Type-II singularities and long-time convergence of rotationally symmetric Ricci flows

Francesco Di Giovanni

Principal Supervisor: Jason Lotay

Subsidiary Supervisor: Lars Louder

A dissertation submitted in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

of

University College London.

Department of Mathematics

University College London

I, Francesco Di Giovanni, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

Abstract

In this thesis we study singularity formation and long-time behaviour of families of cohomogeneity one Ricci flows.

In Chapter 1 we analyse the Ricci flow on $\mathbb{R}^{n+1}$, with $n \geq 2$, starting at some complete bounded curvature $SO(n+1)$-invariant metric g_0. We prove that the solution develops a Type-II singularity and converges to the Bryant soliton after scaling if g_0 has no minimal hyperspheres and is asymptotic to a cylinder. This proves a conjecture by Chow and Tian about Perelman's standard solutions. Conversely, we show that if g_0 has no minimal hyperspheres but its curvature decays at infinity, then the solution is immortal.

In Chapter 2 we study the Ricci flow on $\mathbb{R}^4$ starting at an $SU(2)$-cohomogeneity one metric g_0 whose restriction to any hypersphere is a Berger metric. We prove that if g_0 has no necks and is bounded by a cylinder, then the solution develops a global Type-II singularity and converges to the Bryant soliton when suitably dilated at the origin. This is the first example in dimension $n > 3$ of a *non*-$SO(n)$-invariant Type-II flow converging to an $SO(n)$-invariant singularity model. We also give conditions for the flow to be immortal and prove that if the solution is Type-I and controlled at spatial infinity, then there exist minimal 3-spheres for times close to the maximal time.

In Chapter 3 we focus the analysis on the class of immortal Ricci flows derived in Chapter 2. We prove that if the initial metric has bounded Hopf-fiber, curvature controlled by the size of the orbits and opens faster than a paraboloid in the directions orthogonal to the Hopf-fiber, then the flow converges to the Taub-NUT metric in the Cheeger-Gromov sense in infinite time. We also obtain a uniqueness result for Taub-NUT in a class of collapsed ancient solutions.

Acknowledgements

I wish to thank my supervisor Jason Lotay for taking me up as a PhD student, for suggesting an interesting direction of research and for the constant support and encouragement. For being an incredible listener, for his kindness, and for his guidance which helped me enormously when taking difficult decisions. I owe much of my personal and professional development to you.

I thank Michael Bronstein for believing in my ideas, for always offering new and innovative ways to think about problems, for the many inspiring conversations and for his contagious curiosity. I am very looking forward to continuing working with you in the future.

Thanks to all my PhD friends which helped me cope with the two main enemies of being a TA at UCL: demonstrating activities and political correctness. Thanks to Thomas and Udhav for the nonsense and the laughs, the seminars and researching in geometry in general have been much more enjoyable because of you. Special thanks to Johnny and Carmen for letting me be the Harry Potter of the gang, surely one of the highlights of my PhD. Thanks to the amazing football group as well!

Thanks to the friends of the Principato. It is always a pleasure going back to Italy to discuss our secession plans: I hope that this thesis is going to help us somehow to get that nuclear weapon deterrent we so desperately need. Special thanks to Giacomo for being my favourite sore looser at Munchkin, to our Sire for guiding us in the darkness, to Simone for showing us that the bourgeoisie is a state of mind, to Rux for being the mastermind behind many of our expansion plans, and to Cir for being a constant life inspiration. Most importantly, thanks to Rossana for making clear each time that I am her favourite.

Thanks to the survivors Stefano, Carla and Federica for being witnesses of the dark ages.

Special thanks to Stefano, for all our chats throughout these years, I have always felt I could talk to you about any subject, from by PhD ups and downs, to founding a new religion based on our high-school teacher being a deity.

Thanks to my grandmother for being that cool person I would never be able to become, and for my grandfather for all the politics and football rants.

I want to thank my parents: my mother for trying without success to lead me to the path of salvation, and my father for trying without success to lead me to the path of temptations. Thank you for your love and for believing in me each day.

Thanks to my amazing sister who seems to struggle to acknowledge my resemblance to Charles Leclerc and who has been a precious source of memes for so many years.

Thanks to Sir Pito, a wonderful companion.

Finally, Giulia: thank you for being the one I will never have words to thank you with.

Contents

Overview and main results

Motivations

In light of Klein's Erlangen program, it is only natural that the study of symmetries has played a fundamental role in geometry and hence in physics. According to this approach, homogeneous spaces occupy a special position in the class of Riemannian manifolds since their isometry groups act transitively. This rich structure generally comes with an intrinsic rigidity, which motivates the need to consider more flexible geometries. In this regard, manifolds admitting isometry groups whose associated orbit spaces are 1-dimensional - known as *cohomogeneity one manifolds* - have often proved to both enjoy the algebraic tractability typical of homogeneous spaces and exhibit a more varied spectrum of geometric phenomena. Although summarizing the role of cohomogeneity one manifolds is beyond the purpose of this section, we mention a few highlights before focusing on the Ricci flow aspect of the problem.

Page [1978] found the first example of an inhomogeneous Einstein metric with positive scalar curvature by considering cohomogeneity one metrics on $\mathbb{CP}^2\sharp\overline{\mathbb{CP}}^2$. Many other examples of cohomogeneity one Einstein manifolds with positive scalar curvature were found afterwards - for instance Sakane [1986], Page and Pope [1987] - and we emphasize the class discovered by Böhm [1998] on low-dimensional spheres. In a slightly different direction, Grove and Ziller [2002] showed that any cohomogeneity one manifold admits a complete invariant metric with non-negative Ricci curvature.

Cohomogeneity one manifolds have also played a crucial role in the analysis of geometries with special holonomy. Gibbons and Pope [1979] initially considered the problem of finding cohomogeneity one hyperkähler 4-manifolds, which was ultimately solved by Atiyah and Hitchin [1985]. The first examples of complete G_2–metrics were found by Bryant and Salamon [1989], who constructed a 1–parameter family of cohomogeneity one complete metrics on the total space of a vector bundle over S^3, S^4 and $\mathbb{CP}^2$ respectively. Foscolo and Haskins [2017] later discovered the first complete inhomogeneous nearly Kähler 6–manifolds by showing that S^6 and $S^3 \times S^3$ admit cohomogeneity one nearly Kähler structures.

Symmetries have contributed significantly to the understanding of Hamilton's Ricci flow as well. Since the Ricci flow is invariant under diffeomorphism, whenever the problem admits a unique solution in some class, it follows that symmetries of the input data persist along the solution. Accordingly, homogeneous and cohomogeneity one Ricci flows have been analysed by several authors. In the homogeneous setting we now have a clear picture of both finite-time singularity formation and long-time behaviour Lauret [2013], Lafuente [2015], Böhm [2015], Böhm et al. [2017], Böhm and Lafuente [2018]. In particular, any finite-time singularity of a homogeneous Ricci flow is Type-I and the Riemann curvature and scalar curvature are comparable at the maximal time, in a precise way. Similarly, immortal homogeneous Ricci flows are always Type-III and their blow-down is fairly understood. Therefore homogeneous Ricci flows are never Type-II and exhibit a strong rigidity. The main goal of this thesis consists in providing evidence that for cohomogeneity one Ricci flows one should instead expect Type-II singularities to develop relatively easily - in the non-compact setting - both in the *finite* and *infinite* time case.

The special family of rotationally symmetric manifolds has been particularly studied in the class of cohomogeneity one spaces - in fact, Brendle [2020] showed that any 3-dimensional non-compact κ-solution is rotationally symmetric. The classical analysis of Type-I singularities modelled on neckpinches was performed by Angenent and Knopf [2004] for spherically symmetric Ricci flows on S^n. Moreover, until recently, the only known examples of Type-II singularities in dimension $n \geq 3$ were $SO(n)$-invariant Gu and Zhu [2008], Angenent et al. [2015], Wu [2014]. However, such Type-II Ricci flows need to start at initial metrics satisfying very precise asymptotics. In Chapter 1 we show that on $\mathbb{R}^{n+1}$ the maximal complete, bounded curvature Ricci flow evolving from an

$SO(n+1)$-invariant metric *without* minimal hyperspheres always encounters a finite-time Type-II singularity modelled by the Bryant soliton whenever the profile function of the initial metric is bounded (cylindrical asymptotics) Di Giovanni [2021b].

