HOW (NOT) TO CUT YOUR CHEESE

ABSTRACT. It is well known that a line can intersect at most $2n-1$ unit squares of the $n \times n$ chessboard. Here we consider the 3-dimensional version: how many unit cubes of the 3-dimensional cube $[0,n]^3$ can a hyperplane intersect?

1. INTRODUCTION.

Imagine that you have 1000 pieces of processed cheese of identical size, each of the form of a perfect cube with edge length 1 cm and wrapped up in very thin paper. Everything is packed neatly into a box of edge length 10 cm.

Here comes a tomboy who is allowed to cut the box with a thin knife and she wants to destroy as many pieces of cheese as possible. What is the maximum number of pieces that she can destroy?

Let us restate the general problem in a mathematical way. Let $n \geq 2$ be an integer and consider a cube of edge length n that is divided into n^3 unit cubes in the usual way. What is the maximum number $m(n)$ of unit cubes that can be cut by a single plane? For us a cut means that the plane contains some interior points of the corresponding unit cube, (i.e., not just parts of the wrapping paper of the processed cheese.)

Our main result is that $m(n)$ is asymptotic to $\frac{9}{4}n^2$.

Theorem 1. *For every n we have $m(n) \leq \frac{9}{4}n^2+2n+1$. Moreover, for n large enough, $m(n) \geq \frac{9}{4}n^2+n-5$ for even n and $m(n) \geq \frac{9}{4}n^2+n-\frac{17}{4}$ for odd n, and $m(2)=7$, $m(3)=19$, and $m(4)=35$.*

Thus the tomboy can destroy at least 230 and at most 246 pieces of the processed cheese. This is about one fourth of them, which is quite a lot. The exact number is unknown.

The proof of the upper bound is in Section 4 and of the lower one in Section 5. A different approach is needed for $n=2,3,4$ which is explained in Section 6.

In higher dimensions the analogous question is to determine the maximal number of unit cubes in the $[0,n]^d$ cube that a hyperplane can hit. In a companion paper [3] it is shown that this maximal number is $V_d n^{d-1}(1+o(1))$, where V_d is a well-defined constant depending only on d. This constant is related to a famous result of Ball [2] and has been determined recently by Aliev [1]. As it turns out, there is a *maximizer hyperplane* that is very close to the one with equation $\sum_1^n x_i = dn/2$.

2000 *Mathematics Subject Classification.* Primary 52B20, secondary 11H06.
Key words and phrases. Lattices, polytopes, lattice points in convex bodies.

Here a maximizer hyperplane is a hyperplane that cuts the maximum possible number of unit cubes in $[0,n]^d$. It is called a maximizer plane, or simply a maximizer when $d = 3$.

2. The planar case and some preliminaries.

To solve the analogous problem in two dimensions is much easier. The solution is well known (and is probably folklore): a line can intersect at most $2n-1$ unit squares of the $n \times n$ chessboard, and this bound is the best possible. Variants of this statement have become olympiad problems in several countries. József Beck used a slightly stronger version of this fact in a well-known and influential paper [4]. Our key idea for proving Theorem 1 is to estimate $m(n)$ via the intersection of the maximizer plane P with certain planes. Before making this precise, let us explore the case of two dimensions.

Let B_n be the $n \times n$ chessboard, naturally divided into n^2 unit squares. Thus $B_n = [0,n]^2$. Let $0 < a < b < 1$ be fixed and consider the line L defined by $ax + by = t$. We are interested in the number $m(L)$ of unit squares in B_n that are cut by L.

Let $H_i = \{(x,i) : 0 \le x \le n\}$, $i = 0,1,\dots,n$ be the horizontal line segments on the chessboard. Similarly, let $V_j = \{(j,y) : 0 \le y \le n\}$, $j = 0,1,\dots,n$ be the vertical segments. Note that H_0, V_0, H_n, V_n form the perimeter of B_n. Suppose that we move on L starting from some point (x,y) with $x < 0$ toward B_n; see Figure 1. We go across V_0 or H_n and then enter the first unit square. Moving along L we keep entering new unit squares exactly when hitting a horizontal or vertical segment H_i or V_j with $1 \le i,j \le n-1$. Eventually we hit either H_0 or V_n and then leave the chessboard. This argument shows that L cannot cut more than $1 + 2(n-1) = 2n-1$ unit squares. We will need the following version of this statement.

