Flows on Shimura varieties

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I, Michele Giacomini, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the work.

Abstract

The aim of this thesis is to study both holomorphic and algebraic flows on Shimura varieties. The first part of the thesis studies holomorphic flows, the main result is a hyperbolic analogue of the Bloch-Ochiai Theorem in the context of mixed Shimura varieties. This extends previous results of Ullmo and Yafaev for co-compact pure Shimura varieties. The proof follows the template set by the hyperbolic Ax-Linedmann-Weierstrass theorem of using the Pila-Wilkie counting theorem together with some volume inequalities to prove our result. The heart of the proof consists of two volume inequalities, first one for the intersection of a definable set with hyperbolic balls in Hermitian symmetric domains of non-compact type. The second for the intersection of a definable portion of a holomorphic curve with a fundamental domain for the action of a congruence group on a Hermitian symmetric domain of non-compact type.

In the second part we study totally geodesic subvarieties of mixed Shimura varieties and algebraic flows. We show that contrary to the case of pure Shimura varieties, there is in general no inclusion either way between the concept of weakly special and totally geodesic subvariety in the mixed setting. Then we report an argument communicated by N. Mok which shows that unlike in the pure case there are totally geodesic submanifolds of a mixed Shimura variety that are not homogeneous. Finally we use these results on totally geodesic subvarieties to state and prove a generalisation of results of Ullmo and Yafaev on algebraic flows on pure Shimura varieties to the mixed case. The proof follows the pure case and uses a theorem of Ratner in arithmetic dynamics.

Impact Statement

In 2008 Pila and Zannier presented a new proof of the Manin-Mumford Conjecture concerning torsion points in subvarieties of abelian varieties. This new proof utilised tools from the theory of o-minimal structures and paved the way for a new approach to the André-Oort conjecture and related conjectures. The main geometric input in this strategy is a functional transcendence result proved by Ax in the '70s, this is a functional analogue of the classical Lindemann-Weierstrass theorem in transcendence theory. This application sparked a new interest in transcendence results of Ax-Lindemann-Weierstrass type; it was noted that these can also be studied using o-minimal techniques. The first part of this thesis pushes forward the use of these o-minimal techniques in the study of the uniformisation map of mixed Shimura varieties, generalising and extending previous results on the topic. We also believe that one of the fundamental steps in the proof of the main result of this first part, a volume bound for the intersection of definable sets with a hyperbolic ball in Hermitian symmetric spaces of non-compact type, to be of independent interest.

The second part of the thesis studies totally geodesic subvarieties of mixed Shimura varieties with a view towards the analysis of algebraic flows. Here we show that most of the useful properties of totally geodesic subvarieties that are true in the case of pure Shimura varieties do not generalise to the mixed context. We then prove an analogue in the mixed case of previous results on algebraic flows in pure Shimura varieties. We believe the results on totally geodesic subvarieties may facilitate a better understanding the geometry of mixed Shimura varieties and their uniformising space and advance our knowledge of flows on these spaces.

Part of the research presented in this thesis has been published in peer reviewed journals, we plan on publishing and making available on the ArXiv the part on totally geodesic subvarieties of mixed Shimura varieties as well.

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Contents

Introduction

The aim of this thesis is to study holomorphic and algebraic flows on mixed Shimura varieties. The thesis is divided into three separate parts using different methods, in the first we study holomorphic flows, in the second we analyse totally geodesic subvarieties of mixed Shimura varieties, and the last contains results on algebraic flows.

Universal family of abelian varieties

Let $g > 0$ and $J = \left(\begin{smallmatrix} 0 & -\mathrm{Id}_g \\ \mathrm{Id}_g & 0 \end{smallmatrix}\right)$ be the standard symplectic form on $\mathbb{R}^{2g}$. Consider the pair $(\mathrm{GSp}_{2g}, \mathcal{H}_g)$, where GSp_{2g} is the algebraic $\mathbb{Q}$-group

$$\left\{h \in \mathrm{GL}_{2g} \,|\, h^t J h = \lambda_h J \text{ for some } \lambda_h \in \mathbb{G}_m\right\}$$

and $\mathcal{H}_g$ is the Siegel upper half-space of complex symmetric $g \times g$ matrices of positive definite imaginary part. The group $\mathrm{GSp}_{2g}(\mathbb{R})^+$, the connected component in the archimedean topology of $\mathrm{GSp}_{2g}(\mathbb{R})$ containing the identity element, acts transitively on $\mathcal{H}_g$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} .Z = (AZ+B)(CZ+D)^{-1}.$$

This pair corresponds to the moduli space of principally polarized abelian varieties in the sense that quotients of $\mathcal{H}_g$ by arithmetic subgroups of $\mathrm{GSp}_{2g}(\mathbb{R})^+$ are moduli spaces for principally polarized abelian varieties, possibly with some additional structure. When this moduli space is fine one can attach to it a universal family of abelian varieties.

It turns out it is possible to describe the universal family in a similar way. To achieve this we need to enlarge the pair $(\mathrm{GSp}_{2g}, \mathcal{H}_g)$ and define a new pair $(P_{2g}, \mathcal{X}_g)$ as follows:

- P_{2g} is the semi-direct product of $\mathrm{GSp}_{2g} \ltimes \mathbb{G}_a^{2g}$ via the standard action of GSp_{2g} on $\mathbb{G}_a^{2g}$.
- The space $\mathcal{X}_g$ is the product $\mathcal{H}_g \times \mathbb{R}^{2g}$ with the natural transitive action of $P_{2g}(\mathbb{R})^+$ defined by

$$(h,v).(z,w) = (h.z, v + h.w)$$

  for $(h,v) \in P_{2g}(\mathbb{R})^+$ and $(z,w) \in \mathcal{X}_g$.

It is possible to define a natural complex structure on $\mathcal{X}_g$ by remembering how $\mathcal{H}_g$ is connected to the moduli space of principally polarized abelian varieties. A point $Z \in \mathcal{H}_g$ corresponds to an abelian variety isogenous to $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$, where the complex structure is given by the identification

$$\begin{aligned} \mathbb{R}^{2g} &\to \mathbb{C}^g \\ (a,b) &\mapsto Za + b \end{aligned}$$

we can use this to get an identification

$$\begin{aligned} \mathcal{H}_g \times \mathbb{R}^{2g} &\to \mathcal{H}_g \times \mathbb{C}^g \\ (Z,(a,b)) &\mapsto (Z, Za + b) \end{aligned}$$

and get a complex structure on $\mathcal{X}_g$. We are now in a similar situation as above, that is, we have a pair $(P, \mathcal{X})$ of a complex analytic space with a transitive action of the real points of an algebraic $\mathbb{Q}$-group. At this point, similarly to the case of $(\mathrm{GSp}_{2g}, \mathcal{H}_g)$ we may take the quotient of $\mathcal{X}_g$ by a sufficiently small arithmetic subgroup of $P(\mathbb{R})^+$, this turns out to be the total space of the universal family of abelian varieties over the corresponding fine moduli space.

General mixed Shimura data

A general connected mixed Shimura datum is a pair $(P,\mathcal{X})$ that behaves similarly to $(P_{2g},\mathcal{X}_g)$, see Section 1.2 for precise definitions. For instance P is a $\mathbb{Q}$-group with a uniquely determined subgroup U such that $P(\mathbb{R})^+U(\mathbb{C})$ acts transitively on $\mathcal{X}$. In the case of $(P_{2g},\mathcal{X}_g)$ the subgroup U is trivial. One further example of mixed Shimura variety for which the subgroup U is non-trivial is the canonical ample line bundle over the universal family of abelian varieities.

One other feature of mixed Shimura data we will need in this introduction is the ability of identifying $\mathcal{X}$ with a period domain for mixed Hodge structures, this allows us to embed $\mathcal{X}$ into a projective algebraic variety as an open semi-algebraic set[1].

Given a connected mixed Shimura datum $(P,\mathcal{X})$ an associated mixed Shimura variety M is the quotient $\Gamma\backslash\mathcal{X}$, where Γ is a congruence subgroup of P. The quotient map $\mathcal{X}\to M$ is called the uniformisation map and will be denoted by unif.

A morphism of mixed Shimura data $(P,\mathcal{X})\to(Q,\mathcal{Y})$ is map $\mathcal{X}\to\mathcal{Y}$ that comes from a homomorphism of the $\mathbb{Q}$-groups P and Q.

By a theorem of Pink, mixed Shimura varieties have a canonical structure of algebraic varieties such that all Shimura morphisms are algebraic.

One last definition we will need is the concept of weakly special subvariety of a mixed Shimura variety, here we will define the concept only for $(P_{2g},\mathcal{X}_g)$.

Definition. A subset Y of $\mathcal{X}$ is called *weakly special* if there exists a connected mixed Shimura sub-datum $(Q,\mathcal{Y})$ of $(P_{2g},\mathcal{X}_g)$, a normal subgroup N of Q and a point $y\in\mathcal{Y}$ such that

$$Y=N(\mathbb{R})^+y.$$

Let M be a mixed Shimura variety associated with the datum $(P,\mathcal{X})$, a subvariety of M is called weakly special if it is the image of a weakly special subset of $\mathcal{X}$ under the uniformisation map.

[1] Se p. 15

The original motivation for this definition comes from the fact that in the pure case weakly special subvarieties correspond to non-rigid families of abelian varieties.

Holomorphic flows

The main inspiration, as well as the techniques used in the analysis of holomorphic flows comes form the study of results of Ax-Lindemann-Weierstrass type for mixed Shimura varieties, together with the Bloch-Ochiai theorem for abelian varieties. Below we recall the statements and history of these two results before stating the main results obtained in this part.

Hyperbolic Ax-Lindemann-Weierstrass theorem

We are interested in functional analogues of the classical Lindemann-Weierstrass theorem on the transcendence of values of the exponential function.

Theorem (Lindemann-Weierstrass Theorem cf. [Lan66])**.** *Let* $\alpha_1,\dots,\alpha_n$ *be algebraic numbers linearly independent over* $\mathbb{Q}$ *then* $\exp(\alpha_1),\dots,\exp(\alpha_n)$ *are algebraically independent.*

A functional analogue of this result is due to Ax.

Theorem (Ax-Linedmann-Weierstrass cf. [Ax71; Ax72])**.** *Let* Z *be an irreducible affine algebraic variety over* $\mathbb{C}$ *and* $f_1,\dots,f_n \in \mathbb{C}[Z]$ *be regular functions on* Z*. Assume that* $f_1,\dots,f_n$ *are linearly independent over* $\mathbb{Q}$ *modulo constants, that is, no* $\mathbb{Q}$*-linear combination of the* f_i *is constant. Then the functions* $\exp(f_1)\dots\exp(f_n)$ *are algebraically independent over* $\mathbb{C}$*.*

Remark. This is only a special case of Ax's result. The general theorem is the functional analogue of Schanuel's conjecture that states that given $\alpha_1,\dots,\alpha_n \in \mathbb{C}$, $\mathbb{Q}$-linearly independent the transcendence degree of $\mathbb{C}[\alpha_1,\dots,\alpha_n,\exp(\alpha_1),\dots,\exp(\alpha_n)]$ is at least n.

It is possible to restate the Ax-Lindemann-Weierstrass theorem in more geometric terms, this makes it easier to generalise it to different settings. This geometric restatement is originally due to Pila and Zannier (cf. [PZ08]).

Theorem (Geometric Ax-Lindemann-Weierstrass)**.** *Let Z be an irreducible algebraic subvariety of $\mathbb{C}^n$ and denote*

$$\mathrm{unif} = (\exp, \ldots, \exp) : \mathbb{C}^n \to \mathbb{C}^{*n}.$$

Then the Zariski closure of $\mathrm{unif}(Z)$ *is a translate of an algebraic sub-torus of* $\mathbb{C}^{*n}$*.*

We can extract the main features of this result as follows. We have two algebraic varieties $\overline{S}$ and S together with a transcendental map $\mathrm{unif} : \overline{S} \to S$. As the map unif is transcendental, a priori, there is no clear relationship between the algebraic structures on $\overline{S}$ and S via the map unif. In the above case however the Ax-Lindemann-Weierstrass theorem implies that the class of bi-algebraic subvarieties is non empty. In this context an irreducible subvariety of $\overline{S}$ is called *bi-algebraic* if its image under unif is also irreducible algebraic in S. Similarly bi-algebraic subvarieties in S are images of bi-algebraic subvarieties of $\overline{S}$.

In this form the statement can be easily generalised to abelian varieties. This has been proven by Ax.

Theorem (Ax-Lindemann-Weierstrass for abelian varieties cf. [Ax72])**.** *Let A be a complex abelian variety and* $\mathrm{unif} : \mathbb{C}^n \to A$ *be the universal covering map. Let V be an irreducible algebraic subvariety of $\mathbb{C}^n$, then the Zariski closure of the image* $\mathrm{unif}(V)$ *is a translate of an algebraic subgroup of A.*

Given a mixed Shimura datum $(P, \mathcal{X})$ and an associated mixed Shimura variety, we are interested in a generalisation of this result to the map $\mathrm{unif} : \mathcal{X} \to S$. This is not as straightforward as in the case of abelian varieties, as $\mathcal{X}$ in general has only the structure of a complex analytic set and not of an algebraic variety. It is possible to overcome this difficulty by embedding $\mathcal{X}$ in an algebraic variety. As mentioned above, using the interpretation of mixed Shimura varieties as period domains for certain variations of mixed Hodge structures it is possible to embed $\mathcal{X}$ as a semi-algebraic subet of a flag variety.

This allows us to define irreducible algebraic subvarieties of $\mathcal{X}$ as components of intersections of irreducible subvarieties of the ambient algebraic variety with $\mathcal{X}$. It turns out that this definition of algebraic subvariety of $\mathcal{X}$ allows us to generalise the statement of the Ax-Lindemann-Weierstrass theorem. The other ingredient we are missing in this case is the analogue of translates of algebraic subgroups, in the case of mixed Shimura varieties this role is filled in by weakly special subvarieties.

Theorem (mixed Ax-Lindemann-Weierstrass cf. [Gao17])**.** *Let* $(P, \mathcal{X})$ *be a connected mixed Shimura datum and* S *be an associated mixed Shimura variety. Let* $\mathrm{unif} : \mathcal{X} \to S$ *be the complex uniformisation map. Let* Y *be an irreducible algebraic subset of* $\mathcal{X}$*. Then the Zariski closure of* $\mathrm{unif}(Y)$ *is a weakly special subvariety.*

The general form and setting of this result are due [PT13; PT14] for the cases of the moduli space of principally polarized abelian varieties, [UY14b; KUY16] for the case of a general pure Shimura variety and finally to [Gao17] for the general case of a mixed Shimura variety.

Statements of Ax-Lindemann-Weierstrass type have been widely studied in recent years for their connection to the Zilber-Pink conjecture and related results. The first example of this use was given by Pila and Zannier in [PZ08] where the authors present a new strategy of proof for the Manin-Mumford conjecture. Recall that the Manin-Mumford conjecture states that given an abelian variety A and an algebraic subvariety Y, the Zariski closure of the set of torsion points of A contained in Y is a finite union of translates of abelian subvarieties of A by torsion points. The Pila-Zannier strategy is roughly divided into two parts: a geometric part consisting of the Ax-Lindemann-Weierstrass theorem, and a number theoretic part counting Galois conjugates of torsion points of A. These two ingredients are used together with the Pila-Wilkie theorem on counting rational points in definable sets to get the result. This procedure has been generalised to give a strategy of proof for the analogous statement in the case of Shimura varieties, the André-Oort conjecture. See the

[Gao17] for further information and references on this topic.

o-minimal structures and hyperbolic Ax-Lindemann-Weierstrass

In this section we aim to give a general intuition for the concept of an o-minimal structure and briefly explain their role in the proof of the hyperbolic Ax-Lindemann-Theorem. For more precise statements and definitions see Section 1.3

An o-minimal structure over the field of real numbers is, roughly speaking, a collection of *tame* subsets of $\mathbb{R}^n$ for each n, for a precise definition see Definition 1.37. The tameness of the sets contained in an o-minimal structure, which are called *definable*, has many important implications; for instance we will use the fact that every definable set has a finite number of connected components.

The smallest possible example of o-minimal structure is the collection of semi-algebraic sets. A semi-algebraic set is a finite union of subsets of $\mathbb{R}^n$ that can be defined by a finite number of polynomial equations and inequalities. On the other hand, the finiteness of connected components implies that there is no o-minimal structure where the sine function is definable.

The application of o-minimal structures to the study of mixed Shimura varieties is made possible by the fact that there is an o-minimal structure where a suitable restriction of the uniformisation map of a mixed Shimura variety is definable.

The main result about o-minimal structures that we will use is the Pila-Wilkie theorem. This states,roughly, that most rational points in a definable set are contained in its positive dimensional semi-algebraic subsets.

We now briefly recall the main steps in the proof of the Ax-Lindemann-Weierstrass theorem for $\mathbb{G}_m$; these steps are similar in the hyperbolic case, although the individual proofs are harder. We start by observing that proving the Ax-Lindemann-Weierstrass theorem is equivalent to proving that given an algebraic subvariety Y of $\mathbb{G}_m^n$, maximal algebraic subvarieties Z of $\mathrm{unif}^{-1}(Y)$ are weakly special. Then we fix a definable fundamental domain $\mathscr{F}$ for the action of $\mathbb{Z}^n$ on $\mathbb{C}^n$ and proving that Z intersects many translates of $\mathscr{F}$ by small elements in $\mathbb{Z}^n$. The second step is to define a subset Σ of $\mathbb{C}^n$, such that translates of $\mathscr{F}$ that intersect Z give raise to rationa points in Σ, and prove that this is definable is a suitable o-minimal structure. The definability of Σ depends on the definability of the uniformisation map restricted to $\mathscr{F}$ and the definability of Y and Z. The next step is to prove that the counting result in the first step implies that Σ contains many integer points. Now we can apply the Pila-Wilkie theorem to Σ to obtain a positive dimensional semi-algebraic set contained in Σ. The existence of this semi-algebraic set and the definition of Σ imply that the stabiliser of Z in $\mathbb{C}^n$ is positive dimensional, which can be used to prove that Z is weakly special. A complete account of this proof can be found in Appendix A.

Bloch-Ochiai theorem

The Bloch-Ochiai theorem [Kob98, Chapter 9, 3.9.19] is a classical theorem in Nevanlinna theory.

Theorem (Bloch-Ochiai)**.** *Let A be an abelian variety and $f : \mathbb{C} \to A$ a non-constant holomorphic map. Then the Zariski closure of $f(\mathbb{C})$ is a translate of an abelian subvariety.*

Observing that $\mathbb{C}$ is simply connected, we can lift the map f to $\widetilde{f} : \mathbb{C} \to \mathbb{C}^n$ and use this to reformulate the theorem as follows: given an abelian variety A with its uniformisation map $\mathrm{unif} : \mathbb{C}^n \to A$ and a holomorphic curve $\widetilde{f}(\mathbb{C})$ in $\mathbb{C}^n$, the Zariski closure of $\mathrm{unif}(\widetilde{f}(\mathbb{C}))$ is a translate of an abelian subvariety of A. This reformulation resembles closely the statement of the Ax-Lindemann-Weiertrass theorem, so one might expect that similar techniques to the ones presented in the previous section may also be applied in this case. This however is not the case, the main issue in this context is that the image of a general holomorphic function may not contain any unbounded definable sets, this effectively prevents one to prove the definability of the set Σ mentioned in the previous section. Similar questions were analysed by Ullmo and Yafaev in

[UY17], where the authors are able to get only partial results for this precise reason.

Holomorphic flows on mixed Shimura varieties

The main issue in the application of o-minimal techniques to the study of the Bloch-Ochiai theorem is the need for an unbounded definable set. Ullmo and Yafaev noted that this difficulty can be overcome when considering analogous statements in the case of pure Shimura varieties. This is due to the fact that the space uniformising a pure Shimura variety can be realised as a bounded symmetric domain, which allows one to consider only definable sets also in the more general context of holomorphic flows. In [UY18b], the authors prove the following theorem.

Theorem. *Let $(G,\mathcal{X})$ be a connected pure Shimura datum. Embed $\mathcal{X}$ as a bounded symmetric domain in $\mathbb{C}^n$. Let Γ be an arithmetic subgroup of G such that the associated pure Shimura variety $\Gamma\backslash\mathcal{X}$ is compact. Let $f:\mathbb{C}\to\mathbb{C}^n$ be a holomorphic map such that $f(\mathbb{C})\cap\mathcal{X}\neq\emptyset$. Then the components of the Zariski closure of* $\mathrm{unif}(f(\mathbb{C})\cap\mathcal{X})$ *are weakly special.*

The aim of the first part of this thesis is to generalise this result to the case of mixed Shimura varieties. To do this we first introduce some notation. Let $(P,\mathcal{X})$ be a connected mixed Shimura datum and Γ an arithmetic subgroup of P. Let M be the mixed Shimura variety associated to the above data and $\mathrm{unif}:\mathcal{X}\to M$ be the complex uniformisation map. Let $\pi:P\to G$ be the projection modulo the unipotent radical and let $(G,\mathcal{X}_G)$ be the corresponding pure Shimura datum. We identify $\mathcal{X}$ with $\mathcal{X}_G\times\mathbb{C}^m$. Using the Harish-Chandra embedding of $\mathcal{X}_G$ we identify $\mathcal{X}$ as a subset of $\mathbb{C}^N\times\mathbb{C}^m$. Our main result is.

Theorem. *Let $f:\mathbb{C}\to\mathbb{C}^N\times\mathbb{C}^m$ be a holomorphic function such that the composition of f with the projection to $\mathbb{C}^N$ is non constant and the image of f intersects $\mathcal{X}$. Then the Zariski closure of* $\mathrm{unif}(f(\mathbb{C}))$ *is a weakly special subvariety of M.*

Remark. We restrict to holomorphic functions that are transverse to the fibres of the projection to the pure part, otherwise we would need to reprove the Bloch-Ochiai theorem that, as we mentioned above, cannot be proven using the present methods.

To prove this result we first use the same strategy of proof of the Ax-Lindemann-Weierstrass theorem to deduce that the Zariski closure of $\mathrm{unif}(f(\mathbb{C})\cap\mathcal{X})$ contains a Zariski dense set of weakly special subvarieties, then use a consequence of the Ax-Lindemann-Weierstrass theorem to deduce that it must be weakly special itself.

To be able to apply o-minimal techniques to the problem, we need to start by replacing the set $f(\mathbb{C})$ with a smaller definable subset Z such that the Zariski closure of $\mathrm{unif}(f(\mathbb{C}))$ and $\mathrm{unif}(Z)$ are the same. Another requirement for the set Z is that $\pi(Z)$ must hit the boundary of $\mathcal{X}_G$, this is the equivalent of the unboundedness condition in the abelian case. This can be achieved using the fact that the symmetric space $\mathcal{X}_G$ has a realisation as a bounded symmetric domain in $\mathbb{C}^N$ using the Harish-Chandra embedding theorem.

