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Large time behavior for a compressible two-fluid model with algebraic pressure closure and large initial data

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Abstract
In this paper, we consider a compressible two-fluid system with a common velocity field and algebraic pressure closure in dimension one. Existence, uniqueness and stability of global weak solutions to this system are obtained with arbitrarily large initial data. Making use of the uniform-in-time bounds for the densities from above and below, exponential decay of weak solution to the unique steady state is obtained without any smallness restriction to the size of the initial data. In particular, our results show that degeneration to single-fluid motion will not occur as long as in the initial distribution both components are present at every point.

Keywords: compressible two-fluid model, weak solutions, large time behavior
Mathematics Subject Classification numbers: 76T10, 35D30, 35B40, 74G25.

1. Introduction

In multi-component flows the presence of topologically complex interphase separating the components is a great difficulty from physical as well as mathematical point of view. However, in most of engineering applications precise description of motion of each component of interphase are not rarely needed and only the averaged macroscopic description is important. We will focus on the averaged two-component model derived in the monograph of Ishii and Hibiki in its in viscid form [1]. We refer the interested reader to [2] for concise overview of

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various modelling and mathematical aspects related to such models. In the present paper we immediately assume that the two components of the flow share a common velocity field and that their pressures are equal (algebraic pressure closure). We obtain the following one-dimensional system of partial differential equations:

$$\partial_t(\alpha_\pm\varrho_\pm) + \partial_x(\alpha_\pm\varrho_\pm u) = 0, \tag{1.1}$$

$$\partial_t[(\alpha_+\varrho_+ + \alpha_-\varrho_-)u] + \partial_x[(\alpha_+\varrho_+ + \alpha_-\varrho_-)u^2] + \partial_x p = \mu\partial_{xx}u, \tag{1.2}$$

$$\alpha_+ + \alpha_- = 1, \quad \alpha_\pm \geqslant 0, \tag{1.3}$$

$$p = p_+ = p_-, \tag{1.4}$$

with the space variable $x \in \Omega := (0, 1)$ and the time variable $t \in (0, T)$. Here, α_+ and α_- are the volumetric rates of the two fluids; ϱ_+ and ϱ_- are the two mass densities; u is the common velocity field, and $\mu > 0$ is the viscosity coefficient. The two internal pressures are given by

$$p_+ = \varrho_+^{\gamma_+}, \quad p_- = \varrho_-^{\gamma_-}, \tag{1.5}$$

for adiabatic exponents $\gamma_\pm > 1$. The algebraic closure (1.4) is only one of possible choices, but it seems to be well accepted in the physics community. For more thorough discussion on this and other choices we refer to [1, 2].

We restrict ourselves to the case of Dirichlet boundary conditions for the velocity:

$$u|_{x=0,1} = 0. \tag{1.6}$$

Following [3], we introduce the notation

$$R = \alpha_+\varrho_+, \quad Q = \alpha_-\varrho_-, \quad Z = \varrho_+, \tag{1.7}$$

and reformulate (1.1)–(1.5) to

$$\partial_t R + \partial_x(Ru) = 0, \tag{1.8}$$

$$\partial_t Q + \partial_x(Qu) = 0, \tag{1.9}$$

$$\partial_t[(R+Q)u] + \partial_x[(R+Q)u^2] + \partial_x Z^{\gamma_+} = \mu\partial_{xx}u, \tag{1.10}$$

again supplemented with the Dirichlet boundary conditions (1.6) for the velocity and the initial conditions:

$$(R, Q, u)|_{t=0} = (R_0, Q_0, u_0).$$

Without loss of generality, we may assume that

$$\int_\Omega (R_0 + Q_0)(\xi)\mathrm{d}\xi = 1. \tag{1.11}$$

Due to the algebraic closure (1.4), Z is an implicit function of R and Q interrelated by

$$Q = \left(1 - \frac{R}{Z}\right)Z^\gamma, \quad \gamma := \frac{\gamma_+}{\gamma_-}, \tag{1.12}$$

$$R \leqslant Z. \tag{1.13}$$

The same model, but in semi-stationary Stokes regime, has been recently investigated by Bresch, Mucha and the third author in the three-dimensional setting. They proved the global-in-time existence of weak solutions without any restriction on the initial data. Similar result for the general Navier–Stokes system, with generalized equation of state was later obtained by Novotný and Pokorný [4]. Earlier results in this spirit concern existence of weak solutions to very particular two-component models including the fluid model of atmospheric flow with transport of potential temperature [5], and the hydrodynamic limit of Vlasov–Fokker–Planck system modelling suspension of the particles in the compressible fluid [6]. For other related results in case of one-dimensional two-fluid models, including density dependent coefficients or the so-called drift-flux model, we refer to [7], to the works of Evje *et al* [8–12] and to the recent overview paper [13].

The present paper is, as far as we know, the first attempt to provide some more information about quantitative properties of weak solutions to this system. In order to investigate the large time behavior of solutions, we furthermore rewrite (1.8)–(1.10) in Lagrangian coordinates. To do this, we make the change of variables

$$y := \int_0^x (R + Q)(\xi, t)\mathrm{d}\xi, \quad s := t.$$

Observing that y is nondecreasing with respect to x and, in light of (1.8), (1.9) and (1.11),

$$\int_\Omega (R + Q)(\xi, t)\mathrm{d}\xi = \int_\Omega (R_0 + Q_0)(\xi)\mathrm{d}\xi = 1,$$

we find that $y \in [0, 1]$ when $x \in [0, 1]$. Based on this property, we deduce from (1.6) that the velocity obeys the Dirichlet boundary conditions in the Lagrangian coordinates as well.

As a consequence,

$$\partial_t \tau = \partial_y u, \tag{1.14}$$

$$\partial_t (Q\tau) = 0, \tag{1.15}$$

$$\partial_t u = \partial_y \left(\mu \frac{\partial_y u}{\tau} - Z(R, Q)^{\gamma +} \right), \tag{1.16}$$

where $y \in \Omega, t \in (0, T)$ and

$$\tau := \frac{1}{R + Q}.$$

It follows readily that R satisfies

$$\partial_t (R\tau) = 0.$$

By introducing the two time-independent quantities

$$c_+(y) := \frac{R_0}{R_0 + Q_0}, \quad c_-(y) := \frac{Q_0}{R_0 + Q_0},$$

which are called concentrations of the components, we realize then the pressure takes the form

$$p = p(y, \tau) = Z^{\gamma_+}\left(\frac{c_+(y)}{\tau}, \frac{c_-(y)}{\tau}\right).$$

Hence, we may reformulate (1.14)–(1.16) in a more convenient form

$$\begin{cases} \partial_t \tau = \partial_y u, \\ \partial_t u = \partial_y \left[\mu \dfrac{\partial_y u}{\tau} - Z^{\gamma_+}\left(\dfrac{c_+(y)}{\tau}, \dfrac{c_-(y)}{\tau}\right)\right]. \end{cases} \tag{1.17}$$

The equation (1.17) are reminiscent of 1D viscous, barotropic, compressible Navier–Stokes system with pressure depending on both the specific volume and the Lagrangian mass coordinate. In [14] Zlotnik showed the existence, uniqueness and Lipschitz continuous dependence on the initial data of weak solutions to 1D barotropic (or heat-conductive) flow in Lagrangian mass coordinate; regularity of weak solutions was also considered therein. We refer to [15] for asymptotic behavior of weak (or strong) solutions to the two-scale equations of a viscous compressible barotropic medium. The main ideas, in particular the asymptotic analysis, of the aforementioned papers play a crucial role in our model system.

The equation (1.17) are supplemented with the initial and boundary conditions as follows:

$$(\tau, u)|_{t=0} = (\tau_0, u_0), \tag{1.18}$$

$$u|_{y=0,1} = 0, \tag{1.19}$$

and we denote

$$\tau_0 := \frac{1}{R_0 + Q_0}. \tag{1.20}$$

This paper is mainly devoted to the large time behavior of weak solutions to (1.17)–(1.19) with large initial data. Existence, uniqueness and stability of weak solutions are obtained by making full use of the specific structure of the equations. Unlike in the three-dimensional regime [3, 4], we prove the existence of weak solutions by approximation based on the strong solutions. Then the stability of weak solutions is verified by adapting the arguments for single-fluid equations [16, 17]. The key step in the asymptotic analysis is to show uniform-in-time bounds on the densities from above and below. Due to the complicated form of the pressure, classical methods used in [18–22] cannot be applied here. However, thanks to the structure of the pressure, we are able to adapt the argument from [23] so as to obtain the two-sided bounds; see lemma 4.1. Based on these bounds, we show the exponential decay of weak solution by choosing suitable test functions in the momentum equation and making another use of the structure of the pressure.

Notation. We denote $\Omega_t := \Omega \times (0, t)$; L^p stands for the Lebesgue space $L^p(\Omega)$ with the norm $\|\cdot\|_{L^p}$; H^1 denotes the Sobolev space $W^{1,2}(\Omega)$. H^1_0 is the subspace of H^1 with vanishing trace. Further we denote

$$L^\infty(0, T; L^2) := \left\{ f : (0, T) \to L^2 \text{ strongly measurable} : \operatorname{ess}\sup_{t\in(0,T)} \|f(\cdot, t)\|_{L^2} < \infty \right\}.$$

For $f = f(y, t)$, we define

$$\|f\|_{V_2(\Omega_T)} := \|f\|_{L^\infty(0,T;L^2)} + \|\partial_y f\|_{L^2(\Omega_T)}.$$

Before stating our main results, we specify the meaning of weak solutions.

Definition 1.1. Let R_0, Q_0, u_0 satisfy

$$0 < \underline{R_0} \leqslant R_0 \leqslant \overline{R_0} < \infty, \tag{1.21}$$

$$0 < \underline{Q_0} \leqslant Q_0 \leqslant \overline{Q_0} < \infty, \tag{1.22}$$

$$u_0 \in L^2. \tag{1.23}$$

A triple (R, Q, u) is said to be a weak solution of (1.17)–(1.19) on Ω_T provided that

- (R, Q, u) belongs to the spaces

$$0 < R(y,t),\ Q(y,t), \quad \text{a.e. in } \Omega_T,$$

$$\left(R, Q, \frac{1}{R}, \frac{1}{Q}\right) \in L^\infty(\Omega_T),$$

$$(\partial_t R, \partial_t Q) \in L^2(\Omega_T), \quad u \in V_2(\Omega_T),$$

- The initial-boundary conditions

$$(R, Q, u)|_{t=0} = (R_0, Q_0, u_0), \quad u|_{y=0,1} = 0,$$

are satisfied in the sense of traces,
- The equation

$$\partial_t \tau = \partial_y u,$$

makes sense in $L^2(\Omega_T)$ and the equalities

$$R = \frac{c_+(y)}{\tau}, \quad Q = \frac{c_-(y)}{\tau}, \quad \tau = \frac{1}{R+Q},$$

hold true a.e. in Ω_T,
- The momentum equation is understood in the sense of distributions, i.e.,

$$\int_{\Omega_T} \left\{ u\partial_t\phi - \left[\mu\frac{\partial_y u}{\tau} - Z^{\gamma+}\left(\frac{c_+(y)}{\tau}, \frac{c_-(y)}{\tau}\right)\right]\partial_y\phi \right\} \mathrm{d}y\mathrm{d}s = 0,$$

for any $\phi \in C_c^\infty(\Omega_T)$.

Remark 1.1. Given (R_0, Q_0) in (1.21) and (1.22), we tacitly assume that Z_0 satisfies

$$\begin{cases} Q_0 = \left(1 - \dfrac{R_0}{Z_0}\right) Z_0^\gamma, \\ R_0 \leqslant Z_0. \end{cases} \tag{1.24}$$

Clearly, the positive lower bound of Z_0 follows from $(1.24)_2$ and moreover $R_0 < Z_0$ a.e. in Ω in accordance with $(1.24)_1$. To get the upper bound of Z_0 we again make use of $(1.24)_1$. Indeed, suppose on the contrary that $Z_0 > \max\left\{2\overline{R_0}, (2\overline{Q_0})^{1/\gamma}\right\}$, then we would have

$$\overline{Q_0} \geqslant Q_0 = \left(1 - \frac{R_0}{Z_0}\right) Z_0^\gamma \geqslant \frac{1}{2} Z_0^\gamma > \overline{Q_0},$$

which is a contradiction. Therefore Z_0 must be bounded from above, more precisely

$$Z_0 \leqslant \max\left\{2\overline{R_0}, (2\overline{Q_0})^{1/\gamma}\right\}.$$

The main results of this paper are the following two theorems. The first one is concerned with Lipschitz continuous dependence on the initial data of weak solutions.

Theorem 1.1. *Let $\gamma_\pm > 1$ and (1.21)–(1.23) be satisfied. Then there exists a unique global-in-time weak solution to (1.17)–(1.19). Moreover, if (R, Q, u) and $(\widetilde{R}, \widetilde{Q}, \widetilde{u})$ are two weak solutions on Ω_T corresponding to the initial data (R_0, Q_0, u_0) and $(\widetilde{R_0}, \widetilde{Q_0}, \widetilde{u_0})$, respectively, then*

$$\begin{aligned}&\left(\|R-\widetilde{R}\|_{L^\infty(\Omega_T)} + \|Q-\widetilde{Q}\|_{L^\infty(\Omega_T)} + \|u-\widetilde{u}\|_{V_2(\Omega_T)}\right)\\ &\quad\leqslant C\left(\|R_0-\widetilde{R_0}\|_{L^\infty} + \|Q_0-\widetilde{Q_0}\|_{L^\infty} + \|u_0-\widetilde{u_0}\|_{L^2}\right),\end{aligned} \tag{1.25}$$

where C is a positive constant depending on the lower and upper bounds of $(R_0, Q_0, \widetilde{R_0}, \widetilde{Q_0})$, the L^2-norms of $(u_0, \widetilde{u_0})$, $\mu, \gamma_\pm$, and T.

The second theorem gives the large time behavior of weak solutions. More precisely, we show the asymptotic decay of weak solutions to $(R_\infty, Q_\infty, u_\infty)$—the unique steady state for problem (1.17)–(1.19) given implicitly by

$$\begin{cases} Q_\infty = c_-\tau_\infty^{-1}, \quad R_\infty = c_+\tau_\infty^{-1}, \\ \tau_\infty := (R_\infty + Q_\infty)^{-1}, \\ u_\infty = 0, \quad Z_\infty^{\gamma_+} = C_\star, \\ Q_\infty = \left(1 - \dfrac{R_\infty}{Z_\infty}\right) Z_\infty^{\gamma}, \quad R_\infty \leqslant Z_\infty, \\ \displaystyle\int_\Omega \tau_\infty \mathrm{d}y = \int_\Omega \tau_0 \mathrm{d}y. \end{cases} \tag{1.26}$$

Here, $C_\star$ is the positive constant uniquely determined by $R_0, Q_0, \gamma_\pm$ and the conservation of mass $(1.26)_5$.

Theorem 1.2. *Let (R, Q, u) be the unique weak solution to (1.17)–(1.19) provided by theorem 1.1. Then, for any $t \geqslant 0$, it holds*

$$\|(R-R_\infty, Q-Q_\infty, u-u_\infty)\|_{L^2} \leqslant C_1 \exp(-C_2 t). \tag{1.27}$$

Here, C_1 and C_2 are positive constants depending on the initial data, $\mu, \gamma_\pm$, but independent of time.

Remark 1.2. Given suitably regular initial data, i.e.,

$$(R_0, Q_0) \in L^\infty, \quad (\partial_y R_0, \partial_y Q_0) \in L^\infty, \quad u_0 \in H_0^1,$$

$$0 < R_0(y), \quad Q_0(y), \quad \text{for any } y \in \Omega,$$

it can be shown, adapting the arguments from [23, 24], that

$$\|(R-R_\infty, Q-Q_\infty, u-u_\infty)\|_{H^1} \leqslant C_1 \exp(-C_2 t).$$

Remark 1.3. Theorems 1.1 and 1.2 are obtained based on the hypothesis that the two components of flows share the same velocity field. However, it is not straightforward how to generalise these results to system with two different velocities. First of all, it is not clear how to transform such system from Eulerian to Lagrangian coordinates. Secondly, the pressure term in the two-velocity system would not be in the conservative part, causing that the energy estimates and all the good properties of the pressure would be much harder to obtain.

The rest of this paper is structured as follows. In section 2.1 we show global existence and uniqueness of strong solutions to (1.17)–(1.19). In section 2.2 we prove the existence of global weak solutions via approximation based on regular solutions corresponding to regularized initial data and the weak convergence method. In section 3 we verify the stability of weak solutions. In section 4, we obtain the exponential decay of weak solution to the unique steady state in L^2-norm with large initial data.

2. Global existence of weak solutions

2.1. Global well-posedness to (1.17)–(1.19)

In this subsection, we prove global existence and uniqueness of strong solutions to (1.17)–(1.19) with large data. This will be useful in construction of weak solutions. Throughout this section, we denote by C a generic positive constant depending on the initial data, $\mu, \gamma_\pm$ and T.

Proposition 2.1. *Let* (1.21) *and* (1.22) *be satisfied. Assume that*

$$(R_0, Q_0) \in H^1, \quad u_0 \in H_0^1. \tag{2.1}$$

Then there exists a unique global strong solution (R, Q, u) *to* (1.17)–(1.19). *Furthermore, for any* $0 < T < \infty$*, it holds that*

$$C^{-1} \leqslant R(y,t), Q(y,t), Z(y,t) \leqslant C, \quad \text{for any } (y,t) \in \overline{\Omega_T}, \tag{2.2}$$

$$\|u\|_{L^\infty(0,T;L^2)} + \|\partial_y u\|_{L^2(\Omega_T)} \leqslant C, \tag{2.3}$$

$$\|\partial_t R\|_{L^2(\Omega_T)} + \|\partial_t Q\|_{L^2(\Omega_T)} \leqslant C. \tag{2.4}$$

Local-in-time existence and uniqueness of strong solutions to (1.17)–(1.19) is proved by the classical method based on the linearization of the problem and Banach fixed point theorem. We refer to [25, 26] for similar calculations. Therefore, it remains to derive sufficient global *a priori* estimates so as to extend the local solution globally.

We start by giving the conservation of mass and the elementary energy inequality. To simplify the expression, we define

$$\alpha := \frac{R}{Z}. \tag{2.5}$$

Lemma 2.1. *Let* (R, Q, u) *be a smooth solution to* (1.17)–(1.19) *on* $\overline{\Omega_T}$ *with regular initial data* (2.1)*, then we have*

$$\int_\Omega \tau(y,t)\mathrm{d}y = \int_\Omega \tau_0(y)\mathrm{d}y, \quad \text{for any } t \in [0,T], \tag{2.6}$$

$$\sup_{0\leqslant t\leqslant T}\int_\Omega\left(\frac{1}{2}u^2+\frac{(c_+)^{\gamma_+}}{\gamma_+-1}(\alpha\tau)^{-\gamma_++1}+\frac{(c_-)^{\gamma_-}}{\gamma_--1}[(1-\alpha)\tau]^{-\gamma_-+1}\right)\mathrm{d}y$$
$$+\mu\int_{\Omega_T}\frac{(\partial_y u)^2}{\tau}\mathrm{d}y\mathrm{d}s\leqslant C. \tag{2.7}$$

Proof. (2.6) follows from $(1.17)_1$ immediately after integration over Ω. To show (2.7), we adopt to the Lagrangian coordinates the technique of pressure decomposition in [3]. One deduces from (1.4), (1.7), and (2.5) that

$$\left(\frac{R}{\alpha}\right)^{\gamma_+}=\left(\frac{Q}{1-\alpha}\right)^{\gamma_-}. \tag{2.8}$$

Thus we can decompose the pressure as

$$Z^{\gamma_+}=\alpha\left(\frac{R}{\alpha}\right)^{\gamma_+}+(1-\alpha)\left(\frac{Q}{1-\alpha}\right)^{\gamma_-}. \tag{2.9}$$

Multiplying $(1.17)_2$ by u and integrating by parts yields

$$\begin{aligned}0&=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^2\mathrm{d}y-\int_\Omega Z^{\gamma_+}\partial_y u\mathrm{d}y+\mu\int_\Omega\frac{(\partial_y u)^2}{\tau}\mathrm{d}y\\&=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^2\mathrm{d}y-\int_\Omega Z^{\gamma_+}\partial_t\tau\mathrm{d}y+\mu\int_\Omega\frac{(\partial_y u)^2}{\tau}\mathrm{d}y.\end{aligned} \tag{2.10}$$

The second term on the right-hand side of (2.10) can be computed, with the help of (2.9), through

$$\begin{aligned}-\int_\Omega Z^{\gamma_+}\partial_t\tau\mathrm{d}y&=-\int_\Omega\alpha\left(\frac{R}{\alpha}\right)^{\gamma_+}\partial_t\tau\mathrm{d}y-\int_\Omega(1-\alpha)\left(\frac{Q}{1-\alpha}\right)^{\gamma_-}\partial_t\tau\mathrm{d}y\\&=-\int_\Omega\left(\frac{c_+}{\alpha\tau}\right)^{\gamma_+}\alpha\partial_t\tau\mathrm{d}y-\int_\Omega\left(\frac{c_-}{(1-\alpha)\tau}\right)^{\gamma_-}(1-\alpha)\partial_t\tau\mathrm{d}y\\&=-\int_\Omega\left(\frac{c_+}{\alpha\tau}\right)^{\gamma_+}\partial_t(\alpha\tau)\mathrm{d}y+\int_\Omega\left(\frac{c_+}{\alpha\tau}\right)^{\gamma_+}\tau\partial_t\alpha\mathrm{d}y\\&\quad-\int_\Omega\left(\frac{c_-}{(1-\alpha)\tau}\right)^{\gamma_-}\partial_t[(1-\alpha)\tau]\mathrm{d}y+\int_\Omega\left(\frac{c_-}{(1-\alpha)\tau}\right)^{\gamma_-}\tau\partial_t(1-\alpha)\mathrm{d}y\\&=-\int_\Omega\left(\frac{c_+}{\alpha\tau}\right)^{\gamma_+}\partial_t(\alpha\tau)\mathrm{d}y-\int_\Omega\left(\frac{c_-}{(1-\alpha)\tau}\right)^{\gamma_-}\partial_t[(1-\alpha)\tau]\mathrm{d}y\\&=\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{(c_+)^{\gamma_+}}{\gamma_+-1}(\alpha\tau)^{-\gamma_++1}\mathrm{d}y+\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{(c_-)^{\gamma_-}}{\gamma_--1}[(1-\alpha)\tau]^{-\gamma_-+1}\mathrm{d}y,\end{aligned} \tag{2.11}$$

where we have used (2.8) in the fourth equality. Thus, combining (2.10) and (2.11) gives rise to (2.7). This finishes the proof of lemma 2.1. □

By virtue of lemma 2.1 and the specific mathematical structure of the equations, we are able to show the upper and lower bounds for R and Q. This plays a crucial role in the proof of proposition 2.1.

Lemma 2.2. *Let the assumptions of lemma 2.1 be satisfied. Then*

$$C^{-1} \leqslant R(y,t), Q(y,t), Z(y,t) \leqslant C, \quad \textit{for any } (y,t) \in \overline{\Omega_T}. \tag{2.12}$$

Proof. We observe first that the positiveness of R and Q follow from the method of characteristics and the regular initial data. Given positive R and Q, there exists a unique Z satisfying (1.12) and (1.13). This fact can be justified easily and we refer to lemma 2.1 in [3] for the details. That is, Z can be regarded as a function of R and Q. Furthermore, due to $R = \frac{c_+}{\tau}$ and $Q = \frac{c_-}{\tau}$, Z can be also seen as a function of y and τ, i.e., $Z = Z\left(\frac{c_+(y)}{\tau}, \frac{c_-(y)}{\tau}\right)$.

Rewriting (1.12) as

$$c_-\tau^{-1} = \left(1 - \frac{c_+\tau^{-1}}{Z}\right) Z^\gamma. \tag{2.13}$$

Differentiating both sides of (2.13) with respect to τ leads to

$$-c_-\tau^{-2} = \gamma Z^{\gamma-1}\partial_\tau Z - \left(c_+\tau^{-1}(\gamma-1)Z^{\gamma-2}\partial_\tau Z - c_+\tau^{-2}Z^{\gamma-1}\right),$$

or equivalently,

$$\partial_\tau Z = -\frac{c_-\tau^{-2} + c_+\tau^{-2}Z^{\gamma-1}}{\gamma Z^{\gamma-1} - c_+\tau^{-1}(\gamma-1)Z^{\gamma-2}}. \tag{2.14}$$

The denominator is positive as we have

$$\begin{aligned} \gamma Z^{\gamma-1} - c_+\tau^{-1}(\gamma-1)Z^{\gamma-2} &= Z^{\gamma-2}[\gamma Z - (\gamma-1)R] \\ &= Z^{\gamma-2}[\gamma(Z-R) + R] \geqslant Z^{\gamma-2}R > 0, \end{aligned} \tag{2.15}$$

due to (1.13), assumption about the smoothness of the solution, and remark 1.1. Therefore,

$$\partial_\tau(Z^{\gamma_+}) = \gamma_+ Z^{\gamma_+-1}\partial_\tau Z < 0, \tag{2.16}$$

which means that the pressure is decreasing with respect to τ. Based on this crucial observation, the two-sided bounds of τ are proved by the same method as Zlotnik [14] (see also Antontsev *et al* [27]). The details are omitted here. The two-sided bounds of Z follows in the same manner as remark 1.1. □

With lemma 2.2 at hand, high-order energy estimates are obtained in a routine manner. Thus, the local-in-time solution can be extended globally. Uniqueness of solutions is proved by the classical energy method. This finishes the proof of proposition 2.1.

2.2. Existence of weak solutions

The main task of this subsection is to construct global-in-time weak solutions to (1.17)–(1.19) using approximation based on regular solutions. We start from regularizing the initial data (R_0, Q_0, u_0) in such a way that $\{(R_0^\varepsilon, Q_0^\varepsilon, u_0^\varepsilon)\}_{\varepsilon>0}$ satisfy

$$(R_0^\varepsilon, Q_0^\varepsilon) \in C^2(\overline{\Omega}), \ C^{-1} \leqslant R_0^\varepsilon, Q_0^\varepsilon \leqslant C, \quad u_0^\varepsilon \in C_c^2(\Omega),$$

$$(R_0^\varepsilon, Q_0^\varepsilon, u_0^\varepsilon) \to (R_0, Q_0, u_0) \text{ strongly in } L^2 \text{ as } \varepsilon \to 0.$$

Moreover, we define

$$\tau_0^\varepsilon := (R_0^\varepsilon + Q_0^\varepsilon)^{-1}, \quad \tau_\varepsilon := (R_\varepsilon + Q_\varepsilon)^{-1}.$$

Therefore, it follows from proposition 2.1 that there exists a unique global strong solution $(R_\varepsilon, Q_\varepsilon, u_\varepsilon)$ to (1.17)–(1.19) with initial data $(R_0^\varepsilon, Q_0^\varepsilon, u_0^\varepsilon)$. Furthermore, from proposition 2.1 we conclude the following uniform-in-ε estimates:

$$C^{-1} \leqslant R_\varepsilon(y,t), Q_\varepsilon(y,t) \leqslant C, \quad \text{for any } (y,t) \in \overline{\Omega_T}, \tag{2.17}$$

$$\|u_\varepsilon\|_{L^\infty(0,T;L^2)} + \|\partial_y u_\varepsilon\|_{L^2(\Omega_T)} \leqslant C, \tag{2.18}$$

$$\|\partial_t R_\varepsilon\|_{L^2(\Omega_T)} + \|\partial_t Q_\varepsilon\|_{L^2(\Omega_T)} \leqslant C. \tag{2.19}$$

From (2.17)–(2.19) it follows that there exists a subsequence of $\{(R_\varepsilon, Q_\varepsilon, u_\varepsilon)\}_{\varepsilon>0}$, not relabeling, such that as $\varepsilon \to 0$,

$$(R_\varepsilon, Q_\varepsilon) \to (R, Q) \,\text{weakly} - * \,\text{in}\, L^\infty(\Omega_T), \tag{2.20}$$

$$u_\varepsilon \to u \,\text{weakly} - * \,\text{in}\, L^\infty(0,T;L^2), \tag{2.21}$$

$$(\partial_t R_\varepsilon, \partial_t Q_\varepsilon, \partial_y u_\varepsilon) \to (\partial_t R, \partial_t Q, \partial_y u) \text{ weakly in } L^2(\Omega_T), \tag{2.22}$$

for some limit triple (R, Q, u), and we infer from (2.17)–(2.22) that

$$C^{-1} \leqslant R(y,t), Q(y,t) \leqslant C, \quad \text{for a.e. } (y,t) \in \Omega_T, \tag{2.23}$$

$$\|u\|_{L^\infty(0,T;L^2)} + \|\partial_y u\|_{L^2(\Omega_T)} \leqslant C, \tag{2.24}$$

$$\|\partial_t R\|_{L^2(\Omega_T)} + \|\partial_t Q\|_{L^2(\Omega_T)} \leqslant C. \tag{2.25}$$

The weak convergence results (2.20)–(2.22) are not sufficient to pass to the limit in (1.17)–(1.19), in particular, in the strongly nonlinear pressure function. For the moment we only know that Z_ε is the unique solution of

$$Q_\varepsilon = \left(1 - \frac{R_\varepsilon}{Z_\varepsilon}\right) Z_\varepsilon^\gamma, \quad R_\varepsilon \leqslant Z_\varepsilon.$$

To identify the pressure term, it suffices to verify that the pointwise limit of $\{Z_\varepsilon\}_{\varepsilon>0}$ is the unique solution of

$$Q = \left(1 - \frac{R}{Z}\right) Z^\gamma, \quad R \leqslant Z,$$

for which we need the strong convergence of the sequence $\{Z_\varepsilon\}_{\varepsilon>0}$. In fact, since $Q_\varepsilon = Q_0^\varepsilon \tau_0^\varepsilon \tau_\varepsilon^{-1}$, $R_\varepsilon = R_0^\varepsilon \tau_0^\varepsilon \tau_\varepsilon^{-1}$, Z_ε can also be regarded as $Z_\varepsilon = Z_\varepsilon(Q_0^\varepsilon, R_0^\varepsilon, \tau_0^\varepsilon, \tau_\varepsilon)$. Therefore the strong convergence of $\{Z_\varepsilon\}_{\varepsilon>0}$ will follow from that of $\{\tau_\varepsilon\}_{\varepsilon>0}$. The necessary compactness property in space is provided by the following lemma.

Lemma 2.3. *For any* $0 < h < 1$, *there holds*

$$\|\Delta_h \tau_\varepsilon\|_{L^\infty(0,T;L^2)} \leqslant C(\|\Delta_h R_0\|_{L^2} + r\|\Delta_h Q_0\|_{L^2} + h), \tag{2.26}$$

where $\Delta_h F(y) := F(y+h) - F(y)$ *is the translation in spatial variable with the step* h.

Proof. By setting

$$\sigma_\varepsilon := \mu \frac{\partial_y u_\varepsilon}{\tau_\varepsilon} - Z_\varepsilon^{\gamma+},$$

it follows that

$$\tau_\varepsilon = D_\varepsilon \left(\tau_0^\varepsilon + \frac{1}{\mu} \int_0^t D_\varepsilon^{-1}(y,s) \left(\tau_\varepsilon Z_\varepsilon^{\gamma+} \right)(y,s) \mathrm{d}s \right), \tag{2.27}$$

where

$$D_\varepsilon(y,t) := \exp\left(\frac{1}{\mu} \int_0^t \sigma_\varepsilon(y,s) \mathrm{d}s \right).$$

By definition it holds that

$$\begin{aligned} \Delta_h \tau_\varepsilon(y,t) = \Delta_h D_\varepsilon(y,t) &\left(\tau_0^\varepsilon(y+h) + \frac{1}{\mu}\int_0^t D_\varepsilon^{-1}\left(\tau_\varepsilon Z_\varepsilon^{\gamma+}\right)(y+h,s)\mathrm{d}s \right) \\ &+ D_\varepsilon(y,t)\left(\Delta_h \tau_0^\varepsilon(y) + \frac{1}{\mu}\int_0^t D_\varepsilon^{-1}(y+h,s)\Delta_h\left(\tau_\varepsilon Z_\varepsilon^{\gamma+}\right)(y,s)\mathrm{d}s \right) \\ &- D_\varepsilon(y,t)\left(\frac{1}{\mu}\int_0^t D_\varepsilon^{-1}(y+h,s) D_\varepsilon^{-1}(y,s)\left(\tau_\varepsilon Z_\varepsilon^{\gamma+}\right)(y,s)\Delta_h D_\varepsilon(y,s)\mathrm{d}s \right). \end{aligned} \tag{2.28}$$

Thanks to (2.17), we have

$$C^{-1} \leqslant Z_\varepsilon(y,t), \quad D_\varepsilon(y,t) \leqslant C, \quad \text{for any } (y,t) \in \overline{\Omega_T}. \tag{2.29}$$

The delicate issue is to compute $\Delta_h\left(\tau_\varepsilon Z_\varepsilon^{\gamma+}\right)$. In fact,

$$\Delta_h\left(\tau_\varepsilon Z_\varepsilon^{\gamma+}\right) = Z_\varepsilon^{\gamma+}\Delta_h\tau_\varepsilon + \tau_\varepsilon(y+h,t)\Delta_h Z_\varepsilon^{\gamma+}. \tag{2.30}$$

Furthermore,

$$\Delta_h Z_\varepsilon = (\partial_{Q_0^\varepsilon} Z_\varepsilon)\Delta_h Q_0^\varepsilon + (\partial_{R_0^\varepsilon} Z_\varepsilon)\Delta_h R_0^\varepsilon + (\partial_{\tau_0^\varepsilon} Z_\varepsilon)\Delta_h \tau_0^\varepsilon + (\partial_{\tau_\varepsilon} Z_\varepsilon)\Delta_h \tau_\varepsilon.$$

Subsequent differentiations of (2.13) with respect to $Q_0^\varepsilon, R_0^\varepsilon, \tau_0^\varepsilon$, and τ_ε give rise to

$$\partial_{Q_0^\varepsilon} Z_\varepsilon = \frac{\tau_0^\varepsilon \tau_\varepsilon^{-1}}{\gamma Z_\varepsilon^{\gamma-1} - R_0^\varepsilon \tau_0^\varepsilon \tau_\varepsilon^{-1}(\gamma-1) Z_\varepsilon^{\gamma-2}},$$

$$\partial_{R_0^\varepsilon} Z_\varepsilon = \frac{\tau_0^\varepsilon \tau_\varepsilon^{-1} Z_\varepsilon^{\gamma-1}}{\gamma Z_\varepsilon^{\gamma-1} - R_0^\varepsilon \tau_0^\varepsilon \tau_\varepsilon^{-1}(\gamma-1) Z_\varepsilon^{\gamma-2}},$$

$$\partial_{\tau_0^\varepsilon} Z_\varepsilon = \frac{Q_0^\varepsilon \tau_\varepsilon^{-1} + R_0^\varepsilon \tau_\varepsilon^{-1} Z_\varepsilon^{\gamma-1}}{\gamma Z_\varepsilon^{\gamma-1} - R_0^\varepsilon \tau_0^\varepsilon \tau_\varepsilon^{-1}(\gamma-1) Z_\varepsilon^{\gamma-2}},$$

$$\partial_{\tau_\varepsilon} Z_\varepsilon = -\frac{Q_0^\varepsilon \tau_0^\varepsilon \tau_\varepsilon^{-2} + R_0^\varepsilon \tau_0^\varepsilon \tau_\varepsilon^{-2} Z_\varepsilon^{\gamma-1}}{\gamma Z_\varepsilon^{\gamma-1} - R_0^\varepsilon \tau_0^\varepsilon \tau_\varepsilon^{-1}(\gamma-1) Z_\varepsilon^{\gamma-2}}.$$

In view of (2.15), (2.17) and (2.29), it follows that

$$\left\| \left(\partial_{Q_0^\varepsilon} Z_\varepsilon, \partial_{R_0^\varepsilon} Z_\varepsilon, \partial_{\tau_0^\varepsilon} Z_\varepsilon, \partial_{\tau_\varepsilon} Z_\varepsilon \right) \right\|_{L^\infty(\Omega_T)} \leqslant C. \tag{2.31}$$

Consequently, we conclude from (2.17) and (2.18) and (2.28)–(2.31) that

$$\|\Delta_h\tau_\varepsilon(y,t)\|_{L^2} \leqslant C\left(\|\Delta_h D_\varepsilon(y,t)\|_{L^2} + \|\Delta_h\tau_0^\varepsilon\|_{L^2}\right.$$
$$\left. + \int_0^t \left(\|\Delta_h D_\varepsilon(y,s)\|_{L^2} + \|\Delta_h\tau_\varepsilon(y,s)\|_{L^2} + \|\Delta_h Q_0^\varepsilon\|_{L^2} + \|\Delta_h R_0^\varepsilon\|_{L^2}\right) \mathrm{d}s\right)$$
$$\leqslant C\left(h\|u_\varepsilon - u_0^\varepsilon\|_{L^\infty(0,T;L^2)} + \|\Delta_h Q_0^\varepsilon\|_{L^2} + \|\Delta_h R_0^\varepsilon\|_{L^2} + \int_0^t \|\Delta_h\tau_\varepsilon(y,s)\|_{L^2}\mathrm{d}s\right)$$
$$\leqslant C\left(h + \|\Delta_h Q_0\|_{L^2} + \|\Delta_h R_0\|_{L^2} + \int_0^t \|\Delta_h\tau_\varepsilon(y,s)\|_{L^2}\mathrm{d}s\right). \tag{2.32}$$

Finally, (2.26) follows from (2.32) immediately by invoking Gronwall's inequality. The proof of lemma 2.3 is thus finished. □

Based on lemma 2.3, the relation $\partial_t\tau_\varepsilon = \partial_y u_\varepsilon$ and (2.18), we see

$$\|\tau_\varepsilon(\cdot + h, \cdot + s) - \tau_\varepsilon\|_{L^\infty(0,T-s;L^2)} \leqslant C\left(\|\Delta_h Q_0\|_{L^2} + \|\Delta_h R_0\|_{L^2} + h + s^{\frac{1}{2}}\right),$$

for any $0 < h < 1$, $0 < s < T$. This particularly implies the strong convergence of $\{\tau_\varepsilon\}_{\varepsilon>0}$ to τ in $L^2(\Omega_T)$ and furthermore in $L^p(\Omega_T)$ for any $1 \leqslant p < \infty$. Consequently, it holds that

$$Q_\varepsilon \to c_-\tau^{-1}, \quad R_\varepsilon \to c_+\tau^{-1}, \quad \text{a.e. in } \Omega_T,$$

which yields

$$Q = c_-\tau^{-1}, \quad R = c_+\tau^{-1}, \quad \text{a.e. in } \Omega_T. \tag{2.33}$$

Recalling that $Z_\varepsilon = Z_\varepsilon(Q_0^\varepsilon, R_0^\varepsilon, \tau_0^\varepsilon, \tau_\varepsilon)$, we find Z_ε converges to some limit function Z almost everywhere. Upon passing to the limit in the relations

$$Q_\varepsilon = \left(1 - \frac{R_\varepsilon}{Z_\varepsilon}\right) Z_\varepsilon^\gamma, \quad R_\varepsilon \leqslant Z_\varepsilon,$$

we conclude from (2.29) that $\{Z_\varepsilon\}_{\varepsilon>0}$ converges to Z strongly in $L^p(\Omega_T)$ for any $1 \leqslant p < \infty$ and Z solves exactly

$$Q = \left(1 - \frac{R}{Z}\right) Z^\gamma, \quad R \leqslant Z. \tag{2.34}$$

This finishes the proof of existence of a weak solution.

3. Stability of weak solutions

In the present section, we show Lipschitz continuous dependence on the initial data of weak solutions, i.e., we prove our first main theorem 1.1. We remark that the proof relies on the structure of the equations. As a preliminary step, we state the following lemma, the proof of which is omitted as it is similar to relevant results from [17, 24]. Throughout this section, various positive constants are denoted by the same symbol C depending on the initial data, $\mu, \gamma_\pm$ and T.

Lemma 3.1. *Let* (R, Q, u) *be a weak solution to* (1.17)–(1.19). *Then*

$$\tau(y,t) = \exp\left(\frac{1}{\mu}\int_0^t \sigma(y,s)\mathrm{d}s\right)$$

$$\times \left(\tau_0 + \frac{1}{\mu} \int_0^t \exp\left(-\frac{1}{\mu} \int_0^{\xi} \sigma(y,s) ds \right) \left(\tau Z^{\gamma+} \right)(y,\xi) \mathrm{d}\xi \right), \tag{3.1}$$

and

$$\int_0^t \sigma(y,s)\mathrm{d}s = (\mathcal{I}(u-u_0))(y,t) + \int_0^t \langle \sigma(\cdot,s)\rangle \mathrm{d}s, \tag{3.2}$$

where

$$\sigma(y,t) := \left(\mu \frac{\partial_y u}{\tau} - Z^{\gamma+} \right)(y,t),$$

$$\mathcal{I}f(y) := \int_0^y f(\xi)\mathrm{d}\xi - \left\langle \int_0^y f(\xi)\mathrm{d}\xi \right\rangle, \quad \langle f \rangle := \int_\Omega f(y)\mathrm{d}y.$$

To verify the stability estimate (1.25) from theorem 1.1, we follow the arguments in [16, 24]. Let us start from introducing the following notation:

$$\begin{cases} (\Delta\tau, \Delta R, \Delta Q, \Delta u) := (\tau - \widetilde{\tau}, R - \widetilde{R}, Q - \widetilde{Q}, u - \widetilde{u}), \\ (\Delta\tau_0, \Delta R_0, \Delta Q_0, \Delta u_0) := (\tau_0 - \widetilde{\tau_0}, R_0 - \widetilde{R_0}, Q_0 - \widetilde{Q_0}, u_0 - \widetilde{u_0}), \\ \Delta\sigma := \sigma - \widetilde{\sigma}, \ \widetilde{\sigma} := \mu \dfrac{\partial_y \widetilde{u}}{\widetilde{\tau}} - (\widetilde{Z})^{\gamma+}, \\ D := \exp\left(\dfrac{1}{\mu} \displaystyle\int_0^t \sigma(y,s)\mathrm{d}s \right), \quad \widetilde{D} := \exp\left(\dfrac{1}{\mu} \displaystyle\int_0^t \widetilde{\sigma}(y,s)\mathrm{d}s \right), \\ \widetilde{\varrho} := \widetilde{R} + \widetilde{Q}, \quad \Delta\varrho := \varrho - \widetilde{\varrho}. \end{cases}$$

Recalling that $Q = c_-\tau^{-1}, R = c_+\tau^{-1}$, one has in light of uniform bounds for R, Q from below and above, i.e., (2.23), that

$$\begin{aligned} |\Delta R| &\leqslant C\left(|\Delta R_0| + |\Delta\tau_0| + |\Delta\tau|\right) \leqslant C\left(|\Delta R_0| + |\Delta Q_0| + |\Delta\tau|\right); \\ |\Delta Q| &\leqslant C\left(|\Delta Q_0| + |\Delta\tau_0| + |\Delta\tau|\right) \leqslant C\left(|\Delta R_0| + |\Delta Q_0| + |\Delta\tau|\right). \end{aligned} \tag{3.3}$$

Consequently, in order to estimate $L^\infty(\Omega_T)$-norm of ΔR and ΔQ, it suffices to control $\|\Delta\tau\|_{L^\infty(\Omega_T)}$. This is the key step in proving stability of weak solutions. We follow the idea in [16, 17] to accomplish this goal.

Lemma 3.2. *Let the assumptions of theorem 1.1 be fulfilled, then we have*

$$\begin{aligned} \|\Delta\tau\|_{L^\infty(\Omega_t)} \leqslant C\big(&\|\Delta R_0\|_{L^\infty} + \|\Delta Q_0\|_{L^\infty} + \|\Delta u_0\|_{L^2} \\ &+ \|\Delta u\|_{L^\infty(0,t;L^2)} + \|\partial_y(\Delta u)\|_{L^2(\Omega_t)}\big) \end{aligned} \tag{3.4}$$

for any $t \in (0, T]$.

Proof. It follows from (3.1) that

$$\Delta\tau = D\left\{ \Delta\tau_0 + \frac{1}{\mu}\int_0^t \left(\tau Z^{\gamma+}\left(D^{-1} - (\widetilde{D})^{-1} \right) + \frac{\tau Z^{\gamma+} - \widetilde{\tau}(\widetilde{Z})^{\gamma+}}{\widetilde{D}} \right) \mathrm{d}s \right\}$$

$$+\left(D-\widetilde{D}\right)\left(\widetilde{\tau}_0+\frac{1}{\mu}\int_0^t\frac{\widetilde{\tau}(\widetilde{Z})^{\gamma+}}{\widetilde{D}}\mathrm{d}s\right). \tag{3.5}$$

Similarly to (2.29), it holds that

$$C^{-1}\leqslant\left(Z,\widetilde{Z},D,\widetilde{D}\right)(y,t)\leqslant C,\qquad\text{for a.e. }(y,t)\in\Omega_T. \tag{3.6}$$

Indeed, the upper bound of Z and $\widetilde{Z}$ is derived by the same argument as (2.29). Based on (2.23) and (3.6), we observe that

$$\begin{aligned}\left|\tau Z^{\gamma+}-\widetilde{\tau}(\widetilde{Z})^{\gamma+}\right| &\leqslant C\left(|\Delta\tau|+\|\partial_\tau Z\|_{L^\infty(\Omega_T)}|\Delta\tau|+\|\partial_{Q_0}Z\|_{L^\infty(\Omega_T)}|\Delta Q_0|\right.\\ &\quad\left.+\|\partial_{R_0}Z\|_{L^\infty(\Omega_T)}|\Delta R_0|+\|\partial_{\tau_0}Z\|_{L^\infty(\Omega_T)}|\Delta\tau_0|\right)\\ &\leqslant C\left(|\Delta\tau|+|\Delta R_0|+|\Delta Q_0|\right),\end{aligned} \tag{3.7}$$

where we used a version of (2.31) for the limit functions. Therefore, we deduce from (3.5)–(3.7) that

$$|\Delta\tau|\leqslant C\left(|\Delta R_0|+|\Delta Q_0|+\int_0^t\left(\left|\int_0^s\Delta\sigma\mathrm{d}\xi\right|+|\Delta\tau|\right)\mathrm{d}s\right)+C\left|\int_0^t\Delta\sigma\mathrm{d}s\right|;$$

whence

$$\|\Delta\tau(\cdot,t)\|_{L^\infty}\leqslant C\left(\|\Delta R_0\|_{L^\infty}+\|\Delta Q_0\|_{L^\infty}+\|\int_0^s\Delta\sigma\mathrm{d}\xi\|_{L^\infty(\Omega_t)}+\int_0^t\|\Delta\tau(\cdot,s)\|_{L^\infty}\mathrm{d}s\right). \tag{3.8}$$

The rest of the proof follows the same lines as [24], and we write down the details only for the convenience of the reader. First, from the identity (3.2) and Hölder's inequality we obtain

$$\|\int_0^s\Delta\sigma\mathrm{d}\xi\|_{L^\infty(\Omega_t)}\leqslant\left(\|\mathcal{I}(\Delta u_0)\|_{L^\infty}+\|\mathcal{I}(\Delta u)\|_{L^\infty(\Omega_t)}+\|\Delta\sigma\|_{L^2(\Omega_t)}\right). \tag{3.9}$$

It remains to bound $\Delta\sigma$. Notice that we have

$$\Delta\sigma=\mu\frac{\partial_y(\Delta u)}{\tau}+\mu(\Delta\varrho)\partial_y\widetilde{u}-\left(Z^{\gamma+}-(\widetilde{Z})^{\gamma+}\right),$$

and so, as for (3.7) we obtain

$$|\Delta\sigma|\leqslant C\left(|\partial_y(\Delta u)|+|\Delta\tau|(|\partial_y\widetilde{u}|+1)+|\Delta R_0|+|\Delta Q_0|\right).$$

It follows that

$$\begin{aligned}\|\Delta\sigma\|_{L^2(\Omega_t)}&\leqslant C\left(\|\partial_y(\Delta u)\|_{L^2(\Omega_t)}+\|\Delta R_0\|_{L^\infty}+\|\Delta Q_0\|_{L^\infty}\right.\\ &\quad\left.+\int_0^t(\|(\partial_y\widetilde{u})(\cdot,s)\|_{L^2}+1)\|\Delta\tau(\cdot,s)\|_{L^\infty}\mathrm{d}s\right).\end{aligned} \tag{3.10}$$

Since from (2.24) we deduce that $\int_0^T\|\partial_y\widetilde{u}\|_{L^2}^2\mathrm{d}s\leqslant C$, and therefore we can put together (3.9) and (3.10), and apply Gronwall's inequality to (3.8) to deduce (3.4). The proof of lemma 3.2 is thus finished. □

In order to use (3.4) to conclude (1.25), we need the estimates for Δu. In fact, standard energy estimate for parabolic equation [28] gives

Lemma 3.3. *For any $t \in (0, T]$, it holds that*

$$\|\Delta u\|_{L^\infty(0,t;L^2)} + \|\partial_y(\Delta u)\|_{L^2(\Omega_t)} \leqslant C\left(\|\Delta u_0\|_{L^2} + \|\Delta R_0\|_{L^\infty} + \|\Delta Q_0\|_{L^\infty}\right.$$
$$\left. + \left\| \left(\|(\partial_y \widetilde{u})(\cdot, s)\|_{L^2} + 1\right) \|\Delta \tau(\cdot, s)\|_{L^\infty} \right\|_{L^2((0,t))}\right). \tag{3.11}$$

Having this, (1.25) follows by suitable combination of lemmas 3.2 and 3.3. For the sake of brevity, we omit the details and refer the reader to [24] for similar steps. Clearly, (1.25) implies the uniqueness of weak solutions and so the proof of theorem 1.1 is complete. □

4. Large time behavior of weak solution

In this section, we show the exponential decay of weak solution in L^2-norm. The classical methods to handle the large time behavior of the one-dimensional single-phase Navier–Stokes equations [20–22] are not readily applicable to our two-fluid model system. In [23], the author developed a new technique to treat one-dimensional viscous barotropic gas with nonmonotone pressure. Of great importance in [23] is to obtain the uniform-in-time bounds of the density from above and below. It turns out that the idea can be adapted to our two-fluid model. As a matter of fact, it has already been successfully adapted before to the case of one-dimensional nonresistive magneto hydrodynamic equations [24].

4.1. Two-sided bounds for R and Q

To begin with, we notice that the estimates in lemma 2.1 are uniform-in-time. Then we have the following lemma, which is essential for the proof of theorem 1.2. Throughout this section we use C and C_i to denote generic positive constants depending on the initial data, μ, $\gamma_\pm$, while independent of time.

Lemma 4.1. *Let (R, Q, u) be the unique weak solution to* (1.17)–(1.19) *ensured by theorem* 1.1. *Then*

$$C^{-1} \leqslant R(y,t),\ Q(y,t) \leqslant C, \quad \text{for a.e. } (y,t) \in \Omega_\infty. \tag{4.1}$$

Proof. From (2.33) and the assumptions on the initial data (1.21) and (1.22) one sees that verification of (4.1) requires only to show the two-sided bounds for τ. By adapting the arguments in [23] (see also [24]), this follows from lemma 2.1 and the three items below.

- $$0 < C_1 \leqslant \int_\Omega \tau Z^{\gamma_+} \mathrm{d}y \leqslant C_2 < \infty,$$
- Z^{γ_+} is sufficiently large if τ is sufficiently small,
- Z^{γ_+} is sufficiently small if τ is sufficiently large.

As a consequence, it remains to check that the three items above are satisfied. By the identity of pressure decomposition (2.9) and (2.33), it holds that

$$\int_\Omega \tau Z^{\gamma_+} \mathrm{d}y = \int_\Omega \left(\tau\alpha\left(\frac{R}{\alpha}\right)^{\gamma_+} + \tau(1-\alpha)\left(\frac{Q}{1-\alpha}\right)^{\gamma_-}\right) \mathrm{d}y$$

$$= \int_\Omega \left((c_+)^{\gamma_+}(\alpha\tau)^{-\gamma_++1} + (c_-)^{\gamma_-}[(1-\alpha)\tau]^{-\gamma_-+1}\right) \mathrm{d}y$$
$$\leqslant C_2,$$

where we have used the energy estimate (2.7). Clearly, we conclude from the definition of α, i.e., (2.5), and Jensen's inequality that

$$\begin{aligned}
\int_\Omega \tau Z^{\gamma_+}\mathrm{d}y &= \int_\Omega \left(\tau\alpha\left(\frac{R}{\alpha}\right)^{\gamma_+} + \tau(1-\alpha)\left(\frac{Q}{1-\alpha}\right)^{\gamma_-}\right)\mathrm{d}y \\
&\geqslant \int_\Omega \alpha^{-\gamma_++1}\tau^{-\gamma_++1}(c_+)^{\gamma_+}\mathrm{d}y \\
&\geqslant C\int_\Omega \tau^{-\gamma_++1}\mathrm{d}y \\
&\geqslant C\left(\int_\Omega \tau\mathrm{d}y\right)^{-\gamma_++1} \\
&\geqslant C_1.
\end{aligned}$$

Suppose now that τ is small, i.e., $R+Q$ is large and we consider two possible cases. If R is large, then Z^{γ_+} is also large due to $R \leqslant Z$. If, on the other hand, Q is large, then also Z is large. Indeed, otherwise, we would arrive at a contradiction in the relation

$$Q = \left(1 - \frac{R}{Z}\right) Z^{\gamma}.$$

The third item is verified by using similar observation as above. We refer to [23] and lemma 5.3 in [24] for the remaining details. □

Remark 4.1. The key observations in lemma 4.1 are as follows. Firstly, the pressure term is a function with variables y and τ by virtue of (1.12) and (1.13), tending to infinity as τ goes to zero and tending to zero as τ goes to infinity. Secondly, the two internal pressures satisfy γ-laws. This leads to a positive lower bound of the integral $\int_\Omega \tau Z^{\gamma_+}\mathrm{d}y$; while the upper bound is obtained by the energy inequality. In this way, the arguments in [23, 24] are naturally adapted.

4.2. Exponential decay

In this subsection, we prove the exponential decay of weak solution in L^2-norm by adapting the ideas from [23, 24]. It should be emphasized that the structure of pressure function is crucial for a modification of these arguments to work.

Step 1. Let $(R_\infty, Q_\infty, u_\infty)$ be the unique steady state for problem (1.17)–(1.19) given by (1.26). Thanks to $(1.26)_3$, we rewrite the momentum equation $(1.17)_2$ as

$$\partial_t u + \partial_y\left(Z^{\gamma_+} - Z_\infty^{\gamma_+}\right) = \mu\partial_y\left(\frac{\partial_y u}{\tau}\right). \tag{4.2}$$

Since Z is a function of y and τ, it follows that $Z_\infty = Z(y, \tau_\infty)$. Therefore, testing (4.2) by u and integrating by parts yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^2\mathrm{d}y + \int_\Omega \left(Z^{\gamma_+}(y,\tau_\infty) - Z^{\gamma_+}(y,\tau)\right)\partial_y u\mathrm{d}y + \mu\int_\Omega \frac{(\partial_y u)^2}{\tau}\mathrm{d}y = 0.$$

Using the continuity equation (1.17)$_1$, one has

$$\int_\Omega (Z^{\gamma+}(y,\tau_\infty) - Z^{\gamma+}(y,\tau))\,\partial_y u \mathrm{d}y = \int_\Omega (Z^{\gamma+}(y,\tau_\infty) - Z^{\gamma+}(y,\tau))\,\partial_t \tau \mathrm{d}y$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega G(y,\tau,\tau_\infty)\mathrm{d}y,$$

where we denoted

$$G(y,\tau,\tau_\infty) := \int_{\tau_\infty}^{\tau} (Z^{\gamma+}(y,\tau_\infty) - Z^{\gamma+}(y,\xi))\,\mathrm{d}\xi.$$

Thus we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}u^2 + G(y,\tau,\tau_\infty)\right)\mathrm{d}y + \mu\int_\Omega \frac{(\partial_y u)^2}{\tau}\mathrm{d}y = 0. \tag{4.3}$$

Step 2. The key step in obtaining the exponential decay is to show that

$$C^{-1}(\tau - \tau_\infty)^2 \leqslant G(y,\tau,\tau_\infty) \leqslant C(\tau - \tau_\infty)^2. \tag{4.4}$$

The main observation is as follows. By setting

$$F(\tau) := -\int_{\tau_\infty}^{\tau} Z^{\gamma+}(y,\xi)\mathrm{d}\xi,$$

$G(y,\tau,\tau_\infty)$ is reformulated as

$$G(y,\tau,\tau_\infty) = F(\tau) - F(\tau_\infty) - F'(\tau_\infty)(\tau - \tau_\infty). \tag{4.5}$$

Therefore, in order to deduce (4.4) it is enough to estimate the second derivative of $F(\tau)$. To this purpose we use the expression for $\partial_\tau Z$ from (2.14) to get

$$\partial_\tau(Z^{\gamma+}) = -\gamma_+ Z^{\gamma+-1}\frac{c_-\tau^{-2} + c_+\tau^{-2}Z^{\gamma-1}}{\gamma Z^{\gamma-1} - c_+\tau^{-1}(\gamma-1)Z^{\gamma-2}}. \tag{4.6}$$

As in (2.15) we first observe that the denominator is strictly positive. Moreover, in spirit of remark 1.1, we infer from (4.1) and the relation $Q = \left(1 - \frac{R}{Z}\right)Z^\gamma$ that

$$C^{-1} \leqslant Z(y,t) \leqslant C, \quad \text{for a.e. } (y,t) \in \Omega_\infty, \tag{4.7}$$

which together with lower and upper bound for τ implies boundedness of the numerator of (4.6).

The remaining arguments follow largely the ones from [23, 24]. We incorporate the detailed proof for the sake of completeness.

Step 3. Let $0 < \varepsilon < 1$ and

$$K(y,t) := \int_0^y (\tau(\xi,t) - \tau_\infty(\xi))\mathrm{d}\xi.$$

Testing (4.2) by εK gives rise to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varepsilon u K \mathrm{d}y - \varepsilon\int_\Omega (Z^{\gamma+}(y,\tau) - Z^{\gamma+}(y,\tau_\infty))\,(\tau - \tau_\infty)\mathrm{d}y$$

$$- \varepsilon \int_\Omega u^2 \mathrm{d}y + \varepsilon \int_\Omega \mu \frac{\partial_y u}{\tau} (\tau - \tau_\infty) \mathrm{d}y = 0. \tag{4.8}$$

From (4.3) and (4.8) we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} u^2 + G(y, \tau, \tau_\infty) + \varepsilon u K \right) \mathrm{d}y \\ &\quad + \mu \int_\Omega \frac{(\partial_y u)^2}{\tau} \mathrm{d}y - \varepsilon \int_\Omega \left(Z^{\gamma+}(y, \tau) - Z^{\gamma+}(y, \tau_\infty) \right) (\tau - \tau_\infty) \mathrm{d}y \\ &\quad = \varepsilon \int_\Omega u^2 \mathrm{d}y - \varepsilon \int_\Omega \mu \frac{\partial_y u}{\tau} (\tau - \tau_\infty) \mathrm{d}y. \end{aligned} \tag{4.9}$$

It follows from (4.1), (4.6) and (4.7) that

$$\int_\Omega \left(Z^{\gamma+}(y, \tau) - Z^{\gamma+}(y, \tau_\infty) \right) (\tau - \tau_\infty) \mathrm{d}y \geqslant C_1 \int_\Omega (\tau - \tau_\infty)^2 \mathrm{d}y. \tag{4.10}$$

With the help of Cauchy–Schwarz's inequality and (4.1), we find

$$\left| \int_\Omega \mu \frac{\partial_y u}{\tau} (\tau - \tau_\infty) \mathrm{d}y \right| \leqslant \frac{C_2}{2C_1} \mu \int_\Omega \frac{(\partial_y u)^2}{\tau} \mathrm{d}y + \frac{C_1}{2} \int_\Omega (\tau - \tau_\infty)^2 \mathrm{d}y; \tag{4.11}$$

$$\int_\Omega u^2 \mathrm{d}y \leqslant C_3 \mu \int_\Omega \frac{(\partial_y u)^2}{\tau} \mathrm{d}y. \tag{4.12}$$

Using (4.10)–(4.12), (4.9) implies

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} u^2 + G(y, \tau, \tau_\infty) + \varepsilon u K \right) \mathrm{d}y \\ &\quad + \frac{C_1 \varepsilon}{2} \int_\Omega (\tau - \tau_\infty)^2 \mathrm{d}y + \left(1 - \frac{C_2 \varepsilon}{2C_1} - C_3 \varepsilon \right) \int_\Omega \mu \frac{(\partial_y u)^2}{\tau} \mathrm{d}y \leqslant 0. \end{aligned} \tag{4.13}$$

Step 4. Due to the definition of K, it holds that

$$\left| \int_\Omega \varepsilon u K \mathrm{d}y \right| \leqslant \frac{\varepsilon}{2} \int_\Omega u^2 \mathrm{d}y + \frac{\varepsilon}{2} \int_\Omega (\tau - \tau_\infty)^2 \mathrm{d}y. \tag{4.14}$$

Based on (4.14), after choosing ε suitably small, we see

$$\begin{aligned} C^{-1} (\|\tau - \tau_\infty\|_{L^2}^2 + \|u\|_{L^2}^2) &\leqslant \int_\Omega \left(\frac{1}{2} u^2 + G(y, \tau, \tau_\infty) + \varepsilon u K \right) \mathrm{d}y \\ &\leqslant C (\|\tau - \tau_\infty\|_{L^2}^2 + \|u\|_{L^2}^2), \end{aligned}$$

where we essentially used the property (4.4) from step 2. Combining the above with (4.13) leads to

$$\|\tau - \tau_\infty\|_{L^2} + \|u\|_{L^2} \leqslant C \exp(-Ct), \tag{4.15}$$

for any $t \geqslant 0$.

Step 5. Finally, the exponential decay of $\|R - R_\infty\|_{L^2}$ and $\|Q - Q_\infty\|_{L^2}$ is a direct consequence of (4.15) and the relations

$$Q = c_-\tau^{-1}, \quad R = c_+\tau^{-1};$$

$$Q_\infty = c_-\tau_\infty^{-1}, \quad R_\infty = c_+\tau_\infty^{-1}.$$

The proof of theorem 1.2 is complete. □

Remark 4.2. We observe that the exponential decay of Z follows from that of τ. Indeed,

$$\|Z(y,\tau) - Z(y,\tau_\infty)\|_{L^2} \leqslant \| \partial_\tau Z\|_{L^\infty} \|\tau - \tau_\infty\|_{L^2} \leqslant C \exp(-Ct),$$

in light of (4.6), (4.7) and (4.15).

Remark 4.3. The strategy adopted in this paper is strong enough to show existence, stability and exponential decay of global weak solution to two-fluid models with more general form of pressure considered for example in [4]. In particular, the two-fluid model with pressure satisfying γ-laws could be included.

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