FIRS UNDER FIELD EXTENSIONS

Vitor de Oliveira Ferreira

Ph. D. Thesis
1998

Department of Mathematics
University College London

Supervisor: Prof. Paul M. Cohn

Abstract

Our main aim is to investigate the relation between free ideal rings (firs) which are algebras over a field and the rings obtained from them by a commutative field extension of scalars. Having in mind the monoid of projectives of these rings, we prove that commutative monoids with distinguished element which are conical and have the UGN property are strongly embedded in their coproduct and that the coproduct inherits some properties of its factors. This result, in conjunction with Bergman's coproduct theorems, is used to establish links between coproducts of skew fields and the rings obtained from them by extension of scalars. The notion of a power-free ideal ring is explored when looking at coproducts and, more generally, at rings obtained by matrix reduction of coproducts. We also look at firs of the form $R = F_k\langle X \rangle$, where F is either a finite Galois extension or a simple purely inseparable extension of k. These firs, when tensored with F over k, give rise to rings that are no longer firs, but they are very close to being full matrix rings over firs in the sense that the adjunction to R of a single inverse (in the purely inseparable case) or of finitely many inverses (in the Galois case) originates a ring which under the same extension of scalars is a full matrix ring over a fir. The universal field of fractions of a fir obtained by this construction is just the skew field component of the simple artinian ring obtained by extending the scalars of the universal field of fractions of R.

Contents

Acknowledgments

I would like to thank my supervisor, Prof. P. M. Cohn, for his help and support during the preparation of this work.

During the research on which this thesis is based, I was financially supported by a grant from CNPq - Brasília/Brazil, ref. 200228/95-1.

V. O. F.

Introduction

The concept of a *free ideal ring* (fir) was first introduced by Cohn in the study of rings which can be embedded in skew fields. Later the class of all rings embedded in skew fields which preserve full matrices was named the class of Sylvester domains and firs are therefore seen as special cases of Sylvester domains. The first chapter of this thesis summarizes all the results needed in the rest of the work concerning the theory of firs and Sylvester domains, monoid of projectives and the Bergman coproduct theorems. Section 1.4 contains preliminary results on the direction of the main proposal of this thesis to which no convenient reference was available.

It has been known that the centres of firs which are not principal ideal domains are fields—so that these rings can be regarded as algebras over a field. A natural question arising from this observation is that of the description of the structure of the rings obtained from firs which are algebras over a field by extension of the ground field. We call rings obtained from this process *extended firs*. Since extended firs can be regarded as subrings of their extended universal field of fractions, in the case of finite ground field extensions, they are subrings of simple artinian rings. In Chapter 4 we use this information and try to see how far they are from being matrix rings themselves.

To obtain information on the structure of extended firs we look at the module categories over these rings. A way of measuring the distance of an extended fir from being a fir is by measuring the distance of its modules from being free. To do so, we first look at the global dimension of the extended firs. This provides us with the information on how far modules are from being projective. The second step is to look at the monoid of projectives of such rings. This will tell us how far projectives are from being free.

In the case of separable extensions of the ground field of a fir, the answer to the first question is easily found: extended firs are hereditary. This is the contents of Proposition 1.4.4. The results quoted in the rest of this paragraph are new. Chapter 2 deals with the second question. In this chapter a theory of commutative monoids with distinguished element is developed and special attention is given to coproducts in this category. The main result in this chapter is Theorem 2.1.3 which states that under certain conditions, the coproduct in this category is faithful and separating. This proves useful when considering rings obtained from ground field extensions of firs which are coproducts of "simpler" firs. By "simpler", is meant firs whose extensions have a better known structure. Theorem 2.2.2, used in conjunction with Bergman's theorem on the monoid of projectives of a coproduct of rings, tells us that properties of the monoid of projectives of the extended factors of the coproduct are preserved in the coproduct.

The main tool for studying coproducts of rings are Bergman's coproduct theorems (Theorems 1.3.2 and 1.3.4). These theorems are strong enough to answer the questions on the global dimension and the monoids of projectives of extended coproducts of skew fields, a large class of examples of firs. Extended coproducts of skew fields are dealt with in Chapter 3. In particular, we prove that in some cases, the extended ring is a *power free ideal ring* (pfir) (Proposition 3.2.2)—a new concept that generalizes the concept of a fir. The coproduct theorems also prove useful in the study of extensions of firs obtained by matrix reductions of "simpler" firs. An example is Corollary 3.3.5 which states that the matrix reduction of a skew field under a central field extension is a pfir.

Tensor rings, another large class of examples of firs, are studied in Chapter 4. Theorems 4.2.12 and 4.3.14 are original and the most important of this chapter. Using the fact that an extended fir is a subring of a matrix ring, we enlarge the original fir in order to obtain a matrix ring after the extension. This is always possible, because if R is a ring which can be embedded in a skew field U and both have the same centre k, then, if E/k is a finite commutative field extension, $R_E = R \otimes_k E$ can be embedded in U_E, which is a simple artinian ring, thus a matrix ring. For each matrix unit e_{ij} of U_E, there is a natural number s and elements $u_r \in U$ and $\lambda_r \in E$, $r = 1, \ldots, s$ such that

$$e_{ij} = \sum_{r=1}^{s} u_r \otimes \lambda_r.$$

So if we take S to be a ring between R and U which contains all the elements u_r described above, we find that S_E is also isomorphic to a matrix ring. In chapter 4, we apply this idea to two special tensor rings and extensions; we look at $F_k\langle x\rangle \otimes_k F$, where F is a finite commutative field extension of k. In Section 4.2 we take F/k to be Galois and in the following section we take F to be a simple purely inseparable extension of k. It is proved that the adjunction of only a finite number of inverses to R is enough to produce the matrix units. We develop full calculations of generators and relations of S_E to conclude that, in these specific cases, S_E is in fact a matrix ring over a fir (Theorems 4.2.12 and 4.3.14).

A convention used in this thesis is that all rings are associative, but not necessarily commutative, and have a unit element, different from zero, which is preserved by homomorphisms, inherited by subrings and acts unitally on modules.

Commutative rings which are fields will be referred to as commutative fields. Noncommutative fields will be called, in general, skew fields. Occasionally, when no possibility of confusion may arise, the adjectives "commutative" and "skew" will be dropped. This is the case when we speak of the universal field of fractions of a fir, for instance. We recall that a ring is called *semisimple* if it is isomorphic to a finite direct product of matrix rings over skew fields.

Finally, maps will we written on the left and composed accordingly, that is, from right to left, with a unique exception in Section 4.2, where the exponential notation is used for automorphisms.

List of notation

In the list below R, S will stand for arbitrary rings, P a right module over R, D a skew field, K a sub-skew field of D, X a set, k a commutative field, S a k-algebra and E a commutative field extension of k.

$\mathbb{N}$ The set of non-negative integers

P^m Direct sum of m copies of P

R^m Free right R-module on m generators

$R \times S$ Direct product of R and S

$D_K\langle X\rangle$ Free D_K-ring on X, p. 14

$*_R R_i$ Coproduct of the family of R-rings (R_i), p. 24

$R_1 *_R R_2$ Coproduct of the R-rings R_1 and R_2

${}^m R^n$ Set of all $m \times n$ matrices over R

R_Σ Localization of R at a set of matrices Σ, p. 15

$Z(R)$ Centre of R

$\mathcal{P}(R)$ Monoid of projectives of R, p. 16

$\mathbf{1}_M$ Distinguished element of the commutative monoid M, p. 17

$\coprod_{\mathbb{N}} M_i$ Coproduct of the family of commutative monoids with distinguished element (M_i), p. 18

$M \coprod_{\mathbb{N}} N$ Coproduct of the commutative monoids with distinguished element M and N

$\mathfrak{M}_n(R)$ Ring of $n \times n$ matrices over R

A_E Tensor product $A \otimes_k E$

$pd_R(P)$ Projective dimension of P

$r.gl.dim(R)$ Right global dimension of R

$E(\rho)$ Monoid congruence generated by the relation ρ

$M/E(\rho)$ Quotient of the monoid M by the congruence $E(\rho)$

$c \to d$ Elementary transition, p. 35

$\varinjlim C_i$ Direct limit of the direct set (C_i)

$M \times N$ Direct product of the commutative monoids with distinguished element M and N

$\mathfrak{W}_n(T; R)$ $n \times n$ matrix reduction of the R-ring T, p. 60

$[E : k]$ Degree of the extension E/k

$\mathcal{L}_a, \mathcal{R}_a$ Left and right multiplications by $a \in A$

$\mathcal{M}(A)$ Multiplication algebra of A

$End_k(A)$ Set of all the k-linear endomorphism of A

$ker(f)$ Kernel of the morphism f

$Gal(E/k)$ Galois group of the extension E/k

δ_{ij} Kronecker delta

$E \setminus k$ Set of elements of E outside k

$\binom{n}{i}$ Number of subsets of size i of a set of size n

Chapter 1

Background

1.1 Firs and Sylvester domains

In this section, we will present the definitions and main results related to the theory of firs and Sylvester domains and their properties of embeddability into skew fields. The presentation will follow Cohn (cf. [6] and [9]). Most theorems will be stated without proofs and will be included just for completeness of the whole text.

We remind the reader that all rings are associative, have a unit element, different from zero, which is preserved by homomorphisms, inherited by subrings and acts unitally on modules.

1.1.1 Firs and semifirs

We start by giving some definitions which characterize rings with some restrictive properties about ranks of free modules. We will be working with finitely generated right modules over R. All definitions below, although stated in terms of right modules, will not contain the modifier "right", because a ring satisfying one of them will surely satisfy the dual definition, which is obtained by the duality between the category of finitely generated projective right R-modules and the category of finitely generated projective left R-modules given by the functor $Hom_R(\cdot, R_R)$.

Let R be a ring. R is said to have *invariant basis number* (IBN) if, for all $m, n \in \mathbb{N}$, $R^m \cong R^n$ implies $m = n$; R is said to have *unbounded generating number* (UGN) if for all $m, n \in \mathbb{N}$, $R^m \cong R^n \oplus K$ implies $m \geq n$; R is said to be *weakly finite* if for all $n \in \mathbb{N}$, $R^n \cong R^n \oplus K$ implies $K = 0$.

We will also have to deal with rings which have some restrictions on the behaviour of some (or all) of its finitely generated projective modules. A ring R is said to be an *Hermite ring* if for all $m, n \in \mathbb{N}$, $P \oplus R^m \cong R^n$ implies $n \geq m$ and $P \cong R^{n-m}$. R is said to be *power-free* if it has IBN and for each finitely generated projective module P there exists $n \in \mathbb{N}$, which depends on P, such that P^n is free. R is said to be *projective trivial* if there exists a projective module P such that for every finitely generated projective module Q, there exists a unique $n \in \mathbb{N}$, which depends on Q, such that $Q \cong P^n$. Finally, R is said to be *projective free* if it has IBN and every finitely generated projective module is free.

We say that a ring R has the *cancellation of projectives property* if for any finitely generated projective modules P, P', Q,

$$P \oplus Q \cong P' \oplus Q \ \text{ implies } \ P \cong P'.$$

The relations among the above properties of rings are illustrated by the following diagram, where a move downwards means a more restrictive property.

$$\begin{tikzcd}
 & \text{IBN} \arrow[d, dash] & \\
 & \text{UGN} \arrow[d, dash] & \\
 & \text{Weakly finite} \arrow[dl, dash] \arrow[dr, dash] & \\
\text{Power free} \arrow[ddr, dash] & & \text{Hermite} \arrow[d, dash] \\
 & & \text{Cancellation} \arrow[dl, dash] \\
 & \text{Projective trivial} \arrow[d, dash] & \\
 & \text{Projective free} &
\end{tikzcd}$$

Projective trivial rings are very closely related to projective free rings, as the following proposition shows.

Proposition 1.1.1. *For any ring R, the following properties are equivalent:*

(a) *R is a full matrix ring over a projective free ring;*

(b) *R is Morita equivalent to a projective free ring;*

(c) *R is projective trivial.*

Proof. See [6, Th. 0.4.6, p. 18]. □

We now come to the definitions of firs and semifirs.

Definition. *A ring R is a* semifir *if it has IBN and every finitely generated right ideal is free.*

The above definition is equivalent to its dual—reason why we speak of semifirs instead of right semifirs. (Cf. [6, Ch. 1])

Definition. *A ring R is a* right (left) fir *if it has IBN and every right (left) ideal is free. A right and left fir is called a* fir.

We can give more homological characterizations of firs and semifirs. And this is the content of the following theorem.

Theorem 1.1.2. *Let R be a ring.*

(a) *R is a semifir if and only if it is weakly semihereditary and projective free.*

(b) *R is a right fir if and only if it is right hereditary and projective free.*

Proof. See [6, Th. 1.4.1 and Cor. 1.4.2, pp. 77-78]. □

In particular, we have

Theorem 1.1.3. *For any ring R the following conditions are equivalent:*

(a) *R is a full matrix ring over a semifir (resp. fir);*

(b) *R is Morita-equivalent to a semifir (resp. fir) ;*

(c) *R is weakly semihereditary (resp. hereditary) and projective free.*

Proof. See [6, Th. 1.4.4, p. 78 and Ex. 1.4.3, p. 79]. □

The first examples of firs are skew fields and principal ideal domains; in fact, principal left ideal domains are always left firs. As nontrivial examples of firs, we look at two large classes of rings. First, let D be a skew field and K a subfield of D. Then the free D_K-ring $D_K\langle X\rangle$ on a set X is a fir. $D_K\langle X\rangle$ is also called the *D_K-tensor ring on X* (cf. [6, p. 61]). This is the contents of Th. 2.6.2 on p. 114 of [6] (cf. also Corollary 1.3.7 below). Free associative algebras over arbitrary fields are included in this class of examples and hence they are firs.

Next, we look at coproducts of skew fields. Given a family of skew fields $(D_i)_{i\in I}$ all containing a common subfield K, we can consider their coproduct in the category of K-rings, denoted by $*_K D_i$. As will be stated later, coproducts of firs amalgamating a common subfield are firs. So $*_K D_i$ is a fir (see Proposition 1.3.5 and Corollary 1.3.6 below).

In this thesis we shall mainly be interested in firs which are also algebras over a commutative field. These firs form a large subclass of the class of all firs. In fact, Cohn proves in [6, Cor. 6.3.4, p. 311] that the centre of a right fir which is not a principal right ideal domain is a commutative field.

1.1.2 Universal fields of fractions and Sylvester domains

Given a ring R, a *field of fractions* of R is a skew field K given together with an embedding $R \hookrightarrow U$ such that K is generated as a field by the image of R. Cohn in [9, Ch. 4] gives necessary and sufficient conditions for a ring to have a field of fractions. In contrast to the commutative case, a field of fractions of a noncommutative ring does not have to be unique up to isomorphism. So in this context we speak of the *universal field of fractions* of a ring, which is unique up to isomorphism (cf. Sec. 7.2 of [6]). We shall be interested in rings which have a universal field of fractions with an extra property. We start with some definitions. Given a ring R, and $m, n \in \mathbb{N}$, let ${}^mR^n$ denote the set of all $m \times n$ matrices over R.

Definition. *Let R be a ring and A an $m \times n$ matrix over R. The* inner rank *of A is defined to be the least r such that*

$$A = BC$$

where $B \in {}^mR^r$ and $C \in {}^rR^n$. The inner rank of A is denoted by $\rho(A)$.

Clearly, if $A \in {}^mR^n$, then $\rho(A) \leq min(m,n)$.

Definition. *Let R be a ring and A an $m \times n$ matrix over R; then A is said to be* left full *if $\rho(A) = m$,* right full *if $\rho(A) = n$ and* full *if $m = n$ and $\rho(A) = n$.*

Definition. *A ring R is called a* Sylvester domain *if for any $A \in {}^mR^r$, $B \in {}^rR^n$*

$$AB = 0 \quad \textit{implies} \quad \rho(A) + \rho(B) \leq r.$$

We also need to define the concept of *localization* for arbitrary noncommutative rings. Let R be a ring and Σ a set of square matrices over R. A ring homomorphism f from R into a ring S is called Σ*-inverting* if the image of each matrix in Σ is an invertible matrix in S. We will call a ring R_Σ the *localization* of R at Σ if R_Σ is a universal target of Σ-inverting homomorphisms from R. The existence and uniqueness of such a ring is given by the following

Theorem 1.1.4. *Given a ring R and a set Σ of square matrices over R, there exists a ring R_Σ and a homomorphism $\lambda : R \longrightarrow R_\Sigma$ which is Σ-inverting and if $f : R \longrightarrow S$ is a Σ-inverting homomorphism, there exists a unique homomorphism $g : R_\Sigma \longrightarrow S$ such that $g\lambda = f$.*

Proof. See Th. 4.1.3 in [9]. □

That is, g is a unique homomorphism making the diagram

$$\begin{tikzcd} R \arrow[r, "\lambda"] \arrow[dr, "f"'] & R_\Sigma \arrow[d, "g"] \\ & S \end{tikzcd}$$

commute. We can also look at localization at sets of non-square matrices, in which case, Theorem 1.1.4 is still valid, but this will be of no use for us in this thesis.

Note that λ is an epimorphism but not, in general, a monomorphism. It is also worth noting that if R is an algebra over a commutative ring k, then the homomorphism λ is also k-linear, for, if we denote the centre of R by $Z(R)$, then, since R_Σ is a localization of R, $\lambda(Z(R)) \subseteq Z(R_\Sigma)$.

Localization does not depend on the order that inverses of matrices are adjoined. That is, if R is a ring and Σ_1, Σ_2 are two sets of matrices over R, then

$$R_{\Sigma_1 \cup \Sigma_2} \cong (R_{\Sigma_1})_{\Sigma_2'}$$

where Σ_2' is the image of Σ_2 in R_{Σ_1}. This isomorphism is a direct consequence of Theorem 1.1.4.

The following result states that Sylvester domains are exactly the rings that have a universal field of fractions over which every full matrix is invertible.

Theorem 1.1.5. *Let R be a ring. Then the following are equivalent:*

(a) *R is a Sylvester domain;*

(b) *the localization R_Φ at the set of all full matrices is a skew field.*

Moreover, R_Φ is then the universal field of fractions of R.

Proof. See Th. 7.5.10 on p. 417 of [6] and Th. 4.5.8 on p. 181 of [9]. □

Since every semifir (and, therefore, every fir) is a Sylvester domain (cf. [6, Prop. 5.5.1, p. 253]), we have

Corollary 1.1.6. *Let R be a fir. Then R has a universal field of fractions U such that every full matrix over R can be inverted in U.* □

1.2 Monoids with distinguished element

In the study of the category of all finitely generated projective right modules over an arbitrary ring R, it is often useful to look at the monoid of projectives of R. This is defined to be the monoid $\mathcal{P}(R)$ whose elements are isomorphism classes of finitely generated projective right modules with operation induced by the direct sum. $\mathcal{P}(R)$ is a commutative monoid in which we have a distinguished element given by the isomorphism class of R as a right R-module.

This situation motivates an abstract definition of the category of commutative monoids with distinguished element.

In this section, if x is an element of an additive commutative monoid and n is a natural number, the notation $n.x$ will stand for the sum of n terms equal to

x, that is

$$n.x = \underbrace{x + \ldots + x}_{n \text{ terms}}.$$

1.2.1 The category of commutative monoids with distinguished element

Let $\mathcal{C}$ be the category whose objects are pairs (M, φ), where M is an (additive) commutative monoid and $\varphi : \mathbb{N} \longrightarrow M$ is a monoid morphism from the monoid of the natural numbers into M. If $\varphi \neq 0$, then (M, φ) is said to be *genuine*. In the applications, we will be mainly interested in genuine commutative monoids with distinguished element, but we can carry out the theory without assuming this restrictive property. If (M, φ) and (L, ψ) are objects of $\mathcal{C}$, then a morphism from (M, φ) to (L, ψ) is a monoid morphism $f : M \longrightarrow L$ that makes the diagram

$$\begin{tikzcd} M \arrow[rr, "f"] & & L \\ & \mathbb{N} \arrow[ul, "\varphi"] \arrow[ur, "\psi"'] & \end{tikzcd}$$

commute.

Each object (M, φ) of $\mathcal{C}$ will have, then, a distinguished element given by $\mathbf{1}_M = \varphi(1)$. Therefore, morphisms in $\mathcal{C}$ are just monoid morphisms which preserve the distinguished element.

Given an object (M, φ_M) of $\mathcal{C}$ and a positive integer r, we can define a new object $(\frac{1}{r}M, \varphi_{\frac{1}{r}M})$ in the category such that $\frac{1}{r}M$ is the same as M as a commutative monoid, but in $\frac{1}{r}M$ we regard $r.\mathbf{1}_M$ as its distinguished element. For convenience, we denote the elements of $\frac{1}{r}M$ by $\frac{1}{r}m$, for $m \in M$, so that the operation in $\frac{1}{r}M$ is given by

$$\frac{1}{r}m + \frac{1}{r}m' = \frac{1}{r}(m + m'),$$

where the sum on the right-hand side is to be taken in M. With this convention, the associated morphism $\varphi_{\frac{1}{r}M} : \mathbb{N} \longrightarrow \frac{1}{r}M$ is given by $\varphi_{\frac{1}{r}M}(n) = \frac{1}{r}\varphi_M(r.n)$. And thus the distinguished element $\mathbf{1}_{\frac{1}{r}M}$ is just $\frac{1}{r}(r.\mathbf{1}_M)$.

In the next subsection, we will use the construction of $\frac{1}{r}M$ from M in order to make a convenient use of the equivalence between the category of finitely generated projective modules over a ring and the respective category over the $r \times r$ matrix ring of that ring.

Given two objects $(M,\varphi),(N,\psi)$ of $\mathcal{C}$, a *coproduct* of M and N is an object $M\coprod_{\mathbb{N}} N$ given with morphisms $f : M \longrightarrow M\coprod_{\mathbb{N}} N$ and $g : N \longrightarrow M\coprod_{\mathbb{N}} N$ such that if K is any object in $\mathcal{C}$, given any morphisms $\overline{f} : M \longrightarrow K$, $\overline{g} : N \longrightarrow K$, there exist a unique morphism $h : M\coprod_{\mathbb{N}} N \longrightarrow K$ such that $hf = \overline{f}$ and $hg = \overline{g}$. This definition is equivalent to the definition of the pushout of the morphisms $\varphi : \mathbb{N} \longrightarrow M$, $\psi : \mathbb{N} \longrightarrow N$ in the category of commutative monoids. Therefore, the coproduct of M and N in $\mathcal{C}$ is just a monoid $M\coprod_{\mathbb{N}} N$ given with monoid morphisms f, g which preserve distinguished elements and are universal with this property. It is also customary to call $M\coprod_{\mathbb{N}} N$ the *free product* of M and N amalgamating the respective images of $\mathbb{N}$.

Coproducts always exist in $\mathcal{C}$. In fact, $M\coprod_{\mathbb{N}} N$ is just the commutative monoid generated by M and N whose relations are those of M and N and the relation that identifies $\mathbf{1}_M$ with $\mathbf{1}_N$.

Given an arbitrary family of commutative monoids with distinguished elements $(M_i,\varphi_i)_{i\in I}$, we can also consider the coproduct of the whole family. This will be denoted by $\coprod_{\mathbb{N}} M_i$.

We will come back to coproducts in $\mathcal{C}$ in Chapter 2.

1.2.2 Monoids of projectives

Let us recall the definition, given in the introduction, of the monoid of projectives of an arbitrary ring. Let R be a ring and consider the set $\mathcal{P}(R)$ whose elements are isomorphism classes of finitely generated projective right R-modules. So every finitely generated projective right R-module P defines an element $[P]$ of $\mathcal{P}(R)$ and $[P] = [Q]$ if and only if P is isomorphic to Q. $\mathcal{P}(R)$ can be made into a commutative monoid by defining in it the following binary operation

$$[P] + [Q] = [P \oplus Q].$$

This definiton does not depend on the choice of the representatives P and Q, because the direct sum preserves isomorphic modules. This operation is clearly commutative and defines an identity element in $\mathcal{P}(R)$ given by the zero module. Also, $\mathcal{P}(R)$ has a distinguished element given by $e = [R]$. It will be shown later that $\mathcal{P}$ is in fact a functor from the category of rings to the category $\mathcal{C}$.

Note that, to be absolutely precise, we should be speaking of the monoid of projective *right* modules of R. But, by the duality between the category of finitely generated projective right R-modules and the category of finitely generated projective left R-modules, given by the functor $Hom_R(\ \cdot\ , R_R)$, the corresponding monoids are isomorphic (even as monoids with distinguished elements), so we can restrict ourselves to right modules.

As an example, take R to be a skew field; then $\mathcal{P}(R) \cong \mathbb{N}$, because in this case finitely generated (projective) modules are just finite-dimensional vector spaces over R.

Given any ring R, the monoid of projectives of R, $\mathcal{P}(R)$, has the following properties:

(1) if $p, q \in \mathcal{P}(R)$ are such that $p + q = 0$ then $p = q = 0$;

(2) for every element $p \in \mathcal{P}(R)$ there exist $q \in \mathcal{P}(R)$ and a positive integer r such that $p + q = r.e$. This is because $p = [P]$ for some finitely generated projective R-module P, thus a direct summand of a free module of finite rank.

A monoid satisfying condition (1) above is called *conical.*

In 1974, Bergman proved that any *finitely generated* commutative monoid with a distinguished element satisfying conditions (1) and (2) occurs as the monoid of projectives of a ring. In precise terms,

Theorem 1.2.1 (Bergman). *Let k be any commutative field and M a finitely generated commutative monoid with a distinguished element $e \neq 0$ such that:*

(i) *for every $x, y \in M$ such that $x + y = 0$ we have $x = y = 0$,*

(ii) *for every $x \in M$ there exist $y \in M$ and $n \in \mathbb{N}$ such that $x + y = n.e$.*

Then there exists a k-algebra R which is right and left hereditary, such that $\mathcal{P}(R) \cong M$ as monoids with distinguished elements.

Proof. See [4, Th. 6.2]. □

We can restate some of the definitions of Section 1.1 in terms of the monoid of projectives of a ring.

Proposition 1.2.2. *Let R be a ring. Then*

(a) *R has IBN if and only if the map $\mathbb{N} \longrightarrow \mathcal{P}(R)$ is injective;*

(b) *every finitely generated R-module is free if and only if the map $\mathbb{N} \longrightarrow \mathcal{P}(R)$ is surjective.*

Thus, R is projective free if and only if $\mathbb{N} \longrightarrow \mathcal{P}(R)$ is an isomorphism.

Proof. Let $\varphi : \mathbb{N} \longrightarrow \mathcal{P}(R)$ be the canonical morphism from $\mathbb{N}$ to $\mathcal{P}(R)$ sending 1 to $e = [R]$.

(a) Suppose that R has IBN and that $\varphi(n) = \varphi(m)$ in $\mathcal{P}(R)$ for some $n, m \in \mathbb{N}$. This implies $R^n \cong R^m$. By IBN, $n = m$. Thus, φ is injective. Conversely, suppose that φ injective. Since $R^n \cong R^m$ implies $\varphi(n) = n.e = m.e = \varphi(m)$, by injectivity of φ, it follows that $n = m$. Therefore, R has IBN.

(b) If every finitely generated projective module is free, given $p = [P]$ in $\mathcal{P}(R)$, there exists $n \in \mathbb{N}$ such that $P \cong R^n$. So, $p = n.e = \varphi(n)$. Therefore, φ is surjective. Conversely, if φ is surjective, given any finitely generated projecive module P, there exists $n \in \mathbb{N}$ such that $n.e = \varphi(n) = [P]$, that is, $P \cong R^n$.

The last statement follows immediately from (a) and (b). □

Proposition 1.2.3. *Let R be a ring and let $e = [R]$ be the distinguished element of $\mathcal{P}(R)$. Then R*

(a) *has UGN if and only if $m.e + p = n.e$, for $m, n \in \mathbb{N}$ and $p \in \mathcal{P}(R)$, implies $n \geq m$;*

(b) *is weakly finite if and only if $n.e = n.e + p$, for $n \in \mathbb{N}$ and $p \in \mathcal{P}(R)$, implies $p = 0$;*

(c) *is Hermite if and only if $m.e + p = n.e$, for $m, n \in \mathbb{N}$ and $p \in \mathcal{P}(R)$, implies $n \geq m$ and $p = (n - m).e$;*

(d) *is power free if and only if R has IBN and for every $p \in \mathcal{P}(R)$ there exist $n = n(p), r = r(p) \in \mathbb{N}$ such that $n.p = r.e$.*

(e) *is projective trivial if and only if R has IBN and there exists $x \in \mathcal{P}(R)$ such that for every $p \in \mathcal{P}(R)$, there exists $n = n(p)$ such that $p = n.x$.* □

The proofs are direct applications of the definitions of the terms involved, as is the proof of the following consequence.

Corollary 1.2.4. *A ring R is projective trivial if and only if $\mathcal{P}(R) \cong \frac{1}{r}\mathbb{N}$ for some $r \in \mathbb{N}$.* $\square$

We say that a ring R has the *cancellation of projectives property* if $\mathcal{P}(R)$ satisfies the cancellation law, that is, if $p + q = p' + q$ for $p, p', q \in \mathcal{P}(R)$ implies $p = p'$.

Proposition 1.2.5. *Let R be a ring and denote by e the distinguished element $[R]$ of $\mathcal{P}(R)$. Then R satisfies the cancellation of projectives property if and only if $p + e = p' + e$ in $\mathcal{P}(R)$ implies $p = p'$.*

Proof. We just have to prove that the condition is sufficient. Let p, p', q be elements of $\mathcal{P}(R)$ such that $p + q = p' + q$. There exist $q' \in \mathcal{P}(R)$ and $n \in \mathbb{N}$ such that $q + q' = n.e$. So, $p + n.e = p + q + q' = p' + q + q' = p' + n.e$. By induction on n, $p = p'$. $\square$

Let R, S be rings and $f : R \longrightarrow S$ a ring homomorphism. Then f induces a morphism $f_* = \mathcal{P}(f)$ of monoids with distinguished element from $\mathcal{P}(R)$ into $\mathcal{P}(S)$ in the following way.

$$\begin{array}{rcccc} f_* & : & \mathcal{P}(R) & \longrightarrow & \mathcal{P}(S) \\ & & [P] & \longmapsto & [P \otimes_R S], \end{array}$$

where the right S-module $P \otimes_R S$ is obtained by extension of scalars. More precisely, we regard S as a left R-module via

$$\begin{array}{ccc} R \times S & \longrightarrow & S \\ (r, s) & \longmapsto & f(r)s \end{array}$$

and we can form the abelian group $P \otimes_R S$, which, in fact, carries a right S-module structure such that $(x \otimes s)s' = x \otimes ss'$ for all $x \in P$ and $s, s' \in S$. Since P is a finitely generated projective R-module, there exists an R-module Q and $n \in \mathbb{N}$ such that $P \oplus Q \cong R^n$. So, $(P \otimes_R S) \oplus (Q \otimes_R S) \cong (P \oplus Q) \otimes_R S \cong R^n \otimes_R S \cong S^n$.

