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Test vectors for non-Archimedean Godement–Jacquet zeta integrals

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Abstract

Given an induced representation of Langlands type (π, V_π) of $\mathrm{GL}_n(F)$ with F non-Archimedean, we show that there exist explicit choices of matrix coefficient β and Schwartz–Bruhat function Φ for which the Godement–Jacquet zeta integral $Z(s, \beta, \Phi)$ attains the L-function $L(s, \pi)$.

1. *Introduction*

Let F be a non-Archimedean local field with ring of integers $\mathcal{O}$, maximal ideal $\mathfrak{p}$, and uniformiser ϖ, so that $\varpi\mathcal{O} = \mathfrak{p}$ and $\mathcal{O}/\mathfrak{p} \cong \mathbb{F}_q$ for some prime power q. We normalise the absolute value $|\cdot|$ on F such that $|\varpi| = q^{-1}$.

Let (π, V_π) be a generic irreducible admissible smooth representation of $\mathrm{GL}_n(F)$, where F is a non-Archimedean local field. Given a matrix coefficient $\beta(g) = \langle \pi(g) \cdot v_1, \widetilde{v_2} \rangle$ of π, where $v_1 \in V_\pi$ and $v_2 \in V_{\widetilde{\pi}}$, and given a Schwartz–Bruhat function $\Phi \in \mathscr{S}(\mathrm{Mat}_{n\times n}(F))$, we define the Godement–Jacquet zeta integral [**3, 5**]

$$Z(s, \beta, \Phi) := \int_{\mathrm{GL}_n(F)} \beta(g)\Phi(g)|\det g|^{s+\frac{n-1}{2}}\, dg, \tag{1.1}$$

which is absolutely convergent for $\Re(s)$ sufficiently large. The test vector problem for Godement–Jacquet zeta integrals is the following.

Test Vector Problem. Given a generic irreducible admissible smooth representation (π, V_π) of $\mathrm{GL}_n(F)$, determine the existence of K-finite vectors $v_1 \in V_\pi$, $\widetilde{v_2} \in V_{\widetilde{\pi}}$, and a Schwartz–Bruhat function $\Phi \in \mathscr{S}(\mathrm{Mat}_{n\times n}(F))$ such that

$$Z(s, \beta, \Phi) = L(s, \pi).$$

The Archimedean analogue of this problem has been resolved for $F = \mathbb{C}$ by Ishii [**4**] and for $F = \mathbb{R}$ by Lin [**12**]†. For non-Archimedean F, the spherical case is resolved in [**3**, Lemma 6.10]: one takes v_1 and v_2 to be spherical vectors and

$$\Phi(x) = \begin{cases} 1 & \text{if } x \in \mathrm{Mat}_{n\times n}(\mathcal{O}), \\ 0 & \text{otherwise}. \end{cases}$$

We solve the ramified case of this problem.

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†The author has been unable to verify certain aspects of [**12**]. In particular, the functions constructed in [**12**, (6.5) and (6.7)] are defined only on the maximal compact subgroup $K = \mathrm{O}(n)$ of $\mathrm{GL}_n(\mathbb{R})$. For these functions to be elements of certain induced representations of $\mathrm{GL}_n(\mathbb{R})$, they must transform under the action of diagonal matrices $a = \mathrm{diag}(a_1, \ldots, a_n) \in \mathrm{A}_n(\mathbb{R})$ in a specified manner, and this action does not seem to be compatible with the definitions [**12**, (6.5) and (6.7)] when $k \in K$ is taken to be a diagonal orthogonal matrix.

THEOREM 1.2. *Let (π, V_π) be a generic irreducible admissible smooth representation of $\mathrm{GL}_n(F)$ of conductor exponent $c(\pi) > 0$. Let $\beta(g)$ denote the matrix coefficient $\langle \pi(g) \cdot v^\circ, \widetilde{v^\circ} \rangle$, where $v^\circ \in V_\pi$ is the newform of π normalised such that $\beta(1_n) = 1$. Define the Schwartz–Bruhat function $\Phi \in \mathscr{S}(\mathrm{Mat}_{n\times n}(F))$ by*