In the last few years the Ricci flow problem has been analysed on different families of 4-dimensional cohomogeneity one manifolds that are not (necessarily) $SO(4)$-invariant Bettiol and Krishnan [2016], Isenberg et al. [2016, 2019], Appleton [2019], Di Giovanni [2020, 2021a]. In particular, Isenberg et al. [2016, 2019] studied the Ricci flow problem on families of $SU(2)$-invariant metrics on $S^1 \times S^3$ and $S^2 \tilde{\times} S^2$ whose restrictions to any principal orbit are Berger metrics and constructed examples of enhancement of the symmetries around a Type-I singularity. Appleton [2019] considered the Ricci flow in the same symmetry class on the blow-up of $\mathbb{C}^2/\mathbb{Z}_k$ at the origin. When $k = 2$, they showed that, under additional assumptions on the warping coefficients, the Ricci flow develops a finite-time Type-II singularity modelled on the ALE gravitational instanton given by the Eguchi-Hanson metric. Independently, we studied the Ricci flow in the same class of $SU(2)$-invariant metrics on the $\mathbb{R}^4$-topology and we obtained Type-II singularity formation by deriving the first example of symmetry enhancement around a Type-II singularity in dimension $n > 3$. We also characterized under which assumptions the Ricci flow is immortal Di Giovanni [2020] (Chapter 2). We emphasize that the underlying topology plays a role via the boundary conditions entering the Ricci flow analysis.

In Chapter 3 Di Giovanni [2021a] we prove that a large class of the immortal solutions found in Chapter 2 develop a Type-II(b) singularity at infinite-time by converging to the ALF gravitational instanton given by the Taub-NUT metric (in the pointed Cheeger-Gromov sense *without* rescaling). This is the first instance in dimension $n > 3$ of infinite-time convergence to a Ricci-flat non-flat singularity model starting from metrics that need not be close(asymptotic) to the limit one. As a by-product we derived a rigidity result for a family of 4-dimensional ancient solutions that are strongly collapsed and are not $SO(4)$-invariant.

Summary

This thesis is based on the following three articles Di Giovanni [2021b, 2020, 2021a]:

1. *Rotationally symmetric Ricci flow on* $\mathbb{R}^{n+1}$, Adv. Math. 381, 2021.

2. *Ricci flow of warped Berger metrics on* $\mathbb{R}^4$, Calc. Var. 59(162), 2020.

3. *Convergence of Ricci flow solutions to Taub-NUT*, Commun. Partial. Differ. Equ., 2021.

We first briefly review the theory of Ricci flow with an eye on the necessary background and context to present the main results in each chapter.

The Ricci flow

Assume that (M, g) is an n-dimensional Riemannian manifold with Levi-Civita connection ∇. The *Riemann curvature* of g measures the noncommutativity of second derivatives and is defined by

$$\mathrm{Rm}_g(X, Y, W, Z) := g(\nabla_Y \nabla_X W - \nabla_X \nabla_Y W + \nabla_{[X,Y]} W, Z),$$

for any vector field X, Y, W, Z on M. Given a local orthonormal frame $\{e_1, \ldots, e_n\}$, the Ricci tensor can be derived from the Riemann tensor by tracing as

$$\mathrm{Ric}_g(X, Y) := \sum_{i=1}^{n} \mathrm{Rm}_g(X, e_i, Y, e_i).$$

When the metric is clear from the context we usually simply write Rm and Ric for the Riemann and Ricci curvature respectively.
Since in harmonic local coordinates the Ricci tensor satisfies

$$\mathrm{Ric}_{ij} = -\frac{1}{2}\Delta(g_{ij}) + \text{lower order terms},$$

it is tempting to think of the Ricci curvature as an intrinsic nonlinear Laplacian of the metric. In the seminal paper Hamilton [1982], Hamilton introduced the Ricci flow as a geometric evolution equation in which an input Riemannian structure (M, g_0) is evolved according to

$$\partial_t g = -2\mathrm{Ric}_{g(t)}.$$

In fact, the idea of deforming geometric quantities had also appeared in the harmonic heat flow studied by Eells and Sampson [1964] and in the mean curvature flow originally introduced by Brakke [1978].

While in principle alternative ways of evolving metrics could be investigated, the formulation in Hamilton [1982] had two main advantages. First, in line with the local interpretation of the Ricci curvature as a nonlinear Laplacian of the metric, the Ricci flow represents the counterpart to the heat equation at the level of Riemannian structures. Second, the Ricci flow equation is well-behaved from a PDE-point of view - a property Hamilton exploited in their seminal work - although a gradient flow formulation was only found some two decades later by Perelman [2002].

Hamilton [1982] showed that closed, simply-connected 3-manifolds with positive Ricci curvature admit metrics of positive constant curvature and are hence equivalent to 3-spheres in the category of smooth manifolds. This result set the foundations for a program to study the topology of manifolds via deforming Riemannian metrics using the Ricci flow that culminated in Perelman's celebrated solution of the Poincaré conjecture Perelman [2002, 2003a,b]. Other important achievements of the Ricci flow theory are given by the proof of the differentiable sphere theorem by Brendle and Schoen [2009] and the solution of the generalized Smale Conjecture by Bamler and Kleiner [2019].

Below we provide a *brief* overview of some important notions and results in the Ricci flow literature that play a central role in the analysis we present in the main chapters. We refer to Chow and Knopf [2004], Topping [2006], Chow et al. [2007, 2008] and the references therein for a thorough survey on the Ricci flow theory and its properties.

Existence theory

In the case of compact manifolds, the Ricci flow problem admits short-time existence and uniqueness Hamilton [1982], DeTurck [1983]. In the more general setting of non-compact manifolds, short-time existence was achieved by Shi [1989], while Chen and Zhu [2006b] showed that the solution is unique in the class of complete, bounded curvature solutions. Since throughout this thesis we consider Ricci flows on non-compact manifolds, we state the following:

Theorem 0.1 (Shi [1989], Chen and Zhu [2006b]). *Let M be a non-compact manifold equipped with a complete, bounded curvature metric g_0. There exists a unique maximal solution to the Ricci flow starting at g_0 in the class of complete, bounded curvature metrics on M.*

According to the previous result and Shi [1989], given (M, g_0) as in the statement, there exists a unique *maximal* Ricci flow solution $(M, g(t))$ evolving from g_0, with $g(t)$ a complete and bounded curvature metric for any $t \in [0, T)$, where $T < \infty$ if and only if

$$\limsup_{t \nearrow T} \sup_M |\mathrm{Rm}_{g(t)}|_{g(t)} = \infty.$$

We say that the Ricci flow $(M, g(t))$ develops a (finite-time) *singularity* when $T < \infty$, otherwise we call the flow *immortal*.

In the main chapters we are always interested in complete, bounded curvature (maximal) Ricci flow solutions on non-compact manifolds so that the existence theory and the characterization of finite-time singularities apply.

Although this is not related to our work, we mention that extensions to the existence and uniqueness theorem stated above have been investigated with remarkable results in the case of *surfaces* Giesen and Topping [2013], Topping [2015].

Ricci solitons

A *gradient Ricci soliton* is a Riemannian manifold (M, g_0) such that there exist $\lambda \in \mathbb{R}$ and $f : M \to \mathbb{R}$ smooth satisfying

$$\text{Hess}_{g_0} f + \text{Ric}_{g_0} + \frac{\lambda}{2} g_0 = 0.$$

Gradient Ricci solitons give rise to *self-similar solutions* to the Ricci flow (we report the statement in Chow et al. [2007]).

Proposition 0.2. *Given a complete gradient Ricci soliton (M, g_0, f_0, λ), there exists a solution $g(t)$ of the Ricci flow starting at g_0, diffeomorphisms $\varphi(t)$ with $\varphi(0) = id_M$ and functions $f(t)$ with $f(0) = f_0$ defined for all t with $\tau(t) := 1 + \lambda t > 0$, such that*

- $\partial_t \varphi(t)(x) = (\tau(t))^{-1}(\nabla_{g_0} f_0)(\varphi(t)(x))$,
- $f(t) = \varphi(t)^* f_0$,
- $g(t) = \tau(t)\varphi^*(t) g_0$.

Definition 0.1. *We say that the soliton is in canonical form when $\lambda = -1, 0, +1$, which corresponds to the* shrinking, steady, expanding *case respectively.*

As we will see below, Ricci solitons are fundamental objects in the analysis of Ricci flow singularities and constitute fixed points of Perelman's entropy functionals. In particular, shrinking and steady solitons are instances of ancient and eternal Ricci flow solutions respectively.