Therefore, $P \otimes_R S$ is a finitely generated projective S-module. It is clear that $f_*([P])$ depends only on the isomorphism class of P and not on P itself. Moreover, f_* is indeed a morphism of commutative monoids with distinguished element, because the tensor product operation commutes with the formation of direct sums and $f_*([R]) = [R \otimes_R S] = [S]$. In fact, $\mathcal{P}$ is a functor from the category of rings to the category of commutative monoids with distinguished element.

If we start with an injective ring homomorphism $f : R \longrightarrow S$, then f_* may not be injective but, as the following proposition shows, it will be a *faithful* morphism, *i.e.*, if $f_*([P]) = 0$ for some $[P] \in \mathcal{P}(R)$ then $[P] = 0$.

Proposition 1.2.6. *Let R and S be rings and $f : R \longrightarrow S$ an injective ring homomorphism. Then the induced monoid morphism $f_* : \mathcal{P}(R) \longrightarrow \mathcal{P}(S)$ is a faithful morphism.*

Proof. Suppose that $f_*([P]) = 0$ for some $[P] \in \mathcal{P}(R)$. Consider the exact sequence of left R-modules

$$0 \longrightarrow R \xrightarrow{f} S$$

and tensor it up with the projective, and therefore flat, module P. This yields an exact sequence

$$0 \longrightarrow P \longrightarrow P \otimes_R S.$$

Since $[P \otimes_R S] = f_*([P]) = 0$, it follows that P is the zero module or, in other words, that $[P] = 0$. So f_* is faithful. $\square$

An example of the use of the functor $\mathcal{P}$ is the calculation of $\mathcal{P}(\mathfrak{M}_n(R))$ in terms of $\mathcal{P}(R)$. Let R be a ring and $\mathcal{F}R$ denote the category of finitely generated projective right R-modules and similarly with $\mathcal{F}\mathfrak{M}_n(R)$. We know that there is an equivalence between $\mathcal{F}R$ and $\mathcal{F}\mathfrak{M}_n(R)$, given by additive functor $F : \mathcal{F}R \longrightarrow \mathcal{F}\mathfrak{M}_n(R)$ defined by $F(P) = P \times \ldots \times P = \{(x_1, \ldots, x_n) : x_i \in P\}$. The finitely generated projective $\mathfrak{M}_n(R)$-module $F(R)$ of the rows $(r_1, \ldots, r_n)$, where $r_i \in R$, has the property that $F(R)^n \cong \mathfrak{M}_n(R)$, that is, in $\mathcal{P}(\mathfrak{M}_n(R))$, we have $n.[F(R)] = [\mathfrak{M}_n(R)]$. We will prove that $\mathcal{P}(\mathfrak{M}_n(R)) \cong \frac{1}{n}\mathcal{P}(R)$. Consider the map

$$\begin{array}{ccc} \frac{1}{n}\mathcal{P}(R) & \xrightarrow{\Phi} & \mathcal{P}(\mathfrak{M}_n(R)) \\ \frac{1}{n}[P] & \longmapsto & [F(P)]. \end{array}$$

Φ is a well defined map, because F preserves isomorphisms. Also, Φ is a morphism of monoids with distinguished elements, because F preserves direct sums, sends the zero module to the zero module and $\Phi(\frac{1}{n}(n.[R])) = \Phi(\frac{1}{n}[R^n]) = [F(R^n)] = n.[F(R)] = [\mathfrak{M}_n(R)]$. The fact that F defines an equivalence implies that Φ is surjective and injective, so an isomorphism. Therefore, we have

Proposition 1.2.7. *Let R be a ring and n a positive integer. Then $\mathcal{P}(\mathfrak{M}_n(R))$ is isomorphic to $\frac{1}{n}\mathcal{P}(R)$ as commutative monoids with distinguished element.* □

The next result describes the behaviour of $\mathcal{P}$ under Cartesian products.

Proposition 1.2.8. *Let R and S be rings. Then $\mathcal{P}(R \times S)$ is isomorphic to $\mathcal{P}(R) \times \mathcal{P}(S)$.*

Proof. The map

$$\begin{array}{rcl} \mathcal{P}(R) \times \mathcal{P}(S) & \longrightarrow & \mathcal{P}(R \times S) \\ ([P], [Q]) & \longmapsto & [P \times Q] \end{array}$$

is a well defined injective morphism of monoids with a distinguished element. It is also surjective, because any right $R \times S$-module L can be written as the product of the R-module $L(1,0)$ with the S-module $L(0,1)$. □

1.3 Bergman's coproduct theorems

In this section we will state Bergman's theorems on the global dimension of coproducts of rings and on the description of their the monoid of projectives. Proofs will not be given, since they can be found in the original paper [3] and, in a simplified version, in [9, Ch. 5]. For a commented exposition without proofs, see [14, Ch. 2].

After stating the theorems, we will use them in order to construct some examples of firs.

1.3.1 Bergman's theorems

Let K be an arbitrary fixed ring. We define the category of *K-rings* as the category having as objects rings R given with a ring homomorphism $K \longrightarrow R$ and morphisms being commutative triangles of homomorphisms of rings. Equivalently, a ring R is said to be a K-ring if R is also a K-bimodule satisfying the following compatibility laws:

$$r(sx) = (rs)x; \; r(xs) = (rx)s; \; (xr)s = x(rs),$$

for all $r, s \in R$ and $x \in K$. A K-ring R is said to be *faithful* if the natural map $K \longrightarrow R$ is injective. In this category a coproduct of a family $(R_\lambda)_{\lambda\in\Lambda}$, denoted by $R = *_K R_\lambda$, always exists and can be defined as the ring with generators given by the disjoint union of sets of generators of the R_λ and whose relations are the relations in each R_λ as well as the relations derived from the identifications of the actions of K, that is, in $*_K R_\lambda$, we have

$$(r_1 x) r_2 = r_1 (x r_2),$$

for any $r_1 \in R_{\lambda_1}, r_2 \in R_{\lambda_2}, x \in K$. This definition makes R into a K-ring. If, for each $\lambda \in \Lambda$, R_λ is embedded in R, the coproduct R is said to be *faithful*; in this case, R is also known as the free product of the rings R_λ amalgamating the images of K. Moreover, if the images of R_λ and R_μ intersect exactly in K for all $\lambda \neq \mu$, then R is said be *separating*. The following theorem gives sufficient conditions for the coproduct to be faithful and separating.

Theorem 1.3.1 (Cohn). *Let K be any ring and $(R_\lambda)_{\lambda\in\Lambda}$ a family of faithful K-rings such that R_λ/K is free as right K-module for all $\lambda \in \Lambda$. Then the coproduct $*_K R_\lambda$ is faithful and separating.*

For a proof see [9, Th. 5.2.3, p. 213]. □

In the case where K is a skew field, the hypotheses of Theorem 1.3.1 are all satisfied, so the coproduct is faithful and separating.

Now let us move on to Bergman's theorems. The first result relates the global dimension of the coproduct to that of the factors.

Theorem 1.3.2 (Bergman). *Let K be a semisimple ring and $(R_\lambda)_{\lambda\in\Lambda}$ a family of faithful K-rings. Let R denote their coproduct $*_K R_\lambda$. Then*

$$r.gl.dim(R) = sup\{r.gl.dim(R_\lambda) : \lambda \in \Lambda\}$$

if at least one of the R_λ has positive right global dimension. If all R_λ are semisimple, then $r.gl.dim(R) \leq 1$.

This is Cor. 2.5 of [3]. A proof for the case where K is a skew field can also be found in [9, Th. 5.3.5, p. 218]. □

By symmetry, we can obtain the same result for the left global dimensions. Theorem 1.3.2 has an immediate corollary.

Corollary 1.3.3. *Let K be a semisimple ring and $(R_\lambda)_{\lambda\in\Lambda}$ a family faithful K-rings. Suppose that for all $\lambda \in \Lambda$, R_λ is a hereditary ring. Then their coproduct $*_K R_\lambda$ is also a hereditary ring.*

Proof. This is just an application of Theorem 1.3.2 with the fact that a right hereditary ring R is a ring such that $r.gl.dim(R) \leq 1$. □

The next result describes $\mathcal{P}$, the monoid of projectives, for a coproduct.

Theorem 1.3.4 (Bergman). *Let K be a semisimple ring and $(R_\lambda)_{\lambda\in\Lambda}$ a family of faithful K-rings. Let R denote their coproduct $*_K R_\lambda$. Then*

$$\mathcal{P}(R) \cong \coprod_{\mathbb{N}} \mathcal{P}(R_\lambda).$$

This is Cor. 2.8 of [3]. Again, for the case where K is a skew field, see [9, Th. 5.3.8, p. 221]. □

1.3.2 Applications

In this subsection, Theorems 1.3.2 and 1.3.4 will be used to prove that two important constructions of rings provide examples of firs which, under extension of their centres, will be analysed later on. All the results in this subsection can be found in [9], for instance.

Proposition 1.3.5. *Let $(R_\lambda)_{\lambda\in\Lambda}$ be a family of firs containing a common subfield K and denote their coproduct amalgamating K by R. Then R is again a fir.*

Proof. This is Th. 5.3.9 on p. 222 of [9]. We include a proof. By hypothesis, for every $\lambda \in \Lambda$, R_λ is hereditary. So R is hereditary, by Corollary 1.3.3. Also, by hypothesis, for every $\lambda \in \Lambda$, $\mathcal{P}(R_\lambda) \cong \mathbb{N}$. Using Theorem 1.3.4, we conclude that $\mathcal{P}(R) \cong \mathbb{N}$. Therefore, R is hereditary and projective free, so it is a fir. $\square$

We can use the above result to prove that coproducts of skew fields amalgamating a common skew field and tensor rings are firs.

Corollary 1.3.6. *Let $(S_\lambda)_{\lambda\in\Lambda}$ be a family of skew fields containing a common subfield K. Then $*_K S_\lambda$ is a fir.*

Proof. Every skew field is a fir. The result follows by Proposition 1.3.5. $\square$

Corollary 1.3.7. *Let X be any set and D a skew field. Then the ring $R = D_K\langle X\rangle$, where K is a subfield of D, is a fir.*

Proof. Notice that R can be written as

$$R \cong D *_K K\langle X\rangle.$$

Since both D and $K\langle X\rangle$ are firs, by Proposition 1.3.5, R is a fir. $\square$

1.4 Preliminary results

In this section we will present some preliminary results which will be used throughout the thesis. We start with a well known result.

Lemma 1.4.1. *Let U be a skew field with centre k and E/k a finite commutative field extension. Then $U_E = U \otimes_k E$ is isomorphic as an E-algebra to a matrix ring $\mathfrak{M}_r(K)$, where K is a skew field with centre E and r divides $[E : k]$.*

Proof. U_E is simple, because U is central simple and E is simple. Moreover, U_E is artinian, because it is a left vector space over U of finite dimension $[E : k]$. So $U_E \cong \mathfrak{M}_r(K)$ for some $r \geq 1$ and K a skew field. Finally, if S denotes the simple U_E-module, then $S^r \cong U_E$. So, if $s = dim_U S$, we have $rs = [U_E : U] = [E : k]$. $\square$

Given a ring R which can be embedded in a skew field U and which contains the centre k of U, we can use the above lemma to regard $R_E = R \otimes_k E$ as a subring of a full matrix ring over a skew field. We shall explore this idea further in Chapter 4.

The next result concerns the behaviour of localization under extension of the ground field.

Lemma 1.4.2. *Let R be an algebra over a commutative field k, Σ a set of matrices over R and E/k a commutative field extension. Then we have*

$$(R_\Sigma)_E \cong (R_E)_\Sigma .$$

Proof. Let $\lambda : R \longrightarrow R_\Sigma$ be the universal Σ-inverting homomorphism. Regarding Σ as a set of matrices over R_E, let $\mu : R_E \longrightarrow (R_E)_\Sigma$ be the universal Σ-inverting map in this context. Since $\lambda \otimes E$ inverts all matrices in Σ, there exist an E-algebra homomorphism

$$f : (R_E)_\Sigma \longrightarrow (R_\Sigma)_E$$

such that

$$f\mu = \lambda \otimes E. \tag{1.4.1}$$

Let $i : R \longrightarrow R_E$ be the canonical injective homomorphism from R into R_E. Since μi is Σ-inverting, there exists a homomorphism $h : R_\Sigma \longrightarrow (R_E)_\Sigma$, such that

$$h\lambda = \mu i. \tag{1.4.2}$$

This induces an E-algebra homomorphism

$$g : (R_\Sigma)_E \longrightarrow (R_E)_\Sigma .$$

such that

$$gj = h, \tag{1.4.3}$$

where $j : R_\Sigma \longrightarrow (R_\Sigma)_E$ is the canonical injection. The situation is illustrated in the following commutative diagram.

$$
\begin{tikzcd}
 & R \arrow[dl, "i"'] \arrow[dr, "\lambda"] & \\
R_E \arrow[dr, "\lambda\otimes E"] \arrow[ddr, "\mu"'] & & R_\Sigma \arrow[dl, "j"'] \arrow[ddl, "h"] \\
 & (R_\Sigma)_E \arrow[d, shift left, "g"] & \\
 & (R_E)_\Sigma \arrow[u, shift left, "f"] &
\end{tikzcd}
$$

First, note that

$$
\begin{aligned}
fh\lambda &= f\mu i, \text{ by (1.4.2)} \\
&= (\lambda \otimes E)i, \text{ by (1.4.1)} \\
&= j\lambda.
\end{aligned}
$$

So, by uniqueness,

$$fh = j. \tag{1.4.4}$$

We will now show that $g = f^{-1}$. First,

$$
\begin{aligned}
gf\mu i &= gfh\lambda, \text{ by (1.4.2)} \\
&= gj\lambda, \text{ by (1.4.4)} \\
&= h\lambda, \text{ by (1.4.3)} \\
&= \mu i, \text{ by (1.4.2)}.
\end{aligned}
$$

Since all f, g, μ are E-algebra maps, the above implies that $gf\mu = \mu$. By uniqueness, $gf = 1_{(R_E)_\Sigma}$. And

$$
\begin{aligned}
fgj &= fh, \text{ by (1.4.3)} \\
&= j, \text{ by (1.4.4)}.
\end{aligned}
$$

So, $fg = 1_{(R_\Sigma)_E}$. Hence, f is an isomorphism. $\square$

We will now look at coproducts under ground field extensions.

Lemma 1.4.3. *Let k be a commutative field and R, S k-algebras with a common subfield F. If E/k is a commutative field extension, then*

$$(R *_F S)_E \cong R_E *_{F_E} S_E.$$

Proof. Let $i_R : R \longrightarrow R *_F S$ and $i_S : S \longrightarrow R *_F S$ be the canonical homomorphisms. Also denote by $\alpha : R \longrightarrow R_E$, $\beta : S \longrightarrow S_E$ and $\gamma : R *_F S \longrightarrow (R *_F S)_E$ the canonical injections. By definition of these maps, we have

$$\gamma i_R = (i_R \otimes E)\alpha \tag{1.4.5}$$

$$\gamma i_S = (i_S \otimes E)\beta. \tag{1.4.6}$$

If $I_R : R_E \longrightarrow R_E *_{F_E} S_E$ and $I_S : S_E \longrightarrow R_E *_{F_E} S_E$ are the canonical coproduct maps, then, by the coproduct property, there exists an E-algebra homomorphism

$$f : R_E *_{F_E} S_E \longrightarrow (R *_F S)_E$$

such that

$$f I_R = i_R \otimes E \tag{1.4.7}$$

$$f I_S = i_S \otimes E. \tag{1.4.8}$$

To get a map in the other direction, consider the maps $I_S\beta : S \longrightarrow R_E *_{F_E} S_E$ and $I_R\alpha : R \longrightarrow R_E *_{F_E} S_E$. They give rise to a homomorphism

$$h : R *_F S \longrightarrow R_E *_{F_E} S_E$$

such that

$$h i_R = I_R\alpha \tag{1.4.9}$$

$$h i_S = I_S\beta. \tag{1.4.10}$$

And this induces an E-algebra homomorphism

$$g : (R *_F S)_E \longrightarrow R_E *_{F_E} S_E$$

such that

$$g\gamma = h. \tag{1.4.11}$$

So we have the following diagram.

$$
\begin{tikzcd}
R \arrow[r, "i_R"] \arrow[dd, "\alpha"'] & R *_F S \arrow[d, "\gamma"] & S \arrow[l, "i_S"'] \arrow[dd, "\beta"] \\
& (R *_F S)_E \arrow[d, shift left, "g"] & \\
R_E \arrow[r, "I_R"'] \arrow[ur, "i_R \otimes E"] & R_E *_{F_E} S_E \arrow[u, shift left, "f"] & S_E \arrow[l, "I_S"] \arrow[ul, "i_S \otimes E"']
\end{tikzcd}
$$

Now

$$
\begin{aligned}
gfI_R\alpha &= g(i_R \otimes E)\alpha, \text{ by (1.4.7)} \\
&= g\gamma i_R, \text{ by (1.4.5)} \\
&= hi_R, \text{ by (1.4.11)} \\
&= I_R\alpha, \text{ by (1.4.9).}
\end{aligned}
$$

Since all g, f, I_R are all E-algebra homomorphisms, this implies that $gfI_R = I_R$. Similarly, $gfI_S = I_S$. Therefore, by uniqueness, $gf = 1_{R_E *_{F_E} S_E}$. Moreover,

$$
\begin{aligned}
fg\gamma i_R &= fhi_R, \text{ by (1.4.11)} \\
&= fI_R\alpha, \text{ by (1.4.9)} \\
&= (i_R \otimes E)\alpha, \text{ by (1.4.7)} \\
&= \gamma i_R, \text{ by (1.4.5).}
\end{aligned}
$$

Similarly, $fg\gamma i_S = \gamma i_S$. By the coproduct property, this implies that $fg\gamma = \gamma$ and, since f and g are E-linear, it follows that $fg = 1_{(R *_F S)_E}$. Hence, f is an isomorphism. □

The last result in this section is about the behaviour of the global dimension under extension of the ground field.

Proposition 1.4.4. *Let k be a commutative ring and A a hereditary k-algebra. If E is a finite separable extension of k, then A_E is a hereditary E-algebra.*

The proof of the proposition will follow from two well known lemmas which generalize results by Auslander, Reiten and Smalø in [2, Props. III.4.6. and III.4.9, pp. 89-91]. For completeness we include proofs.

Lemma 1.4.5. *Let B be a ring and A a subring with a split exact sequence of A-bimodules*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

where C is projective as a right A-module. Then $r.gl.dim(A) \leq r.gl.dim(B)$.

Proof. By hypothesis, $B \cong A \oplus C$ as A-bimodules. If M is any right A-module,

$$M \otimes_A B \cong M \otimes_A (A \oplus C) \cong (M \otimes_A A) \oplus (M \otimes_A C) \cong M \oplus (M \otimes_A C).$$

So,

$$pd_A(M) \leq pd_A(M \otimes_A B). \tag{1.4.12}$$

Let

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \otimes_A B \longrightarrow 0 \tag{1.4.13}$$

be a minimal projective B-resolution of $M \otimes_A B$ Since C is a projective right A-module, so is B, because $B \cong A \oplus C$. Therefore, (1.4.13) is a projective A-resolution of $M \otimes_A B$. For if P is a projective right B-module, then there exist a right B-module Q and an index set I such that $P \oplus Q \cong B^I \cong (A \oplus C)^I \cong A^I \oplus C^I$, which is a projective right A-module. Hence

$$pd_A(M \otimes_A B) \leq pd_B(M \otimes_A B).$$

With (1.4.12), this yields

$$pd_A(M) \leq pd_A(M \otimes_A B) \leq pd_B(M \otimes_A B) \leq r.gl.dim(B).$$

So, $r.gl.dim(A) \leq r.gl.dim(B)$. □

Lemma 1.4.6. *Let A, B be rings such that B is separable as A-ring and projective as left A-module. Then $r.gl.dim(B) \leq r.gl.dim(A)$.*

First let us recall that to say that an A-ring B is *separable* means that the exact sequence of B-bimodules

$$0 \longrightarrow \Omega \longrightarrow B \otimes_A B \xrightarrow{\mu} B \longrightarrow 0,$$

where μ is the multiplication map, *i.e.* $\mu(x \otimes y) = xy$, and Ω is the kernel of μ, is a split exact sequence.

Proof of Lemma 1.4.6. By hypothesis, $B \otimes_A B \cong B \oplus \Omega$ as B-bimodules. If N is any right B-module, then

$$N \otimes_A B \cong N \otimes_B (B \otimes_A B) \cong N \otimes_B (B \oplus \Omega) \cong N \oplus (N \otimes_B \Omega).$$

So,

$$pd_B(N) \leq pd_B(N \otimes_A B). \tag{1.4.14}$$

Let

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow N \longrightarrow 0$$

be a minimal projective A-resolution of N and tensor it with B over A:

$$\cdots \longrightarrow P_1 \otimes_A B \longrightarrow P_0 \otimes_A B \longrightarrow N \otimes_A B \longrightarrow 0. \tag{1.4.15}$$

Since B is a projective left A-module, (1.4.15) is again exact. Moreover, if P is any projective right A-module and Q is a right A-module such that $P \oplus Q$ is free, say, $P \oplus Q \cong A^I$ then

$$(P \otimes_A B) \oplus (Q \otimes_A B) \cong (P \oplus Q) \otimes_A B \cong A^I \otimes_A B \cong B^I.$$

Thus, $P \otimes_A B$ is a projective right B-module. Hence, (1.4.15) is a projective B-resolution of $N \otimes_A B$. So

$$pd_B(N \otimes_A B) \leq pd_A(N),$$

which, with (1.4.14) gives

$$pd_B(N) \leq pd_B(N \otimes_A B) \leq pd_A(N) \leq r.gl.dim(A).$$

Therefore, $r.gl.dim(B) \leq r.gl.dim(A)$. $\square$

We are now able to prove the proposition.

Proof of Proposition 1.4.4. First we will show that A_E is separable over A. Since E/k is a finite separable field extension, E is separable as k-algebra (cf. [8, Prop. 5.5.6, p. 190]), so there exists a split exact sequence

$$0 \longrightarrow \Omega \longrightarrow E \otimes_k E \longrightarrow E \longrightarrow 0. \qquad (1.4.16)$$

Tensoring (1.4.16) with A over k, we get a split exact sequence

$$0 \longrightarrow A \otimes_k \Omega \longrightarrow A \otimes_k E \otimes_k E \longrightarrow A \otimes_k E \longrightarrow 0.$$

Since $A \otimes_k E \otimes_k E \cong (A \otimes_k E) \otimes_A (A \otimes_k E)$, it follows that A_E is separable over A.

Now we will show that A_E satisfies the conditions on B in Lemmas 1.4.5 and 1.4.6. If $\{v_1, \dots, v_n\}$ is a basis for E as a vector space over k with $v_1 = 1$, then, as A-bimodules,

$$A \otimes_k E \cong A \otimes (\oplus_{i=1}^n k v_i) \cong \oplus_{i=1}^n (A \otimes_k k v_i) \cong \oplus_{i=1}^n A.$$

Since $v_1 = 1$, the canonical map $A \longrightarrow A \otimes_k E$ can be regarded, via the above isomorphism, as the inclusion $A \longrightarrow \oplus_{i=1}^n A$ in the first term. Therefore, $A_E/A \cong \oplus_{i=2}^n A$. So both A_E and A_E/A are projective A-modules on both sides (in fact, they are free modules). Moreover, the sequence

$$0 \longrightarrow A \longrightarrow A_E \longrightarrow A_E/A \longrightarrow 0$$

is a split exact sequence of A-bimodules. Thus, by Lemmas 1.4.5 and 1.4.6,

$$r.gl.dim(A_E) = r.gl.dim(A).$$

□

Chapter 2

Coproducts of commutative monoids with distinguished element

In this chapter we will look back at the category $\mathcal{C}$ of commutative monoids with distinguished element, defined in section 1.2, and describe more precisely the construction of the coproduct in $\mathcal{C}$. We will use this construction to prove results about properties of the factors that are transmitted to the coproduct.

2.1 The construction of the coproduct

Let M, N be commutative monoids with distinguished elements and let $\varphi : \mathbb{N} \longrightarrow M$ and $\psi : \mathbb{N} \longrightarrow N$ be the associated monoid morphisms. Let $P = M \times N$ and consider the monoid $M \coprod_{\mathbb{N}} N$ defined by $P/E(\rho)$, where $E(\rho)$ is the congruence generated by the relation

$$\rho = \{((\varphi(n), 0), (0, \psi(n))) : n \in \mathbb{N}\}.$$

Let $f : M \longrightarrow M \coprod_{\mathbb{N}} N$ be the morphism defined by $f(x) = \overline{(x, 0)}$ where $\overline{(x, 0)}$ stands for the equivalence class of $(x, 0)$ with respect to the congruence $E(\rho)$ in $M \times N$. Similarly, define a morphism $g : N \longrightarrow M \coprod_{\mathbb{N}} N$ by $g(y) = \overline{(0, y)}$. The monoid $M \coprod_{\mathbb{N}} N$ is a commutative monoid with distinguished element, whose associated morphism $\mathbb{N} \longrightarrow M \coprod_{\mathbb{N}} N$ is given by the morphism $f\varphi = g\psi$. It is immediate to see that $M \coprod_{\mathbb{N}} N$ is indeed the coproduct of (M, φ) and (N, ψ) in the category of commutative monoids with a distinguished element.

In the case where the morphisms φ and ψ are injective (adapting [12]), we say that the pair $\{(M, \varphi), (N, \psi)\}$ is *embeddable* in the coproduct $M \coprod_{\mathbb{N}} N$ if the morphisms f and g are monomorphism and *strongly embeddable* if it is embeddable and $f(M) \cap g(N) = f\varphi(\mathbb{N})(= g\psi(\mathbb{N}))$.

In this section we will give sufficient conditions for the coproduct of commutative monoids with distinguished element to have the strong embeddability property.

First let us see how to express equality of elements in $M \coprod_{\mathbb{N}} N$ in terms of the elements of M and N. This is provided by a result of Howie, which we include here. We start with a definition. Let S be an arbitrary multiplicative semigroup and ρ a relation on S, *i.e.* a subset of S^2. Given $c, d \in S$, we say that c is connected to d by an *elementary ρ-transition*, and write $c \rightarrow d$, if

$$c = xay, \quad d = xby$$

for some $x, y \in S^1$, where either (a, b) or (b, a) belongs to ρ and the symbol S^1 stands either for S if S is a monoid or for the monoid obtained from S by adjoining an element 1 such that $x1 = 1x = x$, for every $x \in S$ and $11 = 1$.

Proposition 2.1.1. *Let ρ be a relation on a multiplicative semigroup S and let $a, b \in S$. Then $(a, b) \in E(\rho)$ if and only if either $a = b$ or, for some $m \in \mathbb{N}$, there is a sequence*

$$a = z_1 \rightarrow z_2 \rightarrow \cdots \rightarrow z_m = b$$

of elementary ρ-transitions connecting a to b.

Proof. See [12, Prop. 1.5.9, p. 27]. □

We can interpret Proposition 2.1.1 in the case where S is a commutative monoid and the relation ρ is symmetric in the following way.

Corollary 2.1.2. *Let ρ be a symmetric relation on an additive commutative monoid M and let $a, b \in M$. Then $(a, b) \in E(\rho)$ if and only if either $a = b$ or, for some $m \in \mathbb{N}$, there is a sequence*

$$a = z_1 \to z_2 \to \cdots \to z_m = b,$$

where for each $i = 1, \ldots, m-1$, $z_i \to z_{i+1}$ if and only if $z_i = x_i + a_i$ and $z_{i+1} = x_i + b_i$ for some $x_i \in S$ and $(a_i, b_i) \in \rho$. □

We have seen that given commutative monoids with distinguished element (M, φ) and (N, ψ), to construct their coproduct we define a relation ρ on $M \times N$ by

$$\rho = \{((\varphi(n), 0), (0, \psi(n))) : n \in \mathbb{N}\}.$$

This relation is not symmetric, but since $E(\rho)$, the congruence generated by ρ, is always symmetric, we can consider the relation

$$\rho' = \{((\varphi(n), \psi(m)), (\varphi(m), \psi(n))) : n, m \in \mathbb{N}\}$$

which is obviously symmetric, contains ρ and is contained in $E(\rho)$. So $E(\rho) = E(\rho')$.

So now we can say, using Corollary 2.1.2, that given two elements $\xi = \overline{(x, y)}, \eta = \overline{(x', y')} \in M \coprod_{\mathbb{N}} N$, $\xi = \eta$ if and only if, for some $n \in \mathbb{N}$, there are for each $i = 1, \ldots, n+1$ elements $z_i \in M$ and $t_i \in N$ such that

$$(x, y) = (z_1, t_1) \to (z_2, t_2) \to \cdots \to (z_{n+1}, t_{n+1}) = (x', y').$$

That is, for each $i = 1, \ldots, n$ there are $x_i \in M$, $y_i \in N$ and $a_i, b_i \in \mathbb{N}$ such that

$$\begin{aligned}
&(x, y) = (x_1, y_1) + (\varphi(a_1), \psi(b_1)) \\
&(x_i, y_i) + (\varphi(b_i), \psi(a_i)) = (x_{i+1}, y_{i+1}) + (\varphi(a_{i+1}), \psi(b_{i+1})) \quad , \quad i = 1, \ldots, n-1 \\
&(x', y') = (x_n, y_n) + (\varphi(b_n), \psi(a_n))
\end{aligned}$$

Note that the case $(x, y) = (x', y')$ is also covered: take $n = 1$ and $a_1 = b_1 = 0$.

Rewriting the above system, we can say that $\overline{(x, y)} = \overline{(x', y')}$ in $M \coprod_{\mathbb{N}} N$ if and only if for some $n \in \mathbb{N}$ there are, for each $i = 1, \ldots, n$, $x_i \in M$, $y_i \in N$ and

$a_i, b_i \in \mathbb{N}$ such that

$$\begin{array}{ll} x = x_1 + \varphi(a_1) & y = y_1 + \psi(b_1) \\ x_1 + \varphi(b_1) = x_2 + \varphi(a_2) & y_1 + \psi(a_1) = y_2 + \psi(b_2) \\ \vdots & \vdots \\ x_i + \varphi(b_i) = x_{i+1} + \varphi(a_{i+1}) & y_i + \psi(a_i) = y_{i+1} + \psi(b_{i+1}) \\ \vdots & \vdots \\ x_{n-1} + \varphi(b_{n-1}) = x_n + \varphi(a_n) & y_{n-1} + \psi(a_{n-1}) = y_n + \psi(b_n) \\ x_n + \varphi(b_n) = x' & y_n + \psi(a_n) = y'. \end{array}$$

We will now look at conditions that guarantee the strong embeddability property in the coproduct $M \coprod_{\mathbb{N}} N$. We say that a monoid M is *conical* if $x + x' = 0$ implies $x = x' = 0$ in M. Borrowing a term from the theory of monoids of projectives, we say that a commutative monoid with a distinguished element (M, φ) has the *UGN property* if $\varphi(n) = x + \varphi(m)$, with $n, m \in \mathbb{N}$ and $x \in M$, implies $n \geq m$. Note that the UGN property implies, in particular, that φ is injective. We will show that if both M and N are conical and have the UGN property, then they are strongly embeddable in $M \coprod_{\mathbb{N}} N$.