$$\Phi(x) := \begin{cases} \dfrac{\omega_\pi^{-1}(x_{n,n})}{\mathrm{vol}(K_0(\mathfrak{p}^{c(\pi)}))} & \text{if } x \in \mathrm{Mat}_{n\times n}(\mathcal{O}) \text{ with } x_{n,1}, \dots, x_{n,n-1} \in \mathfrak{p}^{c(\pi)} \text{ and } x_{n,n} \in \mathcal{O}^\times, \\ 0 & \text{otherwise,} \end{cases} \tag{1.3}$$

where ω_π denotes the central character of π and the congruence subgroup $K_0(\mathfrak{p}^{c(\pi)})$ is as in (3.1). Then for $\Re(s)$ sufficiently large,

$$Z(s, \beta, \Phi) = L(s, \pi).$$

2. *Induced representations of Langlands type*

Rather than working with generic irreducible admissible smooth representations, we will work in the more general setting of induced representations of Langlands type; see [**2**, Section 1.5] for further details.

Given representations $\pi_1, \dots, \pi_r$ of $\mathrm{GL}_{n_1}(F), \dots, \mathrm{GL}_{n_r}(F)$, where $n_1 + \dots + n_r = n$, we form the representation $\pi_1 \boxtimes \dots \boxtimes \pi_r$ of $\mathrm{M_P}(F)$, where $\boxtimes$ denotes the outer tensor product and $\mathrm{M_P}(F)$ denote the block-diagonal Levi subgroup of the standard parabolic subgroup $\mathrm{P}(F) = \mathrm{P}_{(n_1,\dots,n_r)}(F)$ of $\mathrm{GL}_n(F)$. We then extend this representation trivially to a representation of $\mathrm{P}(F)$. By normalised parabolic induction, we obtain an induced representation π of $\mathrm{GL}_n(F)$,

$$\pi = \boxplus_{j=1}^{r} \pi_j := \mathrm{Ind}_{\mathrm{P}(F)}^{\mathrm{GL}_n(F)} \boxtimes_{j=1}^{r} \pi_j.$$

When $\pi_1, \dots, \pi_r$ are irreducible and essentially square-integrable, $\pi_1 \boxplus \dots \boxplus \pi_r$ is said to be an induced representation of Whittaker type; such a representation is admissible and smooth. Moreover, if each π_j is of the form $\sigma_j |\det|^{t_j}$, where σ_j is irreducible, unitary, and square-integrable, and $\Re(t_1) \geqslant \dots \geqslant \Re(t_r)$, then π is said to be an induced representation of Langlands type. Every irreducible admissible smooth representation π of $\mathrm{GL}_n(F)$ is isomorphic to the unique irreducible quotient of some induced representation of Langlands type. If π is also generic, then it is isomorphic to some (necessarily irreducible) induced representation of Langlands type.

An induced representation of Langlands type (π, V_π) is isomorphic to its Whittaker model $\mathcal{W}(\pi, \psi)$, the image of V_π under the map $v \mapsto \Lambda(\pi(\cdot) \cdot v)$, where $\Lambda : V_\pi \to \mathbb{C}$ is the unique (up to scalar multiplication) nontrivial Whittaker functional associated to an additive character ψ of F. This is a continuous linear functional that satisfies

$$\Lambda(\pi(u) \cdot v) = \psi_n(u)\Lambda(v)$$

for all $v \in V_\pi$ and $u \in \mathrm{N}_n(F)$, where $\mathrm{N}_n(F)$ denotes the unipotent radical of the standard minimal parabolic subgroup and $\psi_n(u) := \psi(u_{1,2} + u_{2,3} + \dots + u_{n-1,n})$.

An induced representation of Langlands type π is said to be spherical if it has a K-fixed vector, where $K := \mathrm{GL}_n(\mathcal{O})$. Such a spherical representation π must be a principal series representation of the form $|\cdot|^{t_1} \boxplus \dots \boxplus |\cdot|^{t_n}$; furthermore, the subspace of K-fixed vectors must be one dimensional. This K-fixed vector, unique up to scalar multiplication, is called the spherical vector of π. In the induced model of π, the normalised spherical vector is the unique smooth right K-invariant function $f^\circ : \mathrm{GL}_n(F) \to \mathbb{C}$ satisfying