Definition 0.2. *A Ricci flow solution $(M, g(t))$ existing smoothly in the interval $(-\infty, \omega)$, with $\omega \geq 0$, is called ancient. If $\omega = \infty$, the solution is referred to as eternal.*

Examples. Two solitons will frequently appear in the classification of singularity models in the chapters below.

- The shrinking round soliton on the cylinder $\mathbb{R} \times S^n$, which exists for any negative time and extinguishes at some positive *finite* time.

- The Bryant soliton $(\mathbb{R}^n, g_{\text{Bry}}, f_{\text{Bry}})$ which represents, up to homothety, the only complete $SO(n)$-invariant gradient steady soliton on $\mathbb{R}^n$ Bryant [2005].

Blow-up analysis of singularities

One of the key tool developed by Perelman is the no-local collapsing theorem. We first recall the notion of collapsedness as defined in Chow et al. [2007].

Definition 0.3. *Let $\kappa > 0$. A solution to the Ricci flow $(M^n, g(t))_{0\le t<T}$ is said to be strongly κ-collapsed at $(p_0, t_0) \in M^n \times (0, T)$ at scale $\rho > 0$ if*

(i) $|\text{Rm}_{g(t)}(p)|_{g(t)} \le \rho^{-2}$ *for all $p \in B_{g(t_0)}(p_0, \rho)$ and $t \in [\max\{t_0 - \rho^2, 0\}, t_0]$, and*

(ii) $vol_{g(t_0)} B_{g(t_0)}(p_0, \rho) < \kappa \rho^n$.

If instead given $\rho > 0$, for any $t_0 \in [\rho^2, T)$ and any $p_0 \in M^n$ the solution $g(t)$ is not strongly κ-collapsed at (p_0, t_0) at scale ρ, then we say that $(M^n, g(t))$ is (weakly) κ-non-collapsed at scale ρ.

We now state the no-local collapsing theorem as adapted to the non-compact setting in Chow et al. [2007].

Theorem 0.3 (Perelman's no-local collapsing)**.** *Let $(M^n, g(t))_{0\le t<T}$ be a maximal complete Ricci flow solution with $T < \infty$. Assume that*

(i) $\sup_{M\times[0,t_1]}|\text{Rm}| < \infty$ *for any $t_1 < T$, and*

(ii) there exist $r_1 > 0$ and $v_1 > 0$ such that $vol_{g_0} B_{g_0}(p, r_1) \ge v_1$ for all $p \in M$.

Then there exists κ depending on $r_1, v_1, n, T, \sup_{M\times[0,T/2]}|\text{Ric}|$ such that $(M, g(t))$ is weakly κ-non-collapsed at any point $(p_0, t_0) \in M\times(T/2, T)$ and at any scale $\rho < \sqrt{T/2}$.

The importance of the no-local collapsing of the Ricci flow derives from the Cheeger-Gromov-Hamilton's compactness theorem. We refer to Chow et al. [2007] for the definitions of pointed Cheeger-Gromov convergence both in the static case and in the dynamic setting of Ricci flow solutions. Below we let $\text{inj}_g(p)$ denote the injectivity radius of some metric g evaluated at p.

Theorem 0.4 (Hamilton's compactness of Ricci flows)**.** *Let* $(M_j^n, g_j(t), p_j)_{j\in\mathbb{N}}$*, with* $t \in (\alpha, \omega) \ni 0$*, be a sequence of complete pointed Ricci flow solutions. Assume that*

(i) There exists $C > 0$ *such that* $\sup_j \sup_{M_j \times (\alpha,\omega)} |\mathrm{Rm}_j|_j < C$ *and*

(ii) $inj_{g_j(0)}(p_j) \geq \delta > 0$ *for all* j*.*

Then there exists a subsequence $\{j_k\}$ *such that* $(M_{j_k}, g_{j_k}(t), p_{j_k})$ *converges smoothly in the pointed Cheeger-Gromov sense to a complete Ricci flow solution* $(M_\infty, g_\infty(t), p_\infty)$ *defined in* $M_\infty \times (\alpha, \omega)$*.*

Given a maximal Ricci flow solution $(M, g(t))_{0\leq t<T}$, assume that $T < \infty$. Then the curvature does not stay bounded on the manifold as time approaches T. To better understand the nature and type of singularity, one can consider a sequence of space-time points $(p_j, t_j) \in M \times (0, T)$, with $t_j \nearrow T$, such that $\lambda_j := |\mathrm{Rm}|(p_j, t_j) \to \infty$ and study the sequence of *parabolically rescaled* Ricci flows $(M, g_j(t), p_j)$ defined by $g_j(t) = \lambda_j g(t_j + t(\lambda_j)^{-1})$ on $[-\lambda_j t_j, 0]$. In fact, one can pick the sequence in a way that the condition (i) in Hamilton's compactness theorem holds in some time interval I_j, with (I_j) exhausting $(-\infty, 0]$ Hamilton [1995]. Whenever the assumptions in Perelman's no-local collapsing theorem are satisfied, one can extract a uniform lower bound on the injectivity radii as in (ii) of Hamilton's compactness theorem and hence conclude via a diagonal argument that $(M, g_j(t), p_j)$ converges to an ancient solution $(M_\infty, g_\infty(t), p_\infty)_{t\leq 0}$ in the pointed Cheeger-Gromov sense. Any ancient solution arising as the blow-up limit of Ricci flow solutions is called a *singularity model*. By identifying and classifying singularity models one can understand the formation of singularities and the behaviour of the flow for times close to the maximal time.

From the rescaling conditions we derive that any singularity model of a finite-time singular Ricci flow is κ-non-collapsed at any scale, for some $\kappa > 0$. In particular, we have the following

Definition 0.4. *A complete, bounded curvature ancient solution to the Ricci flow which is* κ*-non-collapsed at all scales, non-flat and has nonnegative curvature is called a* κ*-solution.*

The class of κ-solutions play a crucial role in the understanding of singularity formation. Namely, Perelman proved that any 3-dimensional Ricci flow encountering a finite-time singularity is modeled on a κ-solution Perelman [2002]. Recently, Brendle [2020] showed that any non-compact κ-solution in dimension 3 must be $SO(3)$-invariant, meaning that it is either a family of shrinking cylinders or isometric to the Bryant soliton.

Classification of finite-time singularities

A (complete, bounded curvature) Ricci flow solution $(M, g(t))_{0\leq t<T}$ whose maximal time of existence T is finite can be classified as follows Hamilton [1995]:

$$\begin{aligned} \textit{Type-I}: &\quad \sup_{M\times[0,T)} |\mathrm{Rm}_{g(t)}|_{g(t)}(T-t) < \infty, \\ \textit{Type-II}: &\quad \sup_{M\times[0,T)} |\mathrm{Rm}_{g(t)}|_{g(t)}(T-t) = \infty. \end{aligned}$$

Until recently, the main examples of finite-time Ricci flow singularities were of Type-I Hamilton [1982], Angenent and Knopf [2004]. In fact, any shrinking gradient soliton encounters a Type-I singularity at some positive time. Such singularity formation is now well understood thanks to results of Naber [2010] and Enders-Müller-Topping Enders et al. [2011]. Since their characterization of Type-I singularities plays an important role in ruling out formation of neckpinches in Chapters 1 and 2, we report one of the main results in Enders et al. [2011]:

Theorem 0.5 (Naber [2010], Enders et al. [2011])**.** *Let $(M, g(t))$ be a Type-I Ricci flow in $[0,T)$ and let $p \in M$ be such that there is no open set $U \ni p$ with $\sup_{t\in[0,T)} \sup_U |\mathrm{Rm}_{g(t)}| < \infty$. Then for any sequence $\lambda_j \to \infty$, the rescaled Ricci flows $(M, g_j(t), p)$ defined on $[-\lambda_j T, 0)$ by $g_j(t) = \lambda_j g(T + t(\lambda_j)^{-1})$ subconverge to a (normalized) non-flat gradient shrinking soliton in canonical form.*

On the other hand, far less is known about Type-II singularities. In general, for a Type-II Ricci flow satisfying the assumptions of Perelman's no-local collapsing theorem one can pick a blow-up sequence whose associated singularity model is an eternal solution Hamilton [1995]. In terms of structural properties, we have the following:

Theorem 0.6 (Hamilton [1993a]). *An eternal solution $(M, g(t))_{t\in\mathbb{R}}$ to the Ricci flow with nonnegative curvature, positive Ricci curvature and scalar curvature attaining its supremum at some point in space and time, is a gradient steady soliton.*

In general, in dimension larger than three one has to verify directly that an eternal singularity model satisfies the conditions above. We will review below existing examples of Type-II singularities and present new classes of Type-II Ricci flows.