Proposition 1. *Let ℓ be the length of the intersection $L \cap B_n$, and let $s(L)$ denote the number of unit squares cut by L. Then*

$$s(L) \le 1 + \frac{a+b}{\sqrt{a^2+b^2}}\ell.$$

Proof. There are four almost identical cases to consider. Namely, L can enter B_n through V_0 or H_n and leave through V_n or H_0. We only do the computation for the pair V_0, H_0; see Figure 1. The line enters B_n at $(0,v)$ and leaves at $(u,0)$. Then L hits $\lfloor v \rfloor$ of the horizontal segments H_i, $0 < i < n$, and $\lfloor u \rfloor$ of the vertical segments V_j, $0 < j < n$. Note that $\ell = \sqrt{u^2+v^2}$. Thus

$$s(L) = 1 + \lfloor u \rfloor + \lfloor v \rfloor \le 1 + u + v = 1 + \frac{u+v}{\sqrt{u^2+v^2}}\ell.$$

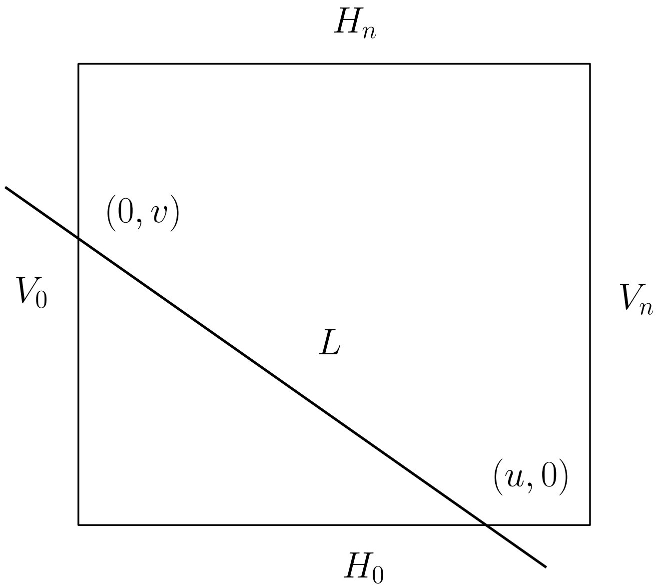

FIGURE 1. Illustration for the proof of Proposition 1.

In view of the equation $ax + by = t$, we have $u = t/a$ and $v = t/b$, which implies that

$$\frac{u+v}{\sqrt{u^2+v^2}} = \frac{a+b}{\sqrt{a^2+b^2}}.$$

□

Now we turn to the 3-dimensional case and prove a simple lower bound on $m(n)$. The better bound from Theorem 1 is given later, in Section 5.

Lemma 1. *If $n \geq 2$ is even then $m(n) \geq \frac{9n^2-8}{4}$, and if $n \geq 1$ is odd then $m(n) \geq \frac{9n^2-5}{4}$.*

Proof. Set $n^* = 3n/2$ for even n and $n^* = (3n-1)/2$ for odd n. Consider the plane P with equation $x + y + z = n^* + \delta$ for some small $\delta > 0$; for instance $\delta = .01$ will do. The unit cube $C(i, j, k) = \{(x, y, z) : i \leq x \leq i+1,\ j \leq y \leq j+1,\ k \leq z \leq k+1\}$ where $i, j, k \in [0, n-1]$ are integers intersects P if and only if $i+j+k < n^* + \delta < i+j+k+3$, which happens if and only if $i+j+k$ equals n^*, n^*-1, or n^*-2 because i, j, k are integers and $\delta > 0$ is small.