$$f^\circ(uag) = f^\circ(g)\delta_n^{1/2}(a)\prod_{i=1}^{n}|a_i|^{t_i}$$

for all $u \in \mathrm{N}_n(F)$, $a = \mathrm{diag}(a_1, \ldots, a_n) \in \mathrm{A}_n(F) \cong F^n$, the subgroup of diagonal matrices, and $g \in \mathrm{GL}_n(F)$, where $\delta_n(a) := \prod_{i=1}^{n}|a_i|^{n-2i+1}$ denotes the modulus character of the standard minimal parabolic subgroup, and normalised such that

$$f^\circ(1_n) = \prod_{i=1}^{n-1}\prod_{j=i+1}^{n}\zeta_F(1 + t_i - t_j), \quad \zeta_F(s) := \frac{1}{1 - q^{-s}}.$$

The normalised spherical Whittaker function W° in the Whittaker model $\mathcal{W}(\pi, \psi)$ is given by the analytic continuation of the Jacquet integral

$$W^\circ(g) := \int_{\mathrm{N}_n(F)} f^\circ(w_n u g)\overline{\psi_n}(u)\, du,$$

where $w_n = \mathrm{antidiag}(1, \ldots, 1)$ is the long Weyl element. The Jacquet integral is absolutely convergent if $\Re(t_1) > \cdots > \Re(t_n)$ [10, Section 3] and extends holomorphically as a function of the complex variables $t_1, \ldots, t_n$ [1]. The Haar measure on $\mathrm{N}_n(F)$ is $du = \prod_{j=1}^{n-1}\prod_{\ell=j+1}^{n} du_{j,\ell}$, where for $u_{j,\ell} \in F$, $du_{j,\ell}$ is the additive Haar measure on F normalised to give $\mathcal{O}$ volume 1. With this normalisation of Haar measures and with ψ an unramified additive character of F, the normalised spherical vector $W^\circ \in \mathcal{W}(\pi, \psi)$ satisfies $W^\circ(1_n) = 1$.

3. *The newform*

For each nonnegative integer m, we define the congruence subgroup $K_0(\mathfrak{p}^m)$ of K by

$$K_0(\mathfrak{p}^m) := \{k \in K : k_{n,1}, \ldots, k_{n,n-1} \in \mathfrak{p}^m\}. \tag{3.1}$$

Theorem 3.2 [8, Théorème (5)]. *Let (π, V_π) be an induced representation of Langlands type of $\mathrm{GL}_n(F)$. Then either π is spherical, so that*

$$V_\pi^K := \{v \in V_\pi : \pi(k) \cdot v = v \text{ for all } k \in K\}$$

is one dimensional, or π is ramified, in which case V_π^K is trivial and there exists a minimal positive integer $m = c(\pi)$ for which the vector subspace

$$V_\pi^{K_0(\mathfrak{p}^m)} := \{v \in V_\pi : \pi(k) \cdot v = \omega_\pi(k_{n,n})v \text{ for all } k \in K_0(\mathfrak{p}^m)\}$$

is nontrivial; moreover, $V_\pi^{K_0(\mathfrak{p}^{c(\pi)})}$ is one dimensional.

Definition 3.3. The vector $v^\circ \in V_\pi^{K_0(\mathfrak{p}^{c(\pi)})}$, unique up to scalar multiplication, is called the newform of π. The nonnegative integer $c(\pi)$ is called the conductor exponent of π, where we set $c(\pi) = 0$ if π is spherical.

For each m, we may view $V_\pi^{K_0(\mathfrak{p}^m)}$ as the image of the projection map $\Pi^m : V_\pi \to V_\pi$ given by

$$\Pi^m(v) := \int_K \xi^m(k)\pi(k) \cdot v\, dk, \tag{3.4}$$

$$\xi^m(k) := \begin{cases} \dfrac{\omega_\pi^{-1}(k_{n,n})}{\mathrm{vol}(K_0(\mathfrak{p}^m))} & \text{if } m > 0 \text{ and } k \in K_0(\mathfrak{p}^m), \\ 1 & \text{if } m = 0 \text{ and } k \in K, \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

Here dk is the Haar measure on the compact group K normalised to give K volume 1. In particular, for any $v \in V_\pi$, we have that

$$\Pi^{c(\pi)}(v) = \langle v, \widetilde{v^\circ}\rangle v^\circ, \tag{3.6}$$

where $v^\circ \in V_\pi^{K_0(\mathfrak{p}^{c(\pi)})}$ and $\widetilde{v^\circ} \in V_{\widetilde{\pi}}^{K_0(\mathfrak{p}^{c(\pi)})}$ are normalised such that $\langle v^\circ, \widetilde{v^\circ}\rangle = 1$.