Classification of infinite-time singularities

The behaviour of a solution existing for all positive times has been classified as follows, similarly to the finite-time case Hamilton [1995]:

$$\begin{aligned}
\textit{Type-II(b)}: \quad & \limsup_{t\nearrow\infty}\left(\sup_M \, t|\mathrm{Rm}_{g(t)}|_{g(t)}\right) = \infty, \\
\textit{Type-III}: \quad & \limsup_{t\nearrow\infty}\left(\sup_M \, t|\mathrm{Rm}_{g(t)}|_{g(t)}\right) < \infty.
\end{aligned}$$

Several instances of Type-III singularities for the Ricci flow have been discovered, both for compact manifolds Lott and Šešum [2014] and non-compact ones Oliynyk and Woolgar [2007]. In fact, some general classification results are available: Bamler [2018] proved that any closed 3-dimensional immortal Ricci flow encounters a Type-III singularity, while Böhm [2015] showed that the same conclusion applies to any immortal *homogeneous* Ricci flow.

Conversely, few examples of TypeII(b)-Ricci flows have been found. We will present them below when discussing the main results of Chapter 3.

Main results

In Chapter 1 we analyse singularity formation for $SO(n+1)$-invariant Ricci flows on $\mathbb{R}^{n+1}$. The highlight consists in proving that a large class of $SO(n+1)$-invariant Ricci flows develop a Type-II singularity modelled on the Bryant soliton. This proves a conjecture by Chow and Tian about Perelman's standard solutions.

Partly motivated by a numerical investigation in Holzegel et al. [2007] and the rigorous analysis in Isenberg et al. [2016], in Chapter 2 we study the Ricci flow on a class of

$SU(2)$-invariant metrics on $\mathbb{R}^4$ whose restriction to any Euclidean hypersphere is a Berger metric. We characterize the formation of finite-time Type-II singularities and, similarly to the Type-I closed setting analysed in Isenberg et al. [2016], we show that the flow acquires further symmetries around a singularity hence obtaining the first example in dimension $n > 3$ of a *non-*$SO(n)$-invariant Type-II flow with an $SO(n)$-invariant singularity model (Bryant soliton). We also provide conditions for the Ricci flow solution to be immortal.

Accordingly, in Chapter 3 we focus on a class of such immortal flows and show that under optimal assumptions these solutions converge to the ALF gravitational instanton given by the Taub-NUT metric in the pointed Cheeger-Gromov sense. As a by-product, we obtain a uniqueness result for Taub-NUT in a class of strongly-collapsed ancient solutions. We now report the main results in each chapter.

Convention. Unless otherwise stated, from now on we refer to an $SO(n)$-invariant n-manifold as *rotationally (spherically) symmetric*.

Chapter 1

The first examples of Type-II singularities in dimension $n \geq 3$ were found by Gu and Zhu [2008], who studied a family of rotationally symmetric metrics on S^{n+1}. Later Angenent, Isenberg and Knopf proved that on S^{n+1} there exist spherically symmetric Ricci flows behaving as degenerate neckpinches Angenent et al. [2015]. Wu [2014] found rotationally symmetric solutions on $\mathbb{R}^{n+1}$ which encounter a Type-II singularity and converge to the Bryant soliton under a strong control on the profile function.

We show that a large class of rotationally symmetric Ricci flows on $\mathbb{R}^{n+1}$ develop a finite time Type-II singularity modelled on the Bryant soliton Bryant [2005].

Theorem 0.7. *Let $(\mathbb{R}^{n+1}, g(t))_{0\leq t<T}$, with $n \geq 2$, be the complete, bounded curvature Ricci flow solution evolving from a rotationally symmetric metric g_0. If $(\mathbb{R}^{n+1}, g_0)$ does not contain minimal hyperspheres and g_0 is asymptotic in C^0 to a round cylinder at infinity, then the solution develops a Type-II singularity at $T < \infty$ and converges to the Bryant soliton in the Cheeger-Gromov sense once suitably rescaled.*

We note that the lack of minimal hyperspheres means that the profile function of the rotationally symmetric metric g_0 is monotone increasing, so that the condition of being asymptotic to a round cylinder is equivalent to the profile function of g_0 being bounded.

As a consequence of the result above, we are able to give an affirmative answer to a conjecture by Chow and Tian [Wu, 2014, Conjecture 1.2] about Perelman's standard solutions. A 3-dimensional version of the conjecture was proved by Ding [2009].

We also study the behaviour of the solution when instead of cylindrical asymptotics we assume a curvature decay at spatial infinity. We recall that Oliynyk and Woolgar [2007] proved that on $\mathbb{R}^{n+1}$ if g_0 is rotationally symmetric, *asymptotically flat* and has no minimal embedded hyperspheres, then the Ricci flow solution starting at g_0 is immortal. We extend this result to initial data that may not be close to the Euclidean metric outside some region.

Theorem 0.8. *Let* $(\mathbb{R}^{n+1}, g(t))_{0\leq t<T}$*, with* $n \geq 2$*, be the complete, bounded curvature Ricci flow solution evolving from a rotationally symmetric metric* g_0 *with curvature decaying at infinity. If* $(\mathbb{R}^{n+1}, g_0)$ *does not contain minimal hyperspheres, then the solution is immortal.*

We finally also relate the formation of Type-I singularities to the existence of sufficiently pinched necks (minimal hyperspheres). We report the result *informally* and we refer to Chapter 1 for the complete and precise statement.

Theorem 0.9. *Let* $(\mathbb{R}^{n+1}, g(t))_{0\leq t<T}$*, with* $n \geq 2$*, be the complete, bounded curvature Ricci flow solution evolving from an asymptotically flat rotationally symmetric metric* g_0 *containing a neck. Assume that* $\mathrm{Ric}_{g_0} > 0$ *is positive around the origin and that* $R_{g_0} \geq 0$ *on* $\mathbb{R}^{n+1}$*. If the neck is sufficiently pinched, in a precise way, then the solution develops a Type-I singularity which is modelled on a family of shrinking cylinders.*

Chapter 2

Only recently explicit examples of *non*-rotationally symmetric Type-II Ricci flows in dimension $n > 3$ have been found Appleton [2019], Stolarski [2019], Di Giovanni [2021b].

We briefly review the family of metrics used as initial data for the flow. Any complete metric g which is both invariant under the cohomogeneity one left-action of $SU(2)$ on $\mathbb{R}^4$

and under rotations of the Hopf-fibres can be diagonalized with respect to a fixed Milnor frame and hence be written, away from the origin, as:

$$g = ds^2 + b^2(s)\,\pi^* g_{S^2(\frac{1}{2})} + c^2(s)\,\sigma_3 \otimes \sigma_3,$$

where s is the g-distance from the origin, $\pi^* g_{S^2(\frac{1}{2})}$ is the pull-back of the Fubini-Study metric under the Hopf map, and σ_3 is the one-form dual to the vector field tangent to the Hopf-fibres. In line with Isenberg et al. [2016] we also assume that $c \leq b$ so that each non-degenerate fiber $\{s\} \times S^3$ is a Berger sphere and we call any such metric *warped Berger*.

Definition 0.5. *We let $\mathcal{G}$ be the set of complete, bounded curvature warped Berger metrics g_0 on $\mathbb{R}^4$ satisfying the following conditions:*

(i) $b_s \geq 0$, $H \geq 0$, *where $H(r)$ is the* mean curvature *of the centred Euclidean sphere of Euclidean radius r with respect to g_0.*

(ii) $\sup_{p\in\mathbb{R}^4} b(p) < \infty$.

We note that the condition in (i) is weaker than asking for both b and c to be monotone and amounts to ruling out formation of necks. We now summarize the main results about Ricci flows in $\mathcal{G}$. Below $\boldsymbol{o}$ and $R_{g(t)}$ denote the origin in $\mathbb{R}^4$ and the scalar curvature of the solution at time t respectively.

