

**On viscous, inviscid and centrifugal instability mechanisms in
compressible boundary layers, including non-linear vortex/wave
interaction and the effects of large Mach number on transition.**

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Nicholas Derick Blackaby,
University College. March 1991.

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Abstract

The stability and transition of a compressible boundary layer, on a flat or curved surface, is considered using rational asymptotic theories based on the large size of the Reynolds numbers of concern. The Mach number is also treated as a large parameter with regard to hypersonic flow. The resulting equations are simpler than, but consistent with, the full Navier Stokes equations, but numerical computations are still required. This approach also has the advantage that particular possible mechanisms for instability and/or transition can be studied, in isolation or in combination, allowing understanding of the underlying physics responsible for the breakdown of a laminar boundary layer.

The nonlinear interaction of Tollmien–Schlichting waves and longitudinal vortices is considered for the entire range of the Mach number; it is found that compressibility has significant effects on the solution properties. The arguments breakdown when the Mach number reaches a certain, large, size due to the ‘collapse’ of the multi-layered boundary layer present and thus we are naturally led on to investigate this new regime where ‘non-parallelism’ must be incorporated in the theory. Also, the effects of compressibility are then more significant, analytically, causing the governing equations to be more complicated, and further analytic progress relies on shorter scales being employed for any perturbations to the basic flow. The numerical solution is discussed along with a non-linear asymptotic solution capturing a ‘finite-time break-up’ of the interactive boundary layer.

This work suggests that for larger Mach numbers the crucial non-linear interaction is between inviscid modes and Görtler vortices and these are discussed in the remaining chapters. The inviscid modes are studied initially with no shock present, before the theory is modified for the inclusion of shock-wave/boundary layer interaction. In the last chapter the Görtler-vortex mechanism for large Mach numbers is considered.

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Chapter 1

General Introduction.

§1.1 BOUNDARY LAYER STABILITY THEORY.

The question why most fluid flows are turbulent rather than laminar has received much attention by investigators during the past hundred years or so. The efficiency of, for example, turbine blades or aircraft wings is reduced significantly by the presence of turbulent flow; thus there has been, and still is, much research aimed at understanding how and why small disturbances in laminar flows can cause the onset of transition to turbulence. The theoretical study of such disturbances (instabilities), of laminar flow (liquid or gaseous) is commonly known as ‘hydrodynamic stability theory’. Our concern in this thesis is with the study of such disturbances situated within thin regions (‘boundary layers’) that form alongside solid surfaces in the presence of high relative speed fluid flow. This aspect of hydrodynamic stability theory is known as ‘boundary layer stability theory’. In particular, we consider high-speed compressible flows; there is currently much interest in such flows because of the desire to design and build faster/more fuel-efficient aircraft and space vehicles.

§1.1.1 Historical background.

Many notable results concerning the instability of inviscid flows were discovered by Rayleigh (1880, 1887, 1913). In these early years, it was usual to consider external fluid flows to be inviscid. This assumption resulted in easier governing equations but the so called ‘no-slip’ condition (no relative tangential motion on solid boundaries) had to be neglected for solutions. Prandtl (1904) first introduced the idea of a thin boundary layer forming adjacent to solid surfaces in the presence of high speed flow. He showed that in these boundary layers it is necessary to take viscosity into account but the no-slip condition could now be satisfied. This boundary layer theory was subsequently developed by Prandtl and other workers. It is convenient here to introduce the concept of the Reynolds number, Re ; this dynamical similarity parameter depends on the free-stream velocity, a typical length-scale and the viscosity of the flow, u_∞^* , L and ν_∞^* respectively.

Reynolds (1883) found that the combination $Re = \frac{u_\infty^* L}{\nu_\infty^*}$ characterizes the flow completely; flows having different u_∞^* , L and ν_∞^* values but the same Reynolds number have the same solution in terms of non-dimensionalised quantities. The size of the Reynolds number is found to be very large for aerodynamical flow, typically $Re \sim 10^6$ or larger. Prandtl and his colleague Blasius showed that boundary layers on a flat plate are of thickness $O(Re^{-\frac{1}{2}})$; thus these layers are thin, consistent with Prandtl's original idea.

At this juncture, it is also convenient to introduce a second parameter of dynamical-similarity parameter, the so-called Mach number,

$$M_\infty = \frac{u_\infty^*}{a_\infty^*};$$

the ratio of the free-stream speed, u_∞^* , to the speed of sound, a_∞^* , in the free-stream. The Mach number measures the compressibility of the flow. Incompressible flows have $a_\infty^* = \infty$ so that $M_\infty = 0$; in this case, if the wall and free-stream temperatures are equivalent, the boundary-layer temperature profile can be taken to be constant and the governing equations are greatly simplified. Compressible flows having $0 < M_\infty < 1$ are commonly referred to as being *subsonic*; whilst those with $M_\infty > 1$ are referred to as being *supersonic*.

The equations governing boundary-layer flow can, in general, only be solved analytically by use of similarity variables, enabling the number of independent variables to be reduced. The first and most celebrated of these similarity solutions was found by Blasius (1908), for the flow induced by the presence of a flat plate in a uniform stream; whilst Von Kármán (1921) derived the form of the velocity field for the flow caused by a rotating disc. Similarity solutions for other incompressible flows have also been found. The first significant contribution for compressible flows was by Von Mises (1927) who found a solution for compressible flow on a flat plate close to the leading edge, in terms of the stream-function. Crocco (1941) solved the equations of motion by treating the downstream co-ordinate and velocity as the independent variables. However, the method usually used to solve the compressible boundary-layer equations is due to Dorodnitsyn (1942) and Howarth (1948). Their solution is based on a stretching of the co-ordinate by means of an integral based on the local temperature. Comprehensive reviews of boundary layer theory can be found in the books by Schlichting (1960) and Stewartson (1964).

We now consider boundary-layer stability. At the turn of the century, viscosity was commonly thought to act only to stabilize the flow: there were many results, mostly due to Lord Rayleigh, concerning the instability of incompressible inviscid flows; in particular, it was believed that flows without inflexional profiles were stable. A few years later, Taylor (1915), ^{and Heisenberg (1924)} Prandtl (1923) independently indicated that viscosity can destabilize a flow that is otherwise stable. These discoveries led to the first viscous theory of boundary-layer instability, due to Tollmien (1929). This theory was further developed by Schlichting (1933a, 1933b, 1935, 1940) and by Tollmien (1935); numerical results were obtained and some predictions concerning the onset of transition to turbulence were made. However, outside of Germany, these theories received little acceptance (see, for instance, Taylor 1938); due mainly to the lack of experimental verification of the theories. However, Schubauer & Skramstad (1947) reported that instability waves had been observed experimentally in a boundary layer; that they were closely connected to transition; and moreover, that there was a close correlation between their observed behaviour and the predicted behaviour from the theories of Tollmien and Schlichting. Thus the latter researchers' viscous, linear, theory of boundary-layer instability was fully vindicated; these instability waves are now referred to as Tollmien-Schlichting waves, in recognition of their contributions.

The next significant advance in the theory of the Tollmien-Schlichting waves followed nearly three decades later when Bouthier (1973)[†] and Gaster (1974) included non-parallel flow effects which are due to boundary layer growth. Their asymptotic method involves a successive approximation procedure, treating $Re^{-\frac{1}{2}}$ as a small parameter. At zeroth order the Orr-Sommerfeld equation (see later) is obtained and must be solved numerically. Non-parallel effects then introduce forcing terms on the right-hand sides of the higher-order equations. An alternative approach to this reliable and efficient procedure was developed by Smith (1979a) who demonstrated that, for high Reynolds numbers, the Tollmien-Schlichting modes could be described by the so called 'triple deck' structure. This short-scaled structure, consisting of three thin layers adjacent to the wall in which classical boundary-layer theory no longer describes the flow (see later subsection), had been independently discovered ten years earlier by Messiter (1970), Neiland (1969) and Stewartson & Williams (1969) to describe the self-induced separation

[†] Bouthier did not compare the same property in theory and experiment.

of flows. We note that the link between the Tollmien–Schlichting modes and the triple-deck scales is, in fact, implied from the work of ^{Heisenberg (1924) and} Lin (1945), who studied the viscous instability of incompressible flows. The remarkable finding by Smith proved to be the catalyst for many subsequent studies of the linear and nonlinear viscous stability properties of various boundary layer flows, based on the triple deck approach, i.e. Bodonyi & Smith (1981); Duck (1985); Goldstein (1984); Hall & Smith (1984, 1989); Smith (1988); Smith & Burggraf (1985); and Smith, Doorly & Rothmayer (1990), to name but a (varied) few.

However, there are other stabilities of boundary layers, apart from the viscous Tollmien–Schlichting modes. We have already noted that the inviscid stability of incompressible flows was first studied over a century ago; the most significant result, that the flow profile necessarily must be inflexional for the flow to be unstable, was found by Lord Rayleigh. The first significant study of the inviscid instability of compressible flows was by Lees & Lin (1946). They extended the theorems of Rayleigh to the case of compressible flows and found that compressible boundary-layer flows are inviscidly unstable, in contrast to their incompressible counterparts. The first study of a compressible boundary-layer is attributed to Küchemann (1938) but his assumptions were extremely restrictive. These restrictions were relaxed somewhat by Lees & Lin who, in addition to their inviscid theory (mentioned above), also considered viscous disturbances, in close analogy with the incompressible asymptotic theory of Lin (1945). The viscous and inviscid linear stability properties of compressible boundary-layer flows have been comprehensively studied by Mack (1965a, 1965b, 1969, 1984) providing a large source of numerical results. Mack finds that the inclusion of compressibility leads to many additional solutions, which he terms ‘higher modes’. He finds that these higher modes are destabilized by ‘cooling’ the wall; several years earlier, Lees (1947) had predicted that cooling the wall acts to stabilize the boundary layer. Balsa & Goldstein (1990), Cowley & Hall (1990) and Smith & Brown (1990) have recently, independently, investigated the inviscid, linear, instabilities of large Mach number flows by asymptotic methods in an attempt to fit analytical theories to some of Mack’s results for inviscid disturbances. The viscous linear instability of compressible boundary-layer flows has been studied asymptotically by Smith (1989) using the compressible version of the triple deck theory.

In the proceeding discussion we have restricted our attention to the instabilities (viscous and inviscid) associated with two-dimensional flows over a flat surface. However there are two further classes of hydrodynamic instabilities, centrifugal and crossflow, appropriate to concave-curved surfaces and three-dimensional flows, respectively. To complicate matters further, each of these instabilities can be ‘inviscid’ or ‘viscous’ in nature. The centrifugal instability mechanism was first identified by Taylor (1923), during laboratory experiments concerned with the motion of fluid between rotating concentric cylinders. The flow is unstable to vortex structures (now referred to as ‘Taylor vortices’) whose axes follow the curvature of the flow streamlines. Hence the flow in curved channels is centrifugally unstable to Taylor vortices; whilst the flow in curved pipes is centrifugally unstable to ‘Dean vortices’ (Dean, 1928).

Our concern in this thesis is with external, boundary-layer, flows and the centrifugal instability of these flows, over a concave wall, was established by Görtler (1940), after whom the particular instability is named. However, the correct governing equations for the linear Görtler vortex instability were not correctly formulated until the papers of Gregory, Stuart & Walker (1955), Floryan & Saric (1979) and El-Hady & Verma (1981), the latter for compressible flows. Moreover, correct numerical results for this instability proved elusive until the paper by Hall (1983), for the incompressible case; correct numerical results for compressible flow were first computed by Wadey (1990). Much analytical progress on the linear and non-linear stability properties of Görtler vortices has been made in the last decade; these are principally due to Hall and colleagues. These asymptotic theories are generally based on the ‘large-wavenumber approximation’, first employed by Meksyn (1950) for the Taylor vortex problem. Stuart (1960) and Watson (1960) showed how non-linear effects could be taken into account, for plane Poiseuille and Couette flows, close to the position of neutral linear stability. Their approach requires the correction to the mean flow to be an order of magnitude smaller than the mean flow itself; however, Hall (1982b) has shown that this is not the case for the Görtler problem. A comprehensive review of the past fifty years’ Görtler vortex instability research can be found in the article by Hall (1990).

The final class of instabilities are found in three-dimensional boundary layers (for which three ortho-normal co-ordinates are necessary to describe the flow).

These instabilities, which can be either viscous or inviscid in nature, are commonly referred to as ‘crossflow vortices’. This important instability mechanism receives little attention in this thesis, principally due to the difficulty of finding solutions to the three-dimensional, compressible, boundary layer equations. Most studies of crossflow instability have considered the boundary-layer above a rotating disc since Von-Kármán’s is an exact solution of the Navier Stokes equations. This self-similar boundary layer was first used for this purpose by Gregory, Stuart & Walker (1955) in their classic paper on three-dimensional boundary-layer instability. Their theory has subsequently been extended by several researchers; notably by Hall (1986) and Bassom & Gajjar (1988). The extension of these theories to compressible flow above a rotating disc constitutes work under progress by several workers. Another exact three-dimensional solution to the (incompressible) Navier Stokes is the ‘swept attachment-line’ boundary layer on an infinite flat plate. The instability of such a boundary-layer was first studied experimentally by Poll (1979); and theoretically by Hall, Malik & Poll (1984). A fuller discussion of the crossflow instability can be found in the recent papers by Hall & Seddougui (1990) and Bassom & Hall (1991), who investigate wave interactions in the swept attachment-line boundary layer and non-stationary crossflow vortex interactions in the boundary layer above a rotating disc, respectively.

Before concluding this discussion on boundary-layer stability theory it is worth recalling the motivation for such studies by several generations of investigators. It was (and still is) hoped and/or believed that an understanding of the linear and non-linear stability properties of laminar boundary-layer flows will lead to an understanding of the factors causing the onset of transition, of these laminar flows, to a turbulent state. The prediction of when transition occurs has been the subject of many papers dating back to Schlichting (1933a).

Many of these ‘predictions’ of transition are based purely on the numerical results of *linear* theory; essentially, it is believed that if a flow is ‘sufficiently’ linearly unstable then transition follows soon afterwards. The most notable example is the so-called ‘ e^N method’, due to Smith & Gamberoni (1956) and Van Ingen (1956), still in routine use today in engineering studies. In this method, if the linear instability theory predicts an e^N -fold increase in the initial amplitude of the disturbance, then transition to turbulence is predicted. Usually the value

$N = 9$ is chosen in this seemingly completely irrational transition criterion. However, this *simple* method would not have remained in routine use for so long had it not had some success in predicting transition. This limited success is usually, ironically, attributed to the (totally dis-regarded) non-linear effects being so powerful that turbulence soon follows linear instability. Further, this method relies on the notion of a ‘unique’ growth rate; for instance, Hall (1983) and Smith (1989) have indicated that such a notion is not tenable, for the general Görtler vortex problem and for the viscous stability of large Mach number flows, respectively. Additionally, any transition predictions based on purely linear theories totally disregard the possibility/probability of *interactions* between these instabilities; the individual ‘participants’ of the interaction may all come from the same ‘group’ of instabilities (eg. multiple-wave interactions) or from different groups (eg. vortex/wave interactions). Individually, based on some linear transition criterion, the individual instabilities may not trigger the onset of transition but *together* they may. These nonlinear interactions are currently causing much excitement amongst theoreticians.

Further aspects of boundary-layer stability theory include (i), the notion of *secondary instabilities* to an initial *primary* disturbance (see, for instance, Hall & Horseman, 1990); (ii) the investigation of *finite-time* and other ‘break-ups’ of interacting unsteady boundary layers (see Smith, 1988); and (iii), the *receptivity* theory. The latter is concerned with the origin of the small disturbances; either due to surface roughness or freestream fluctuations (see, for instance, Goldstein, 1985; Hall, 1990).

Many review articles and books have been written on boundary-layer theory and/or hydrodynamic stability theory: these include the books by Drazin & Reid (1981), Lin (1955), Schlichting (1960), Stewartson (1964); and the articles by Mack (1984)[†], Reid (1965), Reshotko (1976), Smith (1982) and Stuart (1963).

[†] this review paper was the source for many of the historical references cited in the present discussion.

§1.1.2 The inviscid, viscous and centrifugal instabilities.

We now give a few more details concerning the particular instabilities to be considered later in this thesis. Fuller details can be found in the references cited herein or later in the thesis.

Viscous instability theory.

We consider the linear theory first; then the disturbances (Tollmien–Schlichting modes) can generally be written as normal modes, that is the small disturbances are written as proportional to

$$\exp[i(\int \alpha dx + \beta dz - \omega t)],$$

containing all the streamwise (x), spanwise (z) and time (t) variation. The resulting equations, for three-dimensional viscous disturbances of compressible boundary-layer flows, lead to an eighth-order system (see Mack, 1984) which has to be solved numerically. The order of this system is reduced to fourth when the flow is incompressible. In this case, for a 2-D disturbance ($\beta = 0$), the disturbance equations can be combined to yield the familiar Orr–Sommerfeld equation

$$(D^2 - \alpha^2)^2 \bar{v} = iRe[(\alpha \bar{u} - \omega)(D^2 - \alpha^2) - \alpha^2 D^2 \bar{u}] \bar{v},$$

(where Re is the Reynolds number based on boundary-layer thickness) for the normal disturbance velocity amplitude $\bar{v}$. Here $D \equiv \partial/\partial y$ and $\bar{u}$ represents the streamwise velocity profile of the boundary-layer flow. This equation, and the eighth-order system for compressible flows, form the basis for much of the work done/being done in viscous linear stability theory. Later in this thesis we shall refer to ‘Orr–Sommerfeld-type’ solutions for the viscous stability of compressible flows; by this we shall mean solutions based on the numerical ^{solution} of the eighth-order system referred to above. We do so in order to distinguish such solutions from those calculated from an alternative theory, based on the so-called ‘triple deck structure’.

The ‘triple deck theory’, to be described in some detail later in this thesis, provides a far superior theory for investigating both the linear and and non-linear viscosity stability properties (the linear and nonlinear evolution of Tollmien–Schlichting modes) of ‘high-Reynolds-number’ boundary-layer flows. We note

that this high Reynolds number assumption is not very restrictive; it has already implicitly been made by the assumption that a boundary-layer exists — further, recall that typical values are $O(10^6)$, or larger, in practical flow situations. Essentially, we are assuming that the onset of transition in an external boundary-layer flow is a high Reynolds number phenomenon. The unexpected link between the triple-deck structure and the Tollmien–Schlichting disturbances was established by Smith (1979a) who also rationally incorporated the non-parallelism of the boundary-layer flow into his asymptotic theory. More precisely, the triple-deck theory describes modes corresponding to the so-called ‘lower-branch’ of the Orr–Sommerfeld neutral curve; the viscous modes corresponding to its ‘upper-branch’ are described by a very closely related five-tiered, short-scaled asymptotic theory (see Bodonyi & Smith, 1981; Smith & Burggraf, 1985). The viscous modes of a compressible boundary-layer flow have been studied by Smith (1989), using the compressible version of the triple deck theory.

In fact, the triple-deck structure is short-scaled; it has streamwise length of $O(Re^{-\frac{3}{8}})$ and comprises of three, thin, stacked ‘decks’ (the lower, the main and the upper decks) having heights $O(Re^{-\frac{5}{8}})$, $O(Re^{-\frac{1}{2}})$ and $O(Re^{-\frac{3}{8}})$. The resulting governing system of nonlinear equations comprises of the usual (incompressible) boundary-layer equations; however, in this non-classical theory, the pressure is not fixed by the freestream, instead it is driven by the viscous displacement effects, which in turn are driven by the pressure. Fuller details can be found in the comprehensive review of ‘the high Reynolds number theory’ by Smith (1982). The triple-deck-theory approach has the benefit that it allows significant analytical progress to be made for non-linear problems; such investigations are principally due to Smith and colleagues.

Finally, we note that the triple deck structure was discovered ten years before Smith’s classic paper, independently by Stewartson & Williams (1969), Neiland (1969) and Messiter (1970). They were seeking a rational explanation of the (seemingly un-related) problem concerning the ‘self-induced separation’ of a boundary layer from the surface in the absence of external influences. Earlier, Goldstein (1948), has shown that classical boundary layer theory predicted a *singularity* in the solution for separating flow (or flow undergoing ‘reversal’), in the presence of an adverse pressure gradient.

The inviscid instability.

Inviscid instabilities are those not driven by viscous effects, i.e. viscous terms do not enter their governing equations, although the underlying flow may be governed by viscosity. We are chiefly concerned with the inviscid linear instability of two-dimensional boundary-layer flows and this is governed by the compressible form of Rayleigh's equation,

$$\tilde{p}_{yy} - 2\frac{\bar{u}_y}{\bar{u} - c}\tilde{p}_y + \frac{\bar{T}_y}{\bar{T}}\tilde{p}_y - (\alpha^2 + \beta^2) \left[1 - \frac{\alpha^2 M_\infty^2 (\bar{u} - c)^2}{(\alpha^2 + \beta^2)\bar{T}} \right] \tilde{p} = 0,$$

$$\tilde{p}_y(0) = 0, \quad \tilde{p}(\infty) = 0, \quad (1.1.1a - c)$$

for the infinitesimal, wavelike, pressure disturbance $\tilde{p}$. Here $\bar{u}(y)$ and $\bar{T}(y)$ are the streamwise-velocity and temperature profiles of the underlying boundary layer and y is the normal similarity variable for the boundary layer. The constants α, β and c are the scaled wavenumbers and wavespeed, respectively.

It is well known that (1.1.1) cannot have solutions unless a so-called *generalised inflexion point* criterion is satisfied by the boundary-layer flow. This criterion, which requires that

$$\left(\frac{\bar{u}_y}{\bar{T}} \right)_y = 0,$$

for some $y > 0$, can easily be derived by seeking a regular series form for $\tilde{p}(y)$ in the neighbourhood of the critical level, where $\bar{u} \sim c$. When solving the dispersion relation (1.1.1), for $c \equiv c(\alpha, \beta)$ say, care must be taken to choose the correct solution that matches onto the viscous modes in the appropriate limit (see, for instance, the book by Lin, 1955). The incompressible version ($M_\infty = 0, \bar{T} = 1$) was first studied at the end of the nineteenth century by Lord Rayleigh, ^{and later by Fjortoft (1950) and Høiland (1953)} nowadays there is much renewed interest in the compressible case, (1.1.1), including extensions of this theory to other geometries.

The centrifugal instability mechanism.

These instabilities, which take the form of streamwise vortices, are only present in flows over concave surfaces. This surface curvature (if of the appropriate size) leads to an extra term in the y -momentum equation of the 'planar' form of the governing Navier Stokes equations. Associated with this curvature term is a number (named after Taylor, Dean or Görtler; depending on the flow context) which

is a measure of the surface curvature. For illustrative purposes we outline the origin of this ‘extra’ term for incompressible boundary-layer flow over a curved wall defined by

$$y = Re^{-\frac{1}{2}}g(x), \quad (x, y) = L^{-1}(x^*, Re^{\frac{1}{2}}y^*), \quad (1.1.2a, b)$$

where (x^*, y^*) are the physical co-ordinates along and normal to the wall, with associated velocity components $(u^*, v^*) = u_\infty^*(u, Re^{-\frac{1}{2}}v)$, respectively.

The Prandtl transformation,

$$u \rightarrow u, \quad v \rightarrow v + g'(x)u \quad \text{and} \quad y \rightarrow y + g(x)$$

leaves the continuity and x -momentum equations unchanged but the transformed y -momentum equation now contains the term $g''(x)u^2$, due to surface curvature. Note that, in (1.1.2a), the curvature $g(x)$ has been scaled on the boundary-layer thickness. The magnitude of curvature, $|g''(x)|$, is essentially the Taylor/Dean/Görtler number. There have been many studies of centrifugal instabilities, yielding so-called neutral/growth-rate curves which relate this number to the spanwise wavelength of the instabilities (vortices).

§1.2 THE PRESENT THESIS.

The present work will consider some aspects of the viscous, inviscid and centrifugal stability theories for compressible flows. In Chapter 2 we introduce and develop those aspects of compressible boundary-layer necessary for the later chapters. In particular, the choice of constitutive relationship between viscosity and temperature is discussed; this has emerged as a major theme of the present thesis. We investigate how the usual compressible triple-deck scales are modified due to the choice of the more realistic Sutherland’s formula to relate local temperature and viscosity.

In Chapter 3, our concern is with the non-linear interaction of Tollmien-Schlichting (TS) modes and longitudinal vortices in the compressible boundary-layer flow over a flat plate. This work is an extension of the paper by Hall & Smith (1989) for incompressible flow. This interaction is formulated within the framework of the compressible triple-deck theory. We compare the results obtained with those for the incompressible case; it is found that the most significant effect

of compressibility is due to the necessary obliqueness of the three-dimensional TS waves for supersonic flows (Zhuk & Ryzhov, 1981 ; Dunn & Lin, 1955).

In Chapter 4, our concern is with the viscous stability of large-Mach-number, compressible, boundary layer flow. Smith (1989) has shown that the triple-deck theory, governing viscous stability, ‘collapses’ (breaks down) when the Mach number rises to become of the same order as a particular, fractional, power of the Reynolds number. We show that the asymptotic theory describing the so-called ‘upper-branch’ viscous modes also collapses, simultaneously. The resulting, much larger, two-tiered structure is considered; some of the reasons behind the difficulty of finding analytical and numerical solutions are discussed. In the final section, even larger Mach numbers are considered in order to investigate the link between this two-tiered structure (governing viscous instabilities) and the so-called interactive boundary-layer structure, in which the impingement of a shock affects the basic boundary-layer state.

In Chapter 5, we consider the extension of the study of Smith (1988), concerning the probability of ‘finite-time break-ups’ in unsteady, interactive, boundary layers, to the case of compressible flow. Our principal concern is with extending the theory to the two-tiered boundary-layer structure discussed in the previous chapter. The troublesome effects of compressibility result in a complicated ‘critical layer’ analysis. This critical-layer problem remains unsolved, thus preventing definite conclusions although ‘finite-time break-up’ still appears likely and again is related to an inviscid form of Burger’s equation.

In chapter 6, our concern is with the inviscid instability of hypersonic flow over a flat plate. Two cases are considered. Firstly, the stability of the flow far downstream of the leading edge is considered, here the effect of the shock is negligible; the basic boundary-layer flow structure, in this case, was first formulated by Freeman & Lam (1959), based on an idea by Hayes & Probstein (1959). Secondly, the stability of flow close to the leading edge of the plate is considered, here, in the so-called *strong interaction region*, the flow state is strongly affected by an attached shock. This flow was first elucidated by Bush (1966). In both cases it is found that the most unstable linear modes are trapped within thin ‘temperature adjustment layers’, situated immediately above the hot (inner) boundary layers. The small wavenumber behaviour of these modes is considered, asymptotically.

Finally, in Chapter 7, we consider the co-existence of the centrifugally-driven Görtler instabilities and the inviscid Rayleigh modes, in hypersonic boundary-layers on concave walls; the latter instabilities having been previously considered in isolation, in Chapter 6. The Görtler instability in hypersonic flows is formulated for Sutherland-fluids; we find that the most dangerous modes are inviscid in character and that they are also trapped within the thin temperature adjustment layer. In Section 4 we discuss how the presence of a strongly non-linear vortex state will affect (modify) the Rayleigh instability properties. Both types of modes are considered: those associated with the hot boundary layer and those associated with the temperature adjustment layer. In the last section we note that ‘vortex/wave’ interaction is likely.

Presentations and reports.

The work contained in Chapter 3 was first presented at *EUROMECH 261* held at ISITEM, Nantes, France in June 1990. The work contained in Sections 4 and 5 of Chapter 6 was first presented at the 32nd British Theoretical Mechanics Colloquium (B.T.M.C.) held at St. Andrews in April 1990; the work contained in the whole of Chapter 6 forms the basis of the ICASE Report no. 90-40 (Blackaby, Cowley & Hall, 1990). An earlier version of some of the work described in Section 5 of Chapter 7 was presented at the 31st B.T.M.C. held at Exeter in April 1989; whilst the work described in Section 2 of the same chapter forms the basis for part of the ICASE Report no. 90-85 (Fu, Hall & Blackaby, 1990).

Chapter 2

Compressible boundary layer theory

§2.1 INTRODUCTION.

In this chapter we introduce and develop those aspects of compressible boundary-layer theory necessary for the later chapters. For a more complete discussion of some of these aspects the reader is referred to the excellent book by Stewartson (1964), as well as the many references cited later in this chapter.

Later in this section we discuss the physical assumptions that need to be made/are usually taken when considering the governing (compressible) Navier-Stokes equations, before discussing the choice of constitutive relationship between viscosity and temperature. The appropriate choice of viscosity-temperature relation is a major theme of this thesis; we choose one that is more physically appropriate over large temperature variations than that usually chosen by theoreticians. A brief discussion of the atmosphere then follows, prompted by the need for typical values of the temperature therein, to feed into the theories before obtaining quantitative results.

In §2.2 we formulate the boundary layer equations for non-interactive steady flows (no shock effects). These are investigated in some detail in Chapter 6 but in this chapter we are content with merely deducing those properties (mostly concerning the ‘wall values’) directly related to the triple-deck scales discussed in the following section. In §2.3 the compressible triple-deck structure is discussed and particular attention is paid to the ‘new’ scales made necessary by our choice of the temperature-viscosity relation. This mathematical structure, which describes the major viscous stability properties, is shown to enlarge into a longer, two-tiered structure as the Mach number increases to a certain large size (but smaller than that for which the effects of a shock on the base flow are significant). The same (qualitative) structure had been deduced by Smith (1989) but for a different (linear) viscosity-temperature relation.

§2.1.1 The parameters of dynamical similarity

We consider the boundary layer due to high-speed uniform flow of a compressible ^{fluid} over a flat plate. Suppose that L is the distance from the leading edge, and

u_∞^* , a_∞^* , ρ_∞^* and μ_∞^* , are the velocity, speed of sound, density and shear viscosity of the free stream flow, then we assume that the Reynolds number,

$$Re = \frac{\rho_\infty^* u_\infty^* L}{\mu_\infty^*}, \quad (2.1.1a)$$

is large. This is not unreasonable as one is already assuming the presence of a boundary layer. The second important parameter is the Mach number,

$$M_\infty = \frac{u_\infty^*}{a_\infty^*}, \quad (2.1.1b)$$

which we take to be $O(1)$ for the time being. Later we shall consider large Mach numbers and then we must further assume that Re is much greater than M_∞ (or some power of it). Again this is not unreasonable when one recalls that in practical circumstances a large Reynolds number is typically of size $O(10^6)$, whilst a large Mach number is much smaller, typically $O(10^1)$ in magnitude.

§2.1.2 Formulation

We adopt a non-dimensionalisation based on coordinates $L\underline{x}$ (where x is in the direction of flow and y is normal to plate), velocities $u_\infty^* \underline{u}$, time Lt/u_∞^* , pressure $\rho_\infty^* u_\infty^{*2} p$, density $\rho_\infty^* \rho$, temperature $T_\infty^* T$, and shear and bulk viscosities $\mu_\infty^* \mu$ and $\mu_\infty^* \mu'$ respectively, where the subscript ∞ denotes the value of the quantity in the free-stream. On the assumption that the fluid is a perfect gas with a constant ratio of specific heats γ , the governing equations of the flow are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0, \quad (2.1.2a)$$

$$\rho \frac{D\underline{u}}{Dt} = -\nabla p + \frac{1}{Re} [2\nabla \cdot (\mu \underline{e}) + \nabla((\mu' - \frac{2}{3}\mu)\nabla \cdot \underline{u})], \quad (2.1.2b)$$

$$\rho \frac{DT}{Dt} = (\gamma - 1)M_\infty^2 \frac{Dp}{Dt} + \frac{1}{Pr Re} \nabla \cdot (\mu \nabla T) + \frac{(\gamma - 1)M_\infty^2}{Re} \Phi, \quad (2.1.2c)$$

$$\gamma M_\infty^2 p = \rho T, \quad (2.1.2d)$$

where

$$e_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad (2.1.3a)$$

$$\Phi = 2\mu \underline{e} : \underline{e} + (\mu' - \frac{2}{3}\mu)(\nabla \cdot \underline{u})^2, \quad (2.1.3b)$$

and Pr is the constant Prandtl number. For more details on the derivation of these compressible Navier-Stokes equations see Stewartson (1964). Note that (2.1.2d) gives $\gamma M_\infty^2 p_\infty = 1$: the non-dimensionalised pressure in the free-stream is not unity. These equations can easily be generalised, for instance, to account for a non-zero external pressure-gradient and/or real gas effects such as dissociation.

§2.1.3 Real and ideal gases.

The atmosphere is not a real gas and strictly the above equations should be more complicated due to the inclusion of the fact that air (in the atmosphere) is a cocktail of numerous gaseous atoms and molecules. In particular, within a hot boundary layer these will be dissociating and chemically reacting. Further we shall see later in this chapter that the upper atmosphere is polluted, from natural (e.g. volcanic activity) and man-made (e.g. the exhaust fumes from supersonic aircraft) sources, and that its thermodynamic properties depend on latitude, the nature of the surface below and on the season. This thesis is primarily a mathematical study and, as customary in such studies, the ‘idealness’ of the fluid (air) is assumed to be ‘fairly high’, allowing analytical and numerical solutions.

Real gas effects can be incorporated into the governing equations by appealing to the ‘Kinetic Theory of Gases’ or to the more general theory of ‘Statistical Mechanics’. The significant change in our governing equations would be a modification of the gas law, (2.1.2d), especially when the boundary layer is hot. Also one would obtain better models for the specific heats, viscosity and thermal diffusivity. For a much fuller discussion the reader is referred to the plethora of books concerned with the ‘Kinetic Theory of Gases’ (i.e. Jeans, 1940; Loeb, 1934; Chapman & Cowling, 1970 - see also the paper by Lighthill, 1957), whilst many books on viscous flow theory (i.e. Pai, 1956; Schlichting, 1960; Stewartson, 1964) contain very good introductory accounts.

The advent of super-computers coupled with ever increasing sophisticated numerical techniques, the exhaustion of ideal-gas problems and interest in real gas effects as a new source of problems for researchers and/or because they are genuinely felt necessary for a more realistic theory, the desire to build/improve supersonic and hypersonic vehicles, and other reasons, is leading to significant advances towards more realistic theories and thus a better theoretical understanding

of the physical processes governing the transition of a laminar flow to a turbulent state.

§2.1.4 The ratio of specific heats and the Prandtl number.

In this study we are assuming that physical quantities such as the specific heat capacities, c_p and c_v , at constant pressure and constant volume respectively, and their ratio

$$\gamma = \frac{c_p}{c_v}, \quad (2.1.4)$$

are constant. By this we mean that although we may choose different ‘constant’ values for γ , for different flow situations (see later chapters), the variation of γ with temperature (i.e normal distance from plate) is small enough to be neglected in our mathematical formulation. The same is true for the third parameter of compressible flow, the Prandtl number

$$Pr = \frac{\mu^* c_p}{k^*}, \quad (2.1.5)$$

where k is the thermal diffusivity; note that we are evaluating it using the free-stream values of the relevant quantities. The Prandtl number is a measure of the ratio of heat conduction and viscous stress mechanisms at play in the boundary layer.

The constant values of γ and Pr need to be chosen as they cannot be scaled out of the governing equations. What values should be chosen? Once values have been chosen we effectively have lost the powerful generality of our results. Such a situation does not arise in the corresponding incompressible theory.

On fifth page of his book, Stewartson (1964) writes

“...the value of the constant [Pr] is a function of the paper quoted ...”,

which, in the opinion of the present author, is still appropriate today. The same can be said for γ . Moreover there appears to be reluctance by some authors to explicitly state the actual values they have used; whereas some authors seems to choose values just to be different from, or even perhaps, in an attempt to outdo, fellow researchers; whilst others use ‘actual’ data from experimental measurements in their numerical codes. We choose, for the time being, the values $\gamma = 1.4$ and $Pr = 0.72$, these being the most commonly used for air - however in much of the

theory that follows it is advantageous to take the ‘so-called’ model value, that is $Pr = 1$, leading to much simplified analysis and numerics. The value of γ is known to decrease as the temperature increases (see Stewartson, 1964, page 9) and we make use of this fact in later chapters.

§2.1.5 The constitutive relationship between viscosity and temperature.

Apart from boundary conditions, the governing equations (2.1.2a-d) need to be closed by an additional relation expressing the viscosity in terms of the other thermodynamic quantities. It is generally assumed that the viscosity is solely dependent on the temperature and the best constitutive formula (for an ideal gas away from very low temperatures) is

$$\mu = \left(\frac{1 + S}{T + S} \right) T^{\frac{3}{2}}, \quad S = \frac{\bar{C}}{T_{\infty}^*}, \quad (2.1.6a, b)$$

known as Sutherland’s formula (Sutherland, 1893 -for a discussion of his theory see, for example, Loeb, 1934; Jeans, 1940; Chapman & Cowling, 1970). This formula takes into account that molecules, as well as being attractors of one another at short distances, have impenetrable hard kernels (centres). Here $\bar{C}$ is another constant whose quoted values, again, vary from author to author: Stewartson notes that $\bar{C}$ is about 110°K whereas the value of 117°K is used by Rosenhead *et al.* Note also the dependence of S on T_{∞}^* , the free-stream temperature- this effectively means that T_{∞}^* cannot be scaled out of our governing system of equations if Sutherland’s formula is employed. Moreover it means that we have to decide, as theoreticians, what value of T_{∞}^* to choose for our calculations, leading to a further loss of generality. We consider the choice of T_{∞}^* in the next subsection.

There are two simpler viscosity-temperature relations that are commonly used. The first are the so-called ‘power laws’

$$\mu \propto T^{\omega} \quad (2.1.7)$$

where $0.5 < \omega < 1.0$. These have been shown to be very good interpolation formulae for moderate temperature variations. Note that for, $T \gg 1$, Sutherland’s formula has a power-law form, at leading order, with $\omega = \frac{1}{2}$. The second, even simpler, relation is known as Chapman’s law

$$\mu = CT \quad (2.1.8)$$

where C is another constant. Recently the rationality of such an assumption has been questioned (for example by Prof. F.T. Smith, 1987; and more critically by Prof. P. Hall, 1989, during independent, private discussions with the author): the constant C can only match at *one* of the two boundaries (the plate and the freestream). Denoting the (nondimensionalised) temperature and shear viscosity at the wall (plate) by T_w and μ_w , respectively, then C is generally ‘evaluated at the wall’, that is

$$C = \frac{\mu_w}{T_w}, \quad (2.1.9)$$

where T_w is known (theoretically or experimentally) and $\mu_w = \mu_w(T_w)$ follows from Sutherland’s formula, for instance.

At best, Chapman’s law is a reasonable interpolation formula over small temperature variations. Despite the fact that it is not a rational approximation to Sutherland’s formula, particularly for large temperature variations, it is commonly used by theoreticians, as a model, due to its linear form resulting in simpler equations. In this thesis Sutherland’s formula is taken to relate viscosity to temperature as there are often large temperature variations. We shall see how, by evaluating the constant C at the plate using Sutherland’s formula, compressible triple-deck theory, conventionally formulated using Chapmans law (see for instance, Stewartson & Williams, 1969; Stewartson, 1974; Smith, 1989), can be modified to include the effect of large wall temperatures for large M_∞ . The first two papers contain the same sentence suggesting that the scalings given therein should generalise easily to account for whatever viscosity law might be chosen; we show that this is indeed the case, later in this chapter.

§2.1.6 The free-stream temperature: what value should be chosen?

In the last sub-section it was noted that if one chooses to employ Sutherland’s formula to relate the viscosity to the temperature then the free-stream temperature T_∞^* cannot be scaled out of the resulting theory. Thus a value for it must be chosen if one wants quantitative results. There appear to be three sensible approaches in his choice. Firstly, one may want to compare a new approach with existing results and so one would choose T_∞^* appropriately (that is if a value has been quoted in the latter); secondly, one may want to correlate theory with experiments (performed

in a wind tunnel, say) and thus T_{∞}^* (and the other constants γ and Pr) should be chosen to relate to experimental conditions; lastly, one may simply wish to use a typical physical value of T_{∞}^* in the atmosphere. In fact this approach turns out to be not as simple as it sounds and as there is no discussion of this point, to the author's knowledge, to date in the immediate (compressible flow) literature, we spend a little time discussing the temperature profile of the atmosphere.

The concept of a 'standard atmosphere' could be appealed to to supply the desired information regarding the temperatures in the atmosphere. The basis of these models (see for example Rosenhead *et al*, 1952, page 124; Pai, 1956, page 15.) is that, in the lower atmosphere, the air temperature decreases linearly with altitude until a level is reached where the temperature remains constant at about 217°K. There are problems though: there are only very simplistic models; the available data may be out-of-date and the actual model values differ slightly from source to source.

At this juncture, it is worth remembering that in thermodynamics the temperature must be 'absolute', that is such that its zero point corresponds to the state of no internal energy of the constituent atoms and molecules. In addition, the use of 'imperial' (British) and 'metric' units for measurement has resulted in no less than four commonly quoted units: degrees Celcius, Fahrenheit, Kelvin and Rankine. If we denote by C, F, K, R the (same) temperature measured in the above units, respectively, then the conversion formulae are given by

$$K = \begin{cases} C + 273.15, \\ 0.55556F + 255.37, \\ 0.55556R. \end{cases} \quad (2.1.10)$$

The present work is formulated with temperature measured in degrees Kelvin.

On the grounds that the 'standard atmosphere' models quoted in books on compressible, laminar boundary layer flow appear too simplistic and because they may also be a need to consider higher altitudes than those modelled, it was decided to turn to the literature on the physical nature of the atmosphere. The consultation of some of the several recent and interesting books and publications concerning atmospheric dynamics (for example, Wallace & Hobbs, 1977; Kellogg & Mead, 1980; Wells, 1986; McIntyre, 1990) highlights how idealistic the notion of a 'standard atmosphere' really is. In these references the 'standard atmosphere' is

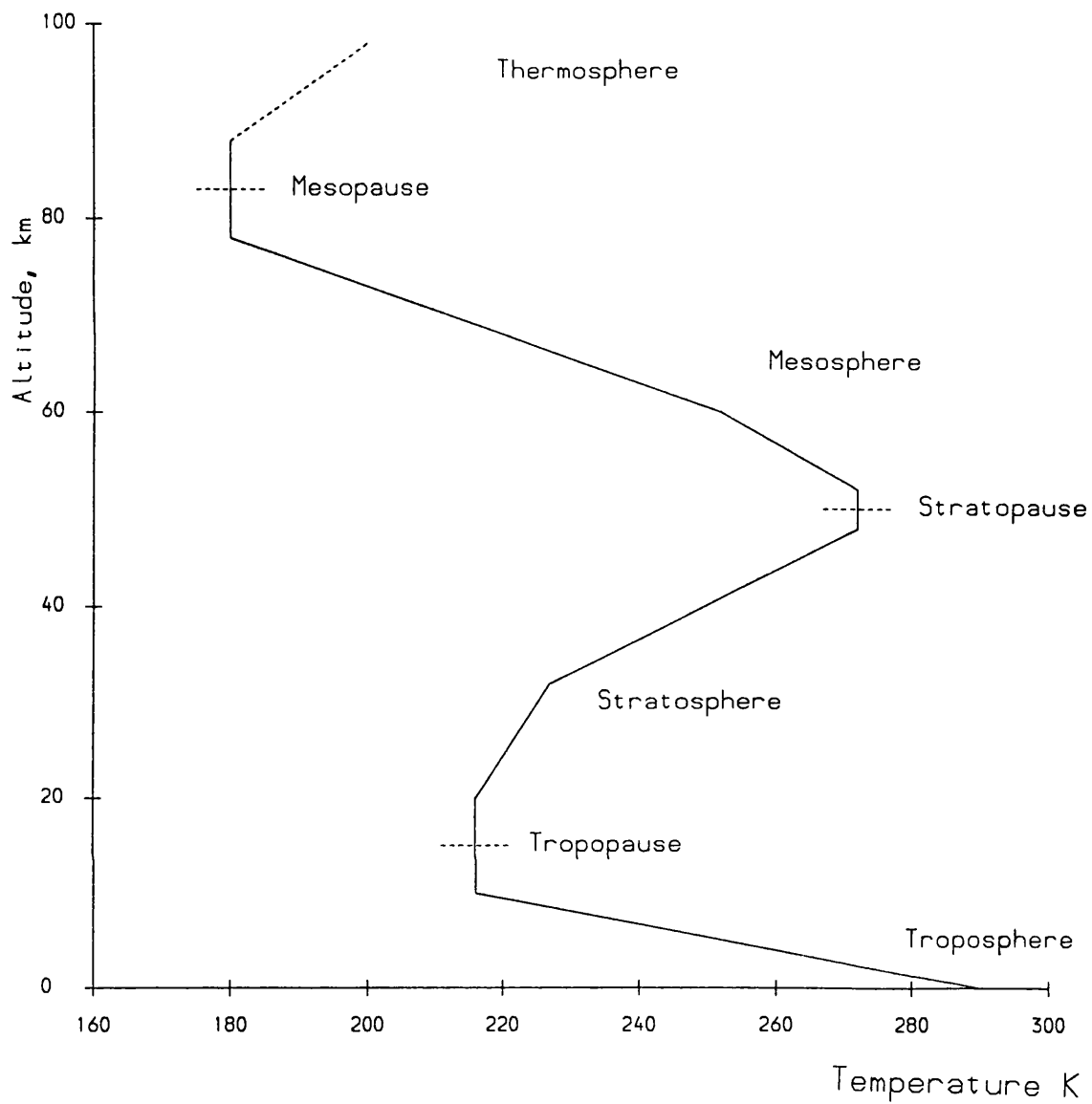


Figure 2.1. The vertical temperature profile of the United States' 'standard atmosphere'†.

† The author is grateful to Dr. P. Lewis for her help in producing this figure.

considered as consisting of four layers (see Figure 2.1), each having its own distinctive properties. It is clear that the choice of T_{∞}^* , to be used in the theories, clearly depends on whereabouts in the atmosphere we are concerned with. Supersonic aircraft generally fly in the stratosphere, above the troposphere: the main residence of water vapour and ice particles (clouds) in the atmosphere. The lower third of the stratosphere is very stable in the sense that there is very little vertical mixing. It is here where the temperature remains constant at about 217°K, the value quoted earlier from the simpler model found in some compressible-viscous-laminar-flow literature.

The lack of mixing referred to above results in the stratosphere acting as a 'reservoir' for certain types of atmospheric pollution. Here the residence time of aerosols is of the order of 1-2 years, compared with about 10 days in the troposphere adjacent to the Earth's surface. The origin of such pollution varies eg. (i) dust from volcanic eruptions (ii) debris from past nuclear explosions (iii) exhaust fumes from high-flying supersonic aircraft. The latter may initiate photo-chemical reactions resulting in the reduction of the high ozone (O_3 - a tri-atomic gas) concentration in the stratosphere which screens the Earth's surface from harmful ultra-violet radiation (although we note that there are probably more significant culprits of ozone depletion such as, for example, chloro-fluoro-carbons and other harmful chemical re-agents and catalysts currently being released/discharged into the atmosphere from some land-based industrial plants and waste-disposal sites). Above this stable layer we see that the temperature fluctuates widely with altitude. In addition to the dependence of temperature on altitude, it also depends on several other factors such as altitude, nature of the Earth's surface below, season and year.

Summarising, the choice of T_{∞}^* needs careful consideration. The notion of global (general) quantitative results, from one choice, is not sensible (a remark also made by Stewartson, 1964). The above brief review of the dynamics of the atmosphere has also shown that a study of real gas effects should really be generalised to include 'real atmosphere effects'. Finally we note that the aerosols (pollutants) mentioned above could trigger transition (perturb the laminar flow critically) on vehicles flying in the stratosphere. At lower altitudes (in the troposphere) there is

also the additional factors of water vapour (leading to the formation of ice on aircraft bodies) as well as the, generally unavoidable, build up of insect debris on the aircraft surface during take-off and landing. Here we are touching on the theories of roughness effects and ‘receptivity’ for predicting/understanding the transition from laminar flow to a turbulent state. These interesting and important theories are beyond the scope of this thesis in which it is assumed that infinitesimally small disturbances are (initially) present and we do not concern ourselves on their origin.

§2.2 NON-INTERACTIVE STEADY FLOWS

We now turn to the mathematical problem of solving the governing equations for a steady laminar boundary layer in a compressible fluid. Note that we are assuming that $Re \gg 1$, but that $M_\infty \sim O(1)$ at present. The similarity solution and its large Mach number properties are investigated in more detail in Chapter 6, where a slightly different notation is used.

§2.2.1 The similarity solution

The boundary layer equations can be recovered by first substituting

$$\zeta = Re^{\frac{1}{2}} \int_0^y \rho dy, \quad v = Re^{-\frac{1}{2}} V, \quad (2.2.1a, b)$$

where the Dorodnitsyn-Howarth variable, ζ , is introduced for convenience, and then taking the limit $Re \rightarrow \infty$.

For steady two-dimensional flow over a flat plate, a similarity solution to these equations exists†. With

$$\eta = \frac{\zeta}{\sqrt{2x}}, \quad u = \psi_\zeta, \quad \rho V = -(\psi_x + \zeta_x \psi_\zeta),$$

$$\psi = \sqrt{2x} f(\eta), \quad T \equiv T(\eta), \quad \rho \equiv \rho(\eta), \quad \mu \equiv \mu(\eta), \quad p = \frac{1}{\gamma M_\infty^2}, \quad (2.2.5a - h)$$

the governing similarity equations are found to be

$$\rho T = 1 \equiv \gamma M_\infty^2 p_\infty, \quad (2.2.3a)$$

$$f f_{\eta\eta} + \left(\frac{\mu}{T} f_{\eta\eta} \right)_\eta = 0, \quad (2.3.6b)$$

† if there is no external pressure-gradient and the boundary conditions are independent of $\mathbf{x}$.

$$fT_\eta + \frac{1}{Pr} \left(\frac{\mu}{T} T_\eta \right)_\eta + (\gamma - 1) M_\infty^2 \frac{\mu}{T} f_{\eta\eta}^2 = 0, \quad (2.2.3c)$$

$$\mu = \begin{cases} CT & \text{- Chapmans' law} \\ \left(\frac{1+S}{T+S} \right) T^{\frac{3}{2}} & \text{- Sutherland's formula} \end{cases} \quad (2.2.3d, e)$$

subject to the boundary conditions

$$f(0) = f_\eta(0) = 0, \quad f_\eta(\infty) = T(\infty) = 1, \quad (2.2.3f)$$

and

$$T(0) = T_w \text{ (fixed wall - temperature), or } T_\eta(0) = 0 \text{ (insulated wall)}. \quad (2.2.3g)$$

The form of (2.6a) is a result that, for a non-interactive steady flow (no-shock), the normal gradient of the pressure is constant, i.e the pressure takes its non-dimensionalised free-stream value.

§2.2.2 Wall shear of base flow: the model Chapman-fluid

The joint assumption that the Prandtl number is unity and that Chapman's viscosity law holds is popular in theoretical studies of laminar boundary layers in compressible fluids. These lead to the simplifications

$$ff_{\eta\eta} + Cf_{\eta\eta\eta} = 0, \quad (2.2.4a)$$

$$T = 1 + [(n - 1) + \frac{1}{2}(\gamma - 1)M_\infty^2(n + f_\eta)(1 - f_\eta)], \quad (2.2.4b)$$

where

$$n = \frac{T_w}{T_{ins}}, \quad T_{ins} = 1 + \frac{(\gamma - 1)}{2} M_\infty^2. \quad (2.2.5a, b)$$

The quantity T_{ins} is the wall temperature for the the case of an insulated wall and the fraction n is a measure of heat transfer at the wall. An insulated wall corresponds to $n = 1$, when

$$T = 1 + \frac{(\gamma - 1)}{2} M_\infty^2 (1 - f_\eta^2), \quad (2.2.6)$$

whilst $n < 1$ corresponds to wall cooling.

Blasius' equation, so familiar in the theory of incompressible flow, can be recovered by writing

$$f = \sqrt{\frac{C}{2}} f_B(\eta_B), \quad \eta_B = \sqrt{\frac{2}{C}} \eta \quad (2.2.7)$$

so that f_B satisfies

$$\frac{1}{2} f_B f_B' + f_B''' = 0, \quad f_B(0) = f_B'(0) = 0, \quad f_B' = 1. \quad (2.2.8a - d)$$

Recalling the well-known properties of the Blasius function

$$f_B(\eta_B) = \frac{\hat{\lambda}_B \eta_B^2}{2} + \dots, \quad \hat{\lambda}_B \simeq 0.3221\dots, \quad (2.2.9a, b)$$

for $\eta_B \ll 1$, we see that the shear at the wall of the basic flow is

$$\begin{aligned} u_y|_{y=0} &= \left(f_{\eta\eta} \cdot \frac{d\eta}{d\zeta} \cdot \frac{d\zeta}{dy} \right) \Big|_{y=0} \\ &= \hat{\lambda}_B \rho_w \sqrt{\frac{Re}{Cx}} \\ &= \frac{\lambda_B}{T_w} \sqrt{\frac{Re}{C}}. \end{aligned} \quad (2.2.10)$$

where we have written $\lambda_B = \hat{\lambda}_B x^{-\frac{1}{2}}$.

Note that if we relax our assumptions to allow for general Pr, γ and n values the effect is *only* felt through T_w in the above expression for the wall-shear.

§2.2.3 Wall shear of base flow: Sutherland-fluid

We now suppose that the viscosity is related to the temperature by the more realistic Sutherland's law. We make no assumptions on the values of S, γ, Pr, M_∞ and wall cooling coefficient n . The temperature T appears in the equation for f and so we have to solve two coupled (fifth-order) ordinary differential equations to obtain the f and T profiles, for *each* choice of the parameters $(Pr, \gamma, S, n, M_\infty)$.

Recall that we require $f(0) = f_\eta(0) = 0$; defining

$$\hat{\lambda}_S = f_{\eta\eta}(0), \quad (2.2.11a)$$

we see that

$$f(\eta) = \frac{\hat{\lambda}_S}{2} \eta^2 + \dots, \quad \eta \ll 1. \quad (2.2.11b)$$

The first crucial point to note here is that

$$\hat{\lambda}_S = \hat{\lambda}_S(Pr, \gamma, S, n, M_\infty)$$

in contrast to the simple value of $\hat{\lambda}_B$ for Chapman's law. Thus for the present case of a Sutherland-fluid, the wall-shear of the base flow is now

$$\begin{aligned} u_y|_{y=0} &= \frac{\hat{\lambda}_S}{T_w} \sqrt{\frac{Re}{2x}} \\ &= \frac{\lambda_S}{T_w} \sqrt{\frac{Re}{2}} \end{aligned} \quad (2.2.13)$$

where $\lambda_S = \hat{\lambda}_S x^{-\frac{1}{2}}$.

The second crucial point is that C is usually treated to be $O(1)$, but for large Mach number (see later)

$$\hat{\lambda}_S \sim \hat{\lambda}_S^H M_\infty^{\frac{1}{2}} + \dots, \quad \hat{\lambda}_S^H \sim O(1). \quad (2.2.14)$$

The same resulting $M_\infty^{\frac{1}{2}}$ factor to the wall-shear can be obtained, in an 'ad-hoc fashion', from the Chapman-formulation by evaluating C at the wall: ^{we} will see later that $T_w \sim M_\infty^2$ so that $C = \frac{\mu_w}{T_w} \sim T_w^{-\frac{1}{2}} \sim M_\infty^{-1}$, and hence the result.

It is important to note that the two expressions for the wall shear cannot be equated however C is calculated: the first expression has been derived from assuming Chapman's law holds everywhere and so ^{correctly evaluating C} ~~at~~ at the wall does not make it right, despite the fact that such a ~~fix~~ is sufficient to capture the 'missing' $M_\infty^{\frac{1}{2}}$ factor for large M_∞ . It is important to remember that $\hat{\lambda}_S$ bears no relation to $\hat{\lambda}_B$: in the next section we give new triple deck scalings that scale out the *correct* wall shear, not some 'fixed-up' approximation of it based on Chapman's law everywhere apart from one position. Note the mathematical symmetry

$$\frac{\lambda_B}{\sqrt{C}} \rightarrow \frac{\lambda_S}{\sqrt{2}}, \quad (2.2.15)$$

of the two expressions for the wall-shear. This replacement is used in the next section (to remove λ_B) when we show how the new Sutherland-scalings can be simply picked out from the Chapman law ones. This transformation does not completely remove C from the scalings, it simply removes that due to the wrong wall shear being used; the remaining C factors result from scaling the ratio $\frac{\mu_w}{\rho_w}$ out of the lower deck equations: see later.

The incompressible limit has, in particular, no temperature variation across the boundary layer and we can easily see that then the two approaches are equivalent: they both give $\frac{\mu}{T} \equiv 1$ so that Blasius' equation is obtained from both viscosity-temperature relations.

§2.3 COMPRESSIBLE TRIPLE DECK THEORY

The triple-deck structure has been discussed in the introduction. Here we pay particular attention on the compressible version. In past years there was more interest in supersonic than incompressible flows. In the following subsection we review the literature; beginning with a brief overview and then go on to discuss particular papers in more detail.

§2.3.1 Significant advances in the theory.

The advent of quicker aircraft resulted in the desire to understand why a supersonic boundary layer separated, despite no external influence (ie. *before* the impingement of a shock on the boundary layer). Significant progress was made by Lighthill (1950, 1953), but more than fifteen years passed before Neiland (1969) and Stewartson & Williams (1969) independently showed how these ideas could form the basis of a rational solution to self-induced separation- the (supersonic) triple-deck; tribute must also be paid to Messiter (1970) who also, independently, discovered the triple deck structure (for the case of incompressible flow; also addressed by Stewartson, 1969). There then followed much activity based on the new, revolutionally structure, principally by Stewartson and his co-workers throughout the world. These studies are reviewed by Stewartson (1974). There then appears to have been a slow down in such studies, in favour of incompressible studies of flow through pipes and channels, etc, to see if the new theory could shed light on problems that the old theory could not - it could be argued that such problems are more appealing to researchers because of their possible physiological applications. In addition, there was much interest in the properties of critical layers and in fitting an asymptotic description to the upper-branch of the Orr-Sommerfeld neutral curve to complement the triple-deck theory describing the lower branch.

This was followed by interest in the nonlinear evolution of disturbances and, recently, interactions involving waves and vortices. The paper by Smith (1989) has now regenerated interest in compressible triple deck theory.

We now give a few more details of some of the studies mentioned above and mention a few more. Lighthill (1950) showed that a purely inviscid theory is inadequate to explain the phenomenon of self-induced separation and shortly afterwards he introduced the notion of an internal viscous layer (Lighthill, 1953). This was the crucial step towards understanding the phenomenon and is generally regarded as a classic work. The full implications of this paper were finally realised, independently, by Neiland (1969) and Stewartson & Williams (1969). The former is written in Russian and although an English translation is available, we shall concentrate our discussion on the latter principally as it is more readily accessible (outside the Soviet Union).

The paper by Stewartson & Williams is another classic and a personal favourite. It begins with a full review of related studies up to that time, continues by deriving the two-dimensional, steady supersonic triple deck scalings and nonlinear equations (the consistency of the approach is established *a posteriori*), before discussing their numerical method and solutions. The paper concludes with a comparison of the new results with those of previous studies. The review by Stewartson (1974) gives a different presentation, as well as reviewing the large amount of research work that resulted from the original paper. A year later Brown, Stewartson & Williams (1975) investigated the hypersonic-boundary-layer-flow, occurring immediately behind a shock, to link together the known pressure-displacement laws of supersonic and hypersonic flows. This paper, like many others of that era, use Chapman's viscosity law even though the more realistic Sutherland's formula leads to a quite different formulation (see, for example, Bush, 1966). The assumption that γ is asymptotically close to unity has to be made.

The next notable contribution was the comprehensive review of 'high Reynolds number flow' theory by Smith (1982). The motivation of the Reynolds-number-powers found in the triple deck scales (common to incompressible and compressible theory) is discussed in Section 3 of this review, whilst Section 4 contains an overview of previous work on compressible boundary-layer flows. Another review, by Ryzhov (1984), contains a note concerning the necessary oblique-ness of neutral

Tollmien-Schlichting waves as the Mach number increases (see Zhuk & Ryzhov, 1981). It also provides many, other helpful references to work by researchers in the Soviet Union. In addition the paper considers the unsteady, three-dimensional triple-deck, later studied by Smith (1989). The latter considers eigenrelations, resulting from a linear stability analysis, and their consequences on stability. A significant result is that non-parallel effects are important for large Mach numbers when, at the same time, the whole triple deck structure collapses. Related work was carried out independently by Duck (1990) who also investigated numerically the nonlinear development of the Tollmien-Schlichting waves.

The last-quoted paper by Smith has been a catalyst for several related studies, including some of the present work presented in this thesis. Such studies include an extensive investigation of the transonic regime (Bowles & Smith, 1989); axisymmetric flows (Duck & Hall, 1989, 1990); hypersonic flow over wedge with shock fitted into the upper-deck (Cowley & Hall, 1990); effects of wall cooling on stability properties (Seddougui, Bowles & Smith, 1989); and the asymptotic description of compressible upper-branch modes (Gajjar & Cole, 1989).

§2.3.2 The Sutherland-fluid triple-deck scales.

In previous sections we saw that the expression for the wall-shear of the basic flow was modified by the use of Sutherland's formula rather than Chapman's law. This will have to be incorporated into the triple-deck scales.

The wall-shear does not account for all of the C -factor appearing in each of the Chapman-law-scalings; the remaining C -factors are due to the rescaling of the momentum equations of the lower deck so that the coefficient of the viscous term is unity. The thinness of the lower-deck, adjacent to the wall, results in the temperature, density and viscosity there all being effectively constant (at leading order), taking their wall values - see Stewartson & Williams (1969, page 191). In this thin layer, the equations appear incompressible apart from coefficients involving ρ_w and μ_w which can be, and are, scaled out. For example, the x -momentum equation has the form

$$\frac{\partial u}{\partial t} = u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho_w} \frac{\partial p}{\partial x} + \frac{\nu_w}{Re} \frac{\partial^2 u}{\partial y^2}. \quad (2.3.1)$$

Here $\nu_w = \frac{\mu_w}{\rho_w}$ is the wall value of the kinematic viscosity- it is this quantity that leads to the additional C - factors mentioned above. We consider this quantity to see how the use of Sutherland's alters things. In fact

$$\nu_w \equiv \frac{\mu_w}{\rho_w} = \mu_w T_w = \begin{cases} CT_w^2 & \text{-Chapman's law} \\ \left(\frac{1+S}{T_w+S} \right) T_w^{\frac{5}{2}} & \text{-Sutherland's formula} \end{cases} \quad (2.3.2a, b)$$

which suggests the replacement

$$C \rightarrow \left(\frac{1+S}{T_w+S} \right) T_w^{\frac{1}{2}} \equiv \frac{\mu_w}{T_w}, \quad (2.3.3)$$

for the remaining C -factors in the scalings, to yield the required Sutherland-law-scalings. Alternatively they could be derived from 'first principles' following the method of Stewartson & Williams (1969). Note that (2.2.3) merely means that we are evaluating the Chapman constant at the wall via Sutherlands formula- this can be thought of as defining the value of C .

As an example, we consider the streamwise 'short' x -scale. The scales for a *general* viscosity law are given by Stewartson & Williams, although they then immediately chose Chapman's viscosity law for definiteness. The scalings corresponding to the latter have generally been used ever since; their generalisation to the unsteady, three-dimensional case can be found, for instance, in Smith (1989). In the last paper, the x - scaling is written in the form

$$x - x_0 = Re^{-\frac{3}{8}} K_1 X, \quad (2.3.4)$$

where the scaling K_1 , with respect to our notation and non-dimensionalisation, has the value

$$\begin{aligned} K_1 &= \frac{C^{\frac{3}{8}} T_w^{\frac{3}{2}}}{\lambda_B^{\frac{5}{4}} (M_\infty^2 - 1)^{\frac{3}{8}}} \\ &\equiv \left(\frac{\lambda_B}{\sqrt{C}} \right)^{-\frac{5}{4}} \cdot C^{-\frac{1}{4}} \cdot \frac{T_w^{\frac{3}{2}}}{(M_\infty^2 - 1)^{\frac{3}{8}}} \end{aligned} \quad (2.3.5)$$

which, making the replacements described above, transforms to

$$\begin{aligned} &\left(\frac{\lambda_S}{\sqrt{2}} \right)^{-\frac{5}{4}} \cdot \left(\frac{\mu_w}{T_w} \right)^{-\frac{1}{4}} \cdot \frac{T_w^{\frac{3}{2}}}{(M_\infty^2 - 1)^{\frac{3}{8}}} \\ &\equiv \left(\frac{\lambda_S}{\sqrt{2}} \right)^{-\frac{5}{4}} \mu_w^{-\frac{1}{4}} T_w^{\frac{7}{4}} (M_\infty^2 - 1)^{-\frac{3}{8}} \end{aligned} \quad (2.3.6)$$

The other scalings can be similarly transformed and the complete set follows, starting with the length and time scales (common to all three decks)

$$[x - x_0, z - z_0] = Re^{-\frac{3}{8}} K_1[X, Z], \quad (2.3.7a, b)$$

$$K_1 = \left(\frac{\lambda_S}{\sqrt{2}} \right)^{-\frac{5}{4}} \mu_w^{-\frac{1}{4}} T_w^{\frac{7}{4}} (M_\infty^2 - 1)^{-\frac{3}{8}}, \quad (2.3.7c)$$

$$t = Re^{-\frac{1}{4}} \left(\frac{\lambda_S}{\sqrt{2}} \right)^{-\frac{3}{2}} \mu_w^{-\frac{1}{2}} T_w^{\frac{3}{2}} (M_\infty^2 - 1)^{-\frac{1}{4}} \hat{t}, \quad (2.3.7d)$$

where (x_0, z_0) corresponds to the location of the initial disturbance of the laminar base-flow. In the viscous sublayer, or lower deck,

$$y = Re^{-\frac{5}{8}} \left(\frac{\lambda_S}{\sqrt{2}} \right)^{-\frac{3}{4}} \mu_w^{\frac{1}{4}} T_w^{\frac{5}{4}} (M_\infty^2 - 1)^{-\frac{1}{8}} Y, \quad (2.3.8a)$$

and to leading order

$$u = Re^{-\frac{1}{8}} \left(\frac{\lambda_S}{\sqrt{2}} \right)^{\frac{1}{4}} \mu_w^{\frac{1}{4}} T_w^{\frac{1}{2}} (M_\infty^2 - 1)^{-\frac{1}{8}} U, \quad (2.3.8b)$$

$$v = Re^{-\frac{3}{8}} \left(\frac{\lambda_S}{\sqrt{2}} \right)^{\frac{3}{4}} \mu_w^{\frac{3}{4}} T_w^{-\frac{1}{4}} (M_\infty^2 - 1)^{-\frac{1}{8}} V, \quad (2.3.8c)$$

$$w = Re^{-\frac{1}{8}} \left(\frac{\lambda_S}{\sqrt{2}} \right)^{\frac{1}{4}} \mu_w^{\frac{1}{4}} T_w^{\frac{1}{2}} (M_\infty^2 - 1)^{-\frac{1}{8}} W, \quad (2.3.8d)$$

$$p - p_\infty = Re^{-\frac{1}{4}} \left(\frac{\lambda_S}{\sqrt{2}} \right)^{\frac{1}{2}} \mu_w^{\frac{1}{2}} T_w^{-\frac{1}{2}} (M_\infty^2 - 1)^{-\frac{1}{4}} P. \quad (2.3.8d)$$

The thinness of this viscous sub-layer results in the density being effectively constant, taking it's wall value at leading order,

$$\rho = \rho_w + \begin{cases} O(Re^{-\frac{1}{8}}) & : \text{Fixed wall temperature} \\ O(Re^{-\frac{1}{4}}) & : \text{Insulated wall,} \end{cases} \quad (2.3.8e)$$

where the size of the correction term is implied from the limiting form of the main deck solution (see later discussion). Recall that the scalings have been introduced to normalize the resultant governing equations, which from the Navier-Stokes equations are

$$U_X + V_Y + W_Z = 0, \quad (2.3.9a)$$

$$U_T + UU_X + VU_Y + WU_Z = -P_X + U_{YY}, \quad (2.3.9b)$$

$$W_T + UW_X + VW_Y + WW_Z = -P_Z + W_{YY}, \quad (2.3.9c)$$

in the lower deck, with the y -momentum equation yielding

$$P_Y = 0. \quad (2.3.9d)$$

Note that these are merely the unsteady, 3-D incompressible boundary-layer equations. The principal boundary conditions are

$$U = V = W = 0 \quad \text{at} \quad Y = 0, \quad (2.3.9e)$$

$$U \sim Y + A(X, Z, T), \quad W \rightarrow O(Y^{-1}), \quad \text{as} \quad Y \rightarrow \infty, \quad (2.3.9f, g)$$

for no slip at the solid surface and for matching with the main deck, $-A$ representing the unknown relative displacement. The main deck has

$$y = Re^{-\frac{1}{2}} \mu_w^{\frac{1}{2}} T_w^{\frac{1}{2}} \tilde{y}, \quad (2.3.10)$$

and merely transmits small displacement effects across the boundary layer as well as smoothing out the induced velocity component w , in the form

$$\begin{aligned} u &= U_0(\tilde{y}) + Re^{-\frac{1}{8}} \left(\frac{\lambda_S}{\sqrt{2}} \right)^{-\frac{3}{4}} \mu_w^{-\frac{1}{4}} T_w^{\frac{3}{4}} (M_\infty^2 - 1)^{-\frac{1}{8}} AU'_0(\tilde{y}) + \dots, \\ v &= -Re^{-\frac{1}{4}} \left(\frac{\lambda_S}{\sqrt{2}} \right)^{\frac{1}{2}} \mu_w^{\frac{1}{2}} T_w^{-\frac{1}{2}} (M_\infty^2 - 1)^{\frac{1}{4}} A_X U_0(\tilde{y}) + \dots, \\ w &= Re^{-\frac{1}{4}} \left(\frac{\lambda_S}{\sqrt{2}} \right)^{\frac{1}{2}} \mu_w^{\frac{1}{2}} T_w^{\frac{1}{2}} (M_\infty^2 - 1)^{-\frac{1}{4}} DR_0(0)/(R_0(\tilde{y})U_0(\tilde{y})) + \dots, \\ p - p_\infty &= Re^{-\frac{1}{4}} \left(\frac{\lambda_S}{\sqrt{2}} \right)^{\frac{1}{2}} \mu_w^{\frac{1}{2}} T_w^{-\frac{1}{2}} (M_\infty^2 - 1)^{-\frac{1}{4}} P(X, Z, T) + \dots, \\ \rho &= R_0(\tilde{y}) + Re^{-\frac{1}{8}} \left(\frac{\lambda_S}{\sqrt{2}} \right)^{-\frac{3}{4}} \mu_w^{-\frac{1}{4}} T_w^{\frac{3}{4}} (M_\infty^2 - 1)^{-\frac{1}{8}} AR'_0(\tilde{y}) + \dots. \end{aligned} \quad (2.3.11a - e)$$

Here $U_0(\tilde{y})$, $R_0(\tilde{y})$ are the streamwise-velocity and density base-flow profiles, respectively, of the steady, non-interactive boundary-layer. Note that for an insulated wall (i.e. no external cooling) $R'_0(0) = 0$, so that there is no need for the density

disturbance in the lower-deck to be as large as $O(Re^{-\frac{1}{8}})$. However if the wall is kept at a fixed temperature then $R'_0(0) \neq 0$ and so the disturbance size in the lower-deck must be $O(Re^{-\frac{1}{8}})$, the same as the streamwise velocity u -disturbance (Stewartson, 1974). Finally, note that $R_0(0) \equiv \rho_w$.

The unknown function satisfies $D_X = -P_Z$ from the spanwise momentum balance and (2.3.9g) shows the jet-like response in the cross-flow due to the spanwise variation in the pressure, the velocity w reaching its maximum amplitude inside the lower deck. This feature is crucial to understanding the interaction between two oblique Tollmien-Schlichting modes and a longitudinal vortex (Hall & Smith, 1989; see also next chapter).

The third, upper, deck then occurs where

$$y = Re^{-\frac{3}{8}} K_1 (M_\infty^2 - 1)^{-\frac{1}{2}} \bar{y} \quad (2.3.12)$$

and

$$[u, v, w, p] = [1, 0, 0, p_\infty] + Re^{-\frac{1}{4}} K_2 [\bar{u}^{(2)}, \bar{v}^{(2)} (M_\infty^2 - 1)^{\frac{1}{2}}, \bar{w}^{(2)}, \bar{p}^{(2)}] + \dots,$$

$$K_2 = \left(\frac{\lambda_S}{\sqrt{2}} \right)^{\frac{1}{2}} \mu_w^{\frac{1}{2}} T_w^{-\frac{1}{2}} (M_\infty^2 - 1)^{-\frac{1}{4}}, \quad (2.3.13a - e)$$

together with similar perturbations of the uniform density and temperature. These yield the supersonic potential-flow equation and main matching conditions, for zero incident wave,

$$(M_\infty^2 - 1) [\bar{p}_{XX}^{(2)} - \bar{p}_{\bar{y}\bar{y}}^{(2)}] - \bar{p}_{ZZ}^{(2)} = 0, \quad (2.3.14a)$$

$$\bar{p}^{(2)} \rightarrow 0 \quad \text{as} \quad \bar{y} \rightarrow \infty, \quad (2.3.14b)$$

$$\bar{p}^{(2)} \rightarrow P, \quad -\bar{p}_{\bar{y}}^{(2)} \rightarrow A_{XX}, \quad \text{as} \quad \bar{y} \rightarrow 0^+. \quad (2.3.14c, d)$$

Subject to suitable farfield conditions of boundedness, the nonlinear problem for U, V, W, P, A in (2.3.9a - e) is closed, therefore, by the pressure-displacement law, between P and A , implied by (2.3.14a - c) controlling $\bar{p}^{(2)}(X, \bar{y}, Z, T)$. This law can be expressed in the form of a double integral but the above formulation turns out to be more convenient.

§2.3.3 Linear stability: the eigenrelation for Tollmien–Schlichting waves.

We turn now to the linearized instability properties that were studied by Smith (1989) for the Chapman-scalings- this paper is followed closely, where possible. With a relatively small disturbance of order h and a normal-mode decomposition, so that

$$(U, V, W, P, A) = (Y, 0, 0, 0, 0) + \{h(\tilde{U}, \tilde{V}, \tilde{W}, \tilde{P}, \tilde{A})E + c.c.\} + O(h^2) \quad (2.3.15)$$

with “c.c.” denoting complex conjugate and

$$E = \exp[i(\alpha X + \beta Z - \Omega T)] \quad (2.3.16)$$

where α, β are the normalised wavenumbers and Ω is the normalised frequency, the governing equations (2.3.9a-c) reduce to

$$i\alpha\tilde{U} + \tilde{V}_Y + i\beta\tilde{W} = 0, \quad (2.3.17a)$$

$$-i\Omega\tilde{U} + i\alpha Y\tilde{U} + \tilde{V} = -i\alpha\tilde{P} + \tilde{U}_{YY}, \quad (2.3.17b)$$

$$-i\Omega\tilde{W} + i\alpha Y\tilde{W} = -i\beta\tilde{P} + \tilde{W}_{YY}, \quad (2.3.17c)$$

subject to

$$\tilde{U} = \tilde{V} = \tilde{W} = 0 \quad \text{at} \quad Y = 0, \quad (2.3.17d)$$

$$\tilde{U} \rightarrow \tilde{A}, \tilde{W} \rightarrow 0, \quad \text{as} \quad Y \rightarrow \infty. \quad (2.3.17e)$$

Here the supersonic interaction in (2.3.14a – c) yields

$$\tilde{P}^{(2)} = h\tilde{P} \exp[-\{\beta^2/(M_\infty^2 - 1) - \alpha^2\}^{\frac{1}{2}}\tilde{y}]E + c.c. \quad (2.3.18a)$$

provided†

$$\text{Real}\{\beta^2/(M_\infty^2 - 1) - \alpha^2\}^{\frac{1}{2}} > 0. \quad (2.3.18b)$$

Hence the displacement law between $\tilde{P}, \tilde{A}$ is

$$\{\beta^2/(M_\infty^2 - 1) - \alpha^2\}^{\frac{1}{2}}\tilde{P} = \alpha^2\tilde{A} \quad (2.3.18c)$$

† so that solutions decay as $\tilde{y} \rightarrow \infty$. If the inequality is reversed, the solutions are merely bounded and only stable Tollmien–Schlichting waves are possible — see discussion by Duck (1990).

from (2.3.14d). Following the standard analyses we obtain from (2.3.17a – c) the solution

$$(\alpha\tilde{U} + \beta\tilde{W})_Y = \tilde{B}Ai(\xi^T) : \xi^T = (i\alpha)^{\frac{1}{3}}Y + \xi_0^T, \quad \xi_0^T = -i^{\frac{1}{3}}\Omega/\alpha^{\frac{2}{3}} \quad (2.3.19a)$$

where Ai denotes the Airy function and $\tilde{B}$ is an unknown constant. The no-slip and displacement conditions require

$$\tilde{B}(i\alpha)^{\frac{1}{3}}Ai'(\xi_0^T) = i(\alpha^2 + \beta^2)\tilde{P} \quad \text{and} \quad \tilde{B}(i\alpha)^{-\frac{1}{3}}\kappa(\xi_0^T) = \alpha\tilde{A}, \quad (2.3.19b, c)$$

in turn, where

$$\kappa = \int_{\xi_0}^{\infty} Ai(q)dq.$$

The combination of (2.3.18c), (2.3.19b, c) yields the eigenrelation

$$(i\alpha)^{\frac{1}{3}}(\alpha^2 + \beta^2) = (Ai'/\kappa)(\xi_0^T) \left\{ \frac{\beta^2}{(M_{\infty}^2 - 1)} - \alpha^2 \right\}^{\frac{1}{2}}, \quad (2.3.20)$$

between α, β and Ω for the normal modes.

Spatial instability properties correspond to Ω, β kept real and α in general complex, whilst temporal instability corresponds to real α, β and complex Ω . The case of neutral stability, where all of α, β and Ω are real occurs for $\xi_0 = -d_1 i^{\frac{1}{3}}$ and $(Ai'/\kappa)(\xi_0) = d_2 i^{\frac{1}{3}}$ where

$$d_1 \simeq 2.2972 \quad \text{and} \quad d_2 \simeq 1.0006. \quad (2.3.21)$$

The dependence of these neutral conditions on β and M_{∞} is discussed by Smith (1989), who also considers the asymptotics of several limiting cases.

The constraint

$$\frac{\beta}{\alpha} > \sqrt{M_{\infty}^2 - 1} \quad (2.3.22)$$

holds for neutral or temporal instability/stability waves, in view of (2.10b), meaning that the directions of such waves lie outside the Mach cones at any particular point, ie.

$$\theta > \tan^{-1} \left[\sqrt{M_{\infty}^2 - 1} \right] \quad \text{where} \quad \tan(\theta) = \frac{\beta}{\alpha}. \quad (2.3.23a, b)$$

Note that $\theta = 0$ corresponds to propagation in the streamwise direction, whereas increased θ , due to the above restriction, means that the propagation direction

becomes more oblique. Thus neutral, supersonic Tollmien-Schlichting modes are necessarily three-dimensional in nature. This result appears in the review paper by Ryzhov (1984), and the earlier paper by Zhuk & Ryzhov (1981); it was also derived by Smith (1989) and found by Duck (1990)—see also Dunn & Lin (1955).