We write W° for the newform in the Whittaker model $\mathcal{W}(\pi,\psi)$ normalised such that $W^\circ(1_n) = 1$, where ψ is an unramified additive character; we also normalise $v^\circ \in V_\pi$ and the Whittaker functional Λ such that $\Lambda(v^\circ) = W^\circ(1_n) = 1$. Note that if π is spherical, then the newform in the Whittaker model is precisely the normalised spherical Whittaker function.

A key property of W° is the fact that it is a test vector for certain Rankin–Selberg integrals.

Theorem 3.7 (Jacquet–Piatetski-Shapiro–Shalika [**8**, Théorème (4)], Jacquet [**7**], Matringe [**13**, Corollary 3.3]). *Let π be an induced representation of Langlands type, and let $W^\circ \in \mathcal{W}(\pi,\psi)$ denote the newform in the Whittaker model. Then for any spherical representation of Langlands type π' of $\mathrm{GL}_{n-1}(F)$ with normalised spherical Whittaker function $W'^\circ \in \mathcal{W}(\pi',\overline{\psi})$, the $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$ Rankin–Selberg integral*

$$\Psi(s, W^\circ, W'^\circ) := \int\limits_{\mathrm{N}_{n-1}(F)\backslash \mathrm{GL}_{n-1}(F)} W^\circ \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W'^\circ(g) |\det g|^{s-\frac{1}{2}}\, dg \tag{3.8}$$

is equal to the Rankin–Selberg L-function $L(s, \pi\times\pi')$.

Here the Haar measure on $\mathrm{GL}_n(F)$ is that induced from the Iwasawa decomposition $\mathrm{GL}_n(F) = \mathrm{N}_n(F)\mathrm{A}_n(F)K$, namely $dg = du\, \delta_n^{-1}(a)\, d^\times a\, dk$, where $d^\times a = \prod_{i=1}^n d^\times a_i$ with the multiplicative Haar measure on $F^\times$ given by $d^\times a_i = \zeta_F(1)|a_i|^{-1}\, da_i$.

Theorem 3.9 [**11**, Theorem 2.1.1]. *Let π be an induced representation of Langlands type, and let $W^\circ \in \mathcal{W}(\pi,\psi)$ denote the newform in the Whittaker model. Then for any spherical representation of Langlands type π' of $\mathrm{GL}_n(F)$ with normalised spherical Whittaker function $W'^\circ \in \mathcal{W}(\pi',\overline{\psi})$, the $\mathrm{GL}_n \times \mathrm{GL}_n$ Rankin–Selberg integral*

$$\Psi(s, W^\circ, W'^\circ, \Phi^\circ) := \int\limits_{\mathrm{N}_n(F)\backslash \mathrm{GL}_n(F)} W^\circ(g) W'^\circ(g) \Phi(e_n g) |\det g|^s\, dg \tag{3.10}$$

is equal to the Rankin–Selberg L-function $L(s,\pi\times\pi')$, where $e_n := (0,\dots,0,1) \in \mathrm{Mat}_{1\times n}(F)$ and $\Phi^\circ \in \mathscr{S}(\mathrm{Mat}_{1\times n}(F))$ is given by

$$\Phi^\circ(x_1,\dots,x_n) := \begin{cases} \dfrac{\omega_\pi^{-1}(x_n)}{\mathrm{vol}(K_0(\mathfrak{p}^{c(\pi)}))} & \text{if } c(\pi) > 0, x_1,\dots,x_{n-1} \in \mathfrak{p}^{c(\pi)}, \text{ and } x_n \in \mathcal{O}^\times, \\ 1 & \text{if } c(\pi) = 0 \text{ and } x_1,\dots,x_n \in \mathcal{O}, \\ 0 & \text{otherwise.} \end{cases}$$

4. *A propagation formula*

We now present a propagation formula for spherical Whittaker functions. This is a recursive formula for a $\mathrm{GL}_n(F)$ Whittaker function in terms of a $\mathrm{GL}_{n-1}(F)$ Whittaker function.