Theorem 0.10. *Let $(\mathbb{R}^4, g(t))_{0\leq t<T}$ be the maximal complete, bounded curvature solution to the Ricci flow starting at some $g_0 \in \mathcal{G}$. Then the solution encounters a Type-II singularity at some $T < \infty$ and the following conditions hold:*

(i) (The Bryant soliton appears at the origin.) *There exists $t_j \nearrow T$ such that the rescaled Ricci flows $(\mathbb{R}^4, g_j(t), \boldsymbol{o})$ defined by $g_j(t) \doteq R_{g(t_j)}(\boldsymbol{o})g(t_j + (R_{g(t_j)}(\boldsymbol{o}))^{-1}t)$ converge to the Bryant soliton in the Cheeger-Gromov sense.*

(ii) (The singularity is global.) *For any $p \in \mathbb{R}^4$ we have*

$$\limsup_{t\nearrow T} \left(|\mathrm{Rm}_{g(t)}|_{g(t)}(p)\right) = \infty.$$

(iii) (Type-I blow-up at infinity.) *For any $t_j \nearrow T$ there exist a sequence $\{p_j\}$ and $\alpha > 0$ such that $d_{g_0}(\boldsymbol{o}, p_j) \to \infty$, $(T - t_j)R_{g(t_j)}(p_j) \leq \alpha$, and the rescaled Ricci flows $(\mathbb{R}^4, g_j(t), p_j)$ defined by $g_j(t) \doteq R_{g(t_j)}(p_j)g(t_j + (R_{g(t_j)}(p_j))^{-1}t)$ converge to the self-similar shrinking cylinder in the Cheeger-Gromov sense.*

Next, we show that the long-time property is satisfied by a class of warped Berger metrics whose curvature decays at infinity.

Definition 0.6. *We let $\mathcal{G}_\infty$ be the set of complete warped Berger metrics g on $\mathbb{R}^4$ with* positive injectivity radius *and satisfying the following conditions:*

(i) $b_s \geq 0$, $H \geq 0$.

(ii) $|\mathrm{Rm}_g|_g(s) \to 0$ *as* $s \to \infty$ *and there exist* $\mu > 0$ *and* $s_0 > 0$ *such that* $c(s) \geq \mu$ *for any* $s \geq s_0$.

We prove the following:

Theorem 0.11. *Any complete, bounded curvature Ricci flow starting in $\mathcal{G}_\infty$ is immortal.*

We finally relate the formation of Type-I singularities to minimal 3-spheres.

Theorem 0.12. *Let $(\mathbb{R}^4, g(t))_{0 \leq t < T}$ be the complete, bounded curvature Ricci flow evolving from a warped Berger metric g_0 with positive injectivity radius and curvature decaying at infinity. If $g(t)$ develops a Type-I singularity at $T < \infty$, then there exists $\delta > 0$ such that $(\mathbb{R}^4, g(t))$ contains minimal embedded 3-spheres for any $t \in [T - \delta, T)$.*

Chapter 3

If a Ricci flow solution develops a Type-III singularity and converges smoothly *without rescaling* in the Cheeger-Gromov sense to some limit, then such limit must be flat. Since Ricci-flat metrics represent fixed points for the flow, it is tempting to investigate when Ricci-flat *non*-flat metrics appear as Type-II(b) singularity models of immortal Ricci flows. In this regard, few results are available and most of them are stability properties Haslhofer and Müller [2014], Deruelle and Kröncke [2020].

In a slightly different direction, Marxen [2019] recently generalized earlier work of Hamilton to prove that if (N, g_N) is closed and Ricci-flat, then a class of warped product solutions to the Ricci flow $(\mathbb{R} \times N, g(t))$, of the form $g(t) = k^2(r,t)dr^2 + f^2(r,t)g_N$, converge to $(\mathbb{R} \times N, dr^2 + c^2 g_N)$, for some $c > 0$, whenever the initial condition is asymptotic to the target Ricci-flat metric. One of the main contributions of Chapter 3 consists in proving that a large class of metrics in the set $\mathcal{G}_\infty$ as defined above converge to the gravitational instanton given by the Taub-NUT metric Hawking [1977]

$$g_{\mathsf{TNUT}} = \frac{1}{16}\left(1 + \frac{2m^{-1}}{r}\right) dr^2 + \frac{r^2}{4}\left(1 + \frac{2m^{-1}}{r}\right) \pi^* g_{S^2(\frac{1}{2})} + \frac{m^{-2}}{1 + \frac{2m^{-1}}{r}} \sigma_3 \otimes \sigma_3,$$

for some parameter m which we call the *mass* of g_{TNUT} and which measures the inverse of the *finite* size of the Hopf-fiber at spatial infinity. We note that the stability result in Deruelle and Kröncke [2020] does *not* apply to the Taub-NUT metric which is not ALE.

We first introduce the class $\mathcal{G}_{\mathsf{AF}}$.

Definition 0.7. *The class $\mathcal{G}_{\mathsf{AF}}$ consists of all complete warped Berger metrics g on $\mathbb{R}^4$ with monotone warping coefficients satisfying*

$$\sup_{p \in \mathbb{R}^4} \left(d_g(\boldsymbol{o}, p)\right)^{2+\epsilon} |\mathrm{Rm}_g|_g(p) < \infty,$$

for some $\epsilon > 0$. A metric $g \in \mathcal{G}_{\mathsf{AF}}$ is called asymptotically flat.

We prove that the long-time behaviour of Ricci flows starting in $\mathcal{G}_{\mathsf{AF}}$ only depends on the mass - i.e. the inverse of the size of the Hopf-fiber at spatial infinity.

In the following we say that a solution converges to a Ricci-flat metric g_∞ on $\mathbb{R}^4$ in the pointed Cheeger-Gromov sense as $t \nearrow \infty$ if for any $t_j \nearrow \infty$ the sequence $(\mathbb{R}^4, g_j(t), \boldsymbol{o})$, defined by $g_j(t) = g(t_j + t)$, converges to $(\mathbb{R}^4, g_\infty, \boldsymbol{o})$ in the pointed Cheeger-Gromov sense.

Theorem 0.13. *Let $(\mathbb{R}^4, g(t))_{t \geq 0}$ be the maximal solution to the Ricci flow starting at some $g_0 \in \mathcal{G}_{\mathsf{AF}}$ and let the mass of g_0 be $(\sup_{\mathbb{R}^4} c)^{-1}$.*

(i) If g_0 has positive mass m, then $g(t)$ encounters a Type-II(b) singularity and converges to the Taub-NUT metric of mass m in the pointed Cheeger-Gromov sense as

$t \nearrow \infty$.

(ii) *If g_0 has zero mass, then the solution encounters a Type-III singularity and converges to the Euclidean metric in the pointed Cheeger-Gromov sense as $t \nearrow \infty$.*

The result above and its generalization below provide a rigorous frame for addressing the questions raised in Holzegel et al. [2007] on the $\mathbb{R}^4$-topology.

Appleton [2018] proved that on $\mathbb{R}^4$ there exists a warped Berger gradient steady soliton with monotone coefficients, bounded Hopf-fiber and coefficient b in the directions orthogonal to the Hopf-fiber opening as fast as a paraboloid in $\mathbb{R}^3$. Namely, the soliton satisfies the asymptotics:

$$c(s) \sim \text{constant}, \quad b(s) \sim \sqrt{s}.$$

Consequently, we cannot expect initial data opening with arbitrary speed to converge to g_{TNUT} along the flow. We show that the soliton represents a sort of lower barrier for the convergence to Taub-NUT in infinite-time. Namely, let us consider the following sets.

Definition 0.8. *For all $0 \leq k < 1$, the class $\mathcal{G}_k$ consists of all complete warped Berger metrics g with monotone coefficients satisfying:*

(i) $0 < \liminf_{s\to\infty} b_s\, b^k(s) \leq \limsup_{s\to\infty} b_s\, b^k(s) < \infty$,

(ii) $\sup_{p\in\mathbb{R}^4} \left(b^2|\mathrm{Rm}_g|_g\right)(p) < \infty$,

(iii) $\sup_{p\in\mathbb{R}^4} c(p) < \infty$.

We note that if $g \in \mathcal{G}_k$, then the warping coefficient b grows like $s^{\frac{1}{k+1}}$, meaning that the projection of g on the base space via the Hopf-map opens faster than a paraboloid in $\mathbb{R}^3$. In particular, it follows that any metric in $\mathcal{G}_{\mathsf{AF}}$ with positive mass is also in $\mathcal{G}_0$. Below we still call *mass* the inverse of the size of the Hopf-fiber at spatial infinity.