§2.3.4 The large Mach number limit

One of the limiting cases, of the eigen-relation (2.3.20), that Smith (1989) went on to investigate was the so-called *hypersonic limit* when M_∞ is assumed to be large. This case leads to some interesting consequences for the whole compressible triple-deck structure governing the first (Tollmien-Schlichting) - modes under consideration. For $M_\infty \gg 1$, he found that the main features revolve around the regime where

$$(\alpha, \beta, \Omega) \sim (M_\infty^{-\frac{3}{2}} \tilde{\alpha}, M_\infty^{-\frac{1}{2}} \tilde{\beta}, M_\infty^{-1} \tilde{\Omega}) + \dots, \quad (2.3.24)$$

where $\tilde{\alpha}, \tilde{\beta}$ and $\tilde{\Omega}$ are $O(1)$, and the eigenrelation reduces to, at leading order,

$$i^{\frac{1}{3}} \tilde{\alpha}^{\frac{1}{3}} \tilde{\beta}^2 = \left(\frac{Ai'}{\kappa} \right) (\tilde{\xi}_0^T) (\tilde{\beta}^2 - \tilde{\alpha}^2)^{\frac{1}{2}}, \quad \tilde{\xi}_0^T = -\frac{i^{\frac{1}{3}} \tilde{\Omega}}{\tilde{\alpha}^{\frac{2}{3}}}. \quad (2.3.25)$$

Before considering the consequences of the regime (2.3.24), we must return to the base, steady 2-D flow to investigate the dependency of the wall-values on the Mach number. For unity Prandtl number ($Pr = 1$) and moderate-to-no (external) wall cooling (so that $n \sim O(1)$), it can be seen from equation (2.2.4b)- which holds for both temperature-viscosity relations of concern- that $T_w \sim M_\infty^2$. Let us now consider arbitrary Prandtl number, but still assume that $n \sim O(1)$ (or more precisely, larger than any inverse power of Re or M_∞ necessary). The governing equations for f and T are (2.2.3). We see from the energy equation (2.2.3c) that, for $M_\infty \gg 1$ and $\gamma - 1 \sim O(1)$, we probably need to rescale T, μ, f, η . The boundary layer has been defined using the Reynolds number (2.2.1a,b) and now we must investigate how this should be divided into sub-layers in the asymptotic description for large Mach number. Such an investigation for the Chapman-law-case has recently been carried out simultaneously, and independently, by Smith & Brown (1989), Cowley & Hall (1990) and Balsa & Goldstein (1990)- the latter concerning the closely related problem involving shear layers ; see also Hall &

Fu (1989). We do not give many details as these papers are based on the model Chapman's law which we have seen is a very idealistic assumption to make when there are such temperature variations. The main result is that the majority of the boundary layer is hot ($T \sim M_\infty^2$) and that the temperature adjusts to its free-stream value in an asymptotically thin layer that is distinct from, and moreover asymptotically a long way from, the wall (plate). This is quite different from the boundary-layer structure that results from the more realistic Sutherland's law that was found/rediscovered by Professor P. Hall (1989- private discussion with the author) -see the 'forgotten' paper of Freeman & Lam (1959)[†]; and those of Luniev (1959) and Bush (1966) which concern the 'interactive' boundary layer (in which a shock alters the boundary layer structure beneath).

We now briefly go through the arguments for the Sutherland case (see Chapter 6 for a fuller discussion): in this case there is an inner (thermal) layer (this can be seen *a posteriori*) where

$$\xi = M_\infty^{a_1} \eta \sim O(1)$$

and

$$f = M_\infty^{a_2} f_0(\xi) + \dots, T = M_\infty^{a_3} T_0(\xi) + \dots, \mu = M_\infty^{a_4} \mu_0(\xi) + \dots.$$

Here $a_1, \dots, a_4$ are constants to be determined. In this thin wall layer the temperature is hot and so Sutherland's formula reduces to the power-law form

$$\mu \propto T^{\frac{1}{2}} \quad (2.3.26)$$

at leading order, thus we require $2a_4 = a_3$. Balancing all terms in equations (2.2.3b,c) yields two more conditions, $a_2 + \frac{a_3}{2} - a_1 = 0$ and $-a_2 + \frac{3}{2}a_3 - 3a_1 = 2$. The fourth condition necessary to solve for the four unknowns stems from matching to the remainder of the boundary layer that lies above this 'thermal' sub-boundary-layer. In particular we require that $f_\eta \sim O(1)$ as we approach the top of the wall layer, which yields the condition $a_2 + a_1 = 0$. Solving for $a_1, \dots, a_4$ yields the leading order properties

[†] in fact, this paper and the motivating investigation by Hayes & Probstein (1959) are referred to by Stewartson in his book.

$$\xi = M_\infty^{\frac{1}{2}} \eta \sim O(1)$$

and

$$f = M_\infty^{-\frac{1}{2}} f_0(\xi) + \dots, T = M_\infty^2 T_0(\xi) + \dots, \mu = M_\infty^1 \mu_0(\xi) + \dots, \quad (2.3.27a-d)$$

in the wall-layer (verifying the assumption that there is such a layer if one chooses to use Sutherland's formula). The resulting, scaled equations in the wall layer are

$$\begin{aligned} f_0 f_{0\xi\xi} + \left(\frac{f_{0\xi\xi}}{T_0^{\frac{1}{2}}} \right)_\xi &= 0, \\ f_0 T_{0\xi} + \frac{1}{Pr} \left(\frac{T_{0\xi}}{T_0^{\frac{1}{2}}} \right)_\xi + (\gamma - 1) \frac{f_{0\xi\xi}^2}{T_0^{\frac{1}{2}}} &= 0, \\ \mu_0 &= (1 + S) T_0^{\frac{1}{2}} \end{aligned} \quad (2.3.28a-c)$$

subject to the boundary conditions

$$f_0(0) = f_{0\xi}(0) = 0, \quad f_{0\xi}(\infty) \rightarrow 1, \quad T_0(\infty) \rightarrow 0, \quad (2.3.28d-g)$$

and

$$\begin{aligned} T_0(0) &= \lim_{M_\infty \rightarrow \infty} \left(\frac{T_w}{M_\infty^2} \right) \quad (\text{fixed wall - temperature}) \\ \text{or } T_{0\xi}(0) &= 0 \quad (\text{insulated wall}), \end{aligned} \quad (2.3.28h)$$

for f_0 , T_0 and μ_0 . From these equations it is easily to see that the temperature decays algebraically- it is this, instead of the exponential decay of the Blasius function, that leads to the inner boundary layer rather than the distinct adjustment layer of the Chapman-law-theory. However, when we revert back to the 'physical' y -variable (rather than the Dorodnitsyn-Howarth variable ζ) then the adjustment layer becomes much thinner than the hot (thermal) boundary layer below, and the cool free-stream above. This is basically due to the 'normal'-behaviour of the density, ρ , which occurs in the Dorodnitsyn-Howarth transformation (2.2.1a). See Figure 2.2.

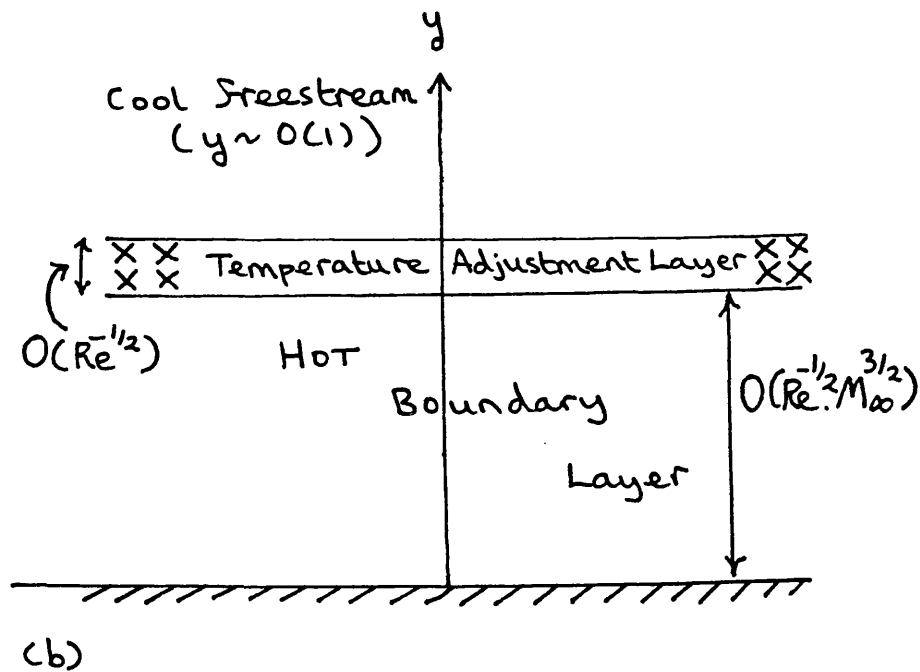
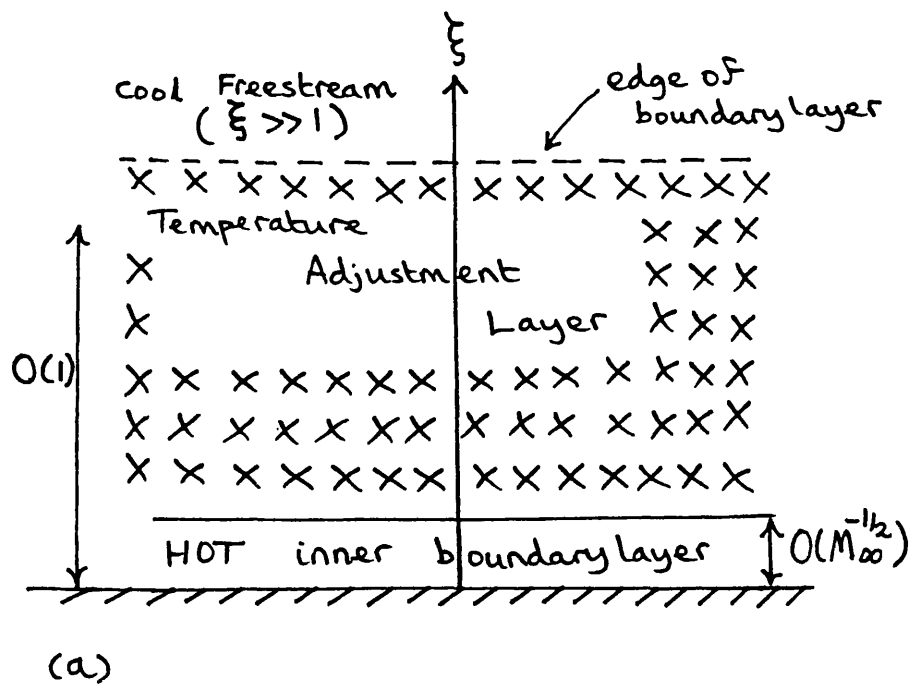


Figure 2.2. The large Mach number form of the boundary layer for flow over a flat plate, far from the leading edge: (a), in terms of the Howarth–Dorodnitsyn variable ξ ; and (b), in terms of the physical y -variable. Note that both are for the same fluid satisfying Sutherland’s temperature–viscosity formula.

The energy equation can be integrated to give

$$T_0(\xi) = \int_{\xi}^{\infty} \left\{ \left(\frac{f_0''}{T_0^{\frac{1}{2}}} \right)^{Pr-1} f'' [B + \int_0^{\xi_2} \frac{Pr(\gamma-1)(f'')^{2-Pr}}{T^{\frac{1}{2}(1-Pr)}} d\xi_1] \right\} d\xi_2, \quad (2.3.29)$$

where the value of B is determined from the temperature condition being imposed at the wall. As T appears in the right-hand side, it is not very helpful (cf the Chapman-law-case) – in fact one has to calculate T_0 numerically at the same time as f_0 .

We can now easily pick out the sizes

$$T_w \sim M_{\infty}^2, \quad \mu_w \sim M_{\infty}. \quad (2.3.30)$$

Also

$$\lambda_S \equiv f_{\eta\eta}(0) \sim M_{\infty}^{\frac{1}{2}} \lambda_S^H, \quad \lambda_S^H = f_{0\xi\xi}(0), \quad (2.3.31a, b)$$

as noted earlier, i.e. $\lambda_S \sim M_{\infty}^{\frac{1}{2}}$: in contrast to the fixed wall-shear value, λ_B , of the Chapman-case.

Let us now return to the effects of large Mach number on our new triple-deck scalings. Recently, Smith (1989) found that when M_{∞} increased to $O(Re^{\frac{1}{10}})$ the whole triple deck structure (based on Chapman-scalings and assuming that $C \sim O(1)$ always) collapses into a two-layer structure, comprising of a viscous boundary layer and inviscid upper deck, and that the x -scale becomes $O(1)$ suggesting that effects of nonparallelism cannot be ignored. In the current work we see that the use of Sutherland's formula has modified the triple-deck scalings, as well as the large Mach number properties of their component parts and so we investigate what changes result in the large M_{∞} conclusions of Smith (1989). In drawing these conclusions there are some subtleties in the argument which are highlighted below.

In the current context, it is obvious that there will be M_{∞} factors, in the scales, resulting from the factors λ_S , T_w , μ_w that appear 'explicitly' in the triple-deck scalings. However, additional M_{∞} factors arise 'implicitly' in most of the scales, due to the Mach number appearing in the eigenrelation. Earlier we noted that,

$$(\alpha, \beta, \Omega) \sim (M_{\infty}^{-\frac{3}{2}} \tilde{\alpha}, M_{\infty}^{-\frac{1}{2}} \tilde{\beta}, M_{\infty}^{-1} \tilde{\Omega}) + \dots,$$

for $M_\infty \gg 1$, which infers that, in this ‘hypersonic’ limit, the triple-deck scales (X, Z, T) need rescaling themselves. Moreover, these scalings for (α, β, Ω) result in most of the (already scaled) triple-deck quantities needing to be rescaled (w.r.t. M_∞). These include, in particular, the lower- and upper- deck normal variables, Y and $\bar{y}$, respectively.

First consider the solution to the linearized lower-deck equations, used to derive the eigenrelation. Note that the normal variable actually used is

$$\xi^T = (i\alpha)^{\frac{1}{3}} Y + \xi_0^T, \quad (2.3.32)$$

where ξ^T and ξ_0^T are assumed to be $O(1)$. Thus we see that

$$Y \sim \alpha^{-\frac{1}{3}}, \quad \sim M_\infty^{\frac{1}{2}} \quad (2.3.33)$$

for large Mach number. Now let us see how the upper-deck variable, $\bar{y}$, behaves in this limit. This can be easily deduced from (2.3.18a) which suggests that the upper-deck variable should, strictly, be $\bar{\bar{y}}$, say, where

$$\bar{\bar{y}} = \left[\frac{\beta^2}{M_\infty^2 - 1} - \alpha^2 \right]^{\frac{1}{2}} \bar{y} \sim O(1). \quad (2.3.4)$$

Thus $\bar{y} \sim \alpha^{-1}$, $\sim M_\infty^{\frac{3}{2}}$ as the Mach number increases.

Following Smith (1989), we see that the unscaled wavelength (with respect to non-dimensionalised x),

$$L_x \sim Re^{-\frac{3}{8}} \lambda_S^{-\frac{5}{4}} \mu_w^{-\frac{1}{4}} T_w^{\frac{7}{4}} (M_\infty^2 - 1)^{-\frac{3}{8}} \alpha^{-1}$$

(say), of linear disturbances described by triple-deck theory, increases in the form

$$\begin{aligned} L_x &\sim Re^{-\frac{3}{8}} \cdot \left(M_\infty^{\frac{1}{2}} \right)^{-\frac{5}{4}} \cdot (M_\infty)^{-\frac{1}{4}} \cdot (M_\infty^2)^{\frac{7}{4}} \cdot (M_\infty^2)^{-\frac{3}{8}} \cdot \left(M_\infty^{-\frac{3}{2}} \right)^{-1} \\ &\sim Re^{-\frac{3}{8}} \cdot M_\infty^{\frac{27}{8}} \end{aligned} \quad (2.3.35)$$

for $M_\infty \gg 1$, where the corresponding forms of λ_S , μ_w , T_w and α have been used. This is different than the result $L_x \sim Re^{-\frac{3}{8}} \cdot M_\infty^{\frac{15}{4}}$ obtained by Smith (1989) from the Chapman-law-formulation. The ‘new’ regime, first found in the latter paper, is encountered when

$$L_x \rightarrow O(1)$$

which now occurs at higher values of the freestream Mach number

$$M_\infty \sim Re^{\frac{1}{9}} \quad (2.3.36)$$

of the Mach number than found from the Chapman-law-formulation.

It is found convenient to define the scaled Mach number

$$m = Re^{-\frac{1}{9}} M_\infty, \quad (2.3.37)$$

noting that the parameter m , (asymptotically) small for supersonic flow, increases to become $O(1)$ in size, in the ‘new’ hypersonic regime.

The steady, two-dimensional non-interactive (thermal) boundary layer and the main deck have the same thickness (a property of the triple-deck structure)

$$y_{\text{main-deck/boundary-layer}} \sim Re^{-\frac{1}{2}} \mu_w^{\frac{1}{2}} T_w^{\frac{1}{2}}.$$

As the Mach number increases

$$\begin{aligned} y_{\text{main-deck/boundary-layer}} &\sim Re^{-\frac{1}{2}} \cdot (M_\infty)^{\frac{1}{2}} \cdot (M_\infty^2)^{\frac{1}{2}} \equiv Re^{-\frac{1}{2}} M_\infty^{\frac{3}{2}} \\ &= Re^{-\frac{1}{2}} \cdot \left(m Re^{\frac{1}{9}}\right)^{\frac{3}{2}} \sim Re^{-\frac{1}{3}} m^{\frac{3}{2}}, \end{aligned} \quad (2.3.38)$$

so that when m increases to become $O(1)$, the (thermal) boundary layer and the main deck have thickness $y \sim Re^{-\frac{1}{3}}$. Simultaneously, the lower-deck (location of the critical layer) expands in thickness like

$$\begin{aligned} Re^{-\frac{5}{8}} \lambda_S^{-\frac{3}{4}} \mu_w^{\frac{1}{4}} T_w^{\frac{5}{4}} (M_\infty^2 - 1)^{-\frac{1}{8}} \alpha^{-\frac{1}{3}} &\sim Re^{-\frac{5}{8}} M_\infty^{\frac{21}{8}} \quad [M_\infty \gg 1] \\ &\sim Re^{-\frac{1}{3}} \end{aligned} \quad (2.3.39)$$

when $m \sim O(1)$, i.e. the lower-deck coalesces with the main deck when $M_\infty \sim Re^{\frac{1}{9}}$.

Meanwhile the upper-deck expands in thickness like

$$\begin{aligned} Re^{-\frac{3}{8}} \lambda_S^{-\frac{5}{4}} \mu_w^{-\frac{1}{4}} T_w^{\frac{7}{4}} (M_\infty^2 - 1)^{-\frac{7}{8}} \alpha^{-1} &\sim Re^{-\frac{3}{8}} M_\infty^{\frac{19}{8}} \quad [M_\infty \gg 1] \\ &\sim Re^{-\frac{1}{9}} \end{aligned} \quad (2.3.40)$$

when $m \sim O(1)$. Hence the y -variation becomes two-tiered ($y \sim Re^{-\frac{1}{3}}$, $y \sim Re^{-\frac{1}{9}}$) and the x -scale is $O(1)$. Similarly, it is easy to show that the z -scale expands to $z \sim Re^{-\frac{1}{9}}$, the time-scale involved rises to $O(1)$ and that the streamwise pressure gradient disturbance, P_X , becomes negligible in the boundary layer.

Thus we have the same structure deduced by Smith (1989), for the Chapman-law- formulation (but note that the underlying basic boundary layer structure is different), but the dimensions have been modified. In addition, the regime is encountered at a larger value of the Mach number. Most of the conclusions of Smith carry straight over, in particular that the maximum growth rate is for very oblique waves at nearly 90° to the free-stream direction; the viscous- inviscid interaction continues (see next chapter), but the waves are now crucially affected by non-parallelism.

Let us now turn and consider what size of the Mach number now, for Sutherland's formula, corresponds to the so-called hypersonic-viscous range where the basic flow past the flat plate changes considerably (it becomes 'interactive') due to a shock impinging on the (thermal) boundary layer. The thickness of this layer $\sim Re^{-\frac{1}{2}} M_\infty^{\frac{3}{2}}$, whereas the position of the shock can be identified by the equation of the so-called Mach lines

$$y = \frac{x}{M_\infty} \quad (2.3.41)$$

in the inviscid region above the boundary layer. As $x \sim O(1)$, we can see that the two (top of boundary layer and position of shock) converge, on one another, when

$$Re^{-\frac{1}{2}} M_\infty^{\frac{3}{2}} \sim M_\infty^{-1} \quad \text{i.e. when} \quad M_\infty \sim Re^{\frac{1}{5}}. \quad (2.3.42)$$

Note that this is also occurs at larger values of the Mach number than the corresponding Chapman-law-result ($M_\infty \sim Re^{\frac{1}{6}}$, assuming that $C \sim O(1)$). More important to the present discussion, we see that the two-tier-structure-regime ($M_\infty \sim Re^{\frac{1}{9}}$) occurs well before the basic boundary layer-flow becomes interactive ($M_\infty \sim Re^{\frac{1}{5}}$).

The above restriction on the size of the Mach number, such that the normal-mode decomposition is rational, reads

$$M_\infty \ll Re^{\frac{1}{9}}, \quad \text{or} \quad M_\infty \ll Re^{\frac{2}{9}}, \quad \text{or} \quad M_\infty \ll Re^{\frac{1}{6}}, \quad (2.3.43)$$

in terms of the local Reynolds numbers $R \equiv Re^{\frac{1}{2}}$ and R_δ based on the (thermal) boundary layer thickness, since

$$R_\delta \simeq Re^{\frac{1}{2}} M_\infty^{\frac{3}{2}} \quad (2.3.44)$$

for large Mach numbers. Note that these restrictions are based on the assumption that the flow over the plate is two-dimensional (more strictly, that there is no significant cross-flow) and that there is little, or no, wall cooling. We would expect similar restrictions for other wall conditions, a remark made by Smith (1989). The latter paper also illustrates the restriction by considering a typical value for the global Reynolds number Re of 2.25×10^6 , corresponding to $R = 1500$. In this case the restriction can be interpreted as meaning that the normal-mode (Orr-Sommerfeld-type) approach holds only for $M_\infty \ll 5.08$ which is slightly less severe than that found from the Chapman-law-formulation: note that the restriction becomes more severe as the Reynolds number decreases. Finally note that, for this chosen value of the Reynolds number, the basic flow over the plate becomes interactive when $M_\infty \sim 18.64$ -this is significantly higher than the corresponding value, $M_\infty \sim 11.45$, obtained from the Chapman-law-formulation.

Let us now, in addition to our typical Reynolds number Re of 2.25×10^6 , also consider the typical range of Mach numbers

$$0 \leq M_\infty \leq 25$$

say, relevant to the limits of current technology and design. We immediately see, from the numbers quoted in the preceding paragraph, that the conventional normal- mode approach of both linear-triple-deck and (the more classical) Orr-Sommerfeld-type theories are only rational for the a relatively small fraction ($\simeq 20\%$) of this range of Mach-numbers, whilst the interactive-boundary-layer theories (dating back to the 1950's) are only strictly applicable to the upper 25% of this range. This therefore leaves about half of the current range of Mach numbers yet to be accounted for. Clearly the new two-tier-structure discovered by Smith (1989), and discussed above, will fill at least some of the 'gap': the question whether it 'fills' all of the gap (i.e. what happens as the scaled Mach number $m \rightarrow \infty$?) is addressed in *Chapter 4*. Concluding the current discussion,

we can see the major significance of the restriction, concerning the validity of the normal-mode approach, found above for a Sutherland-fluid based on the initial work due to Smith (1989).

Above we have derived the $M_\infty \sim Re^{\frac{1}{3}}$, $Re^{\frac{1}{5}}$ regimes from first principles — they can be obtained much more quickly, in an *ad-hoc* fashion, by simply setting $C \equiv \frac{\mu_w}{T_w} \sim M_\infty^{-1}$ in the respective Chapman-law-formulations. In fact, this is how they were first derived by the author. The above derivation, of the condition $M_\infty \sim Re^{\frac{1}{5}}$ on where the basic boundary-layer flow becomes interactive, is based on a well-known argument: it came as no surprise to the author to learn that Luniev (1959) (see also references therein, especially Lees, 1953) had essentially carried out such a derivation over thirty years prior. Luniev used the power-law formulae, $\mu \propto T^\omega$, to relate the wall value of the viscosity to the corresponding value of the temperature, obtaining the result for general ω . Our result immediately follows by setting $\omega = \frac{1}{2}$, corresponding to the high temperature leading order form of Sutherland's formula.

What the author does find surprising, and a view essentially shared by Professor P. Hall (1989 onwards; discussions with present author) when pointing out the deficiencies in the Chapman-law formulation for large temperature variations, is that most theoretical works (including earlier versions of the author's work described in the present thesis) concerning large Mach-number, boundary-layer flows, in the last twenty-or-so years have still used the Chapman-law-formulation, seemingly oblivious of the work carried out in the 1950's and 1960's using more realistic viscosity-temperature relations. See Lees (1953), Luniev (1959), Freeman & Lam (1959) and Bush (1966) as examples of the latter, and Stewartson (1964), Brown, Stewartson & Williams (1975), Brown & Stewartson (1975), Smith (1989), Hall & Fu (1989), Smith & Brown (1989) and Balsa & Goldstein (1990) as examples of the former.

It could, perhaps, be argued that the model assumptions used in the latter papers were justified in order to allow the investigations contained therein to be carried out and/or be presented more simply: the author has found that the Sutherland-formulation leads to complications, particularly with the numerical solutions (see Chapter 6). Whether the earlier papers had been forgotten, ignored, or dis-regarded for whatever reason, it is fair to say that the ideas contained within

them have not, generally, been used or built upon†. This is now being rectified: see, for example, Cowley & Hall (1990); Blackaby, Cowley & Hall (1990); Fu (1990); Fu, Hall & Blackaby (1990); Brown, Cheng & Lee (1990); as well as the present thesis.

Before concluding this section, it is worth making some brief observations and remarks on some closely related topics. The first concerns wall cooling. In this thesis wall cooling is not addressed in any great detail, mainly because we wanted the analysis to be as simple as possible, but also because that aspect has been the topic of research of fellow workers studying high-Reynolds number theory (see, for example, Seddougui, Bowles & Smith, 1989; Brown, Cheng & Lee, 1990). However, the author recognises the necessity to cool the airfoil in order to protect it from the high temperatures produced in the hypersonic boundary-layer (note that $T_w \sim M_\infty^2 \gg 1$; if the wall is an insulator). The next point concerns the large Mach number limit. Recently Cowley & Hall (1990) investigated the effect of incorporating a shock into the triple-deck structure. They considered flow over a wedge and found that the Newtonian assumption ($\gamma - 1 \ll 1$) was necessary to complete their analysis. However the physical relevance of this assumption has not been established. It is interesting to note that, for the case of the triple-deck structure for the interactive boundary-layer, Brown, Cheng & Lee (1990) have shown that, by assuming the wall is significantly cooled, the Newtonian assumption, which is needed for an insulated wall, is no longer necessary.

As well as being an interesting paper in its own right, the paper by Cowley & Hall is, in the author's opinion, most significant for a conclusion not explicitly drawn. Essentially it is shown that by setting the pressure disturbance to be identical to zero at finite (albeit large) $\bar{y}$ (the upper-deck variable) values, rather than at infinity, then the solutions to the 'new' eigenrelation are radically different. Now assume that, for simplicity, there is no shock present. If one determines stability characteristics from the eigenrelation based on the theoretical triple-deck theory

† Stewartson (1974, page 153) remarks that the general features of the interactive flow solution, for a non-linear viscosity-temperature relation, ^{considered} by Bush (1966) and originally suggested by Lees (1955), are "extremely complicated" with many separate regions needing to be considered and interrelated.

then the correct (decay to infinity) boundary condition for the pressure disturbance has already been applied. However, if one solves for the stability properties using an Orr–Sommerfeld–type approach, then infinity must be approximated by some, finite, estimation of the location of the top of the boundary layer. Thus there appears to be the possibility of the latter (numerical) calculations picking up spurious solutions, in addition to the correct one corresponding to decay at infinity.

The next observation concerns the comparisons, made by Smith (1989) and Duck (1990) of their triple-deck results with the Orr–Sommerfeld–type results of Mack (1976) for supersonic, viscous boundary-layer stability. Both of the (former) authors find reasonable qualitative agreement and remark that the quantitative differences could be due to the ‘lowness’ of Mack’s Reynolds number. It should be noted that all three papers use the ‘parallel-flow’ approximation and so the differences between the predictions cannot be attributed to boundary-layer growth. The author has repeated Smith’s comparison but for the Sutherland-formulation (more closely related to Mack’s physical assumptions) and found little change, graphically, from Smith’s predictions. The triple-deck theory predicts that, for $M_\infty \gg 1$, the most unstable modes travel at quite oblique angles which is in disagreement with Mack who finds that the most unstable modes are less oblique.

The final observation relates to the hypersonic-limit of the supersonic non-axisymmetric Tollmien–Schlichting modes studied within the triple-deck framework by Duck & Hall (1990). The form of the appropriate eigenrelation (see next chapter), coupled with the need for spanwise periodicity, results in the ‘new’ regime found for the planar case (where the triple-deck structure collapses and non-parallel effects become important) not occurring (for any large size of the Mach number).

Concluding, the Sutherland-formulation given above rationally establishes the two results, $M_\infty \sim Re^{\frac{1}{9}}$, $Re^{\frac{1}{5}}$ for where non-parallel and shock effects, respectively, are crucial. In addition it provides a sound basis for future investigations concerning the viscous-stability of compressible boundary-layer flows (eg. see next chapter). Also we have seen (due to the coupling of the f_0 - and T_0 - equations) that the wall-shear term λ_S is a function of the physical parameters γ , Pr , S , n , M_∞ (needing to be determined numerically), in contrast to the fixed $\lambda_B \simeq 0.3321$.

The last point, in conjunction with the fact that Sutherland's law leads to different triple-deck scalings, is crucial if one wants to convert results back into actual physical values; the approach of replacing $C \rightarrow \mu_w/T_w$ in the Chapman-law-formulation and then evaluating μ_w by Sutherland's formula, or a power law, is very useful for theoreticians wishing to estimate the sizes of scalings, particularly in the large Mach number limit (see, for example, Cowley & Hall, 1990).

Non-linear Tollmien-Schlichting/vortex interaction in compressible boundary-layer flows.

§3.1. INTRODUCTION

The non-linear interaction between two oblique three-dimensional Tollmien-Schlichting (TS) waves and their induced streamwise (longitudinal)-vortex flow is considered theoretically for a compressible boundary layer. In a recent paper Hall & Smith (1989) considered an incompressible boundary layer, and the present study is an extension of their rational approach. The same theory applies to destabilisation of an incident vortex motion by sub-harmonic TS waves, followed by interaction. The interaction is considered for all ranges of the Mach number. Compressibility has a significant effect on the interaction; principally through its impact on the waves and their governing mechanism.

The motivation for such a study is essentially the same as expressed by Hall & Smith in the introduction to their paper; namely that often in experimental studies of laminar-to-turbulent transition on a flat plate, there appear to be longitudinal vortices co-existing, and interacting, with the viscous TS modes. As there is no concave curvature of the surface, these longitudinal vortices are not driven by surface-curvature (cf. the Görtler vortex studies of Hall, 1982a,b) — instead one could postulate that they are in fact being driven by, and/or interacting with, the (neutral) TS modes. Such experimental studies have been carried out, for instance, by Prof. Y. Aihara and colleagues, in Japan (eg. Aihara & Koyama, 1981; Aihara *et al*, 1985); and by Prof. Y. Kachanov and colleagues, in the Soviet Union (Kachanov, 1990). The reader is referred to the paper by Hall & Smith for a fuller account of relevant experimental findings, as well as supporting computational work (see, for example, Spalart & Yang, 1986). These experimental studies are all for incompressible flow; the author is unaware of any experimental work specifically relevant to this compressible study.

Recently, the origin of streamwise vortices in a turbulent boundary layer has been investigated theoretically by Jang *et al* (1986). The Reynolds number is taken to be finite and their formulation is of the Orr-Sommerfeld-type. They

show that two oblique travelling waves can combine non-linearly to produce a stationary, streamwise vortex — this is essentially the theoretical idea later used Hall & Smith in independent work. However the latter's approach, the approach adopted in this chapter, takes advantage of the feature that the Reynolds numbers of interest in reality are large and indeed the Reynolds number is taken as a large parameter throughout. The non-linear interaction is powerful, starting at quite low amplitudes with a triple-deck structure for the TS waves but a large-scale structure for the induced vortex, after which strong non-linear amplification occurs. Non-parallelism is accommodated within the scales involved.

The non-linear interaction is governed by a partial-differential system for the vortex flow coupled with an ordinary-differential one for the TS pressure. The solutions of these systems depend crucially upon the interaction coefficients which are themselves functions of the Mach number. Additionally, the TS waves are significantly affected by the inclusion of compressibility. It is found that the interaction coefficients, for subsonic flow, do not differ significantly in nature from the incompressible ones, but as the flow becomes supersonic the restriction (for high Reynolds numbers) that the TS waves must be directed outside the local Mach-wave cone (Zhuk & Ryzhov, 1981) excludes a flow solution which is possible for less oblique modes. The flow properties point to the second stages of interaction associated with higher amplitudes.

It is found that the present formulation breaks down as the Mach number becomes large: for then, even when the presence of shock/boundary layer interaction is neglected, the viscous sublayers coalesce to form a single boundary-layer. The structure which is applicable in this hypersonic limit is currently under investigation (see also Chapter 4).

The theoretical idea is basically that, if two low-amplitude TS waves are present (proportional to $E_{1,2} = \exp[i(\alpha X \pm \beta Z - \Omega T)]$ say; see later notation), then nonlinear inertial effects produce the combination $E_1 E_2^{-1} = \exp[2i\beta Z] = E_3$ say, at second order, among other contribution, i.e. a standing-wave or longitudinal-vortex flow is induced. Equally the combination of the vortex and one TS wave provokes the other TS wave.

As we are assuming the Reynolds number to be large, the TS waves are supported by the triple-deck structure (Smith, 1979a,b; 1989) , whilst an extra

sub-boundary layer and a further x -scale are necessary to capture their interaction with the longitudinal vortices. The present vortex/wave interaction mechanism is very similar to that of Hall & Smith (1989); the difference is caused by an error in the latter, found by Blennerhassett & Smith (1991). The ‘corrected’ mechanism still has the induced vortices lying at the top of the lower deck but now the forcing from the TS waves is solely from an inner boundary condition. The wall-shear of the induced vortices modifies the wall-shear of the basic flow at the same order as the latter’s leading-order non-parallel correction. These corrections to the wall shear force secondary TS waves in the lower-deck, whilst the amplitude of the primary TS waves here is governed by an amplitude equation involving these corrections to the wall shear. The behaviour of the primary TS quantities at the top of the lower deck then leads to longitudinal-vortex activity being forced in the sub-layer above, via a boundary condition. Thus the system is truly interactive: the longitudinal vortices are driven by the TS waves, the amplitude of which is determined by an amplitude equation involving a vortex-term. See Figure 3.1.

We consider the interaction for the case of compressible laminar flow over a semi-infinite plate, using the notation and formulation of the previous chapter as our starting point. Results are presented, and conclusions drawn, for subsonic and supersonic flows. However, first the corrected results of the incompressible case are considered so that comparisons can be made with the latter; the incompressible-flow results of Hall & Smith suffer from (at least) two errors, both of which are found to be significant with regard to the quantitative results and resulting predictions.

§3.2 FORMULATION.

§3.2.1 Discussion.

The underlying structure is that of the three-dimensional, compressible Tollmien-Schlichting (TS) waves at large values of the Reynolds number, namely the three-dimensional ‘compressible triple-deck’. This structure has been studied by, in particular, Zhuk & Ryzhov (1981) (see also Ryzhov, 1984), Smith (1989) and Duck (1990)). In the previous chapter this structure was discussed and particular attention was paid to the changes brought about by using Sutherland’s formula

$$\varepsilon \equiv Re^{-1/8}$$

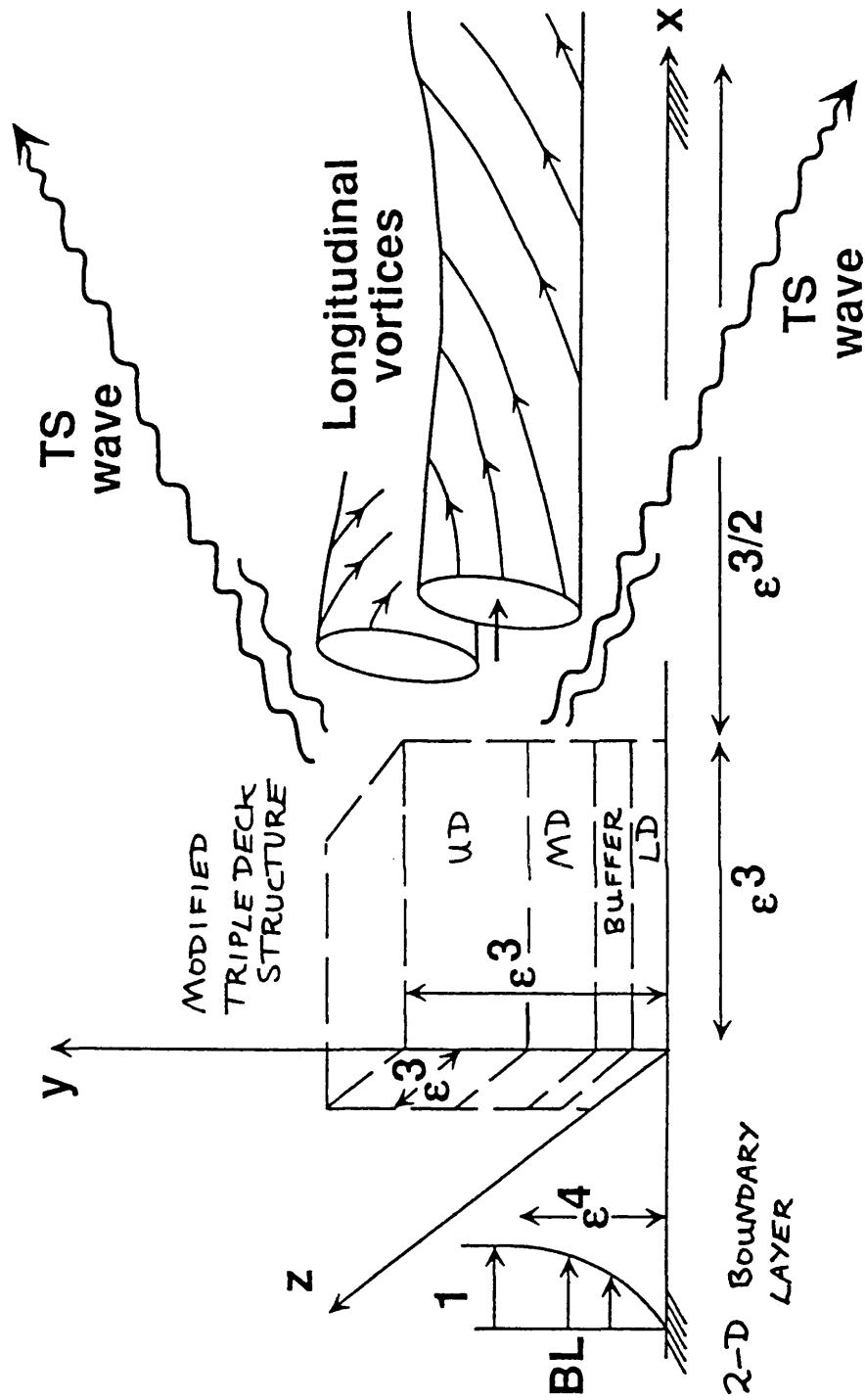


Figure 3.1. Diagram of the 3D TS/vortex structure for nonlinear interaction in a 2D boundary layer. (From Hall & Smith, 1989.)

to relate viscosity and temperature — it is this latter formulation that we shall follow in this chapter.

Recently, Cowley & Hall (1990) and Duck & Hall (1990) have shown that the (former) theories can be adapted to include the effects of a shock for flow over a wedge, and cylindrical geometry, respectively. For definiteness, we assume that the flow is supersonic ($M_\infty > 1$) during the formulation of the interaction equations; the subsonic and other cases follow very similarly. In fact in Section 4 we give an alternative (non-first-principles) derivation of the important interaction coefficients for a ‘general’ Tollmien–Schlichting eigenrelation.

In the next subsection we briefly recap the scales and derivation of the eigenrelation for TS modes, described by the triple-deck structure. In §3.2.3 we describe the modifications necessary to the triple-deck structure in order to support our chosen vortex–wave interaction and derive the necessary scales. This argument follows very closely that of Hall & Smith, although their final choice of scales was not appropriate.

§3.2.2 The 3-D compressible triple-deck equations.

Here we state the scales and resulting equations for equations for completeness — see Chapter 2 for a fuller discussion. These scalings will be referred to when we introduce ‘new’ x - and y - scales for the interaction, as well as when we investigate limiting values of the Mach number, in §3.5.3. In the scalings given below, the Reynolds number is assumed to be large whilst the other factors are taken to be $O(1)$. Recall that although these factors were introduced to normalise the resulting governing equations, the Mach number still remains in the upper-deck’s pressure–disturbance equation and hence it appears in the TS–eigenrelation.

The scales, for $M_\infty > 1$, are (see also §2.3)

$$[x - x_0, z - z_0] = Re^{-\frac{3}{8}} K_1 [X, Z], \quad (3.2.1a, b)$$

$$\text{where } K_1 = \left(\frac{\lambda_S}{\sqrt{2}} \right)^{-\frac{5}{4}} \mu_w^{-\frac{1}{4}} T_w^{\frac{7}{4}} (M_\infty^2 - 1)^{-\frac{3}{8}}, \quad (3.2.1c)$$

$$\text{and } t = Re^{-\frac{1}{4}} \left(\frac{\lambda_S}{\sqrt{2}} \right)^{-\frac{3}{2}} \mu_w^{-\frac{1}{2}} T_w^{\frac{3}{2}} (M_\infty^2 - 1)^{-\frac{1}{4}} \hat{t}, \quad (3.2.1d)$$

where (x_0, z_0) corresponds to the location of the initial disturbance of the laminar base-flow. In the viscous sublayer, or lower deck,

$$y = Re^{-\frac{5}{8}} \left(\frac{\lambda_S}{\sqrt{2}} \right)^{-\frac{3}{4}} \mu_w^{\frac{1}{4}} T_w^{\frac{5}{4}} (M_\infty^2 - 1)^{-\frac{1}{8}} Y, \quad (3.2.2a)$$

and to leading order,

$$u = Re^{-\frac{1}{8}} \left(\frac{\lambda_S}{\sqrt{2}} \right)^{\frac{1}{4}} \mu_w^{\frac{1}{4}} T_w^{\frac{1}{2}} (M_\infty^2 - 1)^{-\frac{1}{8}} U,$$

$$v = Re^{-\frac{3}{8}} \left(\frac{\lambda_S}{\sqrt{2}} \right)^{\frac{3}{4}} \mu_w^{\frac{3}{4}} T_w^{-\frac{1}{4}} (M_\infty^2 - 1)^{-\frac{1}{8}} V,$$

$$w = Re^{-\frac{1}{8}} \left(\frac{\lambda_S}{\sqrt{2}} \right)^{\frac{1}{4}} \mu_w^{\frac{1}{4}} T_w^{\frac{1}{2}} (M_\infty^2 - 1)^{-\frac{1}{8}} W$$

and

$$p - p_\infty = Re^{-\frac{1}{4}} \left(\frac{\lambda_S}{\sqrt{2}} \right)^{\frac{1}{2}} \mu_w^{\frac{1}{2}} T_w^{-\frac{1}{2}} (M_\infty^2 - 1)^{-\frac{1}{4}} P. \quad (3.2.2b - e)$$

The main deck has

$$y = Re^{-\frac{1}{2}} \mu_w^{\frac{1}{2}} T_w^{\frac{1}{2}} \tilde{y}, \quad (3.2.3)$$

and merely transmits small displacement effects across the boundary layer as well as smoothing out an induced spanwise velocity (see later). The third, upper, deck then occurs where

$$y = Re^{-\frac{3}{8}} K_1 (M_\infty^2 - 1)^{-\frac{1}{2}} \bar{y} \quad (3.2.4)$$

and

$$[u, v, w, p] = [1, 0, 0, p_\infty] + Re^{-\frac{1}{4}} K_2 [\bar{u}^{(2)}, \bar{v}^{(2)} (M_\infty^2 - 1)^{\frac{1}{2}}, \bar{w}^{(2)}, \bar{p}^{(2)}] + \dots,$$

$$K_2 = \left(\frac{\lambda_S}{\sqrt{2}} \right)^{\frac{1}{2}} \mu_w^{\frac{1}{2}} T_w^{-\frac{1}{2}} (M_\infty^2 - 1)^{-\frac{1}{4}}, \quad (3.2.5a - e)$$

together with similar perturbations of the uniform density and temperature.

After some manipulation one finds that the Mach number has been scaled out of all but the upper deck equations. The lower deck equations are

$$U_X + V_Y + W_Z = 0,$$

$$U_T + UU_X + VU_Y + WU_Z = -P_X + U_{YY},$$

$$W_T + UW_X + VW_Y + WW_Z = -P_Z + W_{YY},$$

$$P_Y = 0, \quad (3.2.6a - d)$$

to be solved subject to no-slip at the wall,

$$U = V = W = 0, \quad \text{on} \quad Y = 0, \quad (3.2.6e)$$

together with the displacement condition at infinity,

$$U \rightarrow \lambda(Y + A(X, Z, T)), \quad \text{as} \quad Y \rightarrow \infty. \quad (3.2.6f)$$

The displacement, A , is related to the pressure, P , via a pressure–displacement law stemming from matching this solution to the upper deck solutions. Note the ‘incompressible’ appearance of the lower–deck equations.

The upper deck equations lead to the governing equation for the disturbance pressure amplitude $\hat{p}$,

$$(M_\infty^2 - 1)[\hat{p}_{XX} - \hat{p}_{\hat{y}\hat{y}}] - \hat{p}_{ZZ} = 0, \quad (3.2.7a)$$

classically known as the Prandtl–Glauert Equation. This has to be solved subject to

$$\hat{p} \rightarrow P \quad \text{as} \quad \hat{y} \rightarrow 0, \quad \hat{p} \rightarrow 0 \quad \text{as} \quad \hat{y} \rightarrow \infty, \quad (3.2.7b, c)$$

whilst the pressure–displacement law is

$$-\hat{p}_{\hat{y}} \rightarrow A_{XX} \quad \text{as} \quad \hat{y} \rightarrow 0. \quad (3.2.8)$$

The vortex–wave interaction, of concern in this chapter, involves linear Tollmien–Schlichting modes and so we write

$$(U, V, W, P, A) = (\lambda Y, 0, 0, 0, 0) + \{h(\tilde{U}, \tilde{V}, \tilde{W}, \tilde{P}, \tilde{A})E + c.c.\} + O(h^2), \quad (3.2.9a)$$

with

$$E = \exp[i(\alpha X + \beta Z - \Omega T)], \quad (3.2.9b)$$

where α, β are the normalised wavenumbers; Ω is the normalised frequency and $h \ll 1$ is the small linearisation parameter. The solution of the upper-deck equations, with the desired decay as the freestream is approached, results in the following (relatively severe) restriction on possible α and β

$$\text{Real}\left\{\frac{\beta^2}{(M_\infty^2 - 1)} - \alpha^2\right\}^{\frac{1}{2}} > 0, \quad (3.2.10)$$

meaning that the waves must be 3-D and be directed outside of the local Mach-wave cone.

The eigenrelation, for linear supersonic TS-modes, is easily found to be

$$(i\alpha\lambda)^{\frac{1}{3}}(\alpha^2 + \beta^2) = \lambda^2(Ai'/\kappa)(\xi_0)\left\{\frac{\beta^2}{(M_\infty^2 - 1)} - \alpha^2\right\}^{\frac{1}{2}}. \quad (3.2.11a)$$

Here Ai signifies the Airy function,

$$\kappa = \int_{\xi_0}^{\infty} Ai(q) dq \quad \text{and} \quad \xi_0 = -i^{\frac{1}{3}} \frac{\Omega}{(\alpha\lambda)^{\frac{2}{3}}}. \quad (3.2.11b, c)$$

The vortex-wave interaction to be described concerns only neutral modes (that is α, β and Ω are all real) and we note that this occurs for

$$\xi_0 = -d_1 i^{\frac{1}{3}}, \quad (Ai'/\kappa)(\xi_0) = d_2 i^{1/3}$$

where the constants d_1, d_2 have the (well-known) values, $d_1 \simeq 2.297$ and $d_2 \simeq 1.001$.

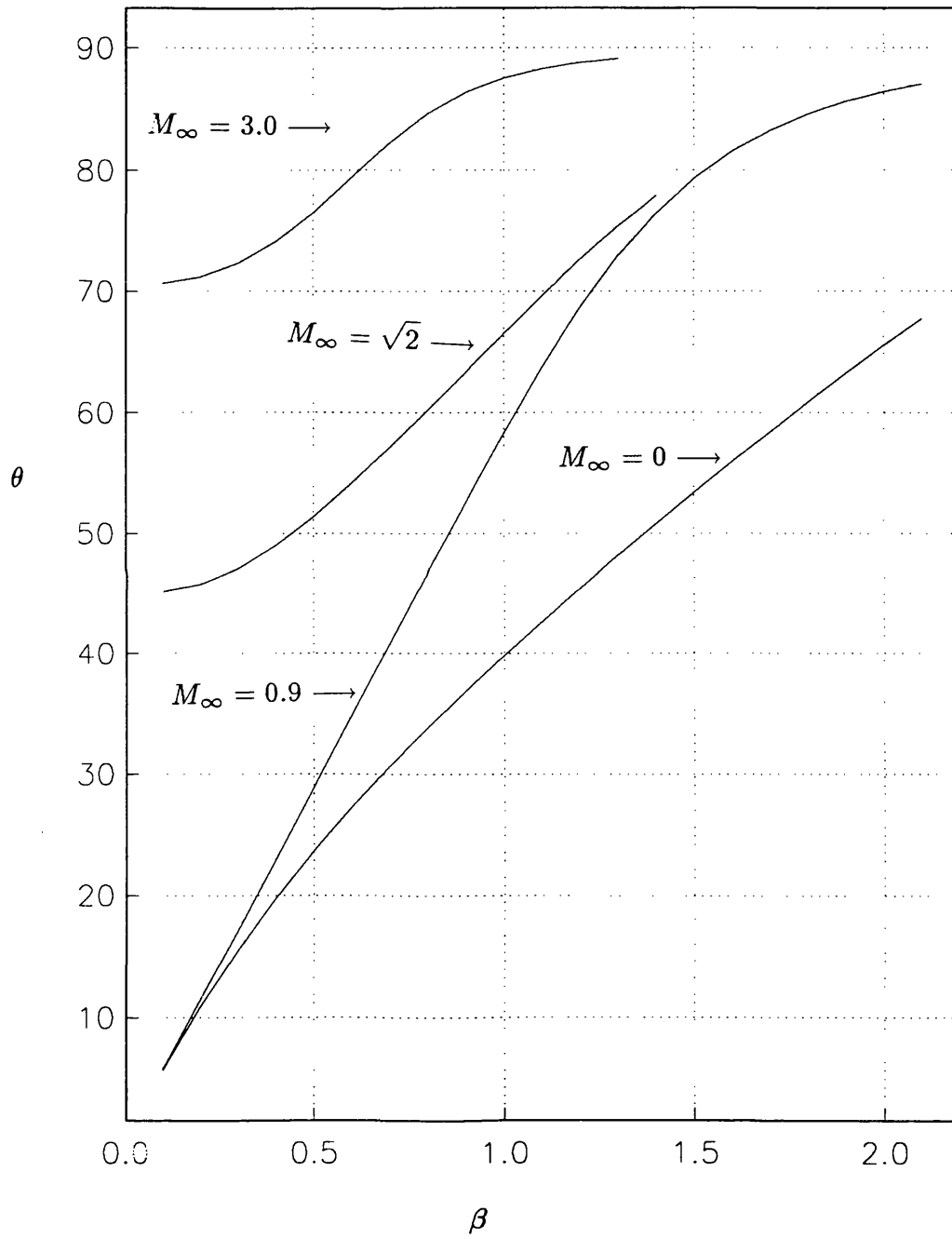
In Figure 3.2 we present the ‘neutral’ solutions of the eigenrelation (3.2.11), and its subsonic counterpart, for a few (illustrative) choices of the Mach number. Here (and hereinafter) the ‘wave-obliqueness-angle’ is defined by

$$0 \leq \theta = \tan^{-1} \left(\frac{\beta}{\alpha} \right) \leq 90^\circ.$$

We see for subsonic values of the Mach number ($M_\infty < 1$) that neutral modes are possible for all wave-angles. However, for increasing supersonic Mach number values ($M_\infty > 1$) the solution properties start to differ noticeably, with only an ever decreasing range of very oblique TS-wave propagation angles, θ , being possible. Thus the restriction (3.2.10), which can be re-written as

$$\theta \geq \tan^{-1} \left[(M_\infty^2 - 1)^{\frac{1}{2}} \right],$$

Figure 3.2. TS wave obliqueness angles θ versus spanwise wavenumbers β , for neutral modes at four values of the Mach number M_∞ .



for real α and β , is clearly evident in this figure. We shall see later, once the interaction has been formulated and numerical values have been calculated for the important interaction coefficients, that this restriction proves to be a more significant ‘compressibility-effect’ on the interaction than the ‘direct’ effect due to the Mach number appearing in the interaction coefficients.

It is beneficial to spend a few moments considering the behaviour of the lower-deck quantities for large- Y as these will be referred to in the next section; they are crucial to the particular vortex-wave interaction that we are going to consider. It can easily be shown that

$$U \sim Y + A + \bar{C}Y^{-1} + O(Y^{-2}), \quad V \sim -YA_X + \bar{D} + O(Y^{-1}),$$

$$\text{and } W \sim \bar{B}Y^{-1} + O(Y^{-2}), \quad \text{as } Y \rightarrow \infty. \quad (3.2.12a - c)$$

Here $\bar{B}$, $\bar{C}$ and $\bar{D}$ are unknown, inter-related functions of X , Z and T (see Stewart and Smith, 1987).

Smith (1989) gives a comprehensive account of the consequences of the above eigenrelation on the stability of the flow to linear TS-modes. Our concern in this chapter is with a vortex-wave interaction based on these length- and timescales. In the next subsection we deduce the size of additional x - and y - scales necessary to capture this interaction.

§3.2.3 The interaction scales.

Here we essentially follow the same argument as Hall & Smith (1989) but with the compressible triple-deck scales given in the last subsection; we appeal to their resulting equations to give an alternative physical interpretation of the interaction mechanism. However, we will take the coupled lower- and upper-deck equations^{as}our starting point, rather than returning to the compressible Navier-Stokes equations; this will lead to the scales appearing simpler.

We have seen in the last subsection that TS-waves are governed by the triple-deck structure, and in particular by the unsteady interactive boundary-layer equations holding in the lower-deck. If the 3D TS- wave amplitudes are comparatively small, say of order $h \ll 1$ relative to fully nonlinear sizes, then (nonlinear) inertial

effects force a vortex motion at relative order h^2 ; the TS-modes are taken to be proportional to

$$E_1 = \exp[i(\alpha X + \beta Z - \Omega T)], \quad E_2 = \exp[i(\alpha X - \beta Z - \Omega T)] \quad (3.2.13a, b)$$

and we see that combinations yield, in particular, induced longitudinal-vortex terms proportional to

$$E_3 = \exp[i(2\beta Z)], \quad (3.2.13c)$$

having only spanwise dependence.

We saw, in the last subsection, that certain lower-deck quantities decay algebraically for large Y ; it can be easily shown that spanwise inertial effects (such as the ' UW_X ' term of the Z -momentum equation) decay slowly like $1/Y^2$ (from (3.2.12)) resulting in the spanwise velocity component of the induced vortex to grow logarithmically like $\ln Y$ (Hall & Smith, 1984; 1989) since the vortex response is predominantly viscous here. Hall & Smith (1989) introduced the concept of a new sub-layer ('the buffer layer') situated within, and at the top of, the lower-deck, along with a longer lengthscale (for amplitude modulation) to dampen down this logarithmic growth. They showed that the main vortex activity was confined to this region.

Before deriving the compressible sizes for $\bar{X}$, the modulation lengthscale, and the buffer where $Y = \delta \bar{Y}$, we briefly mention the link between the x -scales present and non-parallel effects. The triple-deck is a local structure located at non-dimensionalised distance $x = x_0$ from the leading edge. It is short, its length being $O(\epsilon^3 K_1)$ compared to the $O(1)$ development of the underlying boundary layer, and all the X -dependence of the TS-modes is taken to be in the E_1 and E_2 factors. The modulation of the modes is assumed to be on a longer x -scale and thus the eigenrelation (3.2.11) is unaffected. We define this (new) modulation x -scale, $\bar{X}$ say, by

$$x - x_0 = \delta_2 \bar{X} + \epsilon^3 K_1 X, \quad \epsilon^3 K_1 \ll \delta_2 \ll 1, \quad (3.2.14)$$

where δ_2 is to be determined.

The only effect from the base, underlying non-parallel flow felt by the lower-deck equations is the wall shear $\lambda \equiv \lambda(x_0)$. At leading order λ is constant (hence

the linear TS-waves are independent of non-parallel effects) but here we wish to balance the next order (i.e. its correction) term into our interaction equations. A Taylor expansion, about the local station $x = x_0$, gives

$$\lambda = \lambda(x_0) + \delta_2 \bar{X} \lambda_b(x_0) + O(\min[\delta_2^2, \epsilon^3 K_1]) \quad (3.2.15a)$$

where $\lambda_b = \frac{d\lambda}{dx}$, is $O(1)$ and represents the first influence of non-parallelism (streamwise boundary-layer growth). Note that we have multiple-scales in x ; formally we should make the replacement

$$\frac{\partial}{\partial X} \rightarrow \frac{\partial}{\partial X} + \frac{\epsilon^3 K_1}{\delta_2} \frac{\partial}{\partial \bar{X}} \quad (3.2.15b)$$

in the triple-deck equations.

Let us now return to the derivation of the $\bar{Y}$ and $\bar{X}$ scales. We have seen that the size of the spanwise velocity of the induced vortex in the buffer layer is $O(h^2 \ln Y)$, $\sim h^2 \ln \delta$, leading to an induced streamwise velocity of order $\frac{\delta_2}{\epsilon^3 K_1} h^2 \ln \delta$, by continuity (and noting that the modulation is on $\hat{X}$), which alters the basic shear by a relative amount of order $\frac{\delta_2}{\epsilon^3 K_1} \frac{h^2}{\delta} \ln \delta$ and this is the same order as the ‘non-parallel’ λ_b -term if

$$\delta_2 \sim \frac{\delta_2}{\epsilon^3 K_1} \frac{h^2}{\delta} \ln \delta. \quad (3.2.16)$$

Recall that the $\hat{X}$ -modulation was introduced to damp the induced-vortex velocity components in the buffer layer, and so we want the inertial operator, $Y \frac{\partial}{\partial \bar{X}}$, to balance the viscous one, $\frac{\partial^2}{\partial Y^2}$, which immediately implies that

$$\delta^3 \sim \frac{\delta_2}{\epsilon^3 K_1}. \quad (3.2.17)$$

One further relation (between the unknowns h , δ and δ_2) is required and results from balancing the slower $\bar{X}$ -modulation with the second (correction) terms in Taylor series for λ (i.e. balancing λ_b term with $P_{\bar{X}}$ in the x -momentum equation), leading to the balance

$$\delta_2 \sim \frac{\epsilon^3 K_1}{\delta_2} \quad \text{i.e.} \quad \delta_2 \sim \epsilon^{\frac{3}{2}} K_1^{\frac{1}{2}}. \quad (3.2.18a)$$

The other two sizes follow immediately from (3.2.16) and (3.2.17):

$$\delta \sim \epsilon^{-\frac{1}{2}} K_1^{-\frac{1}{6}} \quad \text{and} \quad h \sim \left[\frac{\epsilon^{\frac{5}{2}} K_1^{\frac{5}{6}}}{\ln \delta} \right]^{\frac{1}{2}}. \quad (3.2.18b, c)$$

Note that we are now assigning a size to the small quantity h (cf. other weakly non-linear analyses).

The logarithmic factor occurring in h is important (Blennerhassett & Smith, 1991); it is wrong to dismiss it as unimportant, as did Hall & Smith (1989). We note that δ is large, whilst h and δ_2 are small, as required. We return to a discussion of the implications of these scalings after deriving the interaction equations, and associated coefficients, in the following sections.

§3.3. THE DERIVATION OF THE INTERACTION EQUATIONS

In this section we briefly outline the ‘first-principles’ derivation of the interaction equations and associated coefficients; for more details see Hall & Smith (1989), whose notation we try to adhere to for ease of reference. This approach, though not being the most efficient way to derive the generalised interaction coefficients, has the advantage of clearly illustrating the underlying physical properties and nature of the vortex–wave interaction. We take advantage of their work in that we do not give any details of the main deck solutions — this layer is passive as far as the current vortex–wave interaction is concerned: we work with the lower-deck equations, coupled with the upper-deck’s pressure–disturbance–equation, via the pressure–displacement law. It is important to note that, as far as is possible, the following scalings and expansions are given relative to those of the compressible–triple-deck structure, rather than the primitive Navier–Stokes equations. In this section we assume that the flow is supersonic ($M_\infty > 1$) for completeness; in Section 4 we give an alternative derivation of the interaction-coefficients for a general Tollmien–Schlichting eigenrelation.

§3.3.1 The lower deck.

Following Hall & Smith (1989), but using the ‘new’ scales derived in the last section, we expand the flow quantities as follows

$$\begin{aligned}
U &= \lambda Y + h\tilde{u}^{(1)} + \delta_2[\lambda_3 Y + \bar{X}\lambda_b Y] + h^2\tilde{u}^{(3)} + h\delta_2\tilde{u}^{(e)} + \dots, \\
V &= h\tilde{v}^{(1)} + h^2\tilde{v}^{(3)} + h\delta_2\tilde{v}^{(e)} + \dots, \\
W &= h\tilde{w}^{(1)} + h^2\tilde{w}^{(3)} + h\delta_2\tilde{w}^{(e)} + \dots, \\
P &= h\tilde{p}^{(1)} + h^2\tilde{p}^{(3)} + h\delta_2\tilde{p}^{(e)} + \dots.
\end{aligned} \tag{3.3.1a-d}$$

Here superscript (1) corresponds to the (primary) TS-mode; superscript (3) corresponds to the induced vortex; superscript (e) corresponds to TS-mode forced by the ' $\bar{X}\lambda_b + \lambda_3$ '-terms; and λ_3 is the wall shear of the vortex which is induced (in the buffer).

Further, we decompose the wave-terms into 'oblique' pairs, and all terms are then expressed in terms of E_1 , E_2 or E_3 which are assumed to capture all their X , Z and T dependence. The amplitude functions are then (just) functions of Y and the modulation scale $\bar{X}$ (at most). For example,

$$\begin{aligned}
\tilde{u}^{(1)} &= \tilde{u}_{11}(\bar{X}, Y)E_1 + \tilde{u}_{12}(\bar{X}, Y)E_2 + \text{c.c.}, \\
\tilde{u}^{(e)} &= \tilde{u}_{e1}(\bar{X}, Y)E_1 + \tilde{u}_{e2}(\bar{X}, Y)E_2 + \text{c.c.}, \\
\tilde{u}^{(3)} &= \tilde{u}_{33}(\bar{X}, Y)E_3 + \text{c.c.}, \\
\lambda_3 &= \lambda_{33}(\bar{X})E_3 + \text{c.c.},
\end{aligned} \tag{3.3.2a-d}$$

and similarly for the V, W, A and P terms.

The resulting equations for $(\tilde{u}_{1i}, \tilde{v}_{1i}, \tilde{w}_{1i}, \tilde{p}_{1i})$, $i=1,2$, are merely the linear TS-wave equations studied in Chapter 2 (see also Smith, 1989). The equations for the induced vortex are those found by Hall & Smith†; they are forced by the TS-waves via the nonlinear inertia terms. Here, for completeness of argument, we consider the z -momentum equation as this is crucial to, and an understanding of, the interaction-mechanism which was not quite correctly deduced in the latter paper. In fact, only the E_3 component is of concern and it is found to satisfy

$$\tilde{w}_{33Y} = -i\alpha\tilde{u}_{11}\tilde{w}_{12}^{(cc)} + i\alpha\tilde{u}_{12}^{(cc)}\tilde{w}_{11} + \tilde{v}_{11}\tilde{w}_{12Y}^{(cc)} + \tilde{v}_{12}^{(cc)}\tilde{w}_{11Y} + 2i\beta\tilde{w}_{11}\tilde{w}_{12}^{(cc)}, \tag{3.3.3a}$$

† note however that the definitions of E_1 and E_2 differ between the two studies; the replacement $\beta \rightarrow \frac{\beta}{2}$ is necessary to recover the equations of Hall & Smith

where superscript (*cc*) signifies complex conjugate. It can easily be shown that

$$\tilde{w}_{33} \sim -2i\beta \left(1 - \frac{\beta^2}{\alpha^2}\right) \lambda^{-2} \tilde{p}_{11} \tilde{p}_{12}^{(cc)} \ln Y + O(1), \quad \text{as } Y \rightarrow \infty; \quad (3.3.3b)$$

this follows the solution for the TS-modes, and from (3.2.12), the large- Y asymptotes of the TS-modes. The buffer region, to be discussed in a later subsection, was introduced by Hall & Smith to account for this logarithmic growth — to damp it via amplitude modulation. It is important to note that, following Blennerhassett & Smith, we are not treating the constant of integration as a leading order quantity, thus avoiding the mistake made by Hall & Smith, Smith & Walton (1989) and the present author (when he first considered the vortex-wave interaction under discussion in this chapter). This large- Y asymptote for $\tilde{w}_{33}$ is essentially responsible for the forcing of the vortex in the buffer region, via a boundary condition at the bottom of that region.

Next we consider the TS-modes forced by the non-parallelism of the underlying (growing) boundary-layer flow and by the induced vortex. These are found to satisfy the following equations (having substituted (3.3.1) and (3.3.2) into the lower-deck equations (3.2.6))

$$\begin{aligned} \tilde{u}_X^{(e)} + \tilde{u}_{\bar{X}}^{(1)} + \tilde{v}_Y^{(e)} + \tilde{w}_Z^{(e)} &= 0, \\ \tilde{u}_T^{(e)} + \lambda Y(\tilde{u}_X^{(e)} + \tilde{u}_{\bar{X}}^{(1)}) + (\bar{X}\lambda_b + \lambda_{33})Y\tilde{u}_X^{(1)} \\ + (\bar{X}\lambda_b + \lambda_{33})\tilde{v}^{(1)} + \lambda\tilde{v}^{(e)} + \underline{\lambda_{33}ZY\tilde{w}} &= \\ -\tilde{p}_X^{(e)} - \tilde{p}_{\bar{X}}^{(1)} + \tilde{u}_{Y\bar{Y}}^{(e)}, \\ \tilde{p}_Y^{(e)} &= 0, \end{aligned}$$

and

$$\tilde{w}_T^{(e)} + \lambda Y(\tilde{w}_X^{(e)} + \tilde{w}_{\bar{X}}^{(1)}) + (\bar{X}\lambda_b + \lambda_{33})Y\tilde{w}_X^{(1)} = -\tilde{p}_Z^{(e)} + \tilde{w}_{Y\bar{Y}}^{(e)}. \quad (3.3.4a - d)$$

The ‘underlined’ term, proportional to $\lambda_{33}Z$, is absent in Hall & Smith (1989); this further error, found by the current author, (in addition to that found by Blennerhassett & Smith, 1991, concerning the order of the constant of integration in the large- Y asymptote (3.3.3b); whose correction leads to a simpler set of interaction-equations needing to be solved) results in corrected values for the

important interaction coefficients. In Section 5 we will see that the numerical values of the corrected coefficients, for the incompressible ($M_\infty = 0$) case, are significantly different than those calculated by Hall & Smith from the incorrect formulae, which in turn leads to new conclusions being drawn concerning this vortex–wave interaction in incompressible flows. The current author did not uncover this error until he calculated the interaction coefficients by an alternative method (which was, ironically, suggested by Prof. F.T. Smith; see Section 4) and found some disagreement in one of the interaction coefficients, between the two methods. With hindsight, the author finds it very surprising that this term was initially missed by everyone (including himself); the fact that the induced–vortex modifies the wall–shear at lower order ($\lambda \longrightarrow \lambda + \delta_2 \lambda_3(Z)$) is crucial to the whole interaction–mechanism, cleverly deduced by Hall & Smith.

The interaction–mechanism is most apparent in (3.3.4b); here we see contributions from non–parallelism ($\propto \lambda_b$) and the induced–vortex ($\propto \lambda_{33}$) appearing at the same order as the (primary) TS–wave–quantities’ modulation on $\bar{X}$; this was the motivation for the choice of scales of the previous section. To complete the statement of problem for these TS–modes we must of course consider the upper–deck and the matching process. Recall that in the upper–deck, the M_∞ –dependence manifests itself, and so there the details (of the present study) alter significantly from those of Hall & Smith. In the next sub–section we consider the upper–deck expansions and solutions, then we proceed to reconsider the solution of the lower–deck problem for the ‘forced’ TS–waves which requires a so–called ‘compatibility relation’ to be satisfied. This relation is similar to that found by Hall & Smith but the associated coefficients are now functions of the Mach number (implicitly and explicitly). This relation is in fact a modulation equation for the amplitude of the (primary) TS–modes — similar to those commonly deduced from weakly–nonlinear analyses.

§3.3.2 The upper–deck and pressure–displacement law.

The changes, caused by compressibility, to the corrected incompressible study of Hall & Smith are due to the presence of the Mach number in the Prandtl–Glauert operator, governing the solutions in the upper–deck. We have seen how the TS–wave eigenrelation is altered by compressibility; in addition we expect

the interaction coefficients to be altered by the *explicit* occurrence of the Mach number. Thus we devote more attention to this deck, as well as to the pressure–displacement law, linking the lower– and upper–decks, leading to the compatibility relation. Remember that, at present, we are considering the supersonic case for definiteness; other cases will be considered in the next chapter.

The upper-deck lies outside the boundary layer and linearised potential flow properties are appropriate. Here, the nondimensionalised, velocities and pressure are expand as (cf. (3.2.5))

$$\begin{aligned} u &= 1 + Re^{-\frac{1}{4}} K_2 \left[hu^{(1)} + \dots + h\delta_2 u^{(e)} + \dots \right] + \dots, \\ v &= Re^{-\frac{1}{4}} K_2 (M_\infty^2 - 1)^{\frac{1}{2}} \left[hv^{(1)} + \dots + h\delta_2 v^{(e)} + \dots \right] + \dots, \\ w &= Re^{-\frac{1}{4}} K_2 \left[hw^{(1)} + \dots + h\delta_2 w^{(e)} + \dots \right] + \dots, \\ p &= Re^{-\frac{1}{4}} K_2 \left[hp^{(1)} + \dots + h\delta_2 p^{(e)} + \dots \right] + \dots, \end{aligned} \quad (3.3.5a - d)$$

where

$$K_2 = \left(\frac{\lambda_S}{\sqrt{2}} \right)^{\frac{1}{2}} \mu_w^{\frac{1}{2}} T_w^{-\frac{1}{2}} (M_\infty^2 - 1)^{-\frac{1}{4}}, \quad (3.3.5e)$$

and we have only highlighted the terms of immediate concern to us here in this discussion. Again we decompose into E_1 and E_2 components,

$$p^{(1)} = p_1^{(1)} E_1 + p_2^{(1)} E_2 + \text{c.c.},$$

$$p^{(e)} = p_1^{(e)} E_1 + p_2^{(e)} E_2 + \text{c.c.},$$

and similarly for the other quantities. The boundary conditions are those of decay at infinity, and matching with the boundary layer;

$$p_i^{(1)} \rightarrow 0, \quad p_i^{(e)} \rightarrow 0, \quad \text{as } \hat{y} \rightarrow \infty,$$

$$\text{and } p_i^{(1)} = \tilde{p}_{1i}^{(1)}, \quad p_i^{(e)} = \tilde{p}_{1i}^{(e)}, \quad \text{as } \hat{y} \rightarrow 0^+, \quad (3.3.6a - d)$$

where $i = 1, 2$ in the above and remainder of this subsection. After a little manipulation we soon arrive at the governing equations for the pressures,

$$\mathcal{L} p_i^{(1)} = 0 \quad \text{and} \quad \mathcal{L} p_i^{(e)} = 2i\alpha (M_\infty^2 - 1) p_i^{(1)} \bar{X},$$

where

$$\mathcal{L} = (M_\infty^2 - 1) \partial_{\hat{y}\hat{y}} - [\beta^2 - \alpha^2 (M_\infty^2 - 1)]. \quad (3.3.7a - c)$$

The solutions, which decay at infinity and match to the pressures in the viscous boundary sub-layers, are

$$p_i^{(1)} = \tilde{p}_{1i}^{(1)} \exp \left[-B^{-\frac{1}{2}} \hat{y} \right], \quad p_i^{(e)} = \left(\frac{-i\alpha \tilde{p}_{1i} \hat{y}}{B^{\frac{1}{2}}} + \tilde{p}_{ei}^{(1)} \right) \exp \left[-B^{-\frac{1}{2}} \hat{y} \right],$$

$$\text{where } B = \frac{\beta^2}{(M_\infty^2 - 1)^2} - \alpha^2 \quad (3.3.8a - c)$$

We can now solve for the unknown displacements (steming from the lower-deck),

$$A^{(1)} = A_1^{(1)} E_1 + A_2^{(1)} E_2 + \text{c.c.}, \quad A^{(e)} = A_1^{(e)} E_1 + A_2^{(e)} E_2 + \text{c.c.},$$

by appealing to the pressure-displacement law, central to the triple-deck theory,

$$\begin{aligned} \frac{\partial}{\partial \hat{y}} \left[\tilde{p}_{1i}^{(1)} + h\delta_2 \tilde{p}_{ei}^{(1)} + \dots \right] = \\ \left[\frac{\partial}{\partial X} + h\delta_2 \frac{\partial}{\partial \bar{X}} + \dots \right]^2 \left(A_i^{(1)} + h\delta_2 A_i^{(e)} + \dots \right); \end{aligned} \quad (3.3.9)$$

care must be taken to avoid dropping $O(h\delta_2)$ -terms here, and in subsequent analysis.

As (independently) pointed out by Zhuk & Ryzhov (1981), Smith(1989) and Duck (1990), the restriction,

$$\text{Real}\{B\} > 0, \quad (3.3.10)$$

on the wavenumbers is necessary for the upper deck quantities to remain bounded as free-stream is approached. Note that it is the quantity B that contains the influence of compressibility on the interaction in addition to its influence on the TS-wave problem.

The leading order balance, of (3.3.9), yields the standard $P - A$ relation for the primary modes. The second order balance (at $O(h\delta_2)$) yields the following relation between $A_i^{(e)}$, $A_i^{(1)}$ and $\tilde{p}_{ei}^{(1)}$,

$$B^{\frac{1}{2}} \tilde{p}_{ei}^{(1)} = \alpha^2 A_i^{(e)} - 2i\alpha A_{i\bar{X}}^{(1)} - \frac{i\alpha^3 A_{i\bar{X}}^{(1)}}{B}, \quad (3.3.11)$$

which enables us to procede with the solution of the equations for the forced-TS-mode. These are considered in the next subsection.

§3.3.3 The compatibility relations and the interaction coefficients.

We now return to the lower deck equations to re-consider the primary and forced TS-modes. Those for the primary mode can be solved in terms of the Airy function, leading to the linear eigenrelation (3.2.11). The equations for the forced TS-modes, (3.3.4), are inhomogenous versions of the former, driven by the modulation of the primary modes; the non-parallelism of the underlying boundary-layer; and the wall shear of the induced vortex. The solution of these equations, subject to the appropriate boundary conditions, requires two compatibility relations,

$$a\bar{p}_{11\bar{X}} + b\lambda_b\bar{X}\bar{p}_{11} + c\lambda_{33}\lambda^{-1}\bar{p}_{12}^{(cc)} = 0$$

and

$$a\bar{p}_{12\bar{X}} + b\lambda_b\bar{X}\bar{p}_{12} + c\lambda_{33}^{(cc)}\lambda^{-1}\bar{p}_{11}^{(cc)} = 0, \quad (3.3.12a, b)$$

to be satisfied. These are identical in appearance to those derived by Hall & Smith; however, now the compatibility coefficients a , b and c are now functions of the Mach number. The presence of M_∞ in these coefficients leads to the solution properties for compressible flow (reported in Section 5) differing from those for the incompressible work. In fact, after a little algebraic manipulation, we find that

$$\begin{aligned} a &= \frac{2r_1\gamma D\xi_0\Delta^{-2}}{3\alpha} - iB^{-\frac{1}{2}} \left[\frac{2B}{\gamma} + \frac{B}{3\alpha^2} + 1 \right], \\ b &= -\frac{2r_1\gamma D\xi_0\alpha\Delta^{-5}}{3} - \frac{5B^{\frac{1}{2}}}{3\alpha}, \\ c &= iD\xi_0\Delta^{-2} \left(\frac{2r_1\gamma}{3} + \beta^2 r_2 \right) - \alpha^{-1} B^{-\frac{1}{2}} \left(\frac{5B}{3} - \frac{3\beta^2 B}{\gamma} \right), \end{aligned} \quad (3.3.13a, b)$$

where

$$\begin{aligned} r_1 &= \frac{Ai(\xi_0)}{Ai'(\xi_0)}, \quad r_2 = \frac{\kappa(\xi_0)}{Ai(\xi_0)}, \quad D = 1 + \frac{\xi_0\kappa(\xi_0)}{Ai'(\xi_0)}, \\ \gamma &= \alpha^2 + \beta^2, \quad B = \frac{\beta^2}{M_\infty^2 - 1} - \alpha^2, \quad \Delta = (i\alpha\lambda)^{\frac{1}{3}} \quad \text{and} \quad \xi_0 = -\frac{i^{\frac{1}{3}}\Omega}{(\lambda\alpha)^{\frac{2}{3}}}. \end{aligned} \quad (3.3.14a - g)$$

Remember that these coefficients correspond to supersonic flow — in Section 4 we consider other cases. The quantitative values of these coefficients, and their implications, are discussed in Section 5. First we consider the buffer region where the remaining interaction-equations originate from. In this region the

longitudinal vortex is forced solely by an inner (from the lower-deck beneath) boundary-condition (Blennerhassett & Smith, 1991).

§3.3.4 The buffer layer.

This is the new layer that was introduced into the conventional triple-deck structure by Hall & Smith. Here we take it to lie inside, but at the top off, the lower-deck; in particular it is characterised by $Y \sim \delta \gg 1$. Thus we define the buffer- y variable by

$$\bar{Y} = \delta^{-1}Y, \quad (3.3.15)$$

where $\bar{Y} \sim O(1)$ in the buffer.

As previously mentioned, the interaction's vortex equations come from this sub-layer, but are solely forced by the (primary) TS-modes in the lower-deck via the small- $\bar{Y}$ boundary conditions. Here, the TS-modes merely continue their asymptotic decay from the lower-deck into the main-deck. Again, as M_∞ has been scaled out of all but the upper-deck problem, the forms of the expansions here follow those of Hall & Smith, but they are modified to take account of the inclusion of the neglected logarithmic factor (see Blennerhassett & Smith, 1991).

In this layer, the expansions for the lower-deck quantities[†], are

$$\begin{aligned} U &= \delta \lambda \bar{Y} + \delta_2 \delta \bar{X} \lambda_b \bar{Y} + h^2 \ln \delta \left[\frac{\delta_2}{\epsilon^3 K_1} \right] \hat{u}^{(3)} + \dots + h \hat{u}^{(1)} + \dots \\ &\quad + h \delta^{-1} \hat{u} + \dots + h \delta_2 \hat{u}^{(e)} + \dots, \\ V &= h \delta \hat{v}^{(1)} + h \hat{v} + \delta^2 \epsilon^3 K_1 \left[\frac{-\lambda_b \bar{Y}^2}{2} \right] + h^2 \delta \ln \delta \hat{v}^{(3)} + \dots \\ &\quad + h \delta \delta_2 \hat{v}^{(e)} + \dots, \\ W &= h \delta^{-1} \hat{w}^{(1)} + h \delta^{-2} \hat{w} + h^2 \ln \delta \hat{w}^{(3)} + h \delta_2 \delta^{-1} \hat{w}^{(e)} + \dots, \\ P &= h \hat{p}^{(1)} + h \delta_2 \hat{p}^{(e)} + \dots. \end{aligned} \quad (3.3.16)$$

These expansions are implied mostly by the large- Y forms from the lower deck, and by the desire to pick out our particular interaction. Again, the non-mean-flow terms are decomposed into their E_1, E_2 and E_3 components. The

[†] note that in this chapter the interaction is being formulated with respect to the triple-deck scales and that, further, we are treating the buffer as essentially the upper-part of the lower-deck

TS-parts, denoted by superscripts (1) and (e) for the leading order parts of the primary and forced waves, respectively, and the higher order parts, denoted by double-hat, are not of principal concern here.