As in the case of the hyperbolic Ax-Lindemann-Weierstrass theorem, we continue by proving a counting result showing that the number of translates of a fixed fundamental domain $\mathscr{F}$ by elements $\gamma\in\Gamma$ intersecting $f(\mathbb{C})$ grows polynomially with the height of γ. This is done in two steps, first we give an upper bound on the volume of the intersection between $f(\mathbb{C})$ and a translate of $\mathscr{F}$, then we compare this upper bound with a lower bound on the volume of the intersection between $f(\mathbb{C})$ and an open ball in $\mathcal{X}$ of fixed centre and varying radius. This is the same strategy used in the proof of the hyperbolic Ax-Lindemann-Weierstrass theorem (cf. [KUY16]), however in the present situation we will need to reprove both volume bounds as the results used in the pure case cannot be used in our setting.

With the counting result in place we can use a refinement of the Pila-Wilkie theorem to prove that for for any point in Z there is a positive dimensional connected semi-algebraic set that is contained in $\mathrm{unif}^{-1}(\overline{\mathrm{unif}\,Z}^{\mathrm{Zar}})$.

This then implies the statement on Zariski density of weakly special subvarieties in $\overline{\mathrm{unif}(Z)}^{\mathrm{Zar}}$.

Totally geodesic subvarieties of mixed Shimura varieties

Given a Riemannian manifold M, a submanifold $N \subset M$ is called *totally geodesic* at a point $p \in N$ if all geodesic curves in M through p that are tangent to N are contained in N. N is called totally geodesic if it is totally geodesic at every point. For example in Euclidean space totally geodesic submanifolds are linear spaces. Given a general manifold, one expects to find very few totally geodesic submanifolds; however, in the case of Shimura varieties, and more generally locally symmetric spaces, there are many totally geodesic submanifolds, which can be found by looking at the group action on the corresponding symmetric space.

The aim of this part is to show that some of the properties of totally geodesic submanifolds that are true in the pure case do not transfer to the mixed setting. Our original interest in etudying totally geodesic subvarieties of mixed Shimura varieties comes from an attempt at generalising a result of Ullmo and Yafaev on algebraic flows on pure Shimura varieties to the mixed case. This result will be the topic of the next part.

Moonen in [Moo98] proved that a subvariety of a pure Shimura variety is weakly special if and only if it is totally geodesic; more generally it is possible to prove that given a pure Shimura datum $(G, \mathcal{X}_G)$, all totally geodesic subvarieties of $\mathcal{X}_G$ are orbits under real algebraic subgroups of $G(\mathbb{R})$. We aim to show that both of these properties are not true in the mixed case.

Let $(P, \mathcal{X})$ be a mixed Shimura datum of Kuga type. We will start by showing that there are weakly special subvarieties of $\mathcal{X}$ that are not orbits under subgroups. One first example of this are fibres of the projection map to the pure part $\pi : \mathcal{X} \to \mathcal{X}_G$. This was already observed by Mok in [Mok91] and follows from the computation of some terms in the curvature tensor of $\mathcal{X}$

and the fact that the fibres of π are flat. We will generalise this to prove the following.

Theorem. *Let Y be a weakly special subvariety of $\mathcal{X}$ with non trivial constant part, then Y is not totally geodesic.*

And in another direction.

Theorem. *Let $x \in \mathcal{X}_G$, the only totally geodesic submanifold of $\mathcal{X}$ that contains the fibre $\pi^{-1}(x)$ is $\mathcal{X}$ itself.*

Regarding the second point we report an argument of N. Mok showing that there are geodesic curves in $\mathcal{X}$ that are not orbits under 1-parameter subgroups of $P(\mathbb{R})$. The idea is to consider geodesic curves that are tangent to the fibres π at some point and use the fact that these fibres are not totally geodesic.

These results are dependent on the explicit computations of the curvature terms carried out in Appendix B.

Algebraic flows

This section is dedicated to a result of Ullmo and Yafaev and its extension to Mixed Shimura varieties. This result is somewhat similar to the Ax-Lindemann-Weierstrass theorem but with the Zariski topology replaced by the analytic topology. The switch to the analytic topology presents new difficulties and the result only considers totally geodesic subvarieties of the uniformising space.

In what follows we will always use $(P, \mathcal{X})$ to denote a connected mixed Shimura datum of Kuga type, that is the subgroup U of P is trivial as in the case of $(P_{2g}, \mathcal{X}_g)$ and $(G, \mathcal{X}_G)$ the corresponding connected pure Shimura datum given by taking the quotient of $(P, \mathcal{X})$ by the unipotent radical of P, $\pi : (P, \mathcal{X}) \to (G, \mathcal{X}_G)$ will denote the projection map. Moreover M will denote a connected mixed Shimura variety associated with $(P, \mathcal{X})$ and an arithmetic subgroup Γ of P and S will be the corresponding connected pure Shimura variety.

The result proven by Ullmo and Yafaev is as follows.

Theorem. *Let* $(G, \mathcal{X}_G)$ *be a pure Shimura datum and let* $Z \subset \mathcal{X}_G$ *be a complex totally geodesic subvariety. Then the closure of* $\mathrm{unif}(Z)$ *in the analytic topology is a real weakly special subvariety.*

Real weakly special subvarieties are, as the name suggests, real algebraic subsets of a Shimura variety that come from particular rational algebraic subgroups of G. The precise definition is as follows.

Definition. Let $(G, \mathcal{X}_G)$ be a connected Shimura datum. An algebraic subgroup H of G defined over $\mathbb{Q}$ is said to be *of type* $\mathcal{H}$ if $H/\mathrm{R_u}(H)$ is a non trivial semisimple group with no compact simple factor.

Let $\Gamma \subset G(\mathbb{R})$ be an arithmetic subgroup and $S = Sh_\Gamma(G, \mathcal{X}_G)$. A *real weakly special subvariety* of S is a real analytic subset of S of the form

$$\Gamma \cap H(\mathbb{R})^+ \backslash H(\mathbb{R})^+ . x$$

for a subgroup H of type $\mathcal{H}$ of G and some point $x \in \mathcal{X}$.

Remark. The set $H(\mathbb{R})^+ . x \subset \mathcal{X}$ is a real symmetric space when the intersection of the stabilizer K_x of the point x in $G(\mathbb{R})$ with a Levi subgroup of H is a maximal compact subgroup (cf. [UY18a, p.7]).

The main ingredient in the proof of the above theorem is a result of Ratner about the closure of orbits under subgroups generated by 1-parameter unipotent subgroups together with the description of totally geodesic subvarieties of $\mathcal{X}$ as orbits under suitable subgroups of $G(\mathbb{R})$.

A similar strategy can be adopted in the case of mixed Shimura varieties. As remarked in the previous section, however, there is no description of totally geodesic subvarieties as orbits under real subgroups of P in the mixed case, so we have to start with a suitable orbit under a suitable real subgroup of P.

Theorem. *Let* Z *be a subvariety of* $\mathcal{X}$ *such that* $\pi(Z) \subset \mathcal{X}_G$ *is complex totally geodesic and* Z *is a vector bundle over* $\pi(Z)$ *that is homogeneous under the* *action of a Lie subgroup F of $P(\mathbb{R})^+$. Then the closure with respect to the analytic topology of* $\mathrm{unif}(Z)$ *is a real weakly special subvariety.*

In the setting of mixed Shimura varieties we define real weakly special subvarieties as follows.

Definition. A subvariety Y of a mixed Shimura variety M is a *real weakly special subvariety* if there exists an algebraic subgroup $H \subset P$ defined over $\mathbb{Q}$ such that $\pi(H)$ is of type $\mathcal{H}$ and $Y = \mathrm{unif}(H(\mathbb{R})^+.x)$ for some $x \in \mathcal{X}$.

Structure of the thesis

In Chapter 1 we recall some preliminaries used throughout the thesis. Section 1.1 recalls basic notions concerning families of mixed Hodge structures. Section 1.2 contains basic definitions about mixed Shimura varieties, their weakly special subvarieties and some examples. Section 1.3 recalls the definition of o-minimal structure, along with the main properties we will need.

Chapter 2 contains the main results concerning holomorphic curves in mixed Shimura varieties.

In Chapter 3 we study totally geodesic subvarieties of mixed Shimura varieties and algebraic flows. Sections 3.1 and 3.2 contain the main differential geometric tools we use. In Section 3.3 we study the relationship between totally geodesic, weakly special and homogeneous subvarieties of mixed Shimura varieties.

In Chapter 4 we study algebraic flows on mixed Shimura varieties.

Appendix A reports the o-minimal proof of the Ax-Lindemann-Weierstrass theorem and Appendix B contains some explicit computations of curvature terms for Kuga varieties.

Chapter 1

Preliminaries

In this Chapter we recall some prerequisites. The material is organized as follows. In the first section we start by recalling the definition of mixed Hodge structures and their relationship to linear algebraic groups, we then recall some results about equivariant families of mixed Hodge structures. This first section serves as a motivation for the definition of mixed Shimura varieties, as these can be defined as the classifying spaces for mixed Hodge structures that admit a canonical variation of mixed Hodge structures on them. In the second section we recall some definitions and properties related to mixed Shimura varieties. In the third section we recall the definition of o-minimal structure along with some basic properties we will need and the statement of the Pila-Wilkie counting theorem.

None of the material in this chapter is original, precise pointer to the literature will be given in each section.

1.1 Families of mixed Hodge structures

1.1.1 Definitions

We start by recalling the definition of a mixed Hodge structure, after this we will explain how these are connected to the representations of a particular algebraic torus.

Definition 1.1. Let R be either $\mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$. A *pure R-Hodge structure of weight n* is a pair (V, F), where

- V is a finite dimensional R-module,
- F is a decreasing filtration of $V_{\mathbb{C}} = V \otimes_R \mathbb{C}$ called the *Hodge filtration*.

Subject to the condition

- fro all p $F^p \cap F^{n-p+1} = \{0\}$.

The Hodge filtration determines a decomposition of $V_{\mathbb{C}}$ of the form $V_{\mathbb{C}} = \oplus V^{p,q}$ where $V^{p,q} = \mathrm{F}^p \cap \overline{\mathrm{F}^q}$. By definition $\overline{V^{p,q}} = V^{q,p}$. The set of pairs $(p,q) \in \mathbb{Z}^2$ such that $V^{p,q} \neq 0$ is called the *type* of the Hodge structure.

An *R-mixed Hodge structure* is a triple (V, W, F) where:

- V is a finite dimensional R-vector space,
- W is an increasing filtration of V by R-subspaces called the *weight filtration*,
- F is a decreasing filtration of $V_{\mathbb{C}}$ by complex subspaces called the *Hodge filtration.*

Such that F induces pure Hodge structures on the graded quotients $gr^i_W = W^i/W^{i-1}$.

The *type* of a mixed Hodge structure is the union of the types of its graded quotients.

Example 1.2. We recall some examples that explain the importance of (mixed) Hodge structures.

- The Tate Hodge structure $\mathbb{Z}(n)$ is the only Hodge structure on $(2\pi i)^n\mathbb{Z}$ of weight $-2n$ and type $(-n,-n)$.
- Let V be a $\mathbb{Q}$ vector space of dimension $2d$, giving a complex structure on V is the same as giving a Hodge structure of weight -1 and type $(-1,0),(0,-1)$ on V.
- In general the n-th cohomology group of a Kähler manifold carries a pure Hodge structure of weight n (see [Voi02]).

- Deligne introduced mixed Hodge structures to study the cohomology groups of general algebraic varieties; he proved that the n-th cohomology group of a general (possibly singular and not complete) complex algebraic variety carries a mixed Hodge structure of weight $\leq n$ (cf. [Del71]).

Definition 1.3. The *Deligne torus* is the algebraic group $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_m$[1], in particular $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ and $\mathbb{S}(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$.

The *weight homomoprhism* $w : \mathbb{G}_m \to \mathbb{S}$ is the homomorphism of algebraic groups that on real points corresponds to the inclusion of $\mathbb{R}^\times$ into $\mathbb{C}^\times$.

Notice that the choice of a representation $\mathbb{C}_C \to \mathrm{GL}(V_\mathbb{C})$, for some $\mathbb{Q}$-vector space V, induces a decomposition $V_\mathbb{C} = \oplus_{p,q} V^{p,q}$ where $V_{p,q}$ is the eigenspace for the character (p,q), that is the subspace on which an element $(z_1, z_2) \in \mathbb{C}^\times \times \mathbb{C}^\times = \mathbb{S}(\mathbb{C})$ acts as multiplication by $z_1^{-p} z_2^{-q}$. We can then define two filtrations of V as $W^i(V) = \oplus_{p+q\leq i} V^{p,q}$ and $\mathrm{F}^i V_\mathbb{C} = \oplus_{p\geq i} V^{p,q}$. The following proposition explains when these filtrations induce a mixed Hodge structure and how linear algebraic groups enter the picture.

Proposition 1.4 ([Proposition 1.4 and 1.5, Pin90])**.** *Let P be a linear algebraic group defined over $\mathbb{Q}$, W its unipotent radical, $G = P/W$ the reductive quotient and $\pi : P \to G$ the quotient map. Let $h : \mathbb{S}_\mathbb{C} \to P_\mathbb{C}$ be an homomorphism. Assume the following conditions are satisfied:*

(a) $\pi \circ h : \mathbb{S}_\mathbb{C} \to G_\mathbb{C}$ is already defined over $\mathbb{R}$.

(b) $\pi \circ h \circ w : \mathbb{G}_{m,\mathbb{R}} \to G_\mathbb{R}$ is already defined over $\mathbb{Q}$.

(c) Under the weight filtration on $\mathrm{Lie}(P)$ *defined by* $\mathrm{Ad} \circ h$ *we the pice of weight at most -1 is exactly* $\mathrm{Lie}(W)$.

Then:

1. *For every representation $\rho : P \to \mathrm{GL}(V)$ defined over $\mathbb{Q}$ the homomorphism $\rho \circ h$ defines a rational mixed Hodge structure on V.*

[1] $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}$ indicates the Weil restriction from $\mathbb{C}$ to $\mathbb{R}$ see [Secion 2.i, Mil17]

2. *The weight filtration on V induced by $\rho \circ h$ is invariant under P.*

3. *For any $p \in P(\mathbb{R})W(\mathbb{C})$ the assertion in the previous two points are valid also for $int(p) \circ h$ in place of h. The weight filtration and the Hodge numbers are independent of p.*

Conversely a homomorphism $k : \mathbb{S}_{\mathbb{C}} \to \mathrm{GL}(V_{\mathbb{C}})$ induces a rational mixed Hodge structure on V if and only if there exists a factorisation $k = \rho \circ h$ into a representation $\rho : P \to \mathrm{GL}(V)$ and a morphism $h : \mathbb{S}_{\mathbb{C}} \to P_{\mathbb{C}}$, for some linear algebraic $\mathbb{Q}$-group P, such that h and ρ satisfy the above conditions.

Now that we have an interpretation of Hodge structures in terms of the Deligne torus it is easier to define the notion of morphism of Hodge structures.

Definition 1.5. A *morphism of mixed Hodge sturctures* is a morphism of representations of $\mathbb{S}$. Similarly, the *tensor product* of mixed Hodges structures is the tensor product of representations of $\mathbb{S}$.

With the definition of morphisms and tensors we are now able to define polarizations.

Definition 1.6. Let (V,h) be a pure R-Hodge structure, a *polarisation* on V is a morphism $\psi : V \otimes V \to R(1) = R \otimes \mathbb{Z}(1)$ such that $\psi_h(x,y) = (2\pi i)^n \psi(x, h(i).y)$ is symmetric and positive definite. A *polarized* Hodge structure is a pair of a Hodge structure together with a polarisation. A *graded polarization* of a mixed Hodge structure is a collection of a polarisation for each pure graded quotient.

Example 1.7. A complex torus A of dimension $2d$ is defined as the complex manifold quotient of $\mathbb{C}^d$ by a lattice Λ. The quotient map induces an identification $\Lambda \cong H_1(A,\mathbb{Z})$ and $\mathbb{C}^d \cong H_1(A,\mathbb{R})$. This endows $H_1(A,\mathbb{R})$ with a complex structure and so $H_1(A,\mathbb{Z})$ with a pure Hodge structure of weight -1 and type $(-1,0),(0,-1)$. A classic result in the theory of abelian varieties [Section I.3, Theorem of Lefschetz, Mum70] states that this Hodge structure admits a polarisation precisely when A is an algebraic variety.

1.1.2 Families of Hodge structures

We now introduce families of mixed Hodge structures and present some results which will explain the axioms in the definition of a mixed Shimura variety. From here on we only concentrate on $\mathbb{Q}$-mixed Hodge structures.

Proposition 1.8 ([Proposition 1.7, Pin90]). *Let P be a linear algebraic group over $\mathbb{Q}$ and $\mathcal{H}_W$ a $P(\mathbb{R})W(\mathbb{C})$-conjugacy class in $\mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$. Assume that for some $h \in \mathcal{H}_W$ the conditions in proposition 1.4 are satisfied. Let V be a faithful representation of P and consider the induced map*

$$\phi : \mathcal{H}_W \to \{\textit{rational mixed Hodge structures on } V\} \tag{1.1}$$

(a) There exists a unique structure on $\phi(\mathcal{H}_W)$ of complex manifold such that the Hodge filtration on $V_{\mathbb{C}}$ depends analytically on $\phi(h)$ for $h \in \mathcal{H}_W$. This structure is $P(\mathbb{R})W(\mathbb{C})$ invariant, and $W(\mathbb{C})$ acts analytically on $\phi(\mathcal{H}_W)$.

(b) For any other representation V' of P the analogous map ϕ' factors through ϕ and the Hodge filtration on V' varies analytically with $\phi(h)$.

(c) If in addition V' is faithful, then $\phi(\mathcal{H}_W)$ and $\phi'(\mathcal{H}_W)$ are canonically isomorphic and the isomorphism is compatible with the complex structure.

Definition 1.9. Let X be a complex manifold. A *rational variation of mixed Hodge structure* on X consists of a local system $\mathbb{V}$ of finite dimensional $\mathbb{Q}$ vector spaces on X together with a rational mixed Hodge structure on every fibre such that

(a) The weight filtration is locally constant and the Hodge filtration varies holomorphically.

(b) Let $\mathcal{V}$ be the locally free sheaf associated with the local system $\mathbb{V}$ and let $\mathcal{F}^p\mathcal{V}$ be the sub-sheaves that induce the Hodge filtration on each fibre. For every $p \in \mathbb{Z}$ the canonical connection $\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_X$ maps the sub-sheaf $F^p\mathcal{V}$ to $F^{p-1}\mathcal{V} \otimes \Omega^1_X$. This property is called *transversality.*

Example 1.10. Consider a family of smooth algebraic varieties $Y \to X$. It is a result of Griffiths that the cohomology of the fibres of the family forms a variation of Hodges structures over the base X (cf. [Secion 1.c, Gri68]).

The next proposition explains when the canonical family of Hodge structures on $\mathcal{H}_W$ is in fact a variation.

Proposition 1.11 ([Proposition 1.10, Pin90])**.** *Let P, $\mathcal{H}_W$, V and ϕ be as in Proposition 1.8. Then we have a variation of mixed Hodge structure on V over $\phi(\mathcal{H}_W)$ if and only if for some $h \in \mathcal{H}_W$ the Hodge structure on* Lie(P) *is of type*

$$\{(-1,1),(0,0),(1,-1),\ (-1,0),(0,-1),\ (-1,-1)\}. \tag{1.2}$$

Now we see that it is actually possible to restrict to a smaller conjugacy class of homomorphisms.

Proposition 1.12 ([Proposition 1.16, Pin90])**.** *Assume the condition of the previous proposition is satisfied. Let $U \subset W \subset P$ be the subgroup such that* $\mathrm{Lie}(U) = W_{-1}(\mathrm{Lie}(P))$. *Consider the following stronger version of condition (a) in Proposition 1.4*

(a) $\pi' \circ h : \mathbb{S}_\mathbb{C} \to (P/U)_C$ is already defined over $\mathbb{R}$.

Let $\mathcal{H}$ be the subset of $\mathcal{H}_W$ which satisfies the stricter condition above. Then we have

- *$\mathcal{H}$ is a non empty $P(\mathbb{R})U(\mathbb{C})$ orbit in* $\mathrm{Hom}(\mathbb{S}_\mathbb{C}, P_\mathbb{C})$.
- *given a faithful representation V of P and the associated map ϕ, we have $\phi(\mathcal{H}_W) = \phi(\mathcal{H})$.*

1.2 Mixed Shimura varieties

In this section we introduce mixed Shimura varieties and some related concepts. In the first section we will introduce the definition of mixed Shimura datum and variety and explain the relationship with variations of mixed Hodge structures. In the second we revisit the examples given in the introduction and explain how they fit into the definitions. In the third we introduce special and weakly special subvarieties. In the fourth we recall some definitions and properties of boundary components of pure Shimura varieties.

1.2.1 Mixed Shimura varieties

In this section we introduce Pink's definition of a mixed Shimura variety with particular emphasis on the Hodge theoretic interpretation of the definition in connection to the previous section.

Definition 1.13 ([Definition 2.1, Pin90]). A *mixed Shimura datum* is a triple $(P, \mathcal{X}, h)$ where P is an algebraic group defined over $\mathbb{Q}$, $\mathcal{X}$ is an homogeneous space under $P(\mathbb{R})U(\mathbb{C})$, where U is a $\mathbb{Q}$-subgroup of the unipotent radical of P and $h : \mathcal{X} \to \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$ such that, denoting by $\pi : P \to G = P/W$ and $\pi' : P \to P/U$ the canonical projections, for one (equivalently any) point $x \in \mathcal{X}$ the following are satisfied

(a) The fibres of h consist of finitely many points.

(b) $\pi' \circ h_x : \mathbb{S}_{\mathbb{C}} \to P_{\mathbb{C}}$ is already defined over $\mathbb{R}$.

(c) $\pi \circ h_x \circ w : \mathbb{G}_{m,\mathbb{R}} \to (P/W)_{\mathbb{R}} = G_{\mathbb{R}}$ is a cocharacter of the centre of G.

(d) $\mathrm{Ad}_P \circ h_x$ induces a mixed Hodge structure on the Lie algebra of P of type

$$\{(-1,-1),(-1,0),(0,-1),(-1,1),(0,0),(1,-1)\}. \tag{1.3}$$

(e) The weight filtration on $\operatorname{Lie} P$ is given by

$$W_n = \begin{cases} \{0\} & \text{if } n < -2 \\ \operatorname{Lie} U & \text{if } n = -1 \\ \operatorname{Lie} W & \text{if } n = -1 \\ \operatorname{Lie} P & \text{if } n \geq 0. \end{cases} \tag{1.4}$$

(f) $\pi \circ h_x(\sqrt{-1})$ induces a Cartan involution of $G^{\mathrm{ad}}_{\mathbb{R}}$.