To show, for instance, that $f : M \longrightarrow M \coprod_{\mathbb{N}} N$, defined by $f(x) = \overline{(x, 0)}$, is injective, we must show that $\overline{(x, 0)} = \overline{(x', 0)}$ implies $x = x'$. But this is equivalent to saying that there is a system like the one above, with $y = y' = 0$ and that this system implies $x = x'$. In this particular case, the first equation on the right-hand side of the system would be

$$0 = y_1 + \psi(b_1).$$

If N is conical, this implies $y_1 = \psi(b_1) = 0$ and injectivity of ψ implies $b_1 = 0$. Similarly, the last equation on the right-hand side would be

$$y_n + \psi(a_n) = 0$$

and again, we have $y_n = 0$ and $a_n = 0$. So the fact that $\overline{(x, 0)} = \overline{(x', 0)}$ is equivalent to the existence of element $x_1, \ldots, x_n \in M$, $y_2, \ldots, y_{n-1} \in N$ and

$a_1, \dots, a_{n-1}, b_2, \dots b_n \in \mathbb{N}$ such that

$$\begin{array}{lll}
x = x_1 + \varphi(a_1) & & \\
x_1 = x_2 + \varphi(a_2) & \psi(a_1) = y_2 + \psi(b_2) & \\
\vdots & \vdots & \\
x_i + \varphi(b_i) = x_{i+1} + \varphi(a_{i+1}) & y_i + \psi(a_i) = y_{i+1} + \psi(b_{i+1}) & (2.1.1) \\
\vdots & \vdots & \\
x_{n-1} + \varphi(b_{n-1}) = x_n & y_{n-1} + \psi(a_{n-1}) = \psi(b_n) & \\
x_n + \varphi(b_n) = x'. & &
\end{array}$$

We have the following theorem.

Theorem 2.1.3. *Let (M, φ) and (N, ψ) be commutative monoids with distinguished elements. If both M and N are conical and have the UGN property, then the pair $\{(M, \varphi), (N, \psi)\}$ is strongly embeddable in the coproduct $M \coprod_{\mathbb{N}} N$.*

For the proof we will need some lemmas first. The two following lemmas will be stated with the notation of the system (2.1.1) for $\overline{(x,0)} = \overline{(x',0)}$ shown above.

Lemma 2.1.4. *For $i = 1, \dots, n-2$, $\psi(a_1 + \ldots + a_i) = \psi(b_2 + \ldots + b_{i+1}) + y_{i+1}$.*

Proof. The proof will be by induction on i. For $i = 1$, this is just $\psi(a_1) = y_2 + \psi(b_2)$, the first equation on the right-hand side of the system. Now assume the result valid for some $1 < i < n-2$, we will prove it for $i+1$.

$$\begin{aligned}
\psi(a_1 + \ldots + a_i + a_{i+1}) &= \psi(a_1 + \ldots + a_i) + \psi(a_{i+1}) \\
&= \psi(b_2 + \ldots + b_{i+1}) + y_{i+1} + \psi(a_{i+1}) \\
&= \psi(b_2 + \ldots + b_{i+1}) + y_{i+2} + \psi(b_{i+2}) \\
&= \psi(b_2 + \ldots + b_{i+1} + b_{i+2}) + y_{i+2}.
\end{aligned}$$

This establishes the lemma. □

Corollary 2.1.5. *If N has the UGN property, the system (2.1.1) implies*

(a) *for $i = 1, \dots, n-2$, $a_1 + \ldots + a_i \geq b_2 + \ldots + b_{i+1}$,*

(b) *$a_1 + \ldots + a_{n-1} = b_2 + \ldots + b_n$.*

Proof. (a) By Lemma 2.1.4, $\psi(a_1 + \ldots + a_i) = \psi(b_2 + \ldots + b_{i+1}) + y_{i+1}$. Since N has the UGN property, it follows that $a_1 + \ldots + a_i \geq b_2 + \ldots + b_{i+1}$. (b) By Lemma 2.1.4, $\psi(a_1 + \ldots + a_{n-2}) = \psi(b_2 + \ldots + b_{n-1}) + y_{n-1}$, so

$$\begin{aligned}
\psi(a_1 + \ldots + a_{n-1}) &= \psi(a_1 + \ldots + a_{n-2}) + \psi(a_{n-1}) \\
&= \psi(b_2 + \ldots + b_{n-1}) + y_{n-1} + \psi(a_{n-1}) \\
&= \psi(b_2 + \ldots + b_{n-1}) + \psi(b_n) \\
&= \psi(b_2 + \ldots + b_n).
\end{aligned}$$

Since ψ is injective, it follows that $a_1 + \ldots + a_{n-1} = b_2 + \ldots + b_n$. □

We can now prove Theorem 2.1.3.

Proof of Theorem 2.1.3. We shall use the fact that, for $i = 1, \ldots, n-2$, $a_1 + \ldots + a_i \geq b_2 + \ldots + b_{i+1}$ (Corollary 2.1.5(a)) to write

$$\varphi(a_1 + \ldots + a_i) = \varphi(b_2 + \ldots + b_i) + \varphi(a_1 + \ldots + a_i - b_2 - \ldots - b_i).$$

Following the first column of the system we can write

$$\begin{aligned}
x &= x_1 + \varphi(a_1) \\
&= x_2 + \varphi(a_1 + a_2) \\
&= x_2 + \varphi(b_2) + \varphi(a_1 + a_2 - b_2) \\
&= x_3 + \varphi(a_1 + a_2 + a_3 - b_2) \\
&= \cdots \\
&= x_{n-1} + \varphi(a_1 + \ldots + a_{n-1} - b_2 - \ldots - b_{n-2}) \\
&= x_{n-1} + \varphi(b_{n-1}) + \varphi(a_1 + \ldots + a_{n-1} - b_2 - \ldots - b_{n-1}) \\
&= x_n + \varphi(b_n)\ , \text{ by Corollary 2.1.5 (b).}
\end{aligned}$$

So $x = x'$. Therefore, f is injective.

The proof that g is injective is analogous.

To prove that $f(M) \cap g(N) = f\varphi(\mathbb{N})$ it is enough (in fact equivalent) to prove that a system of the type

$$\begin{array}{ll}
x = x_1 + \varphi(a_1) & \\
x_1 = x_2 + \varphi(a_2) & \psi(a_1) = y_2 + \psi(b_2) \\
\vdots & \vdots \\
x_i + \varphi(b_i) = x_{i+1} + \varphi(a_{i+1}) & y_i + \psi(a_i) = y_{i+1} + \psi(b_{i+1}) \\
\vdots & \vdots \\
x_{n-1} + \varphi(b_{n-1}) = \varphi(a_n) & y_{n-1} + \psi(a_{n-1}) = y_n \\
 & y_n + \psi(a_n) = y
\end{array}$$

implies $x \in \varphi(\mathbb{N})$. Note that, in this case, Corollary 2.1.5 (a) is still valid. So we can proceed as above:

$$\begin{aligned}
x &= x_1 + \varphi(a_1) \\
&= x_2 + \varphi(a_1 + a_2) \\
&= x_2 + \varphi(b_2) + \varphi(a_1 + a_2 - b_2) \\
&= x_3 + \varphi(a_1 + a_2 + a_3 - b_2) \\
&= \cdots \\
&= x_{n-1} + \varphi(a_1 + \ldots + a_{n-1} - b_2 - \ldots - b_{n-2}) \\
&= x_{n-1} + \varphi(b_{n-1}) + \varphi(a_1 + \ldots + a_{n-1} - b_2 - \ldots - b_{n-1}) \\
&= \varphi(a_n) + \varphi(a_1 + \ldots + a_{n-1} - b_2 - \ldots - b_{n-1}).
\end{aligned}$$

So $x \in \varphi(\mathbb{N})$. Hence $f(M) \cap g(N) = f\varphi(\mathbb{N})$. □

It is important to notice that the property of M and N being conical alone with the hypothesis of both φ and ψ being monomorphism do not imply embeddability. The next example shows, in fact, that even for monoids of projectives of rings with IBN, this may not be true.

Example. Let S be the commutative monoid presented by

$$S = \langle a, b \mid 2.b = 2.a,\ 4.a + b = 5.a \rangle,$$

which can be regarded as a monoid with a distinguished element via the morphism

$$\begin{array}{ccc}
\mathbb{N} & \xrightarrow{\alpha} & S \\
1 & \longmapsto & 2.a
\end{array}$$

It can be verified directly that every element $n.a + m.b$ of S, with $n, m \in \mathbb{N}$, can be unambiguously reduced to one of the irreducible elements

$$0,\ a,\ 2.a,\ 3.a, \ldots, b,\ a+b,\ 2.a+b,\ 3.a+b$$

and that $n.a + m.b = 0$ if and only if $n = m = 0$. So the morphism α is injective and S is conical. We can say even more: given any element $n.a+m.b$ of S, we have $2.(n.a+m.b) = n.(2.a)+m.(2.b) = n.(2.a)+m.(2.a) = (n+m)(2.a) = \alpha(n+m)$. Thus, by Bergman's theorem (Theorem 1.2.1), S is isomorphic, as a monoid with a distinguished element, to the monoid of projectives of an algebra R over an arbitrary field. As we have just seen, R has IBN. (In fact, R even has the UGN property.) Our second object is the commutative monoid T presented by

$$T = \langle d, e \mid 2.d + e = d \rangle.$$

This will be considered as a monoid with a distinguished element by means of the following morphism.

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{\beta} & T \\ 1 & \longmapsto & d \end{array}$$

In this case, it is also possible to show directly that every element $n.d + m.e$ of T can be uniquely expressed as one of the following irreducible elements of T:

$$0,\ d,\ 2.d,\ 3.d, \ldots, e,\ 2.e,\ 3.e, \ldots, d+e,\ d+2.e,\ d+3.e, \ldots$$

and that $n.d+m.e = 0$ if and only if $n = m = 0$, so T is conical. Since the integer multiples of d are all distinct in T, β is injective, *i.e.* T has the IBN property. But note that T does not have the UGN property, because $2.d + e = d$. Here it is also the case that T can be regarded as a monoid of projectives of an algebra over a field which has IBN. For given any $n.d + m.e \in T$, $(n.d + m.e) + 2m.d = n.d + m.(2.d + e) = n.d + m.d = (n + m).d = \beta(n + m)$. Let $C = S \coprod_{\mathbb{N}} T$ and $f : S \longrightarrow C$, $g : T \longrightarrow C$ be the canonical morphisms associated with the coproduct. We will show that f is not injective. Indeed:

$$\begin{aligned}
f(3.a) &= f(a) + f(2.a) = f(a) + f\alpha(1) \\
&= f(a) + g\beta(1) = f(a) + g(d) \\
&= f(a) + g(2.d + e) = f(a) + g\beta(2) + g(e) \\
&= f(a) + f\alpha(2) + g(e) = f(a) + f(4.a) + g(e) \\
&= f(5.a) + g(e) = f(4.a + b) + g(e) \\
&= f(b) + f\alpha(2) + g(e) = f(b) + g\beta(2) + g(e) \\
&= f(b) + g(2.d + e) = f(b) + g(d) \\
&= f(b) + g\beta(1) = f(b) + f\alpha(1) \\
&= f(2.a + b).
\end{aligned}$$

But, as we have seen above, the elements $3.a$ and $2.a + b$ are different in S. So $\{(S, \alpha), (T, \beta)\}$ is not embeddable in C and this phenomenon was caused by the absence of the UGN property in T.

Theorem 2.1.3 has a very useful corollary when the monoids are monoids of projectives.

Corollary 2.1.6. *Let R and S be rings with UGN. Then $\{\mathcal{P}(R), \mathcal{P}(S)\}$ is strongly embeddable in the coproduct*

$$\mathcal{P}(R) \coprod_{\mathbb{N}} \mathcal{P}(S).$$

Proof. Monoids of projectives are always conical. Moreover, a ring has UGN if and only if its monoid of projectives has the UGN property. Therefore, by Theorem 2.1.3, the coproduct $\mathcal{P}(R) \coprod_{\mathbb{N}} \mathcal{P}(S)$ has the strong embeddability property. $\square$

Another useful corollary when the monoids involved are monoids of projectives is

Corollary 2.1.7. *Let $r \in \mathbb{N}$ and M and N be two commutative monoids with distinguished element. Suppose that M and N are conical and have the UGN* *property. Then there is an isomorphism of monoids with distinguished element*

$$M \coprod_{\mathbb{N}} N \cong \frac{1}{r}\mathbb{N}$$

if and only if $M \cong \mathbb{N}$ (and, thus, $N \cong \frac{1}{r}\mathbb{N}$) or $N \cong \mathbb{N}$ (and, thus, $M \cong \frac{1}{r}\mathbb{N}$).

Proof. It is clear that the conditions are sufficient for the isomorphism to exist. Let us look at their necessity. As a monoid with a distinguished element, $\frac{1}{r}\mathbb{N}$ is just the free commutative monoid F on one generator x given with a monoid morphism

$$\begin{array}{ccc} \mathbb{N} & \stackrel{\alpha}{\longrightarrow} & F \\ 1 & \longmapsto & r.x \end{array}$$

Let $\beta : \mathbb{N} \longrightarrow M$ and $\gamma : \mathbb{N} \longrightarrow N$ be the monoid morphisms associated to M and N respectively. Let $C = M \coprod_{\mathbb{N}} N$ and denote by $f : M \longrightarrow C$ and $g : N \longrightarrow C$ the canonical coproduct morphisms. Since (M, β) and (N, γ) are conical and have the UGN property, we have, by Theorem 2.1.3, that f and g are injective and $f(M) \cap g(N) = f\beta(\mathbb{N})$. By hypothesis, there exists an isomorphism

$$\Phi : F \longrightarrow C$$

of monoids with distinguished element. Since $\Phi(x) \in C$, there exist $m \in M$ and $n \in N$ such that $\Phi(x) = \overline{(m, n)} = f(m) + g(n)$. Surjectivity of Φ guarantees the existence of $a, b \in \mathbb{N}$ such that

$$\Phi(a.x) = f(m) \quad \text{and} \quad \Phi(b.x) = g(n).$$

So, $\Phi(x) = f(m) + g(n) = \Phi(a.x) + \Phi(b.x) = \Phi((a+b).x)$ and, since Φ is injective, $a + b = 1$. There are two cases to consider: when $a = 0$ and $b = 1$ and when $a = 1$ and $b = 0$. Let us look at the first case, *i.e.* $a = 0$ and $b = 1$. In this case, we have that $f(m) = 0 = f(0)$ and so, because f is injective, $m = 0$. That is, $\Phi(x) = g(n)$. We will now prove that, in this situation, β must be surjective. Indeed, let p be an arbitrary element of M. Since Φ is surjective, there must exist a $c \in \mathbb{N}$ such that $\Phi(c.x) = f(p)$. So,

$$\begin{aligned} f(p) &= \Phi(c.x) \\ &= c.\Phi(x) \\ &= c.g(n) \\ &= g(c.n), \end{aligned}$$

that is, $f(p) \in f(M) \cap g(N) = f\beta(\mathbb{N})$. And, since f is injective, $p \in \beta(\mathbb{N})$. So β is surjective (and injective), providing an isomorphism between M and $\mathbb{N}$. It now follows that $N \cong \frac{1}{r}\mathbb{N}$. The second case ($a = 1$ and $b = 0$) yields an isomorphism between N and $\mathbb{N}$ and has a symmetric proof. $\square$

In particular, if R and S are rings with UGN such that

$$\mathcal{P}(S) \coprod_{\mathbb{N}} \mathcal{P}(R) \cong \mathbb{N},$$

then R and S are projective free rings. So we have the following corollary.

Corollary 2.1.8. *Let K be a skew field and R and S be K-rings. Then $R *_K S$ is a fir if and only if both R and S are firs.*

Proof. In one direction, this is just Proposition 1.3.5. Conversely, suppose that $A = R *_K S$ is a fir. On the one hand, by Theorem 1.3.2, R and S are hereditary rings. On the other hand, by Theorem 1.3.1, R and S can be embedded in A, so both have UGN. Moreover, by Theorem 1.3.4, there is an isomorphism of monoids with distinguished element

$$\mathcal{P}(R) \coprod_{\mathbb{N}} \mathcal{P}(S) \cong \mathbb{N}.$$

Thus, by Corollary 2.1.7, R and S are projective free. Therefore, both R and S are firs. $\square$

2.2 Applications

Having solved the problem of strong embeddability, we can now look at properties of the factors which are preserved in the coproduct. We can use Propositions

1.2.2 and 1.2.3 to define, just as we did with UGN, properties IBN, weak finiteness (WF), Hermite and power freeness (PF) for abstract commutative monoids with distinguished element. That is, if M is a commutative monoid with a distinguished element e, then M has the *IBN property* if $n.e = m.e$ implies $n = m$, the *WF property* if $n.e = n.e + x$ for some $x \in M$ implies $x = 0$, the *Hermite property* if $m.e + x = n.e$ for some $x \in M$ implies $n \geq m$ and $x = (n - m).e$, the *PF property* if it has the IBN property and for every $x \in M$, there exist $n, r \in \mathbb{N}$, both depending on x, such $n.x = r.e$. Finally, a (general) commutative monoid M has the *cancellation property* if $x + y = x + w$ for $x, y, w \in M$ implies $y = w$. It is obvious to see that (UGN $\Longrightarrow$ IBN) and (Hermite $\Longrightarrow$ WF). Moreover, if M is conical, then (PF $\Longrightarrow$ WF). If M is a conical genuine commutative monoid with distinguished element, then we have (WF $\Longrightarrow$ UGN) and (cancellation $\Longrightarrow$ Hermite). Thus, for a conical genuine commutative monoid with distinguished element, we have the same implications represented by the diagram in section 1.1, *i.e.*

$$\text{PF} \Longrightarrow \text{WF} \Longrightarrow \text{UGN} \Longrightarrow \text{IBN}$$

and

$$\text{cancellation} \Longrightarrow \text{Hermite} \Longrightarrow \text{WF} \Longrightarrow \text{UGN} \Longrightarrow \text{IBN}.$$

If $(M, \varphi), (N, \psi)$ are commutative monoids with distinguished element which are both conical and have the IBN property, then an element $\xi = \overline{(x, y)}$ of $M \coprod_{\mathbb{N}} N$, where $x \in M$ and $y \in N$ satisfies

$$\xi = 0 \text{ if and only if } x = 0 \text{ and } y = 0. \tag{2.2.2}$$

Indeed, since $\overline{(0, 0)} = 0$ in the coproduct, we know that for some $n \in \mathbb{N}$, for each

$i = 1, \ldots, n$, there exist $x_i \in M$ and $y_i \in N$ and $a_i, b_i \in \mathbb{N}$ such that

$$\begin{array}{ll}
0 = x_1 + \varphi(a_1) & 0 = y_1 + \psi(b_1) \\
x_1 + \varphi(b_1) = x_2 + \varphi(a_2) & y_1 + \psi(a_1) = y_2 + \psi(b_2) \\
\vdots & \vdots \\
x_i + \varphi(b_i) = x_{i+1} + \varphi(a_{i+1}) & y_i + \psi(a_i) = y_{i+1} + \psi(b_{i+1}) \\
\vdots & \vdots \\
x_{n-1} + \varphi(b_{n-1}) = x_n + \varphi(a_n) & y_{n-1} + \psi(a_{n-1}) = y_n + \psi(b_n) \\
x_n + \varphi(b_n) = x & y_n + \psi(a_n) = y.
\end{array}$$

The first equation on the left gives us, by conicality of M and IBN, $x_1 = 0$ and $a_1 = 0$. The one on the right gives $y_1 = 0$ and $b_1 = 0$. Now, by an induction argument on i, we get $x_i = 0$, $y_i = 0$, $a_i = b_i = 0$ for $1 \le i \le n$. This implies that $x = x_n + \varphi(b_n) = 0$ and $y = y_n + \psi(a_n) = 0$. As a consequence, we get

Proposition 2.2.1. *Let (M, φ) and (N, ψ) be commutative monoids with distinguished elements. If both M and N are conical and have the IBN property, then the pair coproduct $M \coprod_{\mathbb{N}} N$ is conical and has the IBN property.*

Proof. Let C denote the coproduct $M \coprod_{\mathbb{N}} N$. First let us show that C is conical. Let $\xi = \overline{(x, y)}$ and $\eta = \overline{(x', y')}$ be elements of $M \coprod_{\mathbb{N}} N$, where $x, x' \in M$ and $y, y' \in N$. Then, by (2.2.2), $\xi + \eta = \overline{(x + x', y + y')} = 0$ if and only if $x = x' = 0$ and $y = y' = 0$. In other words, $\xi + \eta = 0$ if and only if $\xi = \eta = 0$. Hence C is conical.

To prove IBN, let $r, s \in \mathbb{N}$ be such that $\overline{(\varphi(r), 0)} = \overline{(\varphi(s), 0)}$. We want to show that $r = s$. Indeed, the above identity is equivalent to the existence of a system

$$\begin{array}{ll}
\varphi(r) = x_1 + \varphi(a_1) & 0 = y_1 + \psi(b_1) \\
x_1 + \varphi(b_1) = x_2 + \varphi(a_2) & y_1 + \psi(a_1) = y_2 + \psi(b_2) \\
\vdots & \vdots \\
x_i + \varphi(b_i) = x_{i+1} + \varphi(a_{i+1}) & y_i + \psi(a_i) = y_{i+1} + \psi(b_{i+1}) \\
\vdots & \vdots \\
x_{n-1} + \varphi(b_{n-1}) = x_n + \varphi(a_n) & y_{n-1} + \psi(a_{n-1}) = y_n + \psi(b_n) \\
x_n + \varphi(b_n) = \varphi(s) & y_n + \psi(a_n) = 0.
\end{array}$$

where $x_i \in M$, $y_i \in N$ and $a_i, b_i \in \mathbb{N}$. Since N is conical and has the IBN property, the first and last equations on the right-hand side give us $y_1 = y_n = 0$ and $b_1 = a_n = 0$. So the system can be rewritten as

$$
\begin{array}{ll}
\varphi(r) = x_1 + \varphi(a_1) & \\
x_1 = x_2 + \varphi(a_2) & \psi(a_1) = y_2 + \psi(b_2) \\
\vdots & \vdots \\
x_i + \varphi(b_i) = x_{i+1} + \varphi(a_{i+1}) & y_i + \psi(a_i) = y_{i+1} + \psi(b_{i+1}) \\
\vdots & \vdots \\
x_{n-1} + \varphi(b_{n-1}) = x_n & y_{n-1} + \psi(a_{n-1}) = \psi(b_n) \\
x_n + \varphi(b_n) = \varphi(s). &
\end{array}
$$

The set of equations on the right-hand side, as a simple inductive argument shows, implies that

$$\psi(a_1 + \ldots + a_{n-1}) = \psi(b_2 + \ldots + b_n).$$

By the IBN property of N, we get

$$a_1 + \ldots + a_{n-1} = b_2 + \ldots + b_n. \tag{2.2.3}$$

Now the equations on the left-hand side give, in a similar fashion,

$$\varphi(r + b_2 + \ldots + b_n) = \varphi(s + a_1 + \ldots + a_{n-1}).$$

So, by IBN in M,

$$r + b_2 + \ldots + b_n = s + a_1 + \ldots + a_{n-1}. \tag{2.2.4}$$

Now, (2.2.3) together with (2.2.4) imply $r = s$. Thus, C has the IBN property. $\square$

Although the above proposition guarantees that under conicality, the IBN property of the factors is preserved in the coproduct, we know that just IBN is not enough to guarantee embeddability. We have seen in Theorem 2.1.3 that conicality and UGN were sufficient to guarantee strong embeddability and we will see that, in this case, not only the coproduct contains its factors, but also inherits their UGN property. In fact, other more restrictive properties than UGN are also transmitted from the factors to the coproduct, as the next result shows.

Theorem 2.2.2. *Let (M, φ) and (N, ψ) be commutative monoids with distinguished elements and suppose that both are conical and genuine. Denote by C the coproduct $M \coprod_{\mathbb{N}} N$. If both (M, φ) and (N, ψ) satisfy one of the properties below, then so does C:*

(a) *UGN,*

(b) *WF,*

(c) *Hermite,*

(d) *PF,*

(e) *cancellation.*

Proof. The proofs are similar to the proof of Theorem 2.1.3, so they will be only sketched. For (a), suppose $\overline{(\varphi(r), 0)} = \overline{(x + \varphi(s), y)}$, where $x \in M, y \in N$ and $r, s \in \mathbb{N}$. That is, we have a system

$$\begin{array}{ll}
\varphi(r) = x_1 + \varphi(a_1) & \\
x_1 = x_2 + \varphi(a_2) & \psi(a_1) = y_2 + \psi(b_2) \\
\vdots & \vdots \\
x_i + \varphi(b_i) = x_{i+1} + \varphi(a_{i+1}) & y_i + \psi(a_i) = y_{i+1} + \psi(b_{i+1}) \\
\vdots & \vdots \\
x_{n-1} + \varphi(b_{n-1}) = x_n + \varphi(a_n) & y_{n-1} + \psi(a_{n-1}) = y_n + \psi(b_n) \\
x_n + \varphi(b_n) = x + \varphi(s) & y_n + \psi(a_n) = y.
\end{array}$$

and we must prove that $r \geq s$. This follows because, since N has the UGN property, we have, as in Corollary 2.1.5 (a), $a_1 + \ldots + a_i \geq b_2 + \ldots + b_{i+1}$, for $1 \leq i \leq n-1$. So

$$\begin{aligned}
\varphi(r) &= x_{n-1} + \varphi(a_1 + \ldots + a_{n-1} - b_2 - \ldots - b_{n-2}) \\
&= x_{n-1} + \varphi(b_{n-1}) + \varphi(a_1 + \ldots + a_{n-1} - b_2 - \ldots - b_{n-2} - b_{n-1}) \\
&= x_n + \varphi(a_1 + \ldots + a_{n-1} + a_n - b_2 - \ldots - b_{n-1}) \\
&= x_n + \varphi(b_n) + \varphi(a_1 + \ldots + a_{n-1} + a_n - b_2 - \ldots - b_{n-1} - b_n) \\
&= x + \varphi(s) + \varphi(a_1 + \ldots + a_n - b_2 - \ldots - b_n).
\end{aligned}$$

That is,

$$\varphi(r) = x + \varphi(s + \hat{a} - \hat{b}), \tag{2.2.5}$$

where $\hat{a} = a_1 + \ldots + a_n$ and $\hat{b} = b_2 + \ldots + b_n$. Since M satisfies UGN, $r \geq s + \hat{a} - \hat{b} \geq s$. Therefore C satisfies UGN. Note that we can also use the system above to prove, in the fashion of Lemma 2.1.4, that

$$\psi(\hat{a}) = y + \psi(\hat{b}). \tag{2.2.6}$$

For (b), setting $r = s$ above, we are left to prove that $x = 0$. But this is so, because since M is WF, (2.2.5) implies $x + \varphi(\hat{a} - \hat{b}) = 0$, which implies, by conicality and IBN, $x = 0$ and $\hat{a} = \hat{b}$. So, by (2.2.6) and because N has the WF property, $y = 0$ and, therefore, $\overline{(x, y)} = \overline{(0, 0)} = 0$. Thus, C has the WF property.

For part (c), we must, using the above system and the Hermite property in M and N, prove that $r \geq s$ (which we have already done) and that $\overline{(x, y)} = \overline{(\varphi(r - s), 0)}$. By (2.2.5), it follows that $x = \varphi(r - s - \hat{a} + \hat{b})$. And, by (2.2.6), it follows that $y = \psi(\hat{a} - \hat{b})$. So, setting $d = \hat{a} - \hat{b}$, $\overline{(x, y)} = \overline{(\varphi(r - s - d), \psi(d))} = \overline{(\varphi(r - s - d) + \varphi(d), 0)} = \overline{(\varphi(r - s), 0)}$. Hence, C has the Hermite property.

Part (d) is straightforward: we saw in Proposition 2.2.1 that $M \coprod_{\mathbb{N}} N$ has the IBN property and given $\xi = \overline{(x, y)}$, we know that there are $n, m, r, s \in \mathbb{N}$ such that $n.x = \varphi(r)$ and $m.y = \psi(s)$. So

$$\begin{aligned}
(nm).\xi = (nm).\overline{(x, y)} &= \overline{(m.(n.x), n.(m.y))} \\
&= \overline{(m.\varphi(r), n.\psi(s))} \\
&= \overline{(\varphi(mr + ns), 0)} \\
&= (mr + ns)\overline{(\varphi(1), 0)} = (mr + ns)\mathbf{1},
\end{aligned}$$

where $\mathbf{1}$ is the distinguished element in $M \coprod_{\mathbb{N}} N$.

For part (e), we use a system

$$\begin{array}{ll}
x + z = x_1 + \varphi(a_1) & y + w = y_1 + \psi(b_1) \\
x_1 + \varphi(b_1) = x_2 + \varphi(a_2) & \psi(a_1) = y_2 + \psi(b_2) \\
\vdots & \vdots \\
x_i + \varphi(b_i) = x_{i+1} + \varphi(a_{i+1}) & y_i + \psi(a_i) = y_{i+1} + \psi(b_{i+1}) \\
\vdots & \vdots \\
x_{n-1} + \varphi(b_{n-1}) = x_n + \varphi(a_n) & y_{n-1} + \psi(a_{n-1}) = y_n + \psi(b_n) \\
x_n + \varphi(b_n) = x' + z & y_n + \psi(a_n) = y' + w
\end{array}$$

to represent an equation

$$\overline{(x,y)} + \overline{(z,w)} = \overline{(x',y')} + \overline{(z,w)},$$

where $x, x', z \in M$ and $y, y', w \in N$. Our aim is to show that the existence of such a system implies $\overline{(x,y)} = \overline{(x',y')}$. Indeed, the equations on the left-hand side of the above system yield

$$x + z + \varphi(b_1 + \ldots + b_n) = x' + z + \varphi(a_1 + \ldots + a_n).$$

By cancellation in M, we get

$$x + \varphi(\bar{b}) = x' + \varphi(\bar{a}), \tag{2.2.7}$$

where $\bar{a} = a_1 + \ldots a_n$ and $\bar{b} = b_1 + \ldots + b_n$. The equations on the right-hand side yield

$$y + \psi(\bar{a}) = y' + \psi(\bar{b}). \tag{2.2.8}$$

Suppose that $\bar{a} \geq \bar{b}$ and set $d = \bar{a} - \bar{b}$. Since both M and N have cancellation, (2.2.7) and (2.2.8) become

$$x = x' + \varphi(d), \qquad y + \psi(d) = y'.$$

So there is a system

$$\begin{array}{ll} x = x' + \varphi(d) & y = y \\ x' = x' & y + \psi(d) = y', \end{array}$$

which implies that $\overline{(x,y)} = \overline{(x',y')}$. And, therefore, C has cancellation. If $\bar{a} < \bar{b}$, the argument is analogous. $\square$

A property of a commutative monoid with a distinguished element mimicking projective triviality is not in general preserved by the coproduct construction. For an example, take $M = \frac{1}{s}\mathbb{N}$ and $N = \frac{1}{t}\mathbb{N}$ in Corollary 2.1.7 with $s > 1$ and $t > 1$. As that result shows, as a monoid with distinguished element, $\frac{1}{s}\mathbb{N} \coprod_{\mathbb{N}} \frac{1}{t}\mathbb{N}$ cannot be regarded as a monoid of the kind $\frac{1}{r}\mathbb{N}$ even when $s = t$.