Writing

$$\hat{u}^{(3)} = \hat{u}_{33}E_3 + \text{c.c.},$$

together with similar expansions for $\hat{v}^{(3)}$ and $\hat{w}^{(3)}$, yields the vortex equations

$$\hat{u}_{33}\bar{X} + \hat{v}_{33}\bar{Y} + 2i\beta\hat{w}_{33} = 0,$$

$$\lambda\bar{Y}\hat{u}_{33}\bar{X} + \lambda\hat{v}_{33} = \hat{u}_{33}\bar{Y}\bar{Y},$$

$$\lambda\bar{Y}\hat{w}_{33}\bar{X} = \hat{w}_{33}\bar{Y}\bar{Y}, \quad (3.3.17a - c)$$

which must be solved subject to the following boundary conditions:

$$\hat{u}_{33} \rightarrow \lambda A_{33}, \quad \hat{v}_{33} \rightarrow -\lambda A_{33}\bar{X}\bar{Y}, \quad \hat{w}_{33} \simeq \bar{Y}^{-3}, \quad \text{as } \bar{Y} \rightarrow \infty, \quad (3.3.17d)$$

$$\hat{u}_{33} = \hat{v}_{33} = 0, \quad \hat{w}_{33} = -2i\beta K \lambda^{-2} \tilde{p}_{11} \tilde{p}_{12}^{(cc)}, \quad \text{on } \bar{Y} = 0, \quad (3.3.17e)$$

where, for future convenience, we have defined the important quantity

$$K = \left(1 - \frac{\beta^2}{\alpha^2}\right). \quad (3.3.18)$$

These vortex-equations are discussed in Section 5 within the framework of the whole interaction. Note that the last condition of (3.3.17e) contains all the forcing (due to the nonlinear combination of the (primary) TS-modes in the lower-deck) — this result is identical to that deduced by Blennerhassett & Smith when they considered the corrections necessary to the initial study by Hall & Smith.

§3.4 AN ALTERNATIVE DERIVATION OF THE INTERACTION COEFFICIENTS; GENERAL TS-EIGENRELATION.

In this section we show how the supersonic interaction-coefficients, (3.3.13), can be derived in an alternative, and quicker, manner. Moreover, we generalise the theory to other cases by considering a ‘general’ TS-eigenrelation. Recall that the interaction-coefficients (the coefficients of the compatibility relations) follow from an inhomogenous (forced) form of the equations for the (primary) TS-modes. As the latter leads to the TS-eigenrelation, we expect that by small (appropriate) perturbations of this eigenrelation (corresponding to the inhomogeneities of the equation for the forced TS-modes) we should recover (parts of) the compatibility relations (and hence the desired coefficients), at the order of the small-perturbations. This alternative method, for deriving the interaction-coefficients, was suggested to the author by Prof. F.T. Smith as a means to check the coefficients that had been calculated by solving the forced TS-wave equations, (3.3.4). As mentioned earlier, this ‘checking’ led the author to uncover an error in the work of Hall & Smith, namely the absence of the λ_{33Z} -term. Whilst performing the analysis for the supersonic case, the author realised that it would be far more sensible (and in fact less algebraically tiresome) to consider a ‘general’ TS-eigenrelation and hence derive a general set of interaction coefficients in one go.

The method is much clearer in practice and we simplify the method by splitting the analysis into two parts: firstly, we calculate the ratio of b to a , and then we calculate the ratio of c to a in a separate analysis. Note that we are considering ‘ratios’ rather than the interaction coefficients themselves, as the compatibility relations (3.3.12), are unique ‘modulo’ a multiplicative factor — in fact there are only effectively two independent interaction coefficients, the ratios b/a and c/a .

By a ‘general TS-eigenrelation’, we are assuming one that has the form

$$\frac{Ai'}{\kappa}(\xi_0) = (i\alpha)^{\frac{1}{3}} \lambda^{-\frac{5}{3}} \frac{f(\alpha ; \beta, M_\infty, \dots)}{g(\alpha ; \beta, M_\infty, \dots)}, \quad (3.4.1)$$

where f and g are known functions whose exact form is dependent on factors such as, for example, the Mach number or geometry. In the methods that follow, it is sufficient to treat them as functions of α alone — see Table 3.1.

Table 3.1: The functions f and g for some typical TS-wave eigenrelations. ‡

	Flow type	f	g
1	Incompressible	$\alpha^2 + \beta^2$	$(\alpha^2 + \beta^2)^{\frac{1}{2}}$
2	Subsonic	$\alpha^2 + \beta^2$	$\left(\alpha^2 + \frac{\beta^2}{1 - M_\infty^2}\right)^{\frac{1}{2}}$
3	Supersonic	$\alpha^2 + \beta^2$	$\left(\frac{\beta^2}{M_\infty^2 - 1} - \alpha^2\right)^{\frac{1}{2}}$
4	Hypersonic (no shock)	β^2	$(\beta^2 - \alpha^2)^{\frac{1}{2}}$
5(i)	Hypersonic (with shock) $\beta > \alpha$	β^2	$(\beta^2 - \alpha^2)^{\frac{1}{2}} \coth[(\beta^2 - \alpha^2)^{\frac{1}{2}} y_s]$
5(ii)	Hypersonic (with shock) $\beta < \alpha$	β^2	$(\alpha^2 - \beta^2)^{\frac{1}{2}} \cot[(\alpha^2 - \beta^2)^{\frac{1}{2}} y_s]$
6	Transonic limit	$\alpha^2 + \beta^2$	β
7	Non-axisymmetric flow on axisymmetric surface	$i\alpha \left[1 + \frac{n^2}{\alpha^2 a^2}\right] \times$ $K_n[i\alpha a \sqrt{M_\infty^2 - 1}]$	$\sqrt{M_\infty^2 - 1} K'_n[i\alpha a \sqrt{M_\infty^2 - 1}]$

‡Table 3.1: The functions f and g are given for some typical TS-wave eigenrelations. It should be noted that the scaled wavenumbers α and β , appearing within the table, are not the same for each case; they are the result of different scalings. The actual form of these scalings does not concern us here: the full details of the derivations of these eigenrelations can be found in the paper by Smith (1989), for cases 1-4 and 6; in the paper by Cowley & Hall (1990) for cases 5(i,ii); whilst case 7, in which K_n represents the n^{th} modified Bessel function, is derived by Duck & Hall (1990).

§3.4.1 The ratio b/a .

Firstly we indicate how the interaction coefficients a and b , in terms of f and g , can be deduced elegantly by using a method referred to by Smith (1980) and Hall & Smith (1984). This method involves expanding (perturbing) the eigenrelation about the neutral state and forming an amplitude equation for the pressure.

We suppose that the eigenrelation (3.4.1) is satisfied by $\alpha, \beta, \Omega, \lambda$ and ξ_0 ; and, moreover, that these correspond to a neutral mode. The neutral values of α , λ and ξ_0 are then perturbed by small amounts (signified by ‘overbars’)

$$(\alpha, \lambda, \xi_0) \longrightarrow (\alpha + \bar{\alpha}, \lambda + \bar{\lambda}, \xi_0 + \bar{\xi}_0). \quad (3.4.2)$$

Now, from the definition of κ , (3.2.11b), and the properties of the Airy function, Ai , it follows that

$$\frac{Ai'}{\kappa}(\xi_0) \longrightarrow \frac{Ai'}{\kappa}(\xi_0) \left(1 + \bar{\xi}_0 \left[\xi_0 \frac{Ai_0}{Ai'_0} + \frac{Ai_0}{\kappa_0} \right] + O(\text{overbar} - \text{squared}) \right), \quad (3.4.3)$$

whilst ‘expanding’ ξ_0 yields the following expression for $\bar{\xi}_0$:

$$\frac{\bar{\xi}_0}{\xi_0} = -\frac{2}{3} \left(\frac{\bar{\alpha}}{\alpha} + \frac{\bar{\lambda}}{\lambda} \right). \quad (3.4.4a)$$

Further,

$$(i\alpha)^{\frac{1}{3}} \longrightarrow (i\alpha)^{\frac{1}{3}} \left(1 + \frac{\bar{\alpha}}{3\alpha} + \dots \right), \quad \lambda^{-\frac{5}{3}} \longrightarrow \lambda^{-\frac{5}{3}} \left(1 - \frac{5\bar{\lambda}}{3\lambda} + \dots \right)$$

and

$$\frac{f(\alpha)}{g(\alpha)} \longrightarrow \frac{f(\alpha)}{g(\alpha)} \left[1 + \bar{\alpha} \left(\frac{f'}{f} - \frac{g'}{g} \right) + \dots \right]. \quad (3.4.4b - d)$$

When these expansions are substituted into the eigenrelation, the ‘neutral’ eigenrelation (3.4.1) is recovered. It is at next order, from the overbar-terms, that

the pressure–amplitude equation and the interaction coefficients follow. At this present order we find, after some algebraic simplification,

$$\left[\frac{-2r_1 D\xi_0}{3\alpha} - \frac{g}{f\Delta} \left(\frac{1}{3\alpha} + \frac{f'}{f} - \frac{g'}{g} \right) \right] \bar{\alpha} \check{P} = \left[\frac{2r_1 D\xi_0}{3} - \frac{5g}{3f\Delta} \right] \bar{\lambda} \check{P}, \quad (3.4.5)$$

in which we have now set $\lambda = 1$; also we have multiplied ‘both sides’ by a ‘notional’ pressure amplitude $\check{P}$. Here D, r_1 are as defined by (3.3.14).

The crucial step in the argument is to now equate the neutral–wavenumber–perturbation, $\bar{\alpha}$, with a streamwise derivative,

$$\bar{\alpha} \rightarrow -i \frac{\partial}{\partial \check{X}}, \quad (3.4.6)$$

say, and interpret $\bar{\lambda}$ as the second term of the Taylor expansion of the wall–shear λ about the neutral position, i.e.

$$\bar{\lambda} = \check{X} \lambda_b, \quad (3.4.7)$$

using the previous definition of λ_b (cf. (3.2.15a)).

We can now pick out the coefficients a and b , of $P_{\check{X}}$ and $\lambda_b \check{X}$ respectively, noting that these are unique to a common multiplicative constant of linearity — we are only actually interested in the ratio of the two anyway. We find that (setting this ‘constant of multiplication to be f ’)

$$a = \frac{2fr_1 D\xi_0}{3\alpha\Delta^2} - \frac{ig}{\alpha} \left(\frac{1}{3\alpha} + \frac{f'}{f} - \frac{g'}{g} \right)$$

and

$$b = -\frac{2fr_1 \alpha D\xi_0}{3\Delta^5} - \frac{5g}{3\alpha}. \quad (3.4.8a, b)$$

Thus we have arrived at ‘general’ formulae for the interaction coefficients a and b . We suspend until later the investigation of these formula, including the check on the previous results (3.3.13). First, we outline the alternative method of deriving the formulae for the interaction coefficients a and c ; this is the subject of the next subsection.

§3.4.2 The ratio c/a .

We now turn to the c/a , which we shall see in the next section to be crucial to the possible large- $\bar{X}$ behaviour of the vortex-wave interaction. The interaction coefficients a and c (or, more precisely, the ratio of the two) can also be derived more elegantly (than the ‘first-principles’ method of Section 3) by not assuming harmonic Z -dependence when solving the triple-deck problem for the (primary) linear TS-modes (see, for example, Smith, 1979c; Smith & Walton, 1989; Walton, 1991). Instead of the conventional eigenrelation (of the form (3.4.1)), the eigen-problem consists of a second-order, ordinary differential equation (for the pressure-disturbance $\check{p}$ of the form

$$\begin{aligned} \frac{d^2 \check{p}}{dZ^2} - \frac{\lambda_Z}{\lambda} \left[\frac{3}{2} + \frac{\xi_0}{2Ai_0} (\xi_0 \kappa + Ai_0') \right] \frac{d\check{p}}{dZ} - \alpha^2 \check{\nu} \check{p} \\ = \left(\frac{Ai_0'}{\kappa_0} \right) \frac{(\lambda i \alpha)^{\frac{5}{3}} g(\alpha)}{\alpha^2} \check{p}, \end{aligned} \quad (3.4.9a)$$

to be solved subject to specified boundary conditions (e.g. ‘periodicity’). Note that the possibility that the wall-shear, λ , may be a function of the spanwise variable, Z , has been accounted for. Here, and hereafter, we assume that the function f , appearing in the ‘general’ TS-eigenrelation, is of the form

$$f = \check{\nu} \alpha^2 + \beta^2, \quad (3.4.9b)$$

where $\check{\nu}$ is a constant; we see from Table 3.1, that this form is indeed general enough for the cases 1-6 considered.

The following method (suggested by Prof F.T.Smith) has some similarity to that of §3.4.1; but now we must include the ‘wave- and vortex-factors’, E_1, E_2 and E_3 , in addition to perturbing the ‘neutral’ quantities. We write

$$\begin{aligned} \check{p} &= \check{p}_{01} E_1 + \check{p}_{02} E_2 + \check{\epsilon}(\check{p}_{11} E_1 + \check{p}_{12} E_2) + \cdots + \text{c.c.}, \\ \lambda &\rightarrow \lambda + \check{\epsilon}(\lambda_1 E_3 + \text{c.c.}) + \cdots, \quad \alpha \rightarrow \alpha + \check{\epsilon} \alpha_1 + \cdots \end{aligned}$$

and

$$\xi_0 \rightarrow \xi_0 + \check{\epsilon} \xi_1 + \cdots, \quad (3.4.10a - d)$$

where $\epsilon \ll 1$ is the small expansion parameter. The last two expansions, for α and ξ_0 , follow the corresponding ones of §3.4.1. The perturbation to λ is ‘vortex-like’ — it represents the correction to the wall-shear due to the vortex which has been induced in the buffer-layer. The pressure expansion comprises of the two (primary) oblique TS-waves, each of whose amplitude has been perturbed by a small amount. This perturbed pressure terms correspond to the TS-modes forced by the perturbation to the wall shear.

Once the appropriate forms of the expansions, (3.4.10), are realised, the remaining analysis to derive the general forms for the coefficients, a and c , simply follows that of §3.4.1. When these expansions are substituted into (3.4.9a), the leading order terms (proportional to E_1 and E_2) yield the (familiar) linear TS-eigenrelation (3.4.1), whilst at $O(\epsilon)$, there is a contribution from the ‘ λ_Z ’-term. After a little algebraic manipulation, the terms proportional to $E_3^{\pm 1}$ at this order yield the following relation:

$$a i \alpha_1 \check{p}_{01,2} + c \lambda_1 \lambda^{-1} \check{p}_{02,1}^{(c,c)} = 0, \quad (3.4.11a)$$

where a and c are functions of the ‘leading order’ quantities:

$$a = \frac{2f r_1 D \xi_0}{3\alpha \Delta^2} - \frac{ig}{\alpha} \left(\frac{1}{3\alpha} + \frac{f'}{f} - \frac{g'}{g} \right)$$

and

$$c = i D \xi_0 \Delta^{-2} \left(\frac{2}{3} f r_1 + \beta^2 r_2 \right) - \frac{g}{\alpha f} \left(\frac{5f}{3} - 3\beta^2 \right). \quad (3.4.11b, c)$$

We now equate α_1 with a streamwise operator (cf. (3.4.6)),

$$i \alpha_1 \longrightarrow \frac{\partial}{\partial \bar{X}},$$

and λ_1 is equated with λ_{33} , so that (3.4.11a) transforms into

$$a \frac{d \check{p}_{01,2}}{d \bar{X}} + c \lambda_{33} \lambda^{-1} \check{p}_{02,1} = 0. \quad (3.4.11d)$$

Thus we immediately see that a and c , as given by (3.4.11b,c), are the desired interaction coefficients (to a given multiplicative constant.) In the next section numerical values for these interaction coefficients are given and their physical implications are discussed.

§3.5 THE INTERACTION EQUATIONS.

In the Section 3 the interaction was formulated. The interaction equations can be written

$$\begin{aligned}
 \tau(\bar{X}, \bar{Y}) &= U_{\bar{Y}}(\bar{X}, \bar{Y}), \\
 \tau_{\bar{Y}\bar{Y}}(\bar{X}, \bar{Y}) - \bar{Y}\tau_{\bar{X}}(\bar{X}, \bar{Y}) &= -2i\beta W(\bar{X}, \bar{Y}), \\
 W_{\bar{Y}\bar{Y}} - \bar{Y}W_{\bar{X}} &= 0, \\
 P_{\bar{X}}(\bar{X}) + (c_{1r}\lambda_b\bar{X} + c_{2r}\lambda_{33})P(\bar{X}) &= 0, \\
 \lambda_{33} &= \tau(\bar{X}, 0),
 \end{aligned} \tag{3.5.1a - e}$$

together with the boundary conditions

$$W(\bar{X}, \infty) = \tau(\bar{X}, \infty) = \tau_{\bar{Y}}(\bar{X}, 0) = 0 \quad \text{and} \quad W(\bar{X}, 0) = -2i\beta K P^2(\bar{X}). \tag{3.5.1f - i}$$

Here we have replaced $(\hat{u}_{33}, \hat{v}_{33}, \hat{w}_{33})$ by (U, V, W) ; set $\lambda = 1$, without loss of generality; and, for simplicity, we have taken the two primary TS-modes to have equal pressure-amplitude P (following Hall & Smith);

$$|\tilde{p}_{11}| = |\tilde{p}_{12}| = P. \tag{3.5.2a, b}$$

The (normalised) interaction coefficients

$$c_{1r} = \text{Real}\left(\frac{b}{a}\right) \quad \text{and} \quad c_{2r} = \text{Real}\left(\frac{c}{a}\right), \tag{3.5.3a, b}$$

are crucial to the solution properties; especially the large- $\bar{X}$ behaviour. Real parts have been taken as we are taking λ_{33} to be real; note that we are (primarily) concerned with the magnitude of the disturbance quantities, rather than with their phase.

In the next sub-section we show how they can be reduced to a single ‘integro-differential’ equation by taking Fourier transforms; although, in practice, a numerical solution of the full equations, (3.5.1), proves to be the most efficient method of solution. In §3.5.2 we investigate possible large- $\bar{X}$ behaviours (cf. Hall & Smith, 1989; Smith & Walton, 1990), before investigating the limiting cases of the Mach number tending to unity and infinity, in §3.5.3.

§3.5.1 Solution of interaction equations by Fourier transforms.

Here we show how (3.5.1) can be ‘solved’ by taking Fourier transforms in $\bar{X}$, defined by

$$\mathfrak{F}[f(\bar{X}, \bar{Y})] = f^{(f)}(\omega, \bar{Y}) = \int_{-\infty}^{\infty} f(\bar{X}, \bar{Y}) e^{-i\omega\bar{X}} d\bar{X}.$$

We find that (3.5.1c) transforms to

$$W_{\bar{Y}\bar{Y}}^{(f)}(\omega, \bar{Y}) - i\omega\bar{Y}W^{(f)}(\omega, \bar{Y}) = 0, \quad (3.5.5a)$$

which is immediately recognisable as Airy’s Equation; its solution being

$$W^{(f)}(\omega, \bar{Y}) = F^{(f)}(\omega) \frac{Ai[(i\omega)^{\frac{1}{3}}\bar{Y}]}{Ai[0]}; \quad (3.5.5b)$$

The coefficient of the ‘ Bi ’ having been set to zero so that the boundary condition at infinity can be satisfied. The arbitrary function of integration, $F^{(f)}(\omega)$, is fixed up from the boundary condition (3.5.1i) — more precisely we find that

$$F = -2i\beta KP^2, \quad (3.5.6)$$

where $F(\bar{X})$ the inverse transform of $F^{(f)}(\omega)$.

The transform of (3.5.1b) gives

$$\begin{aligned} \tau_{\bar{Y}\bar{Y}}^{(f)} - i\omega\bar{Y}\tau^{(f)} &= -2i\beta W^{(f)}, \\ &= \frac{-2i\beta F^{(f)}(\omega)}{Ai(0)} Ai[(i\omega)^{\frac{1}{3}}\bar{Y}], \end{aligned} \quad (3.5.7)$$

using (3.5.5b). This forced–Airy–equation is easily solved by appealing to the identity

$$(Ai'(x))'' - x(Ai'(x)) = Ai(x);$$

we soon find that

$$\tau^{(f)}(\omega, \bar{Y}) = \frac{-2i\beta F^{(f)}(\omega)}{Ai(0)(i\omega)^{\frac{2}{3}}} Ai'[(i\omega)^{\frac{1}{3}}\bar{Y}]. \quad (3.5.8)$$

Now, as $\lambda_{33}^{(f)} = \tau^{(f)}(0)$, it follows from (3.5.8) that

$$\lambda_{33}^{(f)}(\omega) = C_1(i\omega)^{-\frac{2}{3}} F^{(f)}(\omega), \quad \text{where } C_1 = -2 \frac{i\beta Ai'(0)}{Ai(0)}. \quad (3.5.9a, b)$$

We need to invert (3.5.9a), to arrive at the desired result for λ_{33} , and to this end we note the standard result

$$\int_0^\infty X^{-\frac{1}{3}} e^{-i\omega X} dX = \Gamma\left(\frac{2}{3}\right) (i\omega)^{-\frac{2}{3}},$$

where Γ is the gamma function. From this we can deduce that the inverse-Fourier-transform of $(i\omega)^{-\frac{2}{3}}$ is

$$\mathfrak{F}^{-1}[(i\omega)^{-\frac{2}{3}}] = \begin{cases} \frac{1}{\Gamma(\frac{2}{3})} \bar{X}^{-\frac{1}{3}}, & \text{if } \bar{X} > 0, \\ 0, & \text{otherwise.} \end{cases} \quad (3.5.10)$$

By employing the *convolution theorem of Fourier transformations*, we can invert $\lambda_{33}^{(f)}$, getting

$$\lambda_{33}(\bar{X}) = \frac{C_1}{\Gamma(\frac{2}{3})} \int_{-\infty}^{\bar{X}} F(\psi) (\bar{X} - \psi)^{-\frac{1}{3}} d\psi.$$

We now suppose that the interaction is ‘initialised’ at $\bar{X} = 0$ and we define functions to be zero for $\bar{X} < 0$ i.e. we can take the above lower limit of integration to be 0 instead of $-\infty$, so that

$$\lambda_{33}(\bar{X}) = \frac{4\beta^2 K Ai'(0)}{\Gamma(\frac{2}{3}) Ai(0)} \int_0^{\bar{X}} P(\psi)^2 (\bar{X} - \psi)^{-\frac{1}{3}} d\psi. \quad (3.5.11)$$

Substituting the last expression into the pressure-amplitude equation, (3.5.1d), yields a nonlinear ‘integro-differential’ equation,

$$\frac{dP}{d\bar{X}} + \left(c_{1r} \lambda_b \bar{X} + \frac{4c_{2r} \beta^2 K Ai'(0)}{\Gamma(\frac{2}{3}) Ai(0)} \int_0^{\bar{X}} P(\psi)^2 (\bar{X} - \psi)^{-\frac{1}{3}} d\psi \right) P = 0, \quad (3.5.12)$$

for the pressure amplitude P . This then has to be solved numerically, by ‘marching’ in $\bar{X}$. Similar equations have been found by Smith & Walton (1989), in their study of vortex-wave interactions.

§3.5.2 Possible limiting forms for large- $\bar{X}$.

We investigate analytically the possible options for the flow solution for large- $\bar{X}$. Hall & Smith found four such options for their system of equations and coefficients — below we see that (considering the incompressible case, $M_\infty = 0$) due to the two amendments to their work (the correction of interaction coefficient c and the inclusion of the logarithmic factor in the scales) one of these options is no longer feasible, namely that of exponential growth. Moreover, there is a swap in the signature required for the crucial quantity Kc_{2r} for the finite-distance-blow-up and the algebraic-growth-to-infinity eventualities to be possible. Thus, the conclusions, drawn later, for the case of zero Mach number are quite different from those found in Hall & Smith. In the next section numerical solutions of the interaction equations will be presented and comparison with these large- $\bar{X}$ predictions are made.

Option I: Finite – distance break – up.

In the vortex-wave study of Hall & Smith, it was shown that a possible, ultimate behaviour of the nonlinear interactive flow, as $\bar{X}$ increases, was that of an algebraic singularity arising at a finite position, say as $\bar{X} \rightarrow \bar{X}_0^-$. It was found that the only TS-forcing, in this case, on the vortex-equations was through the inner boundary condition on W : thus, as the present work does have these terms, this option is still possible but we find that there is a change in sign in the inner boundary condition on W compared to the corresponding condition of Hall & Smith. This is essentially due to the latter authors having to take the logarithm of a small number— in the ‘corrected’ theory this no longer occurs as the logarithm has been built into the interaction scales. This results in a sign change on the required polarity of Kc_{2r} for this option to be possible meaning that that finite-distance-break-up is, in fact, the ‘exception rather than the rule’, a complete reversal to the conclusions of Hall & Smith. Finally, the ‘correction’ of the interaction, c , results in quite different values for the polarity of Kc_{2r} (versus wave-angle θ), with $M_\infty = 0$, to those calculated by Hall & Smith.

We now give a few analytical details for this option. The ‘similarity’ forms proposed by Hall & Smith are (still) appropriate, apart from that for the pressure. As $\bar{X} \rightarrow \bar{X}_0^-$, we propose the following (leading order) behaviours for the interaction quantities:

$$P \sim (\bar{X}_0 - \bar{X})^{-\frac{5}{6}} \tilde{P}(\check{\eta}), \quad W \sim (\bar{X}_0 - \bar{X})^{-\frac{5}{3}} \tilde{W}(\check{\eta}),$$

$$(\tau, \lambda_{33}) \sim (\bar{X}_0 - \bar{X})^{-1} (\tilde{\tau}(\check{\eta}), \tilde{\lambda}_{33}(\check{\eta})) \quad \text{where } \check{\eta} = \bar{Y}(\bar{X}_0 - \bar{X})^{-\frac{1}{3}}. \quad (3.5.13a - e)$$

When these forms are substituted into the interaction equations, (3.5.1), the resulting ‘similarity’ equations can be solved in terms of single and double integrals, as in Hall & Smith. Alternatively, these could be substituted into the integro-differential equation from which the desired result follows more quickly and simply. It is deduced that we require

$$K c_{2r} < 0, \quad (3.5.14)$$

for this option of finite-distance break-up to be a possible large- $\bar{X}$ state of the vortex-wave interaction.

The next option, that we consider in the next subsection, is less ‘catastrophic’, as far as the laminar flow is concerned, with the solution continuing to downstream infinity.

Option II: Algebraic response at infinity.

Again, this option is still possible with our ‘reduced’ (corrected) equations — the terms we have ‘not got’ drop-out from the Hall & Smith-equations in this case. Here we write,

$$P \sim \bar{X}^{\frac{1}{6}} \tilde{\tilde{P}}(\check{\check{\eta}}), \quad W \sim \bar{X}^{\frac{1}{3}} \tilde{\tilde{W}}(\check{\check{\eta}}),$$

$$(\tau, \lambda_{33}) \sim \bar{X} (\tilde{\tilde{\tau}}(\check{\check{\eta}}), \tilde{\tilde{\lambda}}_{33}(\check{\check{\eta}})), \quad \check{\check{\eta}} = \bar{Y} \bar{X}^{-\frac{1}{3}}, \quad (3.5.15a - e)$$

as $\bar{X} \rightarrow \infty$.

It is easy to show, from the integro-differential equation (3.5.12), that we need

$$K c_{2r} > 0 \quad (3.5.16)$$

for this option to be possible. Note that, alternatively, this condition could be derived from the equations, as done so by Hall & Smith, but this method is more complex.

The third large- $\bar{X}$ -option proposed by Hall & Smith (see also Smith & Walton, 1989) was that of an exponential growth as $\bar{X} \rightarrow \infty$. This option is no longer possible† as it depended on direct forcing term in the W -equation that is not present in the ‘corrected equations’. Thus, when the option of finite-distance-break-up is not possible, we do not have the alternative ‘exponential-possibility’ to lead to very quick disturbance-growth. These can be seen to be the only possible options, for the interaction evolution, due to similarity reasons.

However, there is a further option, mentioned by Hall & Smith, that is still feasible; that of decoupling due to linearisation. Here the TS disturbance P becomes very small/negligible, and the vortex flow then grows slowly on its own with downstream variable $\bar{X}$ from its initial upstream state. However, this option is ultimately unstable to the TS-waves since the non-parallel-growth term, proportional to λ_b , will dominate the vortex skin friction λ_{33} .

§3.5.3 The transonic and hypersonic limits.

In this subsection we are interested in whether an asymptotic description of the interaction equations and coefficients is possible in certain limiting cases for the value of the Mach number. There are three obvious limiting cases to consider; here we consider two of them, the third case, corresponding to the incompressible limit, was considered (incorrectly) by Hall & Smith and will be considered in the next section.

The Mach number tending to unity.

In his study of the ‘compressible’ TS-eigenrelation and its properties, Smith (1989) investigated various limiting cases, including those of $M_\infty \rightarrow 1$ and $M_\infty \rightarrow \infty$. Further investigation of the large Mach number case has been carried out by the present author and progress made is reported in the next chapter of this thesis,

† unless $K = 0 \iff \theta = 45^\circ$. For this single value of K the vortex-wave interaction breaks down as there is no forcing. We do not consider this singular case in this study.

whilst the former case, the so-called ‘transonic limit’, has been further investigated by Bowles & Smith (1989) and Bowles (1990).

The ‘transonic limit’ will be considered first. The second case (the so-called ‘hypersonic-limit’) is found to be very similar but we consider it separately because of the further implications that follow from it. Without loss of generality, we suppose that the flow is (just) supersonic and define

$$\hat{m} = (M_\infty^2 - 1)^{\frac{1}{2}}, \quad (3.5.17)$$

and suppose that $\hat{m}$ is small. Smith showed that, in this case, the TS-mode quantities behave like

$$(\alpha, \beta, \Omega) \sim (\hat{m}^{-\frac{3}{4}} \alpha^*, \hat{m}^{-\frac{3}{4}} \beta^*, \hat{m}^{-\frac{3}{4}} \Omega^*) + \dots. \quad (3.5.18)$$

These result in K and the interaction coefficients a, b and c needing to be rescaled. We find that

$$(a, b, c) \sim (\hat{m}^{-\frac{1}{4}} a^*, \hat{m}^{-1} b^*, \hat{m}^{-1} c^*) + \dots,$$

where a^*, b^* and c^* are all $O(1)$ and can be easily found. Thus, the interaction-coefficients appearing in the interaction equations have the following^{ing} behaviour:

$$(c_{1r}, c_{2r}) \sim \hat{m}^{-\frac{3}{4}} (c_{1r}^*, c_{2r}^*) + \dots, c_{1r}^*, c_{2r}^* \sim O(1), \quad (3.5.19)$$

which results in the interaction scales $\bar{X}, \bar{Y}$, the vortex disturbances τ, λ_{33}, W and the TS- pressure amplitude P needing to be modified by the inclusion of $\hat{m}$ -factors;

$$(\bar{X}, \bar{Y}, \tau, \lambda_{33}, W, P) = (\hat{m}^{\frac{3}{8}} \bar{X}^*, \hat{m}^{\frac{1}{8}} \bar{Y}^*, \hat{m}^{\frac{3}{8}} \tau^*, \hat{m}^{\frac{3}{8}} \lambda_{33}^*, \hat{m}^{\frac{7}{8}} W^*, \hat{m}^{\frac{13}{16}} P^*) + \dots. \quad (3.5.20)$$

The last two results can be (and have been) used to check numerical results, for the general supersonic case, by providing a ‘transonic’ asymptote. We do not consider the transonic limit any further—the author is not aware of any vortex-wave formulations for the transonic regime itself.

The large Mach number limit.

Another limiting case that Smith (1989) went on to investigate was the so-called *hypersonic limit* when $M_\infty \gg 1$; this limiting case leads to some interesting

consequences for the whole triple-deck structure (see also Chapters 2 and 4, of the present thesis.) Thus it would be most instructive to consider the same limit here, as our interaction structure is, of course, dependent to a great extent on the underlying triple-deck scales.

First, we recap the results of Smith (see also Chapter 2), before going on to investigate the result of large Mach number on the interaction-coefficients, equations and length-scales; the latter leading to a significant conclusion. For $M_\infty \gg 1$ the main features revolve around the small regime

$$(\alpha, \beta, \Omega) \sim (M_\infty^{-\frac{3}{2}} \tilde{\alpha}, M_\infty^{-\frac{1}{2}} \tilde{\beta}, M_\infty^{-1} \tilde{\Omega}) + \dots, \quad (3.5.21a)$$

where $\tilde{\alpha}, \tilde{\beta}$ and $\tilde{\Omega}$ are $O(1)$; the eigenrelation reduces, at leading order, to

$$i^{\frac{1}{3}} \tilde{\alpha}^{\frac{1}{3}} \tilde{\beta}^2 = \left(\frac{Ai'}{\kappa} \right) (\tilde{\xi}_0) (\tilde{\beta}^2 - \tilde{\alpha}^2)^{\frac{1}{2}}, \quad \tilde{\xi}_0 = -\frac{i^{\frac{1}{3}} \tilde{\Omega}}{\tilde{\alpha}^{\frac{2}{3}}}. \quad (3.5.21b)$$

Note that we have α, β and Ω appear in the interaction coefficients, (3.3.13), so we must now investigate their large- M_∞ properties, in the light of (3.5.21). After a little manipulation we obtain the behaviours

$$a = M_\infty^{\frac{3}{2}} \tilde{a} + \dots, \quad b = \tilde{b} + \dots \quad \text{and} \quad c = \tilde{c} + \dots, \quad (3.5.22a - c)$$

where the leading order coefficients are given by

$$\begin{aligned} \tilde{a} &= \frac{2\tilde{r}_1 \tilde{\beta}^2 \tilde{D} \tilde{\xi}_0 \tilde{\Delta}^{-2}}{\tilde{\alpha}} - i(\tilde{\beta}^2 - \tilde{\alpha}^2)^{-\frac{1}{2}} \left[\frac{\tilde{\beta}^2 - \tilde{\alpha}^2}{3\tilde{\alpha}^2} + 1 \right], \\ \tilde{b} &= -\frac{2\tilde{r}_1 \tilde{\beta}^2 \tilde{D} \tilde{\xi}_0 \tilde{\Delta}^{-5}}{3} - \frac{5(\tilde{\beta}^2 - \tilde{\alpha}^2)^{\frac{1}{2}}}{3\tilde{\alpha}} \\ \text{and } \tilde{c} &= i\tilde{D} \tilde{\xi}_0 \tilde{\Delta}^{-2} \left(\frac{2\tilde{r}_1 \tilde{\beta}^2}{3} + \frac{\tilde{\beta}^2 \tilde{r}_2}{2} \right) - \frac{(\tilde{\beta}^2 - \tilde{\alpha}^2)^{\frac{1}{2}}}{6\tilde{\alpha}}. \end{aligned} \quad (3.5.22d - f)$$

Thus,

$$(c_1, c_2) = M_\infty^{-\frac{3}{2}} (\tilde{c}_1, \tilde{c}_2) + \dots, \quad \text{where } (\tilde{c}_1, \tilde{c}_2) = \left(\frac{\tilde{b}}{\tilde{a}}, \frac{\tilde{c}}{\tilde{a}} \right).$$

We could take further limiting cases such as those described by Smith (i.e. consider the large- and small- $\tilde{\beta}$ limits) but here we are interested to see how these sizes for the interaction-coefficients, coupled with the ‘hypersonic-limit-forms’ of the triple-deck scales, affects the interaction. Recalling that $\alpha \sim c_{1r} \sim c_{2r} \sim$

$M_\infty^{-\frac{3}{2}}$, $\beta \sim M_\infty^{-\frac{1}{2}}$ and $K \sim -M_\infty^2$ as $M_\infty \rightarrow \infty$, we see that there is a need to rescale the quantities appearing in the interaction equations, in this limit. We write

$$(\bar{X}, \bar{Y}, \tau, \lambda_{33}, W, P) = (M_\infty^m \bar{X}, M_\infty^n \bar{Y}, M_\infty^l \bar{\tau}, M_\infty^p \bar{\lambda}_{33}, M_\infty^k \bar{W}, M_\infty^s \bar{P}), \quad (3.5.23a - e)$$

where the unknown powers, m, n, l, p, k and s , will be determined so that the interaction-equations remain as intact as possible, at leading order. In fact, the interaction-equations are recovered intact if we choose

$$l = p = m = k = \frac{3}{4}, \quad n = \frac{1}{4} \quad \text{and} \quad s = -\frac{3}{8}; \quad (3.5.23f - j)$$

giving the *hypersonic-limit-interaction-equations*

$$\begin{aligned} \bar{\tau}_{\bar{Y}\bar{Y}} - \bar{Y}\bar{\tau}_{\bar{X}} &= -2i\bar{\beta}\bar{W}, \quad \bar{W}_{\bar{Y}\bar{Y}} - \bar{Y}\bar{W}_{\bar{X}} = 0, \\ \bar{P}_{\bar{X}} + (\bar{c}_{1r}\lambda_b\bar{X} + \bar{c}_{2r}\lambda_{33})\bar{P} &= 0, \quad \bar{\lambda}_{33} = \bar{\tau}(\bar{X}, 0), \end{aligned}$$

with boundary conditions

$$\bar{W}(\bar{X}, \infty) = \bar{\tau}(\bar{X}, \infty) = \bar{\tau}_{\bar{Y}}(\bar{X}, 0) = 0, \quad \bar{W}(\bar{X}, 0) = 2i\bar{\beta}\bar{K}\bar{P}^2(\bar{X}),$$

where $\bar{K} = \frac{\bar{\beta}^2}{\bar{\alpha}^2}$.

These appear to be exactly the same, in that all the terms are still present, but this hides the fact that the whole multi-layered boundary-layer structure is radically altered as the Mach number increases to such a size that it is the same order as an inverse power of the Reynolds number. It was shown in the previous chapter (see also Smith, 1989) that as $M_\infty \nearrow Re^{\frac{1}{9}}$, the triple-deck streamwise lengthscale, $Re^{-\frac{3}{8}}K_1X$, rises to become $O(1)$ in size; implying that a normal-mode decomposition is no longer rational because of non-parallelism of the underlying, growing boundary-layer is now a leading-order effect. Further, in this limit it was shown that the lower-deck thickens to coalesce with the main-deck.

This collapse of the underlying compressible-triple-deck, as the Mach number increases, will obviously occur for our current concern, the large Mach number behaviour of the vortex-wave interaction described in the earlier sections of

this chapter. However the large Mach number destiny of ^{the}buffer-layer (in particular, its thickness) and the amplitude-modulation scale remain to be established. Intuitively, as the buffer-region is ‘sandwiched’ between the lower- and main-decks which merge into a single viscous layer in this limit, we would also expect the buffer-region to collapse into the same viscous layer. Similarly, as the modulation-scale is ‘sandwiched’ between the triple-deck’s streamwise lengthscale (which emerges as $O(1)$ in this Mach number limit) and the $O(1)$ -lengthscale of the underlying flow, we would expect that the modulation-scale also lengthens to that of the underlying base flow (as $M_\infty \nearrow Re^{\frac{1}{9}}$). We now show that these are in fact the case, by formally considering the large Mach number properties of the scales involved (cf. §2.3.4).

Recall that, in the streamwise direction, we have the multiple scales,

$$\partial_x \longrightarrow \partial_x + \delta_2^{-1} \partial_{\bar{X}} + Re^{\frac{3}{8}} K_1^{-1} \partial_X; \quad (3.5.24)$$

necessary to capture the vortex-wave interaction. The quantities K_1 and δ_2 are as defined by (3.2.1c) and (3.2.18a), respectively. In the large Mach number limit, we have seen that

$$\partial_{\bar{X}} \sim M_\infty^{-\frac{3}{4}} \quad \text{whilst} \quad \partial_X \sim \alpha \sim M_\infty^{-\frac{3}{2}},$$

so that the unscaled lengthscales, L_w and L_v say, of the TS-waves and the modulation of the induced vortices, respectively, are

$$L_w \sim Re^{-\frac{3}{8}} K_1 M_\infty^{\frac{3}{2}} \quad \text{and} \quad L_v \sim \delta_2 M_\infty^{\frac{3}{4}}, \quad \text{both} \ll 1.$$

In the last chapter, we saw that, for the Sutherland temperature-viscosity relation,

$$K_1 \sim M_\infty^{\frac{15}{8}}, \quad (3.5.25)$$

and so

$$L_w \sim Re^{-\frac{3}{8}} M_\infty^{\frac{27}{8}}, \nearrow O(1), \quad \text{as} \quad M_\infty \nearrow Re^{\frac{1}{9}}.$$

As far as the amplitude-modulation scale is concerned, we find that

$$\begin{aligned} L_v \sim \delta_2 M_\infty^{\frac{3}{4}}, & \sim Re^{-\frac{3}{16}} K_1^{\frac{1}{2}} M_\infty^{\frac{3}{4}}, \sim Re^{-\frac{3}{16}} M_\infty^{\frac{27}{16}}, \\ & \nearrow O(1), \quad \text{as} \quad M_\infty \nearrow Re^{\frac{1}{9}}. \end{aligned} \quad (3.5.26)$$

Thus, as predicted earlier, this modulation scale does indeed rise to $O(1)$ -size in this limit of the Mach number.

We now consider the buffer-region. This was found to lie at the top of the lower-deck, where the lower-deck normal-variable $Y = \delta \tilde{Y}$. For large Mach numbers, we have found, (3.5.23), that the buffer-region is characterised by the location where $\tilde{Y}_* = M_\infty^{-\frac{1}{4}} \tilde{Y} \sim O(1)$. Thus the buffer-region lies where

$$\begin{aligned}
Y &\sim \delta M_\infty^{\frac{1}{4}}, \\
&\sim Re^{\frac{1}{16}} K_1^{-\frac{1}{6}} M_\infty^{\frac{1}{4}}, \text{ from (3.2.18b),} \\
&\sim Re^{\frac{1}{16}} M_\infty^{-\frac{1}{16}}, \text{ from (3.2.25),} \\
&\rightarrow O(Re^{\frac{1}{18}}) \sim O(M_\infty^{\frac{1}{2}}), \text{ as } M_\infty \nearrow Re^{\frac{1}{9}}.
\end{aligned}
\tag{3.5.27}$$

Recall that (see (2.3.33)) for large Mach number, the lower-deck variable, Y , also scales on M_∞ ; in fact $Y \sim M_\infty^{\frac{1}{2}}$ — hence from this and (3.5.27) we deduce that the buffer-layer merges with the lower-deck, which in turn coalesces with the main-deck. Thus three sub-boundary-layers, previously present, have all merged into one single viscous layer.

Summarising, when $M_\infty \rightarrow Re^{\frac{1}{9}}$ the four-layered, short-scaled structure underlying the vortex-wave interaction collapses into the two-tiered, long structure found by Smith (1989), discussed in Chapters 2 and 4 of the present thesis.

§3.6. RESULTS, DISCUSSION AND CONCLUSIONS.

This study was motivated by the desire to find out what changes (if any), to the theory, predictions and conclusions of the original work by Hall & Smith, are brought about by the inclusion of compressibility-effects. However, the changes brought about by the correction of the former turn out to be more significant. For this reason, and for later comparison with the results for compressible flows, the results for incompressible flow will also be considered, in §3.6.2. Firstly though, in the following subsection, we show how the interaction equations can be ‘normalised’ so that their solution (for specified initial disturbance(s)) depends merely on two ‘similarity parameters’, each of which can (only) take the value ± 1 (assuming that $K \neq 0$). The numerical solution of these normalised interaction equations

is then briefly discussed, before presenting typical solutions for ‘both’ choices of the second similarity-parameter; the sign of the first being fixed on physical grounds.

§3.6.1 The interaction equations renormalised.

In §3.5.2 we considered possible limiting forms, for solutions to the interaction equations as $\bar{X} \rightarrow \infty$, and found that the sign of the quantity Kc_{2r} was crucial in deciding whether particular limiting forms were, in fact, possible. This suggests that the interaction equations, (3.5.1), can be renormalised. This being desirable, we investigated further and found this was, indeed, the case†.

Writing

$$\begin{aligned}\bar{X} &= |c_{1r}\lambda_b|^{-\frac{1}{2}} X^*, & \bar{Y} &= |c_{1r}\lambda_b|^{-\frac{1}{6}} Y^*, & W &= -2i\beta K|c_3|^{-1} W^*, \\ P &= |c_3|^{-\frac{1}{2}} P^* & \text{and } \tau &= -4\beta^2 K|c_{1r}\lambda_b|^{-\frac{1}{3}} |c_3|^{-1} \tau^*,\end{aligned}\quad (3.6.1a - e)$$

where

$$c_3 = -Kc_{2r} \cdot \frac{4\beta^2}{|c_{1r}\lambda_b|^{\frac{5}{2}}}, \quad (3.6.2)$$

leads to the normalised system

$$W_{Y^*Y^*}^* - Y^* W_{X^*}^* = 0, \quad \tau_{Y^*Y^*}^* - Y^* \tau_{X^*}^* = W^*$$

$$\text{and } P_{X^*}^* + [\text{sgn}(c_{1r}\lambda_b)X^* - \text{sgn}(Kc_{2r})\tau^*(X^*, 0)] P^* = 0, \quad (3.6.3a - c)$$

which must be solved subject to initial conditions (at $X^* = 0$), together with the boundary conditions

$$W^*(X^*, \infty) = \tau^*(X^*, \infty) = \tau_{Y^*}^*(X^*, 0) = 0 \quad \text{and} \quad W^*(X^*, 0) = P^{*2}(X^*). \quad (3.6.3d - g)$$

Here, the function $\text{sgn}(x)$ returns the signature of x :

$$\text{sgn}(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{otherwise;} \end{cases}$$

in (3.6.3c) we have used the fact that $\text{sgn}(c_3) = -\text{sgn}(Kc_{2r})$.

† We note that such a renormalisation is possible because the system, and the pressure-equation in particular, is non-linear.

Thus, the interaction equations (and hence their solutions) are dependent only on the initial conditions imposed; $\text{sgn}(c_{1r}\lambda_b)$ and $\text{sgn}(Kc_{2r})$. In all the numerical calculations carried out, it was found that $c_{1r} > 0$, whilst $\lambda_b < 0$ for a growing ‘Blasius–similarity-^{variable}-type’ boundary layer — appropriate to the present study, if we assume that there is no significant wall-cooling or pressure-gradient effects. We therefore choose $\text{sgn}(c_{1r}\lambda_b) = -1$ and, apart from the initial conditions (which must be consistent with the interaction equations), the only parameter left in the problem is $\text{sgn}(Kc_{2r})$. Thus, with hindsight, it is not surprising that the (predicted) solution properties for, large- $\bar{X}$, depend crucially on the value of $\text{sgn}(Kc_{2r})$. Recall that earlier, in §3.5.2, we noted the following predictions:

$$\text{sgn}(Kc_{2r}) \begin{cases} > 0 & : \text{Algebraic response, as } \bar{X} \rightarrow \infty \\ < 0 & : \text{Finite-distance break-up, as } \bar{X} \rightarrow \bar{X}_0^-, \end{cases} \quad (3.6.4)$$

for the behaviour of the solutions to the interaction equations.

To check these predictions, the normalised system, (3.6.3), was solved numerically; for both possible values of $\text{sgn}(Kc_{2r})$, and for different (consistent) initial conditions. The large- $\bar{X}$ ($X^* \gg 1$) properties of the solutions were found to depend solely on $\text{sgn}(Kc_{2r})$; the initial conditions were found to affect only the initial development of the imposed disturbances. The equations were solved by taking ‘central differences’ in Y^* and ‘forward differences’ in X^* (following the method of Hall & Smith); the appropriate numerical checks were performed.

In Figures 3.3a,b, we present typical results for both values of $\text{sgn}(Kc_{2r})$. In both of these computations the system was initialised at $X^* = -1$ (upstream of the neutral TS point) using

$$P^* = P_0^*, \quad W^* = P_0^{*2}(1 + Y^{*2})\exp[-Y^{*2}], \quad \tau^* = \left(1 - \frac{P_0^{*2}}{2} + Y^{*2}\right)\exp[-Y^{*2}], \quad (3.6.5a - c)$$

with $P_0^* = 0.1$. Note that this initial state, which is compatible with the interaction equations plus boundary conditions, corresponds to a ‘mixed’ wave/vortex state. Moreover, we see from the ‘forcing’ boundary condition (eg. (3.6.3g)) that admissible initial states cannot consist off just the waves alone — longitudinal vortices must be initially be present (see later discussion). It appears to the author that the initial states used by Hall & Smith (see their section 5; particularly figures 2–5) are inconsistent with their system of interaction–equations plus boundary

conditions; they do not appear to satisfy the boundary conditions. In their study of vortex/wave interactions, Smith & Walton (1989) do not comment on the initial conditions they choose.

Returning to Figures 3.3a,b, we see that these numerical results are in full agreement with the theoretical large- X^* predictions, (3.6.4). Thus, in the following subsections, it is sufficient to calculate values of $\text{sgn}(K c_{2r})$ in order to determine the solution properties for large- X^* (of principal concern here, this vortex-wave interaction just possibly being the first stage in a sequence of non-linear theories leading to a plausible theoretical model for the transition processes).

§3.6.2 The incompressible case ($M_\infty = 0$).

In their (first such) study, Hall & Smith considered ‘this’ vortex-wave interaction for incompressible boundary-layer flow. They cleverly deduced the scales and formulated the interaction (as described in earlier sections); unfortunately, they made two errors in their analysis, both of which have a significant effect on the results and conclusions. The first of these errors (most kindly pointed out to the current author by Dr. P. Blennerhassett and Prof. F.T. Smith) leads to a simpler system of interaction-equations (see earlier) as well as leading to changes in the possible large- $\bar{X}$ states and the parameter values for them to be possible. The second (concerning the missing λ_{33Z} -term in the forced TS-mode-equations) was spotted by the current author and leads to a corrected form for c , and hence, a corrected value for the crucial quantity c_{2r} .

The interaction coefficients, a and b , are given by (3.4.8a,b), whilst (3.4.11c) gives the correct form for c — the incompressible case, of course, has $f = \alpha^2 + \beta^2$ and $g = (\alpha^2 + \beta^2)^{\frac{1}{2}}$. In Figure 3.4 we have plotted the resulting numerical values for the important interaction quantities, c_{1r} and c_{2r} , versus TS-wave obliqueness angle θ — recall that, for incompressible flow, all such wave angles are possible. Note that $c_{1r} > 0$ for all θ ; whilst c_{2r} has one zero, at $\theta \simeq 32.21^\circ$. Recalling the definition of K ,

$$K = 1 - \frac{\beta^2}{\alpha^2} = 1 - \tan^2(\theta) \begin{cases} > 0 & : \text{if } \theta < 45^\circ \\ < 0 & : \text{otherwise,} \end{cases}$$

Figure 3.3a. Numerical solution of interaction equations (3.6.3) with $Kc_{2r} < 0$: Finite-distance break-up.

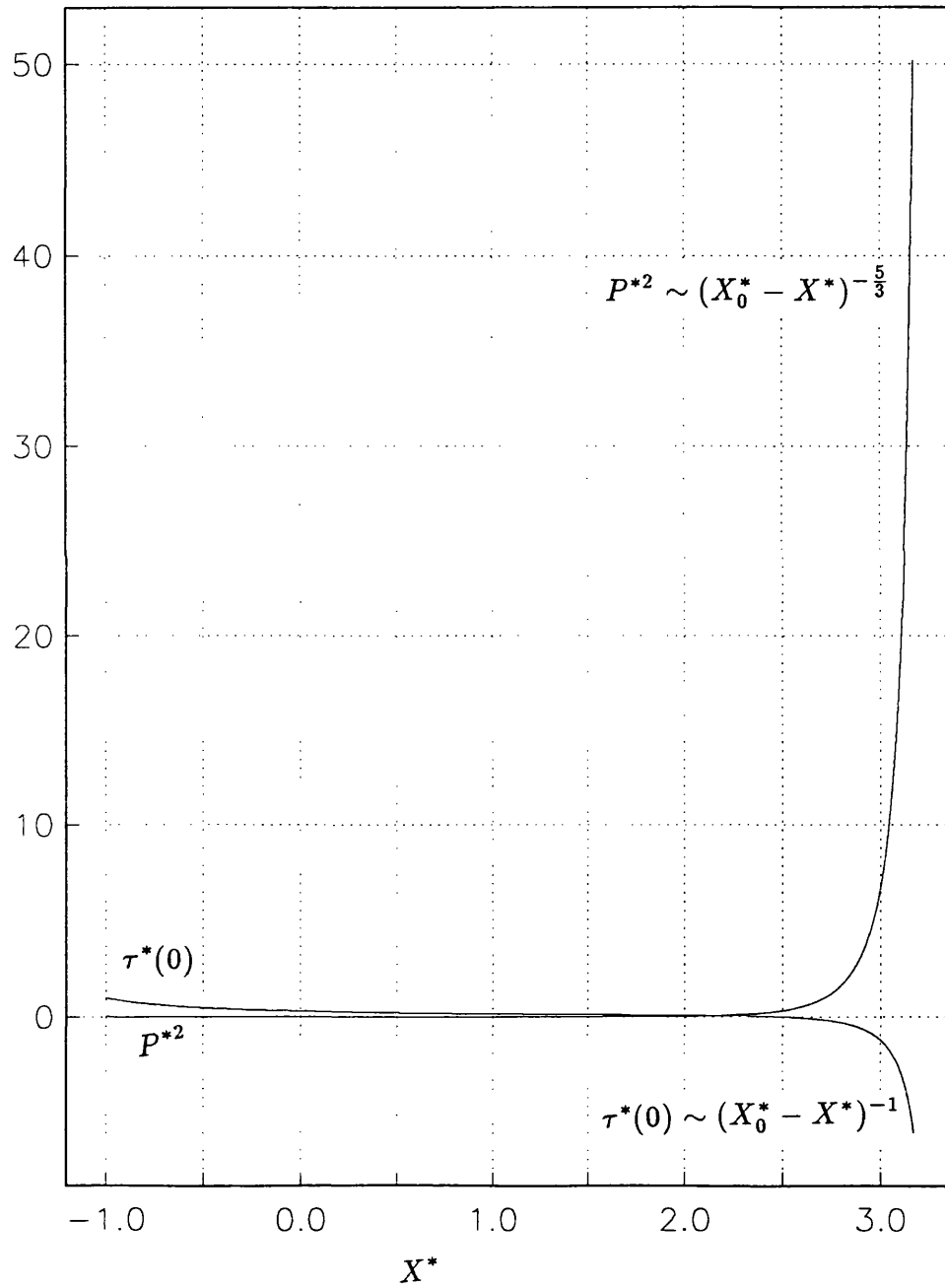


Figure 3.3b. Numerical solution of interaction equations (3.6.3) with $Kc_{2r} > 0$: Algebraic response at infinity.

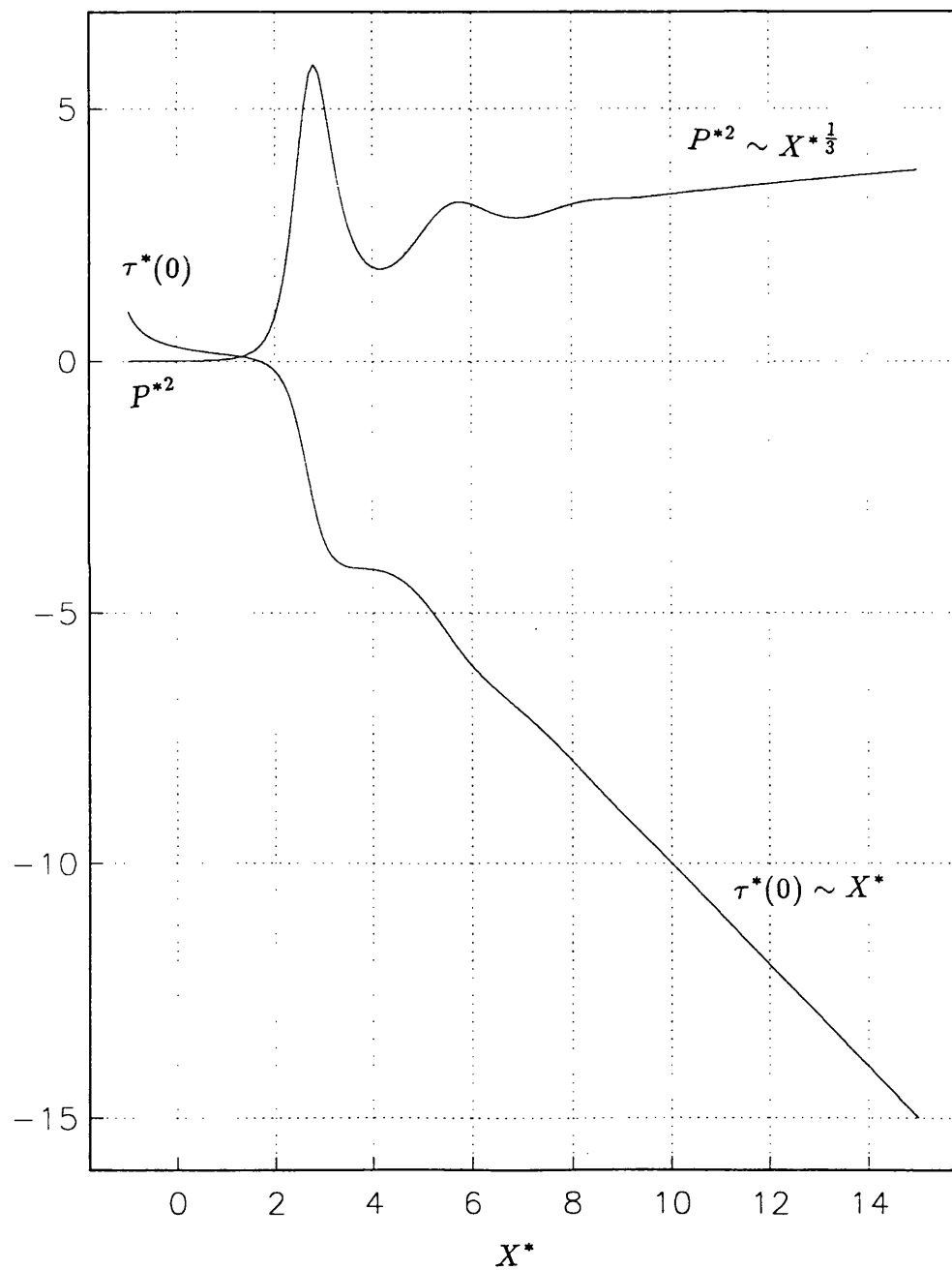
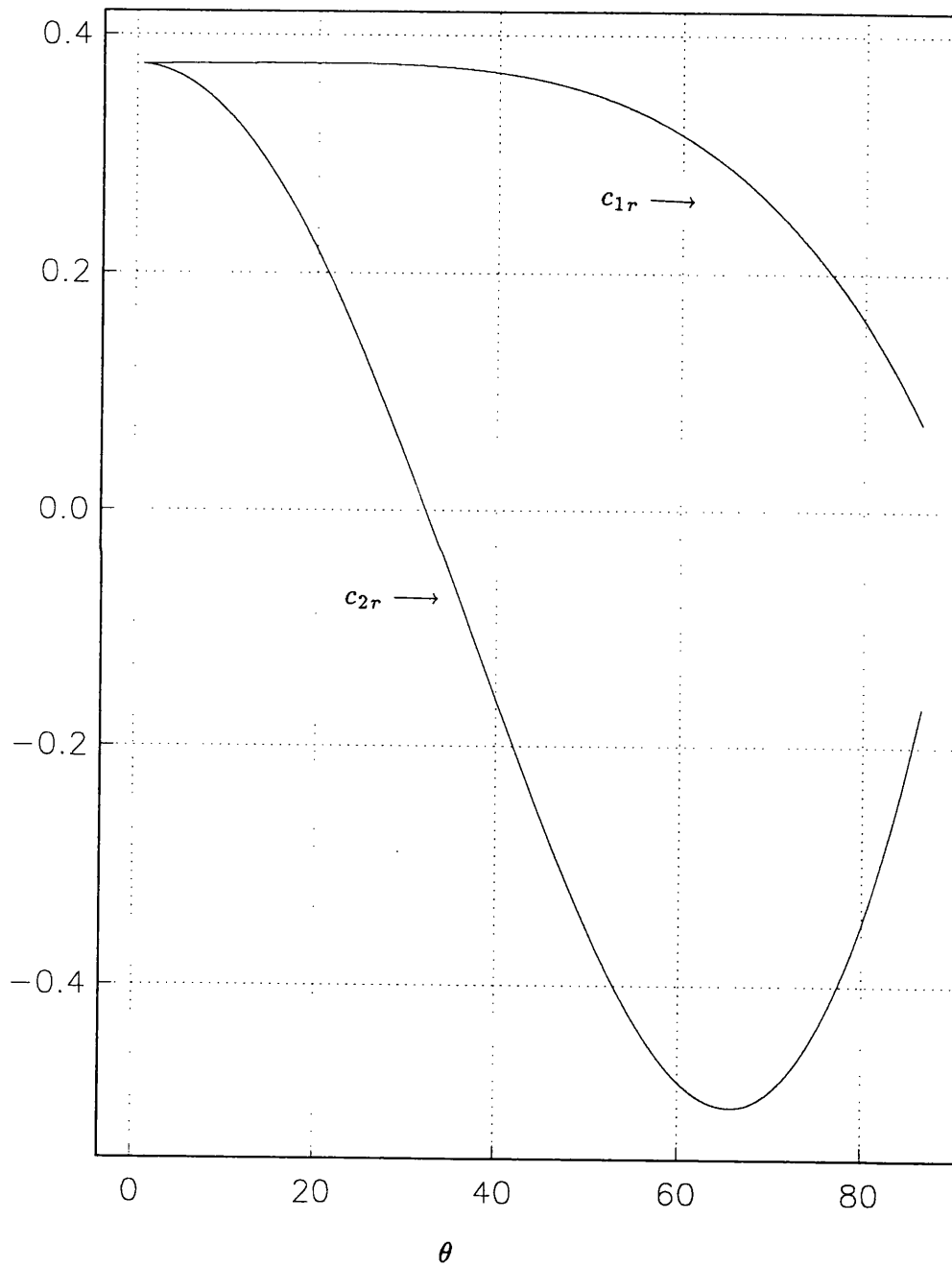


Figure 3.4. The interaction coefficients, c_{1r} and c_{2r} , versus wave-angle θ , for the incompressible case ($M_\infty = 0$).



we see that, when $M_\infty = 0$,

$$\text{sgn}(Kc_{2r}) = \begin{cases} -1 & : \text{if } 32.21^\circ \leq \theta < 45^\circ \\ +1 & : \text{otherwise.} \end{cases}$$

Thus, from this last result and the numerical calculations described in §3.6.1, we deduce the following: (i) if $32.21^\circ \leq \theta < 45^\circ$ then the solution to the interaction equations will ‘blow-up’ in a finite-distance; otherwise (ii) the solutions will grow slowly (far slower than the linear TS-solutions if there were no vortices present), with amplitudes proportional to algebraic powers of X^* , as $X^* \rightarrow \infty$. Note that these conclusions are quite different from those of Hall & Smith (who concluded that the ‘finite-distance break-up’ option was most likely, apart from the small range $45^\circ \leq \theta < \sim 50^\circ$ where an ‘exponential-growth’ option was favoured). We have found that the theoretically-exciting ‘finite-distance break-up’ option is now the exception, rather than the rule.

§3.6.3 The subsonic, supersonic and hypersonic cases.

For subsonic ($M_\infty < 1$) and some supersonic ($1 < M_\infty < \sim 1.15$) flows, the properties of the interaction-coefficients were remarkably similar to those found for the incompressible case i.e. graphs of c_{1r}, c_{2r} against θ appear very similar to Figure 3.4. However, the TS-wave angle restriction

$$\theta \geq \tan^{-1}[(M_\infty^2 - 1)^{\frac{1}{2}}],$$

was found to have a significant effect for ‘more’ supersonic flows — essentially it can be thought of as preventing wave-angles that would allow $\text{sgn}(Kc_{2r}) < 0 \longleftrightarrow$ finite-distance break-up option.

This is illustrated more clearly in Figure 3.5 where all the results are summarised; we see that the $\theta - M_\infty$ plane splits into four regions (labelled I - IV, as shown). Region IV corresponds to the ‘barred’ area, where no neutral TS-modes are possible. We see how the border of this region acts as an ‘abrupt cut-off’ to the larger- M_∞ extent of Region II (finite-distance break-up option). This is so much so that, for Mach numbers above $\sqrt{2}$, the possibility of finite-distance break-up has gone. Thus summarising, in the subsonic case the results are almost identical to the incompressible case; whereas, in general, the finite-distance

Figure 3.5. The regions of the θ - M_∞ plane.

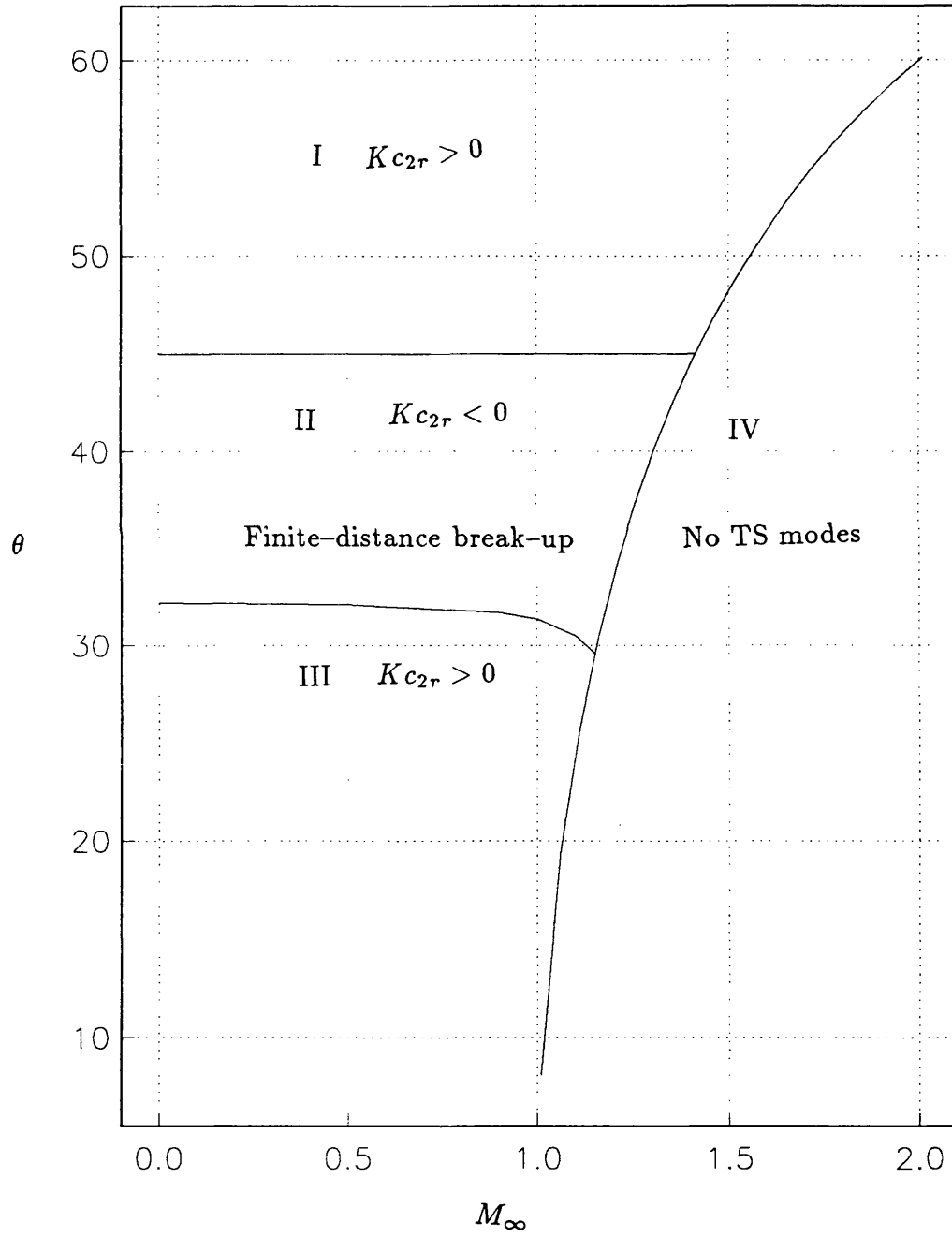
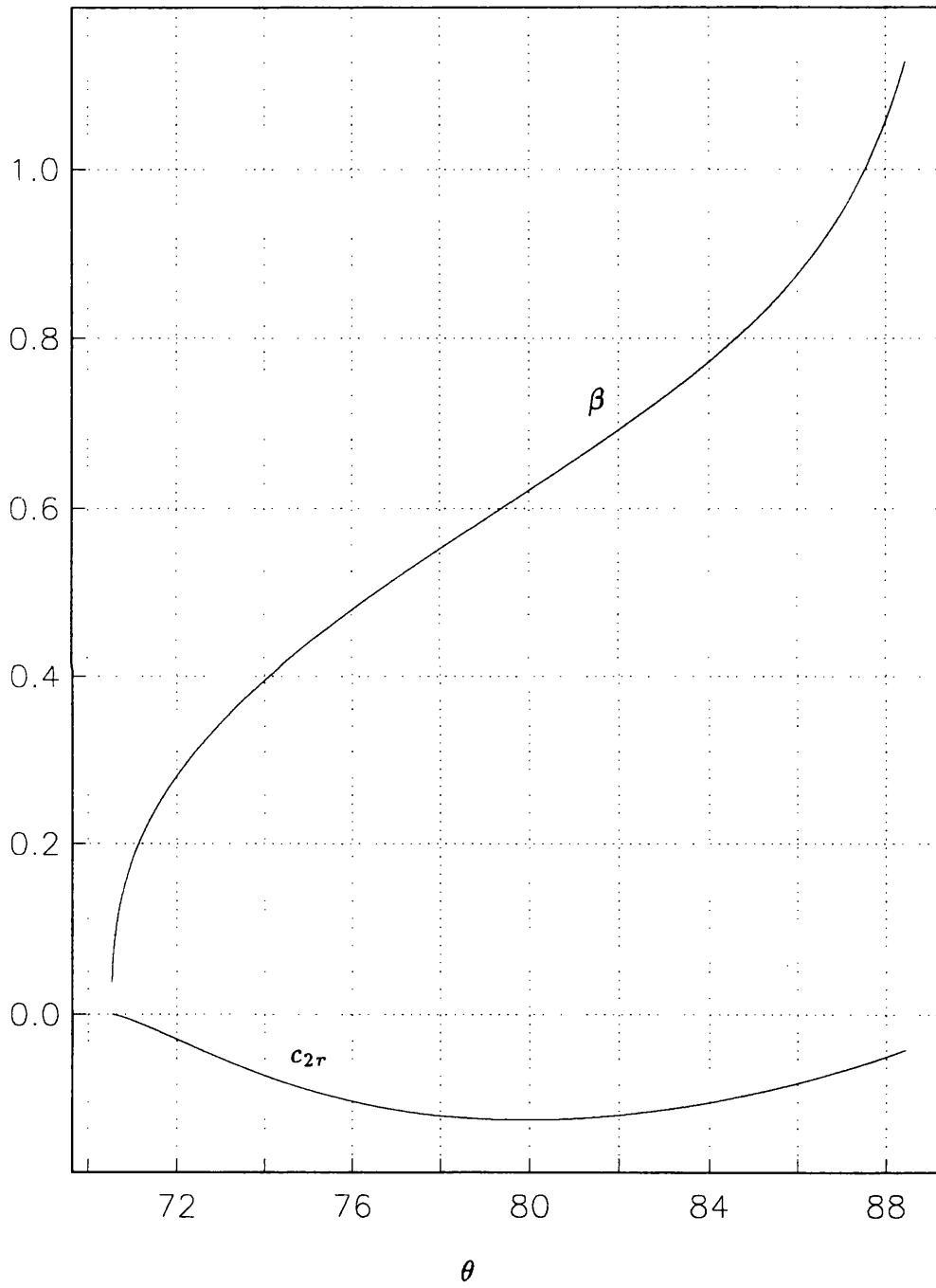


Figure 3.6. Spanwise wavenumber β and the interaction coefficient c_{2r} , versus θ , for $M_\infty = 3$.



break-up eventuality is not possible for supersonic flows, mainly due to the severe cut-off restriction (a $Re \gg 1$ effect). To illustrate this last point, in Figure 3.6 we have plotted c_{2r} versus β and θ — note that (i) there is no zero for c_{2r} , and (ii), the very oblique wave-angles encountered (so that K is always negative and, hence, Kc_{2r} is always positive).

The last set of results that we present are for hypersonic flow over a wedge, as considered by Cowley & Hall (1990), in which a shock is fitted into the upper-deck (at $\bar{y} = \bar{y}_s$, where $\bar{y}$ is the normal-variable of the upper-deck), leading to a modified form of Smith's hypersonic TS-eigenrelation (see the paper by Cowley & Hall for all details of the formulation). In Figure 3.7, we present results for the first (lowest) neutral-curve for the case $\bar{y}_s = 1$ — here $\alpha_{CH}, \beta_{CH} \sim O(1)$ are the α, β of that paper. It is sufficient to note that, in our notation,

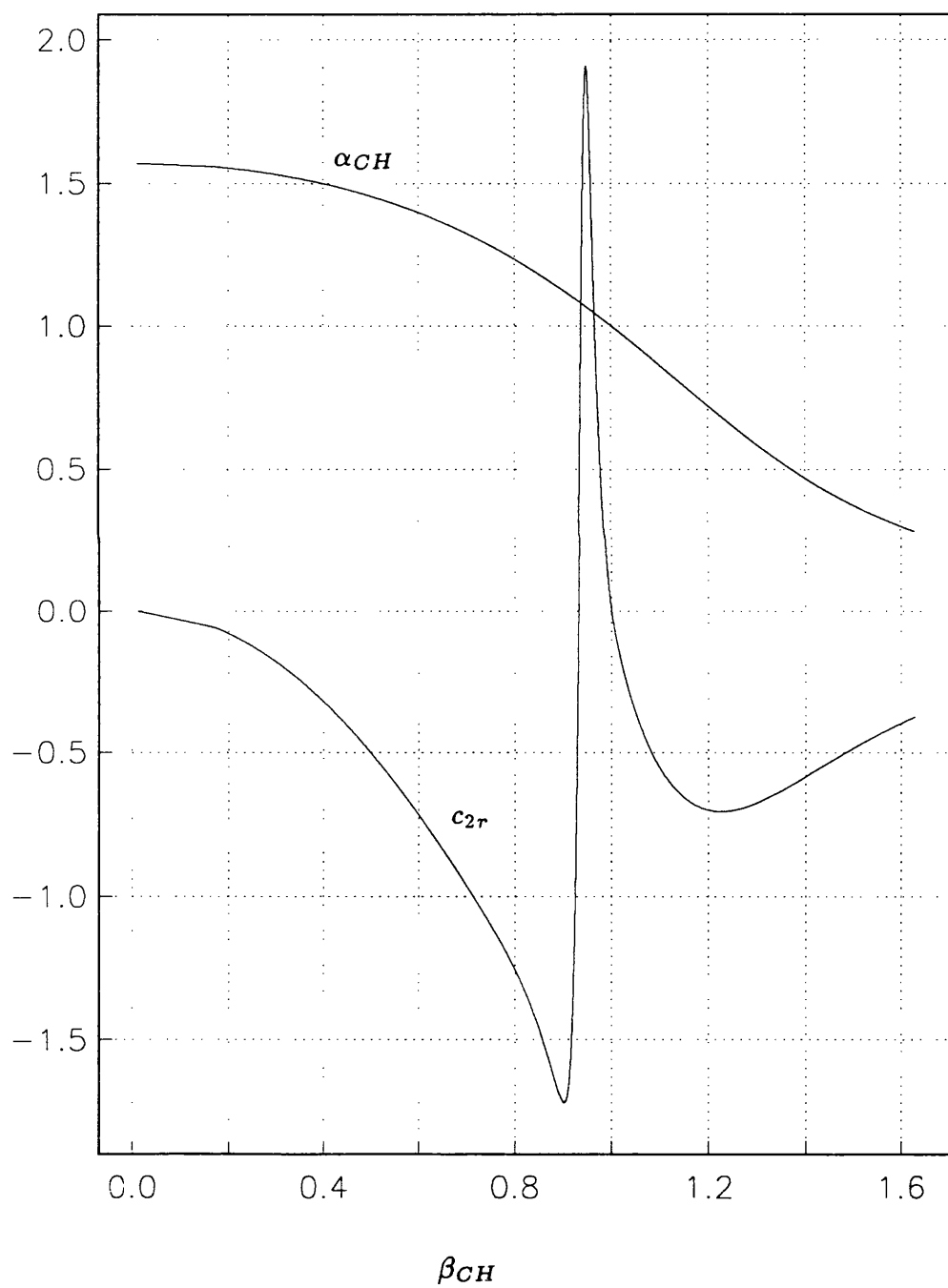
$$\frac{\beta}{\alpha} \sim M_\infty \frac{\beta_{CH}}{\alpha_{CH}} \gg 1,$$

and so the waves they consider are (generally) very oblique. Of particular interest here is the (small) interval where $c_{2r} > 0 \leftrightarrow Kc_{2r} < 0 \leftrightarrow$ finite-distance break-up option; this is an effect of the shock. No such interval is found for the 'higher' neutral curves; this interval appears to be a feature of the 'lowest' neutral curves only (for each choice of $\bar{y}_s$) and corresponds to 'crossing' the 'divide' $\alpha_{CH} = \beta_{CH}$.

Also in Table 3.1, we refer to the case of nonaxisymmetric supersonic flow over an axisymmetric surface, as studied by Duck & Hall (1990). Unfortunately, the author has not found time (as yet) to calculate any quantitative values for the interaction coefficients. However, as the governing TS-eigenrelation is quite different from that for planar supersonic flow, the results are awaited with interest.

Finally, we report that for the 'hypersonic and transonic' limiting cases studied in §3.5.3, the numerical results and the predicted asymptotic behaviours (for the interaction coefficients) were in extremely good agreement.

Figure 3.7 The quantities α_{CH} and c_{2r} , versus β_{CH} , for hypersonic flow over a wedge: $\bar{y}_s = 1$, lowest neutral curve.



§3.6.4 Further discussion and closing remarks.

We conclude this chapter with a few comments; many of the conclusions of Hall & Smith carry over to the present study and so we concentrate on compressibility-related aspects here. In this chapter we have shown that, within the triple-deck framework ($Re \gg 1$), that pairs of small-amplitude Tollmien-Schlichting waves and a longitudinal vortices can interact, leading to mutual growth. We have seen that two possible ‘eventualities’, for the downstream evolution of the interaction, exist; one in which the solutions grow relatively slowly (and ‘respond algebraically’) as $\bar{X} \rightarrow \infty$; whilst the other terminates at a finite-distance in a ‘break-up’. Further, we have seen that the latter is no longer possible, in general, for supersonic flows ($Re \gg 1$). Hall & Smith deduce the scales and interaction structures for the next, higher-amplitude stages resulting from these ‘first’, weakly non-linear vortex/wave interactions—they are currently investigating these ‘strongly non-linear’ interactions.

Note that in the transonic and hypersonic limits the interaction $\bar{X}$ -scale must be rescaled; in the transonic limit this modulation scale shortens, whilst in the hypersonic limit the opposite is true. The investigation of such vortex/wave interactions in transonic and hypersonic flows (not their ‘limits’) should prove interesting — note that the former flow has been studied by Bowles & Smith (1989) and Bowles (1990), whereas the latter flow regime is discussed, at length, in the next chapter. In Chapter 7 we consider an alternative vortex/wave interaction, for hypersonic flow over a curved surface, involving inviscid (Rayleigh-type) modes and Görtler vortices.