(g) G^{ad} possesses no $\mathbb{Q}$-factor of compact type.

(h) $P/P^{der} = Z(G^{\mathrm{ad}})$ is an almost direct product of a $\mathbb{Q}$-split torus with a torus of compact type defined over $\mathbb{Q}$.

Remark 1.14. We will often drop the equivariant map h from the notation of a mixed Shimura datum, unless necessary.

Remark 1.15. As anticipated above, the conditions in the definition of mixed Shimura datum are linked to the previous section. In particular, fixed a rational representation V of P

- Conditions (b) (c) and (e) imply that any point $h \in \mathcal{X}$ induces a mixed Hodge structure on V.
- Condition (h) implies that the weight morphism is defined already over $\mathbb{Q}$ (cf. [1.19, Pin90]).
- Condition (h) implies also that any sufficiently small congruence subgroup of P is contained in P^{der} (cf. [proof of 3.3(a), Pin90]).
- Condition (g) implies that any congruence subgroup of P that is contained in P^{der} is Zariski dense in it (cf. [theorem 4.10, PR94])
- Condition (a) implies that $\mathcal{X}$ has a canonical complex structure invariant under the action of $P(\mathbb{R})U(\mathbb{C})$ and there is an equivariant family of mixed Hodge structures on the constant local system $\mathbb{V}$ with fibre V.

- Condition (d) implies that the equivariant family of mixed Hodge structures on $\mathbb{V}$ obtained in the previous point is indeed a variation.
- The remaining condition (f) implies that the variation is graded polarisable.

Remark 1.16. Condition (h) in the definition is more stringent than the one used in [Pin90], this however does not pose any problems in the current setting as we will be only interested in a single connected component of $\mathcal{X}$.

The connected components of the space $\mathcal{X}$ can be described as follows.

Proposition 1.17 ([2.19, Pin90])**.** *Let $(P, \mathcal{X})$ be a mixed Shimura datum. If P is reductive, every connected component of $\mathcal{X}$ is a Hermitian symmetric domain. In the general case the connected components of $\mathcal{X}$ are holomorphic vector bundles over a Hermitian symmetric domain. The projection map to the Hermitian symmetric domain is induced by the projection $\pi : P \to P/\mathrm{R_u}(P)$ of P modulo its unipotent radical.*

There are two interesting special cases of the definition of mixed Shimura datum.

Definition 1.18 ([Definition 2.1, Gao20])**.** Let $(P, \mathcal{X})$ be a mixed Shimura datum. If P is reductive it is called a *pure Shimura datum*. More generally, if the subgroup U is trivial it is called a mixed Shimura datum of *Kuga type*.

We now recall the definition of mixed Shimura variety.

Definition 1.19 ([Definition 3.1, Pin90])**.** Let $(P, \mathcal{X})$ be a mixed Shimura datum and $K \subset P(\mathbb{A}_f)$ a compact open subgroup. The *mixed Shimura variety* associated to the datum $(P, \mathcal{X})$ and the subgroup K is

$$M_K(P, \mathcal{X}) = P(\mathbb{Q})\backslash \mathcal{X} \times P(\mathbb{A}_f)/K. \tag{1.5}$$

Where $P(\mathbb{Q})$ acts on both factors by multiplication on the left and K acts only on the second factor by multiplication on the right.

As defined mixed Shimura varieties are not connected. The following definition introduces connected mixed Shimura data and varieties, which will be the main point of interest in the following sections.

Definition 1.20 ([Definitions 2.1, 2.4, Pin05])**.** A *connected mixed Shimura datum* is a pair $(P, \mathcal{X}^+)$ where P is an algebraic group over $\mathbb{Q}$ and $\mathcal{X}^+$ is an orbit under the group $P(\mathbb{R})^+U(\mathbb{C})$ which satisfy the conditions in Definition 1.13. A *connected mixed Shimura variety* relative to the datum $(P, \mathcal{X}^+)$ and the congruence subgroup $\Gamma \subset P(\mathbb{Q}) \cap P(\mathbb{R})^+$ is the quotient $\Gamma \backslash \mathcal{X}^+$.

As one might expect, connected mixed Shimura varieties are connected components of mixed Shimura varieties (cf. [3.2, Pin90]).

We now introduce morphism of mixed Shimura data and mixed Shimura varieties. These make Shimura varieties in a category.

Definition 1.21. A *Shimura morphism of mixed Shimura data* $\phi : (P, \mathcal{X}) \to (Q, \mathcal{Y})$ is a homomorphism $\phi : P \to Q$ of algebraic groups over $\mathbb{Q}$ together with the induced map $\mathcal{X} \to \mathcal{Y}$.

A *Shimura morphism of mixed Shimura varieties* is a morphism between mixed Shimura varieties that is induced by a Shimura morphism of the associated mixed Shimura data. Given a morphism $\phi : (P, \mathcal{X}) \to (Q, \mathcal{Y})$ of mixed Shimura data, any morphism between associated mixed Shimura varieties induced by ϕ will be denoted by $[\phi]$

The result below implies that mixed Shimura varieties with their morphisms form a subcategory of the category of complex algebraic varieties.

Proposition 1.22 ([3.3, 9.24, Pin90])**.** *Let S be a connected mixed Shimura variety associated to the datum $(P, \mathcal{X})$ and the subgroup Γ. Then S has a canonical structure of normal complex quasi-projective variety. If moreover Γ is neat S is smooth. Finally, any Shimura morphism between mixed Shimura varieties is algebraic.*

1.2.2 Examples

1.2.2.1 Moduli space of abelian varieties $\mathcal{A}_g$

Here we explain how the pure Shimura datum corresponding to the moduli space of principally polarised abelian varieties is constructed.

The Shimura datum. Let V be a $\mathbb{Q}$ vector space of dimension $2d$ equipped with a symplectic form ψ. Define the group $\mathrm{GSp}(V,\psi)$ to be the subgroup of $\mathrm{GL}(V)$ that preserves ψ up to a scalar. Consider the set $\mathcal{X}$ of complex structures J on V with the following properties

- $J \in \mathrm{GSp}(V,\psi)$
- The bilinear form on V defined by $\psi_J(x,y) = \psi(x,Jy)$ is symmetric and positive or negative definite.

Define $\mathcal{X}^+$, respectively $\mathcal{X}^-$, to be the subsets of $\mathcal{X}$ containing the complex structures J such that ψ_J is positive, respectively negative, definite. There is a bijective and equivariant map between $\mathcal{X}^+$ and the set of symplectic bases $\{e_{\pm i}\}_{i=1,\dots,d}$ for the form ψ defined by taking the complex structure J that exchanges e_{+i} and e_{-i}. This correspondence proves that the group $\mathrm{Sp}(V,\psi)$ of elements in $\mathrm{GSp}(V,\psi)$ preserving ψ acts transitively on $\mathcal{X}^+$, which implies that $\mathrm{GSp}(V,\psi)$ acts transitively on $\mathcal{X}$.

There is a bijective equivariant map

$$\begin{aligned} h : \mathcal{X} &\to \mathrm{Hom}(\mathbb{S}_{\mathbb{R}}, \mathrm{GSp}(V,\psi)_{\mathbb{R}}) \\ J &\mapsto h_J \end{aligned} \tag{1.6}$$

where $h_J(a+ib) = a\,\mathrm{Id} + bJ$. It is now possible to easily verify the axioms of mixed Shimura datum in this case. Consider the triple $(\mathrm{GSp}(V,\psi),\mathcal{X})$. Then axioms (a), (b) and (c) follow directly from the definition of h. Axiom (d) can be proven by decomposing $V_{\mathbb{C}} = V^+ \oplus V^-$ into a direct sum of the subspaces where ψ is, respectively positive and negative definite, then looking at the action of $h_J(\mathbb{S})$ on this decomposition. Axiom (e) is trivial as the weight filtration of Lie(P) consists of only one term in this case. Axiom (f) follows from the second point in the definition of $\mathcal{X}$. The fact that $\mathrm{Sp}(V,\psi)$ is simple implies (g). Finally axiom (h) is true because the centre of $\mathrm{GSp}(V,\psi)$ is itself a $\mathbb{Q}$-split torus.

Cf. [Chapter 6, Mil04] for more details regarding this Shimura datum and its interpretation as moduli space of principally polarised abelian varieties.

Siegel upper half-space. We now restrict to the connected Shimura datum $(\mathrm{Sp}(V,\psi),\mathcal{X}^+)$. The definition of the Shimura datum $(\mathrm{Sp}(V,\psi),\mathcal{X}^+)$ can be made more explicit by fixing a basis for V. In particular we fix a basis so that $V \cong \mathbb{Q}^{2d}$ and ψ has matrix $\left(\begin{smallmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{smallmatrix}\right)$. Rephrasing the construction in terms of Hodge structures, we consider the subspace $\mathrm{F}_0 V_{\mathbb{C}}$ of $V_{\mathbb{C}}$ associated to a complex structure J, that is the eigenspace with eigenvalue $-i$ for J. The requirement that ψ_J be positive definite is equivalent to $\psi(x,\overline{y})$ being a complex inner product on $\mathrm{F}^0 V_{\mathbb{C}}$. This, in particular, implies that $\mathrm{F}_0 V_{\mathbb{C}}$ has intersection zero with the subspaces generated by $e_1,\dots,e_d$ and $e_{d+1},\dots,e_{2d}$, so there is a matrix $\overline{\tau} \in \mathrm{GL}_d(\mathbb{C})$ such that $\mathrm{F}_0 V_{\mathbb{C}} = \left\{ \left(\begin{smallmatrix} \overline{\tau} v \\ v \end{smallmatrix}\right) \text{ for } v \in \mathbb{C}^d \right\}$. The two conditions imposed on ψ in the definition imply that $(2\pi i)\psi$ is a polarisation for V. In particular $\psi : V \otimes V \to \mathbb{Q}(1)$ is a morphism of Hodge structures. This implies that $\psi(\mathrm{F}_0 V_{\mathbb{C}}, \mathrm{F}_0 V_{\mathbb{C}}) = \{0\}$, which gives τ symmetric. Finally the positivity condition on ψ_J implies that $Im(\tau)$ is positive definite.

This discussion identifies $\mathcal{X}$ with the Siegel upper half-space $\mathcal{H}_d$ of complex symmetric $d \times d$ with positive definite imaginary part.

By following through the calculations it can easily be seen that the action induced on $\mathcal{H}_g$ by this identification is the standard linear fractional action of Sp_{2g} on the Siegel upper half-space.

Proposition 1.17 states that $\mathcal{H}_d$ is a Hermitian symmetric space this structure can be realized by equipping the Siegel upper half-space with the metric $\omega = i\partial\overline{\partial}(\log\det\mathrm{Im}(Z))$.

Using a complex basis for V such that ψ has matrix $\left(\begin{smallmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{smallmatrix}\right)$ and carrying out a similar analysis we can get the Harish-Chandra realisation of $\mathcal{X}$ as the bounded symmetric domain D_n^{III}, cf. [Chapter 4, Section 2.3, Mok89] and [Chapter 2, Section 7, Sat80].

1.2.2.2 The universal family of abelian varieties.

As we alluded to in the introduction, the universal family over the fine moduli space of principally polarised abelian varieties can be seen as a mixed Shimura variety. We briefly recall the definition of the associated connected mixed Shimura datum.

Consider the unipotent group $\mathbb{G}_a^{2g}$ with the natural action of Sp_{2g} and construct the semi-direct product $P = \mathbb{G}_a^{2g} \rtimes \mathrm{Sp}_{2g}$. Then, extend $\mathcal{H}_g$ to the vector bundle $\mathcal{X} = \mathbb{R}^{2g} \times \mathcal{H}_g$. Given $(v,h) \in P(\mathbb{R})$ and $(w,x) \in \mathcal{X}$ we can define an action of $P(\mathbb{R})$ on $\mathcal{X}$ as

$$(v,h).(w,x) = (v + h.w, h.x) \tag{1.7}$$

that is, R^{2g} is acting on $\mathcal{X}$ as translation in the vertical direction and $\mathrm{Sp}_{2g}(\mathbb{R})$ is acting via the standard representation on the fibre and the linear fractional action on $\mathcal{H}_g$. At this point we only need to define a complex structure on $\mathcal{X}$, then the axioms in the definition of mixed Shimura datum are satisfied by [Proposition 2.17, Pin90]. The complex structure on $\mathcal{X}^+$ is defined by the following identification

$$\begin{aligned} \mathcal{X}^+ = \mathbb{R}^g \times \mathbb{R}^g \times \mathcal{H}_g &\to \mathbb{C}^g \times \mathcal{H}_g \\ (a,b,Z) &\mapsto (a - \overline{Z}b, Z) \end{aligned} \tag{1.8}$$

and accordingly on $\mathcal{X}^-$.

1.2.3 Special and weakly special subvarieties

In this section we will introduce definitions for special and weakly special subvarieties of mixed Shimura varieties.

Definition 1.23 ([Pin05])**.** Let M be a connected mixed Shimura variety. A subvariety Z of M is called *special* if it is the image of a Shimura morphism

$M' \to M$.

Remark 1.24. Let $(G, \mathcal{X}_G^+)$ be a connected pure Shimura datum and S a pure Shimura variety relative to $(G, \mathcal{X}_G^+)$ and some lattice Γ. In this case special subvariety are usually defined as components of Hecke translates of Shimura subvarieties. Note that in this case the above definition reduces to the usual one (cf. [Lemma 2.1, UY14a]). Indeed if $(H, \mathcal{X}_H^+)$ is a Shimura datum and $i : H \to G$ is a closed immersion, inducing a closed immersion of the data and a closed immersion of $S_H = M_{\Gamma \cap H(\mathbb{Q})}(H, \mathcal{X}_H^+)$ into S, considering a component of the Hecke translate $T_g(S_H)$ of S_H by $g \in G(\mathbb{Q})$ amounts to the same as considering the conjugate immersion gig^{-1}.

Definition 1.25 (cf. [Pin05, p. 266])**.** Let S be a mixed Shimura variety. A point $P \in S$ is *special* if it is a special subvariety of dimension 0.

Remark 1.26 (cf. [Definition 4.10, Pin05])*.* The definition of special point can be restated as follows. Let $(P, \mathcal{X})$ be a mixed Shimura datum. A point $x \in \mathcal{X}$ and its image in any mixed Shimura variety relative to the datum $(P, \mathcal{X})$ is special if and only if there exists a torus $T \subset P$ defined over $\mathbb{Q}$ such that the homomorphism $h_x : \mathbb{S}_{\mathbb{C}} \to P(\mathbb{C})$ factors through $T_{\mathbb{C}}$.

Proposition 1.27 ([11.7, Pin90])**.** *Let S be a mixed Shimura variety and Z a special subvariety. The set of special points of S contained in Z is dense in Z both for the Zariski and the analytic topology.*

Definition 1.28 ([Definition 4.1, Pin05])**.** Let $S = M_\Gamma(P, \mathcal{X})$ be a mixed Shimura variety. A subvariety Z of S is called *weakly special* if there exists a diagram of Shimura morphisms as follows.

$$\begin{array}{ccc} & Q & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle g} \\ Q' & & S \end{array}$$

Such that Z is a component of $g(f^{-1}(P))$ for some point $P \in Q'$.

Proposition 1.29 ([Proposition 4.15, Pin05])**.** *Let M be a mixed Shimura variety. A subvariety of M is special if and only if it is weakly special and contains a special point.*

The following is a classification of the weakly special subvarieties in the Kuga case due to Gao.

Proposition 1.30 (cf. [Corollary 1.2.15, Gao14])**.** *Let M be a mixed Shimura variety of Kuga type, S the associated pure Shimura variety and let $\pi : M \to S$ be the projection morphism. A subvariety Z of M is weakly special if $\pi(Z)$ is weakly special and Z is a translate of an abelian subscheme of $\mathcal{A} = \pi^{-1}(\pi(Z))$ by a torsion section of $\mathcal{A}$ plus a section of the constant part of $\mathcal{A}$.*

1.2.4 Boundary components of pure Shimura varieties

In this section we recall some results about the boundary components of pure Shimura varieties. these will be useful later when analysing the volumes of holomorphic curves near the boundary of the symmetric space associated to a pure Shimura datum.

Let $(G, \mathcal{X})$ be a pure Shimura datum. Recall that in this case $\mathcal{X}$ is a Hermitian symmetric space of non-compact type. Below we recall the Harish-Chandra embedding theorem that allows us to see $\mathcal{X}$ as an analytic open subset in a complex vector space.

Theorem 1.31 (Borel embedding[Theorem 1, Section 3.3, Mok89])**.** *Let $\mathcal{X}$ be a Hermitian symmetric manifold of non-compact type, there is a Hermitian symmetric manifold of compact type $\mathcal{X}^\vee$ called its* compact dual *and an open embedding $\mathcal{X} \to \mathcal{X}^\vee$.*

Remark 1.32. This Theorem can be seen as a purely group theoretical result on the Lie group of biholomorphisms of $\mathcal{X}$. However in the case of Shimura varieties it can also be interpreted in Hodge theoretical terms. Indeed, $\mathcal{X}$ can be seen as a period domain for certain pure Hodge structures. These are constructed in two stages by first fixing a flag variety then imposing the two Riemann bilinear relations. The first relation identifies a closed algebraic subvariety of the flag variety. This is exactly the compact dual of the space $\mathcal{X}$. Then the second relation identifies $\mathcal{X}$ as a semi-algebraic open subset of its compact dual. See [Pea00] for more details.

Theorem 1.33 (Harish-Chandra embedding [Theorem 2, Section 5.2, Mok89])**.** *Let $\mathcal{X}$ a Hermitian symmetric space of non-compact type, $x_0 \in \mathcal{X}$ and $\mathfrak{m}$ the holomorphic tangent space to $\mathcal{X}$ at x_0. There exists a map*

$$\eta : \mathfrak{m} \to \mathcal{X}^\vee \tag{1.9}$$

that is a biholomorphism onto a dense open subset containing $\mathcal{X}$ and such that $\eta^{-1}(\mathcal{X})$ is an open bounded symmetric domain in $\mathfrak{m}$.

From now on we assume $\mathcal{X}$ embedded in $\mathbb{C}^N = \mathfrak{m}$ through the Harish-Chandra embedding.

Definition 1.34. Let $\mathcal{X}$ be a Hermitian symmetric space of non-compact type. A *boundary component* is a maximal analytic sub-manifold of $\partial\mathcal{X}$.

We now recall two results we will need in the following chapters. These were first proven in connection with the study of smooth compactifications of arithmetic varieties, that is quotients of $\mathcal{X}$ by the action of an arithmetic group. Our interest is due to the fact that these results allow us to analyse closely what happens near a fixed boundary component of $\mathcal{X}$ by embedding it as an analytic subset of a Siegel domain of the third kind. Siegel domains of the third kind have a natural Poincaré metric on them that bounds the metric on $\mathcal{X}$ and will allow us to carry out some volume computations, cf. Lemma 2.11 and [Theorem 3.1, Mum77].

Proposition 1.35 ([Chapter III.4, AMRT10])**.** *Given a boundary component $F \subset \overline{\mathcal{X}}$, its normaliser $N(F)$ in G is a parabolic subgroup and can be decomposed as follows*

$$N(F) = \Big(G_h(F) G_l(F) M(F)\Big)\Big(V(F) U(F)\Big),$$

where

- $R(F) = (G_h(F)G_l(F)M(F))$ *is a Levi factor of* $N(F)$ *and the product is direct modulo a finite central group*
- $W(F) = (V(F)U(F))$ *is the unipotent radical of* $N(F)$
- $U(F)$ *is the centre of* $W(F)$ *and is a real vector space*
- $V(F) = W(F)/U(F)$ *is also a real vector space of even dimension* $2l$
- $G_h(F)$ *modulo a finite centre is* $\mathrm{Aut}^0(F)$*, all other factor act trivially*
- $G_l(F)$ *modulo a finite centre acts on* $U(F)$ *by inner automorphisms, the other factors commute with* $U(F)$
- $M(F)$ *is compact*

Proposition 1.36 ([Proposition 3.2 and Lemma 4.2, KUY16])**.** *Fix a boundary component* $F \subset \overline{\mathcal{X}}$*. Define*

$$\mathcal{X}_F = \bigcup_{g \in U(F)_{\mathbb{C}}} g.\mathcal{X} \subset \mathcal{X}^c.$$

There is a holomorphic semi-algebraic isomorphism $j : \mathcal{X}_F \to U(F)_{\mathbb{C}} \times \mathbb{C}^l \times F$*. This isomorphism realises* $\mathcal{X}$ *as a Siegel domain of the third kind*

$$\mathcal{X} \overset{j}{\simeq} \left\{ (x, y, t) \in U(F)_{\mathbb{C}} \times \mathbb{C}^l \times F \mid \mathrm{Im}(x) + l_t(y, y) \in C(F) \right\}$$

where $C(F)$ *is a self-adjoint convex cone in* $U(F)$ *homogeneous under* $G_l(F)$ *and* $l_t : \mathbb{C}^l \times \mathbb{C}^l \to U(F)$ *is a symmetric bilinear form varying real-analytically with* $t \in F$*.*

Let $\Sigma \subset \mathcal{X}$ *be a Siegel set for the action of* Γ*, as above. Then* Σ *is covered by a finite number of open subsets* Θ *having the following properties. For each* Θ *there is a cone* σ *with* $\sigma \subset \overline{C(F)}$*, a point* $a \in C(F)$*, relatively compact subsets*

U', Y' and F' of $U(F)$, $\mathbb{C}^l$ and F respectively such that the set Θ is of the form

$$\Theta \overset{j}{\simeq} \big\{(x,y,t) \in U(F)_{\mathbb{C}} \times \mathbb{C}^l \times F \mid \mathrm{Re}(x) \in U', y \in Y', t \in F' \textit{ and } \mathrm{Im}(x) + l_t(y,y) \in \sigma + a\big\}.$$

1.3 *o*-minimal structures and definability

In this section we recall the definition of o-minimal structure and some properties, we then go on to recall Pila-Wilkie's theorem on rational points in definable sets. We finish the section by describing the definable structure of mixed Shimura data and recalling the Ax-Lindemann-Weierstrass theorem. For more details on the material presented in this section see [Wil15] and references therein for an overview of o-minimal structures and the Pila-Wilkie theorem and [Gao17] for the part on mixed Shimura varieties.