We have seen that given two conical commutative monoids with distinguished element satisfying the UGN property, they are strongly embedded in their coproduct and the coproduct is again conical and has the UGN property. By associativity of the coproduct construction (see [11, Th. 1.3]), it follows that the coproduct of *finitely many* conical commutative monoids with distinguished element satisfying the UGN property is conical, has the UGN property and the factors are strongly embedded in it.

Proposition 2.2.3. *Let $\{(M_1, \varphi_1), \ldots, (M_n, \varphi_n)\}$ be a finite family of commutative monoids with distinguished element. Suppose that, for every $i = 1, \ldots, n$, (M_i, φ_i) is conical and has the UGN property. Let $C = \coprod_{\mathbb{N}} M_i$ and $g_i : M_i \longrightarrow C$ be the coproduct morphisms. Then, for every $i = 1, \ldots, n$, g_i is injective and if $i \neq j$, $g_i(M_i) \cap g_j(M_j) = g_i\varphi_i(\mathbb{N})$. Moreover, C is conical and has the UGN property.* $\square$

Given an arbitrary family $(M_\lambda, \varphi_\lambda)_{\lambda \in \Lambda}$ of commutative monoids with distinguished element, we can regard their coproduct $C = \coprod_{\mathbb{N}} M_\lambda$ as the direct limit of the direct system formed by the coproducts of finite subsets of Λ. More precisely, given i, a finite subset of Λ, let C_i be the coproduct of the monoids M_λ with $\lambda \in i$. If j is another subset of Λ such that $i \leq j$ (that is, $i \subseteq j$), let $\alpha_j^i : C_i \longrightarrow C_j$ be the coproduct morphism from C_i into $C_i \coprod_{\mathbb{N}} C_{j \setminus i} \cong C_j$. Then I is a directed set and (C_i, α_j^i) is a direct system of commutative monoids with distinguished element. Let $(\varinjlim C_i, \beta_i)$ be the direct limit of (C_i, α_j^i). It is easy to see that, as commutative monoids with distinguished element, $\varinjlim C_i \cong C$. Using the fact that, for a fixed $i \in I$, β_i is injective if and only if α_j^i is injective for all $j \geq i$ (see, *e.g.* [10, Lemma 21.2]), we can prove the following result.

Theorem 2.2.4. *Let $\{(M_\lambda, \varphi_\lambda) : \lambda \in \Lambda\}$ be a family of commutative monoids with distinguished element. Suppose that, for every $\lambda \in \Lambda$, $(M_\lambda, \varphi_\lambda)$ is conical and has the UGN property. Then the family $\{(M_\lambda, \varphi_\lambda) : \lambda \in \Lambda\}$ is strongly embeddable in the coproduct $C = \coprod_{\mathbb{N}} M_\lambda$.*

Proof. Let I be the set of all finite subsets of Λ, partially ordered by inclusion (denoted here by $\leq$). Denote by C the coproduct $\coprod_{\mathbb{N}} M_\lambda$. As above, for every $i \in I$, let C_i be the coproduct of the monoids M_λ with $\lambda \in i$ and, for every $j \geq i$ denote by α_j^i the morphism from C_i into C_j. We know that the set I is directed, that the direct system (C_i, α_j^i) has a direct limit $(\varinjlim C_i, \beta_i)$ and that

$C \cong \varinjlim C_i$. By Proposition 2.2.3, the maps α_j^i are all injective, thus all the maps β_i are injective. In particular, for every $\lambda \in \Lambda$, $\beta_\lambda : M_\lambda \longrightarrow C$ (where β_λ stands for $\beta_{\{\lambda\}}$) is injective. Now given $\lambda, \mu \in \Lambda$, let $i = \{\lambda, \mu\}$, so that $\beta_\lambda = \beta_i \alpha_i^\lambda$ and $\beta_\mu = \beta_i \alpha_i^\mu$. Hence

$$\begin{aligned}
\beta_\lambda(M_\lambda) \cap \beta_\mu(M_\mu) &= \beta_i \alpha_i^\lambda(M_\lambda) \cap \beta_i \alpha_i^\mu(M_\mu) \\
&= \beta_i(\alpha_i^\lambda(M_\lambda) \cap \alpha_i^\mu(M_\mu)), \text{ because } \beta_i \text{ is injective} \\
&= \beta_i(\alpha_i^\lambda \varphi_\lambda(\mathbb{N})), \text{ by Proposition 2.2.3} \\
&= \beta_\lambda \varphi_\lambda(\mathbb{N}).
\end{aligned}$$

Therefore, $\{(M_\lambda, \varphi_\lambda) : \lambda \in \Lambda\}$ is strongly embeddable in C. □

For directed limits of monoids, as in the case of modules, all the elements in $\varinjlim C_i$ are of the form $\beta_i(x_i)$, for some $i \in I$ and $x_i \in C_i$. So in this context it is also true that the coproduct will inherit properties shared by all the factors. To see that the Hermite property, for instance, is preserved, suppose that the factors M_λ in the above theorem all have the Hermite property. Denote by $\psi_i : \mathbb{N} \longrightarrow C_i$ the monoid morphism that characterizes C_i as a monoid with a distinguished element and by $\psi : \mathbb{N} \longrightarrow C$ the corresponding morphism for C—so that $\beta_i \psi_i = \psi$, for every $i \in I$. Let

$$\psi(m) + x = \psi(n) \tag{2.2.9}$$

be an equation in C. Take $i \in I$ and $x_i \in C_i$ such that $x = \beta_i(x_i)$. Then (2.2.9) becomes

$$\beta_i(\psi_i(m)) + \beta_i(x_i) = \beta_i(\psi_i(n)). \tag{2.2.10}$$

By injectivity of β_i, (2.2.10) is equivalent to

$$\psi_i(m) + x_i = \psi_i(n).$$

By Theorem 2.2.2, the Hermite property is transmitted from two factors to their coproduct and, thus, by a simple induction, it is transmitted from finitely many factors to their coproduct. So being Hermite is a property of C_i and, therefore,

$n \geq m$ and $x_i = \psi_i(n-m)$. Applying β_i, we get $x = \beta_i(x_i) = \beta_i\psi_i(n-m) = \psi(n-m)$. Hence C has the Hermite property. The other properties mentioned in Theorem 2.2.2 behave in the same way.

Theorem 2.2.5. *Let $\{M_\lambda : \lambda \in \Lambda\}$ be a family of commutative monoids with distinguished element. Suppose that, for every $\lambda \in \Lambda$, M_λ is conical and genuine. Denote by C their coproduct $\coprod_{\mathbb{N}} M_\lambda$. If, for every $\lambda \in \Lambda$, M_λ satisfies one of the properties below, then so does C:*

(a) *UGN.*

(b) *WF,*

(c) *Hermite,*

(d) *PF,*

(e) *cancellation.* $\square$

Chapter 3

Coproducts of skew fields

We saw that coproducts of skew fields amalgamating a common subfield provide a large class of examples of firs. In this chapter, we will look at the effect of submitting these coproducts to an extension of a commutative subfield and obtain a description of their monoid of projectives.

3.1 Coproducts under field extensions

There will be no loss of generality in restricting our analysis to the case of two factors. In this section a monoid with a distinguished element will be called *basic* if it is isomorphic to $\mathbb{N}$. If it is isomorphic to $\frac{1}{n}\mathbb{N}$, for some natural number n, it will be called *parabasic.*

Theorem 3.1.1. *Let C and D be two skew fields having a common subfield K and let $R = C *_K D$. Let k be the centre of K and E a finite separable commutative field extension of k. Then the monoid of projectives of the ring $R_E = R \otimes_k E$ is a coproduct of direct products of parabasic monoids.*

Proof. Let $l = [E : k]$. We know by Lemma 1.4.3 that

$$R_E = (C *_K D) \otimes_k E \cong C_E *_{K_E} D_E.$$

Let us look first at $K_E = K \otimes_k E$. By Lemma 1.4.1, $K_E \cong \mathfrak{M}_r(S)$, where S is a skew field and r a natural number dividing l. Now let Z be the centre of C. We have

$$C_E = C \otimes_k E \cong C \otimes_Z (Z \otimes_k E).$$

Since E/k is separable, it follows that $Z \otimes_k E \cong Z_1 \times \ldots \times Z_m$, where, for each $i = 1, \ldots, m$, Z_i is a finite commutative field extension of Z and $\sum_{i=1}^{m} [Z_i : Z] = l$ (cf. [7, Cor. 5.7.4, p. 194]). So

$$\begin{aligned} C_E &\cong C \otimes_Z (Z \otimes_k E) \\ &\cong C \otimes_Z (Z_1 \times \ldots \times Z_m) \\ &\cong (C \otimes_Z Z_1) \times \ldots \times (C \otimes_Z Z_m). \end{aligned}$$

Since C is a skew field with centre Z and each Z_i is a finite commutative field extension of Z, it follows by Lemma 1.4.1 that, for each $i = 1, \ldots, m$ there exists a natural number s_i, which divides $[Z_i : Z]$, and a skew field G_i such that $C \otimes_Z Z_i \cong \mathfrak{M}_{s_i}(G_i)$. That is,

$$C_E \cong \mathfrak{M}_{s_1}(G_1) \times \ldots \times \mathfrak{M}_{s_m}(G_m). \tag{3.1.1}$$

Similarly, we can write

$$D_E \cong \mathfrak{M}_{t_1}(H_1) \times \ldots \times \mathfrak{M}_{t_n}(H_n),$$

where, for each $i = 1, \ldots, n$, t_i is a natural number and H_i is a skew field. Therefore, R_E is isomorphic to the ring

$$\mathfrak{M}_{s_1}(G_1) \times \ldots \times \mathfrak{M}_{s_m}(G_m) *_{\mathfrak{M}_r(S)} \mathfrak{M}_{t_1}(H_1) \times \ldots \times \mathfrak{M}_{t_n}(H_n).$$

Since $\mathfrak{M}_r(S)$ is simple artinian, by Theorem 1.3.4, we have

$$\begin{aligned} \mathcal{P}(R_E) &\cong \mathcal{P}(\mathfrak{M}_{s_1}(G_1) \times \ldots \times \mathfrak{M}_{s_m}(G_m)) \\ &\qquad \coprod_{\mathcal{P}(\mathfrak{M}_r(S))} \mathcal{P}(\mathfrak{M}_{t_1}(H_1) \times \ldots \times \mathfrak{M}_{t_n}(H_n)) \\ &\cong \mathcal{P}(\mathfrak{M}_{s_1}(G_1)) \times \ldots \times \mathcal{P}(\mathfrak{M}_{s_m}(G_m)) \\ &\qquad \coprod_{\mathcal{P}(\mathfrak{M}_r(S))} \mathcal{P}(\mathfrak{M}_{t_1}(H_1)) \times \ldots \times \mathcal{P}(\mathfrak{M}_{t_n}(H_n)) \\ &\cong \left(\frac{1}{s_1}\mathbb{N} \times \ldots \times \frac{1}{s_m}\mathbb{N}\right) \coprod_{\frac{1}{r}\mathbb{N}} \left(\frac{1}{t_1}\mathbb{N} \times \ldots \times \frac{1}{t_n}\mathbb{N}\right). \end{aligned}$$

□

The proof of the above theorem also provides some information on the relation between r and the s_i's and t_j's. Since K_E is embeddable in C_E, we can regard C_E as an $r \times r$ matrix ring, say $C_E \cong \mathfrak{M}_r(T)$. Since C_E is a semisimple ring, T is semisimple; therefore, there exist natural numbers $r_1, \dots, r_h$ and skew fields $L_1, \dots, L_h$ such that

$$T \cong \mathfrak{M}_{r_1}(L_1) \times \cdots \times \mathfrak{M}_{r_h}(L_h).$$

Thus,

$$\begin{aligned} C_E \cong \mathfrak{M}_r(T) &\cong \mathfrak{M}_r(\mathfrak{M}_{r_1}(L_1) \times \cdots \times \mathfrak{M}_{r_h}(L_h)) \\ &\cong \mathfrak{M}_{rr_1}(L_1) \times \cdots \times \mathfrak{M}_{rr_h}(L_h). \end{aligned}$$

By uniqueness, $h = m$ and, comparing the above expression with (3.1.1) and possibly rearranging the order of the factors, $s_i = rr_i$ and $L_i \cong G_i$. Thus, r divides s_i for every $i = 1, \dots, m$. Similarly, r divides t_j for every $j = 1, \dots, n$. So we get the following result.

Corollary 3.1.2. *With the hypotheses of Theorem 3.1.1,*

$$R_E \cong \left(\mathfrak{M}_{s_1}(G_1) \times \ldots \times \mathfrak{M}_{s_m}(G_m)\right) *_{\mathfrak{M}_r(S)} \left(\mathfrak{M}_{t_1}(H_1) \times \ldots \times \mathfrak{M}_{t_n}(H_n)\right),$$

where for each $i = 1, \dots, m$ and $j = 1, \dots, n$, G_i and H_j are skew fields and r divides both s_i and t_j. □

In the simpler situation where both C and D share the same centre, we can drop the hypothesis on the separability of E over k to get the following

Corollary 3.1.3. *Let C, D be skew fields having a common subfield k as their centres. Let $R = C *_k D$ and E be a finite commutative field extension of k. Then the monoid of projectives of R_E is isomorphic to $\frac{1}{r}\mathbb{N} \coprod_{\mathbb{N}} \frac{1}{s}\mathbb{N}$, for natural numbers r, s. In particular, if $r > 1$ or $s > 1$, R_E is not a fir.*

Proof. This is just an easier version of Theorem 3.1.1. First notice that $k \otimes_k E \cong E$. Moreover, since k is the centre of both C and D there are natural numbers

r, s, which divide $[E : k]$, and skew fields G, H such that $C_E \cong \mathfrak{M}_r(G)$ and $D_E \cong \mathfrak{M}_s(H)$. Therefore, $R_E \cong \mathfrak{M}_r(G) *_E \mathfrak{M}_s(H)$ and

$$\mathcal{P}(R_E) \cong \frac{1}{r}\mathbb{N} \coprod_{\mathbb{N}} \frac{1}{s}\mathbb{N}.$$

We also know that, by Corollary 1.3.3, R_E is hereditary, because both $\mathfrak{M}_r(G)$ and $\mathfrak{M}_s(H)$ are. Thus R_E is a fir if and only R_E is projective free, that is, if and only if $\mathcal{P}(R_E) \cong \mathbb{N}$. But, by Corollary 2.1.7, this is the case if and only if $r = s = 1$. $\square$

Some comments on the nature of the ring R_E obtained in the above corollary are due. $R = C *_k D$ is a fir and, therefore, is embedded in its universal field of fractions U. This embedding induces an embedding of R_E into U_E. By Lemma 1.4.1, there exist n and a skew field L such that $U_E \cong \mathfrak{M}_n(L)$. In the proof of Corollary 3.1.3 we saw that the ring R_E is isomorphic to $\mathfrak{M}_r(G) *_E \mathfrak{M}_s(H)$, so there are two natural ways of regarding R_E as a full matrix ring. First, since $\mathfrak{M}_r(G)$ can be embedded in R_E, we can take the set of $r \times r$ matrix units of $\mathfrak{M}_r(G)$ to be a complete set of matrix units in R_E. Therefore, $R_E \cong \mathfrak{M}_r(T)$, for some ring T. Now, using the embedding $R_E \subseteq U_E$, we can regard U_E as an $r \times r$ matrix ring, say $U_E \cong \mathfrak{M}_r(L')$. Since U_E is simple artinian, $\mathfrak{M}_r(L')$ is simple artinian, hence L' is simple artinian. So there exist a natural number t and a skew field F such that $L' \cong \mathfrak{M}_t(F)$. Thus,

$$\mathfrak{M}_n(L) \cong U_E \cong \mathfrak{M}_r(L') \cong \mathfrak{M}_r(\mathfrak{M}_t(F)) \cong \mathfrak{M}_{rt}(F).$$

By uniqueness, $n = rt$ and $L \cong F$. This implies that $U_E \cong \mathfrak{M}_r(\mathfrak{M}_t(L))$. By definition, the embedding of R_E into U_E preserves the matrix units, so T is embedded in $\mathfrak{M}_t(L)$. If we had started with the embedding of $\mathfrak{M}_s(H)$ into R_E we would have obtained that R_E is isomorphic to a full matrix ring $\mathfrak{M}_s(T')$, where T' is embedded in $\mathfrak{M}_{t'}(L)$ and t' is a natural number such that $st' = n$. Note that in the non-trivial cases, neither T nor T' will be firs, although they are both hereditary rings. This is because

$$\mathfrak{M}_r(T) \cong \mathfrak{M}_r(G) *_E \mathfrak{M}_s(H)$$

implies, by Theorem 1.3.4,

$$\frac{1}{r}\mathcal{P}(T) \cong \frac{1}{r}\mathbb{N} \coprod_{\mathbb{N}} \frac{1}{s}\mathbb{N}.$$

If T is a fir, then $\mathcal{P}(T) \cong \mathbb{N}$ and, by Corollary 2.1.7 we have that $\frac{1}{s}\mathbb{N} \cong \mathbb{N}$, that is, $s = 1$. Conversely, if $s = 1$, then $\frac{1}{r}\mathcal{P}(T) \cong \frac{1}{r}\mathbb{N}$, which implies that $\mathcal{P}(T) \cong \mathbb{N}$. Hence we conclude that T is a fir if and only if $s = 1$. An analogous argument establishes that T' is a fir if and only if $r = 1$.

As we have just seen, in general, the coproduct constructions do not generate matrix rings over firs under ground field extension. But, as the next proposition shows, in some cases this construction can provide examples of matrix rings over firs.

Proposition 3.1.4. *Let C be a skew field with centre k, F a field (commutative or not) containing k and E a finite commutative field extension of k such that $F \otimes_k E$ is a field. Let $R = C *_k F$. Then R is a fir and R_E is isomorphic to a full matrix ring over a fir.*

Proof. R is a fir, because it is a coproduct of two firs amalgamating a common subfield. We know that $R_E \cong C_E *_E F_E$. So $\mathcal{P}(R_E) \cong \mathcal{P}(C_E) \coprod_{\mathbb{N}} \mathcal{P}(F_E)$. Since C_E is simple artinian, $\mathcal{P}(C_E) \cong \frac{1}{n}\mathbb{N}$ for some natural number n and since F_E is a field, $\mathcal{P}(F_E) \cong \mathbb{N}$. Therefore,

$$\mathcal{P}(R_E) \cong \frac{1}{n}\mathbb{N},$$

i.e. R_E is projective trivial. By Corollary 1.3.3, R_E is also hereditary, because both C_E and F_E are. So R_E is isomorphic to a full matrix ring over a fir. $\square$

3.2 Power-free ideal rings

Although coproducts of skew fields having a common centre under finite field extensions do not give rise to projective free rings, they do give rise to power-free rings, *i.e.* rings whose finitely generated projective modules have a power which is free of unique rank, as proved in Corollary 3.1.3. This motivates the following definition.

Definition. *A ring R is called a* left power-free ideal ring, *left pfir for short, if it is left hereditary, power-free and $\mathcal{P}(R)$ is finitely generated as a commutative monoid.*

The term *ideal* appears in the definition because of the following result.

Proposition 3.2.1. *Let R be a ring. Then*

(a) *if R is a left pfir, then every left ideal of R, as a left R-module, has a power which is free of unique rank,*

(b) *if every left ideal of R, as a left R-module, has a power which is free of unique rank, then R is left hereditary and power-free.*

Proof. We will start by proving (a). Let I be a left ideal of R. Since R is left hereditary, I is projective, in fact, I is isomorphic to a direct sum of finitely generated projective left R-modules (cf. [6, Th. 0.3.7, p. 14]). So we can write I as

$$I \cong P_1^{r_1} \oplus \ldots \oplus P_n^{r_n},$$

where the elements $[P_i]$ are the generators of $\mathcal{P}(R)$. Since R is power-free, for each $i = 1, \ldots, n$, there exist an s_i such that $P_i^{s_i}$ is isomorphic to a free module F_i. Let s be the LCM of the s_i's and take t_i's such that $s = s_i t_i$. Then

$$\begin{aligned} I^s &\cong (P_1^{r_1} \oplus \ldots \oplus P_n^{r_n})^s \\ &= P_1^{s_1(t_1 r_1)} \oplus \ldots \oplus P_n^{s_n(t_n r_n)} \\ &\cong F_1^{t_1 r_1} \oplus \ldots \oplus F_n^{t_n r_n}. \end{aligned}$$

So I^s is free. Uniqueness of the rank follows because, by definition, power-free rings have IBN.

For part (b), let I be a left ideal of R. Since I has a power which is free, in particular, I is a projective left R-module. So R is hereditary. For power-freeness, let P be a finitely generated projective left R-module. We must show that there exists a power of P which is free. Since R is left hereditary, P is isomorphic to a finite direct sum of left ideals of R (cf. [6, Cor. 0.3.2, p. 12]), *i.e.*

$$P \cong I_1 \oplus \ldots \oplus I_n,$$

where the I_i's are left ideals of R. By hypothesis, for each $i = 1, \dots, n$, there exists s_i such that $I_i^{s_i}$ is isomorphic to a free module F_i. Let s be the LCM of the s_i's and take t_i such that $s = s_i t_i$ for all i. Then

$$\begin{aligned} P^s &\cong (I_1 \oplus \dots \oplus I_n)^s \\ &\cong I_1^{s_1 t_1} \oplus \dots \oplus I_n^{s_n t_n} \\ &\cong F_1^{t_1} \oplus \dots \oplus F_n^{t_n}. \end{aligned}$$

So P^s is free and, hence, R is power-free. □

We can define right pfirs in a similar way. The unmodified term *pfir* will be used to refer to rings which are left and right pfirs simultaneously.

Recall that, by Proposition 1.2.3(d), a ring R is power-free if and only if it has IBN and for every $p \in \mathcal{P}(R)$, there exist $n, r \in \mathbb{N}$ such that $n.p = r.e$, where e is the distinguished element of $\mathcal{P}(R)$. It is clear that coproducts of parabasic monoids satisfy the same property. This allows us to rephrase Corollary 3.1.3 in the following way.

Proposition 3.2.2. *Let C, D be skew fields having a common subfield k as their centres. Let $R = C *_k D$ and E be a finite commutative field extension of k. Then R_E is a pfir.*

Proof. We saw in the proof of Corollary 3.1.3 that R_E is a hereditary ring. We also know that the monoid of projectives of R_E is isomorphic to the coproduct of two parabasic monoids. So $\mathcal{P}(R_E)$ is finitely generated and R_E is power-free. Therefore, R_E is a pfir. □

3.3 Matrix reduction functor

In this subsection we shall discuss the effect of ground field extensions of firs obtained by matrix reduction.

Let R be a ring and S an R-ring; then the $n \times n$ *matrix reduction* of S, denoted by $\mathfrak{W}_n(S; R)$, is defined to be the centralizer of the matrix units in the R-ring

$$S *_R \mathfrak{M}_n(R).$$

So $\mathfrak{W}_n(S;R)$ is an R-ring which satisfies

$$\mathfrak{M}_n(\mathfrak{W}_n(S;R)) \cong S *_R \mathfrak{M}_n(R),$$

where the above isomorphism sends matrix units to its corresponding matrix units. It is easy to see that $\mathfrak{W}_n$ is a functor from the category of R-rings to itself, called the *$n \times n$ matrix reduction functor.* In fact, it is the left adjoint of the matrix functor $\mathfrak{M}_n$ (cf. [9, p. 43]).

The first result in this subsection shows how to relate a coproduct of a ring and a matrix ring with the matrix reduction functor (cf. Ex. 1.7.6 of [9]).

Lemma 3.3.1. *Let R be any ring and S and T R-rings, then*

$$\mathfrak{M}_n(\mathfrak{W}_n(S *_R T; T)) \cong S *_R \mathfrak{M}_n(T)$$

for every $n \in \mathbb{N}$.

Proof. It is enough to note that $T *_T \mathfrak{M}_n(T) \cong \mathfrak{M}_n(T)$. Because, then,

$$\mathfrak{M}_n(\mathfrak{W}_n(S *_R T; T)) \cong S *_R T *_T \mathfrak{M}_n(T) \cong S *_R \mathfrak{M}_n(T).$$

□

Note that when dealing with R-rings, there is no loss of generality in looking at matrix reductions of coproducts, because we can always write S as $S *_R R$.

When R is taken to be a commutative field and S and T R-algebras, $\mathfrak{W}_n(S *_R T; T)$ is also an R-algebra and we can relate the monoid of projectives of $\mathfrak{W}_n(S *_R T; T)$ with the ones of S and T. More specifically, we can prove a result like Cor. 5.7.7 of [9] in this slightly more general setting.

Proposition 3.3.2. *Let k be a commutative field and S and T arbitrary k-algebras. Then, for every $n \in \mathbb{N}$, $\mathfrak{W}_n(S *_k T; T)$ is a fir if and only if both S and T are firs.*

Proof. Let $A = \mathfrak{W}_n(S *_k T; T)$. Using Lemma 3.3.1, we can write

$$\mathfrak{M}_n(A) \cong S *_k \mathfrak{M}_n(T) \qquad (3.3.2)$$

and, since k is a field, by Theorem 1.3.4,

$$\frac{1}{n}\mathcal{P}(A) \cong \mathcal{P}(S) \coprod_{\mathbb{N}} \frac{1}{n}\mathcal{P}(T). \tag{3.3.3}$$

If both S and T are firs, they are both hereditary and projective free. So by (3.3.2) and Corollary 1.3.3, A is also hereditary and by (3.3.3), A is projective free—so it is a fir. Conversely, if A is a fir, it is, in particular, hereditary, so, by (3.3.2) and Corollary 1.3.3, both S and T are hereditary. Now, since A is a fir, $\mathfrak{M}_n(A)$ is weakly finite and since S and $\mathfrak{M}_n(T)$ are embedded in $\mathfrak{M}_n(A)$ (by Theorem 1.3.1), S is weakly finite and so is T, for the embedding of $\mathfrak{M}_n(T)$ in $\mathfrak{M}_n(A)$, preserving matrix units, induces an embedding of T in A. In particular, both S and T have UGN and we can, then, apply Corollary 2.1.7 to the isomorphism in (3.3.3) to conclude that both S and T must be projective free and, therefore, firs. □

The following lemma establishes that a matrix reduction under field extension is again a matrix reduction.

Lemma 3.3.3. *Let k be a commutative field and S and T arbitrary k-algebras. If E is a commutative field extension of k, then, as E-algebras*

$$\mathfrak{W}_n(S *_k T; T) \otimes_k E \cong \mathfrak{W}_n(S_E *_E T_E; T_E),$$

for every $n \in \mathbb{N}$.

Proof. First tensor $\mathfrak{M}_n(\mathfrak{W}_n(S *_k T; T))$ with E over k to obtain

$$\mathfrak{M}_n(\mathfrak{W}_n(S *_k T; T)) \otimes_k E \cong \mathfrak{M}_n(\mathfrak{W}_n(S *_k T; T) \otimes_k E). \tag{3.3.4}$$

Next, tensor $S *_k \mathfrak{M}_n(T)$ with E over k and use Lemma 1.4.3 to get

$$(S *_k \mathfrak{M}_n(T)) \otimes_k E \cong S_E *_E (\mathfrak{M}_n(T) \otimes_k E) \cong S_E *_E \mathfrak{M}_n(T_E).$$

But, by Lemma 3.3.1, $S_E *_E \mathfrak{M}_n(T_E)$ is isomorphic to $\mathfrak{M}_n(\mathfrak{W}_n(S_E *_E T_E; T_E))$. Therefore,

$$(S *_k \mathfrak{M}_n(T)) \otimes_k E \cong \mathfrak{M}_n(\mathfrak{W}_n(S_E *_E T_E; T_E)). \tag{3.3.5}$$

Combining (3.3.4) with (3.3.5), we get the desired result. □

We saw that Proposition 3.3.2 provided us with a method of obtaining firs from known firs by matrix reduction, but in view of Lemma 3.3.3, this process does not contribute much to our search for firs among extended firs. More precisely,

Corollary 3.3.4. *Let k be a commutative field, S and T be firs which are k-algebras, $n \in \mathbb{N}$ and E be a commutative field extension of k. Let $A = \mathfrak{W}_n(S *_k T; T)$. Then A_E is a fir if and only if both S_E and T_E are firs.*

Proof. Using Lemma 3.3.3, we can write

$$A_E \cong \mathfrak{W}_n(S_E *_E T_E; T_E)$$

and, by Proposition 3.3.2, A_E is a fir if and only if both S_E and T_E are firs. $\square$

But we can still find some pfirs among extended firs obtained by matrix reduction.

Corollary 3.3.5. *Let k be a commutative field, E a finite commutative field extension of k and S a skew field with centre k. Then, for every $n \in \mathbb{N}$, the k-algebra $A = \mathfrak{W}_n(S; k)$ is a fir and A_E is a pfir, but not a fir unless S_E is a skew field.*

Proof. It is clear that A is a fir. Since S is a skew field with centre k, S_E is isomorphic to a matrix ring $\mathfrak{M}_r(K)$, where r divides n and K is a skew field with centre E. We have isomorphisms

$$\mathfrak{M}_n(A_E) \cong \mathfrak{M}_r(K) *_E \mathfrak{M}_n(E) \tag{3.3.6}$$

$$\frac{1}{n}\mathcal{P}(A_E) \cong \frac{1}{r}\mathbb{N} \coprod_{\mathbb{N}} \frac{1}{n}\mathbb{N}. \tag{3.3.7}$$

So, by (3.3.6), A_E is hereditary and, by (3.3.7), $\mathcal{P}(A_E)$ is finitely generated and power free. Hence A_E is a pfir and, by Corollary 2.1.7, a fir if and only if $r = 1$, *i.e.* if and only if S_E is a field. $\square$

Chapter 4

Tensor rings

In this chapter we look at the effect of central extensions on tensor rings. We will show that in some cases the extension itself is enough to provide a matrix ring over a fir, but in other cases a finite localization is necessary to make the extended fir into a matrix ring over a fir.

Firs are always embeddable in skew fields—their universal field of fractions, for instance. Therefore, rings obtained from firs by finite commutative extensions of their centres will be isomorphic to subrings of simple artinian rings, which are matrix rings.

We know, by Proposition 1.4.4, that a ring obtained by a separable extension of the centre of a fir is always hereditary. So to study further homological characteristics of such rings, it is enough to look at their monoid of projectives. We will show that a ring obtained by a finite extension of the centre of a tensor ring can be far from projective free. But, as we shall soon see, in some cases such extensions produce projective trivial rings, providing examples of matrix rings over firs. In Section 4.2, we shall look at a case where projective triviality is not obtained from the extension only and we will resort to a finite localization in order to adjoin to the extended fir the matrix units of its extended universal field of fractions. The result will be a hereditary projective trivial ring, or, a matrix ring over a fir.

In Section 4.3 of this chapter, we shall present an example of a tensor ring that subjected to a purely inseparable extension of its centre gives rise to a projective free ring which is not hereditary. Again, by adjoining to this ring enough inverses so that we get in it the matrix units of its extended universal field of fractions, we shall obtain a matrix ring over a fir.