Theorem 0.14. *Let $(\mathbb{R}^4, g(t))_{t\geq 0}$ be the complete, bounded curvature solution to the Ricci flow starting at some $g_0 \in \mathcal{G}_k$ with mass $m > 0$. Then $g(t)$ converges to the Taub-NUT metric of mass m in the pointed Cheeger-Gromov sense as $t \nearrow \infty$.*

We point out that the result is in some sense optimal because from the existence of the soliton we derive that the theorem does not generalize to initial data opening as fast as a paraboloid in the directions orthogonal to the Hopf-fiber. We deduce the convergence property by showing that any maximal Ricci flow solution starting in $\mathcal{G}_k$ develops a Type-II(b) singularity modelled by an ancient solution satisfying the conditions below.

Definition 0.9. *Let $m > 0$. The class $\mathcal{A}$ consists of all complete, warped Berger ancient solutions to the Ricci flow on $\mathbb{R}^4$ with monotone coefficients and curvature uniformly bounded in the space-time, satisfying*

$$b_s \geq \frac{f\left(\frac{b}{c}\right)}{\frac{b}{c}}$$

$$\sup_{\mathbb{R}^4 \times (-\infty,0]} c = m^{-1}$$

for some continuous positive function f such that $f(z) \to \infty$ as $z \to \infty$.

We note that the class $\mathcal{A}$ describes ancient solutions opening faster than a paraboloid in the directions orthogonal to the Hopf-fiber. We prove a rigidity property.

Theorem 0.15. *The Taub-NUT metric is the only ancient solution in $\mathcal{A}$.*

Once again the result is optimal because the existence of the gradient steady soliton found by Appleton highlights that we cannot drop the requirement on f to diverge in space-time regions where the roundness ratio c/b becomes degenerate. The rigidity result applies to possible *collapsed* infinite-time singularity models and indeed, given $\kappa > 0$, there exist $p \in \mathbb{R}^4$ and $\rho > 0$ such that g_{TNUT} is κ-strongly collapsed at p for all scales larger than ρ.

Chapter 1

Rotationally symmetric Ricci flow on $\mathbb{R}^{n+1}$

In this chapter we study the Ricci flow on $\mathbb{R}^{n+1}$, with $n \geq 2$, starting at some complete, bounded curvature rotationally symmetric metric g_0. We first focus on the case where $(\mathbb{R}^{n+1}, g_0)$ does not contain minimal hyperspheres. We prove that if g_0 is asymptotic to a cylinder, then the solution develops a Type-II singularity and converges to the Bryant soliton after scaling, while if the curvature of g_0 decays at infinity, then the solution is immortal. As a corollary, we prove a conjecture by Chow and Tian about Perelman's standard solutions. We then consider a class of asymptotically flat initial data $(\mathbb{R}^{n+1}, g_0)$ containing a neck and we prove that if the neck is sufficiently pinched, in a precise way, the Ricci flow encounters a Type-I singularity.

1.1 Introduction

We recall that the first examples of Type-II singularities in dimension three or higher were produced by Gu and Zhu [2008], where they studied a family of rotationally symmetric metrics on S^{n+1}. Later Angenent, Isenberg and Knopf proved that on S^{n+1} there exist Ricci flows which behave like degenerate neckpinches, with the singularity modelled on the Bryant soliton Angenent et al. [2015]. Similarly, Wu [2014] found rotationally symmetric solutions on $\mathbb{R}^{n+1}$ which encounter a Type-II singularity and converge to the Bryant soliton Wu [2014]. This result only applies to initial data whose profile function converges uniformly to that of the Bryant soliton near the origin.

We show that a large class of rotationally symmetric Ricci flows on $\mathbb{R}^{n+1}$ develop a finite time Type-II singularity modelled on the Bryant soliton Bryant [2005].

Theorem 1.1. *Let $(\mathbb{R}^{n+1}, g(t))_{0\leq t<T}$, with $n \geq 2$, be the complete, bounded curvature Ricci flow solution evolving from a rotationally symmetric metric g_0. If $(\mathbb{R}^{n+1}, g_0)$ does not contain minimal hyperspheres and g_0 is asymptotic in C^0 to a round cylinder at infinity, then the solution develops a Type-II singularity at $T < \infty$ and converges to the Bryant soliton in the Cheeger-Gromov sense once suitably rescaled.*

According to Theorem 1.1 in the rotationally symmetric case the cylindrical behaviour at infinity determines the dynamics of the flow as long as there are no minimal hyperspheres. In fact, if g_0 does not contain minimal hyperspheres, then g_0 is asymptotic to a round cylinder if and only if its profile function is bounded. In light of this, one might expect that if instead the curvature of g_0 decays at infinity then the solution evolving from g_0 should be immortal. In this direction Oliynyk and Woolgar [2007] proved that on $\mathbb{R}^{n+1}$ if g_0 is rotationally symmetric, with stronger than quadratic decay of the curvature and with no minimal embedded hyperspheres, then the Ricci flow solution starting at g_0 is immortal. Our second result extends the long-time existence property to initial data that need not be close to the Euclidean metric outside a compact region.

Theorem 1.2. *Let $(\mathbb{R}^{n+1}, g(t))_{0\leq t<T}$, with $n \geq 2$, be the complete, bounded curvature Ricci flow solution evolving from a rotationally symmetric metric g_0 with curvature decaying at infinity. If $(\mathbb{R}^{n+1}, g_0)$ does not contain minimal hyperspheres, then the solution is immortal.*

It might be worth comparing the result of Theorem 1.2 with Corollary 5 in Cabezas-Rivas and Wilking [2015], where they show that any n-dimensional Ricci flow starting at a complete metric with nonnegative complex sectional curvature and volume growth faster than r^{n-2} is immortal. Any rotationally symmetric metric on $\mathbb{R}^{n+1}$ has nonnegative curvature operator if and only if has nonnegative Ricci curvature. In particular, the condition of nonnegative complex sectional curvature implies that the radius function is concave and hence that is monotone. As soon as we start the flow, by strong maximum principle we find that there are no minimal hyperspheres - which is equivalent to the strict monotonicity of the radius function - and hence we either have a bounded radius (cylindrical asymptotics) or an unbounded radius (faster than linear volume growth). In particular, in the latter case the curvature decays to zero at spatial infinity meaning that the conditions of Theorem 1.2 are satisfied. Therefore, the previous theorem recovers in particular Corollary 5 in Cabezas-Rivas and Wilking [2015] in the $SO(n+1)$-invariant case on $\mathbb{R}^{n+1}$.

Indeed, we derive a simple application of Theorem 1.1 and Theorem 1.2, consisting in a classification of rotationally invariant Ricci flows with nonnegative bounded curvature.

Corollary 1.3. *Let $(\mathbb{R}^{n+1}, g(t))_{0\leq t<T}$, with $n \geq 2$, be the complete, bounded curvature Ricci flow solution evolving from a rotationally symmetric metric g_0 with nonnegative curvature. Then $T < \infty$ if and only if g_0 is asymptotic in C^0 to the round cylinder $\mathbb{R} \times S^n(r_0)$, for some $r_0 > 0$. In this case T only depends on n and r_0 and the solution develops a global Type-II singularity modelled on the Bryant soliton once suitably dilated.*

Chen and Zhu already proved that any Ricci flow as in Corollary 1.3 develops a *global* singularity at some T only depending on the radius of the cylinder asymptotic to the initial metric [Chen and Zhu, 2006a, Theorem A.1] (see also Proposition 1.22).

The classification of nonnegatively curved rotationally symmetric Ricci flows also leads to a better understanding of standard solutions. Perelman [2003b] introduced the notion of *standard* solutions on $\mathbb{R}^3$ by evolving metrics obtained from gluing a hemispherical cap region to a round cylinder of scalar curvature one. Standard solutions were used to describe the behaviour of the flow after performing surgery. Lu and Tian generalized Perelman's standard solutions by considering Ricci flows on $\mathbb{R}^{n+1}$ starting at some rotationally symmetric metric with nonnegative curvature, sufficiently bounded geometry and asymptotic to a round cylinder (see Definition 1.2). Chow and Tian have conjectured that any standard solution in the sense of Lu and Tian develops a Type-II singularity modelled on the Bryant soliton once suitably dilated [Wu, 2014, Conjecture 1.2]. Wu gave evidence in favour of this conjecture by showing that there exist *some* standard solutions converging to Bryant solitons.

Corollary 1.3 provides an affirmative answer to the Chow-Tian conjecture.

