<\delta$ for $i=1,\dots m-1$).

Finally, we choose φ_m so that $\tilde{u}_m^{(m)}(0)=0$. The existence of such a φ_m again follows from [Sim97, Theorem 3]. In particular, note that by part (b) there, we have (B.6) uniformly over R_m large enough, x large enough, and $\varphi_m\in[0,2\pi/(2\kappa_m)]$. In particular, the Prüfer angles for $\tilde{u}_m^{(m)}$, $\tilde{\phi}_m(x)$ run through a full circle. Therefore, we can choose R_m large enough and φ_m such that the $\tilde{\phi}_m(R_n)$ agrees with the Prüfer angle of the solution to u to $-u''+W_{m-1}(x)u=\kappa_m^2u$, $u(0)=0$ and hence, since $W_{m-1}=\tilde{W}_m$ on $x\leq R_n$, we have that $\tilde{u}_m(0)=0$.

□

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