Finally, in the last section of this chapter we present results on the origin of the universal field of fractions of the firs obtained in the preceding sections.

4.1 Matrix rings over firs

We start with tensor rings that give rise to matrix rings over firs under finite field extensions.

Proposition 4.1.1. *Let D be a skew field containing a subfield K. Suppose that the centres of D and K coincide and denote it by k. Let E be a finite commutative field extension of k. Let X be an arbitrary set and consider the tensor ring $R = D_K\langle X\rangle$. Then $R \otimes_k E$ is isomorphic to a matrix ring over a fir.*

Proof. We can write R as $D *_K K\langle X\rangle$. So

$$R \otimes_k E \cong D \otimes_k E *_{K\otimes_k E} (K \otimes_k E)\langle X\rangle.$$

Since both $D \otimes_k E$ and $K \otimes_k E$ are simple artinian rings, there are skew fields G and H and positive integers r and s such that s divides r and

$$D \otimes_k E \cong \mathfrak{M}_r(G) \qquad\qquad K \otimes_k E \cong \mathfrak{M}_s(H).$$

Therefore,

$$\begin{aligned} R \otimes_k E &\cong \mathfrak{M}_r(G) *_{\mathfrak{M}_s(H)} \mathfrak{M}_s(H)\langle X\rangle \\ &\cong \mathfrak{M}_r(G) *_{\mathfrak{M}_s(H)} \mathfrak{M}_s(H\langle X\rangle). \end{aligned}$$

By Corollary 1.3.3, $R \otimes_k E$ is hereditary. By Theorem 1.3.4, we can also write

$$\mathcal{P}(R \otimes_k E) \cong \frac{1}{r}\mathbb{N} \coprod_{\frac{1}{s}\mathbb{N}} \frac{1}{s}\mathbb{N} \cong \frac{1}{r}\mathbb{N},$$

so $R \otimes_k E$ is projective trivial. Hence $R \otimes_k E$ is isomorphic to an $r \times r$ matrix ring over a fir. □

4.2 Galois case

In this section we will look more closely inside the structure of a specific tensor ring R under a Galois extension of its centre. We will start by showing how to identify the matrix units in the simple artinian ring obtained by the extension of the universal field of fractions of R. We will then adjoin finitely many inverses of elements to R in order to obtain an intermediate ring which, under the same extension, will yield a full matrix ring over a fir.

4.2.1 General theory

We start with some general theory of rings and will be specializing to reach our aim as necessary.

The following is a well known result in the theory of division algebras. For the case $[U : k] < \infty$ this is proved by Jacobson in [13, Th. VII.3, p. 182]. The author refers to Brauer and Albert who proved but did not publish this result and observes that his proof is different from theirs. The proof given here was slightly modified for the infinite dimensional case.

Theorem 4.2.1. *Let U be a skew field with centre k. Suppose U contains a finite Galois extension F of k of degree n generated by an element α. Then there exists an element $y \in U$ such that the set $\{\alpha^i y \alpha^j : i, j = 0, \dots, n-1\}$ is linearly independent over k.*

Proof. In this proof, all tensor products will be over k. Let $\mathcal{M}(U)$ be the multiplication algebra of U, *i.e.* the k-subalgebra of $End_k U$ generated by all left and right multiplications. There is a k-algebra homomorphism

$$\varphi : U \otimes U^\circ \longrightarrow \mathcal{M}(U)$$

such that $\varphi(a \otimes b) = \mathcal{L}_a \mathcal{R}_b$, where $\mathcal{L}_a$ stands for the left multiplication by a and $\mathcal{R}_b$ for the right multiplication by b. Since U is a simple ring with centre k, it follows that $U \otimes U^\circ$ is also simple, by Cor. 7.1.3 of [8], hence φ is injective. In particular, we have an injective homomorphism $\psi = \varphi|_{F \otimes F} : F \otimes F \longrightarrow \mathcal{M}(U)$. Since F is Galois, by Cor. 5.7.5 of [7], $F \otimes F \cong F_1 \times \cdots \times F_n$, where each F_i is isomorphic to F. Let e_i denote the unit element of F_i and $f_i = \psi(e_i) \in \mathcal{M}(U)$. Since ψ is injective and the e_i's are nonzero, it follows that each $f_i \neq 0$, that is, for each $i = 1, \ldots, n$, there exists $y_i \in U$ such that $f_i(y_i) \neq 0$. Now let $y = \sum_{i=1}^n f_i(y_i)$. Then, for each $j = 1, \ldots, n$,

$$\begin{aligned} f_j(y) &= f_j\left(\sum_{i=1}^n f_i(y_i)\right) \\ &= \sum_{i=1}^n f_j f_i(y_i) \\ &= \sum_{i=1}^n \psi(e_j e_i)(y_i) \\ &= \psi(e_j)(y_j) \\ &= f_j(y_j). \end{aligned}$$

Consider now the k-space homomorphism

$$\xi : \mathcal{M}(U) \longrightarrow U$$

defined by $\xi(f) = f(y)$ and the composite k-linear map

$$F \otimes F \xrightarrow{\psi} \mathcal{M}(U) \xrightarrow{\xi} U,$$

which will be denoted by $\eta = \xi\psi$. We will show that η is injective. The k-subspace $ker(\eta)$ of $F \otimes F$ is in fact an ideal, for if $\sum a_i \otimes b_i$ is an element of $ker(\eta)$, then, by definition, $\sum a_i y b_i = 0$. Therefore, for any $a, b \in F$,

$$\begin{aligned} \eta((a \otimes b)(\textstyle\sum a_i \otimes b_i)) &= \eta((\textstyle\sum a a_i \otimes b_i b) \\ &= \textstyle\sum \eta(a a_i \otimes b_i b) \\ &= \textstyle\sum a a_i y b_i b \\ &= a(\textstyle\sum a_i y b_i) b \\ &= 0. \end{aligned}$$

Thus, $(a \otimes b)(\sum a_i \otimes b_i) \in ker(\eta)$ and $ker(\eta)$ is an ideal of $F \otimes F$. Since $F \otimes F \cong F_1 \times \cdots \times F_n$, $ker(\eta)$ must be isomorphic to a sum of the F_i's. But $\eta(e_i) = \xi\psi(e_i) = \xi(f_i) = f_i(y) = f_i(y_i) \neq 0$. So $ker(\eta)$ must be 0. Hence η is injective. This implies that η must send k-linearly independent sets to k-linear independent sets. In particular, since $\{\alpha^i \otimes \alpha^j : i,j = 0 \dots, n-1\}$ forms a basis for $F \otimes F$ over k, the elements $\alpha^i y \alpha^j = \eta(\alpha^i \otimes \alpha^j)$ are linearly independent over k. □

The proof of the above theorem has a corollary. For its proof, we will use a lemma which we state below.

Lemma 4.2.2. *Let U be a skew field, Σ a multiplicative subset of U and K its centralizer in U. If $a_i, b_i \in U^\times$ $(i = 1, \dots, n)$ are such that*

$$\sum_{i=1}^{n} a_i x b_i = 0 \quad \textit{for all } x \in \Sigma,$$

then $a_1, \dots, a_n$ are right linearly dependent over K.

Proof. See [8, Lemma 9.8.1, p. 390]. □

Corollary 4.2.3. *Let R be a ring which has a skew field of fractions U such that both have the same centre k. Suppose R contains a finite Galois extension F of k of degree n generated by an element α. Then there exists an element $x \in R$ such that the set $\{\alpha^i x \alpha^j : i,j = 0, \dots, n\}$ is linearly independent over k.*

Proof. Note that, by the proof of Theorem 4.2.1, in order to find a y satisfying the condition, the only thing we required from the map φ was that it was injective. Therefore, if we prove that the homomorphism

$$\theta : R \otimes R^\circ \longrightarrow \mathcal{M}(R)$$

such that $\theta(a \otimes b) = \mathcal{L}_a \mathcal{R}_b$, for $a, b \in R$, is also injective, we can produce, as we did in Theorem 4.2.1, an injective k-linear map

$$F \otimes F \longrightarrow R$$

sending $c \otimes d$ to cxd, for $c, d \in F$ and some element $x \in R$. And this implies that $\{\alpha^i x \alpha^j : i,j = 0, \dots, n\}$ is linearly independent over k.

To prove that θ is injective, we will use Lemma 4.2.2. First note that, since R is a subring of U, it is a multiplicative subset. Next, since U is generated as a field by R, the centralizer of R in U is just k. Suppose that $ker(\theta) \neq 0$ and pick a nonzero element $\sum_{i=1}^{m} a_i \otimes b_i$ in $\ker(\theta)$ with m minimal. m must be greater than 1, because $a_1 x b_1 = 0$ for all $x \in R$ implies, in particular, that $a_1 b_1 = 0$. Since R is an integral domain, we would have $a_1 = 0$ or $b_1 = 0$. But this is impossible, because we had chosen $a_1 \otimes b_1$ to be nonzero in $ker(\theta)$. By Lemma 4.2.2, $a_1, \ldots, a_m$ are linearly dependent over k. So, by possibly relabelling indices, we can write

$$a_1 = \sum_{i=2}^{m} a_i \lambda_i$$

with $\lambda_i \in k$. But then,

$$\sum_{i=1}^{m} a_i \otimes b_i = \sum_{i=2}^{m} a_i \otimes (\lambda_i b_1 + b_i),$$

contradicting the minimality of m. So $ker(\theta) = 0$, *i.e.* θ is injective. $\square$

One last observation will be necessary before we proceed.

Lemma 4.2.4. *Let k be a commutative field and R a k-algebra. Suppose that R contains a finite Galois extension F of k with basis $\{v_1, \ldots, v_n\}$ and let $G = Gal(F/k)$. Suppose also that there exists $x \in R$ such that the set $\{v_i x v_j : i, j = 1, \ldots, n\}$ is linearly independent over k. Then, for each $\sigma \in G$, the set $\{v_i x v_j^\sigma : i, j = 1, \ldots, n\}$ is linearly independent over k.*

Proof. Consider the following function between k-vector spaces.

$$\begin{aligned} \varphi : F \times \cdots \times F &\longrightarrow R \\ (\xi_1, \ldots, \xi_n) &\longmapsto \textstyle\sum_{i=1}^{n} v_i x \xi_i. \end{aligned}$$

The map φ is obviously linear and is it injective, for if $\varphi(\xi_1, \ldots, \xi_n) = 0$, writing $\xi_i = \sum_j \lambda_{ij} v_j$ with $\lambda_{ij} \in k$, we obtain $\sum_{ij} \lambda_{ij} v_i x v_j = 0$. And this implies, by hypothesis, that $\lambda_{ij} = 0$, therefore, $\xi_i = 0$ for every $i = 1, \ldots, n$. Now take $\sigma \in G$ and suppose that we have an equation

$$\sum_{i,j=1}^{n} \lambda_{ij} v_i x v_j^\sigma = 0$$

with $\lambda_{ij} \in k$. This is equivalent to the relation

$$\varphi\left(\textstyle\sum_j \lambda_{1j} v_j^\sigma, \ldots, \sum_j \lambda_{nj} v_j^\sigma\right) = 0.$$

Since φ is injective, for all i, we have $\left(\sum_j \lambda_{ij} v_j\right)^\sigma = \sum_j \lambda_{ij} v_j^\sigma = 0$. The facts that σ is an automorphism and $\{v_j\}$ linearly independent over k imply that $\lambda_{ij} = 0$, for all i and j. So $\{v_i x v_j^\sigma : i, j = 1, \ldots, n\}$ is linearly independent over k. $\square$

4.2.2 Matrix units

We now start to restrict our objects. We begin by giving the set-up.

Let R be a ring which has a commutative field k as its centre. Suppose that R has a skew field of fractions U with the same centre k and that R contains a finite Galois extension F of k. Let E be a field which is isomorphic to F over k. We will be considering the effect of extending the ring R by E over k. We need some notation. Suppose the degree of the extension F/k is $n > 1$. It is known that F contains a primitive element, *i.e.* there exists $\alpha \in F$ such that $F = k(\alpha)$. Similarly, we can write $E = k(\beta)$, where the isomorphism between E and F takes α to β. Let

$$f(X) = \sum_{i=0}^{n} a_i X^i$$

denote the minimal polynomial of α over k. (It is understood that $a_n = 1$.) Clearly, f is also the minimal polynomial of β over k. Denote by G the Galois group $Gal(F/k) = \{1 = \sigma_1, \dots, \sigma_n\}$ and by G' the group $Gal(E/k) = \{1 = \tau_1, \dots, \tau_n\}$. We know that $G \cong G'$ (because $F \cong E$) and suppose the elements were written in such an order that this isomorphism takes σ_i to τ_i. Finally, write α_i for α^{σ_i} and, similarly, β_i for β^{τ_i}.

Since U_E is simple artinian, it is isomorphic to a matrix ring $\mathfrak{M}_r(K)$, where r divides n and K is a field. We will show explicitly what the matrix units for U_E are. In particular, it will be shown that U_E is in fact an $n \times n$ matrix ring.

Consider the following polynomial over $F \otimes_k E$:

$$\Phi(X,Y) = \sum_{i=1}^{n} a_i \sum_{j=0}^{i-1} X^{i-j-1} Y^j \tag{4.2.1}$$

and note that

$$\Phi(X,Y)(X-Y) = f(X) - f(Y). \tag{4.2.2}$$

This last assertion can be proved by multiplying out the left-hand side.

Some elementary facts about $\Phi(X,Y)$ will be listed in the lemma below.

Lemma 4.2.5. *For the polynomial Φ defined above, the following are true*

(i) $\Phi(\alpha_i, \beta_j)\alpha_i = \Phi(\alpha_i, \beta_j)\beta_j$, *for any* $i, j = 1, \dots, n$;

(ii) $\Phi(X,X) = f'(X)$, *where* $f'(X) = \sum_{i=1}^{n} ia_iX^{i-1}$;

(iii) $\Phi(\alpha_i,\alpha_j) = \delta_{ij}f'(\alpha_i)$;

(iv) *For any* $\sigma \in G$ *and* $i = 1,\dots,n$, $\Phi(\alpha^\sigma,\beta)\alpha_i^\sigma = \Phi(\alpha^\sigma,\beta)\beta_i$.

Proof. (i) follows from (4.2.2), because $\Phi(\alpha_i,\beta_j)(\alpha_i-\beta_j) = f(\alpha_i)-f(\beta_j) = 0$, since all α_i's and β_j's are roots of f. (ii) is a straightforward calculation. (iii) follows from (ii) when $i=j$ and if $i \neq j$, since $\Phi(\alpha_i,\alpha_j)(\alpha_i-\alpha_j) = f(\alpha_i)-f(\alpha_j) = 0$, it follows that $\Phi(\alpha_i,\alpha_j) = 0$. (iv) is true, because since $F = k(\alpha)$, we can write (uniquely) $\alpha_i = \sum_{j=0}^{n-1} b_{ij}\alpha^j$, with $b_{ij} \in k$ and this implies that $\beta_i = \sum_{j=0}^{n-1} b_{ij}\beta^j$. So, $\alpha_i^\sigma = \sum_{j=0}^{n-1} b_{ij}(\alpha^\sigma)^j$ and

$$\begin{aligned}
\Phi(\alpha^\sigma,\beta)\alpha_i^\sigma &= \Phi(\alpha^\sigma,\beta)\sum_{j=0}^{n-1} b_{ij}(\alpha^\sigma)^j \\
&= \sum_{j=0}^{n-1} b_{ij}\Phi(\alpha^\sigma,\beta)(\alpha^\sigma)^j \\
&= \sum_{j=0}^{n-1} b_{ij}\Phi(\alpha^\sigma,\beta)\beta^j, \text{ by (i)} \\
&= \Phi(\alpha^\sigma,\beta)\sum_{j=0}^{n-1} b_{ij}\beta^j \\
&= \Phi(\alpha^\sigma,\beta)\beta_i.
\end{aligned}$$

□

Now, for each $i = 1,\dots,n$, define in $F \otimes_k E$, elements

$$e_i \stackrel{\text{def}}{=} \frac{\Phi(\alpha_i,\beta)}{f'(\alpha_i)} = \frac{\Phi(\alpha_i,\beta)}{f'(\beta)}, \tag{4.2.3}$$

where the last equality follows from Lemma 4.2.5 (i). If we denote by e the element

$$e = \frac{\Phi(\alpha,\beta)}{f'(\beta)} \tag{4.2.4}$$

and introduce the notation

$$e^\sigma \stackrel{\text{def}}{=} \frac{\Phi(\alpha^\sigma,\beta)}{f'(\beta)},$$

for any $\sigma \in G$, we can write $e_i = e^{\sigma_i}$. In particular, $e_1 = e$. Note that since, for every $\sigma \in G$, $e^\sigma \in F \otimes_k E$, which is a commutative subring of R_E, we always have

$$\alpha e^{\sigma} = e^{\sigma}\alpha. \tag{4.2.5}$$

Lemma 4.2.6. *The elements defined in* (4.2.3) *satisfy*

$$\sum_{i=1}^{n} e_i = 1 \quad \textit{and} \quad e_i e_j = \delta_{ij} e_i.$$

Proof. This proof follows a well known argument. We will first prove the second assertion. For each i, e_i is an idempotent, because, by Lemma 4.2.5 (i) and (ii), $\Phi(\alpha_i,\beta)\Phi(\alpha_i,\beta) = \Phi(\alpha_i,\beta)\Phi(\beta,\beta) = \Phi(\alpha_i,\beta)f'(\beta)$. Therefore,

$$\begin{aligned} e_i e_i &= \frac{\Phi(\alpha_i,\beta)}{f'(\beta)}\frac{\Phi(\alpha_i,\beta)}{f'(\beta)} \\ &= \frac{\Phi(\alpha_i,\beta)f'(\beta)}{(f'(\beta))^2} \\ &= \frac{\Phi(\alpha_i,\beta)}{f'(\beta)} \\ &= e_i. \end{aligned}$$

If $i \neq j$, then $\Phi(\alpha_i,\beta)\Phi(\alpha_j,\beta)\alpha_i = \Phi(\alpha_i,\beta)\Phi(\alpha_j,\beta)\beta = \Phi(\alpha_i,\beta)\Phi(\alpha_j,\beta)\alpha_j$. Since $\alpha_i - \alpha_j \neq 0$, it follows that $e_i e_j = 0$ for $i \neq j$. To prove the first assertion, consider the polynomial $P(X) = \sum_{i=1}^{n} \Phi(\alpha_i, X)/f'(\alpha_i)$ over the field F. $P(X)$ is of degree at most $n-1$ (because each $\Phi(\alpha_i, X)$ is) and for all $j = 1, \dots, n$, $P(\alpha_j) = 1$, because $\Phi(\alpha_i,\alpha_j)/f'(\alpha_i) = \delta_{ij}$, by Lemma 4.2.5 (iii). So $P(X) - 1$ is of degree at most $n-1$ and has n distinct roots. Therefore, $P(X)$ must be constant equal to 1, in particular, $\sum_{i=1}^{n} e_i = 1$. $\square$

In R, let $\mathcal{L} = \mathcal{L}(\alpha)$ denote the linear operator defined by left multiplication by α and for each $i = 1, \dots, n$, write $\mathcal{R}_i = \mathcal{R}(\alpha_i)$ for the right multiplication by α_i. Now define the operator $\hat{v}_i = \Phi(\mathcal{L}, \mathcal{R}_i) = \sum_{h=1}^{n} a_h \sum_{j=0}^{h-1} \mathcal{L}^{h-j-1}\mathcal{R}_i^j$. We know, by Corollary 4.2.3 and Lemma 4.2.4, that R contains an element x such that $\{\alpha^i x \alpha_s^j : i,j = 0,\dots,n-1\}$ is linearly independent over k for every $s = 1,\dots,n$. For each $i = 1, \dots, n$, let $v_i = \hat{v}_i(x)$, *i.e.*

$$v_i = \sum_{h=1}^{n} a_h \sum_{j=0}^{h-1} \alpha^{h-j-1} x \alpha_i^j. \tag{4.2.6}$$

Since the a_h are not all zero, v_i is different from zero and these elements satisfy

$$\alpha v_i = v_i \alpha_i, \quad \text{for all } i. \tag{4.2.7}$$

Indeed,

$$\begin{aligned}
(\mathcal{L} - \mathcal{R}_i)\hat{v}_i &= (\mathcal{L} - \mathcal{R}_i)\Phi(\mathcal{L}, \mathcal{R}_i) \\
&= f(\mathcal{L}) - f(\mathcal{R}_i) \\
&= 0,
\end{aligned}$$

because both α and α_i are solutions of $f(X) = 0$ and $\mathcal{L}$ and $\mathcal{R}_i$ commute. Since $v_i = \hat{v}_i(x)$, we have (4.2.7). If we write $\alpha^{\sigma_j} = \alpha_j$ as a sum $\sum_{h=0}^{n-1} b_{jh}\alpha^h$, with $b_{jh} \in k$, as we did in the proof of Lemma 4.2.5, it is easy to see that (4.2.7) implies

$$\alpha^{\sigma_j} v_i = v_i \alpha^{\sigma_j \sigma_i}, \quad \text{for all } i, j. \tag{4.2.8}$$

And, therefore,

$$e^{\sigma_j} v_i = v_i e^{\sigma_j \sigma_i}, \quad \text{for all } i, j. \tag{4.2.9}$$

Since the v_i's are not zero in R, they are invertible over the skew field of fractions U of R. Let S be the subring of U generated by R and v_i^{-1}, for $i = 1, \dots, n$. Over S_E the relations (4.2.8) and (4.2.9) are also valid and, since the v_i's are now invertible, in S_E we have

$$\begin{aligned}
v_i^{-1}\alpha^{\sigma_j} &= \alpha^{\sigma_j\sigma_i} v_i^{-1} \quad \text{and} & (4.2.10)\\
v_i^{-1}e^{\sigma_j} &= e^{\sigma_j\sigma_i} v_i^{-1}, \quad \text{for all } i, j. & (4.2.11)
\end{aligned}$$

It is worth pointing out that (4.2.8), (4.2.9), (4.2.10) and (4.2.11) imply, by replacing σ_j by $\sigma_j\sigma_i^{-1}$, the following other useful relations valid in S_E:

$$\begin{aligned}
\alpha^{\sigma_j\sigma_i^{-1}} v_i &= v_i \alpha^{\sigma_j} \quad \text{and} & (4.2.12)\\
e^{\sigma_j\sigma_i^{-1}} v_i &= v_i e^{\sigma_j}; & (4.2.13)\\
\alpha^{\sigma_j} v_i^{-1} &= v_i^{-1}\alpha^{\sigma_j\sigma_i^{-1}} \quad \text{and} & (4.2.14)\\
e^{\sigma_j} v_i^{-1} &= v_i^{-1} e^{\sigma_j\sigma_i^{-1}} \quad \text{for all } i, j. & (4.2.15)
\end{aligned}$$

Now we are ready to prove

Proposition 4.2.7. *Let R be a ring which has a skew field of fractions U such that both have the same centre k. Suppose R contains a finite Galois extension F of k of degree $n > 1$ and let E be isomorphic to F over k. Then there exist nonzero elements $v_1, \ldots, v_n$ in U such that the subring S of U generated by R and $v_1^{-1}, \ldots, v_n^{-1}$ is such that S_E is isomorphic to a matrix ring $\mathfrak{M}_n(T)$, where T is the centralizer of elements*

$$e_{ij} = v_i^{-1} e v_j \tag{4.2.16}$$

and e is given by (4.2.4).

Proof. If we have in mind Prop. 0.1.1 of [6], the only thing left to prove is that the set $\{e_{ij} | i, j = 1, \ldots, n\}$ forms a complete set of matrix units for S_E. Note that, by (4.2.11), $e_{ii} = e^{\sigma_i} = e_i$, so $\sum_{i=1}^n e_{ii} = 1$, by Lemma 4.2.6. Moreover,

$$\begin{aligned} e_{ij} e_{hl} &= v_i^{-1} e v_j v_h^{-1} e v_l \\ &= v_i^{-1} v_j e_j e_h v_h^{-1} v_l, \text{ by (4.2.9) and (4.2.11)} \\ &= \delta_{jh} v_i^{-1} v_j e_j v_j^{-1} v_l, \text{ by Lemma 4.2.6} \\ &= \delta_{jh} v_i^{-1} e v_l \text{ , by (4.2.14)} \\ &= \delta_{jh} e_{il} \text{ , by (4.2.16).} \end{aligned}$$

And this proves the proposition. □

4.2.3 A special case

We will apply the above results to the case of $R = F_k\langle x\rangle$, *i.e.* the free F_k-ring on one generator. We know that R is a fir, so let U be its universal field of fractions. Note that in this context $x \in R$ certainly satisfies the conclusion of Corollary 4.2.3. The first thing worth mentioning is the fact that although R_E is a hereditary ring, because E/k is separable, it is no longer a fir.

We have

$$R_E \cong (F \otimes_k E) *_E E\langle x\rangle. \tag{4.2.17}$$

One way of seeing this is by writing $R \cong F *_k k\langle x\rangle$ and then applying the tensor product operation to it. Since F/k is Galois and $E \cong F$, it follows that $F \otimes_k E \cong$

$F_1 \times \ldots \times F_n$, where each $F_i \cong F$ (see Cor. 5.7.5 of [7] on p. 194). Therefore, if we apply Bergman's coproduct theorem on the monoid of projectives to (4.2.17), we obtain

$$\begin{aligned}
\mathcal{P}(R_E) &\cong (\mathcal{P}(F_1) \times \ldots \times \mathcal{P}(F_n)) \coprod_{\mathcal{P}(E)} \mathcal{P}(E\langle x\rangle) \\
&\cong (\mathbb{N} \times \ldots \times \mathbb{N}) \coprod_{\mathbb{N}} \mathbb{N} \\
&\cong \mathbb{N} \times \ldots \times \mathbb{N},
\end{aligned}$$

where the Cartesian products above have n terms. Since $n > 1$, R_E is not projective free, therefore, not a fir.

4.2.4 Generators

In this subsection it will be proved that S_E is isomorphic as an E-algebra to the ring $E\langle \alpha, v_i, v_i^{-1} \ {\scriptstyle (i=1,\ldots,n)} | f(\alpha) = 0,\ \alpha v_i = v_i \alpha_i,\ v_i v_i^{-1} = v_i^{-1} v_i = 1\rangle$.

Let us start with

Lemma 4.2.8. *$F_k\langle x\rangle$ is isomorphic to $k\langle \alpha, x | f(\alpha) = 0\rangle$.*

Proof. The map $k\langle t_1, t_2\rangle \longrightarrow F_k\langle x\rangle$ defined as k-linear, with $t_1 \longmapsto \alpha$, $t_2 \longmapsto x$ is a surjective k-algebra homomorphism with kernel generated by $f(t_1)$. □

Corollary 4.2.9. *$F_k\langle x\rangle$ is isomorphic to $k\langle \alpha, v_i \ {\scriptstyle (i=1,\ldots,n)} | f(\alpha) = 0,\ \alpha v_i = v_i \alpha_i\rangle$.*

Proof. By Lemma 4.2.8, we can identify $R = F_k\langle x\rangle$ with $k\langle \alpha, x | f(\alpha) = 0\rangle$. Denote the ring $k\langle t, v_i | f(t) = 0, t v_i = v_i t_i\rangle$ by R', where $t_i = t^{\sigma_i}$ and $\sigma_i \in Gal(k(t)/k)$ which is the same as $Gal(F/k)$. Note that we also have $\Phi(t_i, t_j) = \delta_{ij} f'(t_i)$, where Φ is defined in (4.2.1) (see Lemma 4.2.5). Moreover, (4.2.12), with t substituted for α remains valid, because it is a consequence of the fact that $t v_i = v_i t_i$ alone. It is easy to see that the maps $\phi : R \longrightarrow R'$ and $\psi : R' \longrightarrow R$ defined on generators by $\phi(\alpha) = t$, $\phi(x) = \frac{1}{f'(t)} \sum_{i=1}^{n} v_i$ and $\psi(t) = \alpha$, $\psi(v_i) = \sum_{h=1}^{n} a_h \sum_{j=0}^{h-1} \alpha^{h-j-1} x \alpha_i^j$ are indeed k-algebra homomorphisms. It is clear that $\phi\psi(t) = t$ and that $\psi\phi(\alpha) = \alpha$. We must prove then that $\psi\phi(x) = x$ and that $\phi\psi(v_i) = v_i$ for all $i = 1, \ldots, n$. In other terms, we must prove that

$$x = \frac{1}{f'(\alpha)} \sum_{i=1}^{n} \sum_{h=1}^{n} a_h \sum_{j=0}^{h-1} \alpha^{h-j-1} x \alpha_i^j \tag{i}$$

and

$$v_i = \sum_{h=1}^{n} a_h \sum_{j=0}^{h-1} t^{h-j-1} \left(\frac{1}{f'(t)} \sum_{l=1}^{n} v_l \right) t_i^j. \tag{ii}$$

(i) follows because

$$\begin{aligned}
\sum_{i=1}^{n} \sum_{h=1}^{n} a_h \sum_{j=0}^{h-1} \alpha^{h-j-1} x \alpha_i^j &= \sum_{h=1}^{n} a_h \sum_{j=0}^{h-1} \alpha^{h-j-1} x \left(\sum_{i=1}^{n} \alpha_i^j \right) \\
&= \sum_{h=1}^{n} a_h \sum_{j=0}^{h-1} \alpha^{h-j-1} \left(\sum_{i=1}^{n} \alpha_i^j \right) x.
\end{aligned}$$

The last equality is true because, since $\sum_{i=1}^{n} \alpha_i^j$ is a symmetric polynomial in the α_i's, it is expressible as a polynomial in the elementary symmetric polynomials in the α_i's with coefficients in k. But the elementary symmetric polynomials in the α_i's are just the coefficients of $f(X)$, the minimal polynomial for α over k. Therefore, $\sum_{i=1}^{n} \alpha_i^j$ is expressible as a polynomial in the a_i's with coefficients in k, and thus, belongs to k. Now, continuing the calculations above, we have

$$\begin{aligned}
\sum_{i=1}^{n} \sum_{h=1}^{n} a_h \sum_{j=0}^{h-1} \alpha^{h-j-1} x \alpha_i^j &= \sum_{i=1}^{n} \sum_{h=1}^{n} a_h \sum_{j=0}^{h-1} \alpha^{h-j-1} \alpha_i^j x \\
&= \sum_{i=1}^{n} \Phi(\alpha, \alpha_i) x \\
&= f'(\alpha) x,
\end{aligned}$$

because $\Phi(\alpha, \alpha_i) = 0$ if $\alpha_i \neq \alpha$ and $= f'(\alpha)$ if $\alpha_i = \alpha$. To prove (ii), we proceed as follows

$$
\begin{aligned}
\sum_{h=1}^{n} a_h \sum_{j=0}^{h-1} t^{h-j-1} \left(\frac{1}{f'(t)} \sum_{l=1}^{n} v_l \right) t_i^j &= \\
&= \frac{1}{f'(t)} \sum_{h=1}^{n} a_h \sum_{j=0}^{h-1} \alpha^{h-j-1} \sum_{l=1}^{n} v_l t_i^j \\
&= \frac{1}{f'(t)} \sum_{l=1}^{n} \sum_{h=1}^{n} a_h \sum_{j=0}^{h-1} t^{h-j-1} \left(t^{\sigma_i \sigma_l^{-1}} \right)^j v_l \text{ , by (4.2.12)} \\
&= \frac{1}{f'(t)} \sum_{l=1}^{n} \Phi(t, t^{\sigma_i \sigma_l^{-1}}) v_l \\
&= \frac{1}{f'(t)} f'(t) v_i \text{ , by Lemma 4.2.5 (iii)} \\
&= v_i .
\end{aligned}
$$

□

We now look at the ring S obtained from $R = F_k\langle x \rangle$ by the construction given in Section 4.2.2.