Lemma 4.1. *Let $\pi' = |\cdot|^{t'_1} \boxplus \cdots \boxplus |\cdot|^{t'_n}$ be a spherical representation of Langlands type of $\mathrm{GL}_n(F)$. Then the normalised spherical Whittaker function $W'^\circ \in \mathcal{W}(\pi',\overline{\psi})$*

satisfies

$$W'^{\circ}(g) = |\det g|^{t_1' + \frac{n-1}{2}} \int_{\mathrm{GL}_{n-1}(F)} W_0'^{\circ}(h) |\det h|^{-t_1' - \frac{n}{2}} \times \int_{\mathrm{Mat}_{(n-1)\times 1}(F)} \Phi'\left(h^{-1} \begin{pmatrix} 1_{n-1} & v \end{pmatrix} g\right) \psi(e_{n-1} v) \, dv \, dh, \tag{4.2}$$

where $W_0'^{\circ} \in \mathcal{W}(\pi_0', \overline{\psi})$ *is the normalised spherical Whittaker function of the spherical representation of Langlands type* $\pi_0' := |\cdot|^{t_2'} \boxplus \cdots \boxplus |\cdot|^{t_n'}$ *of* $\mathrm{GL}_{n-1}(F)$ *and* $\Phi' \in \mathscr{S}(\mathrm{Mat}_{(n-1)\times n}(F))$ *is the Schwartz–Bruhat function*

$$\Phi'(x) := \begin{cases} 1 & \text{if } x \in \mathrm{Mat}_{(n-1)\times n}(\mathcal{O}), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let f'° be the normalised spherical vector in the induced model of π', so that

$$f'^{\circ}(1_n) = \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} \zeta_F(1 + t_i' - t_j'), \tag{4.3}$$

$$f'^{\circ}(uag) = f'^{\circ}(g) \delta_n^{1/2}(a) \prod_{i=1}^{n} |a_i|^{t_i'}, \tag{4.4}$$

$$f'^{\circ}(gk) = f'^{\circ}(g) \tag{4.5}$$

for all $u \in \mathrm{N}_n(F)$, $a = \mathrm{diag}(a_1, \ldots, a_n) \in \mathrm{A}_n(F)$, $g \in \mathrm{GL}_n(F)$, and $k \in K$. We claim that f'° is also given by the Godement section

$$f'^{\circ}(g) := |\det g|^{t_1' + \frac{n-1}{2}} \int_{\mathrm{GL}_{n-1}(F)} \Phi'\left(h^{-1} \begin{pmatrix} 0 & 1_{n-1} \end{pmatrix} g\right) f_0'^{\circ}(h) |\det h|^{-t_1' - \frac{n}{2}} \, dh. \tag{4.6}$$

Here $f_0'^{\circ}$ is the normalised spherical vector in the induced model of π_0', so that

$$f_0'^{\circ}(1_{n-1}) = \prod_{i=2}^{n-1} \prod_{j=i+1}^{n} \zeta_F\left(1 + t_i' - t_j'\right), \tag{4.7}$$

$$f_0'^{\circ}(u'a'h) = f_0'^{\circ}(h) \delta_{n-1}^{1/2}(a') \prod_{i=2}^{n} |a_i'|^{t_i'}, \tag{4.8}$$

$$f_0'^{\circ}(hk') = f_0'^{\circ}(h) \tag{4.9}$$

for all $u' \in \mathrm{N}_{n-1}(F)$, $a' = \mathrm{diag}(a_2', \ldots, a_n') \in \mathrm{A}_{n-1}(F)$, $h \in \mathrm{GL}_{n-1}(F)$, and $k' \in \mathrm{GL}_{n-1}(\mathcal{O})$. We then insert the identity (4.6) into the Jacquet integral

$$W'^{\circ}(g) := \int_{\mathrm{N}_n(F)} f'^{\circ}(w_n u g) \psi_n(u) \, du,$$

write $w_n = \left(\begin{smallmatrix} 0 & 1 \\ w_{n-1} & 0 \end{smallmatrix}\right)$ and $u = \left(\begin{smallmatrix} u' & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1_{n-1} & v \\ 0 & 1 \end{smallmatrix}\right)$ for $u' \in \mathrm{N}_{n-1}(F)$ and $v \in \mathrm{Mat}_{(n-1)\times 1}(F)$, and make the change of variables $h \mapsto w_{n-1} u' h$ to obtain the identity (4.2).