When n is even, $i + j + k = n^*$ has a solution in i if and only if j, k are integers with $0 \leq j, k \leq n-1$ and $n^* - (n-1) \leq j + k \leq n^*$, since for each such pair there is a unique integer $i \in [0, n-1]$ with $i+j+k = n^*$. The number of such pairs (j, k) is $n^2 - \binom{n/2+1}{2} - \binom{n/2}{2}$ as one can check easily. Identical counting shows that $i + j + k = n^* - 1$ and $i + j + k = n^* - 2$ have, respectively,

$$n^2 - \binom{n/2}{2} - \binom{n/2+1}{2} \text{ and } n^2 - \binom{n/2-1}{2} - \binom{n/2+2}{2}$$

integer solutions with $0 \leq i, j, k \leq n-1$. Summing these numbers gives $(9n^2-8)/4$. The calculations are analogous in the odd case. $\square$

Assume now that P is a maximizer, i.e., a plane cutting the maximum possible number of unit cubes in K_n where $K_n = [0, n]^3$. For each of these unit cubes C_i fix an interior point R_i that is on P. By the definition of interior point, there is a positive ε_i such that a sphere of radius ε_i centered at R_i is contained entirely in C_i. Set $\varepsilon = \min_i \varepsilon_i$. Translating P by a vector of length less than ε will always result in another plane P' cutting each of the previous unit cubes C_i. Similarly, tilting the plane so slightly that none of the R_i moves ε or more results in a plane P'' cutting the maximum number of unit cubes. This establishes the following result.

Proposition 2. *When proving Theorem 1 we may always assume that the maximizer plane P is in general position, that is, (i) and (ii) hold:*

(i) P does not pass through any vertex of any unit cube nor through the center of K_n.

(ii) P is not parallel to any segment joining two vertices of two, possibly distinct, unit cubes. $\square$

The center of K_n is $Q = (n/2, n/2, n/2)$. We draw a line through Q perpendicular to P. Let R be the intersection of the line and P, and let (a, b, c) be the vector $\overrightarrow{QR}$. By turning K_n (and P) around if necessary we can assume that (a, b, c) is in the positive orthant, that is, a, b, c are all positive. By symmetry, we can assume that $0 < a < b < c$; note that in view of Proposition 2 (ii), equality cannot occur. The equation of the plane is $ax+by+cz = d$. Replacing (a, b, c, d) by $(a/c, b/c, c/c, d/c)$ we may assume that $c = 1$.

Proposition 3. *For any maximizer plane P we have $a + b > 1$.*

Proof. Decompose K_n into n^2 vertical stacks of n unit cubes each, where a vertical stack is just the set of unit squares $C(i, j, k)$, $k = 0, \ldots, n-1$. As $m(n) > 2n^2$ for $n > 2$, P has to cut at least three cubes from some stack. Consequently, there are integers $0 \leq i, j, k, h < n$ with $k + 2 \leq h$ such that P cuts both $C(i, j, k)$ and $C(i, j, h)$. Should $a+b \leq 1$ hold, we infer that, for every pair $(x, y, z) \in P$ and $(x', y', z') \in P$ of interior points of the two cubes,

$$ax + by + z < a(i + 1) + b(j + 1) + k + 1 \leq ai + bj + h < ax' + by' + z',$$

a contradiction. $\square$

3. A formula relying on plane cuts.

Let us define $n + 1$ "floors" of the big cube K_n as $F_i = \{(x, y, i) : 0 \leq x, y \leq n\}$, $i = 0, \ldots, n$. Each F_i can be considered as an $n \times n$ chessboard. Let P be a maximizer plane and set $L_i = F_i \cap P$; so L_i is either a line segment or is empty. Recall that we may assume L_i is not a single point in view of Proposition 2 (i). Proposition 3 implies that either L_0 or L_n is nonempty. Indeed if both $L_0 = L_n = \emptyset$, then the point $(n, n, 0)$ is below P, which implies that $n < an + bn < d$ because $a + b > 1$; and the point $(0, 0, n)$ is above P, yielding $n > d$ which is impossible. By symmetry we may assume that $L_n \neq \emptyset$.

Lemma 2. *There is a maximizer plane P such that either L_0 or L_1 is nonempty.*

Proof. We are done if there is a maximizer cutting both F_n and F_0. Assume there is no such maximizer. Then every maximizer intersecting F_n intersects $F_n, F_{n-1}, \ldots, F_k$ but is disjoint from F_{k-1} for some $k \geq 1$. Now choose such a maximizer, say P, for which k is minimal. We claim that $k = 1$. If not, then the translated plane $P' = P - (0, 0, 1)$ is also a maximizer because if P cuts the unit cube $C \subset K_n$, then P' cuts the unit cube $C - (0, 0, 1) \subset K_n$. As P' does not intersect F_0, it has to intersect F_n. So the maximizer P' intersects $F_n, \ldots, F_k$ and also F_{k-1}, contradicting the minimality of k. $\square$