Other effects which could be incorporated into the present theory include pressure-gradient effects; wall-cooling effects (see Seddougui, Bowles & Smith, 1989); and spanwise-variation (cf Smith & Walton, 1989). Recent investigations of vortex/wave interactions include the papers by Hall & Smith (1988, 1990) and Bennett, Hall & Smith (1991), who consider curved channel flow; whilst Hall & Smith (1991) consider ‘strongly non-linear’ interactions.

Chapter 4

The two-tiered interactive structure governing the viscous stability of supersonic flow in the hypersonic limit.

§4.1. INTRODUCTION

§4.1.1 Introductory discussion.

In this chapter we investigate the two-tiered structure that results from the ‘collapse’ of the triple-deck structure when the Mach number reaches a certain large size. This new structure was first deduced by Smith (1989) from the (supersonic) triple-deck scales based on the Chapman-law formulation.

In Chapter 2 we have seen that the Sutherland-law formulation leads to quantitative alterations (i.e. different length-scales and size of the Mach number for where it occurs), but qualitatively the structure is the same as deduced by Smith. In the subsection below we show how this structure can be deduced by an alternative, simpler physical argument. In the previous chapter we have seen that the mechanism for the weakly nonlinear interaction between Tollmien–Schlichting waves and longitudinal vortices, supported by the triple-deck structure, simultaneously ‘collapses’ along with the latter as $M_\infty \rightarrow Re^{\frac{1}{9}}$. Thus, when the Mach number is of this order, we expect the vortex–wave interaction(s) to be supported by the ‘new’ two-tiered structure. The possibility that such vortex–wave interactions happen in/are responsible for the earlier stages of flow transition obviously necessitates the need for an understanding of the present two-tiered structure.

In §4.2 we formulate the equations governing the flow properties in this region and deduce the pressure–displacement law. In §4.3 we discuss the consequences of the scales and governing equations before going on to consider the linearized problem. The numerical solution is discussed. In §4.4 we look for asymptotic solutions based on the high–frequency approach. We find that the usual methods that work so well for the triple-deck structure do not carry over (at least not easily). In §4.5 we consider even larger Mach numbers and try to deduce the appropriate balances for the governing equations.

§4.1.2 An alternative, physical argument.

Here we show how the conclusions detailed in Chapter 2, concerning the importance of the regime where $M_\infty \sim Re^{\frac{1}{9}}$, can be deduced much more quickly and elegantly by extending a physical argument due to F.T. Smith. This argument also has the advantage that it is not based on a linearised solution of the triple-deck equations: the argument retains ‘nonlinearity’. It serves as a useful verification of the eigen-relation-argument approach due to Smith (1989) (see Chapter 2 for the generalisation to a non-linear temperature–viscosity relation), as well as probably being more instructive to the non-specialist. Further discussion follows, but first the physical argument is outlined.

The formulation is that given in Chapter 2, and the argument is a generalisation of that given by Smith (1982, pp. 222–223). We consider the large Mach number (supersonic) boundary layer flow over a flat plate; in fact we consider a particular streamwise location and suppose that breakaway–separation and stability characteristics are governed by a small (local) three-dimensional three-layered theoretical structure (the ‘triple-deck’), located at that plate–position. We suppose that this structure has length l , spanwise-width k and that the three layers (denoted by I, II and III, say) are characterised by heights $\delta\Delta$, Δ , H (in increasing size), where $l, k, \delta, \Delta, H \ll 1$ are to be determined; see Figure 4.1.

Let us now deduce the unknown lengthscales using purely physical reasoning. The classical boundary layer ‘fills’ layers I and II — the third layer will have inviscid character and is necessary due to the short streamwise lengthscale l . Thus we deduce that, for a power–viscosity law $\mu \propto T^\omega$ ($0.5 \leq \omega \leq 1$),

$$\Delta \sim Re^{-\frac{1}{2}} M_\infty^{1+\omega}, \quad (4.1.1)$$

and that the lengthscales H and k are governed by the supersonic 3-D Prandtl–Glauert pressure–equation:

$$(M_\infty^2 - 1)p_{xx} - p_{yy} - p_{zz} = 0, \quad (4.1.2)$$

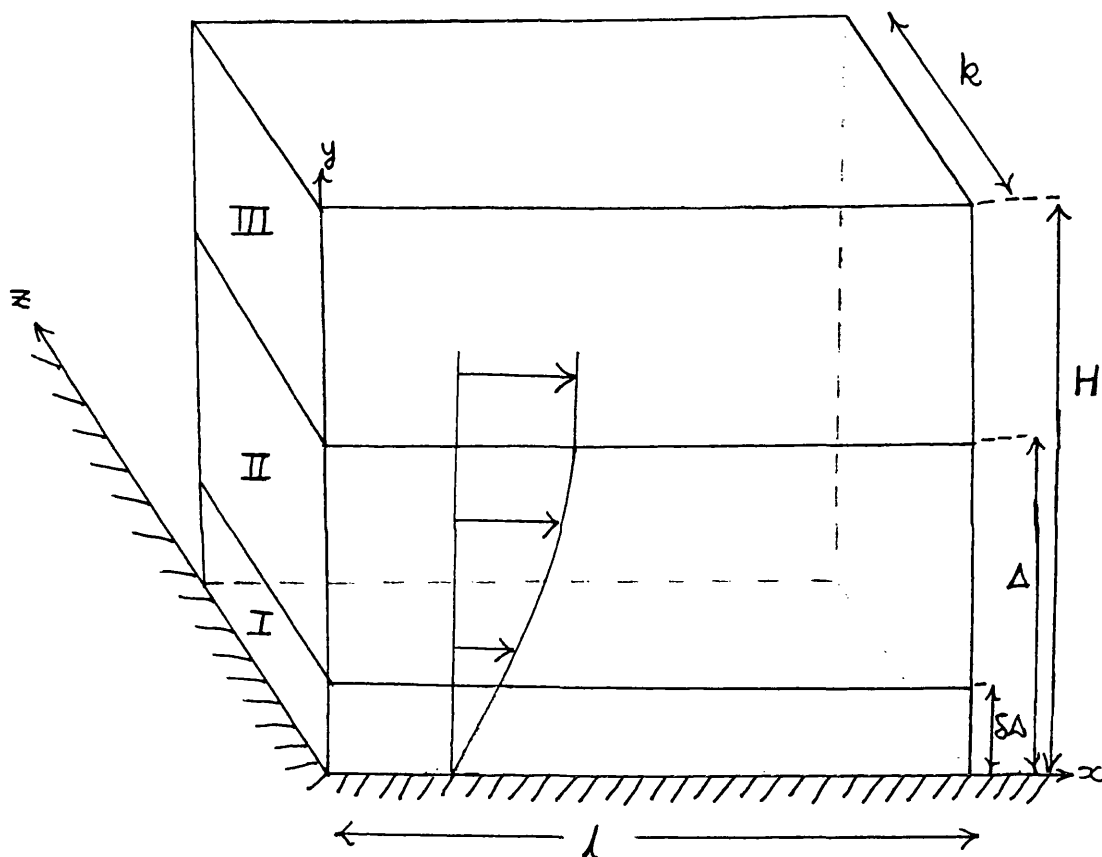


Figure 4.1. The three-layered, short-scaled 'triple deck' structure governing the ('lower-branch') viscous stability and separation properties of $O(1)$ Mach number flow over a flat plate. Layer I: the lower deck; flow is nonlinear and viscous. Layer II: the main deck; flow is rotational but inviscid. Layer III: the upper deck; flow is irrotational and inviscid.

so familiar in the classical inviscid aerodynamic-theories. So, for large Mach number, it immediately† follows that we require

$$H \sim \frac{l}{M_\infty} \sim k, \quad (4.1.3)$$

leaving only l and δ to be determined.

We now suppose that, close to the plate, in layer I, a non-linear viscous response is forced due to an induced pressure gradient. This layer is hot, the temperature being $O(M_\infty^2)$, which in turn fixes the sizes of the viscosity and density in this layer, via the viscosity-relation and gas law respectively. Due to the thinness, $\delta\Delta$, of this layer the oncoming velocity profile is very well approximated by a uniform shear. Thus the streamwise velocity, u , is $O(\delta)$, while balancing inertial and viscous terms requires $\delta = O(l^{\frac{1}{3}})$. The retention, here, of all terms in the continuity equation requires that $v \sim \delta^2 l^{-1} \Delta \sim \delta^{-1} \Delta$ and $w \sim \delta k l^{-1} \sim \delta M_\infty^{-1}$, where v and w are the normal and spanwise velocity components of Layer I.

We now come on to consider the induced-pressure balances in layer I. Balancing its gradient with inertial terms in the x -momentum equation requires (remembering that the density here is $O(M_\infty^{-2})$)

$$p \sim \delta^2 M_\infty^{-2}, \quad (4.1.4a)$$

but the balance in the z -momentum equation requires

$$p \sim \delta^2 M_\infty^{-4}. \quad (4.1.4b)$$

Thus, for large Mach numbers of concern here, we cannot retain both pressure-gradient balances. The streamwise pressure-gradient must be dropped, else the spanwise pressure-gradient would be solely leading order in the z -momentum equation. Note that this is basically an inviscid effect; due to the necessity to retain all terms in the Prandtl Glauert equation of the inviscid layer III. Thus separation and stability process that are governed by this viscous-inviscid interactive structure are three-dimensional in nature. We have seen in Chapter 2 (see also Smith, 1989)

† The p_{zz} balance must be retained for solutions, having harmonic x - and z -dependence, to this linear equation that decay as $y \rightarrow \infty$.

that the same result was deduced from the linear Tollmien-Schlichting mode eigenrelation. Finally, to balance time-derivative at leading order requires $\partial_t \sim u \partial_x$, i.e.

$$t \sim \frac{l}{\delta} \sim \delta^2. \quad (4.1.5)$$

Layer II is displaced in an inviscid manner by an amount $A \sim O(\delta \Delta)$ due to the thickness of layer I, the motion being linearized but rotational, while the leading-order pressure is unaltered across layer II. This displacement effect (i.e. $v \sim A_x$) induces an inviscid pressure response (which in turn satisfies the Prandtl Glauert equation given above) of size

$$p \sim \frac{HA}{l^2} \sim \delta^{-2} M_\infty^{-1} \Delta; \quad (4.1.6)$$

easily deduced from the y -momentum inviscid Euler equation. Again, the final element of the argument is that this pressure size should coincide with that at the plate, in layer I, so that the classical boundary layer theory no longer applies. Thus, we require that

$$\delta^{-2} M_\infty^{-1} \Delta \sim \delta^2 M_\infty^{-4} \quad \leftrightarrow \quad \delta^4 \sim \Delta M_\infty^3. \quad (4.1.7)$$

Substituting for Δ , the height of the classical boundary layer for large Mach number, yields the sizes

$$\delta \sim Re^{-\frac{1}{8}} M_\infty^{\frac{1}{4}(4+\omega)} \quad \text{and} \quad l \sim \delta^3 = Re^{-\frac{3}{8}} M_\infty^{\frac{3}{4}(4+\omega)}, \quad (4.1.8a, b)$$

for the small parameter δ , used in asymptotic expansions, and the short stream-wise lengthscale l , of the inviscid-viscous interaction. This lengthscale and ‘small’ parameter increase in size as the Mach number increases, becoming $O(1)$ when

$$Re^{-\frac{1}{8}} M_\infty^{\frac{1}{4}(4+\omega)} \sim 1 \leftrightarrow M_\infty \sim Re^{1/2(4+\omega)}. \quad (4.1.9)$$

This result is merely a generalisation of those deduced by Smith (1989) for a linear viscosity law, where $\omega = 1$; and in Chapter 2, for Sutherlands formula leading to $\omega = \frac{1}{2}$. Thus we have shown how, via a first-principles physical argument, the important conclusion of Smith (1989), concerning the importance of non-parallel effects for large Mach numbers, can be deduced. Note that δ is closely related to the parameter $\hat{\kappa}$ of the latter paper and the scaled Mach number, m , to be

defined in the next section. This approach has also highlighted the point that, for large Mach numbers, the usual small parameter $\epsilon \equiv Re^{-\frac{1}{8}}$ employed in theoretical studies based on the triple-deck structure, should be replaced by $Re^{-\frac{1}{8}} M_\infty^{\frac{1}{4}(4+\omega)} = \hat{\epsilon}$ say. Thus, as the Mach number increases, $\hat{\epsilon}$ increases and all of the orders of the perturbations increase (but remain ordered) until all orders become $O(1)$ when $\hat{\epsilon} \nearrow O(1)$. So, this regime, where $l \sim O(1)$ signifies the total or absolute collapse of triple-deck theory, in the present context. The new viscous-inviscid interactive structure that results is discussed in the following sections of this chapter.

We now qualify a couple of statements made in the last paragraph. First, by the ‘present context’ we mean supersonic flow over a flat plate as described in Chapter 2. However in other contexts (see, for example Brown, Stewartson & Williams, 1975; Brown, Cheng & Lee, 1990), it is possible to formulate a triple-deck-type structure for much larger Mach numbers, thus appearing to contradict the restriction

$$M_\infty \ll Re^{1/2(4+\omega)} \quad (4.1.10)$$

discussed above. In fact there is no contradiction: these papers indicate that it is possible to ‘fix-up’ l and δ to be small (in the notation used above) by a couple of means: either by requiring $\gamma - 1 \ll 1$ (i.e. the Newtonian approximation), or by a certain, significant wall-cooling. The relevance of the former restriction is doubtful, but the latter may be of practical importance because some wall cooling is almost certainly necessary to protect the wall from very high temperatures. Also, both of these papers consider an interactive basic flow, just behind a shock, and the distance from the leading edge is shown to be significant.

The second comment to be made concerning the preceding paragraph is that results based on the supersonic triple-deck formulation (i.e. the T-S eigenrelation) may be of dubious validity well before the regime where non-parallel effects become important at leading order, because the height δ , of the lower-deck (labelled Layer I in the above) relative to the boundary layer thickness, also increases to $O(1)$. As this layer is no longer so thin as before, firstly, it may not be rational to consider the oncoming velocity profile to be well approximated by a uniform (wall) shear, and secondly, it may also not be rational to still consider this layer as being ‘quasi’-incompressible; more effects of compressibility than just the wall-values of the

basic flow may be necessary. It is fair to conclude that, in general, the triple-deck structure does not survive increasing Mach number.

We end this subsection with a couple of remarks. The physical derivation, described above, may be instructive to the non specialist; highlights points that might be otherwise missed; and we have not had to linearize the problem at any stage, in contrast to the T-S eigenrelation approach. We are cautious in claiming that the same reasoning applies to similar flows; such a generalisation may result in possible significant differences not being immediately realized.

The same type of physical argument certainly will apply to other similar flows but will not necessarily lead to similar conclusions; there are many compressible boundary-layer problems that appear similar to the flat-plate problem, considered here (see also Smith, 1989), but apparently-slight variations, such as wall-cooling (Seddougui, Bowles & Smith, 1989; Brown, Cheng & Lee, 1990); curvature-effects (Duck & Hall, 1990) and including the effects of a shock (Brown, Stewartson & Williams, 1975; Cowley & Hall, 1990; Brown, Cheng & Lee, 1990), can lead to markedly different results and conclusions.

The preceding discussion in this subsection has been motivated by such an apparent ‘over-generalisation’ made by Smith (1982) when he concludes the physical derivation of the arguments behind the steady, two-dimensional incompressible triple-deck scales, by stating that same arguments would carry over to several other boundary-layer flows. However, if he had in fact applied his physical argument to large Mach number supersonic flows, as we have done so above, the facts that (i) normal-modes propagate at increasingly oblique angles, and (ii) the crucial restriction on the validity of the normal mode approach due to non-parallel effects, $M_\infty \ll Re^{\frac{1}{10}}$ - for linear viscosity law⁵, which are deduced from the supersonic Tollmien-Schlichting eigenrelation in Smith (1989), would have been known a few years earlier. The latter paper does not mention that such conclusions could be deduced from a reasonably simple physical argument, rather than having to resort to some subtle asymptotic arguments based on the eigenrelation and the complicated scales involved in its derivation. There are of course many other examples where the effects of compressibility lead to significant changes to the conclusions of the ‘generic’ problem; this example was chosen because of its simple, but illustrative, nature.

§4.1.3 The multi-layered upper-branch structure at large Mach number.

So far in this thesis we have just considered the fate of the lower-branch asymptotic structure (triple-deck) as the Mach number increases. It has been shown that this structure collapses into a much longer, two-tiered structure by two different arguments; one based on the linear Tollmien-Schlichting eigenrelation (see Chapter 2), and the other based on ‘first-principles’-type reasoning (see previous sub-section).

However, there is a second ‘viscous-inviscid’ structure governing the viscous stability of the boundary layer flow—the so-called ‘upper-branch structure’ corresponding to the high-Reynolds-number part of the upper-branch of the Orr-Sommerfeld† neutral curve. Here the so-called ‘critical layer’ is distinct from the wall — the corresponding lower-branch critical layer lies ‘at’ the wall: the lower-deck of the governing triple-deck structure comprises of both the Stokes layer (necessary due to the fast timescale in operation) and the critical layer (due to the unsteady, small disturbance).

As the Mach number becomes large we have seen that the lower-deck (containing the lower-branch critical layer) grows to merge with the main-deck (spanning the classical boundary layer). Thus, as the critical layer for the upper-branch modes lies above that for the lower-branch we would expect the former to be simultaneously “pushed-up” into the heart of the boundary layer (cf the ‘Sandwich-principle’). Hence, without any prior knowledge of the scales and resulting eigenrelation of supersonic upper-branch stability theory, it seems very reasonable to postulate that, (i) the supersonic upper-branch modes must be three-dimensional and travel at ever-increasing oblique angles as the Mach number increases; and, more significantly, (ii) that the governing three-dimensional, five-layered structure collapses into the same long, two-tiered structure, that the triple-deck collapses into, as $M_\infty \rightarrow Re^{\frac{1}{9}}$. The first, a necessary pre-requisite for the second, follows immediately from the fact that the pressure disturbance in the uppermost layer will satisfy the supersonic Prandtl Glauert equation — this inviscid layer plays the same role as the upper-deck does for lower-branch modes, where we have seen

† Strictly, the compressible counterpart of this normal-mode approach

that spanwise z -variation is necessary for the solution to decay as the freestream is approached.

The author is unaware of any theoretical work concerning supersonic upper-branch modes; the corresponding incompressible cases were ‘systematically’ considered by Smith & Bodonyi (1980, 1982a) and Bodonyi & Smith (1981); whilst Gajjar & Cole (1989) studied the subsonic, two-dimensional cases. As implied by the use of plural “cases” above, there is a slight complication inherent in the upper-branch theory: the asymptotic structures and solutions depend crucially on the second terms of the near-wall asymptotes of the basic flow profiles — there are essentially two such cases; one corresponding to Blasius flow (with insulated wall, if appropriate), and the other for flows with ‘pressure-gradient’ (this group includes those with wall-cooling, if appropriate).

So, rather than deriving the scalings for, and deducing the large Mach properties of the supersonic upper-branch eigenrelations, we choose to use an alternative approach that follows the on the lines of the physical argument described in the previous subsection which proved so successful in highlighting the lower-branch properties for large Mach number. The extra layers present result in the argument being more complicated; we outline the argument below — a knowledge of the ‘workings’ of upper-branch structure (see the last four named papers) may benefit the non-expert reader and it is assumed that the previous subsection has been read.

Our starting point is the small, three-dimensional and five-layered structure sketched in Figure 4.2. In labeling the layers (Z1 - Z5) and their dimensions we have tried to conform to the notation of Gajjar & Cole (1989) and the previous subsection. Strictly, the structure consists of four ‘stacked’ layers — Z5, Z3, Z2 and Z1; whilst Z3 is partitioned by the critical layer Z4. All layers lie inside the ‘classical’ boundary-layer except Z1 which lies in the inviscid freestream. We suppose that the structure has stream- and span-wise lengthscales $\bar{l}$ and $\bar{k}$, respectively; that layers Z5, Z3, Z2 and Z1 have (increasing) thicknesses $\delta_2\Delta, \delta_1\Delta, \Delta$ and $\tilde{H}$, respectively; and that $\bar{l}, \bar{k}, \delta_2, \delta_1, \Delta$ and $\tilde{H}$ are all small quantities (to be determined). Further, we assume that there is a fast timescale operating; we write $\frac{\partial}{\partial t} \sim i\Omega$, with $\Omega \gg 1$.

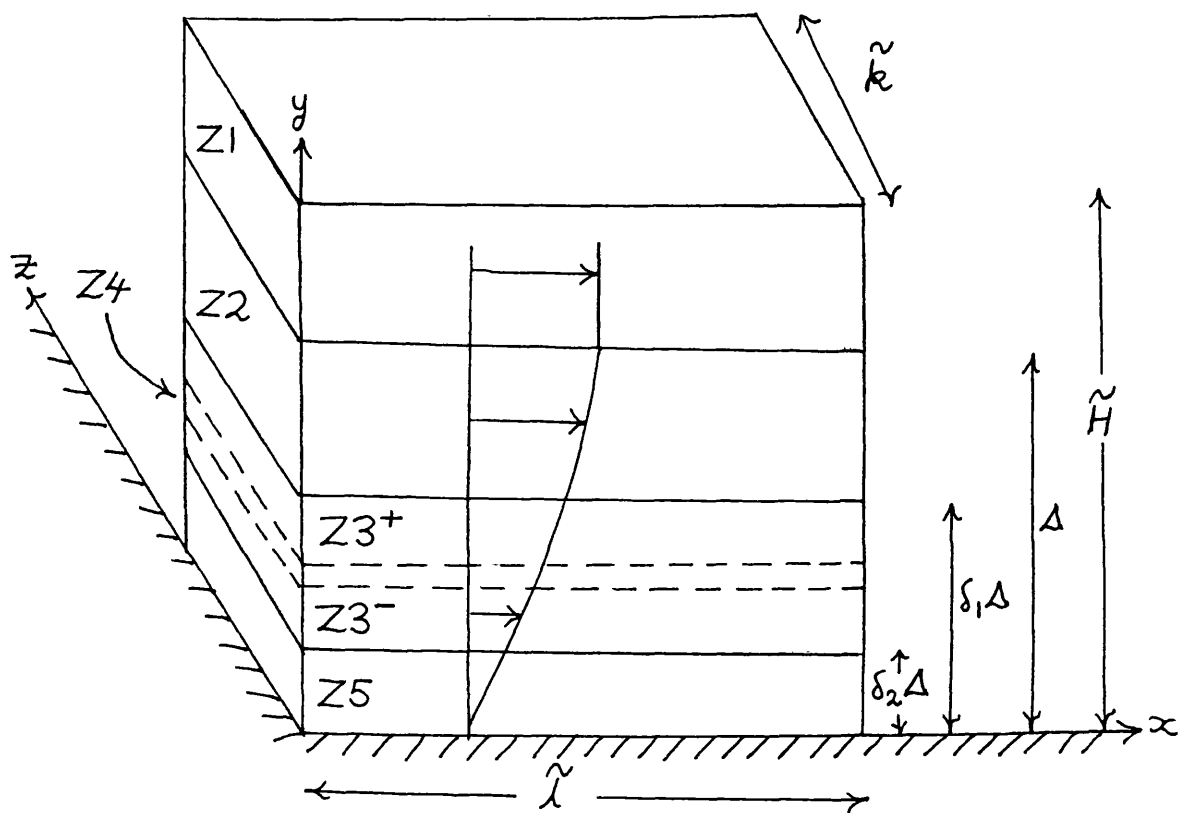


Figure 4.2. The five-layered, short-scaled structure governing the 'upper-branch' stability of $O(1)$ Mach number flow over a flat plate.

Additionally, we assume that the disturbances

$$(u(y), v(y), w(y), p(y), \theta(y), \rho(y), \mu(y)) \exp[i(\tilde{\alpha}\tilde{l}^{-1}x + \tilde{\beta}\tilde{k}^{-1}z - \tilde{\Omega}\Omega t] \\ + c.c., \quad (4.1.11)$$

have purely harmonic dependence in x, z and t ; that they are sufficiently ‘small enough’ that nonlinear effects can be neglected; and that the Mach number is large. Finally, we choose to consider neutral modes, so that the scaled, $O(1)$ wavenumbers and frequency $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\Omega}$ are all real. Note that the short x -scale enables non-parallel effects to be rationally ignored (at the orders of concern here).

In layer Z1, the pressure-disturbance satisfies the supersonic Prandtl Glauert equation and thus it immediately follows that

$$\tilde{k} \sim \tilde{H} \sim \frac{\tilde{l}}{M_\infty}. \quad (4.1.12)$$

Again, Δ is the thickness of the underlying classical boundary-layer i.e.

$$\Delta \sim Re^{-\frac{1}{2}} M_\infty^{1+\omega}, \quad (4.1.13)$$

where ω comes from the viscosity-temperature power-law.

Layer Z5 is merely the standard, so-called, ‘Stokes Layer’ which captures the viscous-effects which are now essentially trapped at the wall due to the effective high-frequency of the disturbance. Balancing unsteady and viscous effects easily gives

$$\delta_2 \sim \Omega^{-\frac{1}{2}}. \quad (4.1.14)$$

It is important to note that in this layer, and indeed throughout all the layers residing inside the classical boundary layer, we do not expect (cf. the lower-branch) the streamwise disturbance-pressure gradient to appear at leading order in the x -momentum equations— basically because of the different stream- and span-wise lengthscales in operation. Thus, wherever possible, we choose to work with the z -momentum equation (in preference to the x -momentum equation), on the basis that we expect all the usual upper-branch balances to remain intact as the Mach number increases. Recall also that, as we are considering $M_\infty \gg 1$, the sizes of the thermodynamic quantities will be proportional to powers of M_∞ in these ‘boundary sub-layers’.

Balancing the unsteady and pressure–gradient terms, in Z5, of the z -momentum equation yields

$$w \sim p M_\infty^3 \Omega^{-1} \tilde{l}^{-1};$$

whilst balancing the u_x, v_y and w_z terms of the continuity equation gives

$$v \sim p M_\infty^4 \Omega^{-\frac{3}{2}} \Delta \tilde{l}^{-2} \quad \text{and} \quad u \sim p M_\infty^4 \Omega^{-1} \tilde{l}^{-1}.$$

So, in Z5, we write $y = \delta_2 \Delta \hat{y}$, with $\hat{y} \sim O(1)$, and the disturbances have the form

$$p = \hat{p}^{(0)} + \dots, \quad v = M_\infty^4 \Omega^{-\frac{3}{2}} \Delta \tilde{l}^{-2} \hat{v}^{(0)} + \dots,$$

$$w = M_\infty^3 \Omega^{-1} \Delta \tilde{l}^{-1} \hat{w}^{(0)} + \dots, \quad u = M_\infty^4 \Omega^{-1} \tilde{l}^{-1} \hat{u}^{(0)} + \dots; \quad (4.1.15a - d)$$

here $\hat{p}^{(0)}, \hat{v}^{(0)}, \hat{u}^{(0)}$ and $\hat{w}^{(0)}$ are $O(1)$. It can easily shown that

$$\hat{v}^{(0)} \rightarrow \hat{v}_{1\infty}^{(0)} \hat{y} + \hat{v}_{0\infty}^{(0)}, \quad \text{as } \hat{y} \rightarrow \infty, \quad (4.1.16)$$

where $\hat{v}_{1\infty}^{(0)}, \hat{v}_{0\infty}^{(0)}$ are $O(1)$ constants.

Thus, as we enter layer Z3, $\hat{y} \sim \frac{\delta_1}{\delta_2} \gg 1$, resulting in the two terms of the large $\hat{y}$ asymptote for $\hat{v}^{(0)}$ occurring at different orders (in terms of the Z3 formulation). The ratio of these two terms has a non-zero imaginary part which must be ‘played-off’ against the imaginary ‘phase-jump’ in v across the critical layer Z4, when crossing from $Z3^-$ to $Z3^+$; ensuring that solutions match properly across the layers — this is essentially the ‘upper-branch mechanism’. We appeal to this notion below.

In layers $Z3^\pm$, the Stokes layer analysis and large $\hat{y}$ asymptote implies that here p and w have the same sizes as they had in Z5 but now v has the form

$$v = M_\infty^4 \Omega^{-\frac{3}{2}} \Delta \tilde{l}^{-2} \left[\frac{\delta_1}{\delta_2} v^{(0)} + \dots + v^{(1,3)} + \dots \right]. \quad (4.1.17)$$

Note that we have only highlighted the terms of interest as far as the upper-branch mechanism is concerned.

The term denoted by $v^{(1,3)}$ is either the second or fourth in the asymptotic expansion, depending on the small- Y forms of the basic-flow, where $Y = \Delta^{-1} y$ is the ‘classical boundary-layer’ variable. In fact, it is the second term if the wall is cooled; the fourth term if the wall is insulated — here we treat these two cases

simultaneously as we are principally interested in the structures' dimensions and not in deriving the eigenrelations. We require this term to 'suffer' the imaginary phase-shift (across the critical layer, Z4) to match with the imaginary part of $\hat{v}^{(0)}$ at the top of the Stokes layer, Z5. This essentially says that we require the ratio $\frac{\delta_1}{\delta_2}$ of first - to - second/fourth terms in the disturbance expansion for v in Z3 to be the same as the corresponding ratio from the small Y basic-flow asymptotes, written in terms of $\bar{Y} = \delta_1^{-1} Y$ (the appropriate scaled normal variable for Z3); grouped and ordered in terms of the small parameter δ_1 (see Gajjar & Cole, 1989 for fuller details).

Thus we require

$$\frac{\delta_1}{\delta_2} \sim \begin{cases} \delta_1^{-1} : & \text{Cooled Wall} \\ \delta_1^{-3} : & \text{Insulated Wall} \end{cases},$$

which, on substituting the known size of δ_2 , yields the first condition

$$\delta_1 \sim \begin{cases} \Omega^{-\frac{1}{8}} : & \text{Cooled wall} \\ \Omega^{-\frac{1}{4}} : & \text{Insulated wall;} \end{cases} \quad (4.1.18a, b)$$

between the unknown quantities Ω, δ_1 and $\tilde{l}$.

Note that in Z3, where we have assumed that the critical layer lies, the basic streamwise velocity still has its near-wall shear form

$$U_B \sim \lambda_1 \delta_1 \bar{Y} + \dots$$

So for the critical layer to actually be in Z3 we require the balance $\partial_t \sim U_B \partial_x$ to hold in this layer, which yields the second condition,

$$\Omega \sim \delta_1 \tilde{l}^{-1}, \quad (4.1.19)$$

between the unknown quantities.

The third condition follows from the fact that the layers Z3 and Z2 transmit the so-called Stokes-layer, viscous displacement effect (proportional to $\hat{v}_{1\infty}^{(0)}$), from Z5, up through to the uppermost layer, Z1, where the disturbance is damped down to zero (cf. the roles of the main- and upper-deck in the triple-deck theory). Returning to the bottom layer Z5, we have shown that

$$v \sim p M_\infty^4 \Omega^{-\frac{3}{2}} \Delta \tilde{l}^{-2} \hat{y} \quad \text{as } \hat{y} \rightarrow \infty,$$

implying that, at leading order,

$$v \sim p M_\infty^4 \Omega^{-\frac{3}{2}} \Delta \tilde{l}^{-2} \frac{\delta_1}{\delta_2}, \quad \text{in layer } Z3^+. \quad (4.1.20)$$

In Z4 the disturbance-solutions will have the standard ‘displacement forms’; in particular

$$u \sim A U_{BY} \quad \text{and} \quad v \sim \frac{\Delta}{\tilde{l}} A U_B, \quad (4.1.21a, b)$$

where the size of the displacement A (already scaled on the small linearization parameter) is to be determined. For small Y we see that

$$v \sim \frac{\Delta}{\tilde{l}} A Y,$$

implying that, at leading order,

$$v \sim \frac{\Delta}{\tilde{l}} A \delta_1, \quad \text{in layer } Z3^-. \quad (4.1.22)$$

Now the leading-order size of the disturbance v is unaltered across the critical layer Z4 and so we can equate (match) it’s sizes in $Z3^\pm$, thus evaluating A in terms of the Stokes-layer pressure disturbance p :

$$A \sim p M_\infty^4 \Omega^{-\frac{3}{2}} \Delta \tilde{l}^{-2} \frac{\delta_1}{\delta_2} \frac{\tilde{l}}{\Delta \delta_1} \sim p M_\infty^4 \Omega^{-1} \tilde{l}^{-1}. \quad (4.1.23)$$

Returning to Z4, we see that

$$v \sim \frac{\Delta}{\tilde{l}} A U_B, \sim \frac{\Delta}{\tilde{l}} A \quad \text{as } Y \rightarrow \infty,$$

thus inducing (via the y -momentum equation) a pressure response in Z5 of size

$$\frac{\Delta}{\tilde{l}} A \cdot \frac{\tilde{H}}{\tilde{l}}. \quad (4.1.24)$$

Again the classical boundary-layer theory no longer holds; instead this pressure response must correspond to the pressure-disturbance in the boundary-layers below. Thus the size of the pressure-disturbance p can be expressed in terms of the displacement A :

$$p \sim \frac{\Delta}{\tilde{l}} A \cdot \frac{\tilde{H}}{\tilde{l}} \sim \frac{\Delta}{\tilde{l}} A M_\infty^{-1}. \quad (4.1.25)$$

So, substituting this expression for p in the previous expression for A finally yields, after a little simplification, the third condition

$$\tilde{l}^2 \Omega \sim \Delta M_\infty^3 \quad (4.1.26)$$

between the unknown quantities Ω, δ_1 and $\tilde{l}$. Solving (4.1.18a,b), (4.1.19) and (4.1.26) gives the sizes

$$\begin{cases} \Omega \sim [\Delta M_\infty^3]^{-\frac{4}{5}}, \quad \tilde{l} \sim [\Delta M_\infty^3]^{-\frac{9}{10}}, \quad \delta_1 \sim [\Delta M_\infty^3]^{-\frac{1}{10}} : & \text{Cooled wall,} \\ \Omega \sim [\Delta M_\infty^3]^{-\frac{2}{3}}, \quad \tilde{l} \sim [\Delta M_\infty^3]^{-\frac{5}{6}}, \quad \delta_1 \sim [\Delta M_\infty^3]^{-\frac{1}{6}} : & \text{Insulated wall.} \end{cases} \quad (4.1.27a, b)$$

Encouragingly, the well-known ‘incompressible’ scales are recovered from (4.1.27a,b) by letting $M_\infty \sim O(1)$, in which case $\Delta \sim Re^{-\frac{1}{2}}$; in particular, the cooled-wall result (which now corresponds to a boundary-layer with non-zero pressure gradient) gives $\tilde{l} \sim Re^{-\frac{9}{20}}$, whilst the insulated-wall case (which now corresponds to standard Blasius flow) gives $\tilde{l} \sim Re^{-\frac{5}{12}}$.

However, for large Mach number, we see that compressibility has a significant effect on the upper-branch scales. Moreover we see that all the Mach-number dependence is contained in the combination “ ΔM_∞^3 ” — this is identical to what we have seen earlier for the triple-deck scales as $M_\infty \rightarrow \infty$. Thus, for both the upper- and lower-branch asymptotic structures we have found the same ‘hypersonic similitude’, namely ΔM_∞^3 . Returning to the upper-branch scales (4.1.27a,b), we see that the whole five-layered, short-scaled structure (for both the insulated-wall and cooled-wall cases) simultaneously collapses into the same two-tiered structure (with streamwise lengthscale $\sim O(1)$) that the lower-branch structure (triple-deck) collapses into, as $\Delta M_\infty^3 \nearrow 1 (\leftrightarrow M_\infty \nearrow Re^{\frac{1}{9}} : \text{Sutherland's viscosity-temperature formula})$.

This result was anticipated at the start of this subsection; we noted that the critical layer corresponding to the upper-branch modes would be forced up into the heart of the boundary-layer, as the Mach number increased, because of the ever-increasing thickness of lower-deck containing the lower-branch critical-layer. Now that we have studied the upper-branch mechanism, this result is even less surprising when one considers the ‘physical’ similarity between the two asymptotic theories describing the viscous stability of a laminar boundary layer

flow. Both concern short-scaled, multi-layered structures in which the boundary-layer pressure disturbance is self-induced and is governed by the displacement effect of a thin viscous layer (the lower-deck, for the lower-branch modes; the Stokes layer Z5, for the upper-branch modes) next to the wall.

So far we have considered linear upper-branch modes. As the disturbance size increases, the first layer to feel the effects of nonlinearity is the critical layer Z4 (see Gajjar & Cole, (1989); and references therein). However, the structure is identical to that illustrated above for linear-modes; the differences that do occur ‘only’ affect the size of the phase shift across the critical layer, resulting in modified eigenvalues. Thus ‘these’ particular nonlinear theories disappear, along with their associated five-layered structure, as $M_\infty \rightarrow Re^{\frac{1}{9}}$. None-the-less, it still may be very profitable to investigate the high Mach-number limit of these nonlinear upper-branch theories to see if they provide (much needed) insight into the stability properties of the resulting two-tiered structure that governs viscous stability.

Theoreticians have mainly concentrated on linear and nonlinear stability characteristics of the lower-branch, viscous modes; these are felt to be the more important in the context of practical airfoil flow. In addition, these modes have the added advantage of a simpler and ‘fruitful’ asymptotic descriptive-structure than the upper-branch modes. This last statement is especially true in the case of compressible flows; the effects of compressibility merely affecting the pressure-displacement law in the triple-deck theory. However we have seen that as the Mach number increases, both of these structures collapse into the same, two-tiered structure which, at first sight, appears to be far less tenable to the existing theoretical techniques because of the $O(1)$ -lengthscale and lack of sub-layers. This initial view of the authors’ was ‘borne out’ by the subsequent months spent considering the governing equations, looking for analytic theories that would provide insight into the solution properties. The remainder of this chapter formulates these ‘troublesome’ governing equations, makes some additional observations concerning physical aspects of relevance to the new slender structure and details why there are ‘troublesome’ with regard to their analytical and numerical solution.

§4.2. FORMULATION

§4.2.1 Introduction.

The underlying formulation is as described earlier, in Chapter 2. This chapter continues where that left off. We try to avoid confusion over names: we essentially have different names for essentially the same region, depending on the size of the Mach number. By upper-deck and upper-tier we mean the same, relatively thin, inviscid layer, adjacent to and above the ‘boundary layer’, responsible for the ‘inviscid reaction’ (via the pressure disturbance) to the viscous displacement from the boundary layer. It plays the same role as the upper deck in conventional triple deck theory.

The lower-tier covers the whole region next to the plate where viscous effects enter both the basic-flow and disturbance equations (these are of course one in the general nonlinear formulation). This (non-linear) region is basically the over-grown lower-deck which has engulfed the (previously inviscid and linear) main deck. It covers the classical boundary layer (of the basic flow) and thus we frequently use this description. It is the upper-tier (deck) which enables the pressure to be unprescribed. See Figure 4.3.

We define the scaled Mach number,

$$m = Re^{-\frac{1}{9}} M_{\infty}. \quad (4.2.1)$$

We assume that $m \sim O(1)$ at present, but do not set it to unity. This complicates the scales, especially in the boundary-layer (lower-tier) where (strictly) we must give dual scales, ensuring that we recover the lower- and main-decks of the (supersonic) triple-deck structure for (asymptotically) small m . The large m limit is studied in a later section. For later reference note that

$$\begin{aligned} m \sim O(1) & \leftrightarrow \text{present two-tiered structure} \\ m \searrow O(Re^{-\frac{1}{9}}) & \leftrightarrow \text{supersonic triple-deck structure} \\ m \nearrow O(Re^{\frac{4}{45}}) & \leftrightarrow \text{interactive basic flow.} \end{aligned} \quad (4.2.2a - c)$$

The $m \gg 1$ work needs a lot more study— see Section 4.5.

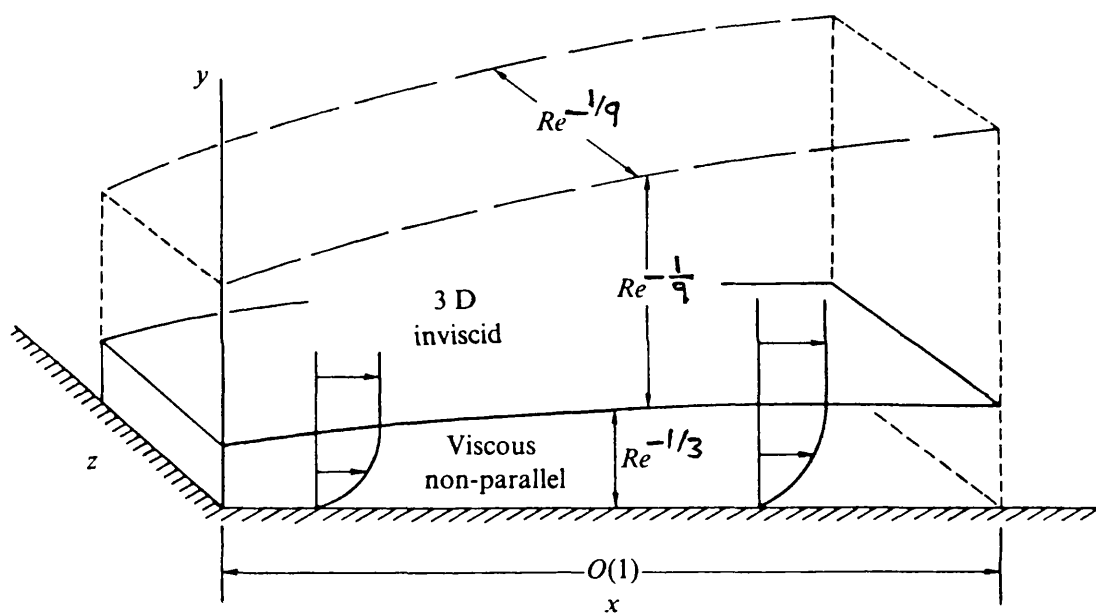


Figure 4.3. The two-tiered structure governing viscous stability when $M_\infty \sim Re^{\frac{1}{9}}$, where non-parallel-flow effects become substantial. (from Smith, 1989.)

The streamwise-lengthscale, spanwise-lengthscale and the timescale, common to both the lower- and upper- tiers,

$$x = m^{\frac{27}{8}} X_1, \quad z = Re^{-\frac{1}{9}} m^{\frac{19}{8}} Z_1 \quad \text{and} \quad t = m^{\frac{9}{4}} \hat{t}, \quad (4.2.3a - c)$$

follow immediately from Chapter 2. These are not appropriate for $m \gg 1$. Note that the multiple-scales in the streamwise direction,

$$\frac{\partial}{\partial x} \rightarrow m^{-\frac{27}{8}} \frac{\partial}{\partial X_1} + \frac{\partial}{\partial x},$$

merge into one as $m \nearrow O(1)$.

§4.2.2 The lower-tier (boundary-layer).

This region, in which viscosity effects are important, comprises of the ‘old’ lower-deck which has grown in thickness (coalescing with the main deck) and now covers the whole ‘classical’ boundary layer. Here we write

$$y = Re^{-\frac{1}{3}} Y_1, \quad Y_1 = \left\{ \begin{array}{l} m^{\frac{21}{8}} Y_L \\ m^{\frac{3}{2}} Y_M \end{array} \right\} \quad \text{where} \quad Y_{L,M} \sim O(1). \quad (4.2.4)$$

Here, and in the following boundary-layer scalings, the upper- m -term in the braces corresponds to the ‘old’ lower-deck whilst the lower term corresponds to the ‘old’ main-deck. The ‘duality’ of the scales is necessary to re-capture the (large Mach number) triple-deck structure for small m ; the case of $m \gg 1$ will be considered in a later section. When considering $m \sim O(1)$ the terms in the braces can be neglected, without any loss of generality.

The scales for the velocities, pressure and thermodynamic quantities are

$$(u, v, w) = \left(m^{\frac{9}{8}} U, Re^{-\frac{1}{3}} m^{\frac{3}{8}} V, Re^{-\frac{1}{9}} m^{\frac{1}{8}} W \right),$$

$$p = \underbrace{\gamma^{-1} Re^{-\frac{2}{9}} m^{-2}}_{P_\infty} + Re^{-\frac{4}{9}} m^{-\frac{7}{4}} P(\hat{t}, X, Z) + Re^{-\frac{8}{9}} m^{-\frac{5}{4}} P_2,$$

$$(\rho, T, \mu) = \left(Re^{-\frac{2}{9}} m^{-2} R, Re^{\frac{2}{9}} m^2 \theta, Re^{\frac{1}{9}} m M \right),$$

where

$$U = \left\{ \begin{array}{c} U_L(\hat{t}, X_1, x, Y_1, Z_1) \\ m^{-\frac{9}{8}} U_0(x, Y_1) + U_M(\hat{t}, X_1, x, Y_1, Z_1) \end{array} \right\},$$

$$V = \left\{ \begin{array}{c} V_L(\hat{t}, X_1, x, Y_1, Z_1) \\ m^{-\frac{9}{8}} V_M(\hat{t}, X_1, x, Y_1, Z_1) + \cdots + m^{\frac{9}{8}} [V_0(x, Y_1) + V_3(\hat{t}, X_1, x, Y_1, Z_1)] \end{array} \right\},$$

$$W = \left\{ \begin{array}{c} W_L(\hat{t}, X_1, x, Y_1, Z_1) \\ m^{\frac{9}{8}} W_M(\hat{t}, X_1, x, Y_1, Z_1) \end{array} \right\},$$

$$P_2 = \left\{ \begin{array}{c} P_{2L}(\hat{t}, X, Y_1, Z) \\ m^{-\frac{27}{8}} P_{2M}(\hat{t}, X, Y_1, Z) \end{array} \right\},$$

and

$$R(\hat{t}, X_1, x, Y_1, Z_1) = R_0(x, Y_1) + \left\{ \begin{array}{c} \left\{ \begin{array}{ll} m^{\frac{9}{4}} R_L(\hat{t}, X_1, x, Y_1, Z_1) & : \theta'_w = 0 \\ m^{\frac{9}{8}} R_L(\hat{t}, X_1, x, Y_1, Z_1) & : \theta_w \text{ fixed} \end{array} \right\} \\ m^{\frac{9}{8}} R_M(\hat{t}, X_1, x, Y_1, Z_1) \end{array} \right\},$$

with similar forms for θ and M .

It is important to note (especially when considering the upper-tier) that p_∞ is not $O(1)$: this is due to our (standard) choice of non-dimensionalisation. All functions depend on $\hat{t}, X_1, Y_1, Z_1$ apart from P . The third term in the pressure expansion, necessary to balance the leading order inertial terms in the y -momentum equation, is crucial in deriving the pressure-displacement law (see later).

These scales can be deduced in a couple of ways; either by 'first -principles', directly from the hypersonic limit of the supersonic triple-deck scales; or by just noting that $u \sim O(1)$, the same as in most 'classical boundary layers' and fixing up the m -scaling and the v, w and p scales such that (i) the expected balances (from the eigenrelation (2.3.25) in the resulting equations are obtained, and (ii) the lower- and main- deck scales are recovered in the appropriate, small- m limit. We shall restrict our discussion to the former approach. As in Chapter 2, the scales arise from 'explicit' contributions (the large-Mach number form of the triple-deck scales) and 'implicit' contributions (due to the wavenumbers and frequency, that appear in the resulting (linearised) triple-deck equations, being dependent on M_∞).

Let us consider these linearised equations, (2.3.17a-e, 2.3.18c). The crucial point to note is that we require

$$h\tilde{U} \sim \tilde{U} \sim Y, \quad (\text{w.r.t. } M_\infty), \quad (4.2.6)$$

to ensure that the critical-layer remains in this layer. Note that we are assuming that inverse-powers of Re are smaller than (relevant) inverse-powers of M_∞ , which in turn are smaller than the small linearisation parameter h .

Once the necessity of the balance, (4.2.6), has been realised, yielding

$$\tilde{U} \sim Y \sim \alpha^{-\frac{1}{3}} \sim M_\infty^{\frac{1}{2}} \quad [M_\infty \gg 1], \quad (4.2.7a)$$

the required ‘implicit’ scalings follow easily. Balancing all terms in the continuity equation (2.3.17a) gives

$$\alpha\tilde{U} \sim \frac{\tilde{V}}{Y} \sim \beta\tilde{W},$$

yielding the required sizes of $\tilde{V}$ and $\tilde{W}$ for large Mach number

$$\tilde{V} \sim \alpha Y \tilde{U} \sim \alpha^{\frac{1}{3}} \sim M_\infty^{-\frac{1}{2}}, \quad \tilde{W} \sim \frac{\alpha\tilde{U}}{\beta} \sim \frac{\alpha^{\frac{2}{3}}}{\beta} \sim M_\infty^{-\frac{1}{2}}. \quad (4.2.7b, c)$$

The size of the pressure can be deduced in a couple of ways, for example from the displacement condition (2.3.17e) we see that

$$\tilde{A} \sim \tilde{U}$$

whilst the pressure-displacement law, (2.3.18c), yields

$$\tilde{P} \sim \alpha\tilde{A}.$$

Combining these last two results gives

$$\tilde{P} \sim \alpha\tilde{U} \sim \alpha^{\frac{2}{3}} \sim M_\infty^{-1} \quad [M_\infty \gg 1]. \quad (4.2.7d)$$

These, together with the large Mach number forms of the ‘explicit’ triple-deck scalings, will yield (the lower-deck version of) the scales, (4.2.5a-d), stated earlier. As an example we derive the scale for the induced pressure, P . From (2.3.8d), (2.3.15)

$$\begin{aligned} p - p_\infty &= Re^{-\frac{1}{4}} \left(\frac{\lambda_S}{\sqrt{2}} \right)^{\frac{1}{2}} \mu_w^{\frac{1}{2}} T_w^{-\frac{1}{2}} (M_\infty^2 - 1)^{-\frac{1}{4}} h \tilde{P} \\ &\sim Re^{-\frac{1}{4}} \cdot \left(M_\infty^{\frac{1}{2}} \right)^{\frac{1}{2}} \cdot (M_\infty)^{\frac{1}{2}} \cdot (M_\infty^2)^{-\frac{1}{2}} \cdot (M_\infty^2)^{-\frac{1}{4}} \cdot M_\infty^{-1} \\ &\sim Re^{-\frac{1}{4}} \cdot M_\infty^{-\frac{7}{4}} \quad [M_\infty \gg 1] \\ &= Re^{-\frac{4}{9}} m^{-\frac{7}{4}} \end{aligned} \quad (4.2.8)$$

The derivation of the lower-deck version of the scales, (4.2.5e-g), for the thermodynamic quantities from the hypersonic limit of the supersonic scales is made very complicated by the need to consider cooled and insulated walls separately. It is easier to deduce them from the supersonic-triple-deck main-deck scales, remembering that the lower-deck and main-deck have coalesced in the current regime and thus we would expect the corresponding two sets of thermodynamic disturbance scales to have tended to the same limit. To determine the ‘lower-deck’ scales completely we also need to fix-up the m -power (the argument of the previous sentence essentially determines the Re scales). This is simply achieved by ensuring that we recapture the lower-deck scales in the appropriate small- m limit. As an example consider the ratio of disturbance to base-flow in the main-deck density expansion, (2.3.11e), †

$$\begin{aligned}
& Re^{-\frac{1}{8}} \left(\frac{\lambda_S}{\sqrt{2}} \right)^{-\frac{3}{4}} \mu_w^{-\frac{1}{4}} T_w^{\frac{3}{4}} (M_\infty^2 - 1)^{-\frac{1}{8}} A \frac{d}{d\bar{y}} \\
& \sim Re^{-\frac{1}{8}} \cdot \left(M_\infty^{\frac{1}{2}} \right)^{-\frac{3}{4}} \cdot (M_\infty)^{-\frac{1}{4}} \cdot (M_\infty^2)^{\frac{3}{4}} \cdot (M_\infty^2)^{-\frac{1}{8}} \cdot M_\infty^{-1} \cdot \left(M_\infty^{-\frac{3}{2}} \right)^{-1} \\
& \sim Re^{-\frac{1}{8}} \cdot M_\infty^{\frac{9}{8}} \quad [M_\infty \gg 1] \\
& = m^{\frac{9}{8}},
\end{aligned}$$

so that when the Mach number increases to size $O(Re^{\frac{1}{9}})$ the disturbance scale has risen to the same order as the base-profile. This, coupled with the above argument concerning the lower- and main- decks having the same limiting properties, accounts for the inclusion of compressibility effects in the viscous-layer of the present two-tier-regime.

The remaining scales follow in the same manner and substituting (4.2.5a-g) into the compressible Navier-Stokes equations, (2.1.2a-d) together with the chosen constitutive relations, yields (setting $h = 1$) the non-linear boundary layer equations,

† The corresponding ratio for the streamwise velocity u is identical.

$$\begin{aligned}
R_{\hat{t}} + (RU)_{X_1} + (RV)_{Y_1} + (RW)_{Z_1} &= 0, \\
R(U_{\hat{t}} + UU_{X_1} + VU_{Y_1} + WU_{Z_1}) &= (MU_{Y_1})_{Y_1}, \\
0 &= -P_{Y_1}, \\
R(W_{\hat{t}} + UW_{X_1} + VW_{Y_1} + WW_{Z_1}) &= -P_{Z_1} + (MW_{Y_1})_{Y_1}, \\
R(\theta_{\hat{t}} + U\theta_{X_1} + V\theta_{Y_1} + W\theta_{Z_1}) &= \frac{1}{Pr} (M\theta_{Y_1})_{Y_1} + (\gamma - 1)MU_{Y_1}^2, \\
R\theta = 1 \quad \text{and} \quad M &= (1 + S)\theta^{\frac{1}{2}}. \tag{4.2.9a - g}
\end{aligned}$$

These must be solved subject to boundary conditions at the wall

$$U = V = W = 0 \quad \text{on} \quad Y_1 = 0$$

with θ or θ_{Y_1} prescribed on $Y_1 = 0$, together with conditions at infinity, essentially

$$U \rightarrow m^{-\frac{9}{8}}, \quad W \rightarrow 0, \quad M, \theta \rightarrow 0 \quad (\text{s.t. } R\theta = 1), \quad \text{as } Y_1 \rightarrow \infty,$$

whilst the normal velocity tends to a constant (of Y_1)

$$V \rightarrow m^{\frac{9}{8}} V_{\infty}(X, Z, \hat{t}), \quad \text{as } Y_1 \rightarrow \infty. \tag{4.2.10}$$

The (so-called) displacement $V_{\infty}(X_1, Z_1, \hat{t})$, defined by (4.2.10d), is related to the (induced pressure) P via a pressure-displacement law that stems from matching the solutions in the upper-deck, to be considered next, with the limiting forms from the boundary layer (cf. standard triple-deck theory). The boundary conditions at infinity will be discussed further in a later subsection (§4.2.4).

Note that this regime was deduced from the linear triple-deck equations (h small); the general nonlinear equations given above can be thought of as the uniform limit of letting $h \rightarrow 1$. We are allowed to do this because h is independent of Mach number scaling. We shall see that the upper-tier equations are still linear. Alternatively, the arguments of §4.1.2 could be appealed to.

§4.2.3 The upper-tier (upper-deck).

This region is very similar to the ‘old’ upper-deck, the main differences being in the dimensions and, more significantly, the effects of unsteadiness now emerge

at leading order. Even when the (viscous) boundary layer equations are nonlinear, the inviscid upper-tier equations are linear- same as triple-deck structure. Here

$$y = Re^{-\frac{1}{9}} m^{\frac{19}{8}} \hat{y} \quad (4.2.11)$$

and the new normal-variable $\hat{y}$ is taken to be $O(1)$. The disturbances to the free-stream state have the form

$$\begin{aligned} [u, v, w, p, \rho, T] = & [1, 0, 0, p_\infty \equiv \gamma^{-1} Re^{-\frac{2}{9}} m^{-2}, 1, 1] + \\ & \left[Re^{-\frac{4}{9}} m^{-\frac{7}{4}} \hat{u}, Re^{-\frac{1}{3}} m^{-\frac{3}{4}} \hat{v}, Re^{-\frac{1}{3}} m^{-\frac{3}{4}} \hat{w}, Re^{-\frac{4}{9}} m^{-\frac{7}{4}} \hat{p}, Re^{-\frac{2}{9}} m^{\frac{1}{4}} \hat{\rho}, Re^{-\frac{2}{9}} m^{\frac{1}{4}} \hat{T} \right] \\ & + \dots \end{aligned} \quad (4.2.12a - f)$$

Notice that the scales are much more straightforward here : this region has always been one and hence there are no dual scales.

Again the first four scales can be deduced from the supersonic triple-deck scales, or merely by inspection, after noting that the role of the upper-tier is to dampen the displacement effect from the boundary-layer by determining/driving the pressure in the latter. Note that the smaller-than-to-be-expected size of the u -disturbance is due to the small size of the streamwise pressure-gradient: this results in there being no contribution from the term $(\rho u)_x$ in the leading order form of the continuity equation quoted below. The remaining disturbance scales can be deduced from the gas law. These are substituted into the (compressible, inviscid) Euler equations, yielding

$$\begin{aligned} \mathcal{D}\hat{u} &= -\hat{p}_{\hat{x}_1}, & \mathcal{D}\hat{v} &= -\hat{p}_{\hat{y}}, & \mathcal{D}\hat{w} &= -\hat{p}_{\hat{z}_1}, \\ \mathcal{D}\hat{\rho} + \hat{v}_{\hat{y}} + \hat{w}_{\hat{z}_1} &= 0 \quad \text{and} \quad \mathcal{D}\hat{p} = \mathcal{D}\hat{p}, \end{aligned} \quad (4.2.13a - e)$$

where the operator

$$\mathcal{D} \equiv \left(m^{\frac{9}{8}} \frac{\partial}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}_1} \right). \quad (4.2.13f)$$

These resulting equation for pressure is

$$\left[\mathcal{D}^2 - \partial_{\hat{y}\hat{y}}^2 - \partial_{\hat{z}_1\hat{z}_1}^2 \right] \hat{p} = 0, \quad (4.2.14a)$$

to be solved subject to the boundary conditions

$$\hat{p} \rightarrow 0 \quad \text{as} \quad \hat{y} \rightarrow \infty, \quad \hat{p} \rightarrow P \quad \text{as} \quad \hat{y} \rightarrow 0^+, \quad (4.2.14b, c)$$

ensuring that the viscous-inviscid interaction is self-contained (no external influences), and that there is a match to the pressure-disturbance in the viscous boundary-layer, respectively.

Note that when $m \sim O(1)$, the generalised Prandtl Glauert equation for the pressure disturbance, $\hat{p}$, contains time-derivatives i.e. the upper-tier (upper-deck) is now unsteady at leading order. We also see that as $m \searrow 0$ these time derivatives drop-out and we recover the standard pressure-equation for supersonic flow. Further, note this equation is different than that derived by Bowles & Smith (1989) in their study of the transonic regime ($M_\infty \ll 1$), even though both equations include time-derivatives.

In the next sub-section we show that the crucial pressure-displacement law can be stated in the form

$$-\frac{\partial \hat{p}}{\partial \hat{y}} \rightarrow DV_\infty(X_1, Z_1, \hat{t}) \quad \text{as } \hat{y} \rightarrow 0, \quad (4.2.15)$$

relating the displacement from the boundary layer, $V_\infty(X_1, Z_1, \hat{t})$, to the pressure P (via the above equation and boundary conditions, (4.2.14), for $\hat{p}$, the upper-deck pressure). As $m \searrow 0$, $\mathcal{D} \rightarrow \frac{\partial}{\partial X}$ and $V_\infty \rightarrow A_X$, so that the previous $P - A$ law, (2.3.14d), for supersonic flow is recovered.

§4.2.4 The pressure-displacement law.

The derivation of this pressure-displacement law essentially follows that of Stewartson & Williams (1969) for the supersonic case. At first sight the problem appears simpler than the former as here there is no main deck to have to solve for and match the pressure and displacement across. However the converse is true due to the apparent ‘mismatches’ in the sizes of the thermodynamic quantities between the two tiers. Strictly this current $M_\infty \sim Re^{\frac{1}{9}}$ -regime two-tier structure should be thought of as comprising of three layers : the current two-tiers but now with a third, thin layer lying in between to match the hot thermal boundary layer to the freestream state. This extra layer basically corresponds to that part of the Re -boundary layer that is not in, i.e. is above, the thermal boundary layer. Here, in the cooler part of the boundary-layer, viscous effects are still important.

The ‘adjustment layer’ has very different properties than the main-deck of conventional triple-deck theory, but there is some analogy between the two because they both play ‘supporting roles’ in the respective viscous-inviscid interaction mechanisms. By this we mean that the coupled set of equations to be solved in both instances involve the boundary-layer equations (from the lower-tier or lower-deck) and a pressure equation (or pressure- displacement law) from the upper-tier/deck: once the equations have been formulated, there is no need to consider the main-deck or the ‘adjustment layer’ in future analytical and/or numerical studies.† Because of this fact we do not give full details of the ‘adjustment layer’ here, we merely point out the particular details necessary to demonstrate that this layer does indeed match together the expansions of the upper- and lower-tiers given earlier , as well as illustrating the origin of the pressure- displacement law quoted in the last subsection.

Consider the y - and z - momentum equations, in the lower tier, as the free-stream is approached ($Y_1 \rightarrow \infty$). Here the leading order balances are

$$\begin{cases} R(V_{Li} + U_L V_{LX_1} + W_L V_{LZ_1}) = -P_{2LY_L} & : \text{Lower-deck scales} \\ R(m^{\frac{9}{8}} V_{Mi} + U_0 V_{MX_1} + m^{\frac{9}{4}} W_M V_{MZ_1}) = -P_{2MY_M} & : \text{Main-deck scales} \end{cases}$$

and

$$\begin{cases} R(W_{Li} + U_L W_{LX} + W_L W_{LZ_1}) = -P_{Z_1} & : \text{Lower-deck scales} \\ R(m^{\frac{9}{8}} W_{Mi} + U_0 W_{MX} + m^{\frac{9}{4}} W_M W_{MZ_1}) = -P_{Z_1} & : \text{Main-deck scales} \end{cases} \quad (4.2.16a - d)$$

Recall that the X_1 , Z_1 and $P \leftrightarrow \hat{p}$ scales are common to both the lower- and upper- tiers, so that

$$W \sim \frac{X_1 P}{R U Z_1} \sim \frac{1}{R U} \quad \text{as } Y_1 \rightarrow \infty.$$

Now, as the ‘adjustment layer’ is crossed (from bottom to top) the thermodynamic quantities adjust rapidly, but smoothly and continuously, to their free-stream sizes (see Chapter 6); in particular

$$R \rightarrow Re^{\frac{2}{9}} m^2 \quad \text{as } Y_1 \rightarrow \infty.$$

† However, we shall see in later chapters that this layer is crucial to, and in fact dominates, the inviscid and Görtler instability mechanisms.

This result, coupled with the fact that $U \rightarrow m^{-\frac{9}{8}}$, as $Y_1 \rightarrow \infty$, implies that

$$W \rightarrow O(Re^{-\frac{2}{9}}m^{-2} \cdot m^{\frac{9}{8}}), \quad \text{as } Y_1 \rightarrow \infty. \quad (4.2.17)$$

Thus W decays across the ‘adjustment layer’, to match on to the smaller w -disturbance size found in the upper-deck. Therefore, for large Y_1 , we can neglect the WV_Z term appearing in the y -momentum equations quoted above, and setting

$$(U, V, R) \equiv (m^{-\frac{9}{8}}, m^{\frac{9}{8}}V_\infty, Re^{\frac{2}{9}}m^2), \quad (4.2.18)$$

so that U, V and R take their limiting, large Y_1 , freestream values) gives

$$Re^{\frac{2}{9}}m^2\mathcal{D}V_\infty = - \left\{ \begin{array}{c} m^{\frac{9}{4}}P_{2LY_L} \\ P_{2MY_M} \end{array} \right\}.$$

The remaining argument closely follows that of Stewartson & Williams (1969): writing the last expression, valid for $Y_1 \gg 1$, in terms of the upper-tier variable $\hat{y}$ gives

$$P_2 = -Re^{\frac{4}{9}} \left\{ \begin{array}{c} m^{-\frac{1}{2}} \\ m^{\frac{23}{8}} \end{array} \right\} \mathcal{D}V_\infty \hat{y} + c_1, \quad \hat{y} \ll 1, \quad (4.2.19)$$

where the unknown function c_1 is independent of $\hat{y}$: its actual value is not needed here. Thus the pressure boundary-layer (lower-tier) expansion (4.2.5d), at the top of the layer, takes the form

$$p = \underbrace{\gamma^{-1}Re^{-\frac{2}{9}}m^{-2}}_{p_\infty} + Re^{-\frac{4}{9}}m^{-\frac{7}{4}} [P(\hat{t}, X, Z) - \mathcal{D}V_\infty \hat{y}] + \dots, \quad (4.2.20)$$

written in terms of the upper-tier variable $\hat{y}$ (cf. Van Dyke’s matching-principle).

Now consider the upper-deck pressure expansion for small $\hat{y}$,

$$p = \underbrace{\gamma^{-1}Re^{-\frac{2}{9}}m^{-2}}_{p_\infty} + Re^{-\frac{4}{9}}m^{-\frac{7}{4}} \left[\hat{p}(0^+) + \hat{y} \frac{\partial \hat{p}}{\partial \hat{y}}(0^+) + \dots \right] + \dots, \quad (4.2.21)$$

where we have written $\hat{p}$ in the form of a Taylor- MacLaurin series about $\hat{y} = 0^+$.

Hence, matching the pressure in the two tiers requires

$$\hat{p} \rightarrow P, \quad \text{and} \quad \frac{\partial \hat{p}}{\partial \hat{y}} \rightarrow -\mathcal{D}V_\infty \quad \text{as} \quad \hat{y} \rightarrow 0^+. \quad (4.2.22)$$

These are the conditions stated in the previous subsection. They prescribe the pressure-displacement relation, crucial to the viscous-inviscid description of self-induced separation, and the stability of, the present supersonic regime of concern.

§4.3. THE TWO-TIER STRUCTURE: PROPERTIES AND RESULTING DIFFICULTIES.

§4.3.1 Introduction and review.

In this section we go on to investigate the properties and characteristics of the two-tier structure formulated in the previous section. Recall that this structure is appropriate for the range of large Mach numbers $M_\infty \sim Re^{\frac{1}{9}}$. The details of the last section were complicated by the fact that we were not specific about the size of the scaled Mach number, $m \sim M_\infty Re^{-\frac{1}{9}}$; the assumption that $m \leq O(1)$ meant that ‘dual’ scales were needed for the boundary layer so that the thinner lower-deck was recovered for small m . In other sections we consider large or small m , but here we are interested in $m \sim O(1)$.

In fact we can choose $m = 1$, without loss of generality, by a suitable redefinition of the lengthscale L (used to non-dimensionalise x, y and z). The viscous-inviscid, nonlinear coupled system, governing self-induced separation and stability properties, that must be solved, is now restated for clarity. The time and streamwise-lengthscale are those of the underlying base-flow, but the spanwise z -variation is characterised by a short scale:

$$(t, x, z) = (t, x, Re^{-\frac{1}{9}} Z_1).$$

The boundary-layer is hot and of classical thickness. The scaled normal variable, $Y_1 = Re^{\frac{1}{3}} y$, is $O(1)$ and the remaining quantities have the following scalings

$$(u, v, w, p - p_\infty) = (U, Re^{-\frac{1}{3}} V, Re^{-\frac{1}{9}} W, Re^{-\frac{4}{9}} P)$$

and

$$(\rho, T, \mu) = (Re^{-\frac{2}{9}} R, Re^{\frac{2}{9}} \theta, Re^{\frac{1}{9}} M).$$

The resulting nonlinear equations are

$$\begin{aligned}
R_t + (RU)_x + (RV)_{Y_1} + (RW)_{Z_1} &= 0, \\
R(U_t + UU_x + VU_{Y_1} + WU_{Z_1}) &= (MU_{Y_1})_{Y_1}, \\
0 &= -P_{Y_1}, \\
R(W_t + UW_x + VW_{Y_1} + WW_{Z_1}) &= -P_{Z_1} + (MW_{Y_1})_{Y_1}, \\
R(\theta_t + U\theta_x + V\theta_{Y_1} + W\theta_{Z_1}) &= \frac{1}{Pr} (M\theta_{Y_1})_{Y_1} + (\gamma - 1)MU_{Y_1}^2, \\
R\theta = 1 \quad \text{and} \quad M &= (1 + S)\theta^{\frac{1}{2}}. \tag{4.3.1a - g}
\end{aligned}$$

These must be solved subject to boundary conditions at the wall

$$U = V = W = 0 \quad \text{on} \quad Y_1 = 0, \tag{4.3.1h - j}$$

with θ or θ_{Y_1} prescribed on $Y_1 = 0$, together with conditions at infinity

$$U \rightarrow 1, \quad V \rightarrow V_\infty(x, Z_1, t), \quad W \rightarrow 0,$$

$$M, \theta \rightarrow 0 \quad \text{s.t.} \quad R\theta = 1 \quad \text{as} \quad Y_1 \rightarrow \infty. \tag{4.3.1k - o}$$

A relation (pressure-displacement law) between the boundary layer pressure, P , and the displacement V_∞ stems from the upper-deck pressure equation and boundary conditions:

$$\begin{aligned}
\left[\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right)^2 - \frac{\partial^2}{\partial \hat{y}^2} - \frac{\partial^2}{\partial Z_1^2} \right] \hat{p} &= 0, \\
\hat{p} \rightarrow P \quad \text{as} \quad \hat{y} \rightarrow 0^+, \quad \hat{p} \rightarrow 0 \quad (\text{or bounded}) \quad \text{as} \quad \hat{y} \rightarrow \infty,
\end{aligned}$$

and

$$\frac{\partial \hat{p}}{\partial \hat{y}} \rightarrow - \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) V_\infty, \quad \text{as} \quad \hat{y} \rightarrow 0^+. \tag{4.3.1p - s}$$

Note that the boundary layer equations are nonlinear, three-dimensional and compressible- although some pressure gradient terms and all the bulk-viscosity terms are absent (at this order). The upper-deck pressure equation is linear but is now three-plus-one dimensional: it is unsteady at leading order. Note the presence of the three physical parameters γ, Pr and S : these need to be chosen before a numerical solution is attempted.

These two coupled sets of equations would be very hard to solve numerically and some analytical progress is only possible by assuming that there are much

faster time-scales operating (see §4.4 and Chapter 5). Even then the familiar (weakly) non-linear theories (eg. Smith, 1979b - one mode, amplitude cubed; Hall & Smith, 1984 - several modes interacting, amplitude cubed; Smith & Burggraf, 1985 - nonlinear evolution of initially linear Tollmien-Schlichting mode; Stewart & Smith, 1987 - resonant triads governed by triple-deck scales; Hall & Smith, 1989 - longitudinal-vortex/TS-wave interaction, amplitude squared - see also Chapter 3) DO NOT carry over (at least, not at all trivially): basically because there is no simple linear eigen-relation. All these theories basically rely on a conventional viscous sublayer of the boundary layer (lower-deck). Note that, even for very high frequencies, the triple-deck structure cannot be recovered when $M_\infty \sim Re^{\frac{1}{9}}$.

As usual when confronted with a ‘new’ nonlinear system, we decide first to investigate the linear stability properties: we hope that it will simplify the problem to one that is more easily solvable. In addition it may be that disturbances of linear size, to the basic flow, lead to the eventual breakdown of laminar boundary layer flows; thus a knowledge of linear stability properties is very useful. However, we shall see in the following pages that we are still left with, in general, a set of partial differential equations to be solved numerically: the fact that $x \sim O(1)$ means that we cannot have harmonic- dependence in x . This results in no linear eigenrelation, so familiar in linear and weakly-nonlinear triple-deck theories, resulting in the need for a more sophisticated non-linear theory to be found. The best prospect seems to be one based on the high-frequency approach, so successful for the triple-deck structure; we discuss progress made in this direction in §4.4. Note that one must also take care over the ‘neglected’ adjustment layer (between the lower and upper tiers, where, in particular, the temperature adjusts to its free-stream value).

§4.3.2 The linearised problem.

We shall now linearise the system (4.3.1a-s) about the conventional steady, two-dimensional, non-interactive, non-parallel base flow

$$(U, V, W, P, \theta) = (U_B(x, Y_1), V_B(x, Y_1), 0, 0, \theta_B(x, Y_1)).$$

Note that V_B , the normal component of the underlying non-parallel base-flow is not leading order in the corresponding triple-deck expansion

$$(U, V, W, P, \theta) = (\lambda Y, 0, 0, 0, \theta_w).$$

Additionally, in the present study we must consider the general base-flow profiles (i.e. for all Y_1) rather than just their near- wall behaviour (cf. the solution of the linearised lower-deck equations in conventional triple-deck theory).

Further, note that $V_\infty \neq 0$ even for the base-flow - there is always a displacement effect for non-parallel boundary-layers; here, however, the choice of scales enables the displacement to drive an induced pressure disturbance in the boundary layer (we have assumed that the base-flow has zero-pressure gradient- if this was relaxed we would still have pressure-effects prescribed rather than self-induced).

As the base-flow is steady and two-dimensional we can choose to look for (linear) disturbances that have harmonic dependence in t and Z_1 , but NOT in x :

$$\begin{aligned} (U, V, W, P, \theta) = & (U_B(x, Y_1), V_B(x, Y_1), 0, 0, \theta_B(x, Y_1)) \\ & + h \left[\left(\tilde{U}(x, Y_1), \tilde{V}(x, Y_1), \tilde{W}(x, Y_1), \tilde{P}(x, Y_1), \tilde{\theta}(x, Y_1) \right) E + c.c. \right] \\ & + O(h^2), \end{aligned} \quad (4.3.2a - e)$$

where $E = \exp i[\beta Z_1 - \Omega t]$, $c.c.$ denotes complex conjugate, $\beta, \Omega \sim O(1)$ and h is a small (linearisation) parameter.

The leading order balances give the equations to be solved for the base flow

$$\begin{aligned} U_{Bx} + V_{BY_1} &= 0, \\ R_B (U_B U_{Bx} + V_B U_{BY_1}) &= (M_B U_{BY_1})_{Y_1}, \\ R_B (U_B \theta_{Bx} + V_B \theta_{BY_1}) &= \frac{1}{Pr} (M_B \theta_{BY_1})_{Y_1} + (\gamma - 1) M_B U_{BY_1}^2, \\ R_B(x, Y_1) &= 1/\theta_B, \quad M_B(x, Y_1) = (1 + S)\theta_B^{\frac{1}{2}}, \end{aligned} \quad (4.3.3a - e)$$

with the usual boundary conditions for the velocities

$$U_B(x, 0) = 0 = V_B(x, 0), \quad U_B(x, \infty) = 1;$$

the temperature at the wall must satisfy either

$$\theta'_B(x, 0) = 0 \quad \text{or} \quad \theta_B(x, 0) = \theta_W(x),$$

for an insulated plate or specified wall temperature, respectively, whilst the need to match with the cooler freestream, above the boundary layer necessitates that

$$\theta_B \rightarrow 0 \quad \text{as} \quad Y_1 \rightarrow \infty.$$

Recall that we are assuming that there is no pressure gradient; these equations can be easily modified to allow for an external pressure-gradient effect (in the x -momentum equation, energy equation and the gas-law). We define the basic displacement, $V_{B\infty}(x)$, by

$$V_{B\infty}(x) \equiv V_B(x, \infty).$$

A similarity solution of these equations exists (and is the appropriate solution) if the wall is an insulator, or if the specified wall temperature is constant (i.e. $\theta_W(x) \equiv \theta_W$). In fact the solution is very closely related to the conventional steady, two-dimensional boundary layer solution (see Stewartson, 1964) which, in turn, is very similar (in the analytical sense) to the classical Blasius solution (Blasius, 1908). See Chapter 6.

At next order we obtain the linear stability equations

$$\begin{aligned} -i\Omega\tilde{R} + (R_B\tilde{U} + \tilde{R}U_B)_x + (R_B\tilde{V} + \tilde{R}V_B)_{Y_1} + i\beta R\tilde{W} &= 0, \\ (-i\Omega\tilde{U} + U_B\tilde{U}_x + \tilde{U}U_{Bx} + \tilde{V}U_{BY_1} + V_B\tilde{U}_{Y_1}) &= \\ \theta_B[M_B\tilde{U}_{Y_1} + \tilde{M}U_{BY_1}]_{Y_1} + \tilde{\theta}[M_BU_{BY_1}]_{Y_1}, \\ 0 &= -\tilde{P}_{Y_1}, \\ (-i\Omega\tilde{W} + U_B\tilde{W}_x + V_B\tilde{W}_{Y_1}) &= -i\beta\theta_B\tilde{P} + \theta_B[M_B\tilde{W}_{Y_1}]_{Y_1}, \\ (-i\Omega\tilde{\theta} + U_B\tilde{\theta}_x + \tilde{U}\theta_{Bx} + \tilde{V}\theta_{BY_1} + V_B\tilde{\theta}_{Y_1}) &= \\ \frac{1}{Pr}\theta_B[M_B\tilde{\theta}_{Y_1} + \tilde{M}\theta_{BY_1}]_{Y_1} + \frac{1}{Pr}\tilde{\theta}[M_B\theta_{BY_1}]_{Y_1} + \\ (\gamma - 1)\theta_B[2M_BU_{BY_1}\tilde{U}_{Y_1} + \tilde{M}U_{BY_1}^2] &+ (\gamma - 1)\tilde{\theta}M_BU_{BY_1}^2, \\ \tilde{R} = -\frac{1}{\theta_B^2}\tilde{\theta} \quad \text{and} \quad \tilde{M} = \frac{(1+S)}{(2\theta_B)^{\frac{1}{2}}}\tilde{\theta}. \end{aligned}$$

(4.3.4a - g)

The boundary conditions at the wall require

$$\tilde{U}(x, 0) = \tilde{V}(x, 0) = \tilde{W}(x, 0) = 0, \text{ and } \tilde{\theta}_{Y_1}(x, 0) = 0 \text{ or } \tilde{\theta}(x, 0) = 0,$$

for adiabatic or cooled wall, respectively. The boundary conditions as $Y_1 \rightarrow \infty$ are

$$\tilde{U} \rightarrow 0, \tilde{W} \rightarrow 0, \tilde{\theta} \rightarrow 0 \text{ and } \tilde{V} \rightarrow \tilde{V}_\infty(x).$$

The displacement perturbation, $\tilde{V}_\infty(x)$ is related to the boundary-layer pressure perturbation, $\tilde{P}$ by a pressure-displacement law stemming from the upper-tier (upper-deck):

$$\frac{\partial \hat{q}}{\partial \hat{y}} \rightarrow - \left(\frac{\partial}{\partial x} - i\Omega \right) \tilde{V}_\infty(x), \text{ as } \hat{y} \rightarrow 0^+, \quad \hat{p} = \hat{q}E + c.c.; \quad (4.3.5)$$

where the upper-deck normal variable, $\hat{y} \sim O(1)$, and its pressure perturbation, $\hat{p}$, satisfies

$$y = Re^{-\frac{1}{9}}\hat{y}, \quad \text{and} \quad p = p_\infty + Re^{-\frac{4}{9}}\hat{p}. \quad (4.3.6a, b)$$

We now show how the resulting, simplified upper-deck system, for $\hat{q}$, can be solved, using Fourier-transform methods, and obtain a ‘closed’ expression relating the (disturbance) displacement to the pressure disturbance. (i.e. one does not have to solve the upper-deck problem numerically). Remember that we have assumed harmonic dependence in t and Z_1 , in particular we have written

$$(P, \hat{p}, V_\infty) = (\tilde{P}, \hat{q}, \tilde{V}_\infty)E + c.c., \quad E = \exp[i(\beta Z_1 - \Omega t)]. \quad (4.3.7)$$

This has reduced the upper-deck pressure-equation to

$$\begin{cases} [\partial_{zz}^2 - \partial_{Y_1}^2 - 2i\Omega\partial_z - \Omega^2 + \beta^2] \hat{q} = 0, \\ \hat{q} = \tilde{P}, \text{ as } \hat{y} \rightarrow 0^+, \quad \hat{q} \rightarrow 0, \text{ as } \hat{y} \rightarrow \infty. \end{cases} \quad (4.3.8)$$

We can solve this system by taking Fourier transforms in x , defined by

$$\bar{f}(k) = \mathfrak{F}(f(x)) = \int_{-\infty}^{\infty} e^{-ikx} f(x) dx, \quad (4.3.9)$$

which further reduces the pressure equation to

$$[-\partial_{Y_1}^2 + \beta^2 - (k - \Omega)^2] \bar{\hat{q}} = 0, \quad \bar{\hat{q}}(k, 0) = \bar{\tilde{P}}, \quad \bar{\hat{q}}(k, \infty) = 0. \quad (4.3.10a - c)$$

We require solutions that decay as $\hat{y} \rightarrow \infty$, leading to the restriction

$$\beta^2 > (k - \Omega)^2; \quad (4.3.11)$$

(cf the wave obliqueness-angle restrictions of Zhuk & Ryzhov, 1981; Ryzhov, 1984; Smith, 1989 and Duck, 1990). Thus the solution is

$$\bar{q} = \bar{P} \exp[-\{\beta^2 - (k - \Omega)^2\} \hat{y}]. \quad (4.3.12)$$

Meanwhile the pressure-displacement condition has been transformed to

$$\bar{q}|_{\hat{y}=0} = \overline{(i\Omega - \partial_x) \tilde{V}_\infty},$$

and substituting for $\bar{q}$ yields

$$\bar{P} = \frac{\overline{(i\Omega - \partial_x) \tilde{V}_\infty}}{\{\beta^2 - (k - \Omega)^2\}^{\frac{1}{2}}}, \quad (4.3.13)$$

the pressure-displacement law in transform-space. To invert this we appeal to a modified form of the well known formula ‘Poisson’s Integral’, namely,

$$J_0(\beta X) = \frac{2}{\pi} \int_0^\beta \frac{\cos kX \, dk}{[\beta^2 - k^2]^{\frac{1}{2}}},$$

where $J_0(X)$ is the Bessel Function of order zero, to deduce that $J_0(\beta X)$ is the *inverse Fourier cosine transform* of the function

$$g(k) = \begin{cases} \frac{1}{[\beta^2 - k^2]^{\frac{1}{2}}}, & \text{if } \beta^2 > k^2, \\ 0, & \text{otherwise.} \end{cases} \quad (4.3.14)$$

Here the Fourier cosine transform is defined by

$$\mathcal{C}(f(x)) = \int_0^\infty \cos kx \, f(x) \, dx.$$

It is very easy to show that if $f(x)$ is an even function then

$$\mathfrak{F}(f) = 2\mathcal{C}(f)$$

i.e. the Fourier transform of an even function is twice its Fourier cosine transform.