1.3.1 Definitions

Definition 1.37. An *o-minimal structure* over the field of real numbers[2] is a sequence $(S_n)_{n\in\mathbb{N}}$ where S_n is collection of subsets of $\mathbb{R}^n$ such that for every $m,n \in \mathbb{N}$ the following are satisfied

(a) Each S_n is closed under finite unions, finite intersections and complements;

(b) The product of sets in S_m and S_n belongs to S_{m+n};

(c) The projection of S_{n+1} onto the first n coordinates belongs to S_n;

(d) The set $\{(x_1,\dots,x_n) :\ x_1 = x_n\}$ belongs to S_n;

(e) The set $\{(x_1,x_2) :\ x_1 < x_2\}$ and the graph of the addition and multiplication functions belong to S_2;

(f) The sets in S_1 are finite unions of intervals and points.

[2] It is possible to define o-minimal structures on any dense linear order without end points, see [Dri98] for a more general definition in this context. Here we insist on the field operations being definable for the nature of the results we are interested in.

A set belonging to one of the S_n is called *definable* (in the structure $(S_n)_{n\in\mathbb{N}}$). A function or a relation is called *definable* is its graph is a definable set.

From now on we will use definable to mean definable in some o-minimal structure, if we need definability in some particular o-minimal structure this will always be specified.

Example 1.38. The smallest example of an o-minimal structure is the collection $\mathbb{R}_{alg}$ of semi-algebraic sets. These are finite unions of sets defined by a finite number of polynomial equations and inequalities. They form an o-minimal structure by a fundamental result of Tarski [Tar48].

There are two larger o-minimal structures we will be interested in: $\mathbb{R}_{\text{an}}$ and $\mathbb{R}_{\text{an,exp}}$. The first is the smallest o-minimal structure such that all functions $f : [0,1]^n \to \mathbb{R}$ analytic in a neighbourhood of $[0,1]^n$ are definable. The second is the smallest o-minimal structure containing the first and the real exponential function. These structures are o-minimal by results of van den Dries, Denef and van den Dries, Wilkie, van den Dries and Miller, cf. [Dri86; DD88; Wil96; DM94].

1.3.2 Cell decomposition

In this section we will introduce one of the most important properties of o-minimal structure: cell decomposition.

We start by defining cells and decompositions.

Definition 1.39 ([(2.3) Chapter 3 Dri98])**.** Let $i_0,\dots,i_n \in \{0,1\}$. A $(i_1,\dots,i_n)$-*cell* is a subset of $\mathbb{R}^n$ defined inductively as follows

- A (0)-cell is a point,
- A (1)-cell is an open interval, possibly unbounded at either or both ends,
- A $(i_1,\dots,i_{n-1},0)$-cell is the graph of a definable function on an $(i_1,\dots,i_{n-1})$-cell,

- A $(i_1, \dots, i_{n-1}, 1)$-cell is defined as the set

$$\begin{aligned} \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid f(x_1, \dots, x_{n-1}) < x_n < g(x_0, \dots, x_{n-1}) \\ \text{and } (x_1, \dots, x_{n-1}) \in C\} \end{aligned} \tag{1.10}$$

  where C is an $(i_1, \dots, i_{n-1})$-cell and f and g are definable functions on C such that $f < g$ on C.

Definition 1.40 ([(2.10) Chapter 3 Dri98]). A *decomposition* of $\mathbb{R}^n$ is a partition of $\mathbb{R}^n$ defined inductively as follows

- A decomposition of $\mathbb{R}$ is a partition of the type

$$\{(-\infty, a_1), (a_1, a_2), \dots, (a_k, \infty), \{a_1\}, \dots \{a_k\}\}, \tag{1.11}$$

- A decomposition of $\mathbb{R}^n$ is a partition of $\mathbb{R}^n$ into finitely many cells $\{A_i\}_{i \in I}$ such that the set of projections $\{\pi(A_i)\}_{i \in I}$ is a decomposition of $\mathbb{R}^{n-1}$; here π is the projection onto the first $n-1$-coordinates.

We can now state the cell decomposition theorem.

Theorem 1.41 (Cell decomposition [(2.11) Chapter 3 Dri98]). *Given definable sets $A_1, \dots, A_l$ in $\mathbb{R}^m$, there is a decomposition of $\mathbb{R}^n$ which partitions each of the A_i.*

One of the consequences of the cell decomposition theorem is the finiteness of connected components of definable sets. In the following chapters we will also need a stronger property: the uniform finiteness of connected components in definable families. To state this we start by recalling the definition of a definable family.

Definition 1.42 ([(3.1) Chapter 3 Dri98]). Let $S \subset \mathbb{R}^m \times \mathbb{R}^n$ be a definable set, the collection $S_a \subset \mathbb{R}^m$ for $a \in \mathbb{R}^n$ given by the fibres of S under the projection $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ is called a *definable famlily.*

Proposition 1.43 ([(3.6) Chapter 3 Dri98])**.** *Let $S_a \subset \mathbb{R}^m$ be a definable family. The number of connected components of S_a is bounded independently of a.*

We close this section with a refinement of the cell decomposition theorem, which states that in certain o-minimal structures it is possible to choose cells defined by real analytic functions.

Definition 1.44. A cell is said to be (real) *analytic* if the definable functions used in its inductive definition can be chosen analytic.

Definition 1.45. The o-minimal structure $\widetilde{\mathbb{R}}$ is said to admit *analytic cell decomposition* if it satisfies the cell decomposition theorem with the additional requirement that the cells can all be chosen analytic.

A result of van den Dries and Miller cf. [DM94, Section 8] implies.

Theorem 1.46 (Analytic cell decomposition)**.** *The o-minimal structures $\mathbb{R}_{\mathrm{an}}$ and $\mathbb{R}_{\mathrm{an,exp}}$ admit analytic cell decomposition.*

1.3.3 The Pila-Wilkie theorem

Fix an o-minimal structure $\widetilde{\mathbb{R}}$ of the field of real numbers. In this section definable will mean definable in the o-minimal structure $\widetilde{\mathbb{R}}$. We will now recall a few definitions before stating the Pila-Wilkie Theorem.

Definition 1.47. Let $p/q \in \mathbb{Q}$ such that $(p,q) = 1$. We define the height $H(p/q) = \max\{p,q\}$. We extend this definition to $\mathbb{Q}^n$ by taking the maximum of the heights of each component.

Definition 1.48. Let $X \subset \mathbb{R}^n$ be any subset. The *algebraic part* of X, denoted X^{alg} is the union of all connected positive dimensional semi-algebraic subsets of X.

Definition 1.49. Let $X \subset \mathbb{R}^n$, the *density function* associated to X is defined as

$$N(X,T) = \#\{x \in X \cap \mathbb{Q}^n : \ H(x) < T\}. \tag{1.12}$$

We are now ready to state the first version of the Pila-Wilkie theorem.

Theorem 1.50 ([PW06, Theorem 1.8])**.** *Let $X \subset \mathbb{R}^n$ be a definable set and $\varepsilon > 0$. There exists a number $c(X, \varepsilon) > 0$ such that for all $T > 1$*

$$N(X \setminus X^{alg}, T) < c(X, \varepsilon) T^{\varepsilon}. \tag{1.13}$$

Remark 1.51. The first observations that led to the proof of the Pila-Wilkie theorem were made by Bombieri and Pila in [BP89] where they studied the set of integer points on the dilation of the graph of a fixed transcendental analytic function. Later Pila proved in [Pil04] and [Pil05] similar results for sub-analytic surfaces.

In the following chapters we will need a strengthening of this result due to Pila.

Definition 1.52. A *definable block* of dimension w and degree d in $\mathbb{R}^n$ is a connected definable set $W \subset \mathbb{R}^n$ of dimension w, regular at every point, such that there is a semi-algebraic subset $A \subset \mathbb{R}^n$ of dimension w and degree $\leq d$ regular at every point, with $W \subset A$.

A *definable block family* of dimension w and degree d is a definable family W such that every non-empty fibre is a definable block of dimension w and degree d.

Theorem 1.53 ([Theorem 3.6 Pil11])**.** *Let $Z \subset \mathbb{R}^n$ be a definable set and fix $\epsilon > 0$. There is a finite number $J = J(Z, \epsilon)$ of definable block families $W^{(j)} \subset \mathbb{R}^n \times \mathbb{R}^m$ and a positive real number $c(Z, \epsilon)$ such that $Z(T)$ is contained in at most $c(Z, \epsilon) T^{\epsilon}$ definable blocks of the form $W^{(j)}_{\eta}$ for some $j = 1, \ldots, J$ and $\eta \in \mathbb{R}^m$.*

Remark 1.54. The above formulation is a consequence of the full theorem proven by Pila which allows the points in Z to be considered algebraic and the Z to vary in a definable family. The main point of this statement is that it allows to find semi-algebraic sets that contain a large number of rational (or in our case integer) points of Z provided we can prove that $N(Z,T)$ grows at least polynomially in T.

1.3.4 Definable structure of mixed Shimura data

In this section we introduce the definable structure of mixed Shimura data. This will allow us to apply o-minimal techniques to study mixed Shimura data in the following chapters.

Let $(P,\mathcal{X})$ be a connected mixed Shimura datum. We start by describing the semi-algebraic structure on $\mathcal{X}$. We accomplish this by realising $\mathcal{X}$ as an open semi-algebraic subset of an algebraic variety $\mathcal{X}^\vee$ called its dual . This is similar to the Borel embedding theorem recalled above in the case of pure Shimura varieties. To define $\mathcal{X}^\vee$ we start by embedding $\mathcal{X}$ in a flag variety as follows. Fix a faithful finite dimensional rational representation V of P. By the definition of mixed Shimura datum, a point $x \in \mathcal{X}$ induces a rational mixed Hodge structure on V, varying x the weight filtration remains constant and the Hodge filtration induces a variation of mixed Hodge structure over $\mathcal{X}$. Fix a point $x_0 \in \mathcal{X}$ and denote by $\mathrm{F}_{x_0}(V)$ the Hodge filtration on V induced by it. The Hodge filtration induces an embedding of $\mathcal{X}$ in the complex variety $\mathcal{G}$ of flags in $V(\mathbb{C})$ of the same type as $\mathrm{F}_{x_0}(V)$. By definition the embedding $\mathcal{X} \to \mathcal{G}$ is equivariant with respect to the action of $P(\mathbb{R})U(\mathbb{C})$ moreover, the inclusion $P(\mathbb{R})U(\mathbb{C}) \subset P(\mathbb{C})$ induces the factorisation

$$\mathcal{X} \to P(\mathbb{C}).x_0 \cong P(\mathbb{C})/\exp(\mathrm{F}^0_{x_0}(\mathrm{Lie}P) \to \mathcal{G}. \tag{1.14}$$

$P(\mathbb{C})/\exp(\mathrm{F}^0_{x_0}(\mathrm{Lie}P)$ is a complex algebraic subvariety of $\mathcal{G}$ and by [1.7, Pin90] the map on the left is an open immersion. Define

$$\mathcal{X}^\vee = P(\mathbb{C})/\exp(\mathrm{F}^0_{x_0}(\mathrm{Lie}P). \tag{1.15}$$

Finally we observe that since the action of $P(\mathbb{C})$ on $\mathcal{X}^\vee$ is algebraic and $P(\mathbb{R})U(\mathbb{C})$ is a semi-algebraic subset of $P(\mathbb{C})$, $\mathcal{X}$ is a semi-algebraic subset of $\mathcal{X}^\vee$.

We have obtained the following result similar to the Borel embedding theorem in the pure case.

Theorem 1.55. *$\mathcal{X}$ can be realised as a semi-algebraic open subset of its dual $\mathcal{X}^\vee$ and the action of $P(\mathbb{R})U(\mathbb{C})$ on $\mathcal{X}$ is semi-algebraic.*

The next piece of information we will need is the definability of the uniformisation map. Fix an arithmetic subgroup Γ of P, since the uniformisation map is periodic under the action of Γ it is necessary to restrict it to a subset of $\mathcal{X}$ to have definability. The following result due to Gao contains all the information we will need.

Theorem 1.56 ([Gao20, Section 10.1])**.** *Let $(P, \mathcal{X})$ be a connected mixed Shimura datum, Γ and arithmetic subgroup of P. There exists a fundamental domain $\mathscr{F}$ for the action of Γ on $\mathcal{X}$ such that the uniformisation map $\mathrm{unif} : \mathcal{X} \to M = \Gamma\backslash\mathcal{X}$ is definable in the o-minimal structure $\mathbb{R}_{an,\exp}$ when restricted to $\mathscr{F}$.*

We can now recall the statement of the Ax-Lindemann-Weierstrass theorem. The fact that $\mathcal{X}$ can be embedded as a open semi-algebraic set of an algebraic variety allows us to talk about semi-algebraic subsets of $\mathcal{X}$.

Theorem 1.57 ([Theorem 1.2, Gao17])**.** *Let $(P, \mathcal{X})$ be a connected mixed Shimura datum, Γ an arithmetic subgroup of P and M the associated mixed Shimura variety with uniformisation map unif. Let $Y \subset M$ be an algebraic subvariety. Let $Z \subset \mathrm{unif}^{-1}(Y)$ be a maximal irreducible semi-algebraic subset; then Z is a weakly special subset of $\mathcal{X}$.*

Below we recall a consequence of the mixed Ax-Lindemann-Weierstrass theorem we will need in the sequel.

Theorem 1.58 ([Theorem 12.2, Gao17])**.** *Let $M = \Gamma\backslash\mathcal{X}$ be a connected mixed Shimura variety associated to the mixed Shimura datum $(P, \mathcal{X})$. Let Y be a Hodge generic irreducible subvariety of S Then there exists $N \lhd P$ such that for the following diagram*

$$\begin{array}{ccc} (P,\mathcal{X}) & \xrightarrow{\ \rho\ } & (P',\mathcal{X}') = (P,\mathcal{X})/N \\ \Big\downarrow{\scriptstyle \mathrm{unif}} & & \Big\downarrow{\scriptstyle \mathrm{unif}'} \\ S & \xrightarrow{\ [\rho]\ } & S' \end{array}$$

the following hold:

- *the union of positive dimensional weakly special subvarieties of S' that are contained in $Y' = \overline{[\rho](Y)}$ is not Zariski dense in Y.*

- $Y = [\rho]^{-1}(Y')$.

Chapter 2

Holomorphic Curves in mixed Shimura varieties

The aim of this chapter is to prove the following result.

Theorem 2.1. *Let $(P, \mathcal{X})$ be a connected mixed Shimura datum and Γ an arithmetic subgroup of P. Let M be the mixed Shimura variety associated to the above data and $\mathrm{unif} : \mathcal{X} \to M$ be the complex uniformisation map. Let $\pi : P \to G$ be the projection modulo the unipotent radical and let $(G, \mathcal{X}_G)$ be the corresponding pure Shimura datum. Identify $\mathcal{X}$ with $\mathcal{X}_G \times \mathbb{C}^m$. Using the Harish-Chandra embedding of $\mathcal{X}_G$ we may identify $\mathcal{X}$ as a subset of $\mathbb{C}^N \times \mathbb{C}^m$. Let $f : \mathbb{C} \to \mathbb{C}^N \times \mathbb{C}^m$ be a holomorphic function such that the composition of f with the projection to $\mathbb{C}^N$ is non constant and the image of f intersects $\mathcal{X}$. Then the Zariski closure of $\mathrm{unif}(f(\mathbb{C}))$ is a weakly special subvariety of M.*

2.1 First reductions and notation

We keep all the notation as in the statement of the Theorem. Start by restricting the datum $(P, \mathcal{X})$ to the smallest special subvariety containing the image of f. As the conclusion of our theorem is invariant under finite coverings, we may assume that the arithmetic subgroup Γ can be written as a semi-direct product $\Gamma = \Gamma_G \ltimes \Gamma_W$, where $\Gamma_W = \Gamma \cap W$ and $\Gamma_G = \Gamma / \Gamma_W$, cf. [PR94, 4.1, Corollary 2]. Moreover, we may assume there is a faithful finite dimensional representation $\rho : P \to \mathrm{GL}(E)$ of P defined over $\mathbb{Q}$ and some lattice $E_{\mathbb{Z}}$ such that $\Gamma = P(\mathbb{Z}) = P(\mathbb{Q}) \cap \mathrm{GL}(E_{\mathbb{Z}})$. With this assumptions, we can give the following definition.

Definition 2.2. Let $\rho : P \to \mathrm{GL}(E)$ be the faithful finite dimensional representation of P fixed above; for any $\gamma \in \Gamma$ write $\rho(\gamma) = (\gamma_{i,j})_{i,j}$. For any $\phi \in End(E_{\mathbb{R}})$ define

$$|\phi|_\infty = \max_{i,j} |\phi_{i,j}| \,. \tag{2.1}$$

Moreover, define the *height* of $\gamma \in \Gamma$ as

$$H(\gamma) = \max(1, |\rho(\gamma)|_\infty).$$

Remark 2.3. Let n be the dimension of E, then for any $\gamma_1, \gamma_2 \in \Gamma$, we have

$$H(\gamma_1\gamma_2) \leq nH(\gamma_1)H(\gamma_2). \tag{2.2}$$

Consider the set $A' = (\pi \circ f)^{-1}(\pi \circ f(\mathbb{C}) \cap \mathcal{X}_G)$; A' is open in $\mathbb{C}$ and $\overline{\pi \circ f(A)} \cap \partial\mathcal{X}_G \neq \emptyset$. It is possible to find some $R > 0$ such that the set $A = A' \cap B_{\mathbb{C}}(0,R)$ has the same property, where $B_{\mathbb{C}}(0,R)$ is the Euclidean ball in $\mathbb{C}$ of centre 0 and radius R.

Define $Z = f(A) \subset \mathcal{X}$, $Y = \overline{\mathrm{unif}(Z)}^{\mathrm{Zar}} \subset M$ and finally $\widetilde{Y}$ the analytic component of $\mathrm{unif}^{-1}(Y)$ that contains Z. As $(P, \mathcal{X})$ is the smallest special subvariety containing the image of f, Y is Hodge generic.

Lemma 2.4.

- *A is bounded definable in* $\mathbb{R}_{\mathrm{an}}$
- *Z is definable in* $\mathbb{R}_{\mathrm{an}}$
- *Assuming* $\overline{\mathrm{unif}(Z)}^{\mathrm{Zar}}$ *is weakly special,* $\overline{\mathrm{unif}(Z)}^{\mathrm{Zar}} = \overline{\mathrm{unif}(f(\mathbb{C}) \cap \mathcal{X})}^{\mathrm{Zar}}$

Proof. The projection π can be viewed as the coordinate projection $\mathbb{C}^N \times \mathbb{C}^m \to \mathbb{C}^m$ so is definable. $f|_{B(0,R)}$ is the restriction of an analytic function to an open ball and by definition can be extended to an open neighbourhood of the closed ball $\overline{B(0,R)}$, so is definable in $\mathbb{R}_{\text{an}}$. When embedding $\mathcal{X}_G \subset \mathbb{C}^m$ via the Harish-Chandra embedding, $\mathcal{X}_G$ is a semi-algebraic set. This shows that all sets and maps used in the definition of A are definable in the o-minimal structure $\mathbb{R}_{\text{an}}$ so also A is.

Z is the image of an $\mathbb{R}_{\text{an}}$-definable set by an $\mathbb{R}_{\text{an}}$-definable map, so is $\mathbb{R}_{\text{an}}$-definable. Let $\widetilde{Z}$ be the analytic component of $\text{unif}^{-1}(\overline{\text{unif}(Z)}^{\text{Zar}})$ containing Z. As we assumed $\overline{\text{unif}(Z)}^{\text{Zar}}$ to be weakly special, $\widetilde{Z}$ is semi-algebraic. By analytic continuation we have $f(\mathbb{C}) \subset \widetilde{Z}^{\text{Zar}} \subset \mathbb{C}^N \times \mathbb{C}^m$, hence the result. $\square$

Finally, fix a fundamental domain $\mathscr{F}$ for the action of Γ on $\mathcal{X}$ and define

$$N_Z(T) = \#\{\gamma \in \Gamma | \gamma.\mathscr{F} \cap Z \neq \emptyset \text{ and } H(\gamma) \leq T\}. \tag{2.3}$$

To recap our notation, we have

- Z is a definable subset of $\mathcal{X}$ such that $\overline{\pi(Z)} \cap \partial\mathcal{X}_G \neq \emptyset$
- $Y = \overline{\text{unif}(Z)}^{\text{Zar}} = \overline{\text{unif}(f(\mathbb{C}) \cap \mathcal{X})}^{\text{Zar}}$ is Hodge generic
- $\widetilde{Y}$ the analytic component of $\text{unif}^{-1}(Y)$ containing Z
- $\mathscr{F}$ a fundamental domain for the action of Γ on $\mathcal{X}$
- $N_Z(T) = \#\{\gamma \in \Gamma | \gamma.\mathscr{F} \cap Z \neq \emptyset \text{ and } H(\gamma) \leq T\}$.

2.2 Proof of Theorem 2.1

We start by stating the main counting result that will be proven in the next section.

Proposition 2.5. *There exist constants $c_1, c_2 > 0$ such that for all $T > 0$ sufficiently large*

$$N_Z(T) \geq c_1 T^{c_2}. \tag{2.4}$$

Using this counting result together with the Pila-Wilkie theorem we can prove that the stabiliser of $\widetilde{Y}$ is large. More precisely we prove

Proposition 2.6. *There exists a positive dimensional semi-algebraic set $X \subset P(\mathbb{R})^+U(\mathbb{C})$, that is not contained in the stabilizer of any point, such that $X.Z \subset \widetilde{Y}$.*

Proof. We start by proving

Lemma. *Consider the sets*

$$\begin{aligned} \Sigma(Y) &= \left\{h \in P(\mathbb{R})^+U(\mathbb{C}) \,|\, \dim(h.Z \cap \mathcal{F} \cap \pi^{-1}(Y)) = \dim(Z)\right\} \\ \Sigma'(Y) &= \left\{h \in P(\mathbb{R})^+U(\mathbb{C}) \,|\, Z \cap h^{-1}.\mathcal{F} \neq \emptyset\right\}. \end{aligned} \tag{2.5}$$

Then

- $\Sigma(Y)$ *is definable in* $\mathbb{R}_{\mathrm{an,exp}}$
- *For all* $h \in \Sigma(Y)$, $h.Z \subseteq \pi^{-1}(Y)$.
- $\Sigma(Y) \cap \Gamma = \Sigma'(Y) \cap \Gamma$.

Proof. The set $\Sigma(Y)$ is definable in $\mathbb{R}_{\mathrm{an,exp}}$ because all sets and maps involved in its definition are.

The second assertion follows from the definition of $\Sigma(Y)$ by analytic continuation.

Finally the equality

$$\Sigma(Y) \cap \Gamma = \Sigma'(Y) \cap \Gamma \tag{2.6}$$

follows from the definition of $\Sigma(Y)$ and the fact that $\pi^{-1}(Y)$ is Γ-invariant. □

We now consider the set

$$\Sigma(Y)(T) = \{\gamma \in \Gamma \cap \Sigma(Y) \,|\, H(\gamma) \leq T\} \tag{2.7}$$

and let $N_{\Sigma(Y)}(T) = \#\Sigma(Y)(T)$. Using $\Sigma(Y) \cap \Gamma = \Sigma'(Y) \cap \Gamma$ and Proposition 2.5 we see that

$$N_{\Sigma(Y)}(T) = N_Z(T) \geq c_1 T^{c_2} \tag{2.8}$$

for some numbers $c_1, c_2 > 0$.