Set $p = 0$ if L_0 is not empty and $p = 1$ otherwise. Then all segments $L_p, L_{p+1}, \ldots, L_n$ are nonempty. Consider the orthogonal projection $(x, y, z) \to (x, y, 0)$. It maps L_i to the line segment L_i^*; we denote their common length by ℓ_i. The points $(x, y, z) \in L_i$ satisfy $ax + by = d - i$, so the points $(x, y, 0) \in L_i^*$ satisfy the same equation (for all $i = p, \ldots, n$). It is easy to check that the distance between L_i^* and L_{i+1}^* is $h := (a^2 + b^2)^{-1/2}$.

Set $W = P \cap K_n$ and let W^* be the orthogonal projection of W onto F_0. Let $r(W)$ denote the number of unit squares in F_0 that have a common interior point with W^*. Further write r_i for the number of unit squares in F_0 cut by L_i^*, $i = p, \ldots, n$.

Lemma 3. *The number $m(P)$ of unit cubes cut by P is*

$$m(P) = r(W) + \sum_{i=p+1}^{n-1} r_i.$$

Proof. By definition, $r(W)$ is the number of vertical stacks cut by P. Whenever P cuts both $C(i, j, k)$ and $C(i, j, k+1)$, then P intersects the top face of $C(i, j, k)$, which is the bottom face $C(i, j, k+1)$. This is equivalent to L_k^* intersecting the unit square $\{(x, y) : i \leq x \leq i+1,\ j \leq y \leq j + 1\}$ on F_0. All unit cubes in a stack cut by P are counted by their bottom face this way in some r_k $(k = p + 1, \ldots, n - 1)$ exactly once, except the bottom-most ones. They are counted in $r(W)$. $\square$

4. Proof of the upper bound in Theorem 1.

Let P be a maximizer plane with equation $ax + by + z = d$. The conditions $0 < a < b < 1$ and $a + b > 1$ imply that $1/\sqrt{2} \leq h =$

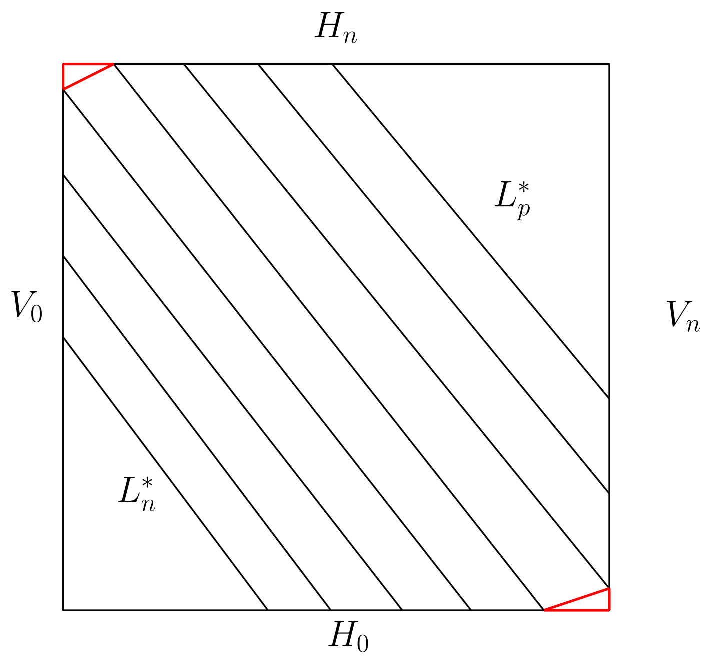

FIGURE 2. The lines L_h^* and the two small triangles.