These last two results imply that

$$\mathfrak{F}^{-1}(g) = \frac{J_0(\beta x)}{2},$$

where the function g is defined above. However we require $\mathfrak{F}^{-1}\{g(k - \Omega)\}$ and this can be easily be evaluated by employing the so-called ‘shift theorem’ for Fourier transforms, yielding

$$\mathfrak{F}^{-1}\{g(k - \Omega)\} = \exp[i\Omega x] \frac{J_0(\beta x)}{2}. \quad (4.3.15)$$

Finally, applying the *convolution theorem* to the transformed pressure-displacement law (with suitable extensions for negative arguments and noting that we require $\xi < x$ from the definition of g) yields

$$\tilde{P} = \int_0^x [(\partial_\xi - i\Omega)V_\infty] e^{i\Omega(x-\xi)} J_0(\beta(x-\xi)) d\xi, \quad (4.3.16)$$

as the final result.

This odd-looking expression, the relation between pressure and displacement, can be partially checked by considering special cases where the form is known already. Setting $\Omega = 0$ yields the correct expression for the steady, 3-D supersonic pressure-displacement law.

The steady 2-D case of Stewartson & Williams (1969) essentially corresponds to putting $\Omega = 0 = \beta$ and $\tilde{V}_\infty = -A_x$. Thus

$$\tilde{P} = \int_0^x \partial_\xi(-A_\xi) \cdot 1 \cdot 1 \cdot d\xi = -\frac{dA}{dx} \quad (4.3.17)$$

which is merely the familiar Ackeret’s Law for 2-D Supersonic case, as required.

Returning to the general case, we see that

$$P = e^{i[\Omega(x-t) - \beta Z_1]} \int_0^x [(\partial_\xi - i\Omega)V_\infty] e^{-i(\Omega-\xi)} J_0(\beta(x-\xi)) d\xi + \text{c.c.}, \quad (4.3.18)$$

which resembles a quasi-parallel form (amplitude is a function of x) for the linear disturbance where the wave travels with phase-speed ~ 1 . Note that here we have implicitly redefined x by a translation, so that the origin corresponds not to the leading edge, but to the position of initial disturbance.

§4.3.3 Comments on the numerical solution of the linearised problem.

Returning to the discussion of the numerical solution of the linearised problem, we see that we are left with a system of partial differential equations, in x and Y_1 , to be solved subject to the usual boundary conditions at, and far from, the wall. In addition, however, the pressure-perturbation in the boundary layer is driven by the displacement effect of the normal velocity-perturbation at the top of the boundary layer, which in turn depends on the pressure-perturbation itself (i.e. there is viscous-inviscid interaction).

Note that there are similarities with the linearised Görtler-vortex equations (see, for example, Hall, 1983 -incompressible version; Spall & Malik, 1989 and Wadey, 1990 -compressible version; Hall & Fu, 1989 and Fu, Hall & Blackaby, 1990 -large Mach number -see also later chapter; Denier, Hall & Seddougui, 1990 and Morris, 1992 -receptivity aspects for incompressible and compressible flows, respectively; Otto (1991)-Taylor vortices in a time-dependent flow, leading to p.d.e.s in y and time). These equations are (usually) formulated for steady vortices (that is, $\Omega = 0$ in our notation) and appear more complicated than those just derived above for viscous-stability properties, that are governed by the two-tier structure. The linearised Görtler-vortex equations do have bulk-viscosity, μ' , contributions as well as a curvature effect term in the y -momentum equation (which must also be solved; the boundary-layer's pressure disturbance is not constant of the normal boundary-layer variable) and spanwise viscous effects. However, they do not have a pressure-displacement law. We claim that the solution properties of these parabolic partial differential equations will provide insight into the, as yet undetermined, solution properties of our linear disturbance equations.

The numerical studies, presented in the above papers, has shown that one does not obtain a unique neutral curve (the normal mode approach has been assumed by several researchers, notably by Floryan & Saric, 1979; they calculated a unique neutral curve that has since been discredited); instead it has been conclusively shown that solution properties depend crucially on initial conditions and solving the right equations (i.e. partial differential equations, initialised by a consistent profile), unless the wavenumber is large; in which case the solutions are easily obtainable analytically using a simple, asymptotic theory.

Thus, the normal mode approach which works reasonably well† for the Tollmien–Schlichting modes of incompressible, subsonic and moderately-supersonic boundary layer flows (see, amongst others, Shen, 1954; Jordinson, 1970; Gaster, 1974; Mack, 1984) is invalid for Görtler vortices, although both are (strictly) governed by partial differential equations: the resolution of this apparent paradox is due to the short-scaled x -dependence of the Tollmien Schlichting modes (shown, using triple-deck theory, by Smith, 1979) rendering validity in the normal-mode approach. However, we have seen that as the Mach number increases to $O(Re^{\frac{1}{6}})$ the viscous modes are no longer short-scaled; the x -scale has risen to $O(1)$ size which implies (see Smith, 1989) that the non-parallelism of the base-flow cannot be ignored, suggesting that some of the supersonic, viscous stability results based on the normal-mode (Orr-Sommerfeld-type) approach (see, for example, including references therein, Mack 1975, 1984, 1986; Malik 1982, 1987) may need checking with results obtained that include the fact that the boundary-layer is not parallel, but growing.

Hence, for the partial-differential equations of concern, governing the viscous linear stability properties, we would expect, in general, not to obtain a unique neutral curve, as the streamwise length-scale of the disturbances is that of the growth of the basic boundary layer flow. Instead, we expect solution properties to be dependent on the imposed initial conditions. In some limiting parameter-space (probably for high-frequency or high-spanwise-wavenumber) we might expect some analytic progress to be possible, giving a unique asymptote to all of the (non-unique) neutral curves.

Let us now be a little more specific by what we mean by neutrality. We choose to study spatial stability properties‡ and so fix β and Ω to be real constants, independent of x (see Hall, 1983). However, once a neutral location has been determined these are rescaled on the x -value of this location giving the neutral

† In recognising that the normal-mode (Orr-Sommerfeld-type) approach leads to reasonable results for predicting the linear viscous-stability of non-highly supersonic boundary-layer flows, we are not accepting that the approach is entirely rational or better than the triple-deck approach, especially at large Reynolds numbers. The triple deck structure also provides a rational base for non-linear theories.

‡ We take the view that the primary instability will be spatial in nature; the secondary instabilities, of the non-linear stages of disturbance growth, are most likely temporal in nature.

values (β_x, Ω_x) , say, representing a point on the neutral curve of the particular chosen initial disturbance profile.

There is some difficulty in deciding what criterion to employ to decide when the numerical solution at a downstream x -position represents a neutral state: note that in normal mode analysis spatial neutrality is simply where the corresponding wavenumber is real. The two main such criteria are (i) when an energy functional of the disturbance is stationary, and (ii) when a measure of disturbance wall-shear is stationary. Different neutrality-criteria lead to different neutral positions (see discussion in Smith, 1979; Fu, Hall & Blackaby, 1990) but we hope that graphically they will be very similar- this is usually the case in incompressible studies but in his compressible study, Wadey (1990) noted that the inclusion of a thermodynamic term into their energy functional ‘causes problems’ and uses the ‘incompressible’ energy functional chosen by Hall (1983).

Spall & Malik (1989) also, independently of Wadey (1990), used an incompressible measure for their compressible study. However, their functional, in contrast to that of the previous author and Hall (1983), includes the Reynolds-number scalings (weighting factors) so that the u -disturbance, being larger, is most significant in the energy measure. They do not say what motivated their (eventual) choice of energy-functional, or investigate the effects of different energy-functionals. Additionally, a further remark on this paper (see chapter 3 of Wadey, 1990) that is relevant to the present viscous-stability study, is that despite solving p.d.e.s, the validity of their results is questionable due to their choices of initial disturbance that do not satisfy the disturbance equations.

The present viscous-stability problem of concern is for large Mach number (hypersonic) flow. The centrifugal (Görtler) instability counterpart are the recent studies by Hall & Fu (1989) and Fu, Hall & Blackaby (1990). These studies, essentially, still employ ‘incompressible’ energy functionals. The present author has some reservations about this approach (see next paragraph) on theoretical grounds, even if it makes no ‘graphical’ difference to the neutral curves. It is a generally accepted, well known fact that different initial disturbances lead to different (i.e. non-unique) neutral curves. However, there appears to be less recognition that there is ‘secondary-uniqueness’, say, present in the problem in that even if the initial disturbance is fixed, different neutrality measures will lead

to different (non-unique) neutral curves. This secondary non-uniqueness problem (which may or may not be significant) will not be cured by receptivity arguments (these essentially determine the relevant initial disturbance); instead there is a need for further thought in deciding what is the best (if it is a tenable idea) neutrality criterion to be used in these ‘marching’ linear-stability calculations.

Note that in hypersonic flow, the temperature disturbance, in the boundary layer, is $O(M_\infty^2)$; much larger than the other, kinematic, disturbances. This suggests that it should not be ignored in determining neutrality. The large (scaling) size of the temperature perturbation, relative to the smaller u -perturbation and even smaller v , w -perturbations, surely suggests that it is most important that the disturbance is neutral ‘with respect to temperature disturbance’; this will minimise the total disturbance energy. Another open question concerns why the total energy of the flow (basic and disturbance) is not considered. Note also that the effect of boundary growth means that the energy of the basic flow is not constant itself, because of viscous effects.

In conclusion of this discussion, the author believes that, for hypersonic flow, the neutrality criterion should be based on a measure (at a sole y -location or an integration of, across the boundary layer) the temperature disturbance. It does not appear rational to be quoting, and applying ‘incompressible’ energy functionals to highly-compressible problems. However, in fairness, the author has not carried out any numerical solutions of this kind (see reasons below) and has thus not experienced the difficulties that may arise from theories that appear reasonable ‘on paper’.

We now close this sub-section with some further discussion and closing remarks. Note that the partial differential equations coupled with the pressure-displacement law has been assumed to be parabolic; this has not been formally verified. The standard numerical solution would involve ‘marching’ the equations, in x , from an initial position with prescribed disturbance which must be consistent with the disturbance equations, until a neutral x -position(s) have been identified from which a point on a neutral curve follows. At each x -station the disturbance profiles are computed. The stepping-forward in x would be using a finite-difference

type method, but the y -solution at each x -station could be calculated by a finite-difference method or a spectral method. The latter method employing, in particular, Chebychev polynomials appears to be more appropriate to the new generation of computers with their parallel processors. Additionally, the author believes that the latter method is crucial in efficiently obtaining accurate solutions for the base flow equations (especially for the special case of the similarity solution)- see Chapter 6 for further discussion. One last remark on possible difficulties concerns the discretisation of closed pressure-displacement law: it would have to be checked for the usual conditions of stability, compatibility and consistency.

As mentioned previously, the numerical solution of the linearised equations derived earlier, governing viscous-stability when $M_\infty \sim O(Re^{\frac{1}{5}})$, has not been attempted. This is due to ^anumber of reasons, in particular: (i) the system to be solved appears complicated and there may be further difficulties in addition to those outlined above, (ii) much time was spent investigating if any simple analytical progress could be made (see next section for a report on such studies), (iii) the (eventual) lack of enthusiasm by everyone directly involved, (iv) the poor computing resources and facilities available, and (v) the apparent lack of expert help and advice available on the numerical methods that the author wished to use.

§4.3.4. Further discussion concerning the two-tiered structure, including curvature effects.

Here we briefly summarize the features distinctive to the new two-tiered structure; noting the main differences compared to the compressible triple-deck structure. The most obvious difference is that non-parallel effects must be incorporated into any theories, as noted by Smith (1989).

However there are two further differences that result in analytical and numerical solutions being harder to compute. The first is the fact that, as the lower-tier covers the whole classical boundary-layer, the general basic-flow velocity profile must be used — not just the wall shear. Secondly, we see that the equations (4.2.9) are ‘truly’ compressible, in contrast to the lower-deck equations of compressible triple-deck theory (2.3.9).

The fact that the timescale has risen to become $O(1)$ is not significant if the base-flow is steady (which we have assumed to be the case throughout this thesis)

but if the base flow does have some time-dependence then special care must be taken. On a similar theme, now that the lengthscale of the asymptotic structure is $O(1)$, much longer than the spanwise scale, the effects of ‘cross-flow’ (spanwise variation of the basic flow) will be much more significant, in the sense that now a much smaller cross-flow will be felt by the governing structure.

Last, but not least, we come to curvature effects. These are now far more significant for essentially the same reason that cross-flow effects are. The ‘long’ $O(1)$ x -scale compared to the small height, $O(Re^{-\frac{1}{9}})$ of the upper-tier (upper-deck) means that less severe (i.e. more significantly reasonable) curvature will affect the (planar) governing equations. The author unfortunately did not have time to investigate this aspect more deeply.

Another effect of (the appropriate) curvature will be to allow the secondary Taylor-Görtler instability of the (primary) nonlinear flow, a generalisation of the study described in Hall & Bennett (1986). Whilst on the subject of secondary instabilities, we would expect the theories of Smith & Bodonyi (1985) and Tutty & Cowley (1986), on Rayleigh-type (inviscid; no curvature necessary) secondary temporal instabilities of the (primary) nonlinear flow, to generalise to the present problem. Note that most of the comments made here for the $M_\infty \sim Re^{\frac{1}{9}}$ regime will still be appropriate for larger Mach numbers because the streamwise length scale must remain $O(1)$ (see §4.5).

§4.4. SOME COMMENTS CONCERNING THE SEARCH FOR ANALYTICAL, ASYMPTOTIC SOLUTIONS: THE ‘HIGH FREQUENCY’ ASSUMPTION.

§4.4.1 Introduction.

The search for analytical, asymptotic theories and solutions to the governing equations of the two-tiered structure proved to be long, difficult and not particularly fruitful. High-frequency asymptotic theories were considered because of their success in triple-deck based theories. Such an approach can also be justified on physical grounds.

The similarities of the present linearised system with that for Görtler vortices has already been noted. This suggests that we should try to find large spanwise

wavenumber ($\beta \gg 1$) theories which prove so successful for the latter. However, these theories do not work here as we have no ‘curvature’ term. Note that the assumption that the frequency is large results in the need to rescale the lengthscales with the large parameter Ω , anyhow.

We have noted earlier in this chapter that we have no linear-eigenrelation; this makes it very tricky to find any weakly-nonlinear theories. The linear and non-linear equations are considered (see also Chapter 5): progress that has been made is described together with discussion of the problems encountered that prevented further progress. It would be nice to have ‘analytic’ asymptotes to compare/check the numerical results with (when the latter have been computed). On the otherhand, perhaps a knowledge of the numerical-solution properties would help out finding analytical theories.

Note the ‘almost-full’ appearance of the governing equations; we appear to have recovered the ‘classical’ boundary-layer equations along with a few modifications. Thus one could easily be excused for assuming that we have essentially ‘gone round in a full circle’ and that the high-frequency theories that ‘work’ for the classical boundary-layer work here as well i.e. that the high-frequency lower- and upper-branch theories can be applied, the actual scales deduced from physical arguments that follow along the lines of those presented in §4.1.2 and §4.1.3. However, with a ^{little} thought, one soon realises that these are not possible — this follows immediately from the results of those subsections i.e. that the frequency necessarily rises to $O(1)$ in such theories, due to large Mach number effects. In other words, once the Mach number is fixed to be $O(Re^{\frac{1}{9}})$, the triple-deck and upper-branch structures are not tenable — they are the two-tiered structure being considered!

In the next subsection we consider the linearised problem. The expansions, scales and mechanism is outlined for two cases of travelling waves, both having $O(1)$ wavespeed (the disturbances travel at roughly the same speed as the underlying base-flow). Note that the classical lower- and upper-branch modes of incompressible, subsonic and moderately supersonic boundary layer flows all have asymptotically small wavespeeds. In Chapter 5 we consider a nonlinear theory based on a fast timescale.

§4.4.2 An outline of two probable high-frequency theories.

In this subsection we deduce two sets of scales for the coupled boundary-layer/upper-tier equations governing the viscous stability properties when $M_\infty \sim Re^{\frac{1}{6}}$. The mechanisms follow closely that for the ‘classical’ upper-branch modes (see §4.1.3). The author did not have time to follow the analyses through, but feels that when the theories outlined below are properly investigated, they should prove to be a useful first step and perhaps even provide a high-frequency asymptote to the (yet to be computed) linear neutral curve. If this is in fact the case, then it should be possible to base further asymptotic theories (e.g. the inclusion of nonlinear effects) on such a description.

Our starting point is the linearised system (4.3.4),(4.3.5),(4.3.8) derived and discussed in the previous section; again, without loss of generality, we take the scaled Mach number $m = 1$. Recall that viscosity acts right across the boundary-layer for the disturbances. However, as soon as one assumes the frequency of the disturbances to be high, the leading-order disturbance equations of the boundary layer will be inviscid in nature (and closely related to the compressible Euler equations) apart from in (at most) two thin layers where viscous effects are important. The first is the so-called Stokes layer, alongside the wall, where the time-derivative balances with the viscous operator. The second is a so-called critical layer and exists if, and where, the wavespeed of the disturbance equals the local speed of the (nondimensionalised) underlying boundary layer. In this section we assume that a critical layer does indeed exist.

Before looking for a high-frequency description, there are three points worth noting. Firstly, stemming from the Stokes layer, there will be a displacement effect in the normal velocity disturbance $\tilde{V}$, which has an imaginary part relative to the leading-order term. Secondly, there will be an imaginary ‘phase-jump’ in $\tilde{V}$ across the (linear) critical-layer. Lastly, we want the pressure disturbance, $\tilde{P}$, in the boundary to be driven by the displacement, $\tilde{V}_\infty$, via the upper-tier equations (4.3.5) and (4.3.8). The ‘classical’ mechanism for upper-branch modes is a high-frequency theory which incorporates the last point whilst, additionally, balancing the two ‘imaginary’ effects (mentioned in the first two points) thus enabling neutral modes to exist.

As our governing equations are very similar to those supporting the (compressible, high-frequency) modes, and because the underlying mathematical structure governing these modes collapses into the two-tiered structure as $M_\infty \nearrow Re^{\frac{1}{9}}$, we look for a similar high-frequency mechanism. Now that we have assumed the frequency to be large, we can (at last) assume harmonic dependence in x i.e.

$$\frac{\partial}{\partial x} \rightarrow i\alpha = i [\Omega^{k_{x0}} \alpha_0 + \Omega^{k_{x1}} \alpha_1 + \dots], \quad (4.4.1)$$

say, where $k_{x0} > k_{x1} > \dots$ are unknown and $\alpha_0 > 1$ so that there is a critical layer. Note that this assumption (of a shortening of the x -scale to $O(\Omega^{-k_{x0}}) \ll 1$) relies on $k_{x0} > 0$ which can be established *a posteriori*. Also we write

$$\beta = \Omega^{k_{x0}} \beta_0 + \Omega^{k_{x1}} \beta_1 + \dots, \quad k_{x0} > k_{x1} > \dots. \quad (4.4.2)$$

Note that the important operator

$$\frac{\partial}{\partial t} + U_B \frac{\partial}{\partial x} = -i\Omega + U_B \frac{\partial}{\partial x} \rightarrow -i\Omega + iU_B [\Omega^{k_{x0}} \alpha_0 + \Omega^{k_{x1}} \alpha_1 + \dots].$$

In fact we choose $k_{x0} = 1$ to retain the balance of both terms at leading order.

Thus

$$\frac{\partial}{\partial t} + U_B \frac{\partial}{\partial x} = i\Omega(\alpha_0 U_B - 1) + iU_B \Omega^{k_{x1}} \alpha_1 + \dots \quad (4.4.3)$$

and so the critical layer resides where $U_B(Y_1) = \frac{1}{\alpha_0} < 1$, at $Y_1 = Y_{1c}$, say.

Let us now consider the thin Stokes layer of height $O(\Omega^{-\frac{1}{2}})$, alongside the wall. Again, as there is no pressure-gradient term in the x -momentum equation, we choose to consider the ‘balances’ of the z -momentum equation. As the equations are linearised, we can consider the sizes of other quantities relative to a typical pressure disturbance $\tilde{P}$. Balancing the time-derivative with the pressure-gradient gives $\tilde{W} \sim \Omega^{k_{x0}-1} \tilde{P}$. Meanwhile, balancing the v_y and w_{Z_1} terms in the equation of continuity gives $\tilde{V} \sim \Omega^{2k_{x0}-\frac{3}{2}}$. From standard Stokes layer theory, we can deduce that as the inviscid region, above the Stokes layer, is approached

$$\tilde{V} \sim O(\Omega^{2k_{x0}-1}) + O(\Omega^{2k_{x0}-\frac{3}{2}});$$

where the second term contains the ‘imaginary’ Stokes-layer displacement effect which we hope will balance with the ‘phase-jump’ across the critical layer, to allow neutral modes.

We now consider the upper-deck. Here we meet another complication, in that the upper-deck height also scales with the frequency. The leading order size of $\tilde{V}$ will be transmitted to the upper-deck where balancing $(\partial_x + \partial_t)v$ and p_y yields $\frac{\partial}{\partial \hat{y}} \sim \Omega^{2k_{z0}-1}$. Balancing the operators $\partial_{\hat{y}}^2$ and $\partial_{Z_1}^2$ in the Prandtl Glauert equation, to ensure a decaying solution, requires $\Omega^{2k_{z0}-1} \sim \Omega^{k_{z0}}$ i.e. $k_{z0} = 1$. This is the furthest that the author reached in the formulation — it is not clear if continuing with this particular argument will lead to neutral modes.

However, if $\alpha_0 = 1$, then the wavespeed

$$\frac{\Omega}{\alpha} = 1 - \Omega^{1-k_{z1}}\alpha_1 + \dots,$$

is asymptotically close to 1 and new sizes for the unknown quantities emerge. Now

$$(\partial_t + U_B \partial_x) = -i\Omega(U_B - 1) + iU_B \alpha_1 \Omega^{k_{z1}} + \dots,$$

and so as the freestream is approached (i.e. as the upper-deck is entered) this operator becomes smaller:

$$(\partial_t + U_B \partial_x) \sim O(\Omega^{k_{z1}}) \text{ as } Y_1 \rightarrow \infty. \quad (4.4.4)$$

For the present case, the Stokes-layer analysis is the same as earlier, but the $\tilde{V}$ -solution in the inviscid regions are proportional to $(1 - \frac{\Omega}{\alpha} U_B) \tilde{P}$, meaning that as the upper-deck is approached

$$\tilde{V} \sim \Omega^{2k_{z0}-1} \cdot (1 - \frac{\Omega}{\alpha} U_B) \tilde{P} \sim \Omega^{2k_{z0}+k_{z1}-2} \tilde{P}.$$

Thus

$$(\partial_x + \partial_t) \tilde{V}_\infty \sim \Omega^{2k_{z0}+k_{z1}-2} \cdot \Omega^{k_{z1}} \tilde{P} \quad (4.4.5)$$

and the pressure-displacement law then requires that

$$\frac{\partial}{\partial \hat{y}} \sim \Omega^{2k_{z0}+2k_{z1}-2}. \quad (4.4.6)$$

Note that we still have two unknowns $2k_{z0}$ and k_{z1} , these are determined by wishing to retain all terms in the Prandtl Glauert equation of the upper deck. Balancing $(\partial_t + \partial_x) \sim \partial_{Z_1}$ yields $k_{z1} = k_{z0}$, whilst balancing $\partial_{Z_1} \sim \partial_{\hat{y}}$ yields $k_{z0} + 2k_{z1} - 2 = 0$. Solving the last two equations gives

$$k_{z1} = \frac{2}{3} = k_{z1}. \quad (4.4.7)$$

Again, no further analytical progress has been made with these scales. However, there are a couple of points that are worth noting. The above argument has resulted in the scales

$$\frac{\partial}{\partial x} \sim \Omega \sim \frac{\partial}{\partial t} \quad \text{and} \quad \frac{\partial}{\partial z} = Re^{\frac{1}{6}} \frac{\partial}{\partial Z_1} \sim Re^{\frac{1}{6}} \Omega^{\frac{2}{3}} \sim M_{\infty} \Omega^{\frac{2}{3}}, \quad (4.4.8a - c)$$

after appealing to (4.2.3) with $m = 1$, whilst the disturbance wavespeed, c^D say, has the form

$$c^D = 1 - O(\Omega^{-\frac{1}{3}}) + O(\Omega^{-\frac{2}{3}}) + \dots. \quad (4.4.8d)$$

Suppose now that the frequency rises so to such a high value that it is proportional to powers of the Mach number. Note that the scales (4.2.3a-c) above are all of the same order when

$$\Omega \sim M_{\infty} \Omega^{\frac{2}{3}} \iff \Omega \sim M_{\infty}^3, \quad (4.4.9a)$$

in which case

$$\frac{\partial}{\partial t} \sim \frac{\partial}{\partial x} \sim \frac{\partial}{\partial z} \sim M_{\infty}^3 \sim Re^{\frac{1}{3}}. \quad (4.4.9b - d)$$

Note that the thickness of the boundary layer (the lower-tier) is $O(Re^{-\frac{1}{3}})$, from (4.2.4); so that

$$\frac{\partial}{\partial y} \sim Re^{\frac{1}{3}} \quad (4.4.9e)$$

in the boundary layer. It follows immediately that, in this higher-frequency limit ($\Omega \nearrow O(Re^{\frac{1}{3}})$) the modes take on classical Rayleigh-type character. That is, these ‘viscous’ modes, as their frequency increases, appear to match onto inviscid modes having wavespeeds of the form

$$c^D = 1 - O(M_{\infty}^{-1}) + \dots. \quad (4.2.9f)$$

However, much more care must be taken since, as $\Omega \nearrow O(M_{\infty}^3)$, the critical layer leaves (the upper part of) the boundary layer — it relocates itself in the adjustment layer lying between the lower- and upper-tiers. This layer is essentially passive in the viscous-stability mechanisms with which we have concerned ourselves so far in this thesis and thus has not received much attention yet. However, the adjustment layer is crucial to an understanding of the inviscid (Rayleigh) and centrifugal (Görtler) stabilities for large Mach numbers (discussed in Chapters 6 and 7, respectively). It is discussed in some detail in Chapter 6.

The higher frequency limiting forms (4.4.9), found above, appear to correspond to the so-called ‘acoustic mode’ scales described in Chapter 6, where it is found that neutral (acoustic) inviscid modes exist having essentially the same scales as (4.4.4). Alternatively, it could be argued that (4.4.9) corresponds to (asymptotically) small- wavelength (inviscid) ‘vorticity-modes’, also described in Chapter 6 — these modes are slightly unstable. However the (neutral) acoustic modes have wavespeed $O(M_\infty^{-2})$ relative to the (nondimensionalised) freestream-speed, 1, and the corresponding relative wavespeed for the (slightly unstable) very-small-wavenumber vorticity modes is $O(M_\infty^{-\frac{8}{7}})$, whereas the limiting relative wavespeed of the (higher frequency) viscous modes is $O(M_\infty^{-1})$, from (4.2.9f). Thus further work is called for to investigate the link of very-high frequency viscous modes with the inviscid modes.

§4.5. LARGE SCALED MACH NUMBER

§4.5.1 Introduction.

When $m \sim O(1) \leftrightarrow M_\infty \sim Re^{\frac{1}{9}}$, the lower-deck has expanded in thickness and coalesced with the main-deck to leave just one viscous boundary layer of classical height. The disturbance length scale has risen to $O(1)$ and so has the time scale. The cross-stream (z) lengthscale and the thickness of the upper-deck have increased to order $Re^{-\frac{1}{9}}$. The disturbance equations in the upper-deck are still linear but are now unsteady. Lastly, the disturbance velocities have grown to become comparable with those of the underlying base flow.

Earlier in this chapter, the expansions, the scales for the range $Re^{-\frac{1}{9}} \ll m \leq O(1)$ were deduced. Note that (trivially) m occurs in these and so one might decide to allow m to grow much larger than $O(1)$, in these scalings, in order to deduce the resulting structure. Firstly, that would be (intuitively) quite irrational because the m -scalings were based on arguments associated with the notion of a triple-deck structure, but, as $m \nearrow O(1)$, this three-tier structure disappears. Moreover, continuing with the approach of increasing the Mach number ($m \gg O(1)$) in the $m < O(1)$ -expansions and scalings results in absurd predictions, for instance: (i) the lower deck continues to grow faster than the main-deck so that the lower-deck would be far thicker than the classical viscous boundary layer! (ii) the velocity sizes

would continue to grow, becoming much larger than the corresponding velocity components of the base !

Thus for $m \gg 1$, the near-plate flow structure must be deduced from ‘first principles’. The form for $m \searrow 1$ is known, but in the following arguments we try not to ‘appeal to hindsight’ - recall that as $m \nearrow Re^{\frac{4}{45}}$, $M_\infty \rightarrow Re^{\frac{1}{5}}$ corresponding to the classical hypersonic-viscous range that has been studied several times, notably by Bush (1966) and Blackaby, Cowley & Hall (1990) - see later chapter. These studies are for steady, two – dimensional flows; to the author’s knowledge, there has been no study of the three-dimensional case in the literature to date.

In this section we aim to deduce the $m \gg 1$ structure solely from the $m \sim O(1)$ (two-tier) structure which, in turn was solely deduced from the compressible triple-deck for smaller (supersonic) Mach numbers. Once the $m \gg 1$ problem has been deduced we then set $m \sim Re^{\frac{4}{45}}$ and compare our predictions with the ‘known properties’, of the previous $M_\infty \sim Re^{\frac{1}{5}}$ studies. In addition we shall note the ‘three-dimensional’ predictions, in this regime.

§4.5.2 Deducing the scales.

First, we suppose that there will be two layers- the boundary layer, above which lies a thin inviscid region - let us call this the upper deck, say. Note that, as $m \nearrow 1$ the ‘need’ for a viscous sub-layer disappears (in general) and thus it is reasonable to assume that, for $m \gg 1$, viscosity continues to act across the whole of the boundary layer, for the disturbances as well as the base flow.

Consider first the boundary layer. When $m \sim O(1)$ the streamwise and normal velocity perturbations have grown to become of the same order as the corresponding base flow quantities and thus, for $m \gg 1$, we take the u and v perturbations to be of the same order as the corresponding base flow quantities i.e.

$$u \sim O(1) \quad \text{and} \quad v \sim Re^{-\frac{1}{3}} m^{\frac{3}{2}} \sim y_B, \quad (4.5.1a - c)$$

where y_B is the classical boundary layer height for large M_∞ compressible flow over a flat plate. Similarly we take the perturbation streamwise (x -) lengthscale to be $O(1)$, corresponding to the non-dimensionalised, plate lengthscale; note that non-parallel effects must be considered in any solution based on the present structure.

We take the base flow's vertical variable, y_B , to be the relevant vertical variable in the boundary layer. Finally, we also take the thermodynamic quantities to be of the same size as the corresponding base ones i.e. in the latter has $\rho \sim M_\infty^{-2} \sim Re^{-\frac{2}{9}} m^{-2}$, for $M_\infty \gg 1$, and this the density-scaling we choose. Similar results for the temperature and the viscosity easily follow.

We are free to choose the timescale, but note that time derivatives will occur in both the boundary-layer and the upper-deck equations, or in neither. Recall that as $m \nearrow 1$ the corresponding timescale for the perturbations became $O(1)$. We wish to include time variation, in our system, and thus choose the time scale to be $O(1)$. Note that the base flow is (conventionally) steady so we have no precedent to follow for the boundary-layer timescale, although $O(1)$ is the logical choice.

Consider now the upper-deck. The vertical height is chosen such that its dimensions are the 'required fit' for a Mach line (see figure 1 of Rizzetta *et al*, 1978). Recall that the gradient of a characteristic of the classical, inviscid high- M_∞ Prandtl Glauert equation, in unscaled, non-dimensionalised variables x and y , is M_∞^{-1} . Therefore the upper-deck y -scale, y_U say, must be $O(M_\infty^{-1})$ to incorporate the inviscid effects i.e.

$$y_U \sim M_\infty^{-1} = m^{-1} Re^{-\frac{1}{9}}. \quad (4.5.2)$$

We assume that the normal velocity perturbation at the top of the boundary layer induces a normal velocity perturbation of the same size in the upper-deck i.e.

$$v \sim Re^{-\frac{1}{3}} m^{\frac{3}{2}}$$

cf. the classical displacement effect at the top of a boundary layer.

In the upper-deck,

$$u = 1 + \delta_1 \hat{u} + \dots, \quad v = Re^{-\frac{1}{3}} m^{\frac{3}{2}} \hat{v} + \dots, \quad w = \delta_2 \hat{w} + \dots$$

$$p = p_\infty + \delta_3 \hat{p} + \dots, \quad \rho = 1 + \delta_4 \hat{\rho} + \dots, \quad \theta = 1 + \delta_4 \hat{\theta}, \quad (4.5.3a - f)$$

where $\delta_1, \delta_2, \delta_3$ and δ_4 remain to be determined. Balancing $\rho_t + u\rho_x$ with ρv_y in the continuity equation requires

$$\delta_4 \sim \frac{v}{y} \sim \frac{Re^{-\frac{1}{3}} m^{\frac{3}{2}}}{Re^{-\frac{1}{9}} m^{-1}} = Re^{-\frac{2}{9}} m^{\frac{5}{2}}, \quad (4.5.4)$$

while balancing $(\rho u)_x \sim p_x$, in the x -momentum equation, yields $\delta_1 = \delta_3$. The choice of δ_2 , for w , is far more troublesome; this is discussed below after we have deduced the size, δ_3 , of the pressure disturbance in the upper-deck. The easiest way of determining δ_3 is from the gas-law, which written in non-dimensionalised variables has the form

$$p = \frac{1}{\gamma M_\infty^2} \rho \theta \equiv p_\infty \rho \theta \equiv \gamma^{-1} Re^{-\frac{2}{9}} m^{-2} \rho \theta. \quad (4.5.5)$$

Balancing second order terms on both sides of the 'gas-law' in the upper-deck requires that

$$\delta_3 \sim Re^{-\frac{2}{9}} m^{-2} \delta_4 = Re^{-\frac{4}{9}} m^{\frac{1}{2}}. \quad (4.5.6)$$

We write the pressure expansion, for the boundary layer, in the form

$$p = \gamma^{-1} Re^{-\frac{2}{9}} m^{-2} + \delta_5 P + \delta_6 P_2 + \dots, \quad (4.5.7)$$

where the small sizes δ_5 and δ_6 are to be determined. A crucial point to note is that we assume that the upper-deck pressure perturbation, $\hat{p}$, has been induced by the boundary layer and hence the leading-order pressure perturbation size, δ_5 , there (in the boundary layer) is also $O(Re^{-\frac{4}{9}} m^{\frac{1}{2}})$, as the leading order pressure disturbance term, P , in the boundary layer is constant of the normal variable. The remaining unknown, δ_6 is easily deduced from considering the y -momentum equation of the boundary layer. Balancing $\rho u v_x \sim p_y$ (at second order) requires that $\delta_6 \simeq Re^{-\frac{8}{9}} m$.

Summarising so far, we know the x, t, y length-scales and the perturbation sizes of quantities in both the viscous boundary layer and the inviscid upper-deck, except the spanwise velocity (w) disturbances in both layers. So these two scales remain to be determined, in addition to the corresponding spanwise lengthscale z . Note the arguments used so far are based on the base-flow properties; the gas law; part of the continuity equation for the upper-deck and the assumption that the pressure disturbance is local to the interaction. We have not considered which terms vanish from the boundary-layer or upper-deck equations yet.

Consider the induced pressure-gradients in the boundary layer stream- and span wise momentum equations. We wish to keep the gradient in at least one of the equations to insure that the structure is still interactive and perturbations can

be self-induced. When $m \sim O(1)$ we saw that p_z is missing from the x -momentum eqn, but p_z is still present. We wish our $m \gg 1$ scalings to match back to this state of affairs as $m \searrow 1$.

Let us now consider the unknown spanwise length and velocity scales. Consider the following investigations:

z – momentum :

$$\text{Balancing } p_z \text{ in boundary layer requires } w_b \sim Re^{-\frac{2}{9}} m^{\frac{5}{2}} / z, \quad (1)$$

$$\text{Balancing } p_z \text{ in upper deck requires } w_u \sim Re^{-\frac{4}{9}} m^{\frac{1}{2}} / z, \quad (2)$$

where w_b and $w_u \sim \delta_2$ are the sizes of w in the boundary-layer and upper-deck respectively, and

Continuity :

$$\text{Balancing } w_z \text{ in boundary layer requires } w_b \sim z, \quad (3)$$

$$\text{Balancing } w_z \text{ in boundary layer requires } w_u \sim Re^{-\frac{2}{9}} m^{\frac{5}{2}} z, \quad (4)$$

Note that when $m \sim O(1)$ all the above are consistent (as expected), but for $m \gg 1$ the conditions above cannot all hold :

$$(1) \text{ and } (3) \text{ yield } w_b \sim Re^{-\frac{1}{9}} m^{\frac{5}{4}} \sim z, \quad (5)$$

whilst

$$(2) \text{ and } (4) \text{ yield } w_u \sim Re^{-\frac{1}{3}} m^{\frac{3}{2}}, \quad z \sim Re^{-\frac{1}{9}} m^{-1}, \quad (6)$$

and clearly (5) and (6) are contradictory on the z -scale. This is not surprising as (1) to (4) represent four equations for just three unknowns w_b, w_u and z . However, it is interesting to note that all four balances are possible in the lower-Mach number interactive structures, the two-tier structure and the triple-deck structure. Note that here, in the present $m \gg 1$ arguments, we have had to base most of the lengthscales on the underlying ‘highly compressible’ base flow and so it is perhaps not surprising that we have to drop (at least) one of the ‘spanwise’ balances.

So we have to settle for (at most) three of the above balances, (1)-(4). The problem that now arises is which one do we drop, and, more interestingly, is there

more than one option? Consider the p_z balances of the z -momentum equations. If we have the balance in the boundary-layer then we also need it in the upper-deck for consistency. Thus only dropping one balance requires that we keep both of the z -momentum-balances which, in turn, means that we have to drop the w_z continuity component in one of the two layers. A quick investigation indicates that option (4) must be ‘sacrificed’; otherwise the terms associated with operator $w\partial_z$ terms become solely leading order in the boundary-layer.

So it appears there is only one sensible option. This option results in dropping balance (4), yielding the unknown sizes

$$w_u \sim Re^{-\frac{1}{3}} m^{-\frac{3}{4}} \sim \delta_2, \quad \text{and} \quad z \sim Re^{-\frac{1}{9}} m^{\frac{5}{4}} \sim w_b. \quad (4.5.8a - d)$$

§4.5.3 Further comments.

In the last subsection we deduced the scales and (interactive) flow structure governing the viscous stability of the boundary-layer flow over a plate. We have seen that (at least one) of the ‘usual’ spanwise balances has to be dropped and argued that there was one choice that seemed more reasonable than other possibilities. We do not quote the equations or pressure-displacement law associated with these scales; instead we note that the structure closely resembles the two-tier structure, for $m \sim O(1)$, discussed earlier in this chapter. However there are a couple of interesting observations of particular note.

The first observation concerns the streamwise pressure gradient, p_x , in the boundary layer; it is already known to be absent (at leading order) in the $m \sim O(1)$ -structure’s x -momentum equation: for $m \gg O(1)$, $\rho u u_x \sim Re^{-\frac{2}{9}} m^{-2}$, whilst $p_x \sim Re^{-\frac{4}{9}} m^{\frac{1}{2}}$. So p_x remains absent in the x -momentum equation for large scaled Mach number. However, as $m \nearrow Re^{\frac{4}{45}} \leftrightarrow M_\infty \nearrow Re^{\frac{1}{5}}$, we see that it re-emerges at leading order.

As mentioned in the introduction to this section, one of the main reasons for investigating the flow structure for large m was to see what is predicted as the classical hypersonic-viscous range ($M_\infty \sim Re^{\frac{1}{5}}$) is approached (from below). We have just observed that in this limit the streamwise pressure-gradient, in the boundary-layer, finally reappears at leading order. Now we go on to investigate

further the implications of this limit. Before doing so, it should be noted that we have not found any (further) new regimes - we find that the large- m study matches straight on to the hypersonic-viscous range.

It is convenient to define a second scaled Mach number, $\hat{\chi}$ say, by

$$M_\infty = Re^{\frac{1}{5}} \hat{\chi} = Re^{\frac{1}{9}} m, \quad (4.5.9)$$

so that $\hat{\chi} \sim O(1)$ for the hypersonic-viscous range. Note that $\hat{\chi}$ is closely related to the so-called ‘hypersonic parameters’ commonly used in theoretical studies (i.e. the ‘ K ’ of Luniev, 1959; the ‘ χ ’ of Brown, Stewartson & Williams, 1975) where there is shock/boundary-layer interaction.

We consider first the dimensions. We have fixed $x \sim O(1)$ but find that (by substituting (4.5.9) into (4.5.1c), (4.5.2) and (4.5.8c)) that

$$y_B \sim Re^{-\frac{1}{5}} \hat{\chi}^{\frac{3}{2}}, \quad y_U \sim Re^{-\frac{1}{5}} \hat{\chi}^{-1}, \quad z \sim \hat{\chi}^{\frac{5}{4}}. \quad (4.5.10a - c)$$

Thus when $\hat{\chi} \nearrow O(1)$, we see that the boundary layer thickness is comparable to that of the upper-deck. This ties in with the interactive boundary-layer structure found by Bush (1966): the latter has a viscous (thermal) boundary layer below an inviscid shock layer, both having the same (physical — not Howarth-Dorodnitsyn) y -scaling ($y \sim Re^{-\frac{1}{5}}$), whilst the $\hat{\chi} \sim O(1)$ limit of the $m \gg 1$ study yields a viscous boundary layer and an inviscid layer above (upper deck) bounded by Mach-lines (shock), both having the same, $y \sim Re^{-\frac{1}{5}}$, scale. Recall that the two-tier and large- m structures strictly comprise of three layers, the third an adjustment layer, inbetween the boundary-layer and the upper-tier (deck), to match the hot conditions near the wall to the free-stream values. This layer corresponds to the transition layer of Bush (1966).

However, the last paper considers two-dimensional flow but we are including spanwise variation. The large- m study has indicated that

$$z \sim \hat{\chi}^{\frac{5}{4}}, \quad \sim O(1) \text{ when } \hat{\chi} \nearrow O(1) \quad (4.5.11)$$

i.e. the spanwise (disturbance) lengthscale grows to, finally, become of size $O(1)$ in this limit.

Thus, as the stream- and span-wise lengthscales are now again of the same order, the effects of cross-flow are less crucial than in the previous, $M_\infty \sim Re^{\frac{1}{9}}$

regime. However, if there is significant, $O(1)$, spanwise variation present (due to strong cross-flow or other factors) then it must be incorporated into any rational solutions, in addition to the effects of non-parallelism and/or unsteadiness of the base-flow which have been crucial since the Mach number rose to $O(Re^{\frac{1}{9}})$. Further, as the structure is long in the x - and z - directions relative to its normal height $Re^{-\frac{1}{5}}$, relatively small wall-curvature (streamwise and/or spanwise) should lead to the equations needing modifying.

Note that the other, less likely, option (for the z -scale) that was rejected would yield $z \sim Re^{-\frac{1}{5}} \sim y$, giving a long ‘toothpaste box-like’ structure. These scales suggest that the structure should support longitudinal vortices, usually associated with transition.

Now let us reconsider the pressure expansions,

$$p = \underbrace{\gamma^{-1} Re^{-\frac{2}{9}} m^{-2}}_{p_\infty} + \begin{cases} Re^{-\frac{4}{9}} m^{\frac{1}{2}} P + Re^{-\frac{8}{9}} m P_2 + \dots & : \text{Boundary Layer} \\ Re^{-\frac{4}{9}} m^{\frac{1}{2}} \hat{p} + \dots & : \text{Upper Deck ,} \end{cases}$$

for $m \gg O(1)$. Note that these are essentially linearised forms about p_∞ . In terms of the hypersonic parameter, $\hat{\chi}$, these become

$$p = \underbrace{\gamma^{-1} Re^{-\frac{2}{5}} \hat{\chi}^{-2}}_{p_\infty} + \begin{cases} Re^{-\frac{2}{5}} \hat{\chi}^{\frac{1}{2}} P + Re^{-\frac{4}{5}} \hat{\chi} P_2 + \dots & : \text{Boundary Layer} \\ Re^{-\frac{2}{5}} \hat{\chi}^{\frac{1}{2}} \hat{p} + \dots & : \text{Upper Deck ,} \end{cases} \quad (4.5.12)$$

so that, as $\hat{\chi} \nearrow O(1)$, the leading order disturbances, P and $\hat{p}$, become of the same order as the base pressure i.e. the expansion becomes nonlinear. Note that the P_2 term still represents a small perturbation; this effectively means that the y -momentum equation gives, at leading order, $P_Y = 0$, so that the pressure is still constant (of Y) across the boundary-layer. The fact that P and $\hat{p}$ have grown to the size of p_∞ suggests that there have been significant changes in the upper-deck equations.

Recall that the upper-deck expansions, for $m \gg 1$, are

$$u = 1 + \delta_3 \hat{u}, \quad v = Re^{-\frac{1}{3}} m^{\frac{3}{2}} \hat{v}, \quad w = \delta_2 \hat{w}$$

$$\rho = 1 + \delta_4 \hat{\rho}, \quad \theta = 1 + \delta_4 \hat{\theta}$$

where we have determined that

$$\delta_2 = Re^{-\frac{1}{3}} m^{-\frac{3}{4}}, \quad \delta_3 = Re^{-\frac{4}{9}} m^{\frac{1}{2}} \quad \text{and} \quad \delta_4 = Re^{-\frac{2}{9}} m^{\frac{5}{2}}. \quad (4.5.13a - c)$$

Note that these are linear perturbations of the base (freestream) flow. Written in terms of the hypersonic parameter these together with the pressure expansion become

$$u = 1 + Re^{-\frac{2}{5}} \hat{\chi}^{\frac{1}{2}} \hat{u}, \quad v = Re^{-\frac{1}{5}} \hat{\chi}^{\frac{3}{2}} \hat{v}, \quad w = Re^{-\frac{2}{5}} \hat{\chi}^{-\frac{3}{4}} \hat{w},$$

$$(\rho, \theta) = (1, 1) + \hat{\chi}^{\frac{5}{2}} (\hat{\rho}, \hat{\theta}), \quad p = \gamma^{-1} Re^{-\frac{2}{5}} \hat{\chi}^{-2} + Re^{-\frac{2}{5}} \hat{\chi}^{\frac{1}{2}} \hat{p}. \quad (4.5.14a - e)$$

Thus we see that, as $\hat{\chi} \rightarrow O(1)$ the density and temperature disturbances rise to become leading order, in addition to those for the pressure and normal velocity. Note however, that the freestream-velocity expansion,

$$u = 1 + Re^{-\frac{2}{5}} \hat{\chi}^{\frac{1}{2}} \hat{u} \equiv 1 + \frac{\hat{\chi}^{\frac{5}{2}}}{M_\infty^2} \hat{u},$$

still represents a small disturbance about the basic state.

The remaining scales can be easily transformed to be in terms of the Mach number by simply replacing $Re^{\frac{1}{5}}$ by $M_\infty \hat{\chi}^{-1}$; in studies of the hypersonic-viscous regime this is the formulation conventionally used. Moreover, we have seen, by taking the large- m limit of the two-tier structure, which in turn, was logically deduced from the familiar triple-deck structure, that the scales for the classical hypersonic-viscous range can be deduced. In addition we have ‘predictions’ for the spanwise quantities,

$$z \sim \hat{\chi}^{\frac{5}{4}} \sim w_b, \quad w_u \sim \frac{\hat{\chi}^{\frac{5}{4}}}{M_\infty^2},$$

which have not been considered in the literature, to date, to the author’s knowledge. We note, however, that there have been several studies of hypersonic flow over axisymmetric bodies (see for example Cross & Bush, 1969; Bush 1970); here the base-flow is very complicated, with several sublayers, and different properties depending on body radius. It is unclear to the author if any z and/or w scalings are implicit in, or can be picked out from, such studies. Finally note that we have shown that the upper-deck (upper-tier) remains linear until $M_\infty \sim Re^{\frac{1}{5}}$; recall that most of the other distinctive properties of the triple-deck structure disappear where $M_\infty \sim Re^{\frac{1}{9}}$. So, although the latter regime is crucial, being where non-parallelism and compressibility become important at leading order, an interactive (viscous-inviscid) structure still exists to describe the separation and stability of supersonic flow at larger Mach numbers.

It should be noted that there are several papers (eg. Neiland, 1970; Rizzetta, Burggraf & Jenson, 1978; Gajjar & Smith, 1983) concerned with the derivation of, or the implications of, the so-called ‘hypersonic triple-deck-structure’, with its simple pressure-displacement law having the simple form (in standard notation)

$$P = -A.$$

This structure, which is short-scaled so that non-parallel effects can be safely ignored, is far simpler than those considered in this chapter. The derivation of this structure and pressure-displacement law relies mainly on the following assumptions: (i) that the hypersonic parameter is large (the presence of a shock and the resulting sublayers is acknowledged) , and (ii) that $\gamma - 1 \ll 1$ i.e. the Newtonian assumption holds. It is unclear, to the author, whether the latter assumption is physically realistic, or just made to make theories simpler or even possible at all. Recall that in the work described in this chapter we make no assumptions on the size of γ , the ratio of specific heat capacities.

The interaction laws of supersonic (Ackeret’s law; free-interaction) and hypersonic flow (given above) were ‘linked’ by Brown, Stewartson & Williams (1975) who deduced the following pressure-displacement law

$$A = - \int_{-\infty}^x P dx - \sigma P, \quad \sigma^4 \propto \frac{\chi}{(\gamma - 1)^2},$$

where the hypersonic parameter $\chi = M_\infty^3 C^{\frac{1}{2}} / Re^{\frac{1}{2}}$ and C is the Chapman constant. It is clear that the limit $\sigma \rightarrow 0$ (requiring $\chi \ll 1$) corresponds to the usual supersonic (free-interaction law) whilst limit $\sigma \rightarrow \infty$ (requiring large χ and/or $\gamma - 1 \ll 1$) yields the hypersonic law given above. It must also be noted that χ is a function of x , the distance from the leading edge, and thus the value of χ can be significantly different at different x -locations, despite the fact that they are all experiencing the same freestream Mach number, M_∞ .

This last point clearly carries over to the present study where, for a constant freestream Mach-number, the value of m and $\hat{\chi}$ varies with x -location, especially when asymptotically close to the leading edge. Recently Brown, Cheng & Lee (1990) have shown that the requirement of the Newtonian assumption, for an ‘hypersonic triple-deck’ structure to be possible, can be replaced by a more physically

realistic assumption concerning the level of wall-cooling.

Related work is by Brown, Khorrami, Neish & Smith (1991) who describe certain features of recent theoretical research into hypersonic flow, concerning boundary layers, shock layers, nozzle flows and hypersonic instability and transition.

Chapter 5

Finite-time break-up in supersonic and larger-Mach-number boundary layers.

§5.1 INTRODUCTION.

In this chapter we are particularly interested in the possibility that finite-time break-up can occur in the nonlinear unsteady interactive boundary-layer equations that can govern instability and transition in the supersonic and larger-Mach-number regimes. Recent work by Brotherton-Ratcliffe & Smith (1987) (hereinafter referred to as B-RS), Smith (1988) (hereinafter referred to as S), Hoyle, Smith & Walker (1990) (hereinafter referred to as HSW) and Hoyle (1991) has indicated that finite-time break-up can occur in any such boundary layer (i.e. one governed by a so-called ‘pressure-displacement’ law), and the present work concentrates on the differences required in the details necessary for the Mach number range of concern. The break-ups proposed in the above are a ‘moderate’ type, yielding a singular pressure gradient, and a ‘severe’ type, associated with a pressure discontinuity. Here we concentrate principally on the former. The present work can be thought of as an extension of the theory of S who showed formally that any 2-D subsonic or supersonic unsteady interactive boundary layers can break-up within a finite time by encountering a nonlinear localized singularity and remarks that the theory carries over to other unsteady interacting boundary layers, to provide a potentially powerful means for transition.

In the present work it is found to be necessary to include spanwise effects and also, later on, varying density, due to the nature of the governing equations. The latter effect is novel to this study (in the present context) but the 3-D effects have begun to be addressed by Hoyle (1991) (see also HSW) who has considered the generalisation of the 2-D study of B-RS for a particular pressure-displacement law which does not result in a “troublesome” critical layer.

The fact that the boundary layer is governed by a pressure-displacement law is crucial to the possibility of the occurrence of ‘finite-time break-up’. In Figure 5.1 the various layers of the multi-structured boundary layer are sketched. Briefly, the flow in the lower deck is governed by the usual boundary-layer equations with

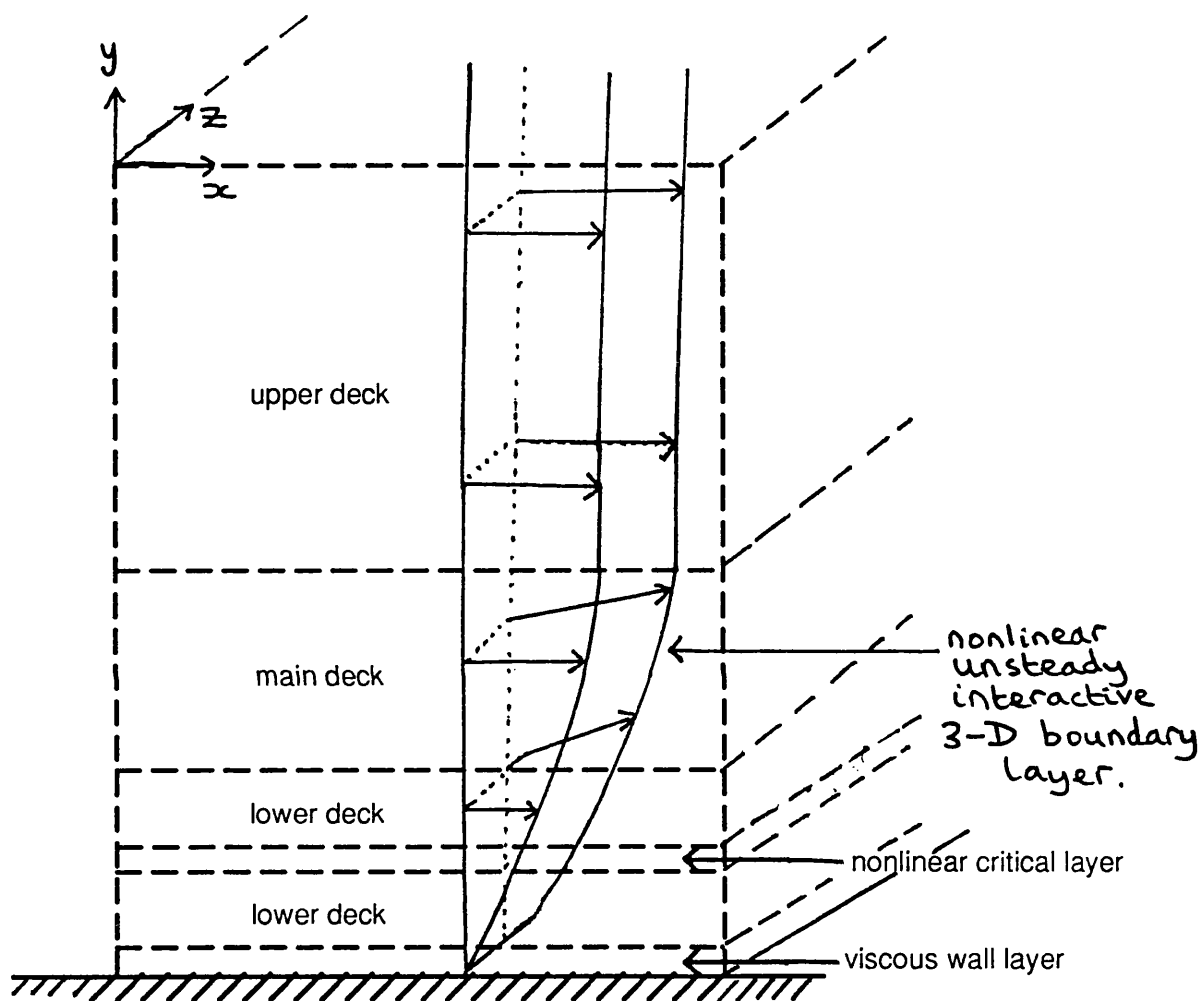


Figure 5.1. The multi-structured 3-D interactive boundary layer.

the important exception that the pressure is *not* prescribed as in classical theory; instead the pressure is driven by a displacement effect transmitted from the lower deck through to the upper deck. Effectively, this means here that a particular form for the pressure (with discontinuity or singular gradient) can be proposed; the analysis (and numerics - Hoyle, 1991; HSW) is required in order to show that such a choice is justified. The theory also involves ‘fast time-scales’, resulting in the usual viscous (Stokes) wall layer, although its effects on the lower-deck solutions enter at higher order and so they are not felt here.

In the next section details of the formulation are kept to a minimum as the method and arguments involved to derive the particular nonlinear pressure equations follow very closely the previously named works, instead we highlight the differences/ complications arising in the present cases. In §5.3 we discuss the $M_\infty \sim Re^{\frac{1}{9}}$ two-tier unsteady interactive boundary layer (see Chapter 4), here the study is truly 3-D and compressible with the analysis far more complicated than the studies mentioned above. Finally, in the last section, we conclude with a brief discussion.

§5.2 SUPERSONIC BOUNDARY-LAYERS, INCLUDING LARGE MACH NUMBER LIMIT.

Following Stewartson (1974), Ryzhov (1984) and Smith (1989), it is now well known that the unsteady lower-deck equations are those of the 3-D incompressible case. S showed that the crucial nonlinear pressure equation during break-up arises from an inviscid form of the lower-deck equations; the viscous wall layer and the upper deck play supporting roles. Thus we concentrate on the lower-deck equations and choose to write the relevant scalings as

$$(u, v, w) = (Re^{-\frac{1}{8}} \hat{m}^{\frac{9}{8}} U, Re^{-\frac{3}{8}} \hat{m}^{\frac{3}{8}} V, Re^{-\frac{1}{8}} \hat{m}^{\frac{3}{8}} W),$$

$$p - p_\infty = Re^{-\frac{1}{4}} \hat{m}^{-\frac{7}{4}} P,$$

$$(x, y, z, t) = (Re^{-\frac{3}{8}} \hat{m}^{\frac{27}{8}} X, Re^{-\frac{5}{8}} \hat{m}^{\frac{21}{8}} Y, Re^{-\frac{3}{8}} \hat{m}^{\frac{19}{8}} Z, Re^{-\frac{1}{4}} \hat{m}^{\frac{9}{4}} T), \quad (5.2.1a - g)$$

leading to the governing unsteady lower deck equations

$$U_T + UU_X + VU_Y + WU_Z = -\hat{m}^{-2}P_X + U_{YY},$$

$$0 = -P_Y,$$

$$W_T + UW_X + VW_Y + WW_Z = -P_Z + W_{YY},$$

and

$$U_X + V_Y + W_Z = 0, \quad (5.2.2a - d)$$

to be solved in conjunction with ‘no-slip’ at the wall and the pressure satisfying a displacement law. We have assumed that the wall is an insulator and that the viscosity is related to the temperature via Sutherland’s formula. Here the parameter $\hat{m}$ is $O(1)$ for moderately supersonic boundary layers, but $\hat{m} \sim M_\infty$ for larger Mach numbers which are of primary concern here. Observe that as M_∞ increases the streamwise pressure gradient term becomes small and therefore the spanwise gradient must play a more important role. We seek a moderate break-up of the local flow at $X = X_s$ as time $T \rightarrow T_s$ and, following the 2-D theory of S, write

$$X - X_s = -c\hat{T} + \hat{T}^{\frac{3}{2}}\xi \quad \text{where} \quad \hat{T} = T_s - T, \quad (5.2.3a, b)$$

where c is a constant to be determined, and the local pressure response expands as

$$p = p_0 + \hat{T}^{\frac{1}{2}}p_1(\xi, \eta) + \cdots. \quad (5.2.3c)$$

In the majority of the lower deck (away from the critical layer)

$$\begin{aligned} u &= U_0(y) + \hat{T}^{\frac{1}{2}}U_1(\xi, \eta, Y) + \hat{T}^{\frac{3}{4}}U_2 + \hat{T}U_3 + \cdots, \\ v &= \hat{T}^{\frac{1}{2}}V_1(\xi, \eta, Y) + \cdots, \end{aligned} \quad (5.2.3d, e)$$

and, assuming that there is no crossflow,

$$w = \hat{T}^b W_1(\xi, \eta, Y) + \cdots,$$

where η is a (local) scaled spanwise variable defined by

$$z - z_s = \hat{T}^a \eta. \quad (5.2.3f, g)$$

The unknown constants a and b are due to the inclusion of three-dimensionality and remain to be determined/chosen. The operator $\frac{\partial}{\partial t}$ is crucial in the derivation of the nonlinear pressure equations, and has the form

$$\frac{\partial}{\partial t} \rightarrow (-\hat{T}^{-\frac{3}{2}}c + \frac{3}{2}\xi\hat{T}^{-1})\frac{\partial}{\partial \xi} + a\eta\hat{T}^{-1}\frac{\partial}{\partial \eta} + \frac{\partial}{\partial t}.$$

This formulation is similar to that of Hoyle (1991) (see also HSW), who chooses $a = \frac{5}{4}$ and $b = \frac{3}{4}$ leading to a second order nonlinear partial differential equation (in ξ and η) for p_1 , but the present study concerns a general basic flow profile. Two possible choices of the pair (a, b) will now be discussed in the following subsections.

§5.2.1 Case I: $a = \frac{5}{4}$ and $b = \frac{3}{4}$

This choice of a and b results in the term $W_{1\eta}$, containing the leading order three-dimensionality effects, occurring with $U_{3\xi} + V_{3\eta}$ in the third order continuity equation; this case can be thought of as being *weakly* three-dimensional in that the leading two orders are unaltered from the two-dimensional theory and it relies on the presence of $\frac{\partial P}{\partial X}$ in the streamwise momentum equation. In the present case (general base flow profile) there is a so-called critical layer, where the speed of the base flow is the same as that of the disturbance, and c would be fixed by $U_0(y)$. Solving in the lower deck (away from the critical layer) leads to the following non-linear partial differential equation for the unknown pressure function $p_1(\xi, \eta)$

$$\hat{m}^{-2}\left(-\frac{1}{2}p_1 + \frac{3}{2}\xi p_{1\xi} + \frac{5}{4}\eta p_{1\eta}\right)\bar{\lambda}_1 + \hat{m}^{-4}p_1 p_{1\xi}\bar{\lambda}_2 = J$$

Here $\bar{\lambda}_1, \bar{\lambda}_2$ are constants, their value depending on U_0 (cf. Smith, 1988) and J is the anticipated jump across the critical layer. Note that only one ‘three-dimensional’ term appears, in contrast to that of Hoyle (1991) and HSW - this is due to the fact that the coefficient of the ‘missing’ term can be shown to be zero. We assume that J can be chosen to be zero, leading to

$$\left(-\frac{1}{2}p_1 + \frac{3}{2}\xi p_{1\xi} + \frac{5}{4}\eta p_{1\eta}\right)\bar{\lambda}_1 + \hat{m}^{-2}p_1 p_{1\xi}\bar{\lambda}_2 = 0 \quad (5.2.4)$$

as the governing equation for the pressure. Note that as $\xi \rightarrow \infty$ (η kept fixed) $|p_1| \sim |\xi|^{\frac{1}{3}}$ and as $\eta \rightarrow \infty$ (ξ fixed) $|p_1| \sim |\eta|^{\frac{2}{5}}$ which match with the flow solution away from the singular point $(x, z) = (x_s, z_s)$. Here we are assuming also that the

solution is smooth, as required: a full numerical solution (see Hoyle, 1991; HSW) is strictly necessary to show this is, in fact, the case.

We can now investigate the effect of increasing M_∞ (i.e. $\hat{m} \rightarrow \infty$) which is the principal concern of this investigation. Letting $\hat{m} \rightarrow \infty$ and keeping all other quantities $O(1)$ leads to

$$-\frac{1}{2}p_1 + \frac{3}{2}\xi p_{1\xi} + \frac{5}{4}\eta p_{1\eta} = 0$$

which, firstly, is no longer nonlinear and, secondly, does have the desired asymptotes. An inner region is required, where $\xi = \hat{m}^{-2}\hat{\xi}$, in order to bring in the nonlinear term, ensuring a solution there. Alternatively we could keep the full nonlinear form everywhere by simply writing $p_1 = \hat{m}^2\hat{p}_1$. We would have to ensure that the p expansion does not break down as $\hat{m} \rightarrow \infty$.

§5.2.2 Case II: $a = \frac{3}{2}$ and $b = \frac{1}{2}$

This choice leads to the term $W_{1\eta}$ (and thus the $\frac{\partial P}{\partial Z}$ -effects) appearing in the leading order form of the continuity equation. The reason for this particular choice are twofold: firstly, as the $\frac{\partial P}{\partial X}$ has $\hat{m}^{-2}$ multiplying it we expect $\frac{\partial P}{\partial Z}$ to be much more dominant/significant, and secondly, this will again lead to a nonlinear second order partial differential equation for p_1 . In fact it is this choice for a and b necessary for the ‘varying density’ case considered in §5.3. One last remark before examining the details concerns the effects of cross-flow, if present. We would then need to modify the w expansion depending on the strength of the cross-flow. We do not consider this here.

For general ξ, η dependence a non-linear partial differential equation for p_1 can be derived, but the determination of the jump requires some involved three-dimensional-critical-layer analysis. Instead here we choose the special case of the skew-direction

$$\Psi = \alpha\xi + \beta\eta$$

for the pressure and other disturbance quantities, working with the skew-velocity

$$\hat{U} = \alpha U + \beta W,$$

resulting in the problem mirroring the two-dimensional theory of S. The resulting non-linear pressure equation is

$$\frac{\hat{\lambda}_1}{2}(3\Psi p_{1\Psi} - p_1) + \left(\frac{\hat{m}^{-2}\alpha^2 + \beta^2}{\alpha}\right)\hat{\lambda}_2 p_1 p_{1\Psi} = 0,$$

a similarity form of the inviscid Burger's equation. Again we consider the large $\hat{m}$ limit. It is important to note that α, β are all $O(1)$ by definition as all $\hat{m}$ dependence has been incorporated into the X and Z scalings (see (5.2.1a-d)). Thus, as $\hat{m} \rightarrow \infty$, the equation reduces to

$$\frac{\hat{\lambda}_1}{2}(3\Psi p_{1\Psi} - p_1) + \left(\frac{\beta^2}{\alpha}\right)\hat{\lambda}_2 p_1 p_{1\Psi} = 0, \quad (5.2.5)$$

illustrating the importance of the inclusion of three-dimensional effects at leading order for large Mach numbers: in the next section we choose this a and b . Finally we note that the breakdown is a three-dimensional effect -the inclusion of spanwise variation is a *necessity* rather than a just a variation on the two-dimensional theory; and for larger Mach numbers density variation must also be included: see next section.

§5.3 LARGER MACH NUMBER: THE TWO-TIER STRUCTURE.

When the Mach number increases to become of size $O(Re^{\frac{1}{9}})$, we see, for example from (5.2.1a-g), that the usual triple-deck structure collapses. The lower deck now envelops the whole classical boundary layer and is no longer incompressible at leading order. The possibility of 'finite-time break-up' is considered for the new two-tiered interactive structure. The lower tier (the classical boundary layer) is predominately inviscid, for the fast timescales considered herein, contains a nonlinear critical layer and there is a thin viscous wall layer beneath next to the wall. Again, for the reasons discussed in the previous section, we concentrate our analysis on this tier.

§5.3.1 The inviscid regions of the lower tier.

Here the governing equations are a simplified form of the so-called Euler equations:

$$\begin{aligned}
 u_t + uu_x + vu_y + wu_z &= 0, \\
 p_y &= 0, \\
 \rho[w_t + uw_x + vw_y + ww_z] &= -p_z, \\
 u_x + v_y + w_z &= 0, \\
 \rho_t + u\rho_x + v\rho_y + w\rho_z &= 0,
 \end{aligned} \tag{5.3.1a - e}$$

to be solved for suitable boundary conditions. The continuity equation (5.3.1d) has been simplified by use of the energy equation (5.3.1e). Note the absence of pressure gradient terms in (5.3.1a), (5.3.1e) but the presence of compressibility terms, equations.

Again we propose a 'moderate' break-up at $(x, z) = (x_s, z_s)$ as time $t \nearrow t_s$ and write

$$x - x_s = -cT + T^{\frac{3}{2}}X, \quad T = t - t_s, \quad z - z_s = T^{\frac{3}{2}}Z$$

so that

$$\frac{\partial}{\partial x} \rightarrow T^{-\frac{3}{2}} \frac{\partial}{\partial X} + \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial z} \rightarrow T^{-\frac{3}{2}} \frac{\partial}{\partial Z} + \frac{\partial}{\partial z}$$

and

$$\frac{\partial}{\partial t} \rightarrow (-T^{-\frac{3}{2}}c + \frac{3}{2}XT^{-1}) \frac{\partial}{\partial X} + \frac{3}{2}ZT^{-1} \frac{\partial}{\partial Z} + \frac{\partial}{\partial t}. \tag{5.3.2a - c}$$

The local expansions, about $x = x_s$, are

$$\begin{aligned}
 u &= u_0(x_s, y) + T^{\frac{1}{2}}u_1(X, y, Z) + T^{\frac{3}{4}}u_2(X, y, Z) + T(u_3(X, y, Z) - cu_{0z}) + \dots, \\
 v &= T^{-1}v_1(X, y, Z) + T^{-\frac{3}{4}}v_2(X, y, Z) + T^{-\frac{1}{2}}v_3(X, y, Z) + \dots, \\
 w &= T^{\frac{1}{2}}w_1(X, y, Z) + T^{\frac{3}{4}}w_2(X, y, Z) + Tw_3(X, y, Z) + \dots, \\
 p &= p_0 + T^{\frac{1}{2}}p_1(X, Z) + T^{\frac{3}{4}}p_2(X, Z) + Tp_3(X, Z) + \dots, \\
 \rho &= \rho_0(x_s, y) + T^{\frac{1}{2}}\rho_1(X, y, Z) + T^{\frac{3}{4}}\rho_2(X, y, Z) + T(\rho_3(X, y, Z) - c\rho_{0x}) + \dots.
 \end{aligned} \tag{5.3.3a - e}$$

The terms u_3X and ρ_3X appearing above are non-parallel corrections of the base flow profile - they could be incorporated into the relevant subscript 3 quantities. Thus nonparallelism of the base-flow is a minor effect due to the short scales

involved; for the same reason the operator $\frac{\partial}{\partial x}$ is too small to be felt at the orders of concern.

Equating powers of T yields the following equations for the leading orders

x – momentum

$$O(T^{-1}) : (u_0 - c)u_{1X} + v_1u_{0Y} = 0,$$

$$O(T^{-\frac{3}{4}}) : (u_0 - c)u_{2X} + v_2u_{0Y} = 0,$$

$$O(T^{-\frac{1}{2}}) : (u_0 - c)u_{3X} + v_3u_{0Y} + \frac{3}{2}Xu_{1X} + u_1u_{1X} \\ + w_1u_{1Z} + v_1u_{1Y} - \frac{1}{2}u_1 + \frac{3}{2}Zu_{1Z} = 0,$$

z – momentum

$$O(T^{-1}) : \rho_0(u_0 - c)w_{1X} = -p_{1Z},$$

$$O(T^{-\frac{3}{4}}) : \rho_0(u_0 - c)w_{2X} = -p_{2Z},$$

$$O(T^{-\frac{1}{2}}) : \rho_0(u_0 - c)w_{3X} + \rho_1(u_0 - c)w_{1X} + \\ \rho_0[\frac{3}{2}Xw_{1X} + u_1w_{1X} + w_1w_{1Z} + v_1w_{1Y} - \frac{1}{2}w_1 + \frac{3}{2}Zw_{1Z}] = -p_{3Z},$$

continuity

$$O(T^{-1}) : u_{1X} + v_{1Y} + w_{1Z} = 0,$$

$$O(T^{-\frac{3}{4}}) : u_{2X} + v_{2Y} + w_{2Z} = 0,$$

$$O(T^{-\frac{1}{2}}) : u_{3X} + v_{3Y} + w_{3Z} = 0,$$

energy

$$O(T^{-1}) : (u_0 - c)\rho_{1X} + v_1\rho_{0Y} = 0,$$

$$O(T^{-\frac{3}{4}}) : (u_0 - c)\rho_{2X} + v_2\rho_{0Y} = 0,$$

$$O(T^{-\frac{1}{2}}) : (u_0 - c)\rho_{3X} + v_3\rho_{0Y} + \frac{3}{2}X\rho_{1X} + u_1\rho_{1X} \\ + w_1\rho_{1Z} + v_1\rho_{1Y} - \frac{1}{2}\rho_1 + \frac{3}{2}Z\rho_{1Z} = 0,$$

whilst y-momentum yields

$$p_{1Y} = p_{2Y} = p_{3Y} = 0.$$

§5.3.2 The inviscid solutions.

Solving the leading order equations gives

$$v_{1X} = -p_{1ZZ}[\Phi(y) + \alpha_1],$$

where

$$\Phi(y) = (u_0 - c) \int_y^\infty \frac{1}{\rho_0(u_0 - c)^2} dy \quad (5.3.4a, b)$$

and α_1 is a constant of integration. The second order solution is very similar, i.e.

$$v_{2X} = -p_{2ZZ}\Phi + \alpha_2,$$

while the other first order quantities can all be expressed in terms of p_1 :

$$u_1 = \frac{\beta^2 u_{0y} \Phi}{\alpha^2 (u_0 - c)} p_1, \quad w_1 = -\frac{\beta}{\alpha \rho_0 (u_0 - c)} p_1$$

$$\text{and } \rho_1 = \frac{\beta^2 \rho_{0y} \Phi}{\alpha^2 (u_0 - c)} p_1. \quad (5.3.5a - c)$$

At third order the nonlinear terms first appear and the equation to be solved for v_3 is found to take the form

$$\rho_0 [(u_0 - c)v_{3y} - v_3 u_{0y}]_X = p_{3ZZ} + \rho_0 F, \quad (5.3.6a)$$

where F contains the nonlinear (inertia terms). In fact

$$F = \left(\frac{3}{2} X u_{1X} + u_1 u_{1X} + w_1 u_{1Z} + v_1 u_{1y} - \frac{1}{2} u_1 + \frac{3}{2} Z u_{1Z} \right)_X$$

$$+ \left(\frac{3}{2} X w_{1X} + u_1 w_{1X} + w_1 w_{1Z} + v_1 w_{1y} - \frac{1}{2} w_1 + \frac{3}{2} Z w_{1Z} \right)_Z$$

$$- \left(\frac{\rho_1}{\rho_0^2} p_{1Z} \right)_Z \quad (5.3.6b)$$

which, in due course, leads to the 2-D nonlinear partial differential equation. Thus

$$v_{3X} = -p_{3ZZ}\Phi(y) - (u_0 - c) \int_y^\infty F \frac{1}{(u_0 - c)^2} dy + \alpha_3, \quad (5.3.6c)$$

and a velocity jump J , in v_3 , is necessary in the above requiring the replacement

$$\alpha_3 \quad [\text{for } u_0 > c] \rightarrow \alpha_3 + J_X \quad [\text{for } u_0 < c]. \quad (5.3.6d)$$

Note that the jump here has been defined to be in v_3 whereas in S the jump is in $-v_3$; this results in an apparent minus sign discrepancy between the two analyses.

In view of the pressure displacement law it can be shown that

$$\alpha_1 = \alpha_2 = \alpha_3 = 0.$$

Two thin layers are needed to make the above solution acceptable, one being a viscous wall layer required for the satisfaction of the no-slip condition at $y = 0$ and the other being a critical layer near the position $y = y_c$, where $u_0 = c$. It is noted here that for self-consistency in the wall layer, the tangential-flow conditions $v_n = 0$ are required due to the smaller sizes in the wall layer. Hence, from the above, the relations

$$\int_0^\infty \frac{1}{\rho_0(u_0 - c)^2} dy = 0 \quad \text{and} \quad \int_0^\infty F \frac{1}{(u_0 - c)^2} dy = J_X \quad (5.3.7a, b)$$

must hold, where the jump effect J is produced by the critical layer and remains to be determined. The first relation serves to determine the effective phasespeed c of the moderate singularity in terms of the profile u_0 and we note that to avoid a contradiction we must have $u_0 = c$ at some positive $y = y_c$ giving rise to the critical layer. The second relation leads to a non-linear equation for the pressure p_1 .

In an attempt to ease the evaluation of the jump J we introduce a shear coordinate

$$\Psi = \alpha X + \beta Z, \quad (5.3.8)$$

where α, β are real, and shear velocities

$$U_n = \alpha u_n + \beta w_n, \quad n = 1, 2, 3 \quad (5.3.9)$$

making the problem appear two-dimensional. The forcing function now takes the form

$$F(\Psi, y) = \frac{\partial}{\partial \Psi} \left[\frac{3}{2} \Psi U_{1\Psi} + U_1 U_{1\Psi} + v_1 U_{1y} - \frac{1}{2} U_1 - \frac{\beta^2 \rho_1 p_{1\Psi}}{\rho_0^2} \right]$$

where

$$U_1 = \frac{\beta^2}{\alpha} \Phi_y p_1. \quad (5.3.10a, b)$$

Note that the dependence of F on first order quantities can be expressed solely in terms of p_1 . If we choose to set $\beta = 0$ then $u_1 = v_1 = w_1 = \rho_1 = 0$; this is anticipatable as we are solely relying on the spanwise derivative of the pressure to drive the disturbances: there is no explicit x -dependence. Also note that compressibility effects modify the equations of previous studies.