So it remains to show that f'° is indeed given by (4.6). We first show that this is an element of the induced model of π', just as in [**6**, Proposition 7.1]. We replace g with $\left(\begin{smallmatrix} 1 & v \\ 0 & u' \end{smallmatrix}\right)\left(\begin{smallmatrix} a_1 & 0 \\ 0 & a' \end{smallmatrix}\right) g$, where $v \in \mathrm{Mat}_{1\times(n-1)}(F)$, $u' \in \mathrm{N}_{n-1}(F)$, $a_1 \in F^{\times}$, and $a' \in \mathrm{A}_{n-1}(F)$. Upon making the change of variables $h \mapsto u'a'h$ and using (4.8), we see that (4.4) is satisfied. Next, we check that f'° given by (4.6) satisfies (4.5), which follows easily from the fact that $\Phi'(xk) = \Phi'(x)$ for all

$x \in \mathrm{Mat}_{(n-1)\times n}(F)$ and $k \in K$. Finally, we confirm the normalisation (4.3). To see this, we use the Iwasawa decomposition $h = u'a'k'$ in (4.6), in which case the Haar measure is $dh = \delta_{n-1}^{-1}(a')\, du'\, d^\times a'\, dk'$. The integral over $\mathrm{GL}_{n-1}(\mathcal{O}) \ni k'$ is trivial. We then make the change of variables $u' \mapsto u'^{-1}$, $a' \mapsto a'^{-1}$, so that

$$f'^{\circ}(1_n) = f_0'^{\circ}(1_{n-1}) \int_{\mathrm{N}_{n-1}(F)} \int_{\mathrm{A}_{n-1}(F)} \Phi'\left(0\ a'u'\right) \prod_{i=2}^{n} |a_i'|^{-t_i'} \delta_{n-1}^{1/2}(a') |\det a'|^{t_1' + \frac{n}{2}}\, d^\times a'\, du',$$

recalling (4.8). Writing $du' = \prod_{j=2}^{n-1} \prod_{\ell=j+1}^{n} du_{j,\ell}'$ and $d^\times a' = \prod_{i=2}^{n} d^\times a_i'$ and making the change of variables $u_{j,\ell}' \mapsto {a_j'}^{-1} u_{j,\ell}'$, this becomes

$$f_0'^{\circ}(1_{n-1}) \prod_{j=2}^{n-1} \prod_{\ell=j+1}^{n} \int_{\mathcal{O}} du_{j,\ell}' \prod_{i=2}^{n} \int_{\mathcal{O}\setminus\{0\}} |a_i'|^{1+t_1'-t_i'}\, d^\times a_i'.$$

The integral over $\mathcal{O} \ni u_{j,\ell}'$ is 1, while the integral over $\mathcal{O} \setminus \{0\} \ni a_i'$ is $\zeta_F(1 + t_1' - t_i')$. Recalling the normalisation (4.7) of $f_0'^{\circ}(1_{n-1})$, we see that (4.3) is indeed satisfied. □

5. *Proof of Theorem* 1.2

Proof of Theorem 1.2. Let π be a ramified induced representation of Langlands type of $\mathrm{GL}_n(F)$, so that $c(\pi) > 0$, and let $\pi' = |\cdot|^{t_1'} \boxplus \cdots \boxplus |\cdot|^{t_n'}$ be an arbitrary spherical representation of Langlands type of $\mathrm{GL}_n(F)$. We insert the identity (4.2) for the normalised spherical Whittaker function $W'^{\circ} \in \mathcal{W}(\pi', \overline{\psi})$ into the $\mathrm{GL}_n \times \mathrm{GL}_n$ Rankin–Selberg integral (3.10). Just as in [**6**, Equation (8.1)], we fold the integration over $\mathrm{N}_{n-1}(F)\backslash\mathrm{N}_n(F) \cong \mathrm{Mat}_{(n-1)\times 1}(F) \ni v$ and make the change of variables $g \mapsto \left(\begin{smallmatrix} h & 0 \\ 0 & 1 \end{smallmatrix}\right) g$. In this way, we find that $\Psi(s, W^\circ, W'^\circ, \Phi^\circ)$ is equal to

$$\int_{\mathrm{N}_{n-1}(F)\backslash\mathrm{GL}_{n-1}(F)} W_0'^{\circ}(h) |\det h|^{s-\frac{1}{2}} \int_{\mathrm{GL}_n(F)} W^\circ\left(\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} g\right) \Phi(g) |\det g|^{s+t_1'+\frac{n-1}{2}}\, dg\, dh, \tag{5.1}$$

with $\Phi(x) := \Phi^\circ(e_n x) \Phi'\left(\left(1_{n-1}\ 0\right) x\right)$ as in (1.3).