Corollary 1.4 (Chow-Tian Conjecture)**.** *Sequences of appropriately scaled standard solutions* (*as defined in Lu and Tian*) *with marked origins converge to Bryant solitons in a suitable sense.*

A three dimensional version of this result was proved by Ding [2009].

We point out that the notion of standard solutions discussed in Lu and Tian does not require the curvature to attain a maximum. If instead we also assume a pinching condition where the radial sectional curvature K_{g_0} is bounded from above by the spherical sectional curvature L_{g_0}, then Corollary 1.3 and the recent classification of (rotationally symmetric) κ-solutions obtained by Brendle [2020] and Li and Zhang [2018] guarantee that by blowing up the standard solution at the origin $\boldsymbol{o} \in \mathbb{R}^{n+1}$ along *any* time sequence approaching the maximal time one obtains the Bryant soliton in the limit.

Corollary 1.5. *Let* $(\mathbb{R}^{n+1}, g(t))_{0\leq t<T}$ *be a standard solution to the Ricci flow* (*as defined in Lu and Tian*) *starting at* g_0. *If* $K_{g_0} \leq L_{g_0}$, *then for any* $t_j \nearrow T$ *the rescaled standard solutions* $(\mathbb{R}^{n+1}, g_j(t), \boldsymbol{o})$ *defined on* $[-R_{g(t_j)}(\boldsymbol{o})t_j, 0]$ *by* $g_j(t) \doteq R_{g(t_j)}(\boldsymbol{o})g(t_j+t/R_{g(t_j)}(\boldsymbol{o}))$ *converge to the Bryant Soliton (up to scaling).*

According to Angenent and Knopf [2004] the conclusion of Theorem 1.2 should generally fail if $(\mathbb{R}^{n+1}, g_0)$ contains minimal hyperspheres. In fact, in Oliynyk and Woolgar [2007] it was expected that if g_0 is asymptotically flat and contains minimal embedded hyperspheres forming a neck region which is sufficiently pinched, then the Ricci flow starting at g_0 develops a Type-I singularity caused by the radius of the neck going to zero in finite-time. Conversely, if the pinching is mild, then the neck should disappear in finite time and the flow should hence be immortal. By extending the analysis in Angenent and Knopf [2004] to $\mathbb{R}^{n+1}$ we are able to confirm such expectation. We consider the Ricci flow evolving from an asymptotically flat metric g_0 of the form

$$g_0 = \xi_0^2(r)dr \otimes dr + \phi_0^2(r)g_{S^n},$$

where g_{S^n} is the constant curvature one metric on S^n. We say that g_0 has a *neck region* if ϕ_0 has a local maximum at some radial coordinate r_1 and a local minimum at some radial coordinate $r_2 > r_1$. Following Angenent and Knopf [2007] the pinching of the neck is then given by the ratio between the radii $\phi_0(r_1)$ and $\phi_0(r_2)$.

The statement below is a weaker version of our result and we refer to Theorem 1.28 for a complete statement containing the cylindrical asymptotics.

Theorem 1.6. *Let $(\mathbb{R}^{n+1}, g(t))_{0\leq t<T}$, with $n \geq 2$, be the complete, bounded curvature Ricci flow solution evolving from an asymptotically flat rotationally symmetric metric g_0 containing a neck region $(r_1, r_2) \times S^n$. Assume that $\mathrm{Ric}_{g_0} > 0$ on the closed Euclidean ball $B(\boldsymbol{o}, r_1)$ and that $R_{g_0} \geq 0$ on $\mathbb{R}^{n+1}$. Let β be defined as*

$$\beta \doteq \inf_{\mathbb{R}^{n+1}} \phi_0^2(L_{g_0} - K_{g_0}),$$

and let $\lambda > 0$ satisfy

$$\lambda^2 > \frac{n+1-2\beta}{n-1} + 1.$$

If $\phi_0(r_1) \geq \lambda\phi_0(r_2)$, then the solution develops a Type-I singularity which is modelled on a family of shrinking cylinders.

Finally, we show that there exist examples of necks that disappear in finite time along the Ricci flow. The next result follows by combining the adaptation of Angenent and Knopf [2004] to $\mathbb{R}^{n+1}$ and the stability result for the Euclidean metric proved by Schnürer, Schulze and Simon in Schnürer et al. [2008].

Proposition 1.7. *There exists $\varepsilon_0 = \varepsilon_0(n) > 0$ such that if g_0 is an asymptotically flat rotationally symmetric metric which has a neck and is ε_0-close to the Euclidean metric on $\mathbb{R}^{n+1}$, then the maximal Ricci flow solution $g(t)$ evolving from g_0 is immortal and the neck disappears in finite time.*

Outline

In Section 1.2 we discuss some preliminaries and we prove a few basic estimates. In Section 1.3 we analyse rotationally invariant Ricci flows on $\mathbb{R}^{n+1}$ with no minimal embedded hyperspheres. We show that by Angenent [1988] no minimal hyperspheres appear along the flow and that the curvature is controlled via lower bounds for the radius ϕ as in Angenent and Knopf [2004]. In Section 1.4 we prove that under the assumptions of Theorem

1.1 the flow develops Type-II singularities. The main ingredients are the characterization of Type I flows in Enders et al. [2011] and the classification of locally conformally flat shrinkers in Zhang [2008]. The appearance of the Bryant soliton follows from Hamilton [1993a] and Cao and Chen [2012], or alternatively from the recent classification of (rotationally symmetric) κ-solutions in Brendle [2020] and Li and Zhang [2018]. We then show that in the setting of Theorem 1.2 any (potential) singularity model is a κ-solution with positive asymptotic volume ratio. According to Perelman [2002], the latter property implies that any Ricci flow as in Theorem 1.2 is immortal. Section 1.5 is devoted to classifying nonnegatively curved rotationally symmetric Ricci flows on $\mathbb{R}^{n+1}$, with focus on studying the singularities of standard solutions. In Section 1.6 we extend the analysis in Angenent and Knopf [2004] to $\mathbb{R}^{n+1}$ to prove Theorem 1.6 (restated in Theorem 1.28) and we outline how the examples of initial data constructed in Angenent and Knopf [2004] may be modified to provide analogous initial data for which Theorem 1.6 applies. We derive cylindrical asymptotics for the neckpinch following Isenberg et al. [2016]. Finally, using Schnürer et al. [2008] we provide examples of initial data with necks that evolve to metrics with no minimal embedded hyperspheres in finite time.

1.2 Preliminaries

Let $n \geq 2$ be an integer. Away from the origin, any rotationally symmetric metric on $\mathbb{R}^{n+1}$ is of the form

$$g = \xi^2(r)\, dr \otimes dr + \phi^2(r) g_{S^n} \tag{1.1}$$

where g_{S^n} is the standard metric of constant curvature one on S^n and ξ, ϕ are smooth functions on $(0, +\infty)$. If we introduce the geometric coordinate s representing the g-distance from the origin, then g extends smoothly to the origin if and only if

$$\lim_{s\to 0} \frac{d^{2k}\phi}{ds^{2k}}(s) = 0, \qquad \lim_{s\to 0} \frac{d\phi}{ds}(s) = 1, \tag{1.2}$$

for any $k \geq 0$. From now on we assume that (1.2) is satisfied so that we can write g as

$$g = ds \otimes ds + \phi^2(s) g_{S^n}. \tag{1.3}$$

In the following we always regard $\phi = \phi(s) = \phi(s(r))$ as a function of the variable r (and of time for solutions to the Ricci flow). The spatial derivative with respect to s is therefore intended to be the vector field

$$\partial_s = \frac{1}{\xi(r)}\partial_r. \tag{1.4}$$

We adopt the same notations as in Angenent and Knopf [2004]. For any metric g of the form (1.3) we denote the sectional curvatures of the 2-planes perpendicular to the fibers $\{r\} \times S^n$ and of the 2-planes tangential to these fibers by K and L respectively. From the rotational symmetry it follows that the curvature of g is entirely described by K and L which are given by

$$K = -\frac{\phi_{ss}}{\phi}, \qquad L = \frac{1-\phi_s^2}{\phi^2}. \tag{1.5}$$

By tracing we get the formulas for the Ricci tensor and the scalar curvature:

$$\mathrm{Ric}_g = -n\frac{\phi_{ss}}{\phi}(ds)^2 + \left(-\phi\phi_{ss} + (n-1)(1-\phi_s^2)\right) g_{S^n}, \tag{1.6}$$

$$R_g = n\left(-2\frac{\phi_{ss}}{\phi} + (n-1)\frac{1-\phi_s^2}{\phi^2}\right). \tag{1.7}$$