The quantity

$$\int_0^\infty F \frac{1}{(u_0 - c)^2} dy, = I_N(\Psi) \quad (\text{say}), \quad (5.3.11a)$$

to be equated with the derivative of the jump J_X , is merely a nonlinear expression in p_1 that can be written in the form

$$I_N = \frac{\partial}{\partial \Psi} \left[\frac{\beta^2}{\alpha} \gamma_1 \left(\frac{3}{2} \Psi p_{1\Psi} - \frac{1}{2} p_1 \right) + \frac{\beta^4}{\alpha^2} (\gamma_2 - \gamma_3) p_1 p_{1\Psi} \right], \quad (5.3.11b)$$

where the real constants γ_1, γ_2 and γ_3 are functions of the basic flow profile;

$$\gamma_1 = \int_0^\infty \Phi_y \frac{1}{(u_0 - c)^2} dy, \quad \gamma_2 = \int_0^\infty (\Phi_y^2 - \Phi \Phi_{yy}) \frac{1}{(u_0 - c)^2} dy$$

and

$$\gamma_3 = \int_0^\infty \Phi \frac{\rho_{0y}}{\rho_0^2 (u_0 - c)^3} dy. \quad (5.3.11c - e)$$

So, for the present skew-coordinates, the nonlinear differential equation for the pressure is

$$\frac{\beta^2}{\alpha^2} \left[\frac{\gamma_1}{2} (3\Psi p_{1\Psi} - p_1) + \frac{\beta^2}{\alpha} (\gamma_2 - \gamma_3) p_1 p_{1\Psi} \right] = J. \quad (5.3.12)$$

The term in the square brackets is closely related to a similarity form of the inviscid Burger's equation, namely

$$\frac{\partial Q}{\partial \hat{t}} = Q \frac{\partial Q}{\partial \hat{x}}. \quad (5.3.13)$$

This result is that found by S and is again anticipatable from B-RS; the present coefficients are more complicated due to varying density. The above result can be seen by writing

$$Q = a_1 \hat{t}^n p_1(\Psi) \quad \text{and} \quad \Psi = \hat{x} \hat{t}^m$$

so that

$$\frac{\partial}{\partial \hat{t}} \rightarrow \frac{\partial}{\partial \hat{t}} + \frac{m\Psi}{\hat{t}} \frac{\partial}{\partial \Psi} \quad \text{and} \quad \frac{\partial}{\partial \hat{x}} \rightarrow \hat{t}^m \frac{\partial}{\partial \Psi}$$

and choosing

$$m = \frac{1}{2}, \quad n = -\frac{3}{2}, \quad \text{and} \quad a_1 = \frac{\beta^2(\gamma_2 - \gamma_3)}{\alpha\gamma_1}.$$

§5.3.3 Near the critical layer at $y \sim y_c$.

Recalling that

$$v_1 = -\frac{\beta^2}{\alpha} p_{1\Psi} (u_0 - c) \int_y^\infty \frac{1}{\rho_0 (u_0 - c)^2} dy,$$

it is clear that special attention is necessary near $y = y_c$ where $u_0 = c$. The basic flow profiles, as power series, about this point are

$$u_0 = c + \sum_{m=1}^{\infty} b_m s^m \quad \text{and} \quad \rho_0 = \sum_{m=0}^{\infty} d_m s^m, \quad (s = y - y_c \ll 1)$$

where

$$b_m = \frac{u_0^{(m)}(y_c)}{m!} \quad \text{and} \quad d_m = \frac{\rho_0^{(m)}(y_c)}{m!}. \quad (5.3.14a - d)$$

We also choose to write, for convenience,

$$\int_y^{\infty} \frac{1}{\rho_0(u_0 - c)^2} dy = I_1 - \int_y^{y_0} \frac{1}{\rho_0(u_0 - c)^2} dy,$$

where

$$I_1 = \int_{y_0}^{\infty} \frac{1}{\rho_0(u_0 - c)^2} dy. \quad (5.3.15a, b)$$

It is easily shown that, near y_c ,

$$\frac{1}{\rho_0(u_0 - c)^2} = \frac{1}{d_0 b_1^2 s^2} \left(1 - \left[\frac{d_1}{d_0} + 2 \frac{b_2}{b_1} \right] s + O(s^2) \right) \quad (5.3.16a)$$

and so, to avoid a logarithmic singularity in Φ , the coefficient of s^{-1} must be zero, that is

$$\frac{d_1}{d_0} + 2 \frac{b_2}{b_1} = 0$$

or, equivalently,

$$\left(\rho_0 u_0'' + \rho_0' u_0' \right) \big|_{y=y_c} = 0 \quad (5.3.16b)$$

i.e. we require that $y = y_c$ is a *point of generalized inflexion* ; it is assumed that this is so. Then it is the second (forcing integral) part of v_3 that causes the first logarithmic irregularity, occurring at that order, leading to the necessity to allow for a jump J anticipated earlier.

Further, we expand

$$\Phi = \sum_{m=0}^{\infty} \phi_m s^m, \quad \text{as} \quad y \rightarrow y_c, \quad (5.3.17)$$

where the early ϕ_m are

$$\phi_0 = \frac{1}{d_0 b_1}, \quad \phi_1 = \frac{b_2 + I_1 d_0 b_1^3}{d_0 b_1^2},$$

$$\phi_2 = \frac{[3d_0^2 b_1 b_3 - 3d_0^2 b_2^2 + (I_1 d_0^3 b_1^3 - 2d_0 d_1 b_1) b_2 + (d_0 d_2 - d_1^2) b_1^2]}{d_0^3 b_1^3}$$

and

$$\begin{aligned} \phi_3 = & (4d_0^3 b_1^2 b_4 + (-2d_0^3 b_1 b_2 + 2I_1 d_0^4 b_1^4 - 2d_0^2 d_1 b_1^2) b_3 \\ & - d_0^2 d_1 b_1 b_2^2 + (d_0^2 d_3 - 2d_0 d_1 d_2 + d_1^3) b_1^3) / (2d_0^4 b_1^4). \end{aligned}$$

The series for U_m and v_m follow immediately and, for convenience, the series

$$\frac{\rho_{0y}}{(u_0 - c)} \Phi = \sum_{m=-1}^{\infty} r_m s^m$$

is defined so that, in particular,

$$\rho_1 = \frac{\beta^2}{\alpha^2} p_1 \sum_{m=-1}^{\infty} r_m s^m. \quad (5.3.18)$$

The initial terms are

$$r_{-1} = \frac{d_1 \phi_0}{b_1}, \quad r_0 = \frac{-(d_1 \phi_0 b_2 + (-2d_2 \phi_0 - d_1 \phi_1) b_1)}{b_1^2}$$

and

$$r_1 = -\frac{(d_1 \phi_0 b_1 b_3 - d_1 \phi_0 b_2^2 + (2d_2 \phi_0 + d_1 \phi_1) b_1 b_2 + (-3d_3 \phi_0 - d_1 \phi_2 - 2d_2 \phi_1) b_1^2)}{b_1^3}.$$

Written as a function of the skew-variable Ψ ,

$$v_3 = \frac{-\beta^2}{\alpha} p_{\Psi} \Phi - \frac{(u_0 - c)}{\alpha} \int_y^{\infty} \frac{\left[\frac{3}{2} \Psi U_{1\Psi} + U_1 U_{1\Psi} + v_1 U_{1y} - \frac{1}{2} U_1 - \frac{\beta^2 \rho_1 p_{1\Psi}}{\rho_0^2} \right]}{(u_0 - c)^2} dy; \quad (5.3.19a)$$

near the critical level, $y \sim y_c$, it has the form

$$v_3 = \frac{c_{-1}}{s} + c_0 + c_{1L\Psi} s \ln s + c_{1\Psi}^{\pm} s = \dots, \quad \text{as } s \rightarrow \pm 0. \quad (5.3.19b)$$

Here c_{-1}, c_0 are (known) expressions involving the basic profile, the former leading to a higher order term in the $\hat{v}_2$ asymptote, and $c_{1\Psi}^{\pm}$ are the unknown coefficients of s either side of the critical layer; the evaluation of the ‘jump’

$$c_{1\Psi}^+ - c_{1\Psi}^-$$

is the principal aim of the next section where the critical layer is analysed. The coefficient of the logarithmic term is central to the current study, its value being

$$\begin{aligned}
c_{1L\Psi} = & - \left(\frac{\beta^2}{\alpha^2 b_1^2} (\phi_2 b_1 - \phi_1 b_2) (p_1 - 3p_1 \Psi) \right. \\
& + \frac{\beta^4}{\alpha^3 b_1^2} p_1 p_1 \Psi (-4\phi_2 \phi_0 b_2 + 2\phi_1^2 b_2 + 6\phi_3 \phi_0 b_1 - 2\phi_1 \phi_2 b_1) \Big) \\
& + \frac{\beta^4}{\alpha^3 d_0^4 b_1^3} p_1 p_1 \Psi \left(d_0^2 (r_{-1} (2b_1 b_3 - 3b_2^2) + 2r_0 b_1 b_2 - r_1 b_1^2) \right. \\
& \left. \left. + d_0 (r_{-1} (2d_2 b_1^2 - 4d_1 b_1 b_2) + 2d_1 r_0 b_1^2) - 3d_1^2 r_{-1} b_1^2 \right) \right).
\end{aligned} \tag{5.3.20}$$

This can be checked, to some extent, with its counterpart in S. The implications of this expression are discussed in the next section, where the critical layer is investigated.

§5.3.4 The nonlinear critical layer.

The appropriate scales and expansions follow very closely from S. The normal variable ν , defined by

$$y - y_c = T^{\frac{1}{4}} \nu, \tag{5.3.21}$$

is $O(1)$ in the critical layer. Now

$$\frac{\partial}{\partial t} \rightarrow (-T^{-\frac{3}{2}} c + \frac{3}{2} X T^{-1}) \frac{\partial}{\partial X} + \frac{3}{2} Z T^{-1} \frac{\partial}{\partial Z} + \frac{\nu}{4} T^{-\frac{1}{4}} \frac{\partial}{\partial \nu}$$

and

$$\frac{\partial}{\partial y} \rightarrow T^{-1} \frac{\partial}{\partial \nu},$$

along with the previous forms for the stream- and span-wise derivatives. The expansions here are

$$\begin{aligned}
u &= c + T^{\frac{1}{4}} \hat{u}_1(X, \nu, Z) + T^{\frac{1}{2}} \hat{u}_2 + T^{\frac{3}{4}} \hat{u}_3 + T \ln T \hat{u}_{4L} + T \hat{u}_4 + \dots, \\
v &= T^{-1} \hat{v}_1(X, \nu, Z) + T^{-\frac{3}{4}} \hat{v}_2 + T^{-\frac{1}{2}} \hat{v}_3 + T^{-\frac{1}{4}} \ln T \hat{v}_{4L} + T^{-\frac{1}{4}} \hat{v}_4 + \dots, \\
w &= T^{\frac{1}{4}} \hat{w}_1(X, \nu, Z) + T^{\frac{1}{2}} \hat{w}_2 + T^{\frac{3}{4}} \hat{w}_3 + T \ln T \hat{w}_{4L} + T \hat{w}_4 + \dots, \\
\rho &= d_0 + T^{\frac{1}{4}} \hat{\rho}_1(X, \nu, Z) + T^{\frac{1}{2}} \hat{\rho}_2 + T^{\frac{3}{4}} \hat{\rho}_3 + T \ln T \hat{\rho}_{4L} + T \hat{\rho}_4 + \dots,
\end{aligned} \tag{5.3.22a - d}$$

with the same expansion for the pressure as in the surrounding inviscid region.

Transforming to the skewed co-ordinates and defining the skew-velocities

$$\hat{U}_n = \alpha \hat{u}_n + \beta \hat{w}_n, \quad n = 1, 2, 3, 4 \quad \text{and} \quad 4L$$

yields the leading order equations in the critical layer

$$d_0[\hat{U}_1 \hat{U}_{1\Psi} + \hat{v}_1 \hat{U}_{1\nu}] = -\beta^2 p_{1\Psi},$$

$$\hat{U}_{1\Psi} + \hat{v}_{1\nu} = 0,$$

$$\hat{U}_1 \hat{\rho}_{1\Psi} + \hat{v}_1 \hat{\rho}_{1\nu} = 0.$$

(5.3.23a – c)

These have to be solved subject to matching conditions stemming from the study of the inviscid solutions near the critical level. These are found to be

$$\hat{U}_1 \rightarrow \alpha b_1 \nu, \quad \hat{v}_1 \rightarrow -\frac{\beta^2 \phi_0 p_{1\Psi}}{\alpha},$$

and

$$\hat{\rho}_1 \rightarrow d_1 \nu + \frac{d_1 \beta^2 \phi_0 p_1}{b_1 \alpha^2 \nu} + \dots, \quad \text{as} \quad \nu \rightarrow \pm\infty. \quad (5.3.23d – f)$$

Solutions for $\hat{U}_1$ and $\hat{v}_1$ follow the standard forms and are simply

$$\hat{U}_1 = \alpha b_1 \nu \quad \text{and} \quad \hat{v}_1 = -\frac{\beta^2 \phi_0 p_{1\Psi}}{\alpha},$$

whereas the solution for $\hat{\rho}_1$ is very similar to those for u_1 and w_1 , had we solved for these. In fact

$$\hat{\rho}_1 = d_1 G(\eta), \quad (5.3.24a)$$

where η is the streamline coordinate (cf. ‘Kelvin’s cats eyes’)

$$\eta = \nu^2 + \frac{2\beta^2 \phi_0}{\alpha^2 b_1} p_1 \quad (5.3.24b)$$

and the function G is undetermined at this order; the boundary conditions on $\hat{\rho}_1$ require that

$$G(\eta) \sim \mp \eta^{\frac{1}{2}}, \quad \text{as} \quad \eta \rightarrow \infty, \quad (5.3.24c)$$

where the ‘–’ sign is taken for $\nu > 0$ and + sign for $\nu \leq 0$.

Note that here, as in S, $p_1 = p_1(\Psi)$, i.e. general Ψ -dependence; Smith & Bodonyi (1982b), for example, have trigonometric dependence for the spatial (x

and z) variables whereas Gajjar & Smith (1985), Cowley (1985), Smith & Bodonyi (1987), for example, have time-dependent streamlines. The previous, present study included, all have *open* streamlines, apart from Smith & Bodonyi (1982b) where some of the streamlines are closed to form ‘cats eyes’.

The second order equations are

$$\begin{aligned}\hat{U}_{2\Psi} + \hat{v}_{2\nu} &= 0, \\ -\frac{1}{4}\hat{U}_1 + \frac{1}{4}\nu\hat{U}_{1\nu} + \frac{3}{2}\Psi\hat{U}_{1\Psi} + \hat{U}_1\hat{U}_{2\Psi} + \hat{U}_2\hat{U}_{1\Psi} + \hat{v}_1\hat{U}_{2\nu} + \hat{v}_2\hat{U}_{1\nu} &= \\ \frac{\beta^2}{d_0^2}(\hat{\rho}_1 p_{1\Psi} - d_0 p_{2\Psi}), \\ \text{and} \\ -\frac{1}{4}\hat{\rho}_1 + \frac{1}{4}\nu\hat{\rho}_{1\nu} + \frac{3}{2}\Psi\hat{\rho}_{1\Psi} + \hat{U}_1\hat{\rho}_{2\Psi} + \hat{U}_2\hat{\rho}_{1\Psi} + \hat{v}_1\hat{\rho}_{2\nu} + \hat{v}_2\hat{\rho}_{1\nu} &= 0.\end{aligned}\tag{5.3.25a - c}$$

The boundary conditions (from matching with the inviscid region) are

$$\hat{U}_2 \sim \alpha b_2 \nu^2 + \frac{\beta^2}{\alpha} \phi_1 p_1 + \dots, \quad \hat{v}_2 \sim -\frac{\beta^2}{\alpha} \phi_1 p_{1\Psi} \nu - \frac{\beta^2}{\alpha} \phi_0 p_{2\Psi},$$

and

$$\hat{\rho}_2 \sim d_2 \nu^2 + \frac{\beta^2}{\alpha^2} r_0 p_1 + \frac{d_1 \beta^2 \phi_0 p_2}{b_1 \alpha^2 \nu} + \dots,$$

as $\nu \rightarrow \pm\infty$. Substituting for the leading order quantities leads to

$$\hat{U}_{2\Psi} + \hat{v}_{2\nu} = 0$$

and

$$\alpha b_1 \nu \hat{U}_{2\Psi} - \frac{\beta^2}{\alpha} \phi_0 p_{1\Psi} \hat{U}_{2\nu} + \hat{v}_2 \alpha b_1 = \frac{\beta^2}{d_0^2} p_{1\Psi} d_1 G(\eta) - \frac{\beta^2}{d_0} p_{2\Psi}. \tag{5.3.26a, b}$$

Note that transforming to the streamline coordinate (Ψ, η) , from (Ψ, ν) requires

$$\frac{\partial}{\partial \Psi} \rightarrow \frac{\partial}{\partial \Psi} + q_\Psi \frac{\partial}{\partial \eta}$$

and

$$\frac{\partial}{\partial \nu} \rightarrow 2\nu \frac{\partial}{\partial \eta} = \pm 2(\eta - q)^{\frac{1}{2}} \frac{\partial}{\partial \eta}$$

where $q = \frac{2\beta^2 p_1 \phi_0}{\alpha^2 b_1}$. Here the '+' sign is taken for ν positive and the '-' sign otherwise; these are the natural coords for the critical layer. The solution for $\hat{U}_2$ is more complicated, being

$$\hat{U}_2 = 2\alpha b_2 \int_q^\eta (\eta - q)^{\frac{1}{2}} G'(\eta) d\eta + h(\Psi)$$

where h is fixed up from matching to the inviscid regions. The solutions for the other quantities at this order have similar forms. The third order equations are

$$\hat{U}_{3\Psi} + \hat{v}_{3\nu} = 0,$$

$$\begin{aligned} & -\frac{1}{2}\hat{U}_2 + \frac{1}{4}\nu\hat{U}_{2\nu} + \frac{3}{2}\Psi\hat{U}_{2\Psi} + \hat{U}_1\hat{U}_{3\Psi} + \hat{U}_2\hat{U}_{2\Psi} + \hat{U}_3\hat{U}_{1\Psi} + \\ & \hat{v}_1\hat{U}_{3\nu} + \hat{v}_2\hat{U}_{2\nu} + \hat{v}_3\hat{U}_{1\nu} = \frac{\beta^2}{d_0^2}\hat{\rho}_2 p_{1\Psi} - \frac{\beta^2}{d_0}p_{3\Psi} - \hat{\rho}_1[-p_{2\Psi} + \frac{\hat{\rho}_1 p_{1\Psi}}{d_0}], \end{aligned}$$

and

$$\begin{aligned} & -\frac{1}{2}\hat{\rho}_2 + \frac{1}{4}\nu\hat{\rho}_{2\nu} + \frac{3}{2}\Psi\hat{\rho}_{2\Psi} + \hat{U}_1\hat{\rho}_{3\Psi} + \hat{U}_2\hat{\rho}_{2\Psi} + \hat{U}_3\hat{\rho}_{1\Psi} + \\ & \hat{v}_1\hat{\rho}_{3\nu} + \hat{v}_2\hat{\rho}_{2\nu} + \hat{v}_3\hat{\rho}_{1\nu} = 0, \end{aligned} \quad (5.3.27a - c)$$

coupled with boundary conditions to match with the outer, surrounding inviscid flow. As far as working out the desired jump the subscript $4L$ equations are not important and neither is the density equation for $\hat{\rho}_4$. The important equations at fourth order are (noting that viscous effects enter at this order- below μ_0 is the value of the base viscosity at y_c)

$$\hat{U}_{4\Psi} + \hat{v}_{4\nu} = 0$$

and

$$\begin{aligned} & -\frac{3}{4}\hat{U}_3 + \frac{1}{4}\nu\hat{U}_{3\nu} + \frac{3}{2}\Psi\hat{U}_{3\Psi} + \hat{U}_1\hat{U}_{4\Psi} + \hat{U}_2\hat{U}_{3\Psi} + \hat{U}_3\hat{U}_{2\Psi} + \hat{U}_4\hat{U}_{1\Psi} + \\ & \hat{v}_1\hat{U}_{4\nu} + \hat{v}_2\hat{U}_{3\nu} + \hat{v}_3\hat{U}_{2\nu} + \hat{v}_4\hat{U}_{1\nu} = \\ & \frac{\mu_0}{d_0}\hat{U}_{1\nu\nu} + \frac{\beta^2}{d_0^2}\hat{\rho}_3 p_{1\Psi} - \frac{\beta^2}{d_0}p_{4\Psi} - \hat{\rho}_2[-p_{2\Psi} + \frac{\hat{\rho}_1 p_{1\Psi}}{d_0}] \\ & - \hat{\rho}_1 \left(\frac{\beta^2}{d_0^2}\hat{\rho}_2 p_{1\Psi} - \frac{\beta^2}{d_0}p_{3\Psi} - \hat{\rho}_1[-p_{2\Psi} + \frac{\hat{\rho}_1 p_{1\Psi}}{d_0}] \right). \end{aligned} \quad (5.3.28a, b)$$

together with matching conditions, to the inviscid region, requiring

$$\hat{U}_4 \sim b_4 \nu^4 + \frac{\beta^2}{\alpha}[3p_1 \phi_3 \nu^2 + 2p_2 \phi_2 \nu] - c_{1L} \ln \nu - (c_{1L} + c_1^\pm) + \dots$$

and

$$\hat{v}_4 \sim -\frac{\beta^2}{\alpha} [p_{1\Psi}\phi_3\nu^3 + p_{2\Psi}\phi_2\nu^2] + c_{1L\Psi}\nu \ln \nu + c_{1\Psi}^\pm \nu + \dots, \quad (5.3.28c, d)$$

as $\nu \rightarrow \pm\infty$. Note the first appearance of logarithms at this order; they have been suppressed at earlier order by the assumption that this critical level coincides with a point of inflexion. The presence of these leads to the jump across the critical layer which still remains to be determined.

§5.4 DISCUSSION.

In the previous section the problem that needs to be solved for the determination of the jump J has been formulated. Unfortunately, despite the use of the skew-coordinate Ψ , the solution requires some involved algebraic manipulation—the calculation of $c_{1L\Psi}$ was only comfortably achieved by use of the computer symbolic manipulation package *MACSYMA*. As no simpler method is apparent we choose to suspend the analysis at its current position: the evaluation of the jump has eluded the current investigation. However, note that in the expressions (5.3.12) and (5.3.20), the proposed nonlinear differential equation for the pressure and the value of $c_{1L\Psi}$ respectively, the pressure terms occur only in linear combinations of

$$p_1 - 3p_{1\Psi}\Psi \quad \text{and} \quad p_1 p_{1\Psi}.$$

This means, assuming that $J = 0$, that $c_{1L\Psi}$, the coefficient of the logarithmic term that results in the jump, is merely proportional to the nonlinear term $p_1 p_{1\Psi}$. Thus we can postulate that $J = 0$ on the basis that the previous analysis, when compared to S, contains no evidence to the contrary. Further one could argue that it is the unsteadiness of the critical layer that is crucial to the whole break-up mechanism which we have not altered from previous studies, the trouble is caused by nonlinear terms, in the critical layer analysis, being more complicated due to compressibility, which (the nonlinear critical layer terms) do not affect the eventual jump anyhow.

Summarising, the present study has indicated that as the Mach number increases the problem becomes truly 3-D and compressible. This results in the analysis becoming very awkward, even when the special case of a skew-direction

is considered. Despite not being able to demonstrate explicitly that break-up can occur we conclude that it is likely. The verification, or otherwise, will only follow from carefully controlled direct numerical simulations of the governing equations (5.3.1a-e), as in Peridier, Smith & Walker (1990), and, perhaps to a much lesser extent, from physical experiments.

Finally we note that the above theory can be generalised by writing

$$x - x_s = -c\hat{T} + \hat{T}^N X \quad \text{and} \quad p - p_0 = \hat{T}^{N-1} p_1(X, Z) + \dots$$

along with similar expansions for other quantities. In particular, the case $N = 1$ corresponding to ‘severe’ break-up (with a discontinuity in the pressure, rather than its gradient) remains to be studied. Note that the above theory should carry over to other related interactive boundary-layer structures such as when a shock is incorporated into the upper deck (Cowley & Hall, 1990), as well as other geometries. Recent related work is by HSW and Peridier *et al* (1990) (see also Peridier, 1989).

Chapter 6

On the inviscid instability of hypersonic flow past a flat plate.

§6.1 INTRODUCTION

In this chapter we are concerned with the inviscid instability of hypersonic boundary-layer flows. In the first instance we will consider flows where the influence of shocks is negligible, and then we will show how the instability problem can be significantly modified by shock effects. The motivation for this and related work on hypersonic boundary-layer instability theory is the renewed interest in hypersonic flight which has been stimulated by plans to build a successor to the Space Shuttle. A primary concern with such vehicles is the question of where transition will occur over a wide range of Mach numbers and whether it can be controlled. At the largest relevant Mach numbers, say Mach 20-25, the extremely high temperatures associated with the flow would destroy the vehicle unless it were cooled, so that it is of interest to identify the effect that the wall temperature has on flow instability.

The purpose of this chapter is to determine the inviscid instability characteristics of physically realistic hypersonic boundary-layer flows. We recall that there is a simple generalization of Rayleigh's (incompressible) inflection point theorem to compressible flows (Lees & Lin, 1946), and that many compressible boundary layers turn out to be inviscidly unstable even though their incompressible counterparts are stable. This is a significant result because the growth rates of inviscid disturbances tend to be much larger than those of viscous or centrifugal instabilities; thus they are prime candidates for causing transition to turbulence in many situations. The modes discussed in this chapter are referred to as generalized-inflection-point modes because the phase speed of the neutral mode is equal to the fluid velocity at the generalized inflection point. Furthermore the eigenfunctions of the fastest growing modes are localized around that inflection point.

For convenience we will concentrate on high-Reynolds-number flow past a flat plate, although many aspects of our analysis are applicable to other boundary-layer flows (e.g. flow past a wedge). Throughout we assume that the fluid viscosity is

Newtonian, and is adequately described by Sutherland's formula. We will also take the Prandtl number to be one. While it is relatively straightforward to relax this restriction, doing so complicates the analysis and numerical work further, and is not thought to alter significantly the qualitative features of the results presented. Also, while we assume that the Mach number is large, the complications arising from real gas effects are not investigated; Fu (1990) considered real gas effects on the Görtler vortex instability mechanism, with no shock present, and concluded that they have little direct influence at the edge of the boundary layer, where the disturbance is concentrated.

Reshotko (1976) and Mack (1987) have reviewed earlier work on the linear instability of high-Reynolds-number compressible flows. Many of these studies are based on the Orr-Sommerfeld equation; for a critique of the mathematical rigor of this approach see Smith (1979a, 1989). Here we examine the linear stability of high-Reynolds-number flows by means of formal asymptotic expansions. For example, Smith (1989) has used such an approach by applying triple-deck theory to lower-branch, viscous, Tollmien-Schlichting modes of compressible boundary layers. Seddougui, Bowles & Smith (1989) have extended this theory to include the effects of severe wall cooling, while Cowley & Hall (1990), hereafter referred to as CH, have shown how such modes can interact with a shock at large Mach number. However, viscous modes have relatively small growth rates, and our main concern will be with the faster growing inviscid modes. As a result, our analysis is based on the Rayleigh equation rather than the triple-deck equations. We note that the third type of instability responsible for boundary-layer transition, the Görtler vortex mode, develops an asymptotic structure at high Mach numbers closely related to that of our inviscid modes (Hall & Fu, 1989; for the Sutherland-formulation, see Fu, Hall & Blackaby, 1990 and Chapter 7).

When a quasi-parallel approximation is formally justifiable because the Reynolds number is large, inviscid modes satisfy the compressible generalisation of Rayleigh's equation. Numerical solutions to this equation have been reported by, *inter alia*, Mack (1984, 1987, 1990) for boundary-layer flows, Jackson & Grosch (1989) for shear flows, and Papageorgiou (1989) for wake flows. For fluids satisfying a Chapman viscosity law, high-Mach-number asymptotic solutions to this equation for the so-called 'acoustic' boundary-layer modes have been obtained by

CH, while Smith & Brown (1990), hereafter referred to as SB, have identified the asymptotic form of the ‘vorticity’ mode - including an exact solution of the governing equation. Balsa & Goldstein (1990) and Papageorgiou (1990) have given asymptotic descriptions for the high-Mach-number inviscid instability of shear-layers and wakes, respectively, assuming a Chapman-law fluid. Though the basic states investigated by SB, Balsa & Goldstein (1990) and Papageorgiou (1990) are different, they found essentially the same most unstable eigenvalue because it corresponds to a disturbance trapped in a thin layer where the overall features of the basic state are unimportant.

In the above mentioned boundary-layer analyses, and also in the hypersonic Görtler vortex instability analysis of Hall & Fu (1989), one of the key asymptotic regions for the case of a Chapman viscosity law is a *logarithmically* thin ‘adjustment’ layer which develops due to the *exponential* decay of the underlying steady temperature field away from the wall. However, Chapman’s viscosity law is not exact, and was introduced as a useful interpolation law which greatly simplified steady boundary-layer calculations; for example Stewartson (1955) hoped that the use of idealised physical properties would help in understanding ‘the behaviour of more realistic fluids’. At the large temperatures typical in hypersonic flows, Chapman’s law differs significantly from the more precise Sutherland’s formula. In fact, because Chapman’s law is simply a linear approximation to the viscosity-temperature dependence of the fluid, it is of questionable validity in the hypersonic limit.

At high Mach numbers the steady temperature field in a Sutherland-formula fluid initially decays algebraically away from the wall, before reverting to exponential decay in an asymptotic region ‘far’ from the wall (e.g. Freeman & Lam 1959). This algebraic decay significantly changes the scalings in the adjustment region; in particular the asymptotic expansions proceed in inverse powers of M_∞ rather than $\sqrt{\log(M_\infty)}$. Moreover the wavelength of the most unstable disturbance varies by a factor of $\sqrt{\log(M_\infty)}$ in the two cases. We note that a similar difference in scalings is evident in the interaction region of steady hypersonic flow past a flat plate. In that case Lee & Cheng (1969) have shown that the shock-heating adjustment layer

is logarithmically thin for Chapman’s viscosity law, whereas for a power-law viscosity formula, and hence for Sutherland’s formula, the scaling for the adjustment layer is algebraic in nature (Bush 1966).

The flows which we consider here are appropriate to different downstream locations for hypersonic flow past a semi-infinite flat plate. In the first instance we shall consider the instability of a non-interactive flow. This is appropriate to large distances downstream of the leading edge of the plate, where the Mach wave, corresponding to the attached shock at the leading edge, has no effect on the flow field. This basic state, and the Rayleigh equation which governs its inviscid instability, are discussed in §6.2. The dispersion relation associated with the Rayleigh equation is then derived in §6.3. We consider the growth rate and show that it is maximum for a vorticity mode. The small-wavenumber limit of the vorticity modes is then investigated and we show how the vorticity mode is related to the acoustic mode.

Then in §6.4 we go on to discuss the basic state in the ‘interaction zone’ further upstream. Since Sutherland’s viscosity formula reduces to a so-called ‘power-law’ at large temperatures, the description of the underlying steady flow in this region is essentially that for a power-law fluid due to Luniev (1959) and Bush (1966). They show that this region occurs where $Re = O(M_\infty^5)$, where Re is the Reynolds number based on distance from the leading edge. The resulting system of equations can only be solved numerically, and we are unaware of any published solutions. Nevertheless it is still possible to consider the instability of the flow in the interactive region, and we derive appropriate (quasi-parallel) stability equations. The ‘strong’ hypersonic interaction limit then corresponds to letting the streamwise variable tend to zero on the $Re = O(M_\infty^5)$ scale. In that limit a similarity solution for the basic flow can be found (Bush, 1966), and a re-scaled Rayleigh equation for the disturbance derived. The solution of that equation is discussed in §6.5. We could have instead considered the weak hypersonic limit further downstream where Bush & Cross (1967) have given an appropriate asymptotic description. We choose to concentrate on the strong-interaction regime because it is, to a certain extent, simpler. Further, if the flow is unstable in this regime it is arguable that

growing disturbances will originate here. Finally in §6.6 we shall draw some conclusions, as well as briefly outlining the main difficulties encountered during the numerical solution of the equations herein.

§6.2 NON-INTERACTIVE STEADY FLOWS

§6.2.1 The similarity solution

The similarity solution to the boundary layer equations has already been formulated in §2.2 and its large Mach number properties were briefly considered in §2.3.4. However, in this chapter it is shown that the adjustment layer is now crucial to understanding the inviscid stability of hypersonic flow (in contrast to the viscous stability) and so here we consider the large Mach number properties of the similarity solution in some detail. Let us now briefly recap their derivation; note the slight change in notation here compared with Chapter 2.

For a shear viscosity obeying Sutherland's formula

$$\mu = \left(\frac{1+S}{T+S} \right) T^{\frac{3}{2}}, \quad S \simeq \frac{110.4}{T_{\infty}^*}. \quad (6.2.1)$$

In the numerical calculations discussed below we took $T_{\infty}^* = 216.9$, leading to the value $S = 0.509$.

The boundary-layer equations are recovered from the (compressible) Navier Stokes equations by substituting

$$\hat{\eta} = Re^{\frac{1}{2}} \int_0^y \rho dy, \quad v = Re^{-\frac{1}{2}} V, \quad (6.2.2a, b)$$

where the Dorodnitsyn-Howarth variable, $\hat{\eta}$, is introduced for convenience, and then taking the limit $Re \rightarrow \infty$.

For steady two-dimensional flow over a flat plate with leading edge at $x = 0$, a similarity solution to these equations exists. With

$$\eta = \frac{\hat{\eta}}{\sqrt{(1+S)x}}, \quad u = \psi_{\hat{\eta}}, \quad \rho V = -(\psi_x + \hat{\eta}_x \psi_{\hat{\eta}}),$$

$$\psi = \sqrt{(1+S)x} f(\eta), \quad T \equiv T(\eta), \quad \rho \equiv \rho(\eta), \quad \mu \equiv \mu(\eta), \quad p = \frac{1}{\gamma M_{\infty}^2}, \quad (6.2.3a-h)$$

the governing equations are found to be

$$\rho T = 1, \quad \frac{1}{2} f f_{\eta\eta} + \left(\frac{T^{\frac{1}{2}}}{T+S} f_{\eta\eta} \right)_{\eta} = 0, \quad (6.2.4a, b)$$

$$\frac{1}{2} f T_{\eta} + \frac{1}{Pr} \left(\frac{T^{\frac{1}{2}}}{T+S} T_{\eta} \right)_{\eta} + \frac{(\gamma-1)M_{\infty}^2 T^{\frac{1}{2}}}{T+S} f_{\eta\eta}^2 = 0, \quad (6.2.4c)$$

subject to the boundary conditions

$$f(0) = f_{\eta}(0) = 0, \quad f_{\eta}(\infty) = T(\infty) = 1, \quad (6.2.4d)$$

and $T(0) = T_w$ (fixed wall – temperature), or $T_{\eta}(0) = 0$ (insulated wall).
(6.2.4e)

For simplicity we will focus attention on $Pr = 1$ †, and denote by T_r the wall temperature when the boundary is insulated. Then, as is well known (e.g. Stewartson 1964), the energy equation can be integrated to yield

$$T = 1 + ((T_b - 1) + \frac{1}{2}(\gamma - 1)M_{\infty}^2(T_b + f_{\eta}))(1 - f_{\eta}), \quad (6.2.5)$$

where $T_w = T_b T_r$ and $T_r = 1 + \frac{1}{2}(\gamma - 1)M_{\infty}^2$.

The solution to (6.2.4) in the limit of large Mach number has been examined by Freeman & Lam (1959). They showed that two asymptotic regions develop, distinguished by the positions where the coordinates η and $\xi = M_{\infty}^{\frac{1}{2}}\eta$, respectively, are order one.

§6.2.2 The high temperature region: $\xi = O(1) \leftrightarrow \eta = O(M_{\infty}^{-\frac{1}{2}})$

In this region we write $f = M_{\infty}^{-\frac{1}{2}} f_0(\xi) + \dots$, then using (6.2.5) it follows from (6.2.4b) that

$$f_0 f_{0\xi\xi} + \left(\frac{8}{\gamma-1} \right)^{\frac{1}{2}} \left(\frac{f_{0\xi\xi}}{(T_b + f_{0\xi})^{\frac{1}{2}} (1 - f_{0\xi})^{\frac{1}{2}}} \right)_{\xi} = 0, \quad (6.2.7a)$$

with

$$f_0(0) = f_{0\xi}(0) = 0, \quad (6.2.7b)$$

† the case of non-zero Prandtl number is discussed briefly, in §6.6.2, in connection with the numerical solution of the boundary-layer equations.

and

$$f_0 \sim \xi + \bar{\Delta} + \frac{24}{(T_b + 1)(\gamma - 1)\xi^3} - \frac{72\bar{\Delta}}{(T_b + 1)(\gamma - 1)\xi^4} + \dots, \text{ as } \xi \rightarrow \infty. \quad (6.2.7c)$$

Throughout the chapter detailed expansions are given for the solutions at the edges of asymptotic regions; this is because the algebraic nature of the expansions often means that more than the leading-order term is required to obtain accurate numerical results. The algebraic decay is of course different from the case of a Chapman-law fluid for which the expansions converge exponentially. The constant $\bar{\Delta}$ must be determined numerically.

For future reference observe: (a) that in this region $T = O(M_\infty^2)$, and hence Sutherland's formula reduces to a power-law form at leading order, and (b) that from (6.2.2a) the boundary-layer thickness in this region is $O(M_\infty^{3/2})$, and is thus relatively large. See Figure 6.1.

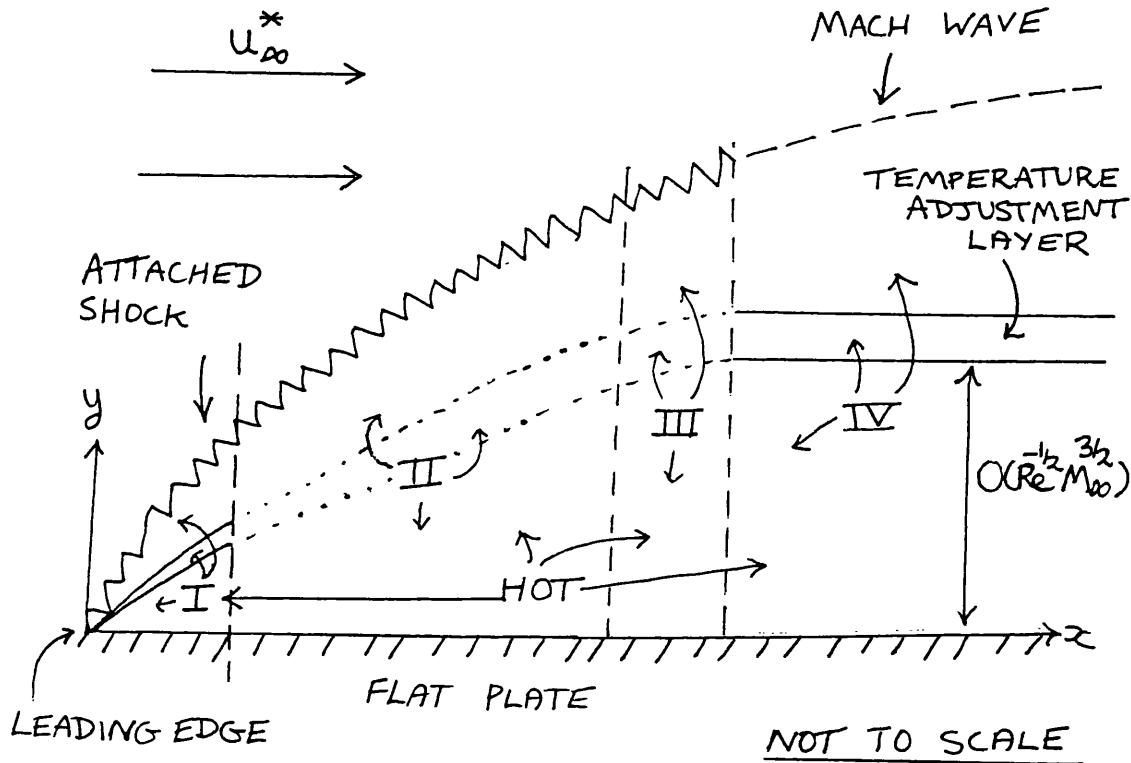


Figure 6.1. The four distinct downstream-regions of hypersonic flow over a flat plate. Region I: the strong interaction zone/region; $x \ll 1$. Region II: $x \sim O(1)$; no asymptotic description. Region III: the weak interaction zone/region; $x \gg 1$. Region IV: 'far downstream', $x \gg 1$; boundary-layer unaffected by Mach wave (shock) at leading orders of concern. Note that in this thesis we only consider Region IV, apart from the second half of this chapter, when we consider Region I. Further, note the so-called temperature adjustment layer (alternatively, sometimes referred to as the viscous transition layer) which 'splits' the flow beneath the shock/Mach wave into three (vertical) layers: the hot, viscous, inner layer; the temperature adjustment layer; and the outermost, cool inviscid layer.

§6.2.3 The temperature adjustment region: $\eta = O(1)$

Here, with

$$f = \eta + \frac{\bar{\Delta}}{M_\infty^{\frac{1}{2}}} + \frac{\bar{f}_1}{M_\infty^2} + \dots, \quad (6.2.8)$$

we find that the small perturbation $\bar{f}_1$ satisfies the nonlinear equation

$$\eta \bar{f}_{1\eta\eta} + 2 \left(\frac{\sqrt{1 - \frac{1}{2}(\gamma - 1)(T_b + 1)\bar{f}_{1\eta}}}{1 + S - \frac{1}{2}(\gamma - 1)(T_b + 1)\bar{f}_{1\eta}} \bar{f}_{1\eta\eta} \right)_\eta = 0, \quad (6.2.9a)$$

subject to

$$\bar{f}_1 \sim \frac{24}{(T_b + 1)(\gamma - 1)\eta^3} \quad \text{as } \eta \rightarrow 0, \quad \text{and} \quad \bar{f}_1 \rightarrow 0 \quad \text{as } \eta \rightarrow \infty. \quad (6.2.9b)$$

In this region $T = O(1)$, and thus the full form of Sutherland's formula holds. Note that although this temperature-adjustment layer is thicker than the high-temperature region in terms of the Dorodnitsyn-Howarth similarity variable η , it follows from (6.2.2a) that in physical coordinates it is a *thin* layer of $O(1)$ thickness sitting at the edge of the much wider $O(M_\infty^{\frac{3}{2}})$ high-temperature boundary layer.

In order to illustrate the underlying velocity profiles, in Figure 6.2a we have plotted the function $f_{0\xi}$ against ξ for the adiabatic case $T_b = 1$ with $\gamma = 1.4$, and then in Figure 6.2b we have plotted the adjustment-layer function $\bar{f}_{1\eta}$ against η . The figures illustrate the respective algebraic and exponential decay of the two velocity profiles away from the wall. It is also worth pointing out that for a Chapman-law fluid the adjustment-layer equation corresponding to (6.2.9a) is linear, and that its solution can be expressed in terms of the exponential function. It is this simplification that enabled SB to spot the exact solution of the neutral vorticity mode in their study of instabilities in this layer. We shall see that an exact solution of the analogous stability equation for Sutherland's formula cannot be found.

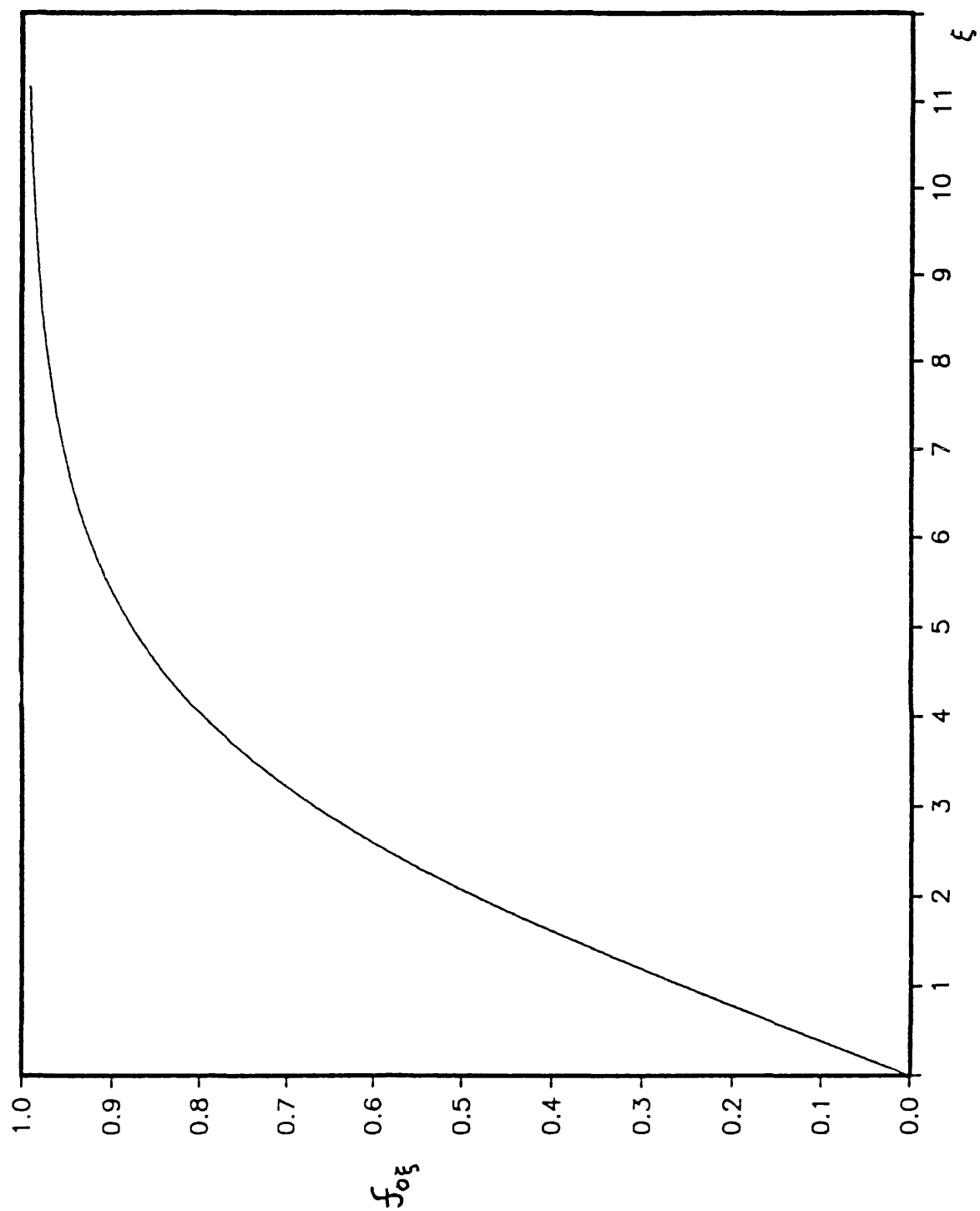


Figure 6.2a. The function $f_{0\xi}$ against ξ for the adiabatic case $T_b = 1$, with $\gamma = 1.4$.

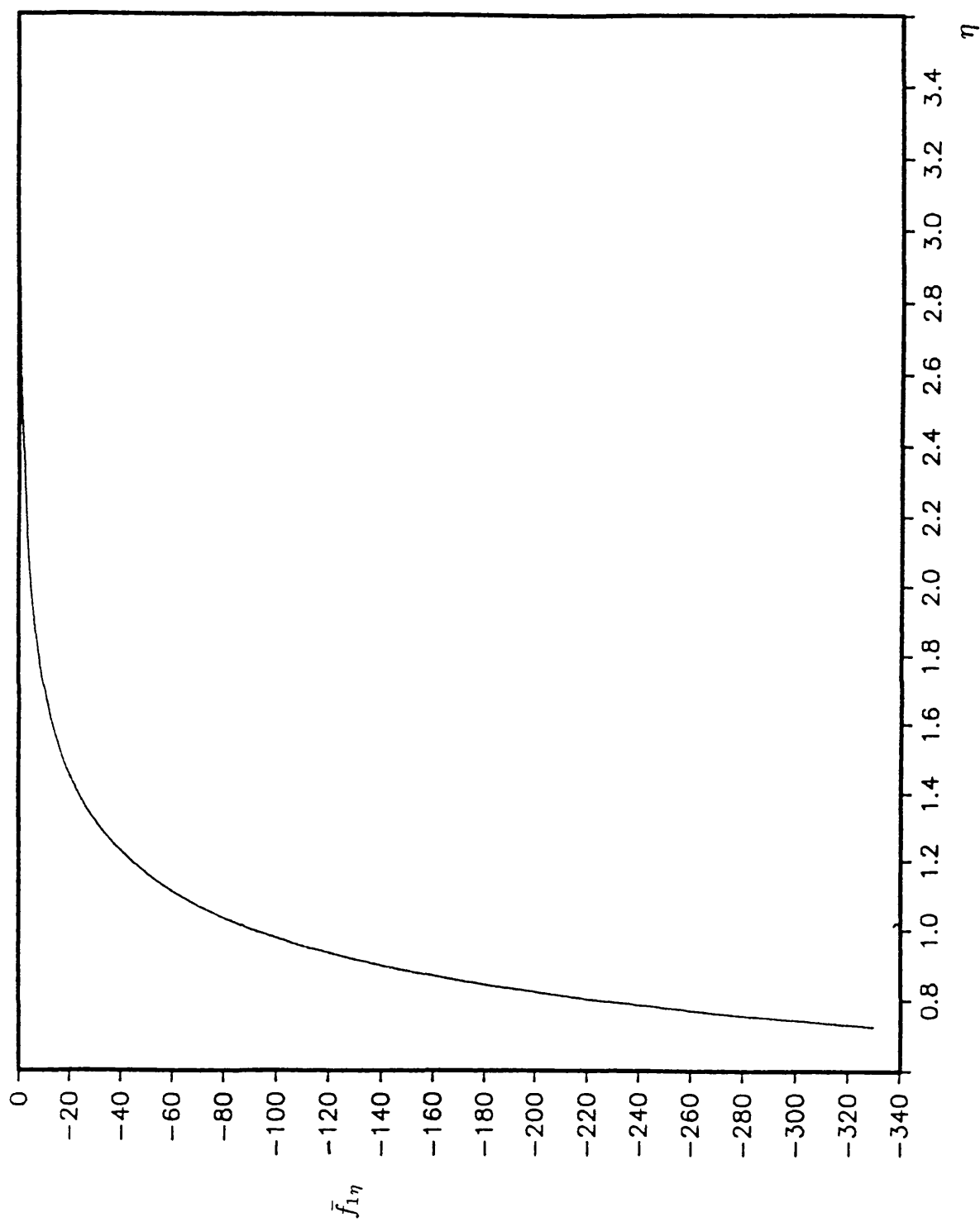


Figure 6.2b. The adjustment layer function $\bar{f}_{1\eta}$ against η .

§6.2.4 Rayleigh's Equation

We now investigate the stability of this non-interactive steady flow. Sufficiently far downstream the quasi-parallel assumption is valid for inviscid instability modes. It is then appropriate to seek perturbations of the form

$$(u, p) = (f'(\eta), \frac{1}{\gamma M_\infty^2}) + \dots + \Delta(\tilde{u}(\eta), \tilde{p}(\eta)) \exp\left(i Re^{\frac{1}{2}} \vartheta(x, z, t)\right) + \dots, \quad (6.2.10a, b)$$

with similar expressions for the other flow quantities. Here Δ is the small disturbance amplitude, and as is conventional we define local wavenumbers, a local frequency and a local wavespeed by

$$(\alpha, \beta, \omega) = \sqrt{(1+S)x} \ (\vartheta_x, \vartheta_z, -\vartheta_t), \quad c = \frac{\omega}{\alpha}. \quad (6.2.10c, d)$$

If Δ is sufficiently small, the pressure perturbation $\tilde{p}$ satisfies the linear, compressible, Rayleigh equation,

$$\frac{d^2 \tilde{p}}{d\eta^2} - \frac{2f''}{f' - c} \frac{d\tilde{p}}{d\eta} - (\alpha^2 + \beta^2) T \left(T - \frac{\alpha^2 M_\infty^2 (f' - c)^2}{(\alpha^2 + \beta^2)} \right) \tilde{p} = 0. \quad (6.2.11a)$$

The conditions that there is no normal velocity at the wall, and that the disturbance is confined to the boundary layer, can be expressed as

$$\tilde{p}' = 0 \text{ on } \eta = 0, \quad \tilde{p} \rightarrow 0 \text{ as } \eta \rightarrow \infty. \quad (6.2.11b)$$

Equation (6.2.11a) and boundary conditions (6.2.11b) specify a temporal stability eigenrelation $c \equiv c(\alpha, \beta)$; alternatively the eigenrelation can be regarded as $\alpha \equiv \alpha(c, \beta)$ from a spatial stability standpoint.

§6.3 THE FAR DOWNSTREAM BEHAVIOUR OF THE INVISCID MODES.

In this section we discuss the asymptotic form of unstable solutions to (6.2.11a) for the region far downstream of the leading edge of the plate. In a previous investigation CH studied the so-called acoustic modes of (6.2.11a) in this region using Chapman's viscosity law. Simultaneously SB investigated the vorticity mode using Chapman's law. The main difference between these two types of modes is that the acoustic modes have $(\alpha, \beta) = O(M_\infty^{-2})$, whilst the vorticity mode, at least close

to the upper branch, has $(\alpha, \beta) = O(\sqrt{2 \log M_\infty^2})$. Moreover the vorticity mode is centred in the adjustment layer at the edge of the boundary layer, whilst the acoustic one is concentrated in the main part of the boundary layer.

However, as indicated above, at high Mach numbers the temperature variations in the boundary layer are large. Thus a *linear* temperature-viscosity law is a bad approximation, and Sutherland's formula should be used to give a better representation of the viscosity. It is then important to see how the asymptotic structures developed by CH and SB change. We shall see that there are significant differences.

In the first instance we derive an asymptotic solution for a vorticity mode of (6.2.11a). We determine the neutral values of α, β and c for this mode, and find the limiting form of the mode when the small wavenumber limit $(\alpha, \beta) \rightarrow 0$ is taken. This limiting solution points to a sequence of distinguished asymptotic limits. Within this sequence the scaling $(\alpha, \beta) = O(M_\infty^{-3/2})$, appropriate to acoustic modes emerges; we therefore discuss these modes as a limiting case of the vorticity mode.

§6.3.1 Modes with wavelengths comparable with the thickness

of the adjustment region: $(\alpha, \beta) = O(1)$. The vorticity modes.

Consider then the solution of (6.2.11a) which has the eigenfunction trapped in the temperature-adjustment layer at the edge of the boundary layer. We seek a solution which has $(\alpha, \beta) = O(1)$, so that the wavelength of the vorticity mode is comparable with the width of the physically-thin adjustment layer. Defining the scaled adjustment-layer function

$$\bar{G} = \frac{1}{2}(T_b + 1)(\gamma - 1)\bar{f}_{1\eta}, \quad (6.3.1a)$$

we easily deduce from (6.2.3), (6.2.5) and (6.2.8), that the velocity field, $\bar{u}$, and temperature field, $\bar{T}$, of the underlying steady flow expand as

$$\bar{u} = 1 + \frac{2\bar{G}(\bar{\eta})}{(\gamma - 1)(T_b + 1)M_\infty^2} + \dots, \quad \bar{T} = 1 - \bar{G} + \dots, \quad (6.3.1b, c)$$

where $\eta = \sqrt{2}\bar{\eta}$, and $\bar{G}$ has the asymptotic behaviour

$$\bar{G} = -\frac{9}{\bar{\eta}^4} + \frac{B}{\bar{\eta}^3 - \sqrt{\bar{\eta}}} + (2S + 1) + \dots \quad \text{as } \bar{\eta} \rightarrow 0, \quad (6.3.2a)$$

$$\bar{G} \rightarrow 0 \quad \text{exponentially as} \quad \bar{\eta} \rightarrow \infty. \quad (6.3.2b)$$

Here B is a constant which has to be calculated numerically.

Next we expand α , β , c and $\bar{p}$ as

$$(\alpha, \beta) = \frac{1}{\sqrt{2}}(\hat{\alpha}, \hat{\beta}) + \dots, \quad c = 1 + \frac{2}{(\gamma - 1)(T_b + 1)M_\infty^2}\hat{c} + \dots, \quad \bar{p} = \hat{p} + \dots, \quad (6.3.3a, b, c)$$

where we have assumed that the disturbance moves downstream with the fluid speed in the adjustment layer. On substituting for $\bar{u}$ and $\bar{T}$ from (6.3.1), and using (6.3.3), we find that the zeroth order approximation to (6.2.11a) in the adjustment layer is the vorticity-mode equation

$$\frac{d^2 \hat{p}}{d\bar{\eta}^2} - \frac{2\bar{G}'}{\bar{G} - \hat{c}} \frac{d\hat{p}}{d\bar{\eta}} - \hat{k}^2(1 - \bar{G})^2 \hat{p} = 0, \quad (6.3.4a)$$

where

$$\hat{k} = (\hat{\alpha}^2 + \hat{\beta}^2)^{\frac{1}{2}}. \quad (6.3.4b)$$

Equation (6.3.4) is to be solved subject to $\hat{p}$ vanishing in the limits $\bar{\eta} \rightarrow 0$ and $\bar{\eta} \rightarrow \infty$, i.e. the disturbance is to be confined to the adjustment layer. For $\bar{\eta} \gg 1$ it follows from (6.3.4) that $\hat{p}$ decays like $\exp(-\hat{k}\bar{\eta})$, whilst for $\bar{\eta} \ll 1$ a WKB solution of (6.3.4) can be expressed in the form

$$\hat{p} \sim \exp\left(-\int \Theta(\bar{\eta})d\bar{\eta}\right), \quad (6.3.5)$$

where

$$\Theta \sim -\frac{9\hat{k}}{\bar{\eta}^4} + \frac{2}{\bar{\eta}} + \frac{B\hat{k}}{\bar{\eta}^3 - \sqrt{\bar{\eta}}} + 2S\hat{k} + \dots \quad \text{as} \quad \bar{\eta} \rightarrow 0. \quad (6.3.6)$$

First, we restrict our attention to the neutral case. The wavespeed $\hat{c}$ is then real and can be evaluated by finding the fluid speed correct to order M_∞^{-2} at the generalized inflection point where

$$\frac{\bar{u}_{\eta\eta}}{\bar{u}_\eta} - \frac{2\bar{T}_\eta}{\bar{T}} = 0, \quad (6.3.7)$$

i.e. where

$$\frac{\bar{G}''}{\bar{G}'} + \frac{2\bar{G}'}{1 - \bar{G}} = 0. \quad (6.3.8)$$

A numerical solution to (6.2.9a) using a Runge-Kutta method shows that this occurs at $\bar{\eta} \simeq 1.604924$, in which case $\hat{c} \simeq -0.993937$. The corresponding real

value of $\hat{k}$ is obtained by integrating (6.3.4) from $\bar{\eta} = 0$ to $\bar{\eta} = \infty$. In order to avoid difficulties at the generalised inflection point the path of integration was deformed into the complex $\bar{\eta}$ -plane by taking a triangular indentation around and below the generalised inflection point[†]. Such a calculation predicts that the neutral wavenumber is $\hat{k} \simeq 0.645065$ in addition to the same neutral wavespeed, $\hat{c}$, value as that calculated from the inflexion-point criterion (6.3.8).

However, of greater significance are the unstable eigenmodes. Figure 6.3 illustrates the dependence of the growth rate, $\text{Im}(\hat{\alpha}\hat{c})$, on the real wavenumber $\hat{\alpha}$ for two-dimensional modes. The maximum temporal growth rate, $\text{Im}(\hat{\alpha}\hat{c}) \simeq 0.256853$, occurs at $\hat{\alpha} \simeq 0.143619$. Further, it follows from the functional form of $\hat{c}$, i.e. $\hat{c} \equiv \hat{c}(\hat{k})$, that three-dimensional modes have smaller growth rates, as do acoustic modes (see Blackaby, Cowley & Hall, 1990 — hereinafter referred to as BCH). Hence this two-dimensional mode is the most unstable inviscid mode for a hypersonic boundary layer. We note that $\hat{c}$ as defined above is independent of T_b and γ , and that the growth rate is obtained from (6.3.3b) by dividing $\text{Im}(\hat{\alpha}\hat{c})$ by $\frac{1}{\sqrt{2}}(T_b + 1)(\gamma - 1)$. Thus wall cooling has a destabilizing effect on the vorticity mode to the extent that the temporal growth rate can be doubled by reducing the wall temperature sufficiently.

In Figure 6.4 we show the eigenfunction of the vorticity mode equation at different values of the wavenumber. This figure indicates that as the wavenumber decreases the eigenfunction starts to expand out of the adjustment layer. Thus, if the wavelength increases sufficiently the possibility arises of the disturbance extending outside the boundary layer, and hence of it interacting with external flow features such as shocks. Further, the form of the growth-rate at small wavenumbers is of interest because for sufficiently small values of the wavenumber the vorticity mode is expected to develop a structure similar to that of the acoustic mode.

[†] the actual shape of the indentation to the contour at the inflexion point is unimportant — in his comprehensive paper concerning the computation of the stability of the laminar compressible boundary layer, Mack (1965; see his figure 2) employs a rectangular indentation.

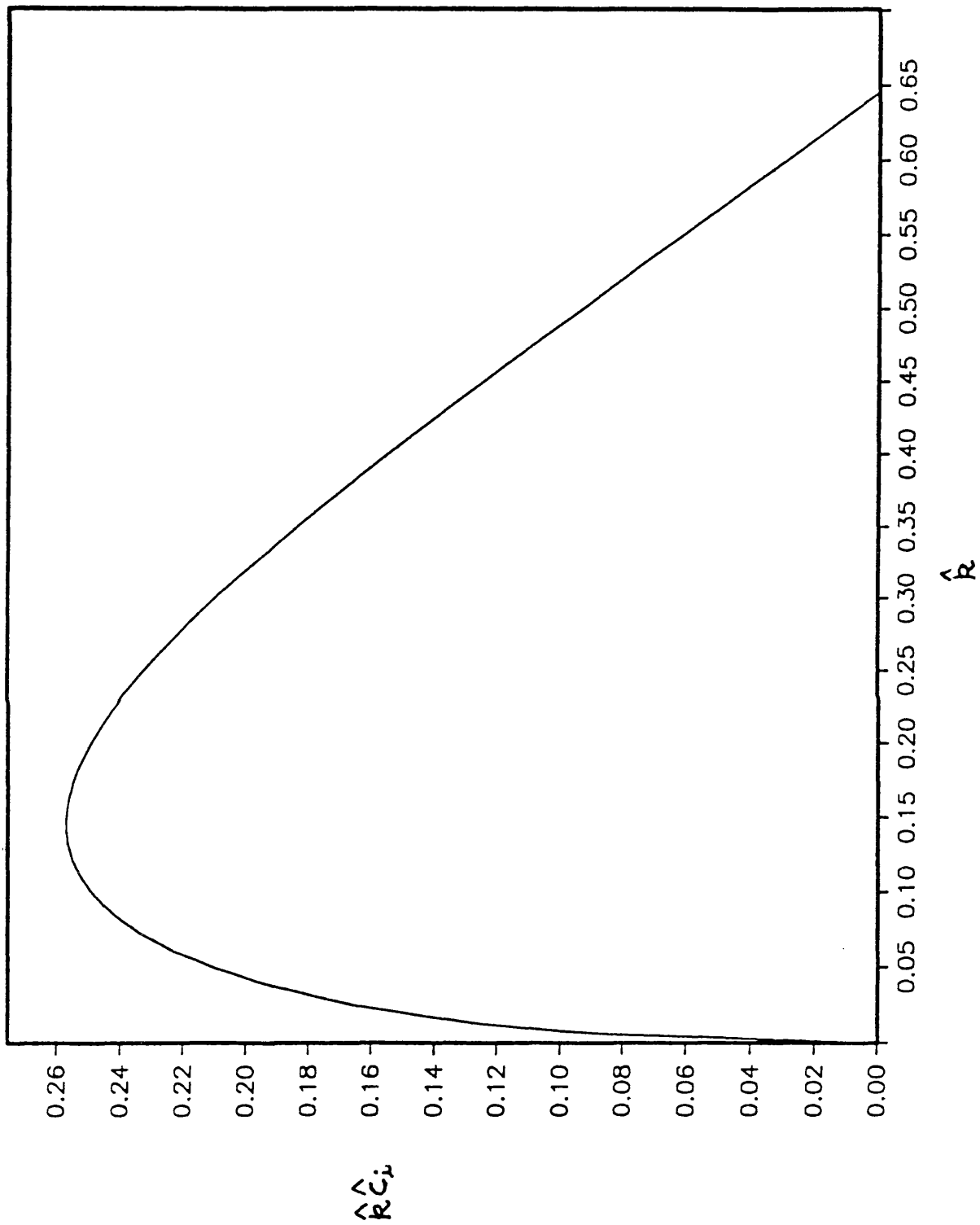


Figure 6.3. The dependence of the vorticity mode growth rate $\hat{k}\hat{c}_i$ on the wavenumber $\hat{k}$.

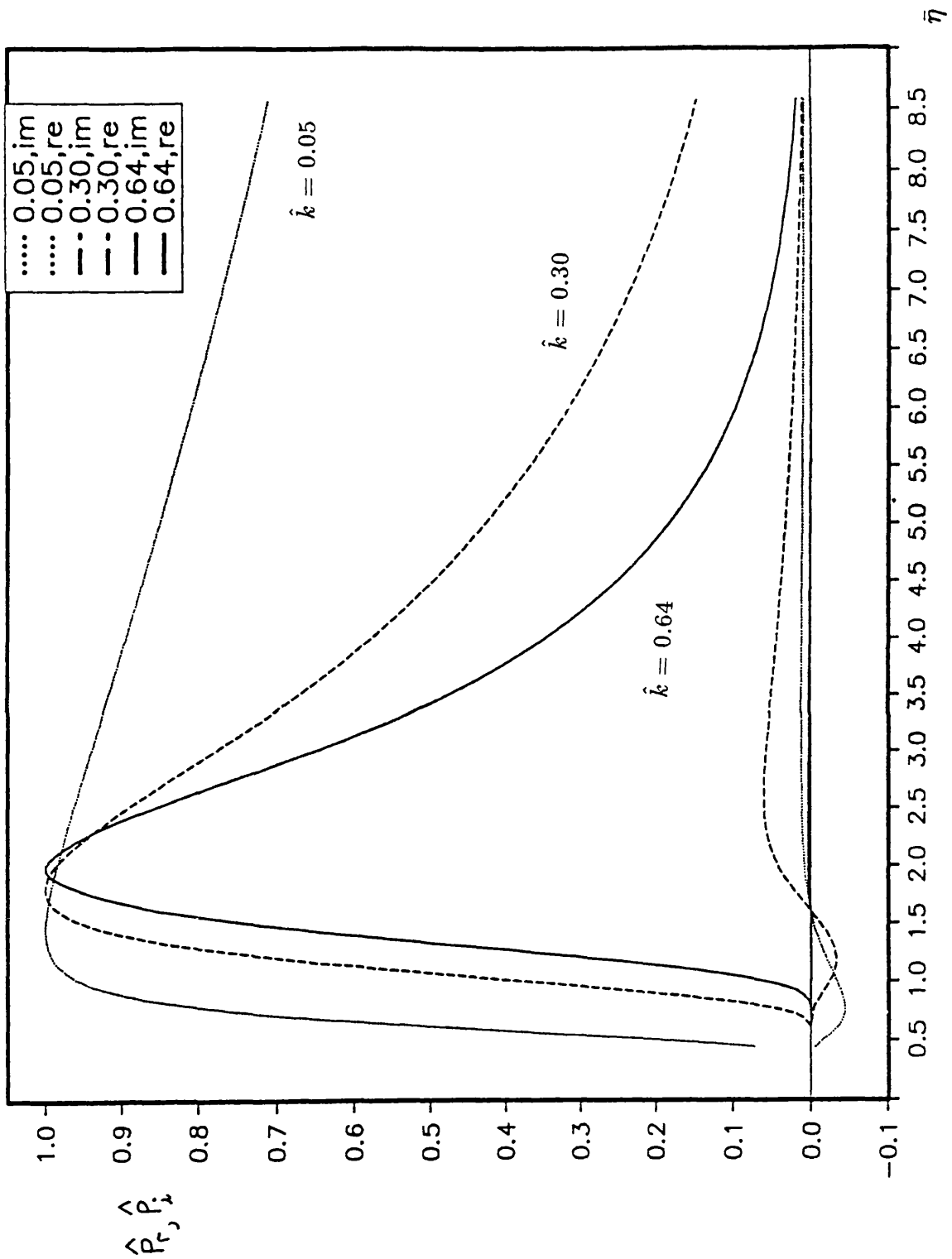


Figure 6.4. The vorticity mode eigenfunction at different values of the wave-number.

§6.3.2 The vorticity mode for small wavenumbers.

The key to understanding the subsequent regimes when $\hat{\alpha}, \hat{\beta}$ are related to inverse powers of the Mach number is to write down the small $\hat{k}$ asymptotic structure of (6.3.4). Figure 6.5 is a schematic illustration of the different regions in $\bar{\eta}$ space which emerge in the limit $\hat{k} \rightarrow 0$. Also shown in this figure is the high temperature wall layer, $\xi = O(1)$. For the moment $\hat{k}$ is not considered to be sufficiently small for it to be $O(M_\infty^{-\phi})$ for some positive ϕ ; it then turns out that region II, and to a lesser extent regions I and IV, are passive. However, at sufficiently small values of $\hat{k}$ the wall-layer structure of the basic state will enter the problem - see §6.3.3 and BCH.

In BCH it is reported that, after some careful numerical calculations at small values of $\hat{k}$, it was deduced that $\hat{c}$ expands in the form

$$\hat{c} = \frac{\hat{c}_1}{\hat{k}^{4/7}} + \frac{\hat{c}_2}{\hat{k}^{3/7}} + \frac{\hat{c}_3}{\hat{k}^{2/7}} + \frac{\hat{c}_4}{\hat{k}^{1/7}} + \dots \quad (6.3.9)$$

The present author has performed his own, independent, numerical calculations for small values of $\hat{k}$ and such a deduction was not obvious. The author was not too surprised at this 'set back' as their predicted small expansion parameter, $\hat{k}^{1/7}$, is not particularly small i.e. when $\hat{k}$ takes the reasonably small value of 0.001, $\hat{k}^{1/7} \simeq 0.37$, so that the asymptote is approached very slowly as $\hat{k} \rightarrow 0$.

Instead, in this subsection we show how the leading-order $\hat{k}$ -power ($-\frac{4}{7}$) and the value of its coefficient ($\hat{c}_1$) can be deduced simultaneously via an alternative argument. It is clear from numerical results that $\hat{c} \rightarrow \infty$ as $\hat{k} \rightarrow 0$ and this is our starting point. In this small wave-number limit the adjustment layer will split into asymptotic regions, as mentioned above.

From (6.2.11b), (6.3.2b) and (6.3.4) we can immediately see that for large $\bar{\eta}$ (but $\hat{k}\bar{\eta}$ small)

$$\hat{p} = \exp(-\hat{k}\bar{\eta}) + \dots = 1 - \hat{k}\bar{\eta} + \frac{\hat{k}^2}{2}\bar{\eta}^2 + \dots, \quad (6.3.10)$$

where an arbitrary multiplicative constant has been set equal to one. Thus, for $\hat{k} \ll 1$ there is a significant region (region I in Figure 6.5) where $\bar{\eta} \sim \hat{k}^{-1} \gg 1$ — this region captures the decay of the disturbance at the top of the adjustment layer. The rapid exponential decay of $\bar{G}$ results in the $\hat{c}$ -terms first occurring at very low order in (6.3.10). This suggests that the next significant asymptotic region

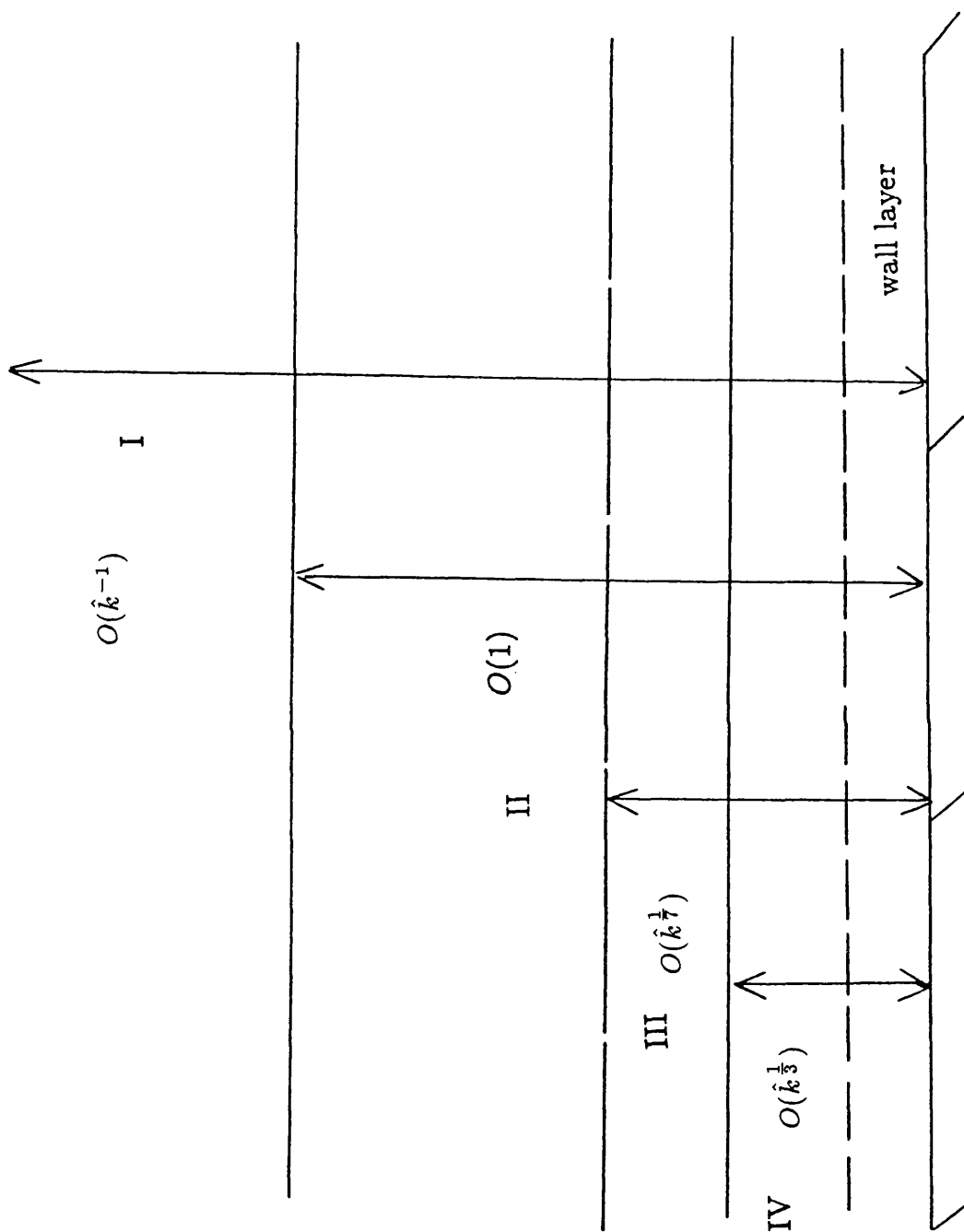


Figure 6.5. The different regions that emerge in the small $\hat{k}$ limit.

(region II) is where $\bar{\eta} \sim O(1)$, so that $\bar{G}_\eta \sim O(1)$. However, it is found to be more profitable to consider the small η limit first.

Considering $\bar{\eta} \ll 1$, the WKB solution, (6.3.5), of (6.3.4) will breakdown when the second (middle) term of (6.3.4a) grows to become leading order. For $\bar{\eta} \ll 1$, (6.3.2a) implies that

$$\frac{\bar{G}_\eta}{\bar{G} - \hat{c}} \frac{d\hat{p}}{d\eta} \sim \frac{\bar{\eta}^{-5}}{(\bar{\eta}^{-4} - \hat{c}) \bar{\eta}} \hat{p} \sim \bar{\eta}^{-2} \hat{p}; \quad (6.3.11)$$

thus the first and second terms automatically balance. The third term of (6.3.2a) has size $\hat{k}^2 \bar{\eta}^{-8} \hat{p}$ and so balancing all three terms requires $\bar{\eta} \sim \hat{k}^{\frac{1}{3}} \ll 1$. Thus we consider an asymptotic region (region III) in the lower part of the adjustment layer where

$$\hat{\xi} \sim \hat{k}^{-\frac{1}{3}} \bar{\eta} \sim O(1). \quad (6.3.12)$$

Here the zeroth order approximation to (6.3.4) is

$$\hat{p}_{\hat{\xi}\hat{\xi}} + \frac{8}{\hat{\xi}} \hat{p}_{\hat{\xi}} - \frac{81}{\hat{\xi}^8} \hat{p} = 0,$$

which must be solved subject to $\hat{p} \rightarrow 0$ as $\hat{\xi} \rightarrow 0$. The appropriate solution has

$$\hat{p} \rightarrow E_0 \quad \text{as} \quad \hat{\xi} \rightarrow \infty, \quad (6.3.13)$$

where E_0 is an order one constant.

The next asymptotic region (region IV) occurs for larger (but still small) values of $\bar{\eta}$ where $\hat{c}$ (also) enters the analysis at leading order. This will be where $\bar{G} \sim \bar{\eta}^{-4} \sim \hat{c} \iff \bar{\eta} \sim \hat{c}^{-\frac{1}{4}} \ll 1$. Obviously we are assuming that $\hat{c}^{-\frac{1}{4}} \gg \hat{k}^{\frac{1}{3}}$, i.e.

$$\hat{c} \ll \hat{k}^{-\frac{4}{3}}, \quad (6.3.14)$$

in this argument — this can be verified *a posteriori*.

The large $\hat{\xi}$ asymptote, (6.3.13), suggests that here we expand

$$\hat{p} = E_0 + \epsilon_1 \tilde{P}_1(\hat{\xi}) + \dots, \quad (6.3.15)$$

where ϵ_1 is a small parameter (to be determined) and

$$\tilde{\xi} = \hat{c}^{\frac{1}{4}} \bar{\eta} \sim O(1). \quad (6.3.16)$$

Note that, at leading order,

$$\hat{p}_{\bar{\eta}\bar{\eta}} \sim \epsilon_1 \hat{c}^{\frac{1}{2}}, \quad \frac{\bar{G}_{\bar{\eta}}}{\bar{G} - \hat{c}} \hat{p}_{\bar{\eta}} \sim \epsilon_1 \bar{\eta}^{-2} \sim \epsilon_1 \hat{c}^{\frac{1}{2}},$$

whilst

$$\hat{k}^2(1 - \bar{G})^2 \hat{p} \sim \hat{k}^2 \bar{\eta}^{-8} E_0 \sim \hat{k}^2 \hat{c}^2.$$

Balancing all these terms at leading order requires

$$\epsilon_1 = O(\hat{k}^2 \hat{c}^{\frac{3}{2}}), \quad (6.3.17)$$

which is small by the previous assumption (6.3.14).