We claim that

$$\Phi(g) = \int_K \xi^{c(\pi)}(k) \Phi(k^{-1} g)\, dk, \tag{5.2}$$

with $\xi^{c(\pi)}$ as in (3.5). Indeed, $\xi^{c(\pi)}(k)$ vanishes unless $k \in K_0(\mathfrak{p}^{c(\pi)})$, in which case $\Phi(k^{-1}g)$ vanishes unless $g \in \mathrm{Mat}_{n\times n}(\mathcal{O})$ with $g_{n,1}, \ldots, g_{n,n-1} \in \mathfrak{p}^{c(\pi)}$ and $g_{n,n} \in \mathcal{O}^\times$. Then as $k^{-1} \in K_0(\mathfrak{p}^{c(\pi)})$, it is easily checked that

$$\omega_\pi^{-1}(e_n k^{-1} g\, {}^t e_n) = \omega_\pi(k_{n,n}) \omega_\pi^{-1}(g_{n,n}),$$

using the fact that $e_n k^{-1} g\, {}^t e_n - e_n k^{-1}\, {}^t e_n g_{n,n} \in \mathfrak{p}^{c(\pi)}$, $e_n k^{-1}\, {}^t e_n k_{n,n} - 1 \in \mathfrak{p}^{c(\pi)}$, and $c(\omega_\pi) \leqslant c(\pi)$. Thus (5.2) follows.

We insert (5.2) into (5.1) and make the change of variables $g \mapsto kg$, so that the integral over $K \ni k$ is

$$\int_K W^\circ\left(\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} kg\right) \xi^{c(\pi)}(k)\, dk = \Lambda\left(\pi\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \cdot \int_K \xi^{c(\pi)}(k) \pi(k) \cdot (\pi(g) \cdot v^\circ)\, dk\right).$$

We note that

$$\int_K \xi^{c(\pi)}(k)\pi(k)\cdot(\pi(g)\cdot v^\circ)\,dk = \Pi^{c(\pi)}(\pi(g)\cdot v^\circ) = \beta(g)v^\circ,$$

where $\beta(g) := \langle \pi(g)\cdot v^\circ, \widetilde{v^\circ}\rangle$, recalling (3.4) and (3.6), so that

$$\int_K W^\circ\left(\begin{pmatrix} h & 0 \\ 0 & 1\end{pmatrix} kg\right)\xi^{c(\pi)}(k)\,dk = \beta(g)W^\circ\begin{pmatrix} h & 0 \\ 0 & 1\end{pmatrix}. \tag{5.3}$$

Combining (5.1) with (5.2) and (5.3), we find that

$$\Psi(s, W^\circ, W'^\circ, \Phi^\circ) = Z(s+t_1', \beta, \Phi)\Psi(s, W^\circ, W_0'^\circ),$$

recalling the definitions (1.1) of the Godement–Jacquet zeta integral and (3.8) of the $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$ Rankin–Selberg integral. From Theorems 3.9 and 3.7,

$$\Psi(s, W^\circ, W'^\circ, \Phi^\circ) = L(s, \pi\times\pi'), \quad \Psi(s, W^\circ, W_0'^\circ) = L(s, \pi\times\pi_0').$$

Moreover, [9, (9.5) Theorem] implies

$$L(s,\pi\times\pi') = L\Big(s, \pi\times|\cdot|^{t_1'}\Big)L(s,\pi\times\pi_0') = L(s+t_1', \pi)L(s, \pi\times\pi_0').$$

Since $L(s,\pi\times\pi_0')$ is not uniformly zero, we conclude that

$$Z(s+t_1', \beta, \Phi) = L(s+t_1', \pi). \qquad \square$$

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