1.2.1 Derived equations

Let g_0 be a complete rotationally symmetric metric on $\mathbb{R}^{n+1}$ of the form (1.3). If g_0 has bounded curvature then there exists a solution $g(t)$ to the Ricci flow starting at g_0 Shi [1989] and this solution is unique in the class of complete solutions with bounded curvature on compact subintervals Chen and Zhu [2006b]. By the Ricci flow diffeomorphism invariance and the uniqueness result in Chen and Zhu [2006b] such solution preserves the rotational symmetry and we may write $g(t)$ as

$$g(t) = \xi^2(r,t)\, dr \otimes dr + \phi^2(r,t) g_{S^n} = ds \otimes ds + \phi^2(s,t) g_{S^n}, \tag{1.8}$$

where $s = s(r,t)$ is the time-dependent $g(t)$-distance from the origin. From (1.6) we derive the evolution equations for ξ

$$\xi_t = n\frac{\phi_{ss}}{\phi}\xi \tag{1.9}$$

and for the radius ϕ

$$\phi_t = \phi_{ss} - (n-1)\frac{1-\phi_s^2}{\phi}. \tag{1.10}$$

Since the geometric variable s depends on time, we have a non-vanishing commutator between ∂_s and ∂_t. By (1.9) we get

$$[\partial_t, \partial_s] = \left[\partial_t, \frac{\partial_r}{\xi(r,t)}\right] = -(\log \xi)_t\partial_s = -n\frac{\phi_{ss}}{\phi}\partial_s. \tag{1.11}$$

Using the commutator formula and (1.10) we compute the equations for the first derivative of ϕ

$$(\phi_s)_t = (\phi_s)_{ss} + \frac{n-2}{\phi}\phi_s(\phi_s)_s + (n-1)\frac{1-\phi_s^2}{\phi^2}\phi_s \tag{1.12}$$

and for its second derivative

$$(\phi_{ss})_t = (\phi_{ss})_{ss} + (n-2)\frac{\phi_s}{\phi}(\phi_{ss})_s - 2\frac{\phi_{ss}^2}{\phi} - (4n-5)\frac{\phi_s^2}{\phi^2}\phi_{ss} + \frac{n-1}{\phi^2}\phi_{ss} - 2(n-1)\frac{\phi_s^2(1-\phi_s^2)}{\phi^3}. \tag{1.13}$$

Similarly to Angenent and Knopf [2004] we introduce the quantity

$$A = \phi^2(L-K) = \phi\phi_{ss} + 1 - \phi_s^2, \tag{1.14}$$

which is a scale-invariant measure of the difference between the spherical sectional curvature L and the radial sectional curvature K. From [Angenent and Knopf, 2004, Lemma 3.1] it follows that the quantity A evolves by

$$A_t = A_{ss} + (n-4)\frac{\phi_s}{\phi}A_s - 4(n-1)\frac{\phi_s^2}{\phi^2}A. \tag{1.15}$$

We also write the expression for the Laplacian along the flow: for any smooth radial map f the Laplacian associated with the solution to the Ricci flow at time t is given by

$$\Delta f = f_{ss} + n\frac{\phi_s}{\phi} f_s. \tag{1.16}$$

1.2.2 Basic estimates

We dedicate the end of this section to proving general bounds for rotationally invariant Ricci flows on $\mathbb{R}^{n+1}$. We first show that we can control the curvature of the Ricci flow solution via lower bounds for the radius ϕ. The following property is analogous to [Angenent and Knopf, 2004, Lemma 7.1].

Lemma 1.8. *Let $(\mathbb{R}^{n+1}, g(t))_{0\leq t<T}$ be the complete, bounded curvature Ricci flow solution evolving from a rotationally symmetric metric g_0. Let $U \subset \mathbb{R}^{n+1}$ and assume that $\phi^2|\mathrm{Rm}_{g(t)}|_{g(t)} \leq C$ along the parabolic boundary of $U \times [0,T)$ for some $C > 0$. Then $\phi^2|\mathrm{Rm}_{g(t)}|_{g(t)} \leq C'$ in $U \times [0,T)$ for some $C' \in [C,\infty)$.*

Proof. It suffices to show that $\phi^2(|L|+|K|) \leq C'$ in $U \times [0,T)$, with K and L as in (1.5). Since $\phi^2 L = 1 - \phi_s^2$ is uniformly bounded along the parabolic boundary of $U \times [0,T)$, by the evolution equation (1.12) we deduce that ϕ_s cannot diverge in $U \times [0,T)$ along a sequence of interior maxima (minima). Explicitly, at any sufficiently large positive maximum we get $\partial_t \phi_s(r_{\max}, t) < 0$.
We now consider the quantity A defined in (1.14). By assumption A is controlled along the parabolic boundary of $U \times [0,T)$. From standard applications of the maximum principle to (1.15) we get that A is then bounded in $U \times [0,T)$. We may conclude that $\phi^2 K = \phi^2 L - A$ is uniformly bounded in $U \times [0,T)$, which completes the proof. □

When the scale-invariant estimate in Lemma 1.8 is satisfied on a given region as long as the solution exists then we can always define a limit (possibly degenerate) radius.

Lemma 1.9. *Let $(\mathbb{R}^{n+1}, g(t))_{0\leq t<T}$ be the complete, bounded curvature Ricci flow solution evolving from a rotationally symmetric metric g_0. Assume that $T < \infty$ and that $\phi^2|\mathrm{Rm}_{g(t)}|_{g(t)} \leq C$, for some $C > 0$, on $U \times [0,T)$, where $U \subset \mathbb{R}^{n+1}$. Then for any $p \in U$ the limit $\lim_{t\nearrow T}\phi(p,t)$ exists finite.*

Proof. From (1.10) we derive that for any $p \in U$ we have

$$|\partial_t(\phi^2)|(p) = |2\phi\phi_{ss} - 2(n-1)(1-\phi_s^2)|(p) = |-2\phi^2 K - 2(n-1)\phi^2 L|(p) \leq C.$$

Therefore the function $\phi(p, \cdot)$ is Lipschitz in $[0, T)$ and the conclusion follows. □

We finally prove that lower bounds for the scale-invariant quantity A defined in (1.14) are preserved along the Ricci flow. In the following $\boldsymbol{o}$ denotes the origin of $\mathbb{R}^{n+1}$.

Lemma 1.10. *Let $(\mathbb{R}^{n+1}, g(t))_{0 \leq t < T}$ be the complete, bounded curvature Ricci flow solution evolving from a rotationally symmetric metric g_0. If $A(\cdot, 0) \geq -\beta$, for some $\beta \geq 0$, then $A(\cdot, t) \geq -\beta$ for any $t \in [0, T)$.*

Proof. We first check that the radius ϕ has a positive lower bound away from the origin.

Claim 1.11. *For any $r_0 > 0$ and $t_0 < T$ there exists a positive δ depending on the solution and on r_0 and t_0 such that $\phi \geq \delta$ in $(\mathbb{R}^{n+1} \setminus B(\boldsymbol{o}, r_0)) \times [0, t_0]$.*

Proof of Claim 1.11. Let $\alpha_0 \doteq \sup|\mathrm{Rm}_{g_0}|_{g_0}$. Given $r_0 > 0$, if $\phi^2(r, 0) \leq (2\alpha_0)^{-1}$ for all $r \geq r_0$ then we have

$$|1 - \phi_s^2|(r, 0) \leq \alpha_0 \phi^2(r, 0) \leq \frac{1}{2},$$

which then implies that $\phi(r, 0) \to \infty$, but that is not possible. Therefore, we deduce that there exists a sequence $r_j \to \infty$ such that $\phi(r_j, 0) > (2\alpha_0)^{-1}$. Assume for a contradiction that there exists a sequence $\hat{r}_j \to \infty$ such that $\phi(\hat{r}_j, 0) \leq (2\alpha_0)^{-1}$. After reordering the sequences, we derive that there exists a sequence of local minima $\tilde{r}_j \to \infty$ such that $\phi(\tilde{r}_j, 0) \leq (2\alpha_0)^{-1}$. It follows that

$$|1 - \phi_s^2|(\tilde{r}_j, 0) \equiv 1 \leq \alpha_0 \phi^2(\tilde{r}_j, 0) \leq \frac{1}{2}.$$

We conclude that $\phi(r, 0) \geq \delta > 0$ for some δ for all $