As the set $\Sigma(Y)$ is definable, we may apply Pila-Wilkie's Theorem 1.53 which implies that for any $\epsilon > 0$ there exists a constant $c_3 > 0$ such that the points in $\Sigma(Y)(T)$ are contained in at most $c_3 T^\epsilon$ definable blocks.

This implies that for any integer k we can find a semi-algebraic set $X \subset \Sigma(Y)$ containing more than k points of Γ; choosing k big enough we may assume that X is not contained in the stabilizer of any point. Finally from the second claim in the Lemma we get that $X.Z \subset \pi^{-1}(Y)$, thus proving the proposition. □

Corollary 2.7. *The union of all weakly special subsets of $\mathcal{X}$ contained in Y is Zariski dense in Y.*

Proof. Let $z \in Z$ be any point, then by Proposition 2.6, the maximal semi-algebraic subset X of $\widetilde{Y}$ containing z has positive dimension. By [PT13, Lemma 4.1], X is a complex algebraic subset of $\widetilde{Y}$ and by the Ax-Lindemann-Weierstrass theorem 1.57 for mixed Shimura varieties it is a weakly special subset of $\mathcal{X}^+$ contained in $\widetilde{Y}$. This implies that $\mathrm{unif}(Z)$ is covered by positive dimensional weakly special subsets of S contained in Y. As $\mathrm{unif}(Z)$ is Zariski dense in Y we get the result. □

We can now complete the proof of the theorem. Since by assumption Y is Hodge generic, we may apply Theorem 1.58 and obtain a normal subgroup N of P with the following properties. Let $\rho : P \to P/N$ denote the quotient map and let $[\rho] : S \to S_{P/N}$ be the associated map on Shimura varieties, let $Y' = [\rho](Y)$. Then the set of weakly special subvarieties of $S_{P/N}$ contained in Y' is not Zariski dense in Y' and $Y = [\rho]^{-1}(Y')$.

By definition of weakly special subvariety we only need to prove that Y' is reduced to a point. Since $Y' = [\rho](Y)$ and $Y = [\rho]^{-1}(Y')$, we have that $[\rho](\mathrm{unif}(Z))$ is Zariski dense in Y'. If, by contradiction, Y' had dimension bigger than zero, the composition $\rho \circ f : \mathbb{C} \to \mathcal{X}^{+\vee}_{P/N}$ would be non constant, so the data $(P/N, \mathcal{X}_{P/N})$ and the map $\rho \circ f$ would satisfy the assumptions of the theorem; we could then apply Corollary 2.7 to this new data to get a set of weakly special subvarieties of $S_{P/N}$ contained in Y' and Zariski dense in it. This contradicts the definition of Y', so Y' must be reduced to a point and Y is weakly special.

2.3 Proof of Proposition 2.5

We first prove the counting result in the pure case, then use this deduce the general case.

2.3.1 Proof in the pure case

We start by proving Proposition 2.5 in the pure case, that is, we assume $(P,\mathcal{X}) = (G,\mathcal{X}_G)$, to stress this we will use the notation $(G,\mathcal{X})$ for the Shimura datum.

As in the Ax-Lindemann-Weierstrass theorem, there are two parts to the proof, first a lower bound on the volume of the intersection of Z with geodesic balls in $\mathcal{X}$. Second an upper bound for the volume of the intersection of Z with translates of the fundamental domain $\mathcal{F}$.

We start by proving the lower bound, this is the analogue of the theorem of Hwang-To used in the proof of the Ax-Lindemann-Weierstrass theorem.

Lemma 2.8. *Let $\widetilde{\mathbb{R}}$ be an o-minimal structure. Assume $\widetilde{\mathbb{R}}$ admits analytic cell decomposition (cf. Definition 1.45). Let U be a connected $\widetilde{\mathbb{R}}$-definable subset of $\mathcal{X}$ of dimension 2 such that $\dim_{\mathbb{R}}(\overline{U} \cap \partial\mathcal{X}) = 1$. Fix a point $x_0 \in \mathcal{X}$. Then there exist real numbers c_1, c_2 such that for any $R > 0$ sufficiently large*

$$Vol(B(x_0,R) \cap U) \geq c_1 \exp(c_2 R), \tag{2.9}$$

where $B(x_0,R)$ is the geodesic ball in $\mathcal{X}$ of centre x_0 and radius R.

Proof. In the course of the proof we will use the following notation:

- $\Delta_{\alpha.\beta} = \{r \exp i\theta | 0 \leq r < 1 \text{ and } \alpha < \theta < \beta\}$ where $0 < \alpha < \beta$ are real numbers,

- $C_{\alpha.\beta} = \{\exp i\theta | \alpha < \theta < \beta\}$,
- $\overline{\Delta_{\alpha,\beta}} = \Delta_{\alpha,\beta} \cup C_{\alpha,\beta}$

Since $\widetilde{\mathbb{R}}$ admits analytic cell decomposition, we may fix an analytic cell U' of dimension 2 such that $\overline{U} \cap \partial\mathcal{X}$ is a connected real analytic curve. This implies that we can find positive real numbers α, β and a real analytic map $\psi : \Delta_{\alpha,\beta} \to U'$ with the following properties:

- $\psi(\Delta_{\alpha,\beta})$ is contained in U',
- ψ extends to a real analytic function in a neighbourhood of $\overline{\Delta_{\alpha,\beta}}$ such that $\psi(C_{\alpha,\beta}) \subset \overline{U'} \cap \partial\mathcal{X}$ is a non-constant real analytic curve.

We now recall two useful lemmas due to Ullmo and Yafaev.

Lemma 2.9 ([UY14b, Lemma 2.4])**.** *Let d_e be the Euclidean metric on $\mathbb{C}^N$ restricted to $\mathcal{X}$ and d_h be the Bergman (hyperbolic) metric on $\mathcal{X}$. There exist positive real numbers a_1, a_2, θ depending only on $\mathcal{X}$ and a choice of $x_0 \in \mathcal{X}$, such that, for all $x \in \mathcal{X}$,*

$$-a_1 \log d_e(x, \partial\mathcal{X}) - \theta \leq d_h(x, x_0) \leq -a_2 \log d_e(x, \partial\mathcal{X}) + \theta \tag{2.10}$$

Lemma 2.10 ([UY14b, Lemma 2.8])**.** *With notation as above, let ω be the Kähler form on $\mathcal{X}$ associated to the Bergman metric and ω_Δ the Poincaré metric on the unit disk. Then we have the following:*

(a) There exists a positive integer s such that, up to changing α and β

$$\psi^*\omega = s\omega_\Delta + \eta \tag{2.11}$$

for a $(1,1)$ form η smooth in a neighbourhood of $C_{\alpha,\beta}$.

(b) Let $d_{e,\Delta}$ be the Euclidean metric on $\mathbb{C}$ restricted to the unit disk. Changing α and β if necessary, there exists $\lambda' > 0$ and $C' > 0$ such that, for all $z \in \Delta_{\alpha,\beta}$,

$$\left| \log d_e(\psi(z), \partial\mathcal{X}) - \lambda' \log d_{e,\Delta}(z, \partial\Delta) \right| \leq C' \tag{2.12}$$

Now given $R > 0$, consider the set

$$I^R_{\alpha,\beta} = \left\{ z \in \Delta_{\alpha,\beta} | c_3 e^{-R-1} \leq d_{\Delta,e}(z, \partial\Delta) \leq c_3 e^{-R} \right\} \tag{2.13}$$

this set is an annular sector inside the unit circle. The main point in considering this set is that as R tends to infinity the hyperbolic distance of $I^R_{\alpha,\beta}$ from the origin tends to infinity and its volume is exponential in R. We now use the two lemmas above to see that, up to changing α and β there exist a constant $c_3 > 0$ such that the image $\psi(I^R_{\alpha,\beta})$ in $\mathcal{X}$ is contained in the geodesic ball $B_{\mathcal{X},h}(x_0, R)$. We are now ready to calculate volumes. Let $R > 0$ be sufficiently large, then

$$\mathrm{Vol}\,(U \cap B(x_0, R)) \geq \int_{I^R_{\alpha,\beta}} \omega_\Delta \geq c_1 \exp(c_2 R) \tag{2.14}$$

for some positive constant $c_1, c_2 > 0$. □

We now address the upper bound.

Lemma 2.11. *There exists a positive constant c_3 such that for any $\gamma \in \Gamma$*

$$\mathrm{Vol}_{\gamma C}(\gamma C \cap \mathscr{F}) \leq c_3, \tag{2.15}$$

where $\mathrm{Vol}_{\gamma C}$ is the volume with respect to the Riemannian metric on γC induced by the metric on $\mathcal{X}$.

Proof. By definition, $\mathscr{F} = J.\Sigma$ for a finite subset $J \subset G(\mathbb{Q})$ and some Siegel subset $\Sigma \subset \mathcal{X}$. Hence it is sufficient to prove the theorem for the Siegel set Σ. In turn, every Siegel set is covered by a finite number of open subsets Θ as in proposition 1.36, so it is sufficient to prove that

$$\mathrm{Vol}_{\gamma C}(\gamma C \cap \Theta) \leq c_4 \tag{2.16}$$

for some constant $c_4 > 0$. Let ω be the natural Kähler form on $\mathcal{X}$, then

$$Vol_{\gamma C}(\gamma C \cap \Theta) = \int_{\gamma C \cap \Theta} \omega \tag{2.17}$$

On $\mathcal{X}_F$ we have the Poincaré metric defined by

$$\omega_F = \sum \frac{dx_1 \wedge d\overline{x_i}}{\mathrm{Im}(x_i)^2} + \sum dy_j \wedge d\overline{y_j} + \sum df_k \wedge d\overline{f_k}. \tag{2.18}$$

By a result of Mumford [Mum77, Theorem 3.1], there is a constant c_5 such that on $\mathcal{X}$

$$\omega \leq c_5 \omega_F. \tag{2.19}$$

Hence

$$Vol_{\gamma C}(\gamma C \cap \Theta) \leq \int_{\gamma C \cap \Theta} \omega_F. \tag{2.20}$$

Now let w be a coordinate between x_i, y_j or f_k, denote by $p_w : \mathcal{X}_F \to \mathbb{C}$ the projection to the w axis. Let $w_0 \in \mathbb{C}$ and $g \in G(\mathbb{R})$ define

$$n_{g.C,w}(w_0) = \text{ number of points in } g.C \cap p_w^{-1}(w_0) \text{ counted with multiplicity.} \tag{2.21}$$

Consider the set

$$W = \left\{ (z_0, g, w_0) \in (U_i \cap B(x_i, R_i)) \times G(\mathbb{R}) \times \mathbb{C} \,|\, g.f(z_0) \in p_w^{-1}(w_0) \right\}. \tag{2.22}$$

Note that the map p_w is the projection on one component from the semi-algebraic set $\mathcal{X}_F$, hence is definable; moreover in [UY14b, Proposition 4.1] Ullmo and Yafaev proved that the action of G on $\mathcal{X}$ is semi-algebraic. Finally, by construction, the function $f|_{U \cap B(x_i, R_i)}$ is definable. This implies the definability of the set W. Now we consider W as a definable family over $G(\mathbb{R}) \times \mathbb{C}$. By proposition 1.43 the number of connected components in a definable family, hence in this case the cardinality of the sets in the family, is uniformly bounded by a constant c_w. We now observe that the fibre of W over a point $(g, w_0) \in G(\mathbb{R}) \times \mathbb{C}$ is the set $f^{-1}(p_w^{-1}(w_0) \cap g.C)$ whose cardinality is exactly $n_{g.C,w}(w_0)$. Hence

$$n_{\gamma C,w}(w_0) \leq c_w \tag{2.23}$$

for all $w_0 \in \mathbb{C}$ and all $\gamma \in \Gamma$. Let c_6 be the maximum of c_w with w equal to x_i,

y_j or f_k, then

$$\begin{aligned}\mathrm{Vol}_{\gamma C}(\gamma C \cap \Theta) \leq c_5 \Bigg(&\sum \int_{p_{x_i}(\Theta)} n_{\gamma C}\left(p_{x_i}^{-1}(x_i)\right) \frac{dx_i \wedge d\overline{x_i}}{\mathrm{Im}(x_i)^2} + \\ &\sum \int_{p_{y_j}(\Theta)} n_{\gamma C}\left(p_{y_j}^{-1}(y_j)\right) dy_j \wedge d\overline{y_j} + \\ &\sum \int_{p_{f_k}(\Theta)} n_{\gamma C}\left(p_{f_k}^{-1}(f_k)\right) df_k \wedge d\overline{f_k}\Bigg) \\ \leq c_5 c_6 \Bigg(&\sum \int_{p_{x_i}(\Theta)} \frac{dx_i \wedge d\overline{x_i}}{\mathrm{Im}(x_i)^2} + \sum \int_{p_{y_j}(\Theta)} dy_j \wedge d\overline{y_j} + \\ &\int_{p_{f_k}(\Theta)} df_k \wedge d\overline{f_k}\Bigg)\end{aligned} \tag{2.24}$$

Now we observe that from the description of Θ, the projection $p_{x_i}(\Theta)$ is contained in a finite union of usual fundamental domains in the upper half plane, which have finite hyperbolic area. Moreover, if w is one of y_j or f_k, then, again from the description of Θ, it follows that $p_w(\Theta)$ is relatively compact in the plane and hence has finite Euclidean area. □

These two results allow us to apply the same strategy used to prove the counting result in the setting of the Ax-Lindemann-Weierstrass theorem also in this case. Below we recall the main steps in the proof.

Lemma 2.12 ([KUY16, Lemma 5.4])**.** *Let $x_0 \in \mathcal{X}$ be a base point. There exists a constant c_7 such that for any $g \in G(\mathbb{R})$ the following inequality holds*

$$\log\left(c_7 \left|g\right|_\infty\right) \leq d(g.x_0, x_0). \tag{2.25}$$

Lemma 2.13 ([KUY16, Lemma 5.5])**.** *Let $\mathscr{F}$ be the fundamental domain for the action of Γ fixed in the previous section. There exists a positive constant c_8 such that for all $\gamma \in \Gamma$ and for all $u \in \gamma\mathscr{F}$*

$$H(\gamma) \leq c_8 \left|u\right|_\infty^n. \tag{2.26}$$

Proof of Proposition 2.5. Choose a base point $x_0 \in C$, let c_7 and c_8 the constants given by Lemma 2.12 and Lemma 2.13 and consider the intersection $C \cap B(x_0, R)$ of C with the geodesic ball of centre x_0 and radius $R = \log\left(\frac{c_7}{c_8{}^{1/n}} T^{1/n}\right)$. On the one hand, we have by Lemma 2.8

$$\mathrm{Vol}_C(C \cap B(x_0, R)) \geq \frac{c_7 c_9}{c_8{}^{1/n}} T^{\frac{c_{10}}{n}}. \tag{2.27}$$

On the other hand, by Lemma 2.12 and Lemma 2.13

$$\begin{aligned} B(x_0, \log R) &\subseteq \{g \cdot x_0 \,|\, g \in G(\mathbb{R}), |g|_\infty \leq {}^{T^{1/n}}/_{c_8^{1/n}}\} \\ &\subseteq \bigcup_{\substack{\gamma \in \Gamma \\ H(\gamma) \leq T}} \gamma \mathcal{F}. \end{aligned} \tag{2.28}$$

Hence, by Lemma 2.11

$$\mathrm{Vol}_C(C \cap B(x_0, \log R)) \leq \sum_{\substack{\gamma \in \Gamma \\ \gamma\mathcal{F} \cap C \neq \emptyset \\ H(\gamma) \leq T}} \mathrm{Vol}_{\gamma^{-1}C}(\gamma^{-1} C \cap \mathcal{F}) \leq N_C(T) c_3 \tag{2.29}$$

We conclude comparing the lower bound and the upper bound

$$\frac{c_7 c_9}{c_8{}^{1/n}} T^{\frac{c_{10}}{n}} \leq N_C(T) c_3 \tag{2.30}$$

□

2.3.2 Reduction to the pure case

The assumption that the restriction of the projection map $\pi : \mathcal{X} \to \mathcal{X}_G$, implies that $f(\mathbb{C})$ us not contained in any fibre of π. Below we prove that it also implies that the set Z is uniformly bounded in the fibres.

Lemma 2.14. *There is a finite subset Λ of Γ_W such that $Z \subset \Lambda \Gamma_G . \mathscr{F}$.*

Proof. As Z is a subset of $f(B(0,R))$ for some $R > 0$ and f is defined on all of $\mathbb{C}$, the maximum modulus principle implies that $f(B(0,R))$ and hence Z are bounded in $\mathbb{C}^{N+m}$. This also implies that the projection of Z to $\mathbb{C}^m$ is bounded, which implies the result. □

Remark 2.15. Notice that since $\overline{(\pi(Z))\cap\partial\mathcal{X}}\neq\emptyset$, we need all of Γ_G to get a set that contains Z in the above lemma.

We now reduce to the pure case.

Lemma 2.16. *Assume Proposition 2.5 is true when $(P,\mathcal{X})$ is a pure Shimura datum, then it is true in general.*

Proof. Let $\Lambda\subset\Gamma_W$ be as in the previous proposition and $k=\max_{\gamma\in\Lambda}H(\gamma)$. Let $\gamma_G\in\Gamma_G$ such that $\gamma_G\mathscr{F}_G\cap Z_G\neq\emptyset$. Then by the previous proposition, there exist $\gamma\in\Gamma$ such that $\pi(\gamma)=\gamma_G$, $\gamma\mathscr{F}\cap Z\neq\emptyset$ and $H(\gamma)<kH(\gamma_G)$. This together with the pure case implies

$$N_Z(T)>\frac{c_1}{k^{c_2}}T^{c_2} \tag{2.31}$$

as wanted. □

Chapter 3

Totally geodesic subvarieties of mixed Shimura varieties

3.1 Differential geometry of Hermitian vector bundles

In this section we recall some notions regarding the differential geometry of holomorphic Hermitian vector bundles over complex manifolds. All the material in this section is classical, proofs can be found for instance in [Kob87].

Let M be a complex manifold and $\mathcal{E}$ a holomorphic Hermitian vector bundle of rank r on M with Hermitian form h.

Definition 3.1.

- We denote by $\mathcal{A}^p(\mathcal{E})$ the p-forms over M with values in E, i.e. $\mathcal{A}^p(\mathcal{E}) = \mathcal{E} \otimes \Omega^p(M)$.

- Given $\alpha \otimes s \in \mathcal{A}^k(\mathcal{E})$ and $\beta \otimes t \in \mathcal{A}^l(E)$, define

$$h(\alpha \otimes s, \beta \otimes t) = h(s,t)\alpha \wedge \overline{\beta} \in \mathcal{A}^{k+l}(\mathcal{E}). \tag{3.1}$$

- A *connection* ∇ on $\mathcal{E}$ is a map $\nabla : \mathcal{A}^0(\mathcal{E}) \to \mathcal{A}^1(\mathcal{E})$ that satisfies the Leibniz rule

$$\nabla(f\sigma) = df \otimes \sigma + f \cdot \nabla\sigma. \tag{3.2}$$

- A connection ∇ on $\mathcal{E}$ is called *metric* if

$$dh(\alpha,\beta) = h(\nabla\alpha,\beta) + h(\alpha,\nabla\beta) \tag{3.3}$$

- Given a connection ∇ on $\mathcal{E}$, define $\nabla : \mathcal{A}^k(\mathcal{E}) \to \mathcal{A}^{k+1}(\mathcal{E})$ by

$$\nabla(\alpha\otimes s) = d\alpha\otimes s + (-1)^k\alpha\wedge\nabla s \tag{3.4}$$

- The curvature Θ of a connection ∇ on $\mathcal{E}$ is $\nabla\circ\nabla : \mathcal{A}^0(\mathcal{E}) \to \mathcal{A}^2(\mathcal{E})$

Similarly to the usual exterior differentiation on a complex manifold, one can decompose a connection ∇ on E as sum of its $(1,0)$ and $(0,1)$ parts.

Lemma 3.2. *There is a unique metric connection ∇ on E such that $\nabla^{(0,1)} = \overline{\partial}$, this is called the* Chern connection *of (E,h).*

Lemma 3.3. *The curvature of the Chern connection of $\mathcal{E}$ is of type $(1,1)$.*

Let $\mathcal{F}$ be a holomorphic sub-bundle of $\mathcal{E}$ and $\mathcal{G}$ the quotient bundle. We are interested in seeing how the Chern connections of $\mathcal{E}$ and $\mathcal{F}$ are related. Let $\nabla_{\mathcal{E}}$ and $\nabla_{\mathcal{F}}$ be the Chern connections on $\mathcal{E}$ and $\mathcal{F}$ respectively. Then, $\sigma_{\mathcal{F}} = \nabla_{\mathcal{E}} - \nabla_{\mathcal{F}}$ is a one form in $\mathcal{A}^1(\mathrm{Hom}(\mathcal{F},\mathcal{E}))$. It can be proven that actually $\sigma_{\mathcal{F}} \in \mathcal{A}^{(1,0)}(\mathrm{Hom}(\mathcal{F},\mathcal{G}))$.

Definition 3.4. σ is called the *second fundamental form* of the sub-bundle $\mathcal{F}$.

Lemma 3.5. *Let $\Theta_{\mathcal{E}}$ and $\Theta_{\mathcal{F}}$ be the curvatures of $\mathcal{E}$ and $\mathcal{F}$ respectively. We have the equality*

$$\Theta_{\mathcal{F}} = \Theta_{\mathcal{E}} - \overline{\sigma}^t\wedge\sigma \tag{3.5}$$

Definition 3.6. Assume M is Kähler. A submanifold $N \subset M$ is called *totally geodesic* if its second fundamental form vanishes.

Remark 3.7. The above definition is equivalent to the usual one in terms of geodesics. That is, $N \subset M$ is totally geodesic if for any point $p \in N$ any geodesic curve through p in M that is tangent to N is contained in N.

3.2 Curvature of Kuga varieties

Let $(P,\mathcal{X})$ be a connected mixed Shimura datum of Kuga type. Recall this implies there is a short exact sequence of algebraic groups defined over $\mathbb{Q}$

$$0 \to V \to P \to G \to 0 \tag{3.6}$$

where V is the unipotent radical of P, additionally V is abelian and the representation of $\mathbb{S}$ on $\mathrm{Lie}(V)$ induced by any $x \in \mathcal{X}$ has type $(-1,0),(0,-1)$. Denote by $(G,\mathcal{X}_G)$ be the associated quotient pure Shimura datum.