In fact $\bar{P}_1$ satisfies

$$\bar{P}_1'' + \frac{72}{\tilde{\xi}(\tilde{\xi}^4 + 9)} \bar{P}_1' = \frac{81 E_0}{\tilde{\xi}^8}$$

with solution

$$\bar{P}_1' = E_0 \left(1 + \frac{9}{\tilde{\xi}^4}\right)^2 \left(\int_{\infty}^{\tilde{\xi}} \frac{81 d\tilde{\xi}}{(\tilde{\xi}^4 + 9)^2} - E_1\right), \quad (6.3.18)$$

where E_1 is the constant of integration; its sign has been chosen for later convenience. We require that $P_1' \rightarrow 0$ as $\tilde{\xi} \rightarrow 0$ to match on with the large $\hat{\xi}$ form of the solution in the previously considered layer; this can only be achieved if

$$\int_0^{\infty} \frac{81 d\tilde{\xi}}{(\tilde{\xi}^4 + 9)^2} + E_1 = 0. \quad (6.3.19a)$$

As $\tilde{\xi} \rightarrow \infty$ we see from (6.3.18) that $\bar{P}_1' \simeq -E_1$ ($\sim O(1)$), so that

$$\hat{p} \sim E_2 - \hat{k}^2 \hat{c}^{\frac{7}{4}} E_1 \bar{\eta}, \quad \text{as } \tilde{\xi} \rightarrow \infty \quad (\bar{\eta} \searrow O(1)), \quad (6.3.19b)$$

where E_2 is an arbitrary constant. We see that this asymptote matches (directly) onto (6.3.10) if $E_2 = 1$ and $\hat{k}^2 \hat{c}^{\frac{7}{4}} E_1 = \hat{k}$, i.e.

$$\hat{c} = E_1^{-\frac{4}{7}} \hat{k}^{-\frac{4}{7}} + \dots.$$

Thus we have found that $\hat{c} \sim \hat{k}^{-\frac{4}{7}}$, as deduced numerically by BCH. Moreover, the quantity E_1 is simply given by (6.3.19a). This definite-integral can be easily evaluated by a standard contour integration to yield the (complex) value of E_1 ; the appropriate choice of seventh root leads to

$$\hat{c}_1 = \left(\frac{8\sqrt{2}}{3\sqrt{3}\pi}\right)^{\frac{4}{7}} \exp\left(\frac{4i\pi}{7}\right) \hat{k}^{-\frac{4}{7}} + \dots, \quad (6.3.20)$$

which corresponds to an unstable mode of (6.2.11a), for small $\hat{k}$.

§6.3.3 Modes with wavelengths comparable with the thickness of the high-temperature region: $\alpha, \beta = O(M_\infty^{-\frac{3}{2}})$, and their link with the acoustic modes.

Now we consider the situation when α, β are so small that region IV in Figure 6.5 merges with the wall layer of the basic state. Since the wall layer is of thickness $M_\infty^{-\frac{1}{2}}$, this occurs when $(\alpha, \beta) \sim M_\infty^{-\frac{3}{2}}$; it is then appropriate to write

$$(\alpha, \beta) = M_\infty^{-\frac{3}{2}}(\alpha_0, \beta_0) + \dots \quad (6.3.21)$$

Since $(1 - c) = O(M_\infty^{-\frac{8}{7}})$, the zeroth order approximation to (6.2.11a) in the wall layer is thus

$$\tilde{p}'' - \frac{2\bar{u}_0'}{\bar{u}_0 - 1}\tilde{p}' - \frac{1}{4}(\gamma - 1)^2(T_b + \bar{u}_0)^2(1 - \bar{u}_0)^2((\alpha_0^2 + \beta_0^2) - \frac{2\alpha_0^2(1 - \bar{u}_0)}{(\gamma - 1)(T_b + \bar{u}_0)})\tilde{p} = 0, \quad (6.3.22)$$

where a dash denotes a derivative with respect to the wall-layer variable $\xi = M_\infty^{\frac{1}{2}}\eta$, and $\bar{u}_0 = f'_0$ is the first term in the expansion of $\bar{u}$ in that layer. The above equation is to be solved subject to $\tilde{p}'(0) = 0$. For large ξ it has the asymptotic solutions

$$\tilde{p} \sim N_0 = \text{constant} \quad \text{and} \quad \tilde{p}\xi^7 \sim N_1 = \text{constant}. \quad (6.3.23)$$

For most choices of α_0, β_0, T_b and γ , the constant N_0 is nonzero, and the structure in layers I, II, III survives intact. Thus for these values of α_0, β_0 the wavespeed c expands as (see (6.3.3), (6.3.9), (6.3.13b) and (6.3.14a))

$$c = 1 + \frac{1}{M_\infty^{\frac{8}{7}}(\gamma - 1)(T_b + 1)(\alpha_0^2 + \beta_0^2)^{\frac{2}{7}}} 2^{\frac{5}{7}}\hat{c}_1 + \dots, \quad (6.3.24)$$

implying that the wave growth rate, $\text{Im}(\alpha c)$, is of order $M_\infty^{-\frac{37}{14}}$.

However, equation (6.3.22) has a countably infinite set of eigenvalues for which the constant $N_0 = 0$. For two-dimensional disturbances with $T_b = 1$ and $\gamma = 1.4$, a numerical solution of (6.3.22) yielded the eigenvalue sequence

$$\alpha_0 = 2.47, 7.17, 12.19, 17.33, 22.54, 27.79, \dots$$

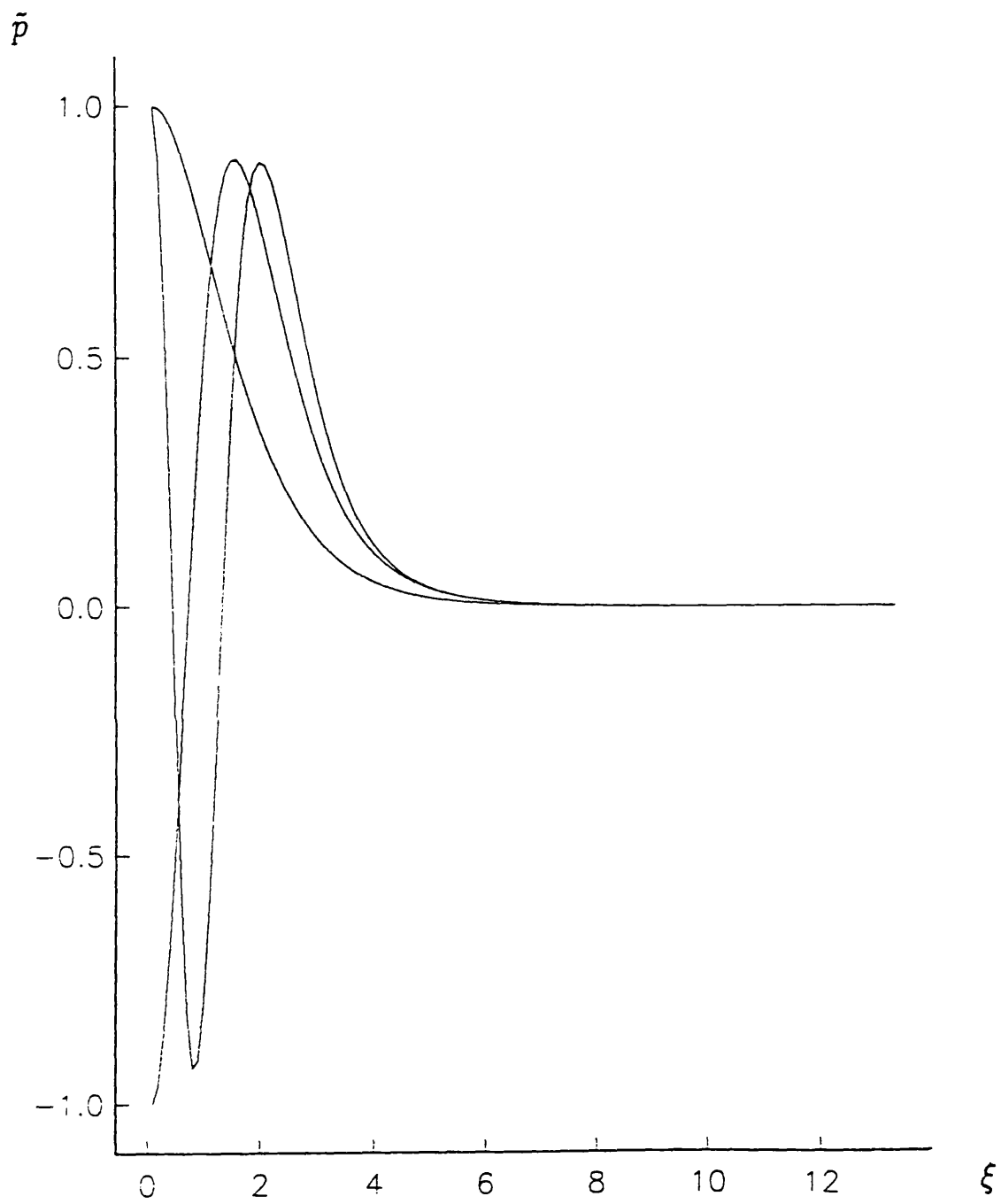


Figure 6.6. The first three acoustic mode eigenfunctions.

The first three eigenfunctions associated with this sequence are shown in Figure 6.6. These eigenvalues indicate the existence of acoustic modes that are the counterpart of those discussed by CH for a Chapman-law fluid. We conclude that at a countable discrete set of points the acoustic modes emerge a special case of the vorticity mode analysis; an outline of the structure of the vorticity mode in the vicinity of an acoustic mode is given in BCH.

In fact it is possible to seek acoustic modes with $(\alpha, \beta) = O(1)$, i.e. with wavenumbers comparable with those chosen in §6.3.1. The eigenfunctions for these modes are again concentrated in the $\xi = O(1)$ wall layer, but they now have a fast variation in this layer which can be described using the WKB method. At certain values of M_∞ these eigenvalues coalesce with the neutral vorticity mode discussed in §6.3.1. An analysis outlined in CH (see also the earlier study by DiPrima & Hall, 1984, of the Taylor problem), and developed in full in SB, can be performed to describe the ‘splitting’ of the eigenvalues in this region. We do not pursue this calculation here. In BCH the main concern is with completing a discussion of the structure of the vorticity mode at all lengthscales, to see if the vorticity mode connects with another neutral state.

To summarise, the main significance of the $(\alpha, \beta) = O(M_\infty^{-3/2})$ asymptotic regime is that it is the stage at which the acoustic modes emerge. However, apart from asymptotically small regions close to the acoustic mode eigenvalues, the small wavenumber structure developed initially in §6.3.1 for $\hat{k} \ll 1$, see (6.3.3), (6.3.9), (6.3.20) and (6.3.24), survives this regime largely intact. We conclude that the *scaled* departure of the complex wavespeed from the unit freestream velocity increases without bound as the wavenumber decreases to zero, thus suggesting that another asymptotic regime will develop. This new regime, where phase and free-stream velocities differ by the order of the sound speed, is discussed in BCH along with two further asymptotic regimes. They found that: (a) the fastest growing modes have $(\alpha, \beta) = O(1)$ and $\text{Im}(\omega) = O(M_\infty^{-2})$, (b) the two-dimensional modes whose influence extends furthest from the boundary have $\alpha = O(M_\infty^{-\frac{7}{4}})$, $\text{Im}(\omega) = O(M_\infty^{-\frac{11}{4}})$ and an $O(M_\infty^{\frac{7}{4}})$ scale normal to the boundary, and (c) the modes whose influence is felt furthest from the boundary are the three-dimensional, highly oblique, lower-branch Tollmien-Schlichting waves studied by Zhuk & Ryzhov (1981), Ryzhov (1984), Smith (1989), Duck (1990), CH and the

present author in Chapters 2 and 4. This completes our asymptotic description of quasi-parallel instability modes of basic flows far downstream of any leading-edge effects.

§6.4. THE INVISCID INSTABILITY PROBLEM IN INTER- ACTIVE BOUNDARY LAYERS

We consider now the hypersonic flow of a Sutherland-formula fluid past an aligned semi-infinite flat plate where leading-edge effects cannot be neglected. We assume that the plate has a sharp leading edge, attached to which is a shock that acts as an upper boundary for disturbances. The steady flow beneath the shock has been studied by Stewartson (1955, 1964), Bush (1966) and others. Apart from some minor differences, our formulation closely follows this earlier work, and so the reader is referred there for a detailed formulation. Only the parts of the solution that we require are outlined below.

In order to obtain a specific formulation, a choice of viscosity law must be made. As indicated in the introduction, mathematical simplifications sometimes arise with the choice of the model Chapman law, and this rather severe, linear approximation is still used in hypersonic shock/boundary-layer research. However, in interactive hypersonic flow no significant complications arise from the use of Sutherland's formula; this is primarily because the viscous layers are regions of high temperature where Sutherland's formula reduces to a power law, $\mu \propto T^{\tilde{\omega}}$, with $\tilde{\omega} = \frac{1}{2}$. The steady hypersonic flow of a general power-law fluid past a flat plate has been studied by Luniev (1959) and Bush (1966).

Figure 6.7 illustrates the distinct asymptotic regions that describe the different parts of the flow field beneath the shock. The lower region is hot and viscous, and of comparable thickness to the cooler, inviscid, upper region. Between these two layers is a thin viscous adjustment region, whose accurate description is vital to the correct formulation of the instability problem. We consider each of these regions in turn.

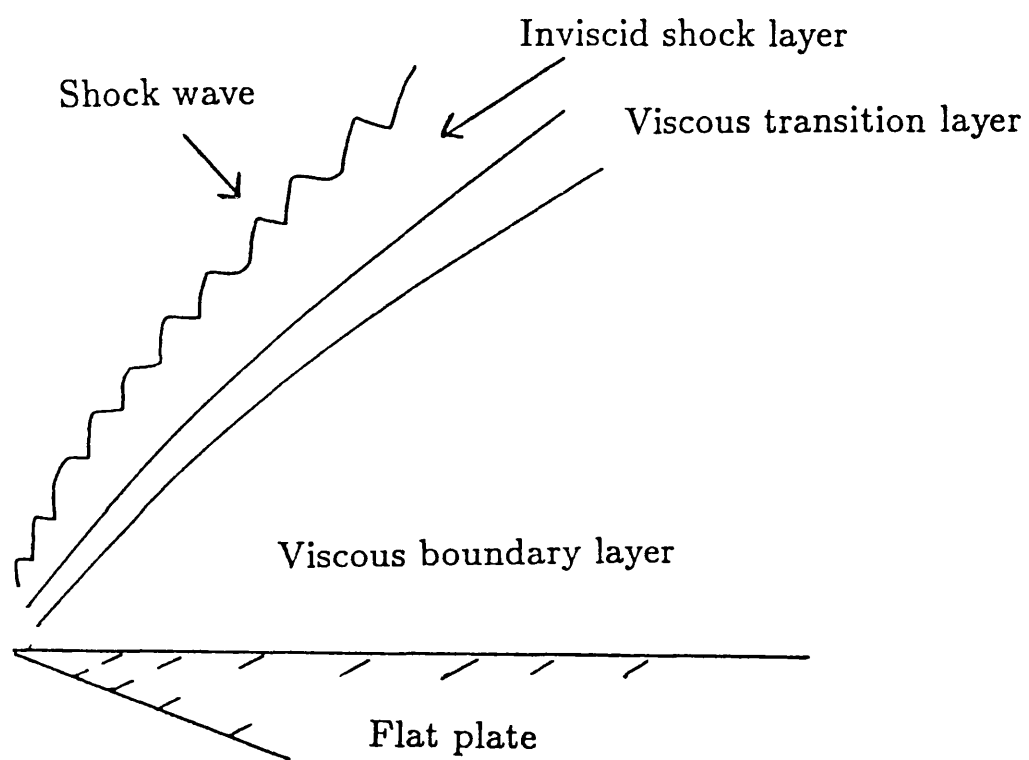


Figure 6.7. The different parts of the flow field in the strong interaction region.

§6.4.1 The upper inviscid region.

Since viscosity is negligible in this region, the choice of viscosity law does not alter the well-known governing equations. As in previous studies we assume that the flow is two-dimensional, and introduce the steady streamfunction ψ defined by

$$\rho u = \psi_y, \quad \rho v = -\psi_x. \quad (6.4.1a, b)$$

Then the velocity, temperature, density and pressure are expanded as

$$(u - 1, v, T, \rho, p) = \left(\frac{u_1}{M_\infty^2}, \frac{v_1}{M_\infty}, T_1, \rho_1, \frac{p_1}{M_\infty^2} \right) + \dots \quad (6.4.2a - e)$$

On substituting into (2.1.2), (2.1.3), and re-writing the equations in terms of Von-Mises coordinates using the scaled streamfunction

$$\tilde{\psi} = M_\infty \psi, \quad (6.4.2f)$$

the leading-order governing equations are found to be

$$v_{1x} = -p_1 \tilde{\psi}, \quad v_{1\tilde{\psi}} = \frac{\partial}{\partial x} \left(\frac{1}{\rho_1} \right), \quad p_1 = E(\tilde{\psi}) \rho_1^\gamma. \quad (6.4.3a - c)$$

The function $E(\tilde{\psi})$ can, in principle, be evaluated from the initial conditions for these hyperbolic equations. The latter are specified at the shock, which is taken to be at $y = M_\infty^{-1} F(x)$, for some unknown function F . Conservation of mass, and the Rankine-Hugoniot relations imply that at the shock

$$\begin{aligned} \tilde{\psi} = F, \quad v_1 &= \frac{2(F'^2 - 1)}{(\gamma + 1)F'}, \quad p_1 = \frac{(2\gamma F'^2 - \gamma + 1)}{\gamma(\gamma + 1)}, \\ \rho_1 &= \frac{(\gamma + 1)F'^2}{2 + (\gamma - 1)F'^2}. \end{aligned} \quad (6.4.4a - d)$$

§6.4.2 The upper inviscid region in the strong-interaction zone.

The solution of (6.4.3), (6.4.4), and the corresponding equations for the lower layers, can be investigated analytically for large and small x using expansion procedures, e.g. Stewartson (1955, 1964), Bush (1966), Bush & Cross (1967), Brown & Stewartson (1975). We will concentrate on the small- x solution valid very close to the leading edge of the plate, although equivalent expressions for large x can be found.

Since the shock is attached to the leading edge, for $x \ll 1$ we assume that $F \propto x^n$. Then a scaling argument based on (6.4.3), (6.4.4), together with the viscous equations (6.4.10-12), shows that if the pressure and normal velocity in the different asymptotic regions are to match then $n = \frac{3}{4}$ (e.g. Stewartson 1964, Bush 1966). The appropriate similarity solution is thus of the form:

$$F = a_1 x^{3/4} + \dots, \quad \tilde{\psi} = a_1 x^{3/4} \bar{\psi}, \quad (6.4.5a, b)$$

$$v_1 = a_1 x^{-1/4} \bar{v}_1(\bar{\psi}) + \dots, \quad p_1 = a_1^2 x^{-1/2} \bar{p}_1(\bar{\psi}) + \dots, \quad \rho_1 = \bar{\rho}_1(\bar{\psi}) + \dots. \quad (6.4.5c - e)$$

On substitution of (6.4.5) into (6.4.3, 4), it follows that

$$E(\tilde{\psi}) \sim e_1 a_1^{\frac{8}{3}} \tilde{\psi}^{-\frac{2}{3}} \quad \text{as} \quad \tilde{\psi} \rightarrow 0, \quad \text{where} \quad e_1 = \frac{9(\gamma - 1)^\gamma}{8(\gamma + 1)^{\gamma+1}}, \quad (6.4.6a, b)$$

and that

$$\bar{v}_1 + 3\bar{\psi}\bar{v}_1' = 4\bar{p}_1', \quad \bar{p}_1(\bar{\psi}) = \frac{e_1}{\bar{\psi}^{2/3}} \bar{\rho}_1^\gamma(\bar{\psi}), \quad \bar{v}_1' = \frac{3\bar{\psi}}{4\bar{\rho}_1^2} \bar{\rho}_1', \quad (6.4.7a - c)$$

where

$$\bar{v}_1 = \frac{3}{2(\gamma + 1)}, \quad \bar{p}_1 = \frac{9}{8(\gamma + 1)}, \quad \bar{\rho}_1 = \frac{\gamma + 1}{\gamma - 1} \quad \text{on} \quad \bar{\psi} = 1. \quad (6.4.7d - f)$$

For given γ a numerical solution can be found for $\bar{\psi} < 1$; in particular $\bar{p}_{10} = \bar{p}_1(0)$ and $\bar{v}_{10} = \bar{v}_1(0)$ can be evaluated. Note that $\bar{\rho}_1 \propto \bar{\psi}^{2/3\gamma}$ as $\bar{\psi} \rightarrow 0$, and thus we require $\gamma > \frac{2}{3}$ if the leading-order solution for v_1 is to have the form (6.4.5c); for a realistic gas this condition is satisfied.

The no-slip boundary condition is not satisfied by (6.4.2a), and thus there is at least one viscous sublayer beneath the present region. We consider next the viscous boundary layer which is immediately adjacent to the surface of the plate.

§6.4.3 The viscous boundary layer.

As is conventional the pressure does not vary significantly across this boundary layer, and it is appropriate to take $T = O(M_\infty^2)$ as in §6.2. It follows from the gas-law, (2.1.2d), and (6.4.1b), (6.4.2e) that $\psi = O(M_\infty^{-3})$, and so we introduce the scaled streamfunction

$$\Psi = M_\infty^3 \psi. \quad (6.4.8)$$

A balance between viscous and inertia forces then demonstrates that the hypersonic parameter, r , defined by

$$Re = rM_\infty^5, \quad (6.4.9)$$

should be taken to be order one (Luniev, 1959)[†]. Recall that the power of the Mach number is six in the definition of the hypersonic parameter for a linear viscosity law. The appropriate expansions of the flow quantities are

$$(u, v, p, \rho, T, \mu) = (U_1, \frac{1}{M_\infty} V_1, \frac{1}{M_\infty^2} P_1, \frac{1}{M_\infty^2} R_1, M_\infty^2 \theta_1, M_\infty \mu_1) + \dots, \quad (6.4.10a-f)$$

with $\mu_1 = (1 + S)\theta_1^{1/2}$.

In order to simplify the analysis, we now assume that the wall is an insulator. Since the Prandtl number is taken to be unity, the energy equation can then be integrated once to obtain

$$\theta_1 = \frac{\gamma - 1}{2}(1 - U_1^2). \quad (6.4.11)$$

We do not expect the relaxation of these assumptions to substantially alter our conclusions, but they allow us to explain better the effect of the shock on the inviscid modes.

The streamfunction is again adopted as an independent variable instead of y . The x -momentum equation in the boundary layer then becomes

$$U_1 U_{1x} = -\frac{(\gamma - 1)(1 - U_1^2)}{2} \frac{P_{1x}}{\gamma P_1} + \frac{\gamma P_1 U_1}{r} \left(\frac{\mu_1}{\theta_1} U_1 U_{1\Psi} \right)_\Psi, \quad (6.4.12)$$

where $P_1 \equiv P_1(x) = p_1(x, 0)$. The boundary conditions on the wall, and the matching conditions with the upper inviscid layer, yield

$$U_1 = 0 \quad \text{on} \quad \Psi = 0, \quad U_1 \rightarrow 1 + O(\Psi^{-4}) \quad \text{as} \quad \Psi \rightarrow \infty, \quad (6.4.13a, b)$$

and

$$v_1(x, 0) = \left(\frac{\gamma - 1}{2\gamma} \right) \frac{\partial}{\partial x} \left(\int_0^\infty \frac{1 - U_1^2}{P_1 U_1} d\Psi \right), \quad (6.4.14)$$

where we have anticipated the fact that the viscous adjustment region plays a passive role as far as leading-order matching is concerned.

[†] By a suitable redefinition of the the lengthscale L we could take $r = 1$ without loss of generality.

§6.4.4 The boundary-layer solution in the strong-interaction zone

For small x we introduce the similarity variable $\bar{\Psi}$, and the velocity function $\bar{U}(\bar{\Psi})$, defined by

$$\Psi = (4\gamma(1+S)\bar{p}_{10})^{1/2} \left(\frac{2}{\gamma-1} \right)^{1/4} \frac{a_1 x^{1/4}}{r^{1/2}} \bar{\Psi}, \quad U_1 = \bar{U}(\bar{\Psi}) + \dots \quad (6.4.15a, b)$$

The boundary-layer equation (6.4.13b) then takes the similarity form

$$-\bar{\Psi} \bar{U} \bar{U}' = \left(\frac{\gamma-1}{\gamma} \right) (1 - \bar{U}^2) + \bar{U} \left(\frac{\bar{U} \bar{U}'}{(1 - \bar{U}^2)^{1/2}} \right)', \quad (6.4.16)$$

with boundary conditions

$$\bar{U}(0) = 0, \quad \bar{U} = 1 - \frac{18\gamma^2}{(3\gamma-1)^2 \bar{\Psi}^4} + \dots \quad \text{as } \bar{\Psi} \rightarrow \infty. \quad (6.4.17a, b)$$

This is a modified Falkner–Skan equation where $\bar{U}$ decays algebraically rather than exponentially for large $\bar{\Psi}$. Substitution of (6.4.15) into (6.4.13) and use of (6.4.5) yields the leading-order coefficient a_1 in the small x expansion for the as yet unknown shock location:

$$a_1^2 = \frac{3(1+S)^{1/2}}{4\bar{v}_{10}} \left(\frac{2(\gamma-1)}{r\gamma\bar{p}_{10}} \right)^{1/2} \left(\frac{\gamma-1}{2} \right)^{1/4} \int_0^\infty \frac{1 - \bar{U}^2}{\bar{U}} d\bar{\Psi}. \quad (6.4.18)$$

The value $a_1^2 = 2.7842533$ was obtained numerically for our chosen parameter values: $S = 0.509$, $\gamma = 1.4$, $r = 1$.

§6.4.5 The viscous adjustment layer.

The existence of this layer can be seen by considering the limiting forms of the temperature at the edges of the upper and lower layers. First, from (2.1.2d) and (6.4.2), (6.4.3c) and (6.4.6) it follows that

$$T \sim T_1 \sim \gamma p_1(x, 0)^{\frac{\gamma-1}{\gamma}} \left(\frac{e_1^3 a_1^8}{\tilde{\psi}^2} \right)^{\frac{1}{3\gamma}} \quad \text{as } \tilde{\psi} \rightarrow 0. \quad (6.4.19)$$

Second, as the edge of the boundary layer is approached from below we see from (6.4.8), (6.4.10e), (6.4.11), (6.4.13b) that

$$T = M_\infty^2 \theta_1 \propto \frac{M_\infty^2}{\bar{\Psi}^4} \propto \frac{1}{M_\infty^6 \tilde{\psi}^4} \quad \text{as } \Psi \rightarrow \infty. \quad (6.4.20)$$

Since (6.4.19) and (6.4.20) do not match, there must be an intermediate asymptotic region where the scaled streamfunction, ζ , defined by

$$\zeta = \tilde{\psi} M_\infty^\lambda = \psi M_\infty^{\lambda+1}, \quad \text{with} \quad \lambda = \frac{9\gamma}{6\gamma-1}, \quad (6.4.21a, b)$$

is order one. Note that the value of λ follows from matching (6.4.19),(6.4.20), and further, that the power of M_∞ is a function of γ , the ratio of specific heat capacities!

Examination of the small $\tilde{\psi}$ limit of the inviscid solution implies that the appropriate expansions in the adjustment layer are

$$(u-1, v, T, p, \rho) = \left(\frac{1}{M_\infty^{\frac{12\gamma-8}{6\gamma-1}}} \tilde{U}_1, \frac{1}{M_\infty} v_1(x, 0), M_\infty^{\frac{6}{6\gamma-1}} \tilde{T}_1, \frac{1}{M_\infty^2} P_1, \frac{1}{M_\infty^{\frac{6}{6\gamma-1}}} \frac{\gamma P_1}{\tilde{T}_1} \right) + \dots, \quad (6.4.22a-e)$$

where we have anticipated the fact that the leading-order contributions to v and p are independent of ζ , and can hence be fixed by matching. When these expansions are substituted into the Navier-Stokes equations, and the energy equation is integrated once, we obtain in terms of Von-Mises co-ordinates

$$\tilde{U}_{1z} = -\frac{\tilde{T}_1 P_{1z}}{\gamma P_1} + \frac{\gamma(1+S)P_1}{r} \frac{\partial}{\partial \zeta} \left(\frac{1}{\tilde{T}_1^{\frac{1}{2}}} \frac{\partial \tilde{U}_1}{\partial \zeta} \right), \quad \tilde{T}_1 = (1-\gamma)\tilde{U}_1. \quad (6.4.23, 24)$$

Equation (6.4.23) is a perturbation form of (6.4.12), and hence in this adjustment layer the high temperature form of Sutherland's formula is still valid. This is in contrast with the shock-free adjustment layer where the full form occurred – see (6.2.9a).

Again, a similarity form of these equations can be found for $x \ll 1$. Using (6.4.5d),(6.4.15),(6.4.17b),(6.4.19) and (6.4.20) it follows that the appropriate similarity variable is s , defined by

$$s = \frac{x^{(\lambda-3)/4} \zeta}{b_1}, \quad \text{where} \quad b_1^2 = \frac{(1+S)\gamma a_1^2 \bar{p}_{10}}{r}. \quad (6.4.25a, b)$$

This is in agreement with Bush's (1966) result. Writing

$$\tilde{U}_1 = x^{\lambda-2} G(s) \quad (6.4.26a),$$

and using (6.4.5d), we find that the governing equation for the flow solution in this crucial region is

$$(\lambda - 2)G - \frac{(3 - \lambda)}{4}sG_s = \frac{(1 - \gamma)}{2\gamma}G + \left(\frac{G_s}{\sqrt{(1 - \gamma)G}} \right)_s. \quad (6.4.26b)$$

From (6.4.22a),(6.4.26a) we expect that $G < 0$.

As $s \rightarrow 0$ we find that G has the asymptotic behaviour

$$G = -G_0s^{-4} + G_1s^{-q} + \dots, \quad \text{where} \quad G_0 = \frac{576\gamma^2}{(\gamma - 1)(3\gamma - 1)^2}. \quad (6.4.27a, b)$$

Thus the adjustment-layer solution matches onto the large $\bar{\Psi}$ form of $\bar{U}$. The coefficient G_1 must be determined from a numerical solution, but the parameter q satisfies a known quadratic equation with coefficients which are functions of γ . With $\gamma = 1.4$ we find that $G_0 \simeq 275.6$ and $q \simeq 0.6267$, so that the correction terms are relatively small.

For large s we find that

$$G \rightarrow -A_0s^{-\frac{2}{3\gamma}} - \frac{8A_0^{\frac{1}{2}}(3\gamma + 1)}{9\gamma(\gamma - 1)^{\frac{1}{2}}(3\gamma - 1)}s^{-2-\frac{1}{3\gamma}} + \dots, \quad (6.4.28)$$

where from matching with the inviscid solution in the upper layer,

$$A_0 = \frac{a_1^2}{\gamma - 1} \left(\frac{\gamma^{3\gamma-1} r \bar{p}_{10}^{3\gamma-4} e_1^3}{(1 + S)} \right)^{\frac{1}{3\gamma}}. \quad (6.4.29)$$

For the choices $r = 1$, $\gamma = 1.4$ and $S = 0.509$, we find $A_0 \simeq 0.744528$. Note that G decays algebraically for large s , in contrast to the rapid exponential decay of the Blasius and the ‘Modified Blasius’ functions which arise in the shock-free, far-downstream cases (the former after employing the Chapman’s law, and the latter from the use of the more realistic Sutherland’s formula).

Now that the leading-order base flow for this region has been identified, we can consider its stability characteristics. In particular we are interested in the linear stability of inviscid modes concentrated (trapped) within this adjustment layer; these are considered in the next subsection.

§6.4.6 The vorticity mode in the strong-interaction zone.

The scalings for these modes appear complicated but follow in a straightforward manner after applying the usual vorticity mode arguments to the flow field discussed above. In particular, the modes should have wavelengths comparable with the physical thickness of the adjustment region, because that is where the generalised inflection point occurs. A Rayleigh analysis using the streamfunction as normal co-ordinate, rather than the related Dorodnitsyn-Howarth variable, suggests that we require $\partial_{\psi\psi} \sim T^2 \check{\alpha}^2$, where $\check{\alpha}$ is the streamwise wavenumber non-dimensionalised using L . From (6.4.21),(6.4.22c) we have $T \sim M_{\infty}^{4\lambda-6}$ and $\psi \sim M_{\infty}^{-\lambda-1}$. We deduce that $\check{\alpha} \sim M_{\infty}^{7-3\lambda} \gg 1$, which indicates that this is a short wavelength mode. The time-scale can be deduced by using the fact that vorticity modes propagate in a moving frame whose velocity is approximately equal to that of the fluid in the adjustment layer – see (6.4.22a). The appropriate ‘fast’ space and time scales are thus

$$X = M_{\infty}^{7-3\lambda}(x - t), \quad Z = M_{\infty}^{7-3\lambda}z, \quad \tau = M_{\infty}^{\lambda-1}t, \quad (6.4.30a - c)$$

which lead to the leading-order multiple-scales transformations

$$\partial_x \rightarrow \partial_x + M_{\infty}^{7-3\lambda} \partial_X, \quad \partial_z \rightarrow M_{\infty}^{7-3\lambda} \partial_Z, \quad \partial_t \rightarrow M_{\infty}^{\lambda-1} \partial_{\tau} - M_{\infty}^{7-3\lambda} \partial_X. \quad (6.4.31a, b)$$

Note that non-parallel effects are $O(\partial_x) \sim O(1)$, and are thus negligible in comparison with the direct growth effects of magnitude $\check{\alpha}(u - 1) = O(M_{\infty}^{\lambda-1}) \gg 1$ ($\lambda = 1.7027$ for our choice of $\gamma = 1.4$). To be consistent with (6.4.30,31) the wavespeed is expanded as

$$c = 1 + \frac{\hat{c}}{M_{\infty}^{4(2-\lambda)}} + \dots \quad (6.4.32)$$

Now that the scales have been deduced, the remainder of the analysis follows the classical inviscid mode approach for formulating the pressure equation describing linear wave-like disturbances. We perform a normal-mode analysis, assume that the infinitesimal pressure disturbance, $\tilde{p}$, is such that

$$\tilde{p} \propto e^{i(\alpha X + \beta Z - \alpha \hat{c} \tau)}, \quad (6.4.33a)$$

and that in the adjustment layer the perturbation velocities, temperature and density scale such that

$$(\tilde{u}, \tilde{v}, \tilde{w}) = O(M_\infty^2 \tilde{p}) , \quad \tilde{T} = O(M_\infty^4 \tilde{p}) , \quad \tilde{\rho} = O(M_\infty^{16-8\lambda} \tilde{p}) . \quad (6.4.33b)$$

After some manipulation, and assuming that $\gamma < \frac{8}{3}$, we obtain a simplified Rayleigh equation, the ‘vorticity mode equation’, for the amplitude of the disturbance pressure, $\hat{p}$:

$$\hat{p}_{\zeta\zeta} - \frac{2\tilde{U}_{1\zeta}}{\tilde{U}_1 - \hat{c}} \hat{p}_\zeta - \frac{(\gamma - 1)^2 \tilde{U}_1^2}{\gamma^2 P_1^2} k^2 \hat{p} = 0 , \quad (6.4.34)$$

where $k^2 = \alpha^2 + \beta^2$, and $\hat{p}$ decays to zero for large and small ζ . This equation describes short-wavelength vorticity modes at any downstream location x in the interaction zone. However, at present numerical solutions of $\tilde{U}_1$ for $x = O(1)$ are not available, and so we again consider the strong-interaction limit $x \ll 1$.

From (6.4.5d),(6.4.25),(6.4.26a) the appropriate small- x dependences for $\hat{c}$ and k are

$$\hat{c} = x^{\lambda-2} \mathcal{C} , \quad k = \frac{\gamma a_1^2 \tilde{p}_{10}}{b_1(\gamma - 1) x^{3(\lambda-1)/4}} \mathcal{K} . \quad (6.4.35a, b)$$

These lead to the following vorticity-mode, pressure-amplitude equation in the strong-interaction zone (cf. (6.3.4a)):

$$\hat{p}_{ss} - \frac{2G_s}{G - \mathcal{C}} \hat{p}_s - \mathcal{K}^2 G^2 \hat{p} = 0 . \quad (6.4.36)$$

This is to be solved subject to $\hat{p}$ vanishing in the limits $s \rightarrow 0$ and $s \rightarrow \infty$, so that the disturbance is again confined to the adjustment layer. The leading-order asymptotes are found to be

$$\hat{p} \rightarrow \hat{p}_0 s^{-2} \exp\left(-\frac{G_0 \mathcal{K}}{3s^3}\right) \quad \text{as } s \rightarrow 0 , \quad (6.4.37a)$$

$$\hat{p} \rightarrow \hat{p}_\infty s^{\frac{1}{3\gamma}} \exp\left(-\frac{3\gamma A_0}{3\gamma - 2} \mathcal{K} s^{\frac{(3\gamma-2)}{3\gamma}}\right) \quad \text{as } s \rightarrow \infty \quad (6.4.37b)$$

where $\hat{p}_0$ and $\hat{p}_\infty$ are constants. Higher order terms in these expressions can be found analytically, and are needed for accurate numerical solutions. We discuss our numerical results, and the asymptotic solution for small wavenumber, in the next section.

§6.5 THE SOLUTION OF THE STRONG-INTERACTION VORTICITY-MODE EQUATION.

§6.5.1 Numerical results.

First we consider the neutral mode; $\mathcal{C}$ is then real and equal to G evaluated at the generalized inflection point where

$$GG_{,s} = 2G_s^2. \quad (6.5.1a)$$

A numerical solution to (6.4.26b) using a Runge-Kutta method shows that this occurs when $\hat{s} \simeq 1.661432$, where the new variable $\hat{s} = \ln s$ is introduced to stretch the co-ordinate in the small- s region where G and $\hat{p}$ vary rapidly. The resulting neutral value of the wavespeed is $\mathcal{C} \simeq -0.633318$. The corresponding real wavenumber $\mathcal{K}$ is obtained from a numerical solution of (6.4.36) using a method similar to that outlined in §6.3.1. The neutral wavenumber was calculated to be $\mathcal{K} \simeq 0.477957$, whilst the corresponding value of $\mathcal{C}$ corresponds with that predicted from the inflexion-point criterion.

Figure 6.8 shows the growth rate for two-dimensional waves, $\text{Im}(\mathcal{K}\mathcal{C})$, plotted against $\mathcal{K}$ for $\gamma = 1.4$, $S = 0.509$, $r = 1$. Observe that the maximum growth rate, $\text{Im}(\mathcal{K}\mathcal{C}) \simeq 0.060918$, occurs at $\mathcal{K} \simeq 0.156100$, and that the growth rate goes to zero as the wavenumber goes to zero. Figure 6.9 illustrates the eigenfunction of the most unstable mode, including the exponential decay of the eigenfunction at both ends of the range of integration.

We now deduce the asymptotic structure of the strong-interaction inviscid mode as the wavenumber tends to zero, for the present scaling. This is sufficient to illustrate an important dependence on γ .

§6.5.2 The small- $\mathcal{K}$ behaviour.

Numerical solutions of the pressure-amplitude equation, (6.4.36), indicate that for $\gamma = 1.4$, $\mathcal{C}$ increases as $\mathcal{K}$ tends to zero. Similar behaviour was found for the far-downstream problem studied in §6.3. Again the leading-order dependence on the wavenumber was not indicated from the numerical solutions, whereas such guidance was not possible here due to a complicated dependence of $\mathcal{C}$ on $\mathcal{K}$.

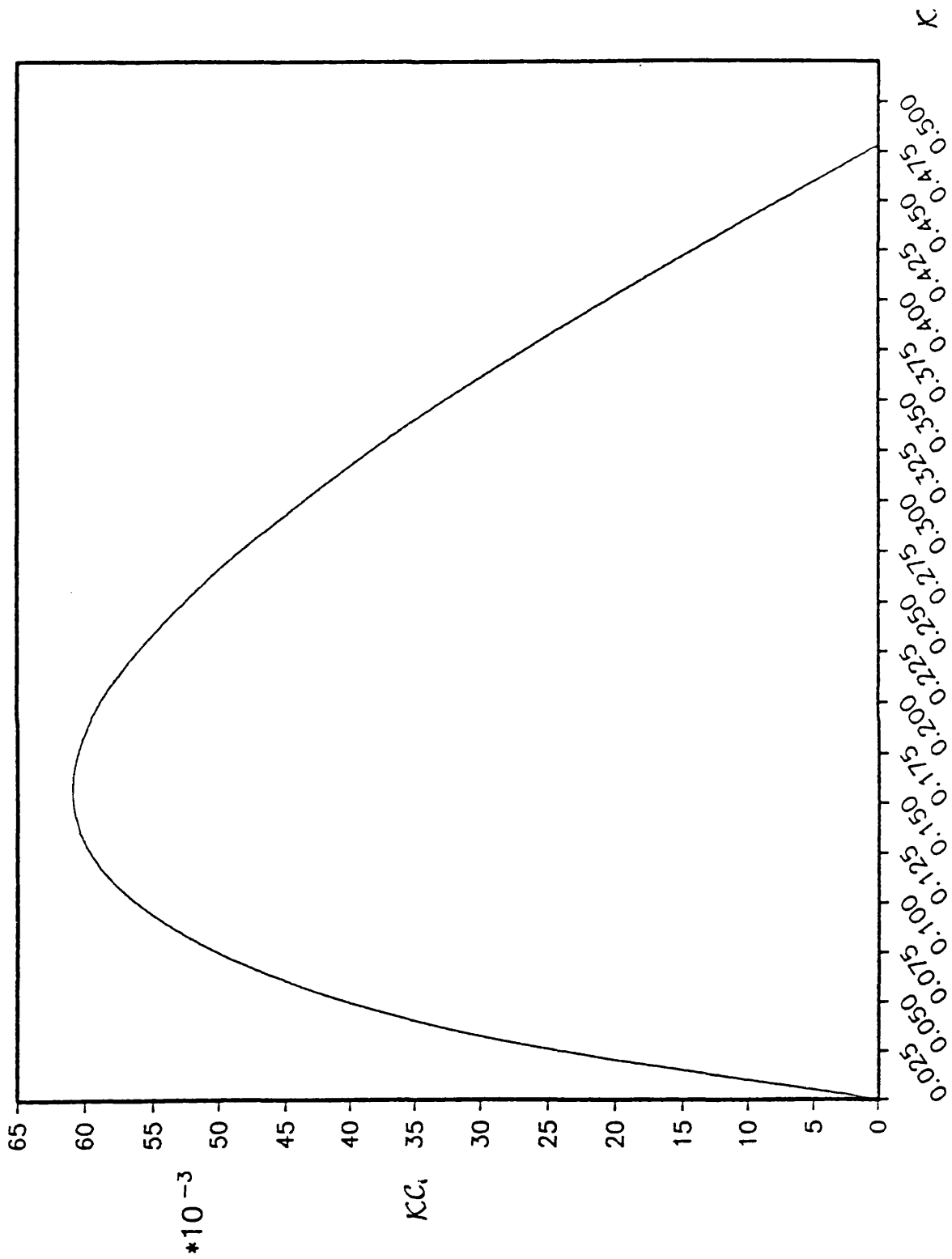


Figure 6.8. The growth rate of the vorticity mode in the strong interaction region.

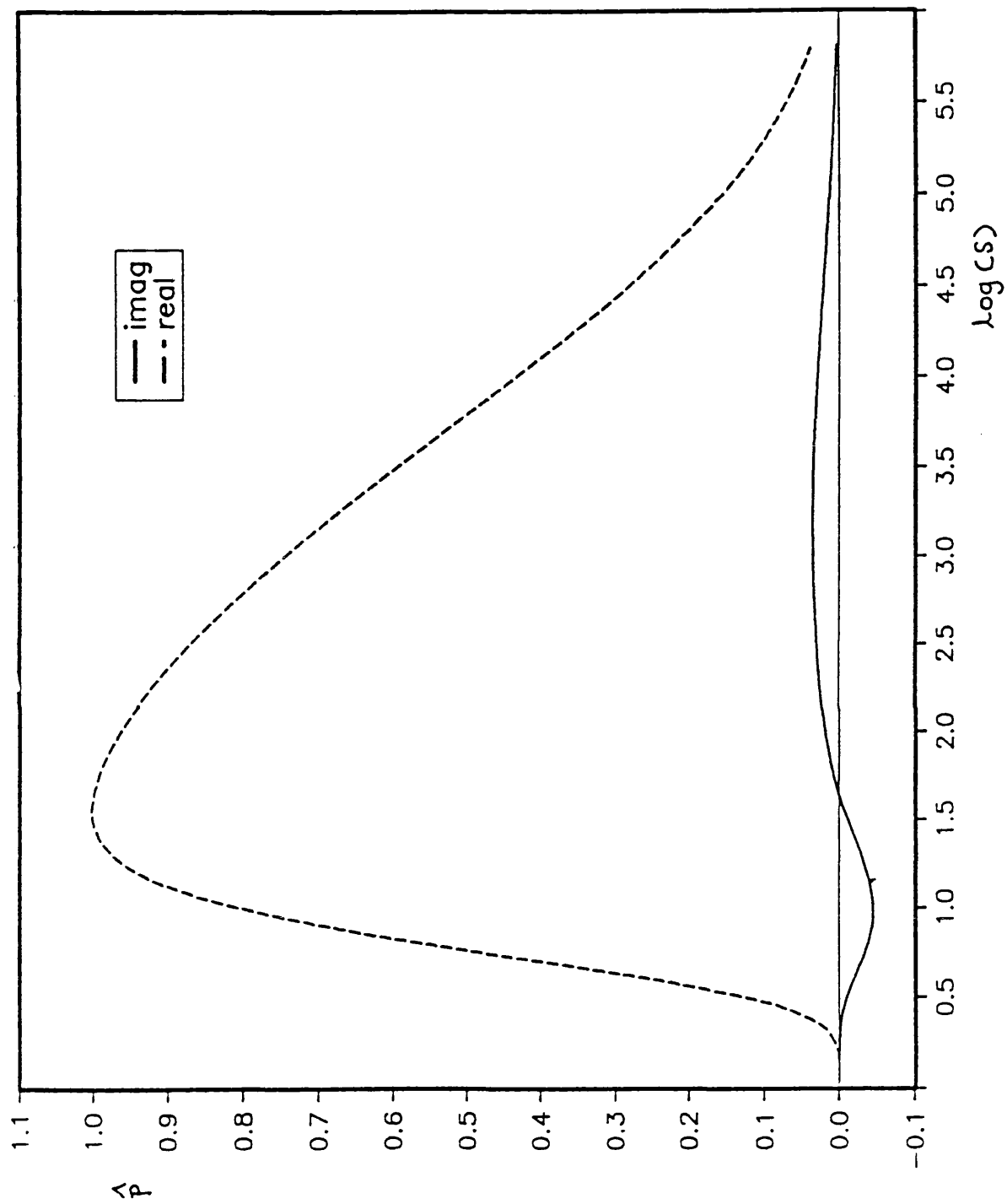


Figure 6.9. The eigenfunction of the most dangerous mode in the strong interaction limit.

As an alternative we note that although the WKB asymptote (6.4.37b) is valid if s is sufficiently large, for small $\mathcal{K}$ this form breaks down in a ‘turning-point’ region where

$$s \sim \mathcal{K}^{-\frac{3\gamma}{3\gamma-2}} \gg 1 . \quad (6.5.2)$$

In this region we define

$$\hat{y} = \mathcal{K}^{\frac{3\gamma}{3\gamma-2}} s, \quad \hat{p} = \hat{P}_0(\hat{y}) + \dots . \quad (6.5.3a, b)$$

Using the large- s asymptotes for G , (4.28), it follows that $\hat{P}_0$ satisfies

$$\hat{P}_0 \hat{y} \hat{y} - A_0^2 \hat{y}^{-\frac{4}{3\gamma}} \hat{P}_0 = 0 . \quad (6.5.4)$$

This equation for $\hat{P}_0$ has an analytic solution involving the modified Bessel function K_ν :

$$\hat{P}_0 = (4A_0\nu)^\nu \bar{D}_0 \hat{y}^{\frac{1}{2}} K_\nu(2A_0\nu \hat{y}^{\frac{1}{2\nu}}) , \quad (6.5.5a)$$

where

$$\nu = \frac{3\gamma}{2(3\gamma-2)} , \quad (6.5.5b)$$

and $\bar{D}_0$ is an arbitrary constant. From the series expansion of the modified Bessel function, e.g. Abramowitz & Stegun (1964), we have that as $\hat{y} \rightarrow 0$

$$\hat{P}_0 = \frac{2^{2\nu-1}\pi\bar{D}_0}{\Gamma(1-\nu)\sin(\nu\pi)} \left(1 - \frac{\Gamma(1-\nu)(A_0\nu)^{2\nu}}{\Gamma(1+\nu)} \hat{y} + \frac{A_0^2\nu^2}{(1-\nu)} \hat{y}^{\frac{1}{\nu}} + \dots \right) , \quad (6.5.6)$$

where Γ is the Gamma function. The ordering of the second and third terms is dependent on the value of γ . For $\gamma = \frac{4}{3}$, i.e. $\nu = 1$, the powers of $\hat{y}$ of these terms are equal, while their coefficients are singular; this indicates the presence of a logarithmic term which requires special treatment (see later).

As $\hat{y} \rightarrow 0$ the expansion (5.3) continues to be valid until contributions from the term proportional to $\hat{p}_s$ in (6.4.36) become significant. The scaling in this region can be deduced by analogy with the corresponding analysis of §6.3.1. In particular we expect $\mathcal{C} \gg 1$ as $\mathcal{K} \rightarrow 0$, and that $s \sim \mathcal{C}^{-1/4}$ in the new asymptotic region (see (6.4.27),(6.4.36)). The appropriate scalings are

$$\bar{y} = \hat{\delta}^{-1/4} s , \quad \hat{p} = \frac{2^{2\nu-1}\pi\bar{D}_0}{\Gamma(1-\nu)\sin(\nu\pi)} (1 + \mathcal{K}^2 \hat{\delta}^{-3/2} \bar{p}_1(\bar{y}) + \dots) , \quad \mathcal{C} = \hat{\delta}^{-1} \hat{\mathcal{C}} + \dots , \quad (6.5.7a - c)$$

where $\hat{\delta} \ll 1$, $\mathcal{K}^2 \hat{\delta}^{-3/2} \ll 1$, and the form of the expansion for $\hat{p}$ follows from the condition that (6.4.36) simplifies to an equivalent equation to (6.3.18). Substitution of (6.5.7) into (6.4.36) and use of (6.4.27) yields

$$\bar{p}_{1\bar{y}} + \frac{8G_0}{\bar{y}(G_0 + \hat{C}\bar{y}^4)} \bar{p}_{1\bar{y}} = \frac{G_0^2}{\bar{y}^8}. \quad (6.5.8)$$

As before this can be integrated to give

$$\bar{p}_{1\bar{y}} = \left(\hat{C} + \frac{G_0}{\bar{y}^4} \right)^2 \left(\int_{\infty}^{\bar{y}} \frac{G_0^2 d\bar{y}}{(\hat{C}\bar{y}^4 + G_0)^2} + \bar{D}_1 \right), \quad (6.5.9)$$

where the constant $\bar{D}_1$ is fixed by matching to the $\hat{y} = O(1)$ region. A straightforward match is possible only if $\gamma > \frac{4}{3}^\dagger$, and consistent choices of $\hat{\delta}$ and $\bar{D}_1$ are then

$$\hat{\delta}^{\frac{7}{4}} = \frac{\Gamma(1+\nu)}{\Gamma(1-\nu)\nu^{2\nu}} \mathcal{K}^{\frac{3\gamma-4}{3\gamma-2}}, \quad \bar{D}_1 = -\frac{A_0^{2\nu}}{\hat{C}^2}. \quad (6.5.10a, b)$$

The eigenrelation for $\hat{C}$ is fixed by considering a further region where $s = O(\mathcal{K}^{1/3})$. The details are identical to those of region IV in §6.3.1, and lead to the condition that $\bar{p}_{1\bar{y}} \rightarrow 0$ as $\bar{y} \rightarrow 0$. From (5.9, 10b), the eigenrelation for $\hat{C}$ is thus (cf. (3.13))

$$\int_0^\infty \frac{G_0^2 d\bar{y}}{(\hat{C}\bar{y}^4 + G_0)^2} + \frac{A_0^{2\nu}}{\hat{C}^2} = 0 \quad \Leftrightarrow \quad \hat{C} = \left(\frac{8\sqrt{2}A_0^{2\nu}}{3\pi G_0^{1/4}} \right)^{\frac{4}{7}} \exp\left(\frac{4i\pi}{7}\right). \quad (6.5.11a, b)$$

This corresponds to an unstable mode of (4.36) such that the complex frequency, $\mathcal{CK}$, tends to zero as $\mathcal{K}$ decreases.

The above derivation only holds if $\gamma > \frac{4}{3}$. If $\gamma = \frac{4}{3}$, then $\nu = 1$, and the small $\hat{y}$ series (6.5.6) is replaced by a formula including logarithms. We now consider the more general case, where

$$\gamma = \frac{4}{3}(1 + \epsilon), \quad |\epsilon| \ll 1. \quad (6.5.12)$$

This small interval, rather than just being of mathematical interest, is, in the opinion of the author, the most relevant on physical grounds. Note that for the standard value, $\gamma = 1.4$, $\epsilon = 0.05$ which certainly represents a small perturbation. Moreover, γ is known to decrease (slightly) as the temperature increases to values typically found in hypersonic boundary-layers, its value being about

[†] in which case the ordering of (6.5.6) is as shown.

1.32 ($\leftrightarrow \epsilon = -0.01$) at $2,000^\circ K$ and 1.29 ($\leftrightarrow \epsilon = -0.0325$) at $4,000^\circ K$ (see page 9 of Stewartson, 1964; and reference therein).

The previous analysis, (6.5.2)-(6.5.6), is still appropriate; now

$$\nu = 1 - \epsilon + O(\epsilon^2),$$

so that (6.5.6) takes the form

$$\hat{P}_0 = 2\bar{D}_0 \left(1 + A_0^2 \hat{y} \left[\frac{\hat{y}^\epsilon - 1}{\epsilon} \right] + O(\epsilon) \right), \quad (6.5.13)$$

at leading order. Note that, by applying L'Hôpital's rule, one can easily deduce that

$$\lim_{\epsilon \rightarrow 0} \left[\frac{\hat{y}^\epsilon - 1}{\epsilon} \right] = \ln \hat{y}.$$

Returning to general $|\epsilon| \ll 1$, we see immediately from (6.5.13) that

$$\hat{P}_{0\hat{y}} \simeq 2\bar{D}_0 A_0^2 \left(\left[\frac{\hat{y}^\epsilon - 1}{\epsilon} \right] + \hat{y}^\epsilon \right), \quad \text{as } \hat{y} \rightarrow 0. \quad (6.5.14)$$

Re-writing this last result in terms of s , noting that (6.5.3a) gives $\hat{y} \simeq \mathcal{K}^{2-\epsilon}s$, yields

$$\hat{P}_{0s} \simeq \mathcal{K}^{2-\epsilon} 2\bar{D}_0 A_0^2 \left(\left[\frac{\mathcal{K}^{(2-\epsilon)\epsilon} - 1}{\epsilon} \right] + 1 + \dots \right), \quad (6.5.15)$$

where only the leading-order constant term, of interest, has been highlighted.

The remainder of the argument follows that for the $\gamma > \frac{4}{3}$ case — again we suppose that $\mathcal{C} \gg 1$ as $\mathcal{K} \rightarrow 0$; (6.5.7b) is replaced by

$$\hat{p} = 2\bar{D}_0 (1 + \mathcal{K}^2 \hat{\delta}^{-\frac{3}{2}} \bar{p}_1(\bar{y}) + \dots) \quad (6.5.16)$$

and $\bar{p}_{1\bar{y}}$ is as given by (6.5.9). Matching constant terms, as $\hat{y} \rightarrow 0$ and $\bar{y} \rightarrow \infty$, in (6.5.15) and (6.5.16) respectively, gives

$$\mathcal{K}^2 \hat{\delta}^{-\frac{3}{2}} \hat{\delta}^{-\frac{1}{4}} \mathcal{C}^2 \bar{D}_1 = \mathcal{K}^{(2-\epsilon)} A_0^2 \left(\left[\frac{\mathcal{K}^{2-\epsilon\epsilon} - 1}{\epsilon} \right] + \dots \right).$$

We choose $\bar{D}_1 = -A_0^2/\mathcal{C}^2$, where $\mathcal{C}$ takes the same value, (6.5.11b), as before. Thus

$$\hat{\delta}^{-\frac{7}{4}} \simeq -\mathcal{K}^{-\epsilon} \left(\left[\frac{\mathcal{K}^{2-\epsilon\epsilon} - 1}{\epsilon} \right] + \dots \right). \quad (6.5.17)$$

It can easily be verified that $\hat{\delta} \ll 1$, $\leftrightarrow \mathcal{C} \gg 1$, for $\mathcal{K} \ll 1$. When $\epsilon \rightarrow 0 \leftrightarrow \gamma \rightarrow \frac{4}{3}$, we see that

$$\hat{\delta}^{-\frac{7}{4}} \simeq -(\ln \mathcal{K}^2) \gg 1. \quad (6.5.18)$$

Thus, for physically relevant values of $\gamma \sim \frac{4}{3}$, we see that the small- $\mathcal{K}$ asymptote for $\mathcal{C}$ has a complicated form which limits further analytic investigations for even smaller $\mathcal{K}$ -values. The case of $\gamma < \frac{4}{3}$ is not considered here; although it is of undoubted mathematical interest, the author has reservations concerning the physical relevance of these values for the ratio of specific heat capacities.

§6.6 DISCUSSION.

§6.6.1 General discussion.

We have investigated the instability of flat plate hypersonic boundary layers to the vorticity mode of instability. This inviscid mode is associated with the generalized inflection point of the basic flow and is thought to be the most dangerous mode of instability of a high Mach number flow. When the mode is neutral the wave propagates downstream with the speed of the fluid at the generalized inflection point. At wavenumbers smaller than the neutral value the mode is unstable and the growth rate attains its maximum value at a finite value of the wavenumber. In the small wavenumber limit the growth rate approaches zero and for the non-interactive boundary layer at sufficiently small wavenumber the vorticity mode spreads out towards the lower boundary and reduces to an acoustic mode at a countable infinite set of wavenumbers. We believe that a similar process happens in the strong-interaction case since there the acoustic mode is correctly described by a quasi-parallel theory there. We did not pursue that calculation here because it would be essentially unchanged from that of §6.3 except that it would be made somewhat more complicated by the necessity of treating the case $\gamma = 4/3$ as a special case in the strong-interaction zone.

We believe that the results we have presented in §6.4 are the first which show the effect of a leading-edge shock on any form of hydrodynamic instability. Interestingly enough the shock does not have a direct influence on the vorticity mode; thus the main effect of the shock is to restructure the boundary layer in the leading-edge region and thereby influence the susceptibility of the flow to inviscid disturbances.

The vorticity mode eigenvalue problem was formulated in the interactive region along the plate at $O(1)$ values of x . However to the authors' knowledge the

basic flow in this regime has not yet been calculated; the numerical problem was set up by Bush (1966) but is sufficiently difficult to have remained unsolved†. Thus we were unable to solve the eigenvalue problem in this regime and therefore choose to consider the strong-interaction regime where a similarity solution for the basic state is available. An alternative to that limit would have been to consider the weak-interaction problem, Bush & Cross (1967), where a different similarity structure holds. We choose to concentrate on the strong-interaction limit because the growth rates there are bigger and if the flow is indeed unstable there the stability of the flow further downstream is possibly of less relevance.

Unfortunately we are unaware of any experimental observations or other theoretical work which we could compare with our results for the strong-interaction regime.

In §6.3 we showed how the acoustic inviscid mode emerges from the small wavenumber description of the vorticity mode. Again it is not possible for us to compare our work with that of previous authors since it appears that the finite Mach number calculations available, mostly due to Mack, have either been carried out using a Chapman viscosity law or a combination of Sutherland's formula and Chapman's law. In fact Mack's calculations were carried out using a combination of the different laws so as to efficiently model the viscosity-temperature structure of the fluid. The fact that the calculations of CH and SB agree so well with Mack's calculations suggests that over the part of the flow where instability took place Chapman's law was being used; in the case of the vorticity mode this is clearly a bad approximation because the mode locates itself in the layer where the basic temperature field varies rapidly.

Further work called for includes: the generalisation to $Pr \neq 1$ (see §6.6.2); the inclusion of real gas effects; a study of the properties of the vorticity-mode equation away from the strong-interaction region; and the generalisation to different geometries, i.e. the very slender axisymmetric bodies studied by Cross & Bush (1969), Bush (1970). The link between the inviscid modes and the viscous Tollmien-Schlichting modes remains to be firmly established; also the extension of the theory to incorporate larger, nonlinear disturbances calls for further study.

† the corresponding problem for Chapman's viscosity law has recently been solved by Brown, Khorrami, Neish & Smith (1991).

In the next Chapter we consider the ‘co-existence’ of inviscid modes and Görtler vortices in (non-interactive) hypersonic boundary-layers; we are particularly interested in investigating the possibility of interactions between them — in addition to the inviscid modes being possible ‘secondary instabilities’ to the mean flow generated by nonlinear Görtler vortices.

Further to the comments made in a previous paragraph concerning the comparison of our results, for non-interactive flows, with the finite Mach number calculations; the author feels that there is a need for such comparisons of like flows i.e. closer links between ‘theoreticians’ and those running general computer-codes for predicting transition-properties. Unfortunately, at present most of the large Mach number analytical theories are for unrealistic flows (so that such theories are simplified — see, for example, the discussion in §6.6.2) whereas the latter codes can in general incorporate, and are usually run for, more realistic flow conditions. Whilst writing this chapter, the author was most grateful to receive a letter (Mack, 1990) from Prof. L.M. Mack (of the J.P.L., California Inst. Tech.) detailing the interesting results that he had obtained from re-running his code, after reading the closely related study, BCH. Based on his newly calculated stability results, he concludes that, at *finite* (but large) values of the Mach number, the flow in the far-downstream region will have its maximum inviscid instability caused by an acoustic mode, not a vorticity mode.

§6.6.2 Some comments on the numerical solutions: why the Prandtl number was chosen to be unity.

Rather than cloud the previous sections with details of the ‘obstacles’ encountered in the calculation of the numerical values quoted therein, such a discussion has been postponed until now; the motivation for such a discussion, at all, follows. Firstly, much time was spent in solving the numerous equations (for the base-flow profiles and for the eigenvalues) — even though these equations are ‘only’ ordinary differential equations, their numerical solution was found to require much more thought and care than, for instance, for Blasius’ equation or Rayleigh’s equation for $O(1)$ Mach numbers. Secondly, there is a need to ‘justify’ our choice of $Pr = 1$ — or, at least, to illustrate why such a choice is attractive. The underlying cause of these difficulties is, essentially, due to the choice of Sutherland’s formula

(to relate viscosity to temperature) which leads to coupled, nonlinear equations for the base-flow profiles. The author wishes to stress that, despite these difficulties, he is very confident in the accuracy of the numerical results which have been presented within this chapter.

We now outline some of the difficulties encountered during the numerical solutions. Consider first, the similarity solution to non-interactive steady flow, first considered in §6.2. For general-Mach-number flows, equations (6.2.4b,c) must be solved for the base-flow profiles. We see that they are coupled — if the linear Chapman viscosity law had been used then the equations would not be coupled; moreover, the equation for f is then simply Blasius' (i.e. it is very easy to solve numerically) and T is simply computed from f , using an analytical solution. Returning to (6.2.4b,c), their solution for $O(1)$ Mach numbers should not cause too many problems — however, their high Mach number form certainly do.

In the high temperature region (we are now considering $M_\infty \gg 1$) we have previously written

$$f = M_\infty^{-\frac{1}{2}} f_0(\xi) + \dots, \text{ where } \xi = M_\infty^{\frac{1}{2}} \eta \sim O(1); \quad (6.6.1a, b)$$

here we additionally need to consider the temperature equation and write

$$T = M_\infty^2 T_0(\xi) + \dots . \quad (6.6.1c)$$

These lead to the high-temperature form of the similarity equations

$$\begin{aligned} \frac{1}{2} f_0 f_{0\xi\xi} + \left(\frac{f_{0\xi\xi}}{T_0^{\frac{1}{2}}} \right)_\xi &= 0, \\ \frac{1}{2} f_0 T_{0\xi} + \frac{1}{Pr} \left(\frac{T_{0\xi}}{T_0^{\frac{1}{2}}} \right)_\xi + \frac{(\gamma-1)}{T_0^{\frac{1}{2}}} f_{0\xi\xi}^2 &= 0, \end{aligned} \quad (6.6.2a, b)$$

which must be solved numerically, subject to the standard boundary conditions on $\xi = 0$; our concern is with the large- ξ behaviour. For a match with the temperature adjustment region we require that

$$f_{0\xi} \rightarrow 1 \text{ and } T_0 \rightarrow 0, \text{ as } \xi \rightarrow \infty . \quad (6.6.3a, b)$$

It is the boundary condition (6.6.3b) which is at the root of the difficulties with the numerical solution — note the $T_0^{-\frac{1}{2}}$ factors in the equations (6.6.2a,b), which can

cause problems, numerically, as $\xi \rightarrow \infty$. It should be stressed that this problem still exists for $Pr = 1$ — however for this particular case the equations decouple and only one equation, (6.2.7a), has to be solved, i.e. the problem is ‘halved’. The author was not able to find transformations for f_0, T_0 and/or ξ to completely remove this difficulty.

Another difficulty with the numerical solution of (6.6.2) also concerns their decay as $\xi \rightarrow \infty$. It can easily be deduced that

$$f_0 = \xi - \bar{\bar{\Delta}} + \frac{\bar{\bar{D}}}{(\xi - \bar{\bar{\Delta}})^{\frac{3}{Pr}}} + \dots,$$

$$T_0 = \frac{36}{Pr^2} \frac{1}{(\xi - \bar{\bar{\Delta}})^4} + \dots, \quad \text{as } \xi \rightarrow \infty, \quad (6.6.4a, b)$$

where $\bar{\bar{\Delta}}$ and $\bar{\bar{D}}$ are constants (but dependent on Pr, S and γ) whose values must be determined numerically. Thus f_0 and T_0 only decay relatively slowly (cf. the exponential decay of the Blasius equation) but this problem can easily be overcome by transforming to a new variable, say $\bar{\xi} = \ln \xi$. Once this transformation is made, and the appropriate care is taken of the $T_0^{-\frac{1}{2}}$ factors as $\xi \rightarrow \infty$, it does not take much effort to ‘solve’ the equations using a Runge–Kutta approach, subject to the boundary conditions on $\xi = 0$, such that (6.6.3) ‘appear’ to be satisfied. Usually such solutions are sufficient (to the accuracy required); however problems occur here as the value of $\bar{\bar{D}}$ must be determined accurately, from the numerical solution, to feed into the boundary condition of the temperature–adjustment–region f –equation. This is unless $Pr = 1$, when $\bar{\bar{D}}$ can be evaluated analytically (see (6.2.7c)). It proved to be impossible to have any confidence in calculated $\bar{\bar{D}}$ –values for the preferred choice $Pr = 0.72$; thus the case of unity Prandtl number was considered instead, when $\bar{\bar{D}}$ is known i.e. the numerical solution of the temperature–adjustment–region base–flow equations can be carried out without first having to solve the troublesome boundary–layer equations. When these boundary–layer equations were solved with unity Prandtl number, the predicted $\bar{\bar{D}}$ varied greatly for small changes in wall–shear values; even though its value was known analytically, it still proved extremely difficult to obtain numerical predictions for $\bar{\bar{D}}$ that were at all accurate. Similar conclusions have been arrived at, independently, by Drs. Y.B. Fu and A.P. Bassom (private communications with

the author, 1989-1990), when they also attempted to solve these equations using Runge–Kutta methods.

The author believes that, despite its simplicity, the Runge–Kutta solution approach to these equations, (6.6.2a,b), is not sufficient to obtain several†, accurate numerical predictions for $\bar{D}$; instead the author believes that a ‘spectral-method’ of numerical solution is required, based on Chebychev polynomials say. Such a solution has not been carried out by the author; despite the very adequate computer facilities available to the author, there was little advice and help available (and forthcoming) on how these resources could be (most appropriately) used for such a spectral-solution. Finally, note that the above discussion applies equally to the solution of the viscous–boundary-layer equations of the interactive flow, considered in §6.4.

We now mention a couple more of the difficulties encountered during the numerical solutions — note that these difficulties are not removed by the unity-Prandtl-number assumption. First we discuss the numerical solution for the base-flow profiles in the adjustment layers. We concentrate on (6.4.26b)–(6.4.29), for the interactive flow case; these being the more complex. Note that, (i) G grows extremely rapidly for $s < 1$ — so much so that a new variable $\hat{s} = \ln s$ was introduced to stretch the co-ordinate, and (ii) G decays very slowly for large s . To make matters worse, the preferred method of solution, by ‘shooting’ from both $s \ll 1$, $s \gg 1$ and matching the solutions at some intermediate s -value, was found not to be possible as the solutions ‘blow up’ if initialised at some $s \gg 1$. It can be shown (analytically) that equation (6.4.26b) does indeed possess a solution which breaks-up at some s -value. This equation was eventually solved numerically by starting from the small- s asymptote and iterating on the value of G_1 , by ‘shooting’ to some $s \gg 1$ and then adjusting G_1 appropriately to ensure that G behaved as required, (6.4.28), as $\hat{s} = \ln s \rightarrow \infty$. In fact, rather than solving for G , the author actually solved for

$$\exp[2\hat{s}] \sqrt{G},$$

using the Runge–Kutta approach. Related discussion, concerning the numerical solution of the second Painlevé transcendent, is due to Rosales (1978).

† for different wall-coolings or S, γ and Pr values.

Lastly, we comment on the (numerical) solution of the vorticity-mode equation, (6.4.36,37), for the interactive boundary-layer case. Difficulties are caused by the behaviour of G for limiting s -values; the fast growth of G , as $s \rightarrow 0$, results in the eigenfunction decaying to zero very rapidly there; whilst the slow decay of G , as $s \rightarrow \infty$, results in the eigenfunctions decaying very slowly — preferably, many terms in the large- s asymptotes should be used here. These difficulties can be overcome; we emphasise that the resulting neutral, real value of $\mathcal{C}$ differed from its predicted value, from the generalised inflexion point criterion (6.5.1), by less than 0.001%; whilst we find numerically that $\mathcal{C}$ becomes large as $\mathcal{K} \rightarrow 0$, as predicted analytically. However, problems are caused by the small- s and large- s asymptotes becoming ‘disordered’ for the very small values of $\mathcal{K}$ that need to be studied to have any chance of identifying the detailed asymptotic behaviour necessary for comparisons with the theories outlined in §6.5.2.

Summarising, the asymptotic theories considered in this chapter will easily generalise to non-unity Prandtl number flows; however the solution of the resulting equations, for quantitative results, will require careful numerical solutions.

Chapter 7

On the co-existence of Görtler vortices and inviscid Rayleigh modes in hypersonic boundary-layer flows.

§7.1 INTRODUCTION.

§7.1.1 Introductory discussion.

In this chapter we are primarily concerned with the centrifugal instability of hypersonic boundary-layer flows — the inviscid instability of such flows was considered in the previous chapter (see also Blackaby, Cowley & Hall, 1990). The motivation for the present study is essentially the same as that as for the latter studies. It is now well established, both experimentally and theoretically, that incompressible and $O(1)$ -Mach-number flows over a concave plate are unstable to longitudinal vortex structures whose axes lie in the streamwise direction (see figure 7.1) — such disturbances are commonly referred to as Görtler vortices.

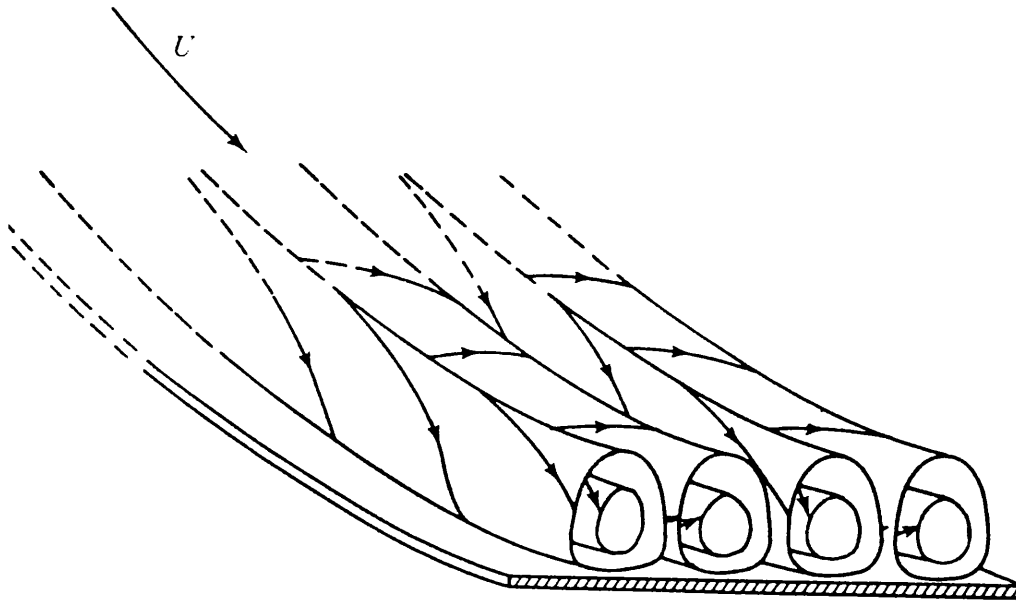


Figure 7.1. The form of the secondary flow which occurs at the onset of centrifugal instability in boundary layers along a concave wall. (From Görtler, 1940.)

A comprehensive account of the theoretical progress made, so far, for the case of incompressible flow can be found in the recent review paper by Hall (1990). The $O(1)$ Mach number case has been recently studied by Wadey (1990) and Spall & Malik (1989) — it is found that there is little significant difference from the incompressible theory. However, the numerical calculations of Wadey (1990) suggest that as the Mach number increases the position where an unstable Görtler vortex locates itself moves towards the edge of the boundary layer. This result is consistent with what we shall describe in this chapter. The case of hypersonic flows was first considered by Hall & Fu (1989), they choose to simplify the analysis by employing a linear viscosity–temperature law. Very recently, this theory has been extended to flows satisfying Sutherlands viscosity formula by Fu, Hall & Blackaby (1990) (hereinafter referred to as FHB) — this paper also presents a first study of ‘real gas effects’ in connection with the Görtler instability. The more realistic viscosity law, chosen by FHB, leads to significant changes in, and introduces complications into, the theory of Hall & Fu (1989), these are outlined in the next section; it is shown that the curvature of the underlying flow is very significant.

The hypersonic boundary layers, being considered here, are also unstable to inviscid Rayleigh–type modes; in §7.3 we recap the formulation of Chapter 6, but extended to non-unity Prandtl number. Thus we expect the Görtler and Rayleigh stabilities to co – exist in such boundary–layer flows, this is our concern in the last two sections. In §7.4 we investigate the modification of the Rayleigh stability properties due to the present of larger–amplitude (nonlinear) vortices — this is closely related to the recent study, concerning the inviscid secondary instabilities of an incompressible strongly nonlinear vortex state, by Hall & Horseman (1990). In §7.5 we make a few comments on the possible interaction of the Görtler and Rayleigh modes — no detailed formulation is given; and no numerical results have been computed.

Recall that, in Chapter 3, we noted that (experimentally) longitudinal vortices have been observed on a flat plate (for $O(1)$ -Mach number flows). These are not caused by centrifugal effects, it is presently believed that they are driven by nonlinear interaction with the viscous Tollmien–Schlichting modes also present. Regardless of the origin of these longitudinal vortices, we note that once, and if,

they have grown strong enough to affect the basic state then they too will modify the Rayleigh stability properties of the boundary—layer flow — alternatively, one may regard the Rayleigh modes as being possible secondary instabilities of these strongly nonlinear longitudinal vortex structures. Another closely related problem, that has yet to be investigated, concerns the co-existence of Görtler and Rayleigh modes in a general, $O(1)$ -Mach-number, compressible boundary layer.

§7.1.2 The Görtler instability in compressible flows.

For later reference, we now briefly outline the formulation of the Görtler vortex equations for general-Mach-number, compressible flows. The notation of previous chapters, especially Chapter 6, will be adhered to where-ever possible.

The basic state.

Consider a compressible boundary layer over a rigid wall of variable curvature $(1/A)\kappa(x^*/L)$, where L is a typical streamwise length scale and A is a lengthscale characterizing the radius of curvature of the wall. We choose a curvilinear coordinate system (x^*, y^*, z^*) with x^* measuring distance along the wall, y^* perpendicular to the wall and z^* in the spanwise direction. The corresponding velocity components are denoted by (u^*, v^*, w^*) and density, temperature and viscosity by ρ^*, T^* and μ^* respectively. The free stream values of these quantities will be signified by a subscript ∞ . We define a curvature parameter δ_G by

$$\delta_G = \frac{L}{A}, \quad (7.1.1)$$

and consider the limit $\delta_G \rightarrow 0$ with the Reynolds number Re defined by

$$Re = \frac{u_\infty^* L \rho_\infty^*}{\mu_\infty^*} \quad (7.1.2)$$

taken to be large so that the Görtler number

$$G = 2Re^{1/2} \delta_G \quad (7.1.3)$$

is $O(1)$. In the following analysis, coordinates (x^*, y^*, z^*) are scaled on $(1, Re^{-1/2}, Re^{-1/2})L$, the velocity (u^*, v^*, w^*) is scaled on $(1, Re^{-1/2}, Re^{-1/2})u_\infty^*$ and other quantities such as ρ^*, T^* , and μ^* are scaled on their free stream values with the only exception that the pressure p^* is scaled on $\rho_\infty^* u_\infty^{*2}$ and the coefficient of

heat conduction k^* is scaled on μ_∞^* . All dimensionless quantities will be denoted by the same letters without a superscript *. Then the Navier-Stokes equations reduce to, at leading orders in $Re \gg 1$,

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_\beta}(\rho v_\beta) &= 0, \\
\rho \frac{Du}{Dt} &= -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}(\mu \frac{\partial u}{\partial y}) + \frac{\partial}{\partial z}(\mu \frac{\partial u}{\partial z}), \\
\rho \frac{Dv}{Dt} + \frac{1}{2} G \kappa u^2 &= -Re \frac{\partial p}{\partial y} + \frac{\partial}{\partial y} \left\{ (\lambda - \frac{2}{3} \mu) \frac{\partial v_\beta}{\partial x_\beta} \right\} + \frac{\partial}{\partial x_\beta}(\mu \frac{\partial v_\beta}{\partial y}) \\
&\quad + \frac{\partial}{\partial y}(\mu \frac{\partial v}{\partial y}) + \frac{\partial}{\partial z}(\mu \frac{\partial v}{\partial z}), \\
\rho \frac{Dw}{Dt} &= -Re \frac{\partial p}{\partial z} + \frac{\partial}{\partial z} \left\{ (\lambda - \frac{2}{3} \mu) \frac{\partial v_\beta}{\partial x_\beta} \right\} + \frac{\partial}{\partial x_\beta}(\mu \frac{\partial v_\beta}{\partial z}) + \frac{\partial}{\partial y}(\mu \frac{\partial w}{\partial y}) + \frac{\partial}{\partial z}(\mu \frac{\partial w}{\partial z}), \\
\rho c_p \frac{DT}{Dt} &= \mu(\gamma - 1) M_\infty^2 \left[\left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial z} \right)^2 \right] + (\gamma - 1) M_\infty^2 \left[1 - \rho \left(\frac{\partial h}{\partial p} \right)_T \right] \frac{Dp}{Dt} \\
&\quad + \frac{1}{Pr} \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + \frac{1}{Pr} \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right), \\
\gamma M_\infty^2 p &= (1 + \alpha_d) \rho T. \tag{7.1.4a-f}
\end{aligned}$$

Here we have used a mixed notation in which (v_1, v_2, v_3) is identified with (u, v, w) and (x_1, x_2, x_3) with (x, y, z) . Repeated suffices signify summation from 1 to 3. The functions λ, k, c_p and h denote in turn the bulk viscosity, the coefficient of heat conduction, the specific heat at constant pressure and the enthalpy per unit mass. The constants γ, M_∞ and Pr are in turn the ratio of specific heats, the Mach number and the Prandtl number defined by

$$\gamma = \frac{c_{v\infty}}{c_{p\infty}}, \quad M_\infty = \frac{u_\infty^{*2}}{\gamma \bar{\mathcal{R}} T_\infty^*} = \frac{u_\infty^{*2}}{a_\infty^2}, \quad Pr = \frac{\mu^* c_{p\infty}}{k^*}, \tag{7.1.5a-c}$$

where $\bar{\mathcal{R}}$ is a gas constant and $a_\infty = \sqrt{\gamma \bar{\mathcal{R}} T_\infty^*}$ is the sound speed in the free stream. The function α_d in the equation of state (7.1.4f) denotes the percentage by mass of the mixture which has been dissociated and in equations (7.1.4), the operator D/Dt is the material derivative and it has the usual expression appropriate to a rectangular coordinate system.