$(a^2 + b^2)^{-1/2} \le \sqrt{2}$. The area of the trapezoid whose vertices are the four endpoints of the segments L_{i+1}^* and L_i^* is $h(\ell_{i+1} + \ell_i)/2$ for $i = p, \dots, n-1$. The area A of W^* is the sum of these areas plus the areas of the two small triangles, as shown in Figure 2. Write A_0 for the sum of the areas of these triangles. Thus

$$
\begin{aligned}
(1) \qquad A &= h\left(\frac{\ell_p + \ell_{p+1}}{2} + \cdots + \frac{\ell_{n-1} + \ell_n}{2}\right) + A_0 \\
&= h\sum_{i=p}^{n} \ell_i + A_0 - \frac{h}{2}(\ell_p + \ell_n).
\end{aligned}
$$

Observe that $r(W)$ is at most the area of W^* plus $r_p + r_n$; see Figure 2. Using Lemma 3 and Proposition 1 we have

$$
\begin{aligned}
m(P) &= r(W) + \sum_{i=p+1}^{n-1} r_i \le A + r_p + r_n + \sum_{i=p+1}^{n-1} r_i \\
&= A + \sum_{i=p}^{n} r_i \le A + (a+b)h\sum_{i=p}^{n} \ell_i + n - p + 1.
\end{aligned}
$$

From equation (1) one can express $h\sum_{i=p}^{n}\ell_i$ as $A - A_0 + h\frac{\ell_p+\ell_n}{2}$. Since $A_0 \geq 0$ this gives

$$\begin{aligned} m(P) &\leq A + (a+b)\left[A - A_0 + h\frac{\ell_p+\ell_n}{2}\right] + n - p + 1 \\ &\leq (a+b+1)A + (a+b)h\frac{\ell_p+\ell_n}{2} + n - p + 1. \end{aligned} \tag{2}$$

This proves that $m(P) \leq (a+b+1)A + 3n + 1$, because $(a+b)h \leq \sqrt{2}$ and $\ell_p, \ell_n \leq \sqrt{2}n$. But our target is a stronger inequality.

We have to estimate A and $\ell_p + \ell_n$, which is a purely geometric problem (actually two problems).

We begin with A, the area of W^*, which is a subset of the square F_0 between two parallel lines, the ones containing L_n^* and L_p^*. Their equations are $ax + by = d - n$ and $ax + by = d - p$ (recall that $c = 1$). These two lines bound a strip S of width nh or $(n-1)h$ depending on whether p equals 0 or 1. We can assume that the width is nh as this may only increase the area of $F_0 \cap S$. The area of this intersection is maximal when the strip is symmetric with respect to the center of F_0. This can be seen by translating the strip and checking how the area changes.

Thus the strip is bounded by the lines with with equations $ax+by = (a+b-1)n/2$ and $ax + by = (a+b+1)n/2$ as a simple computation shows. The first line intersects H_0 and V_0 in points

$$\left(\frac{a+b-1}{2a}n, 0\right) \text{ and } \left(0, \frac{a+b-1}{2b}n\right), \tag{3}$$

and the second line intersects H_n and V_n in points

$$\left(\frac{a-b+1}{2a}n, n\right) \text{ and } \left(n, \frac{-a+b+1}{2b}n\right).$$

Now the area in question is n^2, the area of F_0, minus the area of two congruent right triangles. Direct computation shows then that

$$A \leq n^2 - \left(\frac{a+b-1}{2a}n\right)\left(\frac{a+b-1}{2b}n\right) = n^2\left(1 - \frac{(a+b-1)^2}{4ab}\right).$$

So in order to bound $(1+a+b)A$ we need the following lemma.

Lemma 4. *Let $0 < a \leq b \leq 1$. Then*

$$(a+b+1)\left(1 - \frac{(a+b-1)^2}{4ab}\right) \leq \frac{9}{4}.$$

Proof. The statement is equivalent to

$$4ab(a+b) \leq 5ab + (a+b+1)(a+b-1)^2, \tag{4}$$

which is clearly correct if $4(a+b) \leq 5$. So assume $4(a+b) > 5$. Then (4) holds if and only if

$$[4(a+b)-5]\,ab \leq (a+b+1)(a+b-1)^2.$$

The square bracket on the left hand side is positive, so the last inequality holds if and only if it holds when replacing a and b by $z = (a+b)/2$. In this case (4) becomes