We will start by describing a one parameter family of inequivalent P-invariant Kähler metrics on $\mathcal{X}$ then we will proceed to analyse curvatures.

As a first step we analyse the structure of holomorphic vector bundle on $\mathcal{X}$ in terms of Hodge theory.

The action of $V(\mathbb{R}) \subset P(\mathbb{R})$ on $\mathcal{X}$ gives an identification

$$V(\mathbb{R}) \times \mathcal{X}_G \cong V(\mathbb{R}).(\mathcal{X}_G \times \{0\}) = \mathcal{X}. \tag{3.7}$$

Since V is an abelian unipotent group, the exponential map $\exp : \mathrm{Lie}(V) \to V$ is an isomorphism, which yields an identification

$$\mathrm{Lie}(V)_{\mathbb{R}} \times \mathcal{X}_G \cong \mathcal{X}. \tag{3.8}$$

The axioms of mixed Shimura datum guarantee that the local system $\mathrm{Lie}(V)_{\mathbb{Q}} \times \mathcal{X}_G$ carries a polarised variation of Hodge structures of type $\{(-1,0),(0,-1)\}$. Associated to the local system $\mathrm{Lie}(V)_{\mathbb{R}} \times \mathcal{X}_G$ we have a vector bundle $\widetilde{\mathcal{V}}$ together with the Gauss-Manin connection ∇_{GM} whose local system of horizontal sections is, by definition, $\mathrm{Lie}(V)_{\mathbb{R}} \times \mathcal{X}_G$. The variation of Hodge structures on $\mathrm{Lie}(V)_{\mathbb{Q}} \times \mathcal{X}_G$ defines a holomorphic sub-bundle $F^0\widetilde{\mathcal{V}}_{\mathbb{C}}$ of $\widetilde{\mathcal{V}}_{\mathbb{C}}$ whose fibre over a point $x \in \mathcal{X}_G$ is the term $F^0\,\mathrm{Lie}\,V_{\mathbb{C}}$ in the Hodge filtration of Lie $V_{\mathbb{C}}$ induced by the point x . This discussion gives the following maps

$$\mathcal{X} \to \widetilde{\mathcal{V}} \to \widetilde{\mathcal{V}}_{\mathbb{C}} \to \widetilde{\mathcal{V}}_{\mathbb{C}}/F^0(\widetilde{\mathcal{V}}_{\mathbb{C}}). \tag{3.9}$$

The composition $\mathcal{X} \to \widetilde{\mathcal{V}}/F^0(\widetilde{\mathcal{V}}_{\mathbb{C}})$ gives $\mathcal{X}$ its structure of holomorphic vector bundle mentioned in Proposition 1.17.

Definition 3.8. A section of $\mathcal{X}$ is called *horizontal* if it is the image of a horizontal section for the Gauss-Manin connection.

Remark 3.9. Since $\mathcal{X}_G$ is a Hermitian symmetric domain, it is simply connected; this implies that $\mathcal{X}$ is a trivial holomorphic vector bundle over $\mathcal{X}_G$. We then get two different trivialisations of $\mathcal{X}$, on the one hand $\mathcal{X} = \mathrm{Lie}(V)_{\mathbb{R}} \times \mathcal{X}_G \cong \mathbb{R}^{2g} \times \mathcal{X}_G$, and on the other $\mathcal{X} = \widetilde{\mathcal{V}}_{\mathbb{C}}/F^0(\widetilde{\mathcal{V}}_{\mathbb{C}}) \cong \mathbb{C}^g \times \mathcal{X}_G$. A section of $\mathcal{X}$ is horizontal if it is constant in the first trivialisation. This does not imply in general that horizontal sections are constant also in the second. We will see this effect in practice below when we will explicitly calculate these identifications in the case of the mixed Shimura datum associated to the universal family of abelian varieties.

Finally we note that the axioms of mixed Shimura datum imply that the variation of Hodge structures on $\widetilde{\mathcal{V}}$ is polarisable; fixing a polarisation gives rise to a positive definite P-invariant Hermitian metric on $\mathcal{X}$.

The above construction can be also seen in terms of families of abelian varieties in the following way[1]. Let Γ be a neat arithmetic subgroup of P and $\Gamma_V = \Gamma \cap V_{\mathbb{R}}$. Then $\pi : \mathcal{A} = \Gamma_V \backslash \mathcal{X} \to \mathcal{X}_G$ is an analytic family of abelian varieties over the complex analytic set $\mathcal{X}_G$. We can identify the local system $\mathrm{Lie}(V) \times \mathcal{X}_G$ with $R_1\pi_*\mathbb{Q}_{\mathcal{A}}$, where $\mathbb{Q}_{\mathcal{A}}$ is the locally constant local system on $\mathcal{A}$ with fibre $\mathbb{Q}$. The variation of Hodge structures on $R_1\pi_*\mathbb{Q}_{\mathcal{A}}$ given by the identification

$$R_1\pi_*\mathbb{Q}_{\mathcal{A}} \otimes_{\mathbb{Q}_{\mathcal{X}_G}} \mathcal{O}_{\mathcal{X}_G} \cong \mathcal{H}^1_{dr}(\mathcal{A}/\mathcal{X}_G)^\vee \tag{3.10}$$

[1] A similar discussion can be found in [Gao20].

coincides with the variation of Hodge structures described above over $\mathrm{Lie}(V) \times \mathcal{X}_G$. This gives an identification

$$\widetilde{\mathcal{V}}_{\mathbb{C}}/F^0(\widetilde{\mathcal{V}}_{\mathbb{C}}) \cong \mathcal{H}^1_{dr}(\mathcal{A}/\mathcal{X}_G)^{\vee}/F^0(\mathcal{H}^1_{dr}(\mathcal{A}/\mathcal{X}_G)^{\vee}) \cong \mathrm{Lie}(\mathcal{A}/\mathcal{X}_G). \tag{3.11}$$

So that we get an identification

$$\mathcal{X} \cong \mathrm{Lie}(\mathcal{A}/\mathcal{X}_G). \tag{3.12}$$

The maps we described above have some interesting properties:

- the map $i : \mathrm{Lie}(V) \times \mathcal{X}_G \to \widetilde{\mathcal{V}}_{\mathbb{C}}/F^0(\widetilde{\mathcal{V}}_{\mathbb{C}})$ is real analytic
- the image $i(a \times \mathcal{X}_G)$ is a complex analytic
- i is a group homomorphism when restricted to any fibre.

We can now define the above mentioned Kähler metrics on $\mathcal{X}$. We start by noting that the quotient map $\pi : \mathcal{X} \to \mathcal{X}_G$ induces a short exact sequence of vector bundles over $\mathcal{X}$

$$0 \to \mathcal{W} \to T(\mathcal{X}) \to \pi^* T(\mathcal{X}_G) \to 0. \tag{3.13}$$

Where $\mathcal{W}$ is the kernel of $d\pi$. As remarked above, we have an identification of $\mathcal{X}$ with its structure of holomorphic vector bundle over $\mathcal{X}_G$ with the bundle $\mathcal{V} = \widetilde{\mathcal{V}}_{\mathbb{C}}/F^0(\widetilde{\mathcal{V}}_{\mathbb{C}})$. Fix a polarisation for the variation of Hodge structures on $\widetilde{\mathcal{V}}$, as remarked above, the polarisation induces a G-invariant Hermitian metric on $\mathcal{V}$; denote this metric by h. By translation in the vertical direction, we can identify $\mathcal{W} \cong \pi^*\mathcal{V}$ and pulling back h we obtain a P-invariant Hermitian metric on $\mathcal{W}$. In a similar way, from the canonical Kähler metric on $\mathcal{X}_G$ we obtain a P-invariant Hermitian metric g on $\pi^* T\mathcal{X}_G$. To obtain a P-invariant Hermitian metric on $T\mathcal{X}$ we only need to define a P-invariant splitting $T\mathcal{X} = \mathcal{W} \oplus \pi^* T\mathcal{X}_G$, that is a P-invariant section of $d\pi$. This can be achieved by using the horizontal sections of π defined above; for every point $(Z, w) \in \mathcal{X}$ we identify a vector

$\mu \in \pi^* T(\mathcal{X}_G)_{(Z,w)}$ with $d\sigma(\mu) \in T(\mathcal{X})$ where σ is the only horizontal section through (Z,w). This gives an identification of $\pi^* T(\mathcal{X}_G)$ with a P invariant sub-bundle of $T\mathcal{X}$.

Given this decomposition we can now define a one parameter family of inequivalent Hermitian metrics on $\mathcal{X}$ as follows. Given a tangent vector $\xi \in T\mathcal{X}$ denote by $\xi = h + v$ its decomposition into horizontal and vertical tangent vectors, then define

$$\|\xi\|_t^2 = \|h\|_g^2 + t^2 \|v\|_h^2. \tag{3.14}$$

These metrics are exactly the ones defined in [Sat80, Chapter IV, §8] and the corresponding $(1,1)$-forms ν_t are Kähler.

Example 3.10. We now carry out explicitly the above computations in the case of the mixed Shimura datum $(\mathcal{H}_g \times \mathbb{C}^g, \mathrm{Sp}_{2g} \ltimes \mathbb{G}_a^{2g})$, for a similar discussion see [MT93]. First we need to make explicit the identification $\mathcal{H}_g \times \mathbb{R}^{2g} \to \mathcal{H}_g \times \mathbb{C}^g$. For this we recall that, by definition of $\mathcal{H}_g$, the Hodge structure in the fibre over $Z \in \mathcal{H}_g$ is defined by $H_Z^{0,-1}$ having the columns of the matrix $\binom{Z}{\mathrm{Id}}$ as a basis. So the complex structure on the fibre over $Z \in \mathcal{H}_g \times \mathbb{R}^{2g}$ is given by

$$J_z = \begin{pmatrix} \overline{Z} & Z \\ \mathrm{Id} & \mathrm{Id} \end{pmatrix} \begin{pmatrix} i\mathrm{Id} & 0 \\ 0 & -i\mathrm{Id} \end{pmatrix} \begin{pmatrix} \overline{Z} & Z \\ \mathrm{Id} & \mathrm{Id} \end{pmatrix}^{-1}, \tag{3.15}$$

rearranging the matrices we get

$$J_Z \begin{pmatrix} \overline{Z} & Z \\ \mathrm{Id} & \mathrm{Id} \end{pmatrix} = \begin{pmatrix} \overline{Z} & Z \\ \mathrm{Id} & \mathrm{Id} \end{pmatrix} \begin{pmatrix} i\mathrm{Id} & 0 \\ 0 & -i\mathrm{Id} \end{pmatrix}. \tag{3.16}$$

Which is equivalent to saying that the identification is given by the map

$$\begin{aligned} &\mathcal{H}_g \times \mathbb{R}^{2g} \to \mathcal{H}_g \times \mathbb{C}^g \\ &(Z,(a,b)) \mapsto (Z, Za + b). \end{aligned} \tag{3.17}$$

We put on $\mathcal{H}_g \times \mathbb{R}^{2g}$ the canonical symplectic form, which is invariant under $G = \mathrm{Sp}_{2g}(\mathbb{R})$ by definition. Denoting by (Z,w) coordinates on $\mathcal{H}_g \times \mathbb{C}^g$, the pull-back of this symplectic form gives rise to the Hermitian metric

$$h(w,w') = \overline{w}'^t(\operatorname{Im} Z)^{-1}w. \tag{3.18}$$

This describes the vector bundle $\mathcal{V}$ with its invariant Hermitian metric in the notation used above. Notice that in this case we have an isomorphism $Sym^2(\mathcal{V},h) \cong (T\mathcal{H}_g, g)$, where the isomorphism is given by the map

$$w \odot w' \mapsto \frac{1}{2}\left(w^t w' + w'^t w\right) \tag{3.19}$$

when identifying the tangent space to $\mathcal{H}_g$ with the space of complex symmetric matrices. The one parameter family of Kähler forms defined by these metrics is

$$\nu_t = i\partial\overline{\partial}\left(t^2(\operatorname{Im} w)^t(\operatorname{Im}\tau)^{-1}(\operatorname{Im} w) + \log\det(\operatorname{Im}\tau)^{-1}\right) \tag{3.20}$$

We now analyse the curvature of Kuga varieties. Let $(P,\mathcal{X})$ be a mixed Shimura datum of Kuga type with a given Shimura embedding $(P,\mathcal{X}) \to (\mathrm{Sp}_{2g} \times \mathbb{G}_a^{2g}, \mathcal{H}_g \times \mathbb{C}^g)$. The induced map $\mathcal{X}_G \to \mathcal{H}_g$ is a totally geodesic immersion. Since $\mathcal{X}_G$ is a Hermitian symmetric domain, the centre of the stabilizer $K_x^G \subset G(\mathbb{R})$ of a point $x \in \mathcal{X}_G$ is isomorphic to the circle group S_1, this implies that it is the same as the centre of the stabilizer $K_x^{\mathrm{Sp}_{2g}} \subset \mathrm{Sp}_{2g}(\mathbb{R})$ when considering $x \in \mathcal{X}_G \subset \mathcal{H}_g$. As noted above, the tangent bundle $T\mathcal{H}_g$ is the symmetric square of the vertical bundle $\mathcal{V}_{\mathcal{H}_g}$, hence we can find an element k_x of the centre of $K_x^{\mathrm{Sp}_{2g}}$ that acts as multiplication by -1 on $T_x\mathcal{H}_g$ and as multiplication by i on $\mathcal{V}_{\mathcal{H}_g,x}$. Since the centre of K_x^G coincides with the centre of $K_x^{\mathrm{Sp}_{2g}}$, the element k_x stabilizes $\mathcal{X}_G$ and acts in the same way on $T_x\mathcal{X}_G$ and $\mathcal{V}_{\mathcal{X}_G}$. Using the invariance of the curvature tensor of $\mathcal{X}$ under the action of G we see that it must have the following form

$$T^t_{\alpha\overline{\alpha}\beta\overline{\beta}} = R_{\alpha_h\overline{\alpha_h}\beta_h\overline{\beta_h}} + t^2\left(\Theta_{\beta_v\overline{\beta_v}\alpha_h\overline{\alpha_h}} + \Theta_{\alpha_v\overline{\alpha_v}\beta_h\overline{\beta_h}} + 2\operatorname{Re}\Theta_{\beta_v\overline{\alpha_v}\alpha_h\overline{\beta_h}}\right) + t^4 T_{\alpha_v\overline{\alpha_v}\beta_v\overline{\beta_v}}. \tag{3.21}$$

where the term R indicates the curvature tensor of $\mathcal{X}_G$ and Θ the curvature tensor of the vector bundle $\mathcal{V}$.

3.3 Totally geodesic subvarieties of mixed Shimura varieties

In this section we study totally geodesic subvarieties of mixed Shimura varieties. Let $(P,\mathcal{X})$ be a connected mixed Shimura datum and Γ an arithmetic subgroup of P. Since the action of Γ is properly discontinuous, a subvariety Y of $M = \Gamma\backslash\mathcal{X}$ is totally geodesic if and only if one complex analytic component $\widetilde{Y}$ of $\mathrm{unif}^{-1}(Y)$ is totally geodesic. Hence we analyse totally geodesic subvarieties of the uniformising space $\mathcal{X}$.

3.3.1 The pure case

Proposition 3.11. *Let $\mathcal{X}$ be a Riemannian globally symmetric space given as the quotient G/K of a real semisimple Lie group modulo a compact subgroup. Let $Y \subset \mathcal{X}$ be a totally geodesic subvariety of $\mathcal{X}$. The subgroup $F \subset G$ of isometries preserving Y acts transitively on Y and Y is a Riemannian symmetric space.*

Proof. See [Hel78] Theorem 7.2, Chapter IV and following remark. □

If additionally we know that the pair $(G,\mathcal{X})$ forms a pure Shimura datum we can be more precise on the structure of the subgroup F.

Proposition 3.12 (cf. [UY18a])**.** *Let $(G,\mathcal{X})$ be a pure Shimura datum. Let $Y \subset \mathcal{X}$ be complex a totally geodesic subvariety. Fix a point $y \in Y$. There exists a semisimple real algebraic subgroup F of G without compact factors such that y factors through $F\ N_G(F)$ and $Y = F(\mathbb{R})^+.y$.*

Vice versa, given a semisimple subgroup F of G without compact factors and a point $x \in \mathcal{X}$ such that $x : \mathbb{S} \to G(\mathbb{R})$ factors through $F\ N_G(F)$, the submanifold $F(\mathbb{R})^+.x$ of $\mathcal{X}$ is a complex totally geodesic subvariety.

Proof. First fix a complex totally geodesic subvariety Y of $\mathcal{X}$ and fix a point $y \in Y$. Since Y is a totally geodesic submanifold, there exists a semisimple subgroup F such that $Y = F(\mathbb{R})^+.y$ this is the group of biholomorphisms of Y. We need only prove that F is normalized by $y(\mathbb{S})$, this is the same as proving its Lie algebra is. Finally we note that since $y(\mathbb{S})$ is the Zariski closure of $y(U_1)$, it is sufficient to show that $\mathrm{Lie}(F)$ is invariant under this second group. $\mathrm{Lie}(F)$ can be decomposed as the direct sum of a Lie algebra of compact type $\mathfrak{k}_y$, the Lie algebra of the stabilizer of y plus the tangent space $T_y(Y)$ of Y at y. Let K be the stabilizer of y in G and $\mathfrak{k}$ its Lie algebra. By [Hel78, Ch. VII, p352ff] $y(U_1)$ is in the centre of K and induces the complex structure on $T_y(Y)$. The first fact implies that $\mathfrak{k}_y$ is invariant under $y(U_1)$ and the second plus the fact that Y is a complex submanifold of $\mathcal{X}$ implies that $T_y(Y)$ is.

Conversely let F be a semisimple subgroup of G without compact factors and $x \in \mathcal{X}$ such that x factors through $F\ N_G(F)$. Let $Y = F(\mathbb{R})^+.x\ x(\sqrt{-1})$ is a Cartan involution for G whose action induces the geodesic inversion around x on $\mathcal{X}$. Since F is normalized by $x(\mathbb{S})$, Y is invariant under $x(\sqrt{-1})$, that is Y is symmetric around x, since it is homogeneous it is symmetric and hence totally geodesic. The fact that $x(U_1)$ induces the complex structure on $\mathcal{X}$ and $x(\mathbb{S})$ normalises F implies that Y is a complex submanifold of $\mathcal{X}$. Finally since F is algebraic, Y is complex subvariety of $\mathcal{X}$. □

3.3.2 The mixed case

Let $(\mathcal{X}, P)$ be a mixed Shimura datum of Kuga type with a given Shimura embedding $(\mathcal{X}, P) \to (\mathcal{H}_g \times \mathbb{C}^g, \mathrm{Sp}_{2g} \rtimes \mathbb{G}_a^{2g})$, so that the curvature formula calculated above applies. Let Y be a weakly special subset of $\mathcal{X}$. As usual, denote by $(\mathcal{X}_G, G)$ the quotient pure Shimura datum, π the projection map and by a subscript G the image of subsets of $\mathcal{X}$ by π, for instance $Y_G = \pi(Y)$.

Recall that by Proposition 1.30, given a neat arithmetic subgroup Γ of P, the image $\mathrm{unif}(Y)$ in $S = Sh_\Gamma(\mathcal{X}, \mathcal{P})$ can be described as the translate of an abelian subvariety of $S|_{\mathrm{unif}(Y_G)}$ by a torsion section of S plus a section of the constant part of $S|_{\mathrm{unif}(Y_G)}$. This implies that Y is the translate of a holomorphic sub-bundle of $\mathcal{X}|_{Y_G}$ by a section of $\mathbb{Q}^{2g} \times \mathcal{X}_G$ plus a section of the constant part of $\mathcal{X}|_{Y_G}$. The constant part of $\mathcal{X}|_{Y_G}$ is the largest holomorphic sub-bundle $\mathcal{X}'$ over Y_G such that the restriction of the identification $\mathcal{X} = \mathcal{X}_G \times \mathbb{C}^n \cong \mathcal{X}_G \times \mathbb{R}^{2n}$ to $\mathcal{X}'$ is constant.

Definition 3.13. Let Y' be the holomorphic sub-bundle of $\mathcal{X}|_{Y_G}$ associated to Y. We say that Y has *non trivial constant part* if the constant part of Y' has positive dimension.

Proposition 3.14. *Assume Y has non trivial constant part, then Y is not totally geodesic.*

Proof. Since the identification $\mathcal{X} \cong \mathcal{X}_G \times \mathbb{R}^{2g}$ restricted to the constant part $\mathcal{X}'$ of $\mathcal{X}|_{Y_G}$ is constant, we can extend any constant section of $\mathcal{X}'$ to a horizontal section of $\mathcal{X}$ over $\mathcal{X}_G$. As all sections of $\mathbb{Q}^{2g} \times \mathcal{X}_G$ are horizontal, Y is a translate of a holomorphic sub-bundle of $\mathcal{X}|_{Y_G}$ by a horizontal section. As $V(\mathbb{R})$ acts by translation by horizontal sections and the curvature of $\mathcal{X}$ is invariant under the action of $P(\mathbb{R})$, we may assume that Y is a holomorphic sub-bundle of $\mathcal{X}|_{Y_G}$.

- Let Y' be the constant part of Y, then $Y'_G = Y_G$ and by hypothesis the fibres of $\pi|_{Y'}$ are positive dimensional.
- Let ∇, T^t and $\nabla^{Y'}$, $T^{t,Y'}$ be the Chern connections and curvatures of $\mathcal{X}$ and Y' respectively.
- Let $\sigma_{Y'}$ be the second fundamental form of Y'.
- Let v, w be two vertical tangent vectors to Y' at a point y.

As Y' is constant, the restriction of the Kähler metric ν_t from $\mathcal{X}$ to Y' is a product metric and by definition of ν_t the metric on the fibres of Y' is euclidean. Then by Lemma 3.5

$$0 = T^{t,Y'}(v,\overline{v},w,\overline{w}) = T^t(v,\overline{v},w,\overline{w}) - \|\sigma_{Y'}(v,w)\|^2 = t^4\,|h(v,w)|^2 - \|\sigma_{Y'}(v,w)\|^2. \tag{3.22}$$

This already implies that Y' is not totally geodesic in $\mathcal{X}$; we will now show that it also implies that Y is not totally geodesic in $\mathcal{X}$.

As the fibres of Y' are euclidean $\nabla_v^{Y'} w = 0$. This is true in particular when $Y' = Y = \pi^{-1}(x)$, where the relation

$$\nabla_v w = \nabla_v^{\pi^{-1}(x)} w + \sigma_{\pi^{-1}(x)}(v,w) = \sigma_{\pi^{-1}(x)}(v,w) \tag{3.23}$$

implies that $\nabla_v w$ is a horizontal vector. In the general case, $\nabla_v w = \sigma_{Y'}(v,w) \in (TY')^{\perp}$, as $\nabla_v w$ is a horizontal vector we actually have $0 \neq \sigma_{Y'}(v,w) \in (TY'_G)^{\perp} = (TY_G)^{\perp}$ which implies that also $\sigma_Y(v,w) \neq 0$, so Y is not totally geodesic in $\mathcal{X}$. □

Corollary 3.15. *The fibres of projection map π are weakly special but not totally geodesic.*

Using the explicit computations carried out in Appendix B we can give a more precise result.