Corollary 4.2.10. *S_E is isomorphic to*

$$E\langle \alpha, v_i, v_i^{-1} \ {\scriptstyle (i=1,\dots,n)} | f(\alpha) = 0, \ \ \alpha v_i = v_i \alpha_i, \ \ v_i v_i^{-1} = v_i^{-1} v_i = 1 \rangle .$$

Proof. By Corollary 4.2.9 and the definition of S, the ring S is isomorphic to $k\langle \alpha, v_i, v_i^{-1} \ {\scriptstyle (i=1,\dots,n)} | f(\alpha) = 0, \ \ \alpha v_i = v_i \alpha_i, \ \ v_i v_i^{-1} = v_i^{-1} v_i = 1 \rangle$. □

We know, by Proposition 4.2.7, that S_E is isomorphic to a matrix ring, namely, $\mathfrak{M}_n(T)$, where T is the centralizer of the matrix units $e_{ij} = v_i^{-1} e v_j$ and e is defined in (4.2.4). Prop. 0.1.1 of [6] shows us how to regard the elements in S_E as matrices over T: for each c in S_E, take the matrix $[c] = (c_{ij})$ in $\mathfrak{M}_n(T)$, where c_{ij} is given by

$$c_{ij} = \sum_{\nu=1}^{n} e_{\nu i} c e_{j \nu} . \tag{4.2.18}$$

Since S_E is generated as an E-algebra by α, v_i, v_i^{-1} (see Cor. 4.2.10), T will be generated as an E-algebra by the entries of $[\alpha], [v_i], [v_i^{-1}]$. Let us start with

$[\alpha] = (\alpha_{ij})$.

$$\begin{aligned}
\alpha_{ij} &= \sum_{\nu=1}^{n} e_{\nu i}\alpha e_{j\nu} \\
&= \sum_{\nu=1}^{n} v_\nu^{-1} e v_i \alpha v_j^{-1} e v_\nu \\
&= \sum_{\nu=1}^{n} v_\nu^{-1} v_i e^{\sigma_i} e^{\sigma_j} \alpha v_j^{-1} v_\nu \text{ , by (4.2.9), (4.2.11) and (4.2.5)} \\
&= \delta_{ij} \sum_{\nu=1}^{n} v_\nu^{-1} v_i e^{\sigma_i} \alpha v_j^{-1} v_\nu \text{ , for } e^{\sigma_i} e^{\sigma_j} = \delta_{ij} e^{\sigma_i}.
\end{aligned}$$

By applying the appropriate laws (4.2.8)–(4.2.15), we obtain

$$v_\nu^{-1} v_i e^{\sigma_i} \alpha v_i^{-1} v_\nu = e_\nu \alpha^{\sigma_i^{-1}\sigma_\nu}.$$

Using Lemma 4.2.5 (iv), we find that $e_\nu \alpha^{\sigma_i^{-1}\sigma_\nu} = e_\nu \beta^{\tau_i^{-1}}$, where τ_i was defined as the element of the Galois group of E over k corresponding to σ_i. Therefore,

$$\alpha_{ij} = \delta_{ij} \sum_{\nu=1}^{n} e_\nu \beta^{\tau_i^{-1}} = \delta_{ij} \beta^{\tau_i^{-1}}.$$

So, in matrix form,

$$[\alpha] = \begin{bmatrix} \beta^{\tau_1^{-1}} & 0 & \cdots & 0 \\ 0 & \beta^{\tau_2^{-1}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta^{\tau_n^{-1}} \end{bmatrix}.$$

To look at $[v_h]$, we must first define, for each $h = 1, \dots, n$, an element π_h in the symmetric group S_n which is given by

$$\begin{aligned}
\pi_h \ : \ \{1, \dots, n\} &\longrightarrow \{1, \dots, n\} \\
i &\longmapsto \pi_h(i),
\end{aligned} \tag{4.2.19}$$

where $\pi_h(i)$ is the only element in $\{1, \dots, n\}$ such that $\sigma_{\pi_h(i)} = \sigma_i \sigma_h$.

For a fixed $\nu \in \{1, \dots, n\}$, we have

$$\begin{aligned}
e_{\nu i} v_h e_{j\nu} &= v_\nu^{-1} e v_i v_h v_j^{-1} e v_\nu \\
&= v_\nu^{-1} v_i v_h e^{\sigma_i \sigma_h} e^{\sigma_j} v_j^{-1} v_\nu \\
&= \delta_{\pi_h(i),j} v_\nu^{-1} v_i v_h v_{\pi_h(i)}^{-1} v_\nu e_\nu.
\end{aligned}$$

Therefore,

$$(v_h)_{ij} = \sum_{\nu=1}^{n} e_{\nu i} v_h e_{j\nu} = \delta_{\pi_h(i),j} \sum_{\nu=1}^{n} v_\nu^{-1} v_i v_h v_{\pi_h(i)}^{-1} v_\nu e_\nu.$$

For each $h = 1, \ldots, n$, let Π_h denote the permutation matrix defined by π_h, *i.e.* $\Pi_h = (\delta_{\pi_h(i),j})$. Let $p_{hi} = \sum_{\nu=1}^{n} v_\nu^{-1} v_i v_h v_{\pi_h(i)}^{-1} v_\nu e_\nu$ and

$$P_h = \begin{bmatrix} p_{h1} & 0 & \cdots & 0 \\ 0 & p_{h2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_{hn} \end{bmatrix}.$$

So, $[v_h] = P_h \Pi_h$. Now, for each $h, i = 1, \ldots, n$, set

$$q_{hi} = \sum_{\nu=1}^{n} e_\nu v_\nu^{-1} v_{\pi_h(i)} v_h^{-1} v_i^{-1} v_\nu.$$

With the help of (4.2.9)–(4.2.15), we can see that $q_{hi} = p_{hi}^{-1}$. So P_h is invertible and $[v_h^{-1}] = \Pi_h^{-1} P_h^{-1}$. Note that $p_{h1} = \sum_{\nu=1}^{n} v_\nu^{-1} v_1 v_\nu e_\nu$ for every $h = 1, \ldots, n$. Call this common element p. Similarly, set $q = q_{h1}$. Recall that T, the centralizer of the matrix units of S_E, is generated as an E-algebra by the entries of $[\alpha], [v_i], [v_i^{-1}]$. Since the entries of $[\alpha]$ all belong to E, they are superfluous generators of T as an E-algebra. This sums up to

Proposition 4.2.11. *The ring T, defined as the centralizer of the elements e_{kl} $(k, l = 1, \ldots, n)$ in S_E, is generated as an E-algebra by the elements p, q, p_{ij}, q_{ij}, where $i = 1, \ldots, n$ and $j = 2, \ldots, n$.* □

4.2.5 A matrix ring over a fir

In this subsection, we will prove that the only relations among the generators of T, given by Prop. 4.2.11 are $pq = qp = 1$, $p_{ij}q_{ij} = q_{ij}p_{ij} = 1$. This will finally establish that T is a fir.

Let B be the E-algebra defined by

$$B = E\langle y, z, y_{ij}, z_{ij} \ {\scriptstyle (i=1,\ldots,n, j=2,\ldots,n)} | yz = zy = 1, \ y_{ij} z_{ij} = z_{ij} y_{ij} = 1 \rangle$$

and consider the full $n \times n$ matrix ring over B, $\mathfrak{M}_n(B)$, which is the E-algebra generated by $y, z, y_{ij}, z_{ij}, E_{hl}$ with defining relations

$$
\begin{gathered}
yz = zy = 1 \\
y_{ij}z_{ij} = z_{ij}y_{ij} = 1 \\
yE_{hl} = E_{hl}y \\
y_{ij}E_{hl} = E_{hl}y_{ij} \\
E_{hl}E_{rs} = \delta_{lr}E_{hs} \\
\sum_{h=1}^{n} E_{hh} = 1,
\end{gathered}
$$

where $i, h, l, r, s = 1, \dots, n$ and $j = 2, \dots, n$.

We will show that there exists an isomorphism between S_E and $\mathfrak{M}_n(B)$ which preserves matrix units and, therefore, restricts to an isomorphism between T and B.

Theorem 4.2.12. *Let k be a commutative field and $R = F_k\langle x\rangle$, where F is a finite Galois extension of k of degree $n > 1$. Let E be an extension of k isomorphic to F/k. Denote by U the universal field of fractions of R. Then there exists a subring S of U, obtained from R by adjoining inverses of finitely many elements of R such that S_E is isomorphic to $\mathfrak{M}_n(B)$, where B is the E-algebra defined by*

$$B = E\langle y, z, y_{ij}, z_{ij} \ {\scriptstyle (i=1,\dots,n, j=2,\dots,n)} | yz = zy = 1, \ \ y_{ij}z_{ij} = z_{ij}y_{ij} = 1\rangle$$

Proof. Let S be the ring obtained from $R = F_k\langle x\rangle$ by the construction in the subsection 4.2.2. We will prove that this S satisfies the theorem. First, let

$$\xi : E\langle \alpha, v_i, v_i^{-1} \ {\scriptstyle (i=1,\dots,n)}\rangle \longrightarrow \mathfrak{M}_n(B)$$

be the E-algebra homomorphism induced by the map

$$
\begin{aligned}
\alpha &\longmapsto \sum_{i=1}^{n} \beta^{\tau_i^{-1}} E_{ii} \\
v_h &\longmapsto \left(\sum_{i=1}^{n} y_{hi}E_{ii}\right) \Pi_h \text{ , where } y_{h1} = y \\
v_h^{-1} &\longmapsto \Pi_h^{-1} \left(\sum_{i=1}^{n} z_{hi}E_{ii}\right) \text{ , where } z_{h1} = z
\end{aligned}
$$

and Π_h is the permutation matrix defined by $\Pi_h = \sum_{i,j=1}^{n} \delta_{\pi_h(i),j} E_{ij}$ and π_h is given by (4.2.19). We have, then,

$$\xi(f(\alpha)) = f(\xi(\alpha)) = f\left(\sum_{i=1}^{n} \beta^{\tau_i^{-1}} E_{ii}\right) = \sum_{i=1}^{n} f(\beta^{\tau_i^{-1}}) E_{ii} = 0. \qquad (4.2.20)$$

Moreover, if we write $\alpha^{\sigma_h} = \sum_{j=0}^{n-1} b_{hj}\alpha^j$, with $b_{hj} \in k$, then $\beta^{\tau_h} = \sum_{j=0}^{n-1} b_{hj}\beta^j$ and, so, $\beta^{\tau_h \tau} = \sum_{j=0}^{n-1} b_{hj}(\beta^{\tau})^j$, for any $\tau \in G'$. So

$$\begin{aligned}
\xi(\alpha^{\sigma_h}) &= \xi\left(\sum_{j=0}^{n-1} b_{hj}\alpha^j\right) \\
&= \sum_{j=0}^{n-1} b_{hj}\xi(\alpha)^j \\
&= \sum_{j=0}^{n-1} b_{hj}\left(\sum_{i=1}^{n} \beta^{\tau_i^{-1}} E_{ii}\right)^j \\
&= \sum_{j=0}^{n-1} b_{hj} \sum_{i=1}^{n} (\beta^{\tau_i^{-1}})^j E_{ii} \\
&= \sum_{i=1}^{n} b_{hj} \sum_{j=0}^{n-1} (\beta^{\tau_i^{-1}})^j E_{ii} \\
&= \sum_{i=1}^{n} \beta^{\tau_h \tau_i^{-1}} E_{ii}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\xi(\alpha v_h) &= \xi(\alpha)\xi(v_h) \\
&= \left(\sum_{i=1}^{n} \beta^{\tau_i^{-1}} E_{ii}\right)\left(\sum_{j=1}^{n} y_{hj} E_{jj}\right)\Pi_h \\
&= \sum_{i=1}^{n} \beta^{\tau_i^{-1}} y_{hi} E_{ii}\Pi_h \\
&= \sum_{i=1}^{n} \beta^{\tau_i^{-1}} y_{hi} E_{i,\pi_h(i)}, \qquad (4.2.21)
\end{aligned}$$

because $E_{ii}\Pi_h = E_{i,\pi_h(i)}$. On the other hand,

$$\begin{aligned}
\xi(v_h \alpha_h) &= \xi(v_h)\xi(\alpha_h) \\
&= \left(\sum_{i=1}^{n} y_{hi} E_{ii}\Pi_h\right)\left(\sum_{j=1}^{n} \beta^{\tau_h \tau_j^{-1}} E_{jj}\right) \\
&= \sum_{i=1}^{n} \beta^{\tau_h \tau_{\pi_h(i)}^{-1}} y_{hi} E_{i,\pi_h(i)},
\end{aligned}$$

because $E_{ii}\Pi_h E_{jj} = \delta_{\pi_h(i),j} E_{ij}$. Since, by definition, $\tau_{\pi_h(i)} = \tau_i \tau_h$, we have $\tau_{\pi_h(i)}^{-1} = \tau_h^{-1}\tau_i^{-1}$. Therefore,

$$\xi(v_h \alpha_h) = \sum_{i=1}^{n} \beta^{\tau_i^{-1}} y_{hi} E_{i,\pi_h(i)}. \tag{4.2.22}$$

Equations (4.2.21) and (4.2.22) amount to

$$\xi(\alpha v_h) = \xi(v_h \alpha_h) \text{ , for } h = 1, \ldots, n. \tag{4.2.23}$$

Finally,

$$\begin{aligned} \xi(v_h v_h^{-1}) &= \xi(v_h)\xi(v_h^{-1}) \\ &= \left(\sum_{i=1}^{n} y_{hi} E_{ii} \Pi_h\right) \left(\Pi_h^{-1} \sum_{j=1}^{n} z_{hj} E_{jj}\right) \\ &= \sum_{i=1}^{n} y_{hi} z_{hi} E_{ii} \\ &= 1 \end{aligned} \tag{4.2.24}$$

and, similarly,

$$\xi(v_h^{-1} v_h) = 1. \tag{4.2.25}$$

Because ξ is a homomorphism, (4.2.20), (4.2.23), (4.2.24) and (4.2.25) imply that

$$f(\alpha),\ \alpha v_h - v_h \alpha_h v_h,\ v_h v_h^{-1} - 1,\ v_h^{-1} v_h - 1 \in ker(\xi).$$

Hence, there exists an E-algebra homomorphism $\bar{\xi} : S_E \longrightarrow \mathfrak{M}_n(B)$ defined on generators by

$$\bar{\xi}(\alpha) = \sum_{i=1}^{n} \beta^{\tau_i^{-1}} E_{ii} \text{ , } \bar{\xi}(v_h) = \sum_{i=1}^{n} y_{hi} E_{ii} \Pi_h \text{ , } \bar{\xi}(v_h^{-1}) = \Pi_h^{-1} \sum_{i=1}^{n} y_{hi} E_{ii} \text{ ,}$$

where $y_{h1} = y$, $z_{h1} = z$, for every $h = 1, \ldots, n$.

Note that

$$
\begin{aligned}
\bar{\xi}(e) &= \bar{\xi}\left(\frac{\Phi(\alpha,\beta)}{f'(\beta)}\right) \\
&= \frac{1}{f'(\beta)}\bar{\xi}(\Phi(\alpha,\beta)) \\
&= \frac{1}{f'(\beta)}\sum_{h=1}^{n} a_h \sum_{j=0}^{h-1} \bar{\xi}(\alpha)^{h-j-1}\beta^j \\
&= \frac{1}{f'(\beta)}\sum_{h=1}^{n} a_h \sum_{j=0}^{h-1}\sum_{i=1}^{n} (\beta^{\tau_i^{-1}})^{h-j-1} E_{ii}\beta^j \\
&= \frac{1}{f'(\beta)}\sum_{i=1}^{n} \left(\sum_{h=1}^{n} a_h \sum_{j=0}^{h-1} (\beta^{\tau_i^{-1}})^{h-j-1}\beta^j\right) E_{ii} \\
&= \frac{1}{f'(\beta)}\sum_{i=1}^{n} \Phi(\beta^{\tau_i^{-1}},\beta)E_{ii} \\
&= \frac{1}{f'(\beta)} f'(\beta)E_{11} \\
&= E_{11}.
\end{aligned}
$$

And, therefore,

$$
\begin{aligned}
\bar{\xi}(e_{ij}) &= \bar{\xi}(v_i^{-1} e v_j) \\
&= \bar{\xi}(v_i^{-1})\bar{\xi}(e)\bar{\xi}(v_j) \\
&= \Pi_i^{-1}\sum_{h=1}^{n} z_{ih}E_{hh}E_{11}\sum_{l=1}^{n} y_{jl}E_{ll}\Pi_j \\
&= \Pi_i^{-1} z_{i1}y_{j1}E_{11}\Pi_j \\
&= \Pi_i^{-1}E_{11}\Pi_j \\
&= E_{ij}.
\end{aligned}
$$

That is, the homomorphism $\bar{\xi}$ takes the matrix units of S_E to the ones of $\mathfrak{M}_n(B)$.

Now consider the E-algebra homomorphism

$$
\eta : E\langle y, z, y_{ij}, z_{ij}, E_{hl}\ {}_{(i,h,l=1,\dots,n,j=2,\dots,n)}\rangle \longrightarrow S_E
$$

induced by the map

$$\begin{array}{rcl} y & \longmapsto & p \\ z & \longmapsto & q \\ y_{ij} & \longmapsto & p_{ij} \\ z_{ij} & \longmapsto & q_{ij} \\ E_{hl} & \longmapsto & e_{hl}. \end{array}$$

It is immediate to see that

$$yz-1,\ zy-1,\ y_{ij}z_{ij}-1,\ z_{ij}y_{ij}-1 \in ker(\eta)$$

and, since $\eta(E_{hl}) = e_{hl}$, we also have

$$E_{hl}E_{rs} - \delta_{lr}E_{hs},\ \sum_{i=1}^{n} E_{ii} - 1 \in ker(\eta)$$

and, because $\eta(y) = p$ and $\eta(y_{ij}) = p_{ij}$, which are all in the centralizer of the e_{hl} in S_E, we finally get

$$yE_{hl} - E_{hl}y,\ y_{ij}E_{hl} - E_{hl}y_{ij} \in ker(\eta).$$

Therefore, there exists an E-algebra homomorphism $\bar{\eta} : \mathfrak{M}_n(B) \longrightarrow S_E$ defined on generators by

$$\bar{\eta}(y) = p\ ,\ \bar{\eta}(z) = q\ ,\ \bar{\eta}(y_{ij}) = p_{ij}\ ,\ \bar{\eta}(z_{ij}) = q_{ij}\ ,\ \bar{\eta}(E_{hl}) = e_{hl}\ ,$$

for $i, h, l = 1, \ldots, n$ and $j = 2, \ldots, n$.

The next step is to show that $\bar{\eta} = \bar{\xi}^{-1}$. Indeed,

$$\begin{aligned} \bar{\eta}\bar{\xi}(\alpha) &= \bar{\eta}\left(\sum_{i=1}^{n} \beta^{\tau_i^{-1}} E_{ii}\right) \\ &= \sum_{i=1}^{n} \beta^{\tau_i^{-1}} e_{ii} \\ &= \alpha \end{aligned}$$

and

$$\begin{aligned}
\bar{\eta}\bar{\xi}(v_h) &= \bar{\eta}\left(\sum_{i=1}^{n} y_{hi}E_{ii}\Pi_h\right)\\
&= \bar{\eta}\left(\sum_{i=1}^{n} y_{hi}E_{i,\pi_h(i)}\right)\\
&= \sum_{i=1}^{n} p_{hi}e_{i,\pi_h(i)}\\
&= \sum_{i,\nu=1}^{n} v_\nu^{-1}v_iv_hv_{\pi_h(i)}^{-1}v_\nu e_\nu e_{i,\pi_h(i)}\\
&= v_h\sum_{i=1}^{n} v_{\pi_h(i)}^{-1}v_ie_{i,\pi_h(i)}\\
&= v_h\sum_{i=1}^{n} v_{\pi_h(i)}^{-1}v_iv_i^{-1}ev_{\pi_h(i)}\\
&= v_h\sum_{i=1}^{n} v_{\pi_h(i)}^{-1}v_{\pi_h(i)}e_{\pi_h(i)}\\
&= v_h\sum_{i=1}^{n} e_{\pi_h(i)}\\
&= v_h.
\end{aligned}$$

Thus, $\bar{\eta}\bar{\xi}(v_h^{-1}) = v_h^{-1}$ and, therefore, $\bar{\eta}\bar{\xi} = 1_{S_E}$.

To show that $\bar{\xi}\bar{\eta} = 1_{\mathfrak{M}_n(B)}$, first note that, since $v_h = \sum_{i=1}^{n} p_{hi}e_{i,\pi_h(i)}$, we have $\bar{\xi}(v_h) = \sum_{i=1}^{n}\bar{\xi}(p_{hi})E_{i,\pi_h(i)}$. On the other hand, by definition, $\bar{\xi}(v_h) = \sum_{i=1}^{n} y_{hi}E_{ii}\Pi_h = \sum_{i=1}^{n} y_{hi}E_{i,\pi_h(i)}$. These two equalities imply that $\bar{\xi}(p_{hi}) = y_{hi}$ and, therefore, that $\bar{\xi}(q_{hi}) = z_{hi}$. So

$$\begin{aligned}
\bar{\xi}\bar{\eta}(y) &= \bar{\xi}(p) &&= y\\
\bar{\xi}\bar{\eta}(z) &= \bar{\xi}(q) &&= z\\
\bar{\xi}\bar{\eta}(y_{ij}) &= \bar{\xi}(p_{ij}) &&= y_{ij}\\
\bar{\xi}\bar{\eta}(z_{ij}) &= \bar{\xi}(q_{ij}) &&= z_{ij}\\
\bar{\xi}\bar{\eta}(E_{hl}) &= \bar{\xi}(e_{hl}) &&= E_{hl}.
\end{aligned}$$

Hence, $\bar{\xi}\bar{\eta} = 1_{\mathfrak{M}_n(B)}$. So $\mathfrak{M}_n(B)$ and S_E are isomorphic as E-algebras and we have seen that the isomorphism $\bar{\xi}$ takes the matrix units of S_E to the corresponding matrix units in $\mathfrak{M}_n(B)$. $\square$

Corollary 4.2.13. *S_E is isomorphic to an $n \times n$ matrix ring over a fir.*

Proof. The above isomorphism, between $\mathfrak{M}_n(T) \cong S_E$ and $\mathfrak{M}_n(B)$ preserves matrix units. Therefore, $T \cong B$. Since B is just the group algebra of the free group on $n^2 - n + 1$ letters over E, it follows from Corollary 3 of [5] that B and, therefore T, is a fir. □

4.2.6 Extensions

Theorem 4.2.12 and Corollary 4.2.13 can be extended to the case where $R = F_k\langle X\rangle$ and X is any nonempty set. For this, pick $x \in X$ and write

$$R \cong R' *_k k\langle X'\rangle,$$

where $R' = F_k\langle x\rangle$ and $X' = X \setminus \{x\}$. If we construct S from R in the same manner as we did in Section 4.2.2, we get

$$S_E \cong E\langle \alpha, v_i, v_i^{-1} | f(\alpha) = 0, \alpha v_i = v_i \alpha_i\rangle *_E E\langle X'\rangle.$$

By Theorem 4.2.12 and Corollary 4.2.13, we know that $E\langle \alpha, v_i, v_i^{-1} | f(\alpha) = 0, \alpha v_i = v_i \alpha_i\rangle$ is isomorphic to a matrix ring $\mathfrak{M}_n(T)$, where T is a fir. So

$$S_E \cong \mathfrak{M}_n(T) *_E E\langle X'\rangle. \tag{4.2.26}$$

By Bergman's theorem (Corollary 1.3.3), it follows that S_E is hereditary. Equation (4.2.26) also gives us

$$\mathcal{P}(S_E) \cong \frac{1}{n}\mathbb{N} \coprod_{\mathbb{N}} \mathbb{N}.$$

Hence S_E is hereditary and projective trivial, therefore S_E is isomorphic to a full matrix ring over a fir. Explicitly, $S_E \cong \mathfrak{M}_n(V)$, where $V = \mathfrak{W}_n(T *_E E\langle X'\rangle; T)$.

Corollary 4.2.14. *Let k be a commutative field and $R = F_k\langle X\rangle$, where F is a finite Galois extension of k of degree $n > 1$ and X a nonempty set. Let E be an extension of k isomorphic to F/k. Denote by U the universal field of fractions of R. Then there exists a subring S of U, obtained from R by adjoining inverses of finitely many elements of R, such that S_E is isomorphic to an $n \times n$ matrix ring over a fir.* □

4.3 Purely inseparable case

This section is similar to the previous section in its aim of obtaining from an extended tensor ring, a full matrix ring over a fir. It differs from the earlier section, because we will be looking at a simple purely inseparable extension. We will be also using different methods. As before, we start with a very general situation and go on specializing as necessary.

4.3.1 Recognition of matrix units

We will start by stating a theorem by Agnarsson, Amitsur and Robson.

Theorem 4.3.1. *Let R be a ring containing elements f, a, b such that $f^n = 0$ and $af^{n-1} + fb = 1$. Then the set $\{E_{ij}\}$, given by $E_{ij} = f^{i-1}af^nb^{j-1}$, is a complete set of $n \times n$ matrix units for R.*

Proof. See [1, Th 1.3] □

This theorem will be applied to a ring which contains a purely inseparable extension of a subfield.

Lemma 4.3.2. *Let R be a ring having a field k of characteristic $p > 0$ as its centre. Suppose that R is embeddable in a skew field U and that there exists $\alpha \in R \setminus k$ such that $\alpha^p \in k$. Then there exist nonzero elements $v, t \in R$ such that $\alpha t = t\alpha$ and writing $u = vt^{-1}$, we have*

$$u\alpha - \alpha u = 1.$$

Proof. Let

$$\begin{aligned} \delta : U &\longrightarrow U \\ x &\longmapsto x\alpha - \alpha x \end{aligned}$$

be the inner derivation of U defined by α. Since the characteristic is p and $\alpha^p \in k$, we have that $\delta^p = 0$. Let r be minimal such that $\delta^r(z) = 0$ for all $z \in R$. Then $r > 1$, because α does not belong to the centre k of R. So there exists $x \in R$

such that $\delta^{r-1}(x) \neq 0$. Put $v = \delta^{r-2}(x)$ and $t = \delta(v)$. Note that $t \neq 0$, because $t = \delta^{r-1}(x)$. So t is invertible in U. Moreover, $\delta(t) = \delta^r(x) = 0$ and $\delta(t^{-1}) = 0$. Let

$$u = vt^{-1}.$$

Then

$$\delta(u) = v\delta(t^{-1}) + \delta(v)t^{-1} = tt^{-1} = 1,$$

that is,

$$u\alpha - \alpha u = 1.$$

□

Lemma 4.3.3. *Let k be a commutative field of characteristic $p > 0$ and R a k-algebra containing elements u, ξ such that u is invertible in some ring S which contains R, and these elements satisfy*

$$\xi^p = 0 \tag{4.3.1}$$

$$u\xi - \xi u = 1. \tag{4.3.2}$$

Then the following equation is valid in R:

$$u^{p-1}\xi^{p-1} = (\xi u)^{p-1} - 1. \tag{4.3.3}$$

Proof. Consider the element $y = \xi u$; then in S, $\xi = yu^{-1}$. Now, if we multiply $u\xi - \xi u = 1$ by u on the right, we obtain $uy - yu = u$, or $uy = (y+1)u$, which in S can be written as

$$yu^{-1} = u^{-1}(y+1). \tag{4.3.4}$$

Thus,

$$\begin{aligned}
\xi^{p-1} &= (yu^{-1})^{p-1} \\
&= yu^{-1} \dots yu^{-1}yu^{-1} \\
&= yu^{-1} \dots yu^{-2}(y+1), \text{ by } (4.3.4) \\
&= \cdots \\
&= u^{-(p-1)}(y+(p-1)) \dots (y+2)(y+1) \\
&= u^{-(p-1)}(y^{p-1}-1),
\end{aligned}$$

where the last equality was obtained from the fact that if $0 < m < p$, then $m^{p-1} \equiv 1 (mod\ p)$. Hence

$$u^{p-1}\xi^{p-1} = (\xi u)^{p-1} - 1.$$

□

Corollary 4.3.4. *With the hypothesis of Lemma 4.3.3, the elements*

$$f_{ij} = -\xi^{i-1}u^{p-1}\xi^{p-1}(u(\xi u)^{p-2})^{j-1} \tag{4.3.5}$$

form a complete set of matrix units for R.

Proof. We can rewrite (4.3.3) as

$$-u^{p-1}\xi^{p-1} + \xi(u(\xi u)^{p-2}) = 1.$$

Applying Theorem 4.3.1 to R with $n = p$, $f = \xi$, $a = -u^{p-1}$ and $b = u(\xi u)^{p-2}$ yields that the elements

$$f_{ij} = -\xi^{i-1}u^{p-1}\xi^{p-1}(u(\xi u)^{p-2})^{j-1}$$

form a complete set of $p \times p$ matrix units for R. □

Theorem 4.3.5. *Let R be a ring having a field k of characteristic $p > 0$ as its centre. Suppose that R is embeddable in a skew field U and that there exists $\alpha \in R \setminus k$ such that $\alpha^p \in k$. Let E be an extension of k isomorphic to $k(\alpha)$. Then there exists a nonzero element $t \in R$ such that the subring S of U generated by R and t^{-1} is such that S_E is isomorphic to a full $p \times p$ matrix ring.*

Proof. By Lemma 4.3.2, there exist elements $v, t \in R$ such that if $u = vt^{-1} \in S$, then

$$u\alpha - \alpha u = 1.$$

Let $E = k(\beta)$, where α is sent to β in the isomorphism between E and $k(\alpha)$. In S_E, let $\xi = \alpha - \beta$; so $\xi^p = (\alpha - \beta)^p = \alpha^p - \beta^p = 0$. Since β is in the centre of S_E, it follows that $u\beta = \beta u$ and this implies that

$$u\xi - \xi u = 1.$$

Now, by Corollary 4.3.4, S_E contains a complete set of $p \times p$ matrix units and, hence, is isomorphic to a full matrix ring. □

In the above theorem, we know that the matrix units of S_E are given by (4.3.5). Some simplifications can be made on the expression (4.3.5). For this, some technical lemmas are needed. In the three lemmas below and in the corollary following it, the elements u and ξ are those of the proof of Theorem 4.3.5.