$$8z^3 \leq 5z^2 + (4z^2-1)(2z-1) = 8z^3 + (z-1)^2,$$

which is always true. □

We consider next $\ell_p + \ell_n$. Let u and v be the endpoints of the segment L_p^* with the x-coordinate of v larger than that of u. If $u \in V_0$ and $v \in H_0$, then we replace L_p^* by its reflection, L_p', with respect to the center of F_0. If $u \in H_n$ and $v \in H_0$, then replace L_p^* by L_p', which is its translate by the vector $(n, 0) - v$. The length of L_p' is still ℓ_p, its endpoints lie in H_n and V_n, it is still parallel to L_n^*, and their distance has not decreased. In the same way we can replace L_n^* by a parallel segment L_n' of length ℓ_n whose endpoints lie in H_0 and V_0. Again, the lines of L_p' and L_n' determine a strip, S' say. Observe now that the sum of the lengths of L_p' and L_n' does not change if the strip is moved so that it becomes symmetric with respect to the center of F_0.

We have to consider the cases $p = 0$ and $p = 1$ separately. When $p = 0$, the line containing L_n' has equation $ax + by = d'$ with $d' \leq \frac{a+b-1}{2}n$. The endpoints of L_n' are, similarly to (3), $(\frac{d'}{a}, 0)$ and $(0, \frac{d'}{b})$. Since $0 < a, b < 1$ and $a + b > 1$, we have $0 < \frac{d'}{a}, \frac{d'}{b} \leq \frac{n}{2}$, and then the length of L_n' is at most $\frac{n}{\sqrt{2}}$. The same applies to $L_p' = L_0'$ so $\ell_0 + \ell_n \leq \sqrt{2}n$, and so in equation (2) we have

$$(a+b)h\frac{\ell_0 + \ell_n}{2} + n + 1 \leq 2n + 1.$$

When $p = 1$ the distance between L_1' and L_n' is at least $(n-1)h$. The equation of the line containing L_n' is $ax+by = d'$ with $d' \leq \frac{a+b-1}{2}n + \frac{1}{2}$. A computation similar to the previous one gives that $\ell_1 + \ell_n \leq \sqrt{2}(n+1)$, which implies

$$(a+b)h\frac{\ell_1 + \ell_n}{2} + n \leq 2n + 1.$$

So in both cases we have indeed $m(P) \leq \frac{9}{4}n^2 + 2n + 1$. □

5. Improving the lower bound.

We give an informal description of the improvement. Consider the plane P from Lemma 1; its equation is $x + y + z = n^* + \delta$, where $n^* = 3n/2$ for even n and $(3n-1)/2$ for odd n. Write Z for the set of points (i, j, k) with integer coordinates satisfying $0 \leq i, j, k \leq n-1$. The number of unit cubes hit by P is the number of lattice points

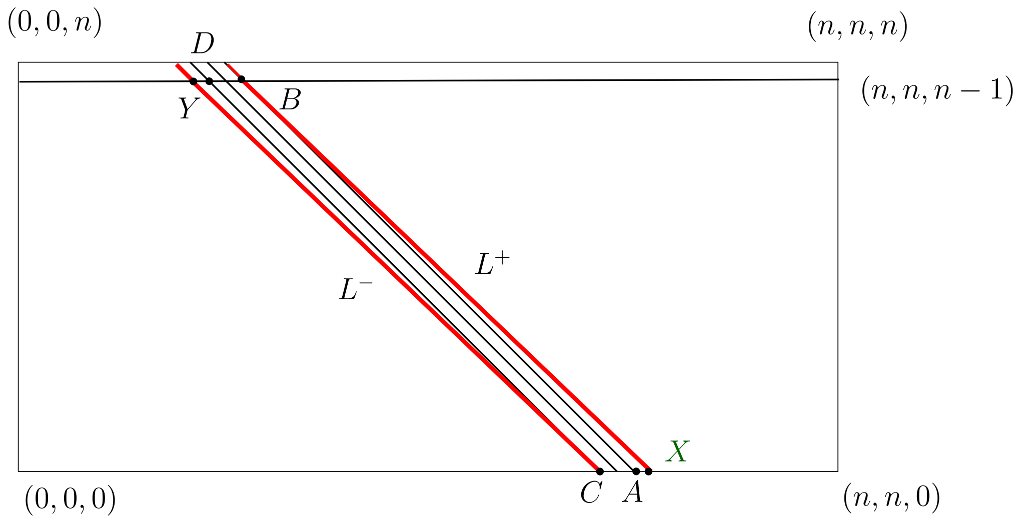

FIGURE 3. Projection in the plane Q.