Proposition 3.16. *The smallest complete totally geodesic submanifold of $\mathcal{X}$ containing one fibre of the map π is $\mathcal{X}$ itself.*

Proof. Let $x \in \mathcal{X}_g$ and $Y = \pi^{-1}(x)$. Here we use the same notation as in Appendix B. From Lemma B.4 we have

$$\Gamma^{z_p}_{w_k,w_m} = \frac{it^2}{2}\left(\delta_{m,p_1}\delta_{k,p2} + \delta_{m,p2}\delta_{k,p_1}\right). \tag{3.24}$$

This means that for any coordinate z_j we can find a pair of vertical vectors v_1, v_2 such that $\nabla_{v_1} v_2$ has a non zero component in the direction of z_j. Let Y' be the smallest complete totally geodesic submanifold of $\mathcal{X}$ containing Y, then at any point in Y the tangent space to Y' must contain the space generated by all of the z_j. Using the completeness of Y' we see it must then be equal to $\mathcal{X}$. □

Next we prove that, in contrast to the pure case, there are totally geodesic submanifolds of $\mathcal{X}$ that are not orbits under subgroups. The argument below is due to N. Mok.

Proposition 3.17. *Let $t > 0$ and $\gamma : (a,b) \to \mathcal{X}$ be a geodesic curve in $(\mathcal{X}, \nu_t)$ parametrised by arc length. Then the image Γ of γ is not contained in any fibre of π.*

Proof. Assume on the contrary that there is some geodesic curve $\gamma : (a,b) \to \mathcal{X}$ parametrised by arc length that is contained in a fibre $\pi^{-1}(x)$ for some $x \in \mathcal{X}$. By completeness of $(\mathcal{X}, \nu_t)$ as a Riemannian manifold, we may extend γ to $(-\infty, \infty)$ and, without loss of generality, we may assume that the extension of γ to $(a-\varepsilon, b+\varepsilon)$ is an embedding. Since the Kähler metric ν_t is real analytic also γ and its extension to $(-\infty, \infty)$ must be real-analytic. After extending γ to $(a-\varepsilon, b+\varepsilon)$, by complexification in local holomorphic coordinates, Γ admits a complexification to a holomorphic curve C. Since Γ is contained in the fibre $\pi^{-1}(x)$ and fibres of π are complex submanifolds of $\mathcal{X}$, also C must be contained in $\pi^{-1}(x)$.

Recall that the complexification C of Γ can be defined as the image of a holomorphic map defined in neighbourhood of the closed interval $[a,b]$ in $\mathbb{C}$. Then, for any point $p = \gamma(\tau) \in \Gamma$ we have $T_p^{\mathbb{R}}(C) = T_p(\Gamma) + JT_p(\Gamma)$, where J denotes the integrable almost complex structure on $\mathcal{X}$. Moreover, $T^{\mathbb{R}}(C)$ is J invariant and is equal to the real part of the holomorphic tangent bundle $T(C) = T^{\mathbb{C}}(C)^{(1,0)}$. So, the tangent vector $\dot{\gamma}$ to γ extends to a real analytic vector field on C such that the vector field $\dot{\gamma} - iJ\dot{\gamma}$ is holomorphic. In what follows we will consider $\dot{\gamma}$ extended to a real-analytic vector field as described above.

Denote by ∇ the metric connection on $(\mathcal{X}, \nu_t)$. Since γ is a geodesic curve, $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$. As $(\mathcal{X}, \nu_t)$ is Kähler, the almost complex structure J is parallel and we have $\nabla_{\dot{\gamma}} J\dot{\gamma} = J\nabla_{\dot{\gamma}}\dot{\gamma} = 0$. It follows that

$$\nabla_{\dot{\gamma}-iJ\dot{\gamma}}\dot{\gamma} - iJ\dot{\gamma} = (\nabla_{\dot{\gamma}}\dot{\gamma} - \nabla_{J\dot{\gamma}}J\dot{\gamma}) - i(\nabla_{J\dot{\gamma}}\dot{\gamma} + \nabla_{\dot{\gamma}}J\dot{\gamma}) = -(\nabla_{J\dot{\gamma}}J\dot{\gamma} + i\nabla_{J\dot{\gamma}}\dot{\gamma}). \tag{3.25}$$

As the connection ∇ is torsion free and the tangent space $T^{\mathbb{R}}(C)$ is J invariant we have

$$\nabla_{J\dot{\gamma}}\dot{\gamma} = \nabla_{J\dot{\gamma}}\dot{\gamma} - \nabla_{\dot{\gamma}}J\dot{\gamma} = [J\dot{\gamma}, \dot{\gamma}] \in \Gamma(C, \mathcal{A}^0(T^{\mathbb{R}}(C))). \tag{3.26}$$

From the J-invariance of $T^{\mathbb{R}}(C)$ we also get

$$\nabla_{J\dot{\gamma}}J\dot{\gamma} = J(\nabla_{J\dot{\gamma}}\dot{\gamma}) \in \Gamma(C, \mathcal{A}^0(T^{\mathbb{R}}(C))). \tag{3.27}$$

Combining the last three displays we obtain

$$\nabla_{\dot{\gamma}-iJ\dot{\gamma}}\dot{\gamma} - iJ\dot{\gamma} \in \Gamma(C, \mathcal{A}^0(T^{\mathbb{C}}(C))). \tag{3.28}$$

Denote by $\sigma^{\mathcal{X}}$ the second fundamental form of C in $\mathcal{X}$. The last equation implies that $\sigma^{\mathcal{X}}(\alpha, \alpha) = 0$ for any $(1,0)$-vector $\alpha \in T_p(C)$ and any $p \in \Gamma$. As a consequence, denoting by S the curvature tensor of C, for $\alpha \neq 0$ we have

$$S_{\alpha,\overline{\alpha},\alpha,\overline{\alpha}} = T^t_{\alpha,\overline{\alpha},\alpha,\overline{\alpha}} - \left\|\sigma^{\mathcal{X}}(\alpha, \alpha)\right\|^2 = t^4 \left|h(\alpha, \alpha)\right|^2 > 0. \tag{3.29}$$

On the other hand, $C \subset \pi^{-1}(x)$ and $E_x = \pi^{-1}(x)$ is flat; denoting σ^{E_x} the second fundamental form of C in E_x we have

$$S_{\alpha,\overline{\alpha},\alpha,\overline{\alpha}} = -\left\|\sigma^{E_x}(\alpha, \alpha)\right\|^2 \leq 0. \tag{3.30}$$

This gives a contradiction, completing the proof. □

Corollary 3.18. *Let $t > 0$, $x \in \mathcal{X}_G$ and $E_x = \pi^{-1}(x)$. Let $p \in E_q$, $0 \neq v \in T^{\mathbb{R}}_p(E_x)$ and $\gamma : (-\infty, \infty) \to \mathcal{X}$ be the unique geodesic curve through parametrized by arc length through p such that $\dot{\gamma}(p) \in \mathbb{R}v$. Then the image Γ of γ is not the orbit under any 1-parameter subgroup of P.*

Proof. Since the vector bundle of vertical tangent vectors $\mathcal{V}$ is P-invariant, any smooth and immersed orbit of p under a 1-parameter subgroup of P whose tangent vector at p is vertical and non-zero must be contained in the fibre E_x. On the other hand, by Proposition 3.17, Γ is not contained in any fibre of π. □

Chapter 4

Algebraic flows

In what follows we will always use $(P, \mathcal{X})$ to denote a connected mixed Shimura datum of Kuga type, as above, $(G, \mathcal{X}_G)$ will denote the corresponding connected pure Shimura datum and π will denote the projection map. M will denote a connected mixed Shimura variety associated with $(P, \mathcal{X})$ and an arithmetic subgroup Γ of P and $[\pi] : M \to S$ the Shimura morphism to the associated pure Shimura variety induced by the projection π.

The aim of this section is to prove the following theorem.

Theorem 4.1. *Let $F \subset P$ be a real algebraic subgroup such that the projection $F_G = \pi(F)$ is a real semisimple Lie group without compact almost simple factors. Let $\Gamma \subset P$ be an arithmetic subgroup and M the associated connected mixed Shimura variety. Let $Z = F(\mathbb{R}).x$ for some point $x \in \mathcal{X}$. The closure of $unif(Z)$ in the analytic topology of M is a real weakly special subvariety.*

Recall that in the setting of mixed Shimura varieties we define real weakly special subvarieties as follows.

Definition 4.2. Let $(G, \mathcal{X}_G)$ be a connected Shimura datum. An algebraic subgroup H of G defined over $\mathbb{Q}$ is said to be *of type $\mathcal{H}$* if $H/\mathrm{R_u}(H)$ is a non trivial semisimple group with no compact simple factor.

A subvariety Y of a mixed Shimura variety M is a *real weakly special subvariety* if there exists an algebraic subgroup $H \subset P$ defined over $\mathbb{Q}$ such that $\pi(H)$ is of type $\mathcal{H}$ and $Y = \mathrm{unif}(H(\mathbb{R})^+.x)$ for some $x \in \mathcal{X}$.

Similarly to the pure case it is possible to prove the following.

Proposition 4.3 (cf. [UY18a, Proposition 2.6])**.** *The Zariski closure of a real weakly special subvariety Y of a mixed Shimura variety M is weakly special.*

Proof. From the definition of real weakly special subvariety we know that there exists an algebraic subgroup H of P defined over $\mathbb{Q}$ such that $Y = \mathrm{unif}(H(\mathbb{R})^+.x)$ for some $x \in \mathcal{X}$.

Let $\widetilde{Y} = H(\mathbb{R})^+.x$, $V = \mathrm{R_u}(P)$ and $H_V = H \cap V$.

We start by analysing the Zariski closure of the Zariski closure of a single fibre of Y. Let $y \in Y$ and $\widetilde{y} \in \widetilde{Y}$ such that $\mathrm{unif}(\widetilde{y}) = y$; consider the fibre $\widetilde{E}_{\widetilde{y}} = H_V(\mathbb{R})^+.\widetilde{y}$ and $E_y = \mathrm{unif}(\widetilde{E}_{\widetilde{y}})$.

Recall that we have an isomorphism of complex vector spaces

$$\begin{aligned} V(\mathbb{R}) &\to \pi^{-1}(\pi(\widetilde{y})) \\ v &\mapsto v.\widetilde{y} \end{aligned} \tag{4.1}$$

where the complex structure on the left is given by identifying $V(\mathbb{R})$ with $\mathrm{Lie}\, V(\mathbb{R})$ via the exponential map, and the complex structure on $\mathrm{Lie}\, V(\mathbb{R})$ induced by the homomorphism $\mathbb{S}(\mathbb{C}) \to P(\mathbb{C})$ associated to the point $\widetilde{y}$. This induces an isomorphism of the fibre A_y of the map $[\pi]$ containing the point y with $\Gamma_V \backslash V(\mathbb{R})$[1]. As the action of $P(\mathbb{R})$ on $\mathcal{X}$ is semi-algebraic, we have that $\widetilde{E}_{\widetilde{y}}$ is a semi-algebraic subset of the fibre $V(\mathbb{R}).\widetilde{y}$. We can now use the Ax-Lindemann-Weierstrass theorem for abelian varieties to conclude that the Zariski closure of E_y is a translate of an abelian subvariety of A_y.

Now we turn back to the whole of Y. We may without loss of generality assume that Y is Hodge generic and fix an Hodge generic point $y \in Y$. By [UY18a, Proposition 2.6] we have that $\overline{Y_G}^{\mathrm{Zar}}$ is a weakly special subvariety, so there exists a normal $\mathbb{Q}$-algebraic subgroup N_G of G such that $\overline{Y_G}^{\mathrm{Zar}} = \mathrm{unif}(N_G(\mathbb{R})^+.\widetilde{y})$, where, as above $\mathrm{unif}(\widetilde{y}) = y$. By the previous part we have

[1] since we are working with data of Kuga type the fibre A_y is an abelian variety, however the isomorphism is not an isomorphism of abelian varieties but only of algebraic varieties; it can be decomposed as an isomorphism of abelian varieties composed with the translation by y.

that there is a normal subgroup $N_V \subset V$ containing H_V such that the Zariski closure of E_y in A_y is $\mathrm{unif}(N_V(\mathbb{R}).\widetilde{y})$.

To show the Zariski closure of Y is weakly special we need to show that N_V is an N_G-module in V so that the extension N of N_V by N_G is a subgroup of P, additionally we need to show that N is normal in P. Note that as N_G is normal in G we only need to show that N_V is a G-module to prove N to be normal, this also implies that N_V is an N_G module. This assertion then follows from the assumption that the point y be Hodge generic. Indeed, from the definition of N_V, $N_V(\mathbb{R})$ is a complex vector subspace of $V(\mathbb{R})$ for the complex structure induced by the point $\widetilde{y}$, this implies that it is invariant under the image of $\mathbb{S}_{\mathbb{C}}$ in $P\mathbb{C}$ by the homomorphism $h_{\widetilde{y}}$ associated to $\widetilde{y}$. In particular N_V is then invariant under the $\mathbb{Q}$-Zariski closure of $h_{\widetilde{y}}(\mathbb{S}(\mathbb{C})) = MT(\widetilde{y})$ which, by assumption, is the whole of P.

□

Remark 4.4. In the above proof we have used the version of the Ax-Lindemann-Weierstrass theorem for abelian varieties that states that given an irreducible semi-algebraic subset[2] $X \subset \mathbb{C}^m$, then the complex Zariski closure of $\mathrm{unif}(X)$ in the abelian variety is a translate of an abelian subvariety.

We recall an important result of Ratner in arithmetic dynamics before stating and proving the main result of this part.

Theorem 4.5 ([Theorem 3 and 4, Rat95])**.** *Let G be a Lie group and $U \subset G$ Lie subgroup generated by* Ad*-unipotent elements. Let Γ be a lattice in G and $x \in \Gamma\backslash G$, then the closure of the orbit $x.U$ in $\Gamma\backslash G$ is homogeneous.*

Theorem 4.6. *Let Z be a subvariety of $\mathcal{X}$ such that $\pi(Z) \subset \mathcal{X}_G$ is totally complex geodesic and Z is a vector bundle over $\pi(Z)$ that is homogeneous under the action of a Lie subgroup F of $P(\mathbb{R})^+$. Then the closure with respect to the analytic topology of* $\mathrm{unif}(Z)$ *is a real weakly special subvariety.*

[2]A semi-algebraic subset is called irreducible if it is not the union of two non-empty relatively closed subsets in the topology induced on it by the Zariski topology og algebraic sets defined over $\mathbb{R}$, cf. [PT13]

Proof. As we noted above the proof follows [UY18a] with very minor modifications.

Let F be the Lie group in the statement of the theorem, then by the assumption on Z we have that $\mathrm{R_u}(F) = F \cap V(\mathbb{R})$ and $Z = F.x$ for $x \in Z$.

The idea is to apply Ratner's theorem to the group F to get that the closure of $\mathrm{unif}(F.x)$ is homogeneous, then prove that it is real weakly special.

To apply Ratner's theorem we first need to check that F is generated by its 1-parameter unipotent subgroups. The assumption on F implies that is is the semi-direct product of its unipotent radical with a semisimple subgroup without compact almost simple factors. The unipotent radical is clearly generated by its 1-parameter unipotent subgroups; the same is true for the semi-simple quotient by [Proposition 7.6, PR94].

We can now apply Ratner's Theorem 4.5 and conclude that the closure of $\Gamma\backslash\Gamma F$ in $\Gamma\backslash P(\mathbb{R})^+$ is homogeneous under a subgroup H of P. From [UY18a] we know that the projection H_G of H to G is $MT(F_G)(\mathbb{R})^+$, where $MT(F_G)$ indicates the smallest connected $\mathbb{Q}$-algebraic subgroup of G whose real points contain F. Moreover we know that the image of $MT(F)(\mathbb{R})^+$ in $\Gamma\backslash P(\mathbb{R})^+$ is closed. These two facts imply that $H = MT(F)(\mathbb{R})^+$.

To complete the proof we only need to recall that the map $\Gamma\backslash P(\mathbb{R})^+ \to \Gamma\backslash\mathcal{X}$ defined by $\Gamma.g \mapsto \mathrm{unif}(g.x)$ is proper (cf. [Pin90, Lemma 1.17]). □

Appendix A

Ax-Lindemann-Weierstrass for $\mathbb{G}_m^n$

In this appendix we give a proof of the Ax-Lindemann-Weierstrass theorem for tori recalling all definitions and result needed. Recall the statement of the Ax-Lindemann-Weierstrass theorem in this context.

Theorem A.1 (Ax-Lindemann-Weierstrass)**.** *Let $Y \subset \mathbb{C}^n$ be an algebraic subvariety and $\pi : \mathbb{C}^n \to (\mathbb{C}^*)^n$ be as in the previous section. Then $\overline{\pi(Y)}^{Zar}$ is a weakly special subvariety of $(\mathbb{C}^*)^n$, where $\overline{V}^{Zar}$ indicates the Zariski closure of V.*

We will actually prove the theorem in a different formulation.

Theorem A.2. *Let $V \subset (\mathbb{C}^*)^n$ be an algebraic subvariety and $\pi : \mathbb{C}^n \to (\mathbb{C}^*)^n$ as above. Let Y be a maximal irreducible algebraic subvariety of $\mathbb{C}^n$ contained in $\pi^{-1}(V)$. Then $\pi(Y)$ is weakly special.*

We start by proving the two formulations are equivalent.

Proof. First, assuming the first version we prove the second. Let $V \subset (\mathbb{C}^*)^n$ be an algebraic subvariety and $Y \subset \mathbb{C}^n$ be a maximal irreducible algebraic subvariety contained in $\pi^{-1}(V)$. Then, by the first formulation of the theorem, $\widetilde{Y} = \overline{\pi(Y)}^{Zar}$ is a weakly special subvariety. Now, since $\widetilde{Y}$ is weakly special, the analytic component of inverse image $\pi^{-1}(\widetilde{Y})$ containing Y is algebraic and, by maximality of Y, it is equal to it. Thus, $\pi(Y) = \widetilde{Y} = \overline{\pi(Y)}^{Zar}$ and $\pi(Y)$ is weakly special.

Vice versa, let $Y \subset \mathbb{C}^n$ be an algebraic subvariety and let $Y' \subset \mathbb{C}^n$ be a maximal algebraic subvariety contained in $\pi^{-1}(\overline{\pi(V)}^{Zar})$ and containing Y. By the second version of the theorem, $\pi(Y')$ is a weakly special subvariety, and, by construction, $\pi(Y) \subset \pi(Y') \subset \overline{\pi(Y)}^{Zar}$. Hence, $\pi(Y') = \overline{\pi(Y)}^{Zar}$ and it is a weakly special subvariety. □

As in the statement, we fix an algebraic subvariety $V \subset (\mathbb{C}^*)^n$ and a maximal complex algebraic subvariety $Y \subset \mathbb{C}^n$ such that $Y \subset \pi^{-1}(V)$. We may and do assume that V is the Zariski closure of $\pi(Y)$. Moreover we let $\mathcal{F}$ be the set $\{(z_1, \dots, z_n) \in \mathbb{C}^n \mid 0 < Re(z_i) < 1\, i = 1, \dots, n\}$; $\mathcal{F}$ is the interior of a fundamental domain for the action of $\mathbb{Z}^n = \ker \pi$ on $\mathbb{C}^n$.

First we recall a result needed later.

Lemma A.3 ([PT13, Lemma 4.1])**.** *Let $Z \subset \mathbb{C}^n$ be a complex analytic set and $X \subset Z$ a connected irreducible real semi-algebraic set*[1]*. Then there is a complex algebraic variety $X' \subset Z$ such that $X \subset X'$*

Recall the definition of height in this context.

Definition A.4. Let $\gamma = (m_1, \dots, m_n) \in \mathbb{Z}^n$, define the *height* of γ as

$$H(\gamma) = \max\{|m_1|, \dots, |m_n|\}. \tag{A.1}$$

We now define and give the most important properties of the definable set central to the proof.

Lemma A.5. *Define*

$$\Sigma = \left\{ z \in \mathbb{C}^n \mid (Y + x) \cap \mathcal{F} \neq \emptyset \text{ and } Y + x \subset \pi^{-1}(V) \right\}. \tag{A.2}$$

[1]See note on page 75

Then there exists a real constant T_0 such that, for all $T > T_0$,

$$\#\{z \in \Sigma \cap \mathbb{Z}^n \mid H(z) < T\} \geq T/2. \tag{A.3}$$

Proof. Since $Y \subset \mathbb{C}^n$ is an affine complex algebraic variety, there is one projection $Y \to \mathbb{C}$ to one of the components of $\mathbb{C}^n$ which is surjective; we may suppose that this projection is the one onto the first factor. Hence, we can find a function $\phi = (\phi_1, \ldots, \phi_n) : [0, \infty) \to Y$ such that the image of ϕ_1 has unbounded real part.

Now consider the set

$$\Gamma_\phi = \{z \in \mathbb{Z}^n \mid \operatorname{Im} \phi \cap (\mathcal{F} + x) \neq \emptyset\} \tag{A.4}$$

and observe that, since $\pi^{-1}(V)$ is $\mathbb{Z}^n$-invariant, we can describe the set $\Sigma \cap \mathbb{Z}^n$ as

$$\{z \in \mathbb{Z}^n \mid (Y + x) \cap \mathcal{F} \neq \emptyset\}. \tag{A.5}$$

Hence $\Gamma_\phi \subset \Sigma \cap \mathbb{Z}^n$.

Each time the image of ϕ crosses the boundary between two sets $\mathcal{F} + z$ and $\mathcal{F} + z'$, with $z, z' \in \mathbb{Z}^n$, the height of x and x' differs by at most one. Thus, the heights of points in Γ_ϕ form a set of consecutive integers.

Since the real part of ϕ_1 is unbounded, the set Γ_ϕ contains points of arbitrary large height. Hence there is some t_0 such that, for any $t > t_0$, Γ_ϕ contains at least one point of height t. Thus we can take $T_0 = 2t_0$. □

Lemma A.6. *The set Σ defined in Lemma A.5 is definable in the o-minimal structure $\mathbb{R}_{\text{an,exp}}$.*

Proof. First we observe that

$$\Sigma = \left\{z \in \mathbb{C}^n \mid (Y + x) \cap \mathcal{F} \neq \emptyset \text{ and } (Y + x) \cap \mathcal{F} \subset \pi^{-1}(V) \cap \mathcal{F}\right\}. \tag{A.6}$$

This is true because both $Y + x$ and $\pi^{-1}(V)$ are analytic sets, $Y + x$ is irreducible and $(Y + x) \cap \mathcal{F}$ is open in $Y + x$. Hence, by analytic continuation, the condition $(Y+x)\cap\mathcal{F}\subset\pi^{-1}(V)\cap\mathcal{F}$ implies that $Y+x\subset\pi^{-1}(V)$.