Lemma 4.3.6. *For any $l, n \in \mathbb{N}$, $u(\xi u + l)^n = (\xi u + (l+1))^n u$.*

Proof. By induction on n. For $n = 0$ the result is trivially true and for $n = 1$ it is an immediate consequence of (4.3.2). Suppose now it is valid for $n \geq 1$ and let us prove it for $n + 1$.

$$\begin{aligned} u(\xi u + l)^{n+1} &= u(\xi u + l)^n(\xi u + l) \\ &= (\xi u + (l+1))^n u(\xi u + l), \text{ by induction hypothesis} \\ &= (\xi u + (l+1))^n(\xi u + (l+1))u, \text{ by case } n = 1 \\ &= (\xi u + (l+1))^{n+1}u. \end{aligned}$$

□

Lemma 4.3.7. *For any $m, n \in \mathbb{N}$, $u^m(\xi u)^n = (\xi u + m)^n u^m$.*

Proof. By induction on m. The induction basis $m = 0$ is trivial. Now, for $m \geq 1$, we have

$$\begin{aligned} u^{m+1}(\xi u)^n &= uu^m(\xi u)^n \\ &= u(\xi u + m)^n u^m, \text{ by induction hypothesis} \\ &= (\xi u + (m+1))^n u^{m+1}, \text{ by Lemma 4.3.6.} \end{aligned}$$

□

Lemma 4.3.8. *For any $l, n \in \mathbb{N}$, $\xi^{p-1}(\xi u + l)^n = l^n \xi^{p-1}$.*

Proof. By induction on n. For $n = 0$, the result is obviously valid. If we suppose it is valid for some $n \geq 1$, then

$$\begin{aligned} \xi^{p-1}(\xi u + l)^{n+1} &= \xi^{p-1}(\xi u + l)^n(\xi u + l) \\ &= l^n \xi^{p-1}(\xi u + l), \text{ by induction hypothesis} \\ &= l^{n+1}\xi^{p-1}, \text{ for } \xi^p = 0. \end{aligned}$$

□

Corollary 4.3.9. *For any $m, n \in \mathbb{N}$, $\xi^{p-1}(u(\xi u)^n)^m = (m!)^n \xi^{p-1} u^m$.*

Proof. By induction on m. For $m = 0$, the result is true. If $m \geq 1$,

$$\begin{aligned}
\xi^{p-1}(u(\xi u)^n)^{m+1} &= \xi^{p-1}(u(\xi u)^n)^m u(\xi u)^n \\
&= (m!)^n \xi^{p-1} u^{m+1} (\xi u)^n \text{ , by induction hypothesis} \\
&= (m!)^n \xi^{p-1} (\xi u + (m+1))^n u^{m+1}, \text{ by Lemma 4.3.7} \\
&= (m!)^n (m+1)^n \xi^{p-1} u^{m+1}, \text{ by Lemma 4.3.8} \\
&= ((m+1)!)^n \xi^{p-1} u^{m+1}.
\end{aligned}$$

□

Now we can state and prove

Theorem 4.3.10. *Let R be a ring having a field k of characteristic $p > 0$ as its centre. Suppose that R is embeddable in a skew field U and that there exists $\alpha \in R \setminus k$ such that $\alpha^p = a \in k$. Let $E = k(\beta)$ be an extension of k isomorphic to $k(\alpha)$ such that $\beta^p = a$. Then there exist $t, v \in R$ such that the elements*

$$e_{ij} = -u^{p-i} \xi^{p-1} u^{j-1}, \tag{4.3.6}$$

where $u = vt^{-1}$ and $\xi = \alpha - \beta$, form a complete set of p^2 matrix units for the E-algebra $S_E = S \otimes_k E$, where S is the subring of U generated by R and t^{-1}.

Proof. We have seen in Theorem 4.3.5 that the elements

$$f_{ij} = -\xi^{i-1} u^{p-1} \xi^{p-1} (u(\xi u)^{p-2})^{j-1}$$

form such a set. By applying Corollary 4.3.9 to each of the f_{ij}'s we can simplify them to

$$f_{ij} = -((j-1)!)^{p-2} \xi^{i-1} u^{p-1} \xi^{p-1} u^{j-1}. \tag{4.3.7}$$

Next, note that (4.3.2) implies

$$u^n \xi = \xi u^n + n u^{n-1}, \tag{4.3.8}$$

for every $n \in \mathbb{N}$. By rewriting (4.3.8) as

$$\xi u^n = u^n \xi - n u^{n-1},$$

we can prove, by induction on l, that

$$\xi^l u^{p-1} \xi^{p-1} = l! u^{p-(l+1)} \xi^{p-1}.$$

Applying the above relation to (4.3.7) and noting that for $j \leq p$,

$$((j-1)!)^{p-2} = \frac{1}{(j-1)!},$$

we get

$$f_{ij} = -\frac{(i-1)!}{(j-1)!} u^{p-i} \xi^{p-1} u^{j-1} = \frac{(i-1)!}{(j-1)!} e_{ij}. \tag{4.3.9}$$

So $e_{ii} = f_{ii}$ and thus $\sum_{i=1}^{p} e_{ii} = 1$. Moreover,

$$\begin{aligned} e_{ij} e_{lm} &= \frac{(j-1)!}{(i-1)!} \frac{(m-1)!}{(l-1)!} f_{ij} f_{lm}, \text{ by (4.3.9)} \\ &= \frac{(j-1)!}{(i-1)!} \frac{(m-1)!}{(l-1)!} \delta_{jl} f_{im} \\ &= \delta_{jl} \frac{(m-1)!}{(i-1)!} f_{im} \\ &= \delta_{jl} e_{im}. \end{aligned}$$

Therefore the e_{ij}'s do form a complete set of matrix units for S_E. □

4.3.2 A special case

We will now apply the results of the preceding subsection to a tensor ring. We will start by fixing our notation.

Let k be a commutative field of prime characteristic $p > 0$ and $F = k(\alpha)$ an extension where $\alpha \in F \setminus k$ and $\alpha^p = a \in k$. Consider the ring $R = F_k\langle x \rangle$, which is a fir and therefore has a universal field of fractions U. Let $E = k(\beta)$ be an extension of k isomorphic over k to F such that the isomorphism sends α to β. We shall need to look at the ring $R_E = R \otimes_k E$.

It is known that U_E is a simple artinian ring, thus isomorphic to a matrix ring over a skew field. More precisely, $U_E \cong \mathfrak{M}_r(K)$, where K is a skew field over E

and r divides $[E:k]=p$. Since $R_E \subseteq U_E$ and R_E is not an integral domain (*e.g.* the element $\xi = \alpha - \beta \in R_E$ is such that $\xi^p = (\alpha-\beta)^p = \alpha^p - \beta^p = a - a = 0$ and ξ is non-zero), it follows that $r = p$.

We shall consider the ring S, obtained from R by localizing enough in order to obtain the matrix units of U_E in S_E, just as we did above.

Since x is a free element in R, it certainly satisfies $\delta^{p-1}(x) \neq 0$, where δ is the inner derivation of U defined by α. We can than construct t and u as in the preceding subsection and apply Theorem 4.3.10 to $R = F_k\langle x\rangle$.

First we point out that R_E is not a fir, because it is not even an integral domain. Writing

$$R \cong F *_k k\langle x\rangle,$$

we can see that the monoid of projectives of R_E is given by

$$\begin{aligned}\mathcal{P}(R_E) &\cong \mathcal{P}(F_E) \coprod_{\mathcal{P}(E)} \mathcal{P}(E\langle x\rangle)\\ &\cong \mathcal{P}(F_E) \coprod_{\mathbb{N}} \mathbb{N}\\ &\cong \mathcal{P}(F_E).\end{aligned}$$

If we write $F \cong k[Y]/(f(Y))$, where $f(Y) = (Y-\alpha)^p$, then we have an exact sequence

$$0 \longrightarrow (f(Y)) \longrightarrow k[Y] \longrightarrow F \longrightarrow 0$$

of k-spaces. Tensoring it up with E over k, we get an exact sequence

$$0 \longrightarrow (f(Y)) \longrightarrow E[Y] \longrightarrow F_E \longrightarrow 0.$$

So $F_E \cong E[Y]/(f(Y))$ and the only maximal ideal of F_E will be $(Y-\alpha)/(f(Y))$. So F_E is a local ring. By Cor. 0.5.5 of [6], F_E is projective free. Thus R_E is a projective free ring which is not a fir. In particular, R_E is not a hereditary ring.

4.3.3 Generators and relations

Using Theorem 4.3.10, we can apply Proposition 0.1.1 in [6] to S_E and conclude that S_E is isomorphic to the matrix ring $\mathfrak{M}_p(T)$, where T is the centralizer in S_E

of the matrix units e_{ij} given by (4.3.6). The proposition mentioned above also tells us that this isomorphism is given by the map

$$\begin{aligned} S_E &\xrightarrow{[\,]} \mathfrak{M}_p(T), \\ f &\longmapsto [f] \end{aligned}$$

where $[f] = (f_{ij})$ with $f_{ij} = \sum_{\nu=1}^{p} e_{\nu i} f e_{j\nu}$.

Our aim will be to prove that T is a fir. In order to do that it will be necessary to look at the images of the generators ξ, x, and t^{-1} of S_E in $\mathfrak{M}_p(T)$.

Let us start by evaluating $[\xi]$. First note that

$$e_{i1}\xi = -u^{p-i}\xi^{p-1}\xi = 0, \text{ for every } i = 1, \ldots, p.$$

For $j > 1$, we have

$$\begin{aligned} e_{ij}\xi &= -u^{p-i}\xi^{p-1}u^{j-1}\xi \\ &= -u^{p-i}\xi^{p-1}(\xi u^{j-1} + (j-1)u^{j-2}), \text{ by (4.3.8)} \\ &= -(j-1)u^{p-i}\xi^{p-1}u^{j-2} \\ &= (j-1)e_{i,j-1}. \end{aligned}$$

Therefore,

$$\begin{aligned} e_{\nu i}\xi e_{j\nu} &= \begin{cases} 0 & \text{if } i = 1 \\ (i-1)e_{\nu,i-1}e_{j\nu} & \text{if } i > 1 \end{cases} \\ &= \begin{cases} 0 & \text{if } i = 1 \\ (i-1)\delta_{i-1,j}e_{\nu\nu} & \text{if } i > 1. \end{cases} \end{aligned}$$

This implies that $\xi_{1j} = 0$ and $\xi_{ij} = \sum_{\nu=1}^{p} e_{\nu i}\xi e_{j\nu} = (i-1)\delta_{i-1,j}$ for $i > 1$. That is,

$$[\xi] = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & p-2 & 0 & 0 \\ 0 & 0 & \cdots & 0 & p-1 & 0 \end{bmatrix}.$$

It will be useful to know also what $[u]$ looks like.

$$\begin{aligned} e_{ij}u &= -u^{p-i}\xi^{p-1}u^{j-1}u \\ &= -u^{p-i}\xi^{p-1}u^{j} \\ &= \begin{cases} e_{i,j+1} & \text{if } j<p \\ u^p e_{i1} & \text{if } j=p, \text{ because } u^p\xi=\xi u^p. \end{cases} \end{aligned}$$

So, for $i<p$, $e_{\nu i}ue_{j\nu}=e_{\nu,i+1}e_{j\nu}=\delta_{i+1,j}e_{\nu\nu}$ and $e_{\nu p}ue_{j\nu}=u^pe_{\nu 1}e_{j\nu}=\delta_{1j}u^pe_{\nu\nu}$. Thus

$$u_{ij}=\begin{cases} \delta_{i+1,j} & \text{if } i<p \\ \delta_{1j}u^p & \text{if } i=p. \end{cases}$$

Note that, because $u^p\xi=\xi u^p$, u^p commutes with the matrix units e_{ij} of S_E. So $u^p\in T$. In matrix form, we can then write

$$[u]=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ u^p & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

In order to avoid confusion with the matrix units of S_E, the p^2 matrix units of $\mathfrak{M}_p(T)$ will be denoted by E_{ij}, *i.e.* E_{ij} is the $p\times p$ matrix which contains a 1 on the (i,j)-entry and zeros elsewhere. We can now rewrite $[\xi]$ and $[u]$ as

$$[\xi]=\sum_{i=1}^{p-1} iE_{i+1,i}\ ,\ [u]=\sum_{i=1}^{p-1}E_{i,i+1}+u^pE_{p1}.$$

To find out what the entries of $[x]$ are, we will look first at $[t]$. We will need a result due to Frobenius.

Proposition 4.3.11. *Let k be a commutative field and A an $n\times n$ matrix over k such that $A^n=0$, but $A^{n-1}\neq 0$. Then, if B is a matrix over k such that $AB=BA$, there exist $\lambda_0,\lambda_1,\ldots,\lambda_{n-1}$ in k such that $B=\sum_{i=0}^{n-1}\lambda_iA^i$.* $\square$

The proposition has the following consequence.

Corollary 4.3.12. *Let k be a commutative field, R a k-algebra and A an $n \times n$ matrix over k such that $A^n = 0$, but $A^{n-1} \neq 0$. Then, if B is a matrix over R such that $AB = BA$, there exist $r_0, r_1, \dots, r_{n-1}$ in R such that $B = \sum_{i=0}^{n-1} r_i A^i$.*

Proof. Write $B = \sum_{j=1}^{m} B_j u_j$, where the B_j's are matrices over k and the u_j's are linearly independent over k. Since $BA = AB$, we have

$$\sum_{j=1}^{m} B_j A u_j = \sum_{j=1}^{m} A B_j u_j,$$

which implies that $B_j A = A B_j$ for all $j = 1, \dots, m$. Now apply Proposition 4.3.11 to A and B_j to get, for each $j = 1, \dots, m$, elements $\lambda_{ji} \in k$, $i = 0, \dots, n-1$ such that

$$B_j = \sum_{i=0}^{n-1} \lambda_{ji} A^i.$$

Thus,

$$B = \sum_{j=1}^{m} B_j u_j = \sum_{j=1}^{m} \sum_{i=0}^{n-1} \lambda_{ji} A^i u_j = \sum_{i=0}^{n-1} r_i A^i,$$

where $r_i = \sum_{j=0}^{m} \lambda_{ji} u_j \in R$. $\square$

Since $[\xi]^p = 0, [\xi]^{p-1} \neq 0$ and $[t][\xi] = [\xi][t]$ and the entries of $[\xi]$ are in the ground field, we can apply Corollary 4.3.12 to $[t]$ and obtain

$$[t] = r_0 I + r_1 [\xi] + \dots + r_{p-1} [\xi]^{p-1},$$

for some r_i in S_E.

Using the fact that $[\xi]^s = \sum_{i=1}^{p-s} \binom{i-1+s}{i-1} s! E_{i+s,i}$, for $s = 1, \dots, p-1$, we can write

$$[t] = \sum_{i=1}^{p} \sum_{j=1}^{i} \binom{i-1}{j-1} b_{i-j} E_{ij}, \tag{4.3.10}$$

where $b_s = s! r_s$. Which, as a matrix, is just

$$[t] = \begin{bmatrix} \binom{0}{0} b_0 & 0 & \cdots & 0 \\ \binom{1}{0} b_1 & \binom{1}{1} b_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \binom{p-1}{0} b_{p-1} & \binom{p-1}{1} b_{p-2} & \cdots & \binom{p-1}{p-1} b_0 \end{bmatrix}.$$

Note that, since t is invertible in S_E and $[t]$ is a triangular matrix with b_0 on the main diagonal, b_0 must be invertible in T. We can therefore write the matrix $[t^{-1}] = [t]^{-1}$ in terms of the b_i's and b_0^{-1}.

We will now proceed to the evaluation of $[x] = (x_{ij})$. The element u was defined to be $u = \delta^{p-2}(x)t^{-1}$. If we define $\Delta : \mathfrak{M}_p(T) \longrightarrow \mathfrak{M}_p(T)$ to be the derivation given by $\Delta(M) = M[\xi] - [\xi]M$, we can obtain some information on the entries of $[x]$ by looking at the image under $[\ ]$ of the equation above in the following way: by definition of u we have

$$[u][t] = \Delta^{p-2}([x]). \tag{4.3.11}$$

The left-hand side of (4.3.11) is easily seen to be

$$[u][t] = \sum_{i=1}^{p-1}\sum_{j=1}^{i+1}\binom{i}{j-1} b_{i-j+1}E_{ij} + b_0 u^p E_{p1}.$$

Bearing in mind that $\Delta^n(M) = \sum_{i=0}^{n}(-1)^i\binom{n}{i}[\xi]^i M[\xi]^{n-i}$ for every $n \geq 0$ and $M \in \mathfrak{M}_p(T)$, we can write the right-hand side of (4.3.11) as

$$\begin{aligned}\Delta^{p-2}([x]) \ = \ & \sum_{i=0}^{p-2}\sum_{j=1}^{p-i}\sum_{m=1}^{i+2}(-1)^i\binom{p-2}{i}\binom{j-1+i}{j-1}\binom{m-1+p-2-i}{m-1} \\ & i!(p-2-i)!\ x_{j,m+p-2-i}E_{i+j,m}.\end{aligned}$$

Note that $\binom{p-2}{i}i!(p-2-i)! = (p-2)!$. And, since $(p-1)! = -1$, it follows that $(p-2)! = 1$. Thus

$$\begin{aligned}\Delta^{p-2}([x]) = & \sum_{i=0}^{p-2}\sum_{j=1}^{p-i}\sum_{m=1}^{i+2}(-1)^i\binom{j-1+i}{j-1} \\ & \binom{m-1+p-2-i}{m-1} x_{j,m+p-2-i}E_{i+j,m}.\end{aligned} \tag{4.3.12}$$

We will now rewrite (4.3.12) in order to make it clearer what the entries of $\Delta^{p-2}([x])$ are. First let h and l be new parameters such that $i + j = h$ and $i + 1 = l$. So (4.3.12) is equivalent to

$$\begin{aligned}\Delta^{p-2}([x]) = & \sum_{l=1}^{p-1}\sum_{h=l}^{p}\sum_{m=1}^{l+1}(-1)^{l-1}\binom{h-1}{h-l} \\ & \binom{m-1+p-l-1}{m-1} x_{h-l+1,m+p-l-1}E_{hm}.\end{aligned} \tag{4.3.13}$$

Let us introduce some notation. Set

$$\tau_{lhm} = (-1)^{l-1}\binom{h-1}{h-l}\binom{m-1+p-l-1}{m-1} x_{h-l+1,m+p-l-1}E_{hm}.$$

We want to interchange the summations over l and h in (4.3.13). A quick analysis shows us that in order to do so, it is necessary to break the expression into two sums: $\sum_{h=1}^{p-1}\sum_{l=1}^{h}\sum_{m=1}^{l+1}\tau_{lhm} + \sum_{l=1}^{p-1}\sum_{m=1}^{l+1}\tau_{lpm}$. That is,

$$\begin{aligned}
&\Delta^{p-2}([x]) = \\
&\sum_{h=1}^{p-1}\sum_{l=1}^{h}\sum_{m=1}^{l+1}(-1)^{l-1}\binom{h-1}{h-l}\binom{m-1+p-l-1}{m-1} x_{h-l+1,m+p-l-1}E_{hm} \\
&+\sum_{l=1}^{p-1}\sum_{m=1}^{l+1}(-1)^{l-1}\binom{p-1}{p-l}\binom{m-1+p-l-1}{m-1} x_{p-l+1,m+p-l-1}E_{pm}.
\end{aligned}$$

We will now interchange the summations over m and l. Again, this will break each term above in two sums:

$$\sum_{h=1}^{p-1}\left(\sum_{m=2}^{h+1}\sum_{l=m-1}^{h}\tau_{lhm} + \sum_{l=1}^{h}\tau_{lh1}\right) + \left(\sum_{m=2}^{p}\sum_{l=m-1}^{p-1}\tau_{lpm} + \sum_{l=1}^{p-1}\tau_{lp1}\right).$$

More explicitly,

$$\begin{aligned}
&\Delta^{p-2}([x]) = \\
&\sum_{h=1}^{p-1}\sum_{m=2}^{h+1}\sum_{l=m-1}^{h}(-1)^{l-1}\binom{h-1}{h-l}\binom{m-1+p-l-1}{m-1} x_{h-l+1,m+p-l-1}E_{hm} \\
&+\sum_{h=1}^{p-1}\sum_{l=1}^{h}(-1)^{l-1}\binom{h-1}{h-l} x_{h-l+1,p-l}E_{h1} \\
&+\sum_{m=2}^{p}\sum_{l=m-1}^{p-1}(-1)^{l-1}\binom{p-1}{p-l}\binom{m-1+p-l-1}{m-1} x_{p-l+1,m+p-l-1}E_{pm} \\
&+\sum_{l=1}^{p-1}(-1)^{l-1}\binom{p-1}{p-l} x_{p-l+1,p-l}E_{p1}.
\end{aligned}$$

Remember that $\binom{p-1}{p-l} = (-1)^{p-l}$. Therefore, the last two terms are equal to

$$\sum_{m=2}^{p}\sum_{l=m-1}^{p-1}\binom{m-1+p-l-1}{m-1} x_{p-l+1,m+p-l-1}E_{pm}$$

and

$$\sum_{l=1}^{p-1} x_{p-l+1,p-l}E_{p1},$$

respectively. Also, if in the first and second terms we replace l by $h-j+1$, in the third term we replace l by $p-j-1$ and in the forth term we replace l by $p-j+1$, we obtain

$$\begin{aligned}
&\Delta^{p-2}([x]) = \\
&\quad \sum_{h=1}^{p-1}\sum_{m=2}^{h+1}\sum_{j=1}^{h-m+2}(-1)^{h-j}\binom{h-1}{j-1}\binom{m-1+p-h+j-2}{m-1}x_{j,m+p-h+j-2}E_{hm} \\
&\quad + \sum_{h=1}^{p-1}\sum_{j=1}^{h}(-1)^{h-j}\binom{h-1}{j-1}x_{j,p-h+j-1}E_{h1} \\
&\quad + \sum_{m=2}^{p}\sum_{j=0}^{p-m}\binom{m-1+j}{m-1}x_{j+2,j+m}E_{pm} \\
&\quad + \sum_{j=2}^{p}x_{j,j-1}E_{p1}.
\end{aligned}$$

One last adjustment will be necessary. For this we will need the following

Lemma 4.3.13. *In characteristic $p>0$, for any N and M non-negative integers, not both zero, such that $N+M\le p$,*

(i) $\displaystyle\binom{N-1+M}{N-1} = (-1)^{N-1}\binom{p-1-M}{N-1}$;

(ii) $\displaystyle\binom{N-1+M}{N-1} = (-1)^{M}\binom{p-N}{M}$.

Proof. Case (i) is as follows:

$$\begin{aligned}
\binom{p-1-M}{N-1} &= \frac{(p-1-M)!}{(N-1)!(p-M-N)!} \\
&= \frac{(p-(M+1))\ldots(p-(M+N-1))(p-(M+N))!}{(N-1)!(p-M-N)!} \\
&= (-1)^{N-1}\frac{(M+1)\ldots(M+N-1)}{(N-1!)} \\
&= (-1)^{N-1}\frac{(N-1+M)!}{(N-1)!M!} \\
&= (-1)^{N-1}\binom{N-1+M}{N-1}.
\end{aligned}$$

Case (ii) is similar:

$$\begin{aligned}\binom{p-N}{M} &= \frac{(p-N)!}{M!(p-N-M)!}\\ &= \frac{(p-N)(p-(N+1))\dots(p-(N+M-1))(p-(N+M))!}{M!(p-N-M)!}\\ &= (-1)^M\frac{N(N+1)\dots(N+M-1)}{M!}\\ &= (-1)^M\frac{(N-1+M)!}{M!(N-1)!}\\ &= (-1)^M\binom{N-1+M}{N-1}.\end{aligned}$$

□

Using Lemma 4.3.13 (i) with $N = m$ and $M = p-h+j-2$ we get $\binom{m-1+p-h+j-2}{m-1} = (-1)^{m-1}\binom{h-j+1}{m-1}$ and putting $N = m, M = j$ in (ii) yields $\binom{m-1+j}{m-1} = (-1)^j\binom{p-m}{j}$. So the first and third term are modified and we can finally write

$$\begin{aligned}\Delta^{p-2}([x]) =&\\ &\sum_{h=1}^{p-1}\sum_{m=2}^{h+1}\sum_{j=1}^{h-m+2}(-1)^{h-j+m-1}\binom{h-1}{j-1}\binom{h-j+1}{m-1}x_{j,m+p-h+j-2}E_{hm}\\ &+\sum_{h=1}^{p-1}\sum_{j=1}^{h}(-1)^{h-j}\binom{h-1}{j-1}x_{j,p-h+j-1}E_{h1}\\ &+\sum_{m=2}^{p}\sum_{j=0}^{p-m}(-1)^j\binom{p-m}{j}x_{j+2,j+m}E_{pm}\\ &+\sum_{j=2}^{p}x_{j,j-1}E_{p1}.\end{aligned} \tag{4.3.14}$$

We know that

$$[u][t] = \sum_{i=1}^{p-1}\sum_{j=1}^{i+1}\binom{i}{j-1}b_{i-j+1}E_{ij} + b_0u^pE_{p1}. \tag{4.3.15}$$

We will use the equation (4.3.11) and the expressions for each of the sides of (4.3.11) given in (4.3.14) and (4.3.15) to determine relations among the x_{ij}'s.

Comparing the entries $(p,1)$ of both matrices, we get

$$b_0u^p = \sum_{j=2}^{p}x_{j,j-1}. \tag{4.3.16}$$

The entries (p, m) with $2 \leq m \leq p$ give us $\sum_{j=0}^{p-m}(-1)^j \binom{p-m}{j} x_{j+2,j+m} = 0$, or

$$\begin{aligned} x_{2p} &= 0 \\ x_{p-m+2,p} &= \sum_{j=0}^{p-m-1} (-1)^{j+p-m+1} \binom{p-m}{j} x_{j+2,j+m}, \text{ for } 2 \leq m \leq p-1. \end{aligned}$$

Replacing $p-m+2$ by h in the second equation above leaves us with the following relations.

$$x_{2p} = 0 \tag{4.3.17}$$

$$x_{hp} = \sum_{j=0}^{h-3} (-1)^{h+j-1} \binom{h-2}{j} x_{j+2,p-h+j+2}, \text{ for } 3 \leq h \leq p. \tag{4.3.18}$$

If we now look at the entries $(h, 1)$ with $1 \leq h \leq p-1$, we get

$$b_h = \sum_{j=1}^{h} (-1)^{h-j} \binom{h-1}{j-1} x_{j,p-h+j-1}. \tag{4.3.19}$$

And the entries $(h, h+1)$ for $1 \leq h \leq p-1$ give

$$b_0 = -x_{1p}. \tag{4.3.20}$$

What should be remarked at this stage is that (4.3.16), (4.3.17), (4.3.18), (4.3.19) and (4.3.20) are all the relations derived from the equation (4.3.11). In fact, using (4.3.17), (4.3.18) and (4.3.19) we can *prove* that all the other entries of the matrices $[u][t]$ and $\Delta^{p-2}([x])$ coincide. This fact will be used later on in this section. It is clear that the entries (h, m) with $2 \leq h \leq p-2$ and $h+2 \leq m \leq p$ are zero both for $[u][t]$ and for $\Delta^{p-2}([x])$. We are left to check that the entries (h, m) with $2 \leq h \leq p-1$ and $2 \leq m \leq h$ are equal, *i.e.* that

$$\binom{h}{m-1} b_{h-m+1} = \sum_{j=1}^{h-m+2} (-1)^{h-j+m-1} \binom{h-1}{j-1} \binom{h-j+1}{m-1} x_{j,m+p-h+j-2}$$

for each $2 \leq h \leq p-1$ and $2 \leq m \leq h$.

We will now proceed to the proof of the above assertion. Denote by ε_{hm} the (h, m)-entry of $\Delta^{p-2}([x])$, with $2 \leq h \leq p-1$ and $2 \leq m \leq h$, which is given in (4.3.14), *i.e.*

$$\varepsilon_{hm} = \sum_{j=1}^{h-m+2} (-1)^{h-j+m-1} \binom{h-1}{j-1} \binom{h-j+1}{m-1} x_{j,m+p-h+j-2}.$$

First, let us verify the assertion for entries of the kind (h,h), with $2 \leq h \leq p-1$. On the one hand, the (h,h)-entry of $\Delta^{p-2}([x])$ is equal to $hx_{1,p-1}$. This can be easily verified by evaluating ε_{hh} and then applying (4.3.17). A direct calculation of the (h,h)-entry of $[u][t]$ gives us the element hb_1. Therefore, we have

$$hx_{1,p-1} = hb_1,$$

for all $h = 2, \ldots, p-1$. But this is just a special case of (4.3.19).

Now let us look at the entries (h,m) with $3 \leq h \leq p-1$ and $2 \leq m \leq h-1$. On the one hand, using (4.3.19), we have

$$\begin{aligned}
\binom{h}{m-1} b_{h-m+1} = & \\
&= \sum_{j=1}^{h-m+1} (-1)^{h-m-j+1} \binom{h}{m-1} \binom{h-m}{j-1} x_{j,m+p-h+j-2} \\
&= (-1)^{h-m} \binom{h}{m-1} x_{1,m+p-h-1} \\
&\quad + \sum_{j=2}^{h-m+1} (-1)^{h+m-j+1} \binom{h}{m-1} \binom{h-m}{j-1} x_{j,m+p-h+j-2}.
\end{aligned} \tag{4.3.21}$$

On the other hand,

$$\begin{aligned}
\varepsilon_{hm} \quad = \quad & \sum_{j=1}^{h-m+1} (-1)^{h-j+m-1} \binom{h-1}{j-1} \binom{h-j+1}{m-1} x_{j,m+p-h+j-2} \\
& + -1 \binom{h-1}{h-m+1} x_{h-m+2,p}.
\end{aligned}$$

Now we apply formula (4.3.18) to the second term of the right-hand side of the equation above. This can be done, because, since $m \leq h-1$, $h-m+2 \geq 3$.

$$\begin{aligned}
\varepsilon_{hm} \quad = \quad & \sum_{j=1}^{h-m+1} (-1)^{h-j+m-1} \binom{h-1}{j-1} \binom{h-j+1}{m-1} x_{j,m+p-h+j-2} \\
& + \sum_{j=0}^{h-m-1} (-1)^{h-m+j} \binom{h-1}{h-m+1} \binom{h-m}{j} x_{j+2,m+p-h+j}.
\end{aligned}$$

If we replace j by $l-2$ in the second term of the right-hand side, we get

$$
\begin{aligned}
\varepsilon_{hm} = & \sum_{j=1}^{h-m+1} (-1)^{h-j+m-1} \binom{h-1}{j-1} \binom{h-j+1}{m-1} x_{j,m+p-h+j-2} \\
& + \sum_{l=2}^{h-m+1} (-1)^{h-m+l} \binom{h-1}{h-m+1} \binom{h-m}{l-2} x_{l,m+p-h+l-2} \\
= & (-1)^{h+m} \binom{h}{m-1} x_{1,m+p-h-1} \\
& + \sum_{j=2}^{h-m+1} (-1)^{h-j+m-1} \left[\binom{h-1}{j-1} \binom{h-j+1}{m-1} \right. \\
& \left. - \binom{h-1}{h-m+1} \binom{h-m}{j-2} \right] x_{j,m+p-h+j-2}.
\end{aligned}
\tag{4.3.22}
$$

We want to compare the extended expression for $\binom{h}{m-1} b_{h-m+1}$ in (4.3.21) with the one for ε_{hm} in (4.3.22). First note that the coefficients of $x_{1,m+p-h-1}$ are equal. It is then sufficient to prove that the other correspondent coefficients are also the same. Indeed,

$$
\begin{aligned}
& \binom{h-1}{j-1} \binom{h-j+1}{m-1} - \binom{h-1}{h-m+1} \binom{h-m}{j-2} = \\
= & \frac{(h-1)!}{(j-1)!(h-j)!} \frac{(h-j+1)!}{(m-1)!(h-j-m+2)!} \\
& - \frac{(h-1)!}{(h-m+1)!(m-2)!} \frac{(h-m)!}{(j-2)!(h-m-j+2)!} \\
= & \frac{(h-1)!}{(j-2)!(m-2)!(h-j-m+2)!} \left[\frac{h-j+1}{(j-1)(m-1)} - \frac{1}{h-m+1} \right] \\
= & \frac{(h-1)!}{(j-2)!(m-2)!(h-j-m+2)!} \frac{(h-j+1)(h-m+1)-(j-1)(m-1)}{(j-1)(m-1)(h-m+1)} \\
= & \frac{(h-1)!}{(j-2)!(m-2)!(h-j-m+2)!} \frac{h(h-m-j+2)}{(j-1)(m-1)(h-m+1)} \\
= & \frac{h!(h-m)!}{(j-1)!(m-1)!(h-j-m+1)!(h-m+1)!} \\
= & \binom{h}{m-1} \binom{h-m}{j-1}.
\end{aligned}
$$

This completes the proof of the claim that the relations (4.3.16), (4.3.17), (4.3.18), (4.3.19) and (4.3.20) imply all the other relations among elements obtained from the matrix identity (4.3.11).