$(i,j,k) \in Z$ lying on three planes P_0, P_1, P_2 with respective equations $x+y+z=n^*$, $x+y+z=n^*-1$, and $x+y+z=n^*-2$, as explained in the proof of Lemma 1.

Project the whole picture onto the plane Q determined by $(0,0,0)$, $(1,1,0)$, and $(0,0,1)$. These three planes project onto three lines L_0, L_1, L_2 in Q; see Figure 3. We consider the case when n is even and larger than 2. We define four subsets of Z: A and C consist of points $(i,j,0)$ with $i+j=3n/2$ and $i+j=3n/2-2$ respectively, and B and D consist of points $(i,j,n-1)$ with $i+j=n/2+1$ and $i+j=n/2-1$, respectively. Their projections onto the plane Q are marked by A,B,C,D in Figure 3. It is clear that $A,B \subset P_0$ and $C,D \subset P_2$.

We introduce two more sets $X,Y \subset Z$: X is the set of points $(i,j,0)$ with $i+j=3n/2+1$, and Y is the set of points $(i,j,n-1)$ with $i+j=n/2-2$; their projections are marked with the same letters. There is a unique plane P^+ containing B and X, and another one, P^-, containing C and Y. Their projections are line segments on Q denoted by L^+ and L^- in Figure 3. Let S be the set of points in K_n between the planes P^+ and P^-. It is clear, for instance from Figure 3, that every lattice point in $Z \cap (P_0 \cup P_1 \cup P_2)$ lies in S. Moreover, the points of X and Y also lie in S.

Proposition 4. *If $(i,j,k) \in S$, then the point $(i+1,j+1,k+1)$ is outside S, above the plane L^+.*

Proof. It suffices to check this for a single point in P^-, say $(n-1, n/2-1, 0) \in C$. Direct verification that can be carried out even in Figure 3 shows that $(n, n/2, 1)$ is above P^+. $\square$

Let P^* be a plane, parallel and only slightly above P^+. The proposition implies that the unit cube $C(i,j,k)$ intersects P^* properly for every $(i,j,k) \in S \cap Z$. Thus P^* cuts all unit cubes that are cut by P plus the unit cubes from X and Y. Since $|X| = n/2 - 2$ and $|Y| = n/2 - 1$, P^* intersects $9n^2/4 + n - 5$ unit cubes.

The case of odd n is similar and is left to the reader. Together we have established the following result.

Theorem 2. *If $n \geq 4$ is even then $m(n) \geq \frac{9}{4}n^2 + n - 5$, and if $n \geq 5$ is odd, then $m(n) \geq \frac{9}{4}n^2 + n - \frac{17}{4}$.* □

In a formal proof, the equation of P^* is $x + y + \frac{n}{n-1}z = n^* + 1 + \varepsilon$ with $0 < \varepsilon < \frac{1}{n+1}$ and the argument just amounts to checking that P^* hits $C(i,j,k)$ for every $(i,j,k) \in Z \cap S$ and for every $(i,j,k) \in X \cup Y$.

The lower bound on $m(n)$ can be further improved by 2 (again for large enough n). To see this, note that the planes P^+ and P^- separate the lattice points in S from the lattice points in $K_n \setminus S$. One can slightly tilt these two planes (while keeping them parallel) so that a new lattice point from $K_n \setminus S$ appears on both of them. We invite the reader to check that this works. Note that the projection of these new planes onto Q is no longer a line.

6. A different approach.

Here we show that $m(2) = 7$, $m(3) = 19$, and $m(4) = 35$. Lemma 1 implies that $m(2) \geq 7$ and $m(3) \geq 19$, while $m(4) \geq 35$ follows from the improved lower bound in Section 5. So we only need to prove the upper bounds.

We assume again that the equation of P is $ax + by + cz = d$ and $0 < a, b, c$ (we do not use $a < b < c = 1$ here). For the upper bound we need the following.