Now we observe that the set above is defined using only analytic functions restricted to compact sets or the exponential function and is thus definable in the o-minimal structure $\mathbb{R}_{\text{an,exp}}$. □

Recall the definition of algebraic part of a subset of $\mathbb{R}^n$ and the statement of the Pila-Wilkie theorem.

Definition A.7. Let $X\subset\mathbb{R}^n$, the algebraic part of X is the set X^{alg} defined as the union of all real semi-algebraic subsets of X of positive dimension.

We are now in place to apply Pila-Wilkie Theorem to the set Σ.

Theorem A.8 ([PW06, Theorem 1.8])**.** *Let $X\subset\mathbb{R}^m$ be a set definable in an o-minimal structure. Let $\varepsilon>0$ There exists a constant c, depending only on X and ε, such that, for all $T>1$,*

$$\#\left\{x\in X\setminus X^{alg}\mid x\in\mathbb{Q}^n \text{ and } H(x)<T\right\}\leq cT^{\varepsilon}. \tag{A.7}$$

We now construct a semi-algebraic set from the properties of the set Σ and the Pila-Wilkie theorem, we then show how to produce from this a semi-algebraic set that stabilises Y.

Corollary A.9. *The set Σ contains a positive dimensional semi-algebraic set.*

Proof. This is a direct consequence of Pila-Wilkie Theorem and of the Lemmas A.5 and A.6. □

Lemma A.10. *If $W\subset\Sigma$ is a connected irreducible semi-algebraic set and $w_0\in W\cap\mathbb{Z}^n$, then $W+Y-w_0\subset Y$.*

Proof. By definition of Σ and the fact that $\pi^{-1}(V)$ is $\mathbb{Z}^n$ invariant, we have $Y\subset Y+W-w_0\subset\pi^{-1}(V)$.

By Lemma A.3, there is a complex algebraic variety $Y'\subset\pi^{-1}(V)$ such that $Y+W-w_0\subset Y'$. Then, by maximality of Y among complex algebraic subvarieties contained in $\pi^{-1}(V)$, we obtain $Y=Y+W-w_0=Y'$. □

Lemma A.11. *Let Θ be the stabiliser of Y in $\mathbb{C}^n$ and $G \subset (\mathbb{C}^*)^n$ the identity component of the stabiliser of V. Then $\pi(\Theta) = G$.*

Proof. We start by showing that $\pi(\Theta) \subset G$. Let $z \in \Theta$, then $Y - z = Y \subset \pi^{-1}(V)$ and, applying π, $\pi(Y) \subset V\pi(z)$. We now recall that, by assumption, $\pi(Y)$ is Zariski dense in V; hence, $\pi(Y) = \pi(Y + z) = \pi(Y)\pi(z)$ is dense in $V\pi(z)$. Together, these observations give us $V = V\pi(z)$. Finally to get that $\pi(\Theta) \subset G$ as desired, we need only observe that Θ is connected, so its image is connected and thus contained in G.

Now we have to show that $G \subset \pi(\Theta)$. Let Θ' be the identity component of $\pi^{-1}(G)$. First, we observe that $\pi(\Theta') = G$, since both are analytic subgroups of $(\mathbb{C}^*)^n$ of the same dimension and $\pi(\Theta') \subset G$. This implies that it is sufficient to show that $\Theta' \subset \Theta$. Since G stabilises V, Θ' stabilises $\pi^{-1}(G)$; hence $Y + \Theta' \subset \pi^{-1}(V)$. But $Y + \Theta'$ is an irreducible complex algebraic subset of V containing Y, so, by maximality of Y, $Y + \Theta' \subset Y$. □

We can now finish the proof of the Ax-Lindemann-Weierstrass Theorem.

Let G and Θ be as in the last Lemma. By Corollary A.9 and Lemma A.10, we have $\dim G > 0$. Then, by Lemma A.11, we have a commutative diagram of quotient maps

$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{\overline{q}} & \mathbb{C}^n/\Theta \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ (\mathbb{C}^*)^n & \xrightarrow[q]{} & (\mathbb{C}^*)^n/G \end{array}$$

Now let $V' = q(V)$ and Y' the closure of $\overline{q}(Y)$ in C^n/Θ. Observe that, since we are taking the quotients by groups stabilising V and Y respectively, we have $q^{-1}(V') = V$ and $\overline{q}^{-1}(Y') = Y$. Moreover, by the maximality of Y follows that Y' is a maximal algebraic subvariety of $\mathbb{C}^n/\Theta$ contained in $\pi'^{-1}(V')$. If we had $\dim Y' > 0$ then we could repeat the argument above to obtain that the stabiliser of V' in $(\mathbb{C}^*)^n/G$ has positive dimension, and, taking the inverse image of this stabiliser under q, we would obtain a subgroup of $(\mathbb{C}^*)^n$ containing properly G and stabilising V. But this contradicts the construction of G.

Hence the image Y' of Y is a point and this implies that $\pi(Y)$ is a translate of the subgroup G.

Appendix B

Explicit computations of Curvature terms

In this chapter we explicitly calculate some of the curvature terms in a Kuga variety. Recall that the Kähler metric on the universal family of abelian varieties can be written as

$$\nu_t = i\partial\overline{\partial}\left(\operatorname{Im} w^t \left(\operatorname{Im} Z\right)^{-1} \operatorname{Im} w + \log\det\left(\operatorname{Im} Z\right)^{-1}\right)$$

Consider a mixed Shimura datum $(P, \mathcal{X})$ of Kuga type with a given Shimura embedding into $(\mathrm{Sp}_{2g} \ltimes \mathbb{G}_a^{2g}, \mathcal{H}_g \times \mathbb{C}^g)$. This embedding induces a totally geodesic embedding $\mathcal{X}_G \to \mathcal{H}_g$, we may assume that $\mathcal{X}$ is the pull-back of $\mathcal{H}_g \times \mathbb{C}^g$ to $\mathcal{X}_G$ by this embedding. Up to the action of Sp_{2g} on $\mathcal{H}_g$, we may assume that $\mathcal{X}_G$ is the submanifold of $\mathcal{H}_g$ identified by setting some of the coordinates equal to 0. With these assumption, the Kähler metric on $\mathcal{X}$ described in 3.2 is the restriction of the metric on $\mathcal{H}_g$ to $\mathcal{X}$.

Our aim is to calculate

$$R\left(v_1, \overline{v_1}, v_2, \overline{v_2}\right)$$

and

$$\nabla_{v_1} v_2$$

for vertical tangent vectors v_1, v_2. The first result is needed in Chapter 4 when proving that weakly special subvarieties of $\mathcal{X}$ with a non-trivial constant part are not totally geodesic, while the second will show that with the assumptions above $\mathcal{X}$ is *not* totally geodesic in $\mathcal{H}_g \times \mathbb{C}^g$.

In the calculations below we will use coordinates z on $\mathcal{X}_G$ and w in the vertical direction of $\mathcal{X}$. Greek letters will denote any of the coordinates z, w. When denoting one of the coordinates z we will often use just one index, e.g. z_j, however these coordinates are actually entries in a matrix the index j is a pair $j = (j_1, j_2)$.

We will denote by $g_{\alpha,\overline{\beta}}$ the metric coefficients, by $\Gamma^{\gamma}_{\alpha,\beta}$ the Christoffel symbols and by $R_{\alpha,\overline{\beta},\gamma,\overline{\varepsilon}}$ the curvature coefficients on $\mathcal{X}$. $g^{\alpha,\beta}$ denotes the coefficients of the inverse of the metric and we use throughout Einstein's convention on summation on repeated indices. We start by recalling some formulas for the connection and curvature coefficients in Kähler manifolds, cf. [Bal06]

$$\begin{aligned}
\Gamma^{\gamma}_{\alpha,\beta} &= g^{\gamma,\overline{\varepsilon}} \frac{\partial g_{\overline{\varepsilon},\beta}}{\partial \alpha} \\
R^{\alpha}_{\beta,\overline{\gamma},\varepsilon} &= -\frac{\partial \Gamma^{\alpha}_{\beta,\varepsilon}}{\partial \overline{\gamma}} \\
R_{\overline{\alpha},\beta,\overline{\gamma},\delta} &= g_{\overline{\alpha},\zeta} R^{\zeta}_{\beta,\overline{\gamma},\varepsilon} \\
R_{\alpha,\overline{\beta},\gamma,\overline{\delta}} &= \overline{R_{\overline{\alpha},\beta,\overline{\gamma},\delta}}
\end{aligned} \tag{B.1}$$

We write the matrix of the metric on $\mathcal{X}$ as a block matrix separating the coordinates z from the coordinates w

$$\left(g_{\alpha,\overline{\beta}}\right) = \begin{pmatrix} g_{z_j,\overline{z_k}} & g_{z_j,\overline{w_m}} \\ g_{w_l,\overline{z_k}} & g_{w_l,\overline{w_m}} \end{pmatrix} = \begin{pmatrix} A & B \\ \overline{B}^t & D \end{pmatrix}.$$

Then

$$
\begin{aligned}
A_{j,k} &= \frac{\partial^2}{\partial z_j \partial \overline{z_k}} \left(\operatorname{Im} w^t \left(\operatorname{Im} Z\right)^{-1} \operatorname{Im} w - \log\det \operatorname{Im} Z \right) \\
B_{j,m} &= \frac{\partial^2}{\partial z_j \partial \overline{w_m}} \left(\operatorname{Im} w^t \left(\operatorname{Im} Z\right)^{-1} \operatorname{Im} w \right) \\
D_{l,m} &= \frac{\partial^2}{\partial w_l \partial \overline{w_m}} \left(\operatorname{Im} w^t \left(\operatorname{Im} Z\right)^{-1} \operatorname{Im} w \right)
\end{aligned}
$$

Using the division in blocks of the metric g we can use the following formula to calculate its inverse

$$
\left(g^{\alpha,\beta}\right) = \begin{pmatrix} \left(A - BD^{-1}B^t\right)^{-1} & -\left(A - BD^{-1}B^t\right)^{-1} BD^{-1} \\ -D^{-1}B^t\left(A - BD^{-1}B^t\right)^{-1} & D^{-1}D^{-1}B^t\left(A - BD^{-1}B^t\right)^{-1} BD^{-1} \end{pmatrix}
$$

We recall some facts that will help us calculate these terms.

Lemma B.1.

- *Let M be a square invertible matrix, then*

$$
\frac{\partial}{\partial x} M^{-1} = -M^{-1} \frac{\partial M}{\partial x} M^{-1}.
$$

and

$$
\frac{\partial \det M}{\partial x} = \det M \cdot \operatorname{Tr}\left(M^{-1} \frac{\partial M}{\partial x} \right)
$$

- *For any j we have*

$$
\frac{\partial \operatorname{Im} Z}{\partial z_j} = -\frac{\partial \operatorname{Im} Z}{\partial \overline{z_j}} = -\frac{i}{2} \frac{\partial Z}{\partial z_j}. \tag{B.2}
$$

- *For all l we have*

$$
\frac{\partial \operatorname{Im} w}{\partial w_l} = \overline{\frac{\partial \operatorname{Im} w}{\partial \overline{w_l}}} = -\frac{i}{2} e_l
$$

Using these formulas and the fact that Z is symmetric we can calculate

Lemma B.2.

$$
\begin{aligned}
A_{j,k} &= \frac{t^2}{2}(\operatorname{Im} w)^t (\operatorname{Im} Z)^{-1} \frac{\partial Z}{\partial z_j} (\operatorname{Im} Z)^{-1} \frac{\partial Z}{\partial z_k} (\operatorname{Im} Z)^{-1} \operatorname{Im} w + \frac{\partial^2 \log\det(\operatorname{Im} Z)^{-1}}{\partial z_j \partial \overline{z_k}} \\
B_{j,m} &= -\frac{t^2}{2}(\operatorname{Im} w)^t (\operatorname{Im} Z)^{-1} \frac{\partial Z}{\partial z_j} (\operatorname{Im} Z)^{-1} e_m \\
D &= \frac{t^2}{2}(\operatorname{Im} Z)^{-1}.
\end{aligned}
$$

We now turn to the inverse of the metric. We start by calculating $BD^{-1}B^t$.

$$
\begin{aligned}
&\left(BD^{-1}B^t\right)_{j,k} = \\
&= \frac{t^2}{2}\left((\operatorname{Im} w)^t (\operatorname{Im} Z)^{-1} \frac{\partial Z}{\partial z_j} (\operatorname{Im} Z)^{-1} e_l\right) \operatorname{Im} Z_{l,m} \left(e_m (\operatorname{Im} Z)^{-1} \frac{\partial Z}{\partial z_k} (\operatorname{Im} Z)^{-1} (\operatorname{Im} w)\right) \\
&= \frac{t^2}{2}(\operatorname{Im} w)^t (\operatorname{Im} Z)^{-1} \frac{\partial Z}{\partial z_j} (\operatorname{Im} Z)^{-1} \frac{\partial Z}{\partial z_k} (\operatorname{Im} Z)^{-1} (\operatorname{Im} w)
\end{aligned}
$$

Using this plus the identities in lemma B.1 we get.

Lemma B.3.

$$
\left(A - BD^{-1}B^t\right)_{j,k} = \frac{\partial^2 \log\det(\operatorname{Im} Z)^{-1}}{\partial z_j \partial \overline{z_k}} = \frac{1}{4}\operatorname{Tr}\left[\frac{\partial Z}{\partial z_k} (\operatorname{Im} Z)^{-1} \frac{\partial Z}{\partial z_j} (\operatorname{Im} Z)^{-1}\right]
$$

We now go on to analyse the Christoffel symbols $\Gamma^{\alpha}_{w_l,w_m}$. From the formulas in equation B.1 we have

$$
\Gamma^{\alpha}_{w_l,w_m} = g^{\alpha,\overline{\beta}} \frac{\partial g_{\overline{\beta},w_m}}{\partial w_l}
$$

As the derivative of D in any of the w_l is the zero matrix, we must have $\beta = z_j$ for some j. We then have two different cases to consider

(a) The case $\alpha = z_k$

(b) The case $\alpha = w_n$

In the first case we get

$$\Gamma^{z_p}_{w_k,w_m} = g^{z_p,\overline{z_r}} \frac{\partial g_{\overline{z_r},w_m}}{\partial w_k}$$

and in the second

$$\Gamma^{w_p}_{w_k,w_m} = g^{w_p,\overline{z_r}} \frac{\partial g_{\overline{z_r},w_m}}{\partial w_k}$$

This can be translated in terms of matrices as

$$\begin{aligned}
\Gamma^{z_p}_{w_k w_m} &= \left(\left(A - BD^{-1}B^t\right)^{-1} \frac{\partial \overline{B}^t}{\partial w_k} \right)_{p,m} \\
\Gamma^{w_p}_{w_k w_m} &= -\left(D^{-1}B^t \left(A - BD^{-1}B^t\right)^{-1} \frac{\partial B^t}{\partial w_k} \right)_{p,m}
\end{aligned}$$

From the expressions for B and $(A - BD^{-1}B^t)$, we have that the derivative of $\Gamma^{z_k}_{w_l,w_m}$ in any of the w_n is zero. This implies that for the curvature terms we have

$$\begin{aligned}
R_{\overline{w_j},w_k,\overline{w_l},w_m} &= g_{\overline{w_j},\alpha} R^{\alpha}_{w_k,\overline{w_l},w_m} \\
&= g_{\overline{w_j},\alpha} \frac{\partial \Gamma^{\alpha}_{w_k,w_m}}{\partial \overline{w_l}} \\
&= g_{\overline{w_j},w_n} \frac{\partial \Gamma^{w_n}_{w_k w_m}}{\partial \overline{w_l}}
\end{aligned}$$

Using the fact that the derivative of both D and $(A - BD^{-1}B^t)$ in any w_l is zero, we can rewrite this equality in terms of matrices as

$$\begin{aligned}
\left(R_{\overline{w_j},w_k,\overline{w_l},w_m}\right)_{\{j,m\}} &= -D \frac{\partial}{\partial w_l} \left(-D^{-1}B^t \left(A - BD^{-1}B^t\right)^{-1} \frac{\partial B}{\partial w_k} \right) \\
&= \frac{\partial B^t}{\partial w_l} \left(A - BD^{-1}B^t\right)^{-1} \frac{\partial B}{\partial w_k}
\end{aligned} \tag{B.3}$$

Lemma B.4. *We have*

$$\left(\frac{\partial B^t}{\partial w_k}\right)_{z_r,w_j} = \frac{it^2}{4} \left[(1 - \delta_{r_1,r_2}) \left(\operatorname{Im} Z\right)^{-1}_{j,r_1} \left(\operatorname{Im} Z\right)^{-1}_{r_2,k} + \left(\operatorname{Im} Z\right)^{-1}_{j,r_2} \left(\operatorname{Im} Z\right)^{-1}_{r_1,k} \right].$$

and

$$\Gamma^{z_p}_{w_k,w_m} = \left(\left(A - BD^{-1}B^t\right)^{-1} \frac{\partial B}{\partial w_k}\right)_{z_p,w_m} = \frac{it^2}{2}\left(\delta_{m,p_1}\delta_{k,p_2} + \delta_{m,p_2}\delta_{k,p_1}\right)$$

Where δ is the Kronecker delta symbol, and the coordinate z_p is identified with the entry $p = (p_1, p_2)$ of matrices in $\mathcal{H}_g$ as remarked at the start of this section.

Proof. As the coordinate z_p corresponds to the entry (p_1, p_2) of matrices in $\mathcal{H}_g$ we have the following formula for the derivative of Z

$$\frac{\partial Z}{\partial z_p} = \frac{\partial \overline{Z}}{\partial \overline{z_p}} = \left((1 - \delta_{p_1,p_2})\,\delta_{s,p_1}\delta_{u,p_2} + \delta_{s,p_2}\delta_{u,p_1}\right)_{\{s,u\}}.$$

Using this and the symmetry of Z we can calculate

$$\begin{aligned}(A - BD^tB)_{z_r,z_s} &= \frac{1}{4}\operatorname{Tr}\left[\frac{\partial Z}{\partial z_s}(\operatorname{Im} Z)^{-1}\frac{\partial Z}{\partial z_r}(\operatorname{Im} Z)^{-1}\right] \\ &= \frac{1}{4}\left[(2 - \delta_{r_1,r_2} - \delta_{s_1,s_2} - \delta_{r_1,r_2}\delta_{s_1,s_2})\,(\operatorname{Im} Z)^{-1}_{s_1,r_2}(\operatorname{Im} Z)^{-1}_{r_1,s_2}\right. \\ &\quad \left. + (2 - \delta_{r_1,r_2} - d_{s_1,s_2})\,(\operatorname{Im} Z)^{-1}_{s_1 r_1}(\operatorname{Im} Z)^{-1}_{r_2,s_1}\right]\end{aligned}$$

and

$$\begin{aligned}\left(\frac{\partial B}{\partial w_k}\right)_{z_r,w_l} &= \frac{it^2}{4} e_l\,(\operatorname{Im} Z)^{-1}\frac{\partial Z}{\partial z_r}(\operatorname{Im} Z)^{-1} e_k \\ &= \frac{it^2}{4}\left[(1 - \delta_{r_1,r_2})\,(\operatorname{Im} Z)^{-1}_{l,r_1}(\operatorname{Im} Z)^{-1}_{r_2,k} + (\operatorname{Im} Z)^{-1}_{l,r_2}(\operatorname{Im} Z)^{-1}_{r_1,k}\right].\end{aligned}$$

It is easy to verify the following relations between the two expressions

- if $s_1 = s_2 = l = k$, then the second expression differs from the first only by a factor of it^2
- if either $l = s_1$ and $m = s_2$, or $m = s_1$ and $l = s_2$, and $s_1 \neq s_2$ then the two expressions differ by a factor of $\frac{it^2}{2}$.

Then, up to a factor, the rows of $\frac{\partial B}{\partial w_k}$ are the same as some of the rows of $A - BD^{-1}B^t$. Using this and the precise relation between the rows of $\frac{\partial B}{\partial w_k}$ and $A - BD^{-1}B^t$ we get the desired result. □

Proposition B.5.

$$R_{\overline{w_j},w_k,\overline{w_l},w_m} = \frac{t^4}{8}\left(\operatorname{Im} Z\right)^{-1}_{m,j}\left(\operatorname{Im} Z\right)^{-1}_{k,l} + \left(\operatorname{Im} Z\right)^{-1}_{j,k}\left(\operatorname{Im} Z\right)^{-1}_{l,m}.$$

In particular

$$R_{\overline{w_j},w_j,\overline{w_l},w_l} = \frac{t^4}{8}\left[\left(\left(\operatorname{Im} Z\right)^{-1}_{j,l}\right)^2 + \left(\operatorname{Im} Z\right)^{-1}_{j,j}\left(\operatorname{Im} Z\right)^{-1}_{l,l}\right] = \frac{t^4}{8}\left(|h(w_j,w_l)|^2 + h(w_j,w_j)h(w_l,w_l)\right).$$

Where in the last equality we are denoting by h the normalised Hermitian form on the vertical tangent bundle as in Section 3.2 and w_j, w_l are the vertical tangent vector corresponding to derivations along the coordinate w_j and w_l respectively.

Proof. The second display follows from the first and the definition of the Hermitian metric h. From equation B.3 we have

$$\left(R_{\overline{w_j},w_k,\overline{w_l},w_m}\right)_{\{j,m\}} = -\frac{\partial B^t}{\partial w_k}(A - BD^{-1}B^t)\frac{\partial B}{\partial \overline{w_l}}$$

Using the formulas in the previous lemma this evaluates to

$$\frac{t^4}{8}\left(\delta_{j,r_1}\delta_{l,r_2} + \delta_{j,r_2}\delta_{l,r_1}\right)\left[(1-\delta_{r_1,r_2})\left(\operatorname{Im} Z\right)^{-1}_{m,r_1}\left(\operatorname{Im} Z\right)^{-1}_{r_2,l} + \left(\operatorname{Im} Z\right)^{-1}_{m,r_2}\left(\operatorname{Im} Z\right)^{-1}_{r_1,l}\right] =$$
$$\frac{t^4}{8}\left(\operatorname{Im} Z\right)^{-1}_{m,j}\left(\operatorname{Im} Z\right)^{-1}_{k,l} + \left(\operatorname{Im} Z\right)^{-1}_{j,k}\left(\operatorname{Im} Z\right)^{-1}_{l,m}.$$

□

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