The identities (4.3.16) and (4.3.20) allows us to write the element u^p in terms of the x_{ij}'s:

$$u^p = -\left(\sum_{j=2}^{p} x_{j,j-1}\right) x_{1p}^{-1}, \tag{4.3.23}$$

because, since b_0 is invertible in T, so is x_{1p}.

What we have found so far is that all of the entries of $[u]$, $[t]$ and $[t^{-1}]$ can be written in terms of the x_{ij}'s and x_{1p}^{-1}. We have also found that there are some relations among the x_{ij}'s which are given by (4.3.17) and (4.3.18). The next theorem and its corollaries claim that (4.3.17) and (4.3.18) together with the fact that x_{1p} is invertible are the only relations among the generators of T.

Theorem 4.3.14. *Let k be a commutative field of prime characteristic $p > 0$ and $F = k(\alpha)$, where $\alpha^p = a \in k$ and $\alpha \notin k$. Let $R = F_k\langle x\rangle$ and denote by U its universal field of fractions. Let E be an extension of k isomorphic to F/k. Then there exists a subring S of U, obtained from R by adjoining the inverse of a single element of R, such that S_E is isomorphic as an E-algebra to the matrix ring $\mathfrak{M}_p(A)$, where A is the E-algebra defined as*

$$A = E\langle y_{ij(i=1,\dots,p,\ j=1,\dots,p-1)}, y_{1p}, y_{1p}^{-1} \mid y_{1p}y_{1p}^{-1} = y_{1p}^{-1}y_{1p} = 1\rangle.$$

Proof. Throughout this proof, indexes i will range between 1 and p and j between 1 and $p-1$. In the proof we shall again use the notation E_{mn}, with $m, n = 1, \dots, p$, to denote the matrix units of $\mathfrak{M}_p(A)$.

We recall that our original ring R was just the free F_k-ring on x, *i.e.* $R = F_k\langle x\rangle$ and that $F = k(\alpha)$, where $\alpha^p = a \in k$. If in R we consider the inner derivation determined by α,

$$\begin{array}{rcccc} \delta_\alpha & : & R & \longrightarrow & R \\ & & f & \longmapsto & f\alpha - \alpha f, \end{array}$$

S can be defined to be the ring obtained from R by adjoining the inverse of $\delta_\alpha^{p-1}(x)$, *i.e.* $S \cong F_k\langle x, t^{-1} \mid \delta_\alpha^{p-1}(x)t^{-1} = t^{-1}\delta_\alpha^{p-1}(x) = 1\rangle$. Using the fact that $\alpha^p = a$, we can also write $S \cong k\langle \alpha, x, t^{-1} \mid \alpha^p = a, \delta_\alpha^{p-1}(x)t^{-1} = t^{-1}\delta_\alpha^{p-1}(x) = 1\rangle$. When we tensor up with $E = k(\beta)$, where $\beta^p = a$, we get $S_E \cong E\langle \alpha, x, t^{-1} \mid \alpha^p = a, \delta_\alpha^{p-1}(x)t^{-1} = t^{-1}\delta_\alpha^{p-1}(x) = 1\rangle$, which becomes, after setting $\xi = \alpha - \beta$,

$$S_E \cong E\langle \xi, x, t^{-1} \mid \xi^p = 0, \delta^{p-1}(x)t^{-1} = t^{-1}\delta^{p-1}(x) = 1\rangle,$$

where $\delta : S_E \longrightarrow S_E$ is given by $\delta(f) = f\xi - \xi f$, which is the same as δ_α, because β centralizes every element of E.

Consider the E-algebra homomorphism $\varphi : E\langle \xi, x, t^{-1}\rangle \longrightarrow \mathfrak{M}_p(A)$ given by

$$\begin{aligned}\varphi(\xi) &= \sum_{j=1}^{p-1} jE_{j+1,j}\\ \varphi(x) &= \sum_{ij} y_{ij}E_{ij} + y_{1p}E_{1p} + \sum_{n=3}^{p} z_n E_{np}\\ \varphi(t^{-1}) &= Z^{-1},\end{aligned}$$

where

$$z_n = \sum_{m=0}^{n-3} (-1)^{m+n-1}\binom{n-2}{m} y_{m+2,m+p-n+2}$$

and Z^{-1} is the inverse of the matrix

$$Z = \sum_{i=1}^{p}\sum_{n=1}^{i}\binom{i-1}{n-1} c_{i-n}E_{in},$$

where $c_0 = -y_{1p}$, $c_n = \sum_{m=1}^{n}(-1)^{n-m}\binom{n-1}{m-1} y_{m,p+m-n-1}$ for $n = 1 \ldots p-1$. Note that Z is indeed invertible in $\mathfrak{M}_p(A)$, because it is a triangular matrix whose main diagonal entries are all equal to $-y_{1p}$, which is a unit in A.

For the sake of simplicity, we will put

$$\Xi = \sum_{j=1}^{p-1} jE_{j+1,j}, \quad Y = \sum_{ij} y_{ij}E_{ij} + y_{1p}E_{1p} + \sum_{n=3}^{p} z_n E_{np}.$$

Since Ξ is a lower triangular matrix, it follows that $\Xi^p = 0$.

Let $\Delta : \mathfrak{M}_p(A) \longrightarrow \mathfrak{M}_p(A)$ be the inner derivation determined by Ξ: $\Delta(M) = M\Xi - \Xi M$, for each $M \in \mathfrak{M}_p(A)$. We have $\varphi\delta(f) = \varphi(f\xi - \xi f) = \varphi(f)\varphi(\xi) - \varphi(\xi)\varphi(f) = \Delta\varphi(f)$, thus

$$\Delta\varphi = \varphi\delta. \tag{4.3.24}$$

Call U the matrix given by

$$U = \sum_{j=1}^{p-1} E_{j,j+1} - \left(\sum_{n=2}^{p} y_{n,n-1}\right) y_{1p}^{-1}E_{p1}.$$

Our aim is to show that all corresponding entries of $\Delta^{p-2}(Y)$ and UZ coincide. It should be noted that the matrices Ξ, Y, Z and U were defined in such a way that some corresponding entries of $\Delta^{p-2}(Y)$ and UZ coincide. These entries are exactly what the relations (4.3.17)–(4.3.20) and (4.3.23) suggested they should be in order to obtain the identity

$$\Delta^{p-2}(Y) = UZ. \tag{4.3.25}$$

In fact, a proof of (4.3.25) is obtained by repeating all the steps of the (long) remark that follows equation (4.3.20), making the appropriate modifications, for example, the entries of $[x] = (x_{ij})$ should be replaced by the entries of Y, etc.

With (4.3.25) in hand and the facts

$$\Delta(U) = U\Xi - \Xi U = 1, \ \Delta(Z) = Z\Xi - \Xi Z = 0,$$

which follow easily from the definition of the matrices above, we can conclude that $\Delta^{p-1}(Y) = \Delta(UZ) = \Delta(U)Z + U\Delta(Z) = Z$. In other words,

$$\varphi(t) = \varphi(\delta^{p-1}(x)) = \Delta^{p-1}\varphi(x) = \Delta^{p-1}(Y) = Z. \tag{4.3.26}$$

The information $\varphi(\xi)^p = 0$ and $\varphi(t) = Z$ is enough to ensure that φ induces a homomorphism $\bar{\varphi} : S_E \longrightarrow \mathfrak{M}_p(A)$ defined on generators by $\bar{\varphi}(\xi) = \Xi, \bar{\varphi}(x) = Y, \bar{\varphi}(t^{-1}) = Z^{-1}$.

A homomorphism $\bar{\psi} : \mathfrak{M}_p(A) \longrightarrow S_E$ in the other direction is given by $\bar{\psi}(E_{mn}) = e_{mn}, \bar{\psi}(y_{1p}) = x_{1p}, \bar{\psi}(y_{ij}) = x_{ij}, \bar{\psi}(y_{ip}^{-1}) = -\sum_{\nu=1}^{p} e_{\nu 1}t^{-1}e_{1\nu}$. We should recall that these elements of S_E were defined above as $e_{mn} = -u^{p-m}\xi^{p-1}u^{n-1}$, where $u = \delta^{p-2}(x)t^{-1}; x_{1p} = \sum_{\nu=1}^{p} e_{\nu 1}xe_{p\nu}$ and $x_{ij} = \sum_{\nu=1}^{p} e_{\nu i}xe_{j\nu}$. Note that $\bar{\psi}$ is well defined, because we know that the e_{mn} form a set of matrix units for S_E (see Theorem 4.3.10) and because the (1,1) entry of the matrix $[t^{-1}]$, *i.e.* the element $b_0^{-1} = \sum_{\nu=1}^{p} e_{\nu 1}t^{-1}e_{1\nu}$, is $-x_{1p}^{-1}$ (see (4.3.20)). So

$$\bar{\psi}(y_{1p}y_{1p}^{-1}) = \bar{\psi}(y_{1p})\bar{\psi}(y_{1p}^{-1}) = x_{1p}(-b_0^{-1}) = x_{1p}(-(-x_{1p}^{-1})) = 1. \tag{4.3.27}$$

(This can also be verified directly by evaluating

$$(\sum_{\nu=1}^{p} e_{\nu 1}xe_{p\nu})(-\sum_{\nu=1}^{p} e_{\nu 1}t^{-1}e_{1\nu})$$

using the fact that since $\delta^{p-1}(x) = t$, we have $\xi^{p-1}x\xi^{p-1} = \xi^{p-1}t$, that is, $\xi^{p-1}x\xi^{p-1}t^{-1} = \xi^{p-1}$.)

We will show that $\bar{\psi} = \bar{\varphi}^{-1}$. Let us start by calculating $\bar{\psi}\bar{\varphi}$. By the definition of the homomorphisms,

$$\begin{aligned}
\bar{\psi}\bar{\varphi}(\xi) &= \bar{\psi}\left(\sum_{j=1}^{p-1} jE_{j+1,j}\right) \\
&= \sum_{j=1}^{p-1} je_{j+1,j}.
\end{aligned}$$

Now recall (4.3.8) which stated that

$$u^l\xi = \xi u^l + lu^{l-1},$$

for every $l \in \mathbb{N}$. Multiplying both sides by ξ^{p-1} on the left yields

$$\xi^{p-1}u^l\xi = l\xi^{p-1}u^{l-1},$$

which implies that for $l = 1, \ldots, p-1$,

$$e_{j,l+1}\xi = le_{jl}.$$

So, $je_{j+1,j} = e_{j+1,j+1}\xi$ and then,

$$\begin{aligned}
\bar{\psi}\bar{\varphi}(\xi) &= \sum_{j=1}^{p-1} je_{j+1,j} \\
&= \left(\sum_{j=1}^{p-1} e_{j+1,j+1}\right)\xi \\
&= (1 - e_{11})\xi \\
&= \xi - e_{11}\xi \\
&= \xi, \text{ because } e_{11}\xi = 0.
\end{aligned}$$

Let us look at $\bar{\psi}\bar{\varphi}(x)$. We know that $x = \sum_{h,m=1}^{p} x_{hm}e_{hm}$, where $x = \sum_{\nu=1}^{p} e_{\nu h}xe_{m\nu}$. First, apply $\bar{\varphi}$ to the defining equation of u, $u = \delta^{p-2}(x)t^{-1}$, obtaining

$$\begin{aligned}
\bar{\varphi}(u) &= \bar{\varphi}(\delta^{p-2}(x)t^{-1}) \\
&= \bar{\varphi}(\delta^{p-2}(x))\bar{\varphi}(t^{-1}) \\
&= \Delta^{p-2}(\bar{\varphi}(x))Z^{-1}, \text{ by (4.3.24) and (4.3.26)} \\
&= \Delta^{p-2}(Y)Z^{-1}.
\end{aligned}$$

So $\bar{\varphi}(u)Z = \Delta^{p-2}(Y) = UZ$ and, since Z is invertible, we get $\bar{\varphi}(u) = U$. Therefore, if we apply $\bar{\varphi}$ to e_{hm}, whose equation is given in terms of u and ξ in (4.3.6), we get

$$\bar{\varphi}(e_{hm}) = -U^{p-h}\Xi^{p-1}U^{m-1}.$$

And, by evaluating the powers of U and Ξ, we finally get

$$\bar{\varphi}(e_{hm}) = E_{hm}.$$

So we have, on the one hand,

$$\begin{aligned} \bar{\varphi}(x) &= \bar{\varphi}\left(\sum_{hm} x_{hm}e_{hm}\right) \\ &= \sum_{hm} \bar{\varphi}(x_{hm})E_{hm} \end{aligned}$$

and, on the other hand, by definition of $\bar{\varphi}$,

$$\bar{\varphi}(x) = \sum_{ij} y_{ij} + y_{1p}E_{1p} + \sum_{n=3}^{p} z_n E_{np}.$$

Thus, by comparing corresponding entries, $\bar{\varphi}(x_{ij}) = y_{ij}$, $\bar{\varphi}(x_{1p}) = y_{1p}$, $\bar{\varphi}(x_{2p}) = 0$ and $\bar{\varphi}(x_{np}) = z_n$ for $n = 3, \dots p$. Therefore,

$$\begin{aligned} &\bar{\psi}\bar{\varphi}(x_{ij}) = \bar{\psi}(y_{ij}) = x_{ij} \\ &\bar{\psi}\bar{\varphi}(x_{1p}) = \bar{\psi}(y_{1p}) = y_{1p} \\ &\bar{\psi}\bar{\varphi}(x_{2p}) = 0 = x_{2p}, \text{ by } (4.3.17), \end{aligned}$$

and

$$\begin{aligned} \bar{\psi}\bar{\varphi}(x_{np}) &= \bar{\psi}(z_n) \\ &= \sum_{l=0}^{n-3}(-1)^{n-l-1}\binom{n-2}{l}\bar{\psi}(y_{l+2,l+p-n+2}) \\ &= \sum_{l=0}^{n-3}(-1)^{n-l-1}\binom{n-2}{l}x_{l+2,l+p-n+2} \\ &= x_{np}, \text{ for } n = 3, \dots, p. \end{aligned}$$

Thus,

$$
\begin{aligned}
\bar{\psi}\bar{\varphi}(x) &= \bar{\psi}\bar{\varphi}\left(\sum_{hm} x_{hm} e_{hm}\right) \\
&= \sum_{hm} x_{hm} \bar{\psi}(E_{hm}) \\
&= \sum_{hm} x_{hm} e_{hm} \\
&= x.
\end{aligned}
$$

Finally, since $\Delta^{p-1}(Y) = Z$, applying $\bar{\psi}$ on both sides and noting that $\bar{\psi}(\Xi) = \xi$ and that $\bar{\psi}(Y) = x$, we get

$$\bar{\psi}(Z) = \bar{\psi}(\Delta^{p-1}(Y)) = \delta^{p-1}(\bar{\psi}(Y)) = \delta^{p-1}(x) = t.$$

Thus, $\bar{\psi}\bar{\varphi}(t^{-1}) = \bar{\psi}(Z^{-1}) = \bar{\psi}(Z)^{-1} = t^{-1}$. Therefore,

$$\bar{\psi}\bar{\varphi} = Id_{S_E}.$$

In the other direction, we have

$$
\begin{aligned}
\bar{\varphi}\bar{\psi}(y_{ij}) &= \bar{\varphi}(x_{ij}) \\
&= y_{ij} \\
\bar{\varphi}\bar{\psi}(y_{1p}) &= \bar{\varphi}(x_{1p}) \\
&= x_{1p} \\
\bar{\varphi}\bar{\psi}(E_{ij}) &= \bar{\varphi}(e_{ij}) \\
&= E_{ij} \\
\bar{\varphi}\bar{\psi}(y_{1p}^{-1}) &= \bar{\varphi}(x_{1p}^{-1}), \text{ by (4.3.27)} \\
&= \bar{\varphi}(x_{1p})^{-1} \\
&= y_{1p}^{-1}.
\end{aligned}
$$

Thus,

$$\bar{\varphi}\bar{\psi} = Id_{\mathfrak{M}_p(A)}.$$

Therefore, $\bar{\varphi}$ constitutes an isomorphism between $\mathfrak{M}_p(A)$ and S_E. $\square$

Corollary 4.3.15. *The centralizer T of the matrix units e_{ij} in S_E is isomorphic to A.*

Proof. Since the isomorphism $\bar{\varphi}$ takes matrix units to matrix units, the restriction of $\bar{\varphi}$ to T is an isomorphism onto A. $\square$

Corollary 4.3.16. *T is a fir.*

Proof. Since T is isomorphic to A, by Corollary 4.3.15, and because

$$A \cong E\langle y_{ij}\rangle *_E E\langle y_{1p}, y_{1p}^{-1} | y_{1p}y_{1p}^{-1} = 1 = y_{1p}^{-1}y_{1p}\rangle,$$

it follows from Proposition 1.3.5 that A, and, therefore, T, is a fir, for both the free algebra $E\langle y_{ij}\rangle$ and the group algebra $E\langle y_{1p}, y_{1p}^{-1} | y_{1p}y_{1p}^{-1} = 1 = y_{1p}^{-1}y_{1p}\rangle$ are firs. The fact that the group algebra over a field of a free group is a fir is Corollary 3 of [5] on p. 68. $\square$

4.3.4 Extensions

Like the Galois case, the results in this section can be extended to a ring of the form $F_k\langle X\rangle$, where X is any nonempty set.

Corollary 4.3.17. *Let k be a commutative field of prime characteristic $p > 0$ and $F = k(\alpha)$, where $\alpha^p = a \in k$ and $\alpha \notin k$. Let $R = F_k\langle X\rangle$, where X is any nonempty set, and denote by U its universal field of fractions. Let E be an extension of k isomorphic to F/k. Then there exists a subring S of U, obtained from R by adjoining the inverse of a single element of R, such that S_E is isomorphic as an E-algebra to a matrix ring $\mathfrak{M}_p(A)$, where A is a fir.*

Proof. Pick an element $x \in X$ and write

$$R \cong R' *_k k\langle X'\rangle,$$

where $R' = F_k\langle x\rangle$ and $X' = X \setminus \{x\}$. After constructing S, we get

$$S_E \cong E\langle \xi, x, t^{-1} \mid \xi^p = 0, \delta^{p-1}(x)t^{-1} = t^{-1}\delta^{p-1}(x) = 1\rangle *_E E\langle X'\rangle.$$

Since $E\langle \xi, x, t^{-1} \mid \xi^p = 0, \delta^{p-1}(x)t^{-1} = t^{-1}\delta^{p-1}(x) = 1\rangle \cong \mathfrak{M}_p(T)$, where T is a fir, it follows that S_E is isomorphic to a $p \times p$ matrix over a fir. Explicitly, $S_E \cong \mathfrak{M}_p(V)$, where $V = \mathfrak{W}_p(T *_E E\langle X'\rangle; T)$. $\square$

4.4 Final remarks

In both the two preceding sections, we obtained a matrix ring over a fir following a very precise construction. We started with a fir R which had a universal field of fractions U such that both shared the same centre k. Then we considered a finite commutative field extension E of k of degree n. By tensoring U with E over k we obtained a full $n \times n$ matrix ring over a skew field K, $U_E \cong \mathfrak{M}_n(K)$. Next, we looked at a subring S of U, containing R, which was obtained by adjunction of inverses. This ring S was such that S_E contained the matrix units of $\mathfrak{M}_n(K)$ and was, therefore, itself isomorphic to a full $n \times n$ matrix ring, say $S_E \cong \mathfrak{M}_n(T)$, where T was the centralizer in S_E of the matrix units. In the two instances studied in Sections 4.2 and 4.3, we were able to prove that T was a fir. In this section, we will be showing that K is the universal field of fractions of T. But we start by noting that the fact that the ring T obtained by this process in Sections 4.2 and 4.3 was a fir is a particularity of the cases studied. It is not true, in general, that rings obtained by this process are always firs, as the next example shows.

Example. Let C be a skew field with centre k and E/k a commutative field extension of degree 2 such that $C_E \cong \mathfrak{M}_2(V)$ for some skew field V (*e.g.* take a skew field C which contains an isomorphic copy of E, cf. [8, Cor. 7.2.12]). Let $R = C^1 *_k C^2$, where $C^i \cong C$ for $i = 1, 2$. We know that R is a fir; let U denote its universal field of fractions. Since R_E is not an integral domain, because it contains a full 2×2 matrix ring, it follows that $U_E \cong \mathfrak{M}_2(K)$, where K is a skew field. Write $C^i_E \cong \mathfrak{M}_2(V_i)$. Since $\mathfrak{M}_2(V_1) \cong C^1_E \subseteq R_E \subseteq U_E$, we can regard U_E as a 2×2 matrix ring over the centralizer of the matrix units of C^1_E, say $U_E \cong \mathfrak{M}_2(G)$. Since U_E is simple artinian, we have $K \cong G$, that is G is a skew field. Now R itself is a subring of U such that R_E contains the matrix units of $U_E \cong \mathfrak{M}_2(G)$, so $R_E \cong \mathfrak{M}_2(T)$. But in this case, $T \cong \mathfrak{W}_2(V_1 *_E \mathfrak{M}_2(V_2); V_1)$, which is not a fir, because $\frac{1}{2}\mathcal{P}(T) \cong \frac{1}{2}\mathbb{N} \coprod_{\mathbb{N}} \frac{1}{2}\mathbb{N}$.

We return to the cases analysed in Sections 4.2 and 4.3, where it was proved that T was a fir. We will start by analysing what happens in a general situation where R_E is itself a fir. Although this is not the case of the two instances of the preceding sections, the study of this situation exemplifies the kind of argument we shall be considering later.

Proposition 4.4.1. *Let R be a Sylvester domain with universal field of fractions U and suppose that the centre k of U is contained in R. Let E be a finite commutative field extension of k. If R_E is a Sylvester domain then U_E is a skew field. In this case, U_E is the universal field of fractions of R_E.*

Proof. Let Φ be the set of all full matrices over R. Then $U = R_\Phi$. Denote by $\lambda : R \longrightarrow U$ the canonical homomorphism from R into U and by $\bar{\lambda} : R_E \longrightarrow U_E$ the induced homomorphism.

Suppose that R_E is a Sylvester domain—so it has a universal field of fractions K. Let $\eta : R_E \longrightarrow K$ the canonical homomorphism of R_E into K which inverts all the full matrices over R_E. Since $[E : k]$ is finite, U_E is a simple artinian ring. Let A be a full matrix over R. Then $\lambda(A)$ is invertible in U, so $\bar{\lambda}(A)$ is invertible in U_E. Since U_E has UGN, $\bar{\lambda}(A)$ is full over U_E, thus A is full over R_E which implies that $\eta(A)$ is invertible over K. Hence η is Φ-inverting and so there exists a unique homomorphism $f : U_E \longrightarrow K$ such that $f\bar{\lambda} = \eta$. Since U_E is simple, f is injective and, because K is a skew field, U_E must be an integral domain, but this implies that U_E is a skew field. We know that $U_E \cong (R_E)_\Phi$, so that $\bar{\lambda}$ is an epimorphism and, therefore U_E is generated as a skew field by $\bar{\lambda}(R_E)$. But we also know that K is generated as a skew field by $\eta(R_E)$. So f must be an isomorphism and this implies that U_E is the universal field of fractions of R_E. □

In particular, if R is a fir and R_E also a fir, U_E will be a skew field—the universal field of fractions of R_E.

Now we look at the case of matrix rings over firs.

Theorem 4.4.2. *Let R be a Sylvester domain with universal field of fractions U and suppose that the centre k of U is contained in R. Let E be a finite commutative field extension of k of degree m. Then U_E is isomorphic to an $r \times r$ matrix ring over a skew field K, where r divides m. Regarding R_E as contained in U_E, let A be a subring of U_E, containing R_E and the matrix units of U_E. The ring A*

is, then, isomorphic to an $r \times r$ matrix ring $\mathfrak{M}_n(T)$. If T is a Sylvester domain, then K is its universal field of fractions.

Proof. That $U_E \cong \mathfrak{M}_r(K)$ where K is a skew field and r divides m is the contents of Lemma 1.4.1. The fact that $A \cong \mathfrak{M}_r(T)$ is a consequence of the presence of the matrix units of U_E in A. We also know that T is the centralizer of the matrix units in A and that T is a subring of K. Now suppose that T is a Sylvester domain and denote its universal field of fractions by V. Let Φ be the set of all full matrices over R, so that $U \cong R_\Phi$. We will show that the homomorphism $R_E \longrightarrow \mathfrak{M}_r(V)$, obtained by composing the map $A \cong \mathfrak{M}_r(T) \longrightarrow \mathfrak{M}_r(V)$, induced by the canonical monomorphism $T \longrightarrow V$ with the inclusion $R_E \longrightarrow A$ is Φ-inverting.

$$\begin{tikzcd}
R \arrow[rr] \arrow[d] & & U = R_\Phi \arrow[d] \\
R_E \arrow[r] & A \arrow[r] & U_E \cong (R_E)_\Phi \\
& \mathfrak{M}_r(T) \arrow[u, phantom, sloped, "\cong"] \arrow[dr] & \mathfrak{M}_r(K) \arrow[u, phantom, sloped, "\cong"] \\
& & \mathfrak{M}_r(V)
\end{tikzcd}$$

Indeed, let M be an $n \times n$ matrix over R belonging to Φ; then M is invertible over $U_E \cong (R_E)_\Phi$. Since R is embedded in $A \cong \mathfrak{M}_r(T)$, we can regard M as an $rn \times rn$ matrix over T, and thus, over K. Because M is invertible over $U_E \cong \mathfrak{M}_r(K)$ as an $n \times n$ matrix, it is invertible over K as an $rn \times rn$ matrix. But this implies that M is full over K, so full over T. That is, M is an $n \times n$ matrix over $\mathfrak{M}_r(T)$ which, regarded as an $rn \times rn$ matrix over T, is full. So M is an invertible $rn \times rn$ matrix over V and, therefore, it is invertible as an $n \times n$ matrix over $\mathfrak{M}_r(V)$. This proves $R_E \longrightarrow \mathfrak{M}_r(V)$ to be Φ-inverting. So there exists a homomorphism $\zeta : \mathfrak{M}_r(K) \cong U_E \cong (R_E)_\Phi \longrightarrow \mathfrak{M}_r(V)$ extending $R_E \longrightarrow \mathfrak{M}_r(V)$. Since $\mathfrak{M}_r(V)$ is simple artinian, we can suppose that ζ preserves matrix units. For it it did not, we would have $\mathfrak{M}_r(V)$ isomorphic to an $r \times r$ matrix ring $\mathfrak{M}_r(V')$, whose matrix units were the images under ζ of the matrix units of $\mathfrak{M}_r(K)$. Since $\mathfrak{M}_r(V)$ is simple artinian, we would have $V \cong V'$. So ζ induces a homomorphism $K \longrightarrow V$, which must be injective, because K is a field. Hence we can regard K as a sub-skew field of V containing T. But since V is a field of fractions of T, we must have K isomorphic to V. $\square$

We can apply the above result to the rings obtained in Section 4.2, for instance. Let us recall the set-up. Let k be a commutative field and F a Galois extension of k of finite degree n. Then $R = F_k\langle x\rangle$ is a fir. Denote its universal field of fractions by U. Let E be an extension of k isomorphic to F. Then U_E is isomorphic to an $n \times n$ matrix ring over a skew field K. By inverting a finite number of elements of R, we obtained a ring S which had the property that S_E contained the n^2 matrix units of U_E. So $S_E \cong \mathfrak{M}_n(T)$, where T is a subring of K. In Theorem 4.2.12, we proved that T was a fir. Now, applying the above theorem to this situation, we conclude that K is the universal field of fractions of T. These ideas can be applied in the same way to the rings of Section 4.3.

Bibliography

[1] G. Agnarsson, S. A. Amitsur and J. C. Robson, Recognition of matrix rings II, *Israel J. of Math.* **96** (1996), 1-13.

[2] M. Auslander, I. Reiten & S. O. Smalø, *Representation Theory of Artin Algebras*, Cambridge Studies in Advanced Mathematics 36, Cambridge University Press, 1995.

[3] G. M. Bergman, Modules over coproducts of rings, *Trans. Amer. Math. Soc.* **200** (1974), 1-32.

[4] G. M. Bergman, Coproducts and some universal ring constructions, *Trans. Amer. Math. Soc.* **200** (1974), 33-88.

[5] P. M. Cohn, Free ideal rings, *J. of Algebra* **1** (1964), 47-69.

[6] P. M. Cohn, *Free Rings and Their Relations*, 2nd Ed., LMS Monographs No. 19, Academic Press, London, 1985.

[7] P. M. Cohn, *Algebra 2*, 2nd Ed., John Wiley & Sons, Chichester, 1989.

[8] P. M. Cohn, *Algebra 3*, 2nd Ed., John Wiley & Sons, Chichester, 1991.

[9] P. M. Cohn, *Skew Fields. Theory of General Division Rings*, Encyclopedia of Mathematics and its Applications, Vol. 57, Cambridge University Press, 1995.

[10] G. Grätzer, *Universal Algebra*, 2nd Ed., Springer-Verlag, New York, 1979.

[11] J. M. Howie, Commutative semigroup amalgams, *J. Austr. Math. Soc.* 8 (1968), 609–630.

[12] J. M. HOWIE, *Fundamentals of Semigroup Theory*, Clarendon Press, Oxford, 1995.

[13] N. JACOBSON, *Structure of Rings*, Colloquium Publications Vol. 37, American Mathematical Society, Providence, RI, 1956.

[14] A. H. SCHOFIELD, *Representations of Rings over Skew Fields*, London Mathematical Society Lecture Notes Vol. 92, Cambridge University Press, 1985.

Index