A similarity solution to these boundary layer equations exists; it is identical to that for the 'flat-plate' boundary-layer equations which was formulated in Chapter

6, where we also considered its large Mach number properties. For completeness, we recap this solution here.

This similarity solution to these equations, for steady two-dimensional flow over the curved plate with leading edge at $x = 0$, has the form

$$u = \bar{u}(\eta) = \psi_{\hat{\eta}}, \quad v = \bar{v}(\eta) = -\frac{1}{\rho}(\psi_x + \hat{\eta}_x \psi_{\hat{\eta}}), \quad \psi = \sqrt{(1+S)x} f(\eta),$$

$$T \equiv \bar{T}(\eta), \quad \rho \equiv \bar{\rho}(\eta), \quad \mu \equiv \bar{\mu}(\eta), \quad p = \bar{p} = \frac{1}{\gamma M_\infty^2} \equiv p_\infty, \quad (7.1.6a - g)$$

where

$$\eta = \frac{\hat{\eta}}{\sqrt{(1+S)x}}, \quad (7.1.7a)$$

and the Dorodnitsyn-Howarth variable

$$\hat{\eta} = \int_0^y \rho dy, \quad (7.1.7b)$$

has been introduced for convenience. Here S is the constant which appears in Sutherland's formula,

$$\mu = \left(\frac{1+S}{T+S} \right) T^{\frac{3}{2}}, \quad S \simeq \frac{110.4}{T_\infty^*};$$

our chosen viscosity-temperature relation. We are assuming here that the fluid is an ideal (one component) gas undergoing no dissociation so that $\alpha_d = 0$. Then we can assume that (i), the specific heats are constants; (ii), the coefficient of heat conduction is linearly related to the shear viscosity and (iii), the enthalpy h is given by $h = c_p T$. These assumptions lead to the results

$$\bar{k} = \bar{\mu}, \quad \bar{c}_p = 1, \quad \bar{\rho} = \frac{1}{\bar{T}}.$$

(Note that all of these quantities have been non-dimensionalized).

The governing equations, for the similarity solution, are then found to be

$$\bar{\rho} \bar{T} = 1, \quad \frac{1}{2} f f_{\eta\eta} + \left(\frac{\bar{T}^{\frac{1}{2}}}{\bar{T} + S} f_{\eta\eta} \right)_\eta = 0,$$

$$\frac{1}{2} f \bar{T}_\eta + \frac{1}{Pr} \left(\frac{\bar{T}^{\frac{1}{2}}}{\bar{T} + S} \bar{T}_\eta \right)_\eta + \frac{(\gamma-1) M_\infty^2 \bar{T}^{\frac{1}{2}}}{\bar{T} + S} f_{\eta\eta}^2 = 0, \quad (7.1.8a - c)$$

subject to the boundary conditions

$$f(0) = f_\eta(0) = 0, \quad f_\eta(\infty) = \bar{T}(\infty) = 1, \quad (7.1.8d)$$

and $\bar{T}(0) = T_w$ (fixed wall – temperature), or $\bar{T}_\eta(0) = 0$ (insulated wall).
(7.1.8e)

Note that (7.1.4c,d) require that the leading-order pressure term is a function of x at most: in the above we have assumed that there is no pressure gradient acting along the streamwise direction and equated the constant basic-state pressure with the freestream value, (7.1.6g). Modifications necessary in (7.1.8) for dissociated gases are discussed in Fu, Hall & Blackaby (1990).

The linear perturbation equations.

We now assume that the flow is perturbed to spanwise periodic stationary vortex structure with constant wavenumber a . The linearized stability equations for these Görtler vortices are then found by linearizing (7.1.4) about the basic state:

$$(u, v, w, T, p) = (\bar{u}, \bar{v}, 0, \bar{T}, p_\infty) + h(U, V, W, T, Re^{-1}P)E + \text{c.c.} + O(h^2), \quad (7.1.9a - e)$$

where $h \ll 1$, $E = \exp(iaz)$ and U, V, W, T, P are functions of x and y .

At $O(h)$ we obtain the linear stability equations

$$\begin{aligned} & \frac{1}{\bar{T}}(\bar{u}U_x + \bar{v}U_y) + (\bar{\mu}a^2 + \frac{\bar{u}_x}{\bar{T}})U - (\bar{\mu}U_y)_y + \frac{1}{\bar{T}}\bar{u}_yV \\ & - \left\{ \frac{1}{\bar{T}^2}(\bar{u}\bar{u}_x + \bar{v}\bar{u}_y) + (\bar{\mu}\bar{u}_y)_y \right\} T - \bar{\mu}\bar{u}_yT_y = 0, \end{aligned}$$

$$\begin{aligned} & \frac{1}{\bar{T}}(\bar{v}_x + \kappa\bar{u}G)U + \frac{2}{3}\bar{\mu}_yU_x - \frac{1}{3}\bar{\mu}U_{xy} - \bar{\mu}_xU_y + \frac{1}{\bar{T}}(\bar{u}V_x + \bar{v}V_y) + (\bar{\mu}a^2 + \frac{\bar{v}_y}{\bar{T}})V - \frac{4}{3}(\bar{\mu}V_y)_y \\ & + P_y - \left[\frac{1}{\bar{T}^2}(\bar{u}\bar{v}_x + \bar{v}\bar{v}_y + \frac{1}{2}\kappa G\bar{u}^2) + \frac{1}{3}\bar{\mu}\bar{u}_{xy} - \frac{2}{3}\bar{\mu}_y\bar{u}_x + \frac{4}{3}(\bar{\mu}\bar{v}_y)_y + \bar{\mu}_x\bar{u}_y \right] T \\ & - \bar{\mu}\bar{u}_yT_x - \left[-\frac{2}{3}\bar{\mu}\bar{u}_x + \frac{4}{3}\bar{\mu}\bar{v}_y \right] T_y + \frac{2}{3}\bar{\mu}_yiaW - \frac{1}{3}ia\bar{\mu}W_y = 0, \end{aligned}$$

$$\begin{aligned} \bar{\mu}_x iaU + \frac{1}{3}\bar{\mu}iaU_x + \bar{\mu}_y iaV + \frac{1}{3}\bar{\mu}iaV_y - iaP - \frac{2}{3}\bar{\mu}(\bar{u}_x + \bar{v}_y)iaT \\ - \frac{1}{\bar{T}}(\bar{u}W_x + \bar{v}W_y) - \frac{4}{3}\mu a^2 W + (\bar{\mu}W_y)_y = 0, \end{aligned}$$

$$\begin{aligned} \frac{2}{\bar{T}^3}(\bar{u}\bar{T}_x + \bar{v}\bar{T}_y)T - \frac{1}{\bar{T}^2}(\bar{u}_x + \bar{v}_y)T - \frac{1}{\bar{T}^2}(\bar{u}T_x + \bar{v}T_y) \\ + \frac{1}{\bar{T}}(U_x + V_y) - \frac{1}{\bar{T}^2}(\bar{T}_x U + \bar{T}_y V) + ia\left(\frac{W}{\bar{T}}\right) = 0, \end{aligned}$$

$$\begin{aligned} \frac{1}{\bar{T}}\bar{T}_x U - 2(\gamma - 1)M_\infty^2 \bar{\mu}\bar{u}_y U_y + \frac{1}{\bar{T}}\bar{T}_y V + \frac{1}{\bar{T}}(\bar{u}T_x + \bar{v}T_y) + \frac{\bar{\mu}}{Pr}a^2 T \\ - \left\{ \frac{1}{\bar{T}^2}(\bar{u}\bar{T}_x + \bar{v}\bar{T}_y) + (\gamma - 1)M_\infty^2 \bar{\mu}\bar{u}_y^2 + \frac{1}{Pr}(\bar{\mu}\bar{T}_y)_y \right\} T - \frac{1}{Pr}\bar{\mu}\bar{T}_y T_y - \frac{1}{Pr}(\bar{\mu}T_y)_y = 0. \end{aligned} \quad (7.1.10a - e)$$

Here $\bar{\mu} = d\bar{\mu}/d\bar{T}$, whilst (U, V, W) , P and T denote the vortex velocity field, pressure and temperature, respectively. Equations (7.1.10) differ from the corresponding equations given in Hall & Fu (1989) only in that the bulk viscosity is taken to be zero here; that assumption is actually implied in that paper.

It was shown by Hall (1982a) that in the incompressible case the neutral curve for small wavelength vortices has $G \sim a^4$ and that the vortices are confined to a layer of depth $a^{-1/2}$ where the flow is locally most unstable. Hall & Malik (1989) extended this approach to the above system for $M_\infty = O(1)$ and wrote

$$G = g_0 a^4 + g_1 a^3 + \dots \quad (7.1.11a)$$

They found that the leading order growth rate $a^2 \delta^*$ has δ^* given by

$$\delta^* = \frac{\bar{\mu}^2}{Pr} + \left(\frac{\bar{u}^2 \bar{T}_y}{2\bar{T}^3} - \frac{\bar{u}\bar{u}_y}{Pr\bar{T}^2} \right) g_0. \quad (7.1.11b)$$

In the neutral case, $\delta^* = 0$ and (3.15) then determines the neutral Görtler number g_0 as a function of η . The most unstable location η^* is where g_0 has its minimum. In Hall & Fu (1989), it is found that when Chapman's law is used, η^* moves away from the wall as the Mach number increases.

§7.1.3 The Rayleigh instability in compressible flows.

In §6.2.4 we formulated Rayleigh's equation for compressible flows, we recap that formulation here for completeness, as well as for later reference. The basic state is that considered in the previous subsection, §7.1.2; we are assuming that x -stations under consideration are sufficiently downstream so that (i), the basic state is non-interactive (not influenced by a shock), and (ii), the quasi-parallel assumption is valid for inviscid instability modes. It is then appropriate to seek perturbations of the form

$$(u, v, w, T, p) = (\bar{u}, \bar{v}, 0, \bar{T}, p_\infty) + \Delta(\tilde{u}(\eta), Re^{\frac{1}{2}}\tilde{v}(\eta), Re^{\frac{1}{2}}\tilde{w}(\eta), \tilde{T}(\eta), \tilde{p}(\eta)) \exp\left(iRe^{\frac{1}{2}}\vartheta(x, z, t)\right) + \dots, \quad (7.1.12a - e)$$

where Δ is the small disturbance amplitude (linearisation parameter). We define local wavenumbers, a local frequency and a local wavespeed by

$$(\alpha, \beta, \omega) = \sqrt{(1+S)x} \ (\vartheta_x, \vartheta_z, -\vartheta_t), \quad c = \frac{\omega}{\alpha}. \quad (7.1.13a - d)$$

If Δ is sufficiently small, the pressure perturbation $\tilde{p}$ satisfies the linear, compressible, Rayleigh equation,

$$\frac{d^2 \tilde{p}}{d\eta^2} - \frac{2f''}{f' - c} \frac{d\tilde{p}}{d\eta} - (\alpha^2 + \beta^2) \bar{T} \left(\bar{T} - \frac{\alpha^2 M_\infty^2 (f' - c)^2}{(\alpha^2 + \beta^2)} \right) \tilde{p} = 0. \quad (7.1.14a)$$

The conditions that there is no normal velocity at the wall, and that the disturbance is confined to the boundary layer, can be expressed as

$$\tilde{p}' = 0 \text{ on } \eta = 0, \quad \tilde{p} \rightarrow 0 \text{ as } \eta \rightarrow \infty. \quad (7.1.14b, c)$$

Equation (7.1.14a) and boundary conditions (7.1.14b, c) specify a temporal stability eigenrelation $c \equiv c(\alpha, \beta)$. The large Mach number features of equation (7.1.14b, c) were considered in Chapter 6 and will be re-considered in §7.3.

§7.2 THE GÖRTLER INSTABILITY IN HYPERSONIC FLOWS.

§7.2.1 The basic state and the location of the vortices.

Before investigating the large Mach number properties of the stability equations, we need to recap the corresponding properties of the basic state; these were considered in Chapter 6. An explicit analytical for the equations (7.1.8) is not possible; however, an asymptotic analysis in the large Mach number limit shows that the boundary layer can be divided into two regions: an inner high temperature region, where $\eta = O(M_\infty^{-\frac{1}{2}})$, and an outer region, where $\eta = O(1)$.

In the inner region, we define the $O(1)$ quantities ξ , T_0 and f_0 by

$$\xi = M_\infty^{\frac{1}{2}}\eta, \quad T_0(\xi) = M_\infty^{-2}\bar{T} \quad \text{and} \quad f_0(\xi) = M_\infty^{\frac{1}{2}}f, \quad (7.2.1a-c)$$

so that (7.1.8b,c) reduce to

$$\begin{aligned} \frac{1}{2}f_0 f_{0\xi\xi} + \left(\frac{f_{0\xi\xi}}{T_0^{\frac{1}{2}}} \right)_\xi &= 0, \\ \frac{1}{2}f_0 T_{0\xi} + \frac{1}{Pr} \left(\frac{T_{0\xi}}{T_0^{\frac{1}{2}}} \right)_\xi + \frac{(\gamma-1)}{T_0^{\frac{1}{2}}} f_{0\xi\xi}^2 &= 0, \end{aligned} \quad (7.2.2a,b)$$

at leading order. These equations must be solved numerically, subject to the conditions

$$\begin{aligned} f_0(0) = f_{0\xi}(0) = 0, \quad T_0(\infty) = 0, \quad f_{0\xi}(\infty) = 1, \\ \begin{cases} T_{0\xi}(0) = 0 & : \text{if the wall is thermally insulated,} \\ T_0(0) = nT_{0w} & : \text{if the wall is under cooling,} \end{cases} \end{aligned}$$

where T_{0w} is the wall temperature scaled on $M_\infty^2 T_\infty$ when the wall is thermally insulated and n is the wall cooling coefficient.

For large ξ , equations (7.2.2a,b) have the asymptotic solutions

$$\begin{aligned} f_0 &= \xi - \bar{\Delta} + \frac{\bar{D}}{(\xi - \bar{\Delta})^{\frac{3}{Pr}}} + \dots, \\ T_0 &= \frac{36}{Pr^2} \frac{1}{(\xi - \bar{\Delta})^4} + \dots, \quad \text{as } \xi \rightarrow \infty, \end{aligned} \quad (7.2.3a,b)$$

where $\bar{\Delta}$ and $\bar{D}$ are constants (but dependent on Pr , S and γ) whose values must be determined numerically.

These asymptotic expressions imply that in the temperature adjustment region, where $\eta = O(1)$,

$$f = \eta - \frac{\bar{\bar{D}}}{M_\infty^{\frac{1}{2}}} + \frac{\bar{f}_1}{M_\infty^{\frac{3}{2Pr} + \frac{1}{2}}},$$

$$\bar{T} = T_1(\eta) + \dots, \bar{\mu} = \mu_1(\eta) + \dots. \quad (7.2.4a - c)$$

On substituting these into (7.1.8b,c), we obtain to leading order

$$\left(\frac{\sqrt{T_1}}{T_1 + S} \bar{f}_{1\eta\eta} \right)_\eta + \frac{1}{2} \eta \bar{f}_{1\eta\eta} = 0,$$

$$\frac{1}{Pr} \left(\frac{\sqrt{T_1}}{T_1 + S} T_{1\eta} \right)_\eta + \frac{1}{2} \eta T_{1\eta} = 0. \quad (7.2.5a, b)$$

These two equations must be solved numerically subject to the matching conditions

$$\bar{f}_1(\eta) \sim \frac{\bar{\bar{D}}}{\eta^{3/Pr}}, \quad T_1 \sim \frac{36}{Pr^2} \frac{1}{\eta^4} + \dots, \quad \text{as } \eta \rightarrow 0, \quad (7.2.5c, d)$$

and the conditions at infinity

$$\bar{f}'_1(\infty) = 0, \quad T_1(\infty) = 1. \quad (7.2.5e, f)$$

We note that whilst the solution of (7.2.5b) is independent of the inner region solution and thus of the conditions at the wall, the function $\bar{f}_1$ is dependent on the inner region solutions through the matching constant $\bar{\bar{D}}$.

Returning to the linear theory for Görtler vortices in compressible flows, we can now consider the location of the most unstable small-wavelength vortices, for large Mach numbers, when the more realistic Sutherland's formula is used. Recall that for these linear, small-wavelength vortices to be neutral we require that $\delta^* = 0$ (see (7.1.11b)) which then determines the (scaled) neutral Görtler number g_0 as a function of y (or, equivalently, η). The most unstable location, η^* say, is where g_0 has its minimum. Using (7.1.11b) with $\delta^* = 0$ to calculate the orders of g_0 in the inner layer, where $\eta = O(M_\infty^{-1/2})$, and the outer temperature adjustment layer, where $\eta = O(1)$, we find that $g_0 = O(M_\infty^{15/2})$ in the former and $g_0 = O(1)$ in the latter. Hence, as with the Chapman-formulation, the temperature adjustment layer is most susceptible to Görtler vortices with wavenumber of order one or larger.

It should be noted, however, the above conclusion is based upon a large wavenumber argument. In fact the wall layer, where $\eta = O(M_\infty^{-1/2})$, is actually

of order $M_\infty^{3/2}$ thickness in terms of the physical variable y . Thus Görtler vortices with wavelength comparable with the boundary layer thickness must be trapped in the wall layer and have $a = O(M_\infty^{-3/2})$. It will be reported later that this wall mode has neutral Görtler number decreasing monotonically and has the centre of Görtler vortex activity moving towards the temperature adjustment layer as the wavenumber increases. Therefore, the minimum Görtler number corresponds to the mode trapped in the temperature adjustment layer and the latter is indeed found the most dangerous mode when the whole range of wavenumbers are considered (see FHB).

It should also be noted that the result $g_0 = O(1)$ for the temperature adjustment layer is obtained by taking the large Mach number limit of the $O(1)$ Mach number results. By doing so we have actually missed a term related to the curvature of the basic state which is not important for the case $M_\infty = O(1)$ and $a \gg 1$, but is important in the large Mach number limit. As we shall show later on, the curvature of the basic state produces an effective negative Görtler number of order $M_\infty^{3/2}$ in the absence of wall curvature so that instability can not occur for $G = O(1)$.

§7.2.2 The strongly unstable inviscid Görtler mode.

Let us first confine our attention to the mode trapped in the temperature adjustment layer. It is easy to show, from (7.1.6), (7.1.7), (7.2.1) and (7.2.4), that in this temperature adjustment layer,

$$\bar{u}\bar{v}_x + \bar{v}\bar{v}_y = -\frac{BM_\infty^{3/2}}{(2x)^{3/2}} + O(1), \quad (7.2.6a)$$

where

$$B \equiv B(\gamma, S, Pr, n) = \int_0^\infty T_0(\xi) d\xi \sim O(1). \quad (7.2.6b)$$

An investigation of the y -momentum equation (7.1.10b) shows that the Görtler number must be of order $M_\infty^{3/2}$ in order to enter the leading order analysis. Thus we write

$$\frac{1}{2}\kappa(x)G = G^*(x)M_\infty^{3/2}, \quad (7.2.7)$$

and for convenience, we also define another function $Q(x)$ by

$$Q(x) = \frac{B}{(2x)^{3/2}}, \quad (7.2.8a)$$

so that

$$\bar{u}\bar{v}_x + \bar{v}\bar{v}_y + \frac{1}{2}\kappa(x)G\bar{u}^2 = (G^* - Q)M_\infty^{3/2} + o(M_\infty^{3/2}). \quad (7.2.8b)$$

With the use of this relation, we can deduce from the perturbation equations (7.1.10) that

$$V = O(M_\infty^{3/4}T), \quad W = O(M_\infty^{3/4}T), \quad P = O(M_\infty^{3/2}T), \quad U = O(M_\infty^{-1}T),$$

and that for fixed η ,

$$\frac{\partial}{\partial x} = O(M_\infty^{3/4}), \quad \text{where } M_{\infty 1} \stackrel{\text{def.}}{=} 1/M_\infty^{\frac{3}{2P^*} + \frac{1}{2}}. \quad (7.2.9a - f)$$

We therefore look for asymptotic solutions of the form

$$(U, V, W, T, P) = \exp\left(M_\infty^{3/4} \int^x \beta(x) dx\right) (M_{\infty 1}^{-1} \check{U}_0(x, \eta), M_\infty^{3/4} \check{V}_0, M_\infty^{3/4} \check{W}_0, \check{T}_0, M_\infty^{3/2} \check{P}_0) + \dots \quad (7.2.10a - e)$$

where $\beta(x)$ is the local growth. On substituting these into (7.1.10) and then equating the coefficients of like powers of M_∞ , we find, at leading order, that $\check{V}_0$ satisfies the differential equation

$$\frac{\partial^2 \check{V}_0}{\partial \eta^2} - \frac{2T_{1\eta}}{T_1} \frac{\partial \check{V}_0}{\partial \eta} - \tilde{k}^2 T_1^2 \check{V}_0 = \frac{\tilde{k}^2}{\sqrt{(1+S)x}\beta^2} (G^* - Q) T_{1\eta} \check{V}_0, \quad (7.2.11a)$$

whilst $\check{T}_0$, $\check{W}_0$ and $\check{P}_0$ are related to $\check{V}_0$ by

$$\check{T}_0 = -\frac{T_{1\eta}}{\sqrt{(1+S)x}T_1\beta} \check{V}_0, \quad ia\check{W}_0 = -\frac{1}{\sqrt{(1+S)x}T_1} \frac{\partial \check{V}_0}{\partial \eta}, \quad P_0 = -\frac{\beta\sqrt{(1+S)x}}{T_1^2 \tilde{k}^2} \frac{\partial \check{V}_0}{\partial \eta},$$

and note that $\check{U}_0$ does not appear in the leading order analysis. Here $\tilde{k} = \sqrt{(1+S)x}$ a is the local wavenumber. Equation (7.2.11a), subject to $\check{V}_0$ vanishing at $\eta = 0, \infty$, is a Sturm-Liouville problem which has solutions if

$$\frac{(G^* - Q)T_{1\eta}}{\beta^2} \leq 0.$$

This means that $\beta^2 \geq 0$ if $G^* \geq Q$, $\beta^2 \leq 0$ if $G^* \leq Q$, (7.2.11b,c)

since $T_{1\eta} < 0$. It then follows that neutral stability ($\beta = 0$) occurs at the position $x = x_n$, where

$$G^*(x_n) = Q(x_n), \quad (7.2.11d)$$

at zeroth order. Therefore, in view of the definitions (7.2.7) and (7.2.8), the neutral Görtler number has the expansion

$$G = \frac{2B}{\kappa(x_n)(2x_n)^{3/2}} M_\infty^{3/2} + \text{higher order correction terms.} \quad (7.2.12)$$

The important point concerning (7.2.12) is that the first term on the right hand side is independent of x_n if the wall curvature varies like $x^{-\frac{3}{2}}$; in the latter situation nonparallel effects dominate and the vortex growth rate is smaller. Thus to determine the higher order correction terms to the neutral Görtler number, we have to distinguish two cases, namely (i) $\kappa(x) \propto x^{-3/2}$, and (ii), $\kappa(x) \not\propto x^{-3/2}$.

Equation (7.2.11a), with appropriate boundary conditions, can also be interpreted as an eigenvalue problem which determines the growth rate $\beta(x)$ at a given value of x corresponding to any wavenumber $\tilde{k}$. It is easy to show analytically that as $\tilde{k} \rightarrow 0$, $\beta^2 \rightarrow 0$ whilst as $\tilde{k} \rightarrow \infty$, $\beta^2 \rightarrow \text{constant}$; these results are borne out by a numerical solution (see FHB).

The inviscid Görtler mode we have described above therefore has growth rate proportional to $M_\infty^{\frac{3}{4}}$ and we refer to it as the strongly unstable inviscid Görtler mode. We note that when $G^* = Q$ the growth rate vanishes. In this case it is necessary to look for evolution of the vortices on a shorter lengthscale in the streamwise direction; that problem will be addressed in §7.2.4 and we shall refer to the inviscid Görtler mode in that regime as the near neutral inviscid Görtler mode.

§7.2.3 Neutral stability with $\kappa(x) \propto (2x)^{-3/2}$.

In the case when the curvature $\kappa(x) \propto (2x)^{-3/2}$, $G^*(x) = Q(x)$ and the $O(M^{3/2})$ term on the right hand side of (7.2.11a) vanishes for all x . Thus, for this special distribution, the curvature of the basic state is exactly counteracted by wall curvature over an $O(1)$ interval in x , in the more general curvature case that is only the case over an asymptotically small interval. An investigation of

the perturbation equations (7.1.10) then reveals that the neutral Görtler number expands as

$$G = \frac{2BM_\infty^{3/2}}{\tilde{\kappa}} + \tilde{G} + o(1), \quad \text{where } \tilde{\kappa} = (2x)^{\frac{3}{2}}\kappa(x), \quad (7.2.13a, b)$$

and that the perturbation quantities have relative orders

$$U = O\left(\frac{1}{M_{\infty 1}}V\right), \quad T = O(V), \quad W = O(V), \quad P = O(V), \quad (7.2.14a - d)$$

where $\tilde{G} = O(M_\infty^0)$ is to be determined. We therefore look for the following form for the solutions for (7.1.10):

$$U = \frac{1}{M_{\infty 1}}\tilde{U}(x, \eta) + \dots, \quad V = \tilde{V}(x, \eta) + \dots, \quad W = \tilde{W}(x, \eta) + \dots,$$

$$P = \frac{1}{\sqrt{(1+S)x}}\tilde{P}(x, \eta) + \dots, \quad T = \sqrt{(1+S)x}\tilde{\theta}(x, \eta) + \dots, \quad (7.2.15a - e)$$

where the insertion of the factor $\sqrt{(1+S)x}$ is purely for convenience.

On substituting these into (7.1.10) and then equating the coefficients of like powers of M_∞ , we obtain, to leading order,

$$\begin{aligned} \frac{\partial}{\partial \eta}\left(\frac{\mu_1}{T_1}\frac{\partial \tilde{U}}{\partial \eta}\right) + \frac{(1+S)}{2}\eta\frac{\partial \tilde{U}}{\partial \eta} - \mu_1 T_1 \tilde{k}^2 \tilde{U} - (1+S)x\frac{\partial \tilde{U}}{\partial x} \\ = \sqrt{2x}\left\{\frac{\tilde{f}_1''}{T_1}\tilde{V} + \left[\frac{\eta}{T_1}\tilde{f}_1'' - \frac{2}{1+S}\frac{\partial}{\partial \eta}\left(\frac{\tilde{\mu}\tilde{f}_1''}{T_1}\right)\right]\tilde{\theta} - \frac{2}{1+S}\frac{\tilde{\mu}\tilde{f}_1''}{T_1}\frac{\partial \tilde{\theta}}{\partial \eta}\right\} \end{aligned}$$

$$\begin{aligned} \frac{\partial \tilde{P}}{\partial \eta} = \frac{\eta T_1'}{T_1}\tilde{V} + \eta\frac{\partial \tilde{V}}{\partial \eta} - s\mu_1 \tilde{k}^2 T_1 \tilde{V} + \frac{4s}{3}\frac{\partial}{\partial \eta}\left(\frac{\mu_1}{T_1}\frac{\partial \tilde{V}}{\partial \eta}\right) \\ + \frac{\tilde{\theta}}{T_1}[\eta^2 T_1' - \frac{s\mu_1 T_1'}{Pr T_1} + \frac{\tilde{\kappa}}{2s}\tilde{G} - \frac{4s}{3}T_1\frac{\partial}{\partial \eta}\left(\frac{\tilde{\mu}\eta T_1'}{T_1}\right)] \\ - \frac{4s}{3T_1}\tilde{\mu}\eta T_1'\frac{\partial \tilde{\theta}}{\partial \eta} - \frac{2s}{3}\tilde{\mu}T_1' i\tilde{k}\tilde{W} + \frac{s}{3}\mu_1\frac{\partial}{\partial \eta}(i\tilde{k}\tilde{W}) - 2x\frac{\partial \tilde{V}}{\partial x}, \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial \eta}\left(\frac{\mu_1}{T_1}\frac{\partial \tilde{W}}{\partial \eta}\right) = -\tilde{\mu}T_1' i\tilde{k}\tilde{V} - \frac{1}{3}\mu_1 i\tilde{k}\frac{\partial \tilde{V}}{\partial \eta} + s^{-1}i\tilde{k}T_1\tilde{P} \\ - \frac{2}{3}\tilde{\mu}\eta T_1' i\tilde{k}\tilde{\theta} - s^{-1}\eta\frac{\partial \tilde{W}}{\partial \eta} + \frac{4}{3}\mu_1 \tilde{k}^2 T_1 \tilde{W} + s^{-1}2x\frac{\partial \tilde{W}}{\partial x}, \end{aligned}$$

$$\frac{\partial}{\partial \eta} \left(\frac{\tilde{V}}{T_1} \right) + \frac{\eta}{T_1} \frac{\partial \tilde{\theta}}{\partial \eta} = -i\tilde{k}\tilde{W} + \left(1 + \frac{\eta T_1'}{T_1} \right) \frac{\tilde{\theta}}{T_1} + \frac{2x}{T_1} \frac{\partial \tilde{\theta}}{\partial x},$$

$$\begin{aligned} \frac{s}{Pr} \frac{\partial}{\partial \eta} \left(\frac{\mu_1}{T_1} \frac{\partial \tilde{\theta}}{\partial \eta} \right) &= -\left(\eta + \frac{s\tilde{\mu}_1 T_1'}{Pr T_1} \right) \frac{\partial \tilde{\theta}}{\partial \eta} + \frac{T_1'}{T_1} \tilde{V} \\ &+ \tilde{\theta} \left[1 + \frac{\eta T_1'}{T_1} - \frac{s}{Pr} \frac{\partial}{\partial \eta} \left(\frac{\tilde{\mu} T_1'}{T_1} \right) + \frac{s\mu}{Pr} T_1 \tilde{k}^2 \right] + 2x \frac{\partial \tilde{\theta}}{\partial x}. \end{aligned} \quad (7.2.16a - e)$$

Here $\tilde{k} = \sqrt{(1+S)x} a$ and $s = 2/(1+S)$. We see that (7.2.16b-e) are independent of $\tilde{U}$ and the latter is determined from (7.2.16a) after $(\tilde{V}, \tilde{W}, \tilde{T}, \tilde{P})$ have been determined.

Since these leading order perturbation equations are parabolic with respect to the variable x , they have to be solved by specifying the perturbation quantities at a given upstream position and then marching downstream. The numerical solution to these equations is reported in FHB, where it is found that neutral stability depends crucially on what initial conditions are imposed and where they are imposed. However, there is a special case, the large wavenumber limit, for which a simple asymptotic solution is possible.

Large wavenumber limit

In the large wavenumber limit, the length scale over which vortices vary is small compared with the lengthscale over which the boundary layer grows. Then we expect that nonparallel effects do not come into the leading order analysis. This is indeed the case, as we show below.

For large $\tilde{k}$, vortices are confined to a thin layer of $O(\epsilon^{1/2})$ thickness centred on $\eta = \eta^*$ where,

$$\epsilon = 1/\tilde{k}, \quad (7.2.17)$$

and where η^* is the most unstable position to be determined in the course of our calculation. We therefore define a new variable ϕ by

$$\phi = \epsilon^{-1/2}(\eta - \eta^*). \quad (7.2.18)$$

An investigation of equations (7.2.16a-e) implies the following asymptotic expansions, as $\epsilon \rightarrow 0$,

$$\tilde{G} = \frac{1}{\epsilon^4} (G_0 + \epsilon^{1/2} G_1 + \epsilon G_2 + \epsilon^{3/2} G_3 + \dots),$$

$$\tilde{U} = (\epsilon \tilde{U}_0 + \epsilon^{3/2} \tilde{U}_1 + \dots)E, \tilde{V} = (\epsilon^{-1} \tilde{V}_0 + \epsilon^{-1/2} \tilde{V}_1 + \dots)E, \tilde{W} = (\epsilon^{-1/2} \tilde{W}_0 + \tilde{W}_1 + \dots)E,$$

$$\tilde{P} = (\epsilon^{-3/2} \tilde{P}_0 + \epsilon \tilde{P}_1 + \dots), \quad \tilde{\theta} = (\epsilon \tilde{\theta}_0 + \epsilon^{3/2} \tilde{\theta}_1 + \dots)E,$$

where

$$E = \exp \left\{ \frac{1}{\epsilon^2} \int^x (\beta_0(\phi) + \epsilon^{1/2} \beta_1(\phi) + \dots) d\phi \right\}, \quad (7.2.19a - f)$$

and where $\tilde{V}_0, \tilde{V}_1$ etc. are functions of ϕ and x . Note that E here represents the fast variation of the perturbation quantities along the streamwise direction whilst the dependence of V_0, W_0 etc. on x represent the slow variation of perturbation quantities due to the nonparallel effect of the boundary layer growth. Here we are only concerned with neutral stability, so we set $\beta_0 = \beta_1 = \dots = 0$. On substituting (7.2.19) into (7.2.16) and then equating the coefficients of like powers of ϵ , we obtain a hierarchy of matrix equations. To leading order, we have

$$\begin{aligned} \tilde{U}_0 &= -\frac{\text{sqrt}2x \tilde{f}_1''}{\mu_1 T_1^2} \tilde{V}_0, \quad \tilde{\theta}_0 = -\frac{Pr T_{11}}{s \mu_{10} T_{10}^2} \tilde{V}_0, \quad i \tilde{W}_0 = -\frac{1}{T_{10}} \frac{\partial \tilde{V}_0}{\partial \phi}, \\ \tilde{P}_0 &= -\frac{4s \mu_{10}}{3 T_{10}} \frac{\partial \tilde{V}_0}{\partial \phi}, \quad \tilde{\kappa} G_0 = -\frac{2s^3 \mu_{10}^2 T_{10}^4}{Pr T_{11}}, \end{aligned} \quad (7.2.20a - e)$$

where $T_{10} = T_1(\eta^*)$, $T_{11} = T_1'(\eta^*)$ and $\mu_{10} = \mu_1(T_{10})$. To next order, we deduce that $G_1 = 0$ and at next order we find that V_0 must satisfy a parabolic-cylinder equation. The centre of vortex activity η^* is determined by the condition that G_0 attains its minimum there:

$$\left(\frac{\partial G_0}{\partial \eta} \right) \Big|_{\eta=\eta^*} = 0. \quad (7.2.21)$$

After solving (7.2.5b) numerically for the basic state temperature T_1 , we then use (7.2.20e) and (7.2.21) to determine η^* and G_0 . The value of the higher order correction terms can similarly be found. We find that

$$\eta^* = 3.455 \quad \text{and} \quad \tilde{\kappa} G_0 = 27.20, \quad (7.2.22a, b)$$

for the choices $S = 0.509, Pr = 0.72, \gamma = 1.4$. Finally, we remark that the above analysis is valid as long as the local wavenumber $\tilde{k} = \sqrt{(1+S)}xa$ is large. This means that the far downstream evolution of Görtler vortices can always be described by the above theory.

§7.2.4 Neutral instability with $\kappa(x) \propto (2x)^{-3/2}$.

When the wall curvature is not proportional to $(2x)^{-3/2}$, the $O(M_\infty^{3/2})$ term on the right hand side of (7.2.8b) only vanishes at the leading order neutral position and its effect will persist in the downstream development of Görtler vortices. An important consequence of such an effect is that non-parallel effects will be important over a larger range of wavenumbers than was the case for the special curvature case. Suppose we measure the order of the wavenumber by writing it as $a = O(M_\infty^\alpha)$. Then FHB show that non-parallel effects continue to be dominant for α up to, and including, $1/4$. For $\alpha > 1/4$, non-parallel effects become negligible compared with viscous effects and an analytical expression can be obtained for the second order correction to the Görtler number expansion. This second order correction becomes of the same order as the leading order term (which, we recall, is due to the curvature of the basic state) when $\alpha = 3/8$.

The $O(1)$ -wavenumber regime—the near neutral inviscid Görtler mode

In the $O(1)$ wavenumber regime, it is convenient to determine the stability properties by considering the evolution of Görtler vortices in the neighbourhood of the leading order neutral position x_n given by (7.2.11d). Thus we shall fix the Görtler number as

$$G = \frac{2B}{(2x_n)^{3/2}\kappa(x_n)} M_\infty^{3/2} \quad (7.2.23)$$

and determine the second order correction, $\tilde{x}_n$ say, to the neutral position x_n so that Görtler vortices with G given by (7.2.23) are neutrally stable at location $x_n + \tilde{x}_n$. Replacing x_n by $x_n - \tilde{x}_n$ in (7.2.23) then gives the appropriate expansion of the Görtler number for vortices neutrally stable at $x = x_n$.

It can be shown that in the neighbourhood of x_n , the second term in the expansion of $\kappa(x)G/2$ will force a growth rate of order $M_\infty^{1/2}$. Hence we shall consider the evolution of Görtler vortices in an $O(M_\infty^{-1/2})$ neighbourhood of x_n by defining a new variable X_I by

$$X_I = (x - x_n)M_\infty^{1/2}, \quad (7.2.24a)$$

and look for asymptotic solutions of the form

$$T = T_I(X_I, \eta) + \cdots, \quad V = M_\infty^{1/2} V_I(X_I, \eta) + \cdots,$$

$$W = M_\infty^{1/2} W_I(X_I, \eta) + \dots, \quad P = M_\infty P_I(X_I, \eta) + \dots \quad (7.2.24b-e)$$

Equation (7.2.8b) becomes

$$\bar{u}\bar{v}_x + \bar{v}\bar{v}_y + \frac{1}{2}\kappa(x)G\bar{u}^2 = EX_I M_\infty + o(M_\infty),$$

where

$$E = \frac{d(G^* - Q)}{dx} \Big|_{x=x_n}. \quad (7.2.25a, b)$$

Note that it is this term that gives rise to a local growth rate of order $M_\infty^{1/2}$. On substituting (7.2.24b-e) into the perturbation equations (7.1.10), we soon find, at leading order, that T_I satisfies the equation

$$\frac{\partial^2}{\partial X_I^2} \left[\frac{\partial^2 T_I}{\partial \eta^2} - \frac{2T_1''}{T_1'} \frac{\partial T_I}{\partial \eta} - [k^2 T_1^2 - T_1 T_1' \left(\frac{1}{T_1^2} - \frac{T_1''}{T_1 T_1'^2} \right)'] T_I \right] - k^2 \frac{EX_I T_1'}{\sqrt{(1+S)x_n}} T_I = 0, \quad (7.2.26)$$

where $k = \sqrt{(1+S)x_n} a$. We can interpret (7.2.26) as the turning point equation associated with the breakdown of the WKB structure in x of the expansions (7.2.10), indeed the evolution equation (7.2.11a) is retrieved from (7.2.26) by taking X_I to be large. The latter equation admits separable solutions of the form

$$T_I(X_I, \eta) = \tilde{\phi}(X_I) \psi(\eta), \quad (7.2.27a)$$

with $\tilde{\phi}$ and ψ satisfying

$$\tilde{\phi}''(X_I) - \omega X_I \tilde{\phi}(X_I) = 0,$$

$$\psi''(\eta) - \frac{2T_1''}{T_1'} \psi'(\eta) - [k^2 T_1^2 - T_1 T_1' \left(\frac{1}{T_1^2} - \frac{T_1''}{T_1 T_1'^2} \right)'] \psi(\eta) - k^2 \frac{ET_1'}{\sqrt{(1+S)x_n} \omega} \psi(\eta) = 0, \quad (7.2.27b, c)$$

where the separation constant ω is to be determined by solving the eigenvalue problem (7.2.27c) subject to appropriate boundary conditions. By a simple substitution $z = X_I \omega^{1/3}$, equation (7.2.27b) reduces to the standard form of Airy's equation $W''(z) - zW(z) = 0$ which has the two independent solutions $Ai(z)$ and $Bi(z)$, so the solution of (7.2.27b) is given by

$$\tilde{\phi}(X_I) = a Ai(\omega^{1/3} X_I) + b Bi(\omega^{1/3} X_I), \quad (7.2.28)$$

where a and b are two constants to be determined by initial conditions.

In the large wavenumber limit, the solution of (7.2.27c) can be written in terms of Hermite polynomials. However, in the $O(1)$ wavenumber regime, this equation has to be solved by a numerical integration, and in general an infinite number of eigenvalues ω_s ($s = 0, 1, \dots$) and eigenfunctions ψ_s can be obtained. Then the general solution of (7.2.26) can be written as

$$T_I = \sum_{s=0}^{\infty} \{a_s Ai(\omega_s X_I) + b_s Bi(\omega_s X_I)\} \psi_s(\eta), \quad (7.2.29a)$$

where a_s and b_s are constants to be fixed by initial conditions at $X_I = 0$. We also find that V_0 is given by

$$V_I = -\frac{\sqrt{(1+S)x_n}T_1}{T_1'} \sum_{s=0}^{\infty} \{a_s Ai'(\omega_s X) + b_s Bi'(\omega_s X)\} \omega_s \psi_s(\eta). \quad (7.2.29b)$$

It is clear that once $T_I(X, \eta)$ and $V_I(X, \eta)$ are specified at $X_I = 0$, the coefficients (a_s, b_s) and hence the evolutionary behaviour of the perturbation field (V_0, W_0, T_0, P_0) will be completely determined.

The correction term to the neutral position can be defined as the position where a certain energy measure has zero growth rate. It is obvious that such a position would depend upon what initial conditions we impose at $X_I = 0$ and what energy measure is employed to monitor the energy growth. In principle then it is an easy matter to determine the local neutral position associated with any initial perturbation, we note however that before growth of the vortices occurs they will have an oscillatory behaviour in X_I since both Airy functions are oscillatory on the negative real axis. Clearly this occurs because the boundary between instability and stability is controlled by inviscid effects in this regime, there is no counterpart to this result in the behaviour of Görtler vortices or, for that matter, Tollmien-Schlichting waves in incompressible flows. We further note that appropriate forms for the initial conditions can be obtained from the receptivity problems associated with wall roughness or free stream disturbances, see Denier, Hall & Seddougui (1990) and Hall (1990). We merely note in passing here that it is reasonable to expect that the type of mode discussed above is more likely to be stimulated by free-stream disturbances since the effect of wall roughness is diminished by the wall layer over which the wall roughness must diffuse before reaching the unstable adjustment layer.

In the present problem, non-parallel effects dominate in the evolution of Görtler vortices mainly through the $O(M_\infty^{3/2})$ curvature of the basic state. As we increase the wavenumber, viscous effects will gradually come into play in the evolution of Görtler vortices and nonparallel effects will become less important. Wavenumbers of order $M_\infty^{1/4}$ are considered in FHB — this is the maximum order at which nonparallel effects are dominant. When the wavenumber is increased further, to $O(M_\infty^{3/8})$ order, nonparallel effects become negligible and viscous effects dominate.

The $O(M_\infty^{3/8})$ wavenumber regime—the parallel viscous mode

When the wavenumber becomes of order $M_\infty^{3/8}$, viscous effects are of the same order as the centrifugal acceleration of the basic state in the determination of the Görtler number, and the leading order inviscid result (7.2.12) has to be modified. We assume that to leading order the Görtler number now expands as

$$G = \frac{2BM_\infty^{3/2}}{\kappa(x_n)(2x_n)^{3/2}} + a^4 g_0. \quad (7.2.30)$$

Here the first term is due to the curvature of the basic state and the second term is due to viscous effects and is to be determined.

For convenience, we introduce a small parameter $\hat{\epsilon}$ and an $O(1)$ constant N by

$$\hat{\epsilon} = \frac{1}{a}, N = M_\infty^{3/2} \hat{\epsilon}^4,$$

so that (7.2.30) can be written

$$G = \left(\frac{2BN}{\kappa(x_n)(2x_n)^{3/2}} + g_0 \right) \frac{1}{\hat{\epsilon}^4}. \quad (7.2.31a - c)$$

To determine the higher order correction terms to the Görtler number expansion, we shall first fix the Görtler number as given by (7.2.31c) and consider the evolution of Görtler vortices in the neighbourhood of the leading order neutral position x_n defined by (7.2.30), aiming at finding the second order correction say $\epsilon \tilde{x}_n$ to the neutral position. As we have remarked at the beginning of the first subsection, replacing x_n by $x_n - \epsilon \tilde{x}_n$ in (7.2.31c) would give the appropriate expansion of the Görtler number for vortices neutrally stable at $x = x_n$.

The vortices under consideration vary on small lengthscales in both x and η directions. In the streamwise direction, their growth rate can be shown to be

$O(1/\hat{\epsilon})$ so that they evolve on an $O(\hat{\epsilon})$ lengthscale. In the η direction, they are confined to an $O(\hat{\epsilon}^{1/2})$ thin viscous layer because of their small wavelength. We therefore define two new variables $\hat{X}$ and ζ by

$$\hat{X} = \frac{x - x_n}{\hat{\epsilon}}, \quad \zeta = \frac{\eta - \eta^*}{\hat{\epsilon}^{1/2}}, \quad (7.2.32a, b)$$

where η^* is the centre of vortex activity and is to be determined.

We now look for asymptotic solutions of the form

$$\begin{aligned} T &= \hat{\epsilon}[\theta_0(\hat{X}, \zeta) + \hat{\epsilon}^{1/2}\theta_1(\hat{X}, \zeta) + \hat{\epsilon}\theta_2(\hat{X}, \zeta) + \cdots], \\ V &= \hat{\epsilon}^{-1}[V_0(\hat{X}, \zeta) + \hat{\epsilon}^{1/2}V_1(\hat{X}, \zeta) + \hat{\epsilon}V_2(\hat{X}, \zeta) + \cdots], \\ W &= \hat{\epsilon}^{-1/2}[W_0(\hat{X}, \zeta) + \hat{\epsilon}^{1/2}W_1(\hat{X}, \zeta) + \hat{\epsilon}W_2(\hat{X}, \zeta) + \cdots], \\ P &= \hat{\epsilon}^{-3/2}[P_0(\hat{X}, \zeta) + \hat{\epsilon}^{1/2}P_1(\hat{X}, \zeta) + \hat{\epsilon}P_2(\hat{X}, \zeta) + \cdots], \end{aligned} \quad (7.2.33a - d)$$

where the relative orders of the perturbation quantities are deduced from the perturbation equations (7.1.10). On inserting these expansions into the perturbation equations (7.1.10), expanding all coefficients there about $x = x_n$ and $\eta = \eta^*$, and then equating the coefficients of like powers of $\hat{\epsilon}$, we obtain a hierarchy of equations. To leading order, the Görtler number g_0 in (7.2.30) is determined as a solvability condition for (V_0, θ_0) and is given by

$$g_0 = -\frac{2\sqrt{(1+S)x_n\mu_{10}^2T_{10}^4}}{Pr\kappa_0T_{11}}; \quad (7.2.34)$$

whilst θ_0 , W_0 and P_0 are related to V_0 by

$$\begin{aligned} \theta_0 &= -\frac{PrT_{11}}{\sqrt{(1+S)x_n\mu_{10}T_{10}^2}}V_0, \quad iW_0 = -\frac{1}{\sqrt{(1+S)x_nT_{10}}}\frac{\partial V_0}{\partial \zeta}, \\ P_0 &= -\frac{4\mu_{10}}{3\sqrt{(1+S)x_nT_{10}}}\frac{\partial V_0}{\partial \zeta}, \end{aligned} \quad (7.2.35a - c)$$

where $T_{10} = T_1(\eta^*)$, $T_{11} = \bar{T}_1'(\eta^*)$, $\mu_{10} = \mu_1(T_{10})$ and $\kappa_0 = \kappa(x_n)$. Note that (7.2.34) is of the same form as (7.2.20e), as we would expect.

At next order, we obtain three expressions for θ_1 , W_1 and P_1 in terms of V_1 and V_0 and the condition that

$$\left(\frac{dg_0}{d\eta}\right)\bigg|_{\eta=\eta^*} = 0, \quad (7.2.36)$$

which implies that η^* is where g_0 attains its minimum.

At one order higher, a solvability condition for (V_2, θ_2) follows which requires that V_0 must satisfy an evolution equation of the form

$$\frac{\partial^2 V_0}{\partial \zeta^2} - \tilde{\gamma} \frac{\partial V_0}{\partial \hat{X}} - \tilde{a} \zeta^2 V_0 + \tilde{b} \hat{X} V_0 = 0 \quad (7.2.37),$$

where $\tilde{\gamma}, \tilde{a}(>0)$ and $\tilde{b}$ are functions of the basic state, evaluated at (x_n, η^*) .

The solutions of (7.2.37) which satisfy the conditions $V_0 \rightarrow 0$ as $|\zeta| \rightarrow \infty$ can be written in terms of parabolic cylinder functions. The neutral position $\tilde{x}_n$ is taken to be the point where $\partial V_0 / \partial \hat{X} = 0$, and it is found that

$$\tilde{x}_n = \frac{\sqrt{\tilde{a}}}{\tilde{b}}. \quad (7.2.38)$$

With the expression for $\tilde{x}_n$ determined, we could now replace x_n by $x_n - \epsilon \tilde{x}_n$ in (7.2.31c) and then expand the two terms on the right hand side up to and including the $O(1/\epsilon^3)$ term, hence obtaining an expression for G corresponding to Görtler vortices which are neutrally stable at position x_n . Note that the above analysis is essentially identical to that presented in 7.2.3 for the special curvature case, in the high-wavenumber limit.

§7.2.5 The wall mode.

We have seen that as the wavenumber becomes large, Görtler vortices become increasingly trapped in the $O(1)$ temperature adjustment layer. Thus the preceding sub-sections are devoted to Görtler vortices which have wavelength of $O(1)$ or smaller and which are trapped in the temperature adjustment layer. Clearly it is possible for vortices of wavelength smaller than the thickness of the transition layer to be excited, far enough downstream the local wavenumber will become comparable to the adjustment layer thickness and the previous analysis will apply. However before this occurs the vortices must be described by an analysis which takes account of the fact that they are of wavelength much larger than the adjustment layer thickness, we shall now address that situation.

We can immediately deduce from the definitions (7.1.7a) and (7.1.7b) that the variation dy , of the physical variable $y^\dagger$, and the variation $d\eta$, of the Blasius-Howarth-Dorodnitsyn similarity variable η , satisfy

$$dy = \sqrt{(1+S)x\bar{T}} d\eta.$$

[†] note that the dimensional normal variable $y^* = Re^{-\frac{1}{2}} Ly$.

The wall layer which corresponds to $\eta = O(M_\infty^{-1/2})$ with $\bar{T} = O(M_\infty^2)$ is therefore actually of $O(M_\infty^{3/2})$ thickness in terms of the physical variable y , whilst the temperature adjustment layer is still of $O(1)$ thickness. Thus a natural scale for larger wavelength vortices is provided by the thickness of the wall layer, the appropriate size of the Görtler number is found by rescaling the vortex wavelength and velocity field by the scales relevant to the wall layer. Such Görtler vortices are referred to as the wall modes and are considered in the present subsection to complete our brief review of the linear stability theory.

Since in the large Mach number limit the boundary layer thickens by $O(M_\infty^{3/2})$, we should rescale (y, z) by a factor $M_\infty^{3/2}$ and the corresponding velocity components likewise. This effectively replaces all “ $Re^{-1/2}$ ”s by “ $Re^{-1/2} M_\infty^{3/2}$ ”. It is therefore appropriate to rescale the Görtler number G and the wavenumber a by defining

$$G_W = M_\infty^{-3/2} G \quad \text{and} \quad a_W = M_\infty^{3/2} a, \quad (7.2.40a, b)$$

where $G_W, a_W \sim O(1)$.

The basic state in the wall layer is (from (7.1.6a-c), (7.1.7) and (7.2.1)) given by

$$\bar{u} = f_0 \xi(\xi), \quad \bar{v} = \frac{M_\infty^{3/2}}{\sqrt{2x}} [-T_0 f_0(\xi) + f_0 \xi \Omega_W(\xi)],$$

where

$$\Omega_W(\xi) = \int_0^\xi T_0 d\xi. \quad (7.2.41a - c)$$

It can be deduced, from the perturbation equations (7.1.10), that the relative scalings of the velocity, pressure and temperature disturbance fields are such that

$$V = O(M_\infty^{3/2} U), \quad W = O(M_\infty^{3/2} U), \quad T = O(M_\infty^2 U), \quad P = O(M_\infty U).$$

Therefore solutions of the form

$$U = U_W(x, Y) + \cdots, \quad V = M_\infty^{3/2} V_W(x, Y) + \cdots, \quad W = M_\infty^{3/2} W_W(x, Y) + \cdots,$$

$$P = M_\infty^2 P_W(x, Y) + \cdots, \quad T = M_\infty^2 \theta_W(x, Y) + \cdots, \quad (7.2.42a - e)$$

are sought. On substituting these into the perturbation equations (7.1.10) and then equating the coefficients of like powers of M_∞ , to leading order a set of partial differential equations are obtained which govern the evolution of Görtler

vortices in the wall layer. the wall layer. Obviously, this set of partial differential equations have to be solved numerically to determine the evolution properties of Görtler vortices in the wall layer. We further note that the downstream velocity component of the perturbation now does not decouple from the other disturbance quantities — these wall modes can be thought of as being the ‘classical’ hypersonic limiting-form of the $O(1)$ Mach number Görtler vortices (which have been studied by, for example, Wadey, 1990; Spall & Malik, 1989).

We conclude this subsection by stating the most important results of FHB’s investigation of the wall mode. They show that the wall layer can support a disturbance trapped in the wall layer with wavelength comparable with the wall layer thickness. This mode is dominated by nonparallel effects and has a neutral Görtler number which is a monotonic decreasing function of the vortex wavenumber; the neutral curves have no ‘right-hand’ branches. In the limit of high vortex wavenumbers the mode takes on a structure essentially identical to that found for the small wavenumber limit of the inviscid Görtler modes of wavelength comparable with the adjustment layer thickness. Moreover in this limit the wall-mode vortices have neutral Görtler number which approaches from above the zeroth order approximation to the neutral Görtler number of the temperature-adjustment-layer modes.

§7.3 THE RAYLEIGH INSTABILITY IN HYPERSONIC FLOWS.

For large Mach numbers, we have seen that the basic boundary-layer state splits into two regions; an inner wall layer, where $\eta \sim M_\infty^{-1/2}$; and an outer layer, the so-called temperature adjustment layer, where $\eta \sim O(1)$. Thus there are two natural choices for the size of the wavelengths α, β (defined in §7.1.3). One choice, to be considered first, has $\alpha, \beta \sim O(1)$ so that the modes have wavelengths comparable with the thickness of the adjustment region — the so-called *vorticity mode* is confined to this region and are thus dependent on the Mach number being large. The vorticity mode has been discussed in some detail in Chapter 6 for the case $Pr = 1$. In §7.3.1 we formulate the governing equation for general Prandtl number. The other choice has $\alpha, \beta \sim M_\infty^{-3/2}$, so that the modes have wavelengths comparable with the thickness of the wall layer ($\equiv$ ‘classical’ boundary layer). These so-called *acoustic modes* were also mentioned in Chapter 6 — in §7.3.2 we formulate the governing equation for general Prandtl number.

§7.3.1 The vorticity mode.

This inviscid Rayleigh mode, which has $(\alpha, \beta) \sim O(1)$, is ‘trapped’ in the temperature adjustment layer, where $\eta \sim O(1)$ — in general they decay to zero before the wall layer is entered. The neutral mode has wavespeed equal to the streamwise velocity of the basic state evaluated at the generalised inflexion point (g.i.p.) which can be shown to lie in this temperature adjustment layer. Thus, in the vicinity of the g.i.p., the basic-state streamwise velocity

$$\bar{u} = f_\eta(\eta) = 1 - \frac{\bar{f}_{1\eta}(\eta)}{M_\infty^{\frac{3}{2Pr} + \frac{1}{2}}} + \dots \equiv 1 - M_\infty^{-1} \bar{f}_{1\eta}(\eta) + \dots,$$

from which it follows that the wavespeed of disturbance of the disturbance (vorticity mode) should be written

$$c = 1 - \frac{c_v}{M_\infty^{\frac{3}{2Pr} + \frac{1}{2}}} + \dots, \quad (7.3.1)$$

where $c_v \sim O(1)$ and is real for the case of neutrality. In fact, the neutral value of $c_v = \bar{f}_{1\eta}(\eta_g)$, where η_g is the location of the g.i.p.. Writing $\bar{p} = \hat{p}_v + \dots$ and substituting from (7.2.4a,b), (7.3.1) for $f, \bar{T}$ and c , we find that the zeroth order approximation to (7.1.14a), in the temperature adjustment layer, is the so-called vorticity mode equation

$$\frac{d^2 \hat{p}_v}{d\eta^2} - \frac{2\bar{f}_1''}{\bar{f}_1' - c_v} \frac{d\hat{p}_v}{d\eta} - (\alpha^2 + \beta^2) T_1^2 \hat{p}_v = 0. \quad (7.3.2)$$

This equation, which holds for all Prandtl numbers, must be solved subject to $\hat{p}$ vanishing in the limits $\eta \rightarrow 0$ and $\eta \rightarrow \infty$, i.e. the disturbance is to be confined to the temperature adjustment layer. In Chapter 6, numerical results $c_v \equiv c_v(\alpha^2 + \beta^2)$ were given for $Pr = 1$ (when $\bar{f}_1$ can be expressed in terms of T_1). Finally we note that only one neutral mode exists, for each set of physical parameters: cf. the acoustic modes, to be discussed in the next subsection.

§7.3.2 The acoustic modes.

Classically, inviscid Rayleigh modes are sought which have wavenumbers comparable with the thickness of the viscous boundary layer. In the present context, the latter corresponds to the inner wall layer, where the physical variable $y = Re^{\frac{1}{2}} L^{-1} y^* \sim M_\infty^{\frac{3}{2}}$. Thus we seek solutions to (7.1.14a) with

$$(\alpha, \beta) = M_\infty^{-\frac{3}{2}} (\alpha_a, \beta_a), \quad (7.3.3)$$

where $\alpha_a, \beta_a \sim O(1)$. The wavespeed again has the form (7.3.1), we write $\tilde{p} = \hat{p}_a + \dots$, and substituting for η, f and $\bar{T}$ (from (7.2.1)) in (7.1.14a) yields the so-called acoustic mode equation

$$\frac{d^2 \hat{p}_a}{d\xi^2} - \frac{2f_{0\xi\xi}}{f_{0\xi} - 1} \frac{d\hat{p}_a}{d\xi} - (\alpha_a^2 + \beta_a^2) T_0 \left\{ T_0 - \frac{\alpha_a^2 (f_{0\xi} - 1)^2}{(\alpha_a^2 + \beta_a^2)} \right\} \hat{p}_a = 0. \quad (7.3.4a)$$

The conditions that there is no normal velocity at the wall, and that the disturbance is, in general, confined to the boundary layer, can be expressed as

$$\hat{p}_{a\xi} = 0 \quad \text{on} \quad \xi = 0, \quad \hat{p}_a \rightarrow 0 \quad \text{as} \quad \xi \rightarrow \infty. \quad (7.3.4b, c)$$

The special case of 2-D disturbance and unity Prandtl number was considered in §6.3.3 where it was reported that an infinite, discrete set of (neutral) eigenvalues α_a were found to exist. This mode has received much less attention than the vorticity mode because its associated unstable modes are far less ‘dangerous’ than those of the vorticity mode. Thus we see that for both the Görtler and Rayleigh instabilities, the most unstable disturbances are located in the temperature adjustment layer.

§7.4 THE LINEAR RAYLEIGH INSTABILITY OF HYPERSONIC FLOWS MODIFIED BY NON-LINEAR GÖRTLER VORTEX EFFECTS.

In the previous two sections we have considered separately, in isolation, the Rayleigh and Görtler instabilities of hypersonic flow over a concave plate. We considered small enough disturbances ($h, \Delta \ll 1$) such that non-linear effects could be ignored. In this section we are still concerned with linear Rayleigh modes and investigate how they are affected as the unstable, linear, Görtler vortices evolve downstream, becoming more and more non-linear in nature until they modify/alter the basic state at leading order.

Very recently, Hall & Horseman (1990) have studied the inviscid *secondary instability* of fully nonlinear vortex structures in growing, incompressible, boundary layers. They show that the strong vortex activity renders the previously inviscidly-stable Blasius flow unstable to linear Rayleigh-type modes. This section can be regarded, in some sense, as an extension of their work to the case of hypersonic

flows — in fact, the study described in §7.4.2 is very closely related to the latter paper.

Returning to hypersonic flow, some theoreticians might also refer to the acoustic and vorticity Rayleigh modes as being possible secondary instabilities of the primary non-linear vortex state. However, in the opinion of the author, this is mis-leading as it suggests that in the absence of nonlinear Görtler vortices such instabilities are not possible. We have seen that the basic state, in the absence of nonlinear Görtler vortices, is still unstable to (linear) Rayleigh modes and thus it is felt more appropriate to refer to the Rayleigh stability properties as being modified by, rather than the Rayleigh modes being secondary instabilities of, the nonlinear Görtler vortex flow. We note that the latter flow state does have another, *proper*, secondary instability in which the thin shear layers bounding the (high-wavenumber) vortex activity are destabilised by so-called wavy modes — the counterpart for incompressible flow was first studied by Hall & Seddougui (1989).

Again, as implicitly assumed in previous sections, we assume that the size, h , of the Rayleigh modes is small enough such that nonlinear combination of these modes will not force the vortices (linear or nonlinear) at the leading orders of concern. That is, the Rayleigh modes do not affect the governing equations for the vortex state — in this section we are not considering a vortex/wave interaction.

We have seen that there are, essentially, two types of Rayleigh and Görtler modes in hypersonic flows (over a concave plate): those trapped in the temperature adjustment layer (the Rayleigh vorticity mode and the ‘non-wall’ Görtler modes) with $O(1)$ wavelengths; and those lying in the high temperature region (the Rayleigh acoustic modes and the Görtler wall mode) with wavelengths of $O(M_\infty^{\frac{3}{2}})$. Thus the nonlinear effects of the non-wall Görtler modes will modify the stability characteristics of the Rayleigh vorticity mode; whilst the nonlinear effects of the Görtler wall modes will modify the stability characteristics of the Rayleigh acoustic modes.

In the following subsections, we consider two illustrative cases; the first involving the temperature-adjustment-layer modes; and the second involving the wall-layer modes.

§7.4.1 The modifications to the vorticity mode due to a nonlinear, high-wavenumber, Görtler vortex state.

For simplicity, we consider the $O(M_\infty^{\frac{3}{8}})$ -wavenumber-parallel-viscous Görtler modes trapped in the temperature adjustment layer (previously discussed in the second part of §7.2.4) — we have seen that the linear stability properties of such modes can be easily described using asymptotic methods. The evolution of these modes immediately downstream of the point of linear instability, (x_n, η^*) , is described by the linear evolution (7.2.37). The weakly nonlinear evolution of these vortices (further downstream) can be described asymptotically, by a simple generalisation of the corresponding incompressible theory due to Hall (1982b). Here the basic state is unaltered, to leading order, and thus there is no vortex-related effect felt by the Rayleigh vorticity mode.

It can be deduced from the weakly non-linear analysis that at positions $O(1)$ downstream of the neutral position, the mean temperature correction and the mean streamwise velocity (due to the non-linear Görtler vortex state) become as large as the corresponding basic state quantities, T_1 and $\bar{f}_1$. This nonlinear theory was first established by Hall & Lakin (1988), for incompressible flow. Thus, as T_1 and $\bar{f}_1$ occur in the Rayleigh vorticity mode equation, (7.3.2), the stability characteristics of the latter can be expected to suffer modification due to the modification of the former by the nonlinear vortex state.

Due to lack of time, the present author has not been able to fully formulate this non-linear vortex state and thus has not calculated either the non-linear vortex state or the modified eigenvalues of the vorticity mode equation. However, one can immediately (intuitively) deduce (from the incompressible theory of Hall & Lakin; and the corresponding compressible theory due to Wadey, 1990) that the ‘new’ basic state quantities f_{1g}, T_{1g}, μ_{1g} , defined by

$$f = \eta - \frac{\bar{\Delta}}{M_\infty^{\frac{1}{2}}} + \frac{\bar{f}_{1g}(x, \eta)}{M_\infty^{\frac{3}{2Pr} + \frac{1}{2}}},$$

$$\bar{T} = T_{1g}(x, \eta) + \dots, \quad \bar{\mu} = \mu_{1g}(x, \eta) + \dots, \quad (7.4.1a - c)$$

will be governed by coupled mean-flow/vortex equations of the form

$$\begin{aligned} \frac{\partial}{\partial \eta} \left(\frac{T_{1g}^{\frac{1}{2}}}{T_{1g} + S} \frac{\partial \bar{f}_{1g}}{\partial \eta} \right) + \eta \frac{\partial \bar{f}_{1g}}{\partial \eta} - 2x \frac{\partial \bar{f}_{1g}}{\partial x} &= \bar{F}_{HL1}(\partial/\partial \eta; \bar{f}_{1g}, T_{1g}, |V_0|^2), \\ \frac{1}{Pr} \frac{\partial}{\partial \eta} \left(\frac{T_{1g}^{\frac{1}{2}}}{T_{1g} + S} \frac{\partial T_{1g}}{\partial \eta} \right) + \eta \frac{\partial T_{1g}}{\partial \eta} - 2x \frac{\partial T_{1g}}{\partial x} &= \bar{F}_{HL2}(\partial/\partial \eta; \bar{f}_{1g}, T_{1g}, |V_0|^2), \\ \left[BN \left(\frac{\kappa(x)}{\kappa(x_n)(2x_n)^{3/2}} - \frac{1}{(2x)^{3/2}} \right) + \frac{1}{2} \kappa g_0 \right] \frac{Pr}{\mu_{1g}^2 T_{1g}^4} \frac{\partial T_{1g}}{\partial \eta} + \sqrt{(1+S)x} &= 0, \\ \text{where} \quad \mu_{1g} &= (1+S) \frac{T_{1g}^{\frac{3}{2}}}{T_{1g} + S}. \end{aligned} \quad (7.4.2a-d)$$

The ‘forcing functions’ $\bar{F}_{HL1,2}$ can be determined analytically — we see that the mean flow is driven by $|V_0|^2$, arising from the nonlinear combination of vortex terms — h has now grown to $O(1)$ size, in (7.1.9). Note the downstream influence of basic-state curvature (the term proportional to B).

In fact, the non-linear vortex state only exists in a finite, growing region bounded above and below by ‘shear layers’ at $\eta = \eta_1, \eta_2$ (say). The locations $\eta_1(x)$ and $\eta_2(x)$ must be determined numerically from a ‘free-boundary’ problem. Outside of the vortex region ($\eta > \eta_1, \eta < \eta_2$) $V_0 \equiv 0$ but the mean-state is still different than if there were no non-linear vortex activity at all.

Note that the mean-flow (basic state) is (i), independent of z -variation, and (ii), still two-dimensional, to orders of concern. Thus, its Rayleigh stability properties are still governed by the ordinary differential equation (7.3.2), but with $(\bar{f}_1, T_1)$ replaced by $(\bar{f}_{1g}, T_{1g})$.

§7.4.2 The necessary modifications to the acoustic modes due to fully non-linear Görtler wall modes.

In this subsection it is assumed that a fully nonlinear vortex state exists in the wall layer — we do not concern ourselves with its evolution, or with the detailed formulation of the governing equations. This flow state is merely the ‘classical’ generalisation to hypersonic flow (in the sense that the wavenumbers are scaled on the boundary-layer thickness, $M_\infty^{\frac{3}{2}}$) of the fully nonlinear, incompressible, vortex flow studied and computed by Hall (1988). Moreover, the ‘generalised’

acoustic modes studied here are merely the hypersonic counterparts of the inviscid secondary instabilities (to this fully non-linear vortex state) investigated by Hall & Horseman (1990).

The underlying flow state, ‘felt’ by the acoustic Rayleigh modes, now has spanwise variation at leading order — it has been completely altered by the vortex activity. The $O(1)$ co-ordinates (x, Y, Z) are now defined by

$$L^{-1}(x^*, y^*, z^*) = (x, Re^{-\frac{1}{2}} M_\infty^{\frac{3}{2}} Y, Re^{-\frac{1}{2}} M_\infty^{\frac{3}{2}} Z), \quad (7.4.3a - c)$$

whilst the $O(1)$ nonlinear vortex velocities, pressure and temperature, $(\bar{U}, \bar{V}, \bar{W}, \bar{T}, \bar{P})$, are defined by

$$(u^*, v^*, w^*)/u_\infty^* = (\bar{U}(x, Y, Z), Re^{-\frac{1}{2}} M_\infty^{\frac{3}{2}} \bar{V}(x, Y, Z), Re^{-\frac{1}{2}} M_\infty^{\frac{3}{2}} \bar{W}(x, Y, Z)),$$

$$T^*/T_\infty^* = M_\infty^2 \bar{T}(x, Y, Z)$$

and

$$p^*/(\rho_\infty^* u_\infty^{*2}) = \underbrace{\gamma^{-1} M_\infty^{-2}}_{p_\infty} + Re^{-1} M_\infty \bar{P}(x, Y, Z). \quad (7.4.4a - e)$$

These forms can be deduced from §7.2.5, where the linear wall modes were considered; alternatively those for the co-ordinates, the velocities and the pressure can be arrived at by replacing $Re^{-\frac{1}{2}}$ (the thickness of the incompressible boundary layer) by $Re^{-\frac{1}{2}} M_\infty^{\frac{3}{2}}$ (the thickness of the hypersonic boundary layer) in the scales of Hall (1988), Hall & Horseman (1990).

We now consider the inviscid stability of the fully non-linear vortex state, represented by (7.4.4a-e). As the leading-order flow state now has z -variation, it is no longer appropriate to seek inviscid modes with harmonic z -dependence. Instead, we perturb this flow state by small disturbances having general z -dependence:

$$(u^*, v^*, w^*)/u_\infty^* = (\bar{U}, \bar{V}, \bar{W}) + \hat{\Delta}[(\hat{u}(Y, Z), \hat{v}(Y, Z), \hat{w}(Y, Z))\hat{E} + \text{c.c.}] + \dots,$$

$$T^*/T_\infty^* = M_\infty^2 \bar{T}(x, Y, Z) + M_\infty^2 \hat{\Delta}[\hat{T}(Y, Z)\hat{E} + \text{c.c.}] + \dots,$$

$$p^*/(\rho_\infty^* u_\infty^{*2}) = \gamma^{-1} M_\infty^{-2} + Re^{-1} M_\infty \bar{P}(x, Y, Z) + M_\infty^{-2} \hat{\Delta}[\hat{p}(Y, Z)\hat{E} + \text{c.c.}] + \dots, \quad (7.4.5a - e)$$

where $\hat{\Delta} \ll 1$; $\hat{E} = \exp\{i\alpha Re^{\frac{1}{2}} M_\infty^{-\frac{3}{2}}(x - ct)\}$, $\alpha, c \sim O(1)$; and the ‘hatted’ quantities represent the inviscid disturbance (Rayleigh mode). The M_∞ -factors

of $\hat{T}$ and $\hat{p}$ are implied from the large Mach number form of the linearised Euler equations for general Mach number. When the expansions (7.4.5) are substituted into the Navier–Stokes equations, we find that the Rayleigh disturbances satisfy

$$\begin{aligned} i\alpha(\bar{U} - c)\hat{u} + \bar{U}_Y\hat{v} + \underline{\bar{U}_Z\hat{w}} &= -i\alpha\bar{T}\hat{p}, \\ i\alpha(\bar{U} - c)\hat{v} &= -\bar{T}\hat{p}_Y, \\ i\alpha(\bar{U} - c)\hat{w} &= -\bar{T}\hat{p}_Z, \\ i\alpha\hat{u} + \hat{v}_Y + \hat{w}_Z + i\alpha(\bar{U} - c)\hat{p} &= 0. \end{aligned} \quad (7.4.6a - d)$$

Note that the z -variation of the underlying flow is only ‘felt’ through the stream-wise velocity component, $\bar{U}$ (see underlined term above).

The disturbance–equations (7.4.6) can be combined to form the *generalised acoustic mode equation*

$$\hat{p}_{YY} + \hat{p}_{ZZ} - \frac{2(\bar{U}_Y\hat{p}_Y + \bar{U}_Z\hat{p}_Z)}{(\bar{U} - c)} + \frac{(\bar{T}_Y\hat{p}_Y + \bar{T}_Z\hat{p}_Z)}{\bar{T}} - \alpha^2 \left[1 - \frac{(\bar{U} - c)^2}{\bar{T}} \right] \hat{p} = 0. \quad (7.4.7a)$$

This partial differential equation must be solved subject to the boundary conditions

$$\hat{p}_Y(0, Z) = 0, \quad \hat{p}(Y, Z) \rightarrow 0 \quad \text{as } Y \rightarrow \infty; \quad (7.4.7b, c)$$

additionally, we require that $\hat{p}$ is periodic in Z (due to the spanwise periodicity of the underlying vortex state). Equation (7.4.7a), together with the boundary and periodicity conditions, specifies the temporal stability eigenrelation $c \equiv c(\alpha)$.

Note that the usual acoustic mode equation, (7.3.4a), is easily recovered (for weaker vortex states) by setting $\bar{U}_Z = 0 = \bar{T}_Z$, transforming to the Howarth–Dorodnitsyn variable and then setting $(\alpha, \frac{\partial}{\partial Z}) = (\alpha_a, i\beta_a)$. Alternatively, the pressure equation of Hall & Horseman (1990) can be easily recovered by setting $\bar{T} = M_\infty^{-2}$ and then letting $M_\infty \rightarrow 0$. The solution to the above eigen–problem requires a (non–trivial) numerical solution in general. However for weaker Z -dependence, corresponding to low wavenumber vortices ($a_W \ll 1$), we expect that some further analytical progress will be possible, along the lines of that described by Hall & Horseman (1990). Finally note that, similarly, we can derive a *generalised vorticity mode equation* to describe the Rayleigh stability of

the strongly-vortex-affected temperature adjustment layer. This has the form of equation (7.4.7a) but now the term in the square brackets is replaced by unity.

§7.5 SOME COMMENTS ON THE INTERACTION OF THE RAYLEIGH AND GÖRTLER MODES.

In the previous sections we have considered, in isolation, the Görtler and Rayleigh instabilities, before considering how the presence of strong vortex activity will modify the Rayleigh stabilities of the basic flow. As the title of this section implies, we shall now investigate the possibility of interaction between the two instabilities.

In the present context, for such a vortex/wave interaction, we require that, (i), the vortex activity ‘affects’ the Rayleigh waves, and (ii), the presence of these Rayleigh modes ‘affects’ the vortex state. In §7.4.1 we considered the high-wavenumber nonlinear vortex state, in the temperature adjustment layer, and found that the eigen-problem for the inviscid (Rayleigh) modes still takes the form of a linear, second-order, differential equation (plus boundary conditions) for the pressure disturbance but now the coefficients are functions of the underlying non-linear vortex state. However, when the vortex state is fully nonlinear (as in §7.4.2) a partial differential equation now replaces the ordinary differential Rayleigh equation.

In contrast to the non-linear vortices (which affect the Rayleigh modes indirectly, via the $O(1)$ mean-flow corrections), linear Rayleigh modes can affect/force the (linear or non-linear) Görtler vortices directly via the nonlinear combination of two modes having opposite spanwise wavenumber (cf. the vortex/wave interaction studied in Chapter 3). This mechanism has the ‘advantage’ that the amplitude of the Rayleigh modes does not have to be so large as to modify the basic flow at leading order.

Again, the full details for all possible cases remain to be formulated. This is principally due to the fact that the Görtler vortex linear instability in hypersonic flows has only very recently been properly understood. Moreover, the resulting infinite set of coupled equations will require a non-trivial numerical solution.

Obviously, we would expect the acoustic Rayleigh modes to interact with the Görtler wall modes; whilst the Rayleigh vorticity mode should interact with the

vortices trapped in the temperature adjustment layer. At first sight, it appears sensible to consider the special case of the interaction involving the large-wavenumber parallel vortex modes trapped in the latter layer — it should be simpler than, and provide insight into, the ‘general’ problem. However, one soon realises that such an interaction (as envisaged above) is not tenable: the inviscid mode would need to have large wavenumber ($\sim O(M_\infty^{\frac{3}{2}})$, for the case of general curvature) and such modes do not exist (see Chapter 6)!

The vorticity mode can ~~also~~ interact with the $O(1)$ -wavenumber Görtler modes, whose linear stability properties are also inviscid in nature (see §7.2.4). The nonlinear evolution of such modes has yet to be studied and thus it is not easy to consider this particular (possible) interaction, at the present time. Thus we turn to the inner, high-temperature region.

The neutral curves for the Görtler wall mode do not have ‘right-hand’ branches, in contrast to their incompressible and $O(1)$ Mach number counterparts; thus the non-linear Hall-Lakin theory cannot be employed in the high-wavenumber limit ($a_W \gg 1$). Thus, we must consider the general non-linear vortex state (discussed in §7.4.2); this exists across the whole wall-layer, with all harmonics of $a_W \sim O(1)$ present. The corresponding vortex state for incompressible boundary-layer flow (which, in some sense, can be thought of as a special case of the present wall-mode equations) has been computed by Hall (1988). The Rayleigh modes appropriate to this interaction are governed by the generalised acoustic mode equation (7.4.7a) — in general they can, at best, be expressed as an infinite Fourier series in Z . As the nonlinear vortex state has the Fourier-series forms, we can expect that the interaction will be controlled by an infinitely-coupled system of equations — the numerical solution of such coupled systems, commonly found in theoretical formulations of strongly non-linear vortex/wave interactions, is very difficult.

The size of $\hat{\Delta}$ necessary for an interaction.

To conclude this section, we deduce the smallest size of the small linearisation parameter $\hat{\Delta} \ll 1$ (the amplitude of the Rayleigh modes) that will lead to the Rayleigh modes affecting (forcing) the strongly non-linear vortex state in the wall-layer i.e. we seek the smallest size of $\hat{\Delta}$ necessary for an interaction.

Consider the term $\frac{\mathbf{u}v_x}{T}$, appearing in the y -momentum equation of the governing Navier Stokes equations. This results in the ‘non-linear’ vortex-term $\bar{T}^{-1}\bar{\mathbf{u}}\bar{V}_x$; this leading-order term occurs at $O(Re^{-\frac{1}{2}}M_\infty^{\frac{5}{2}})$ (from (7.4.3),(7.4.4)). The former will also result in the non-linear combination $\bar{T}^{-1}i\alpha\hat{\mathbf{u}}\hat{v}^{(c.c.)}$ involving the Rayleigh modes, among others. The latter term will contain, also among others, stationary vortex-like terms: those terms proportional to $\hat{E}\hat{E}^{(c.c.)}$. Thus the potential forcing terms (due to the nonlinear combination of the linear Rayleigh modes) arise at $O(\hat{\Delta}^2 Re^{\frac{1}{2}} M_\infty^{-\frac{1}{2}})$ (from (7.4.4d),(7.4.5) and definition of $\hat{E}$). Hence, these potential forcing terms do indeed force the y -momentum, non-linear, vortex equation (and thus completing the proposed interaction) if the previously deduced orders are the same, i.e.

$$\begin{aligned} Re^{-\frac{1}{2}}M_\infty^{\frac{5}{2}} &\sim \hat{\Delta}^2 Re^{\frac{1}{2}} M_\infty^{-\frac{1}{2}}. \\ \leftrightarrow \hat{\Delta} &\sim Re^{-\frac{1}{2}} M_\infty^{\frac{3}{2}} (\ll 1) \end{aligned} \quad (7.5.1).$$

Note that this choice of $\hat{\Delta}$ is small enough such that the equations for the Rayleigh mode are still linear. It is obvious that the other nonlinear inertial terms, including those in the other governing Navier Stokes equations, also lead to ‘forcing’ terms — for our choice, (7.4.8), of $\hat{\Delta}$ it can be shown that all leading-order forcing is due to the material derivative terms $v\partial_y + w\partial_z$. Further it can be shown that there is no direct forcing of the x -momentum equation; moreover this choice of $\hat{\Delta}$ is consistent in the sense that none of these forcing terms are solely leading-order.

The complete formulation of this proposed interaction should not cause too many problems — however the resulting infinite sets of coupled equations will not be easy to solve numerically.

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