Lemma 5. *If $i < i'$, $j < j'$, $k < k'$, then P may cut at most one of the cubes $C(i,j,k)$ and $C(i',j',k')$.*

Proof. Recall that if $(x,y,z) \in C(i,j,k)$ and $(x',y',z') \in C(i',j',k')$, then $x < x'$, $y < y'$, $z < z'$. Thus $ax+by+cz < ax'+by'+cz'$, showing that both points could not simultaneously lie in P. □

Next partition the n^3 unit cubes in K_n into groups. If $0 \leq i,j,k < n$ and $ijk = 0$, then we form the group starting with $C(i,j,k)$. Set $t = \min\{n-1-i, n-1-j, n-1-k\}$ and define the corresponding group

$$\mathcal{G}(i,j,k) = \{C(i,j,k), C(i+1,j+1,k+1), \ldots, C(i+t,j+t,k+t\}.$$

Note that $|\mathcal{G}(i,j,k)| = 1$ if and only if i, j, or k is equal to $n-1$. It is also clear that the number of groups is $n^3 - (n-1)^3$. For an arbitrary (i,j,k) with $0 \leq i,j,k < n$, we have $C(i,j,k) \in \mathcal{G}(i-s, j-s, k-s)$ for

$s = \min\{i, j, k\}$. Consequently these groups cover (actually partition) all n^3 small cubes. Lemma 5 shows that P cuts at most one small cube from each group, so we have the following upper bound.

Lemma 6. *The plane cuts at most $3n^2 - 3n + 1$ small cubes in K_n. Thus $m(n) \leq 3n^2 - 3n + 1$.* □

Note that this follows from Lemma 3 as well: $r(W) \leq n^2$ and each $r_i \leq 2n-1$, so $m(n) \leq n^2 + (n-1)(2n-1) = 3n^2 - 3n + 1$. For $n = 2$ and 3 this bound gives $m(2) \leq 7$ and $m(3) \leq 19$. In the case $n = 4$, $3n^2 - 3n + 1 = 37$. We show next that $35 = 37 - 2$ is the tight upper bound.

Lemma 7. *We have $m(4) \leq 35$.*

Proof. We consider the following six groups, each consisting of a single small cube:

$$\mathcal{G}(3,3,0),\ \mathcal{G}(3,0,3),\ \mathcal{G}(0,3,3),\ \mathcal{G}(3,0,0),\ \mathcal{G}(0,3,0),\ \mathcal{G}(0,0,3).$$

Should P cut 36 or more small cubes, it must cut at least five small cubes in the six groups above. By symmetry we can assume that it cuts the first five of them. Then

$$d > 3a+3b,\ d > 3a+3c,\ d > 3b+3c \text{ and } d < 4a+b+c,\ d < a+4b+c.$$

Adding the last two inequalities gives

$$2d < 5a + 5b + 2c. \tag{5}$$

The sum of the first three inequalities is $3d > 6a + 6b + 6c$; adding to this one the first multiplied by 3 gives that $6d > 15a + 15b + 6c$ which contradicts (5). □

We part with the reader by offering a new question. How many lines are needed in order to cut all unit squares of the $n \times n$ chessboard? The example of n suitably chosen horizontal lines show that n lines suffice. As no line can cut more than $2n - 1$ squares, $n/2$ cannot suffice. We can show a little more, namely, that one needs at least $\frac{23}{45}n$ lines. The truth is probably n.

Acknowledgements. The first author was supported by Hungarian National Research Grants No 131529, 131696, and KKP-133819.

References

[1] Aliev, I. (2020). On the volume of hypercube sections. To appear in *Acta Mathematica Hungarica.* arXiv:2004.00873

[2] Ball, K. M. (1986). Cube slicing in $\mathbb{R}^n$. *Proc. Amer. Math. Soc.*, 97(3): 465–473.

[3] Bárány, I., Frankl, P. (2020). Cells in the box and a hyperplane. Submitted. arXiv:2004.12306

[4] Beck, J. (1983). On the lattice property of the plane and some problems of Dirac, Motzkin and Erdős in combinatorial geometry. *Combinatorica*. 3(3-4): 281–297.

Imre Bárány
Rényi Institute of Mathematics,
13-15 Reáltanoda Street, Budapest, 1053 Hungary
barany.imre@renyi.hu
and
Department of Mathematics
University College London
Gower Street, London, WC1E 6BT, UK

Péter Frankl
Rényi Institute of Mathematics,
13-15 Reáltanoda Street, Budapest, 1053 Hungary
peter.frankl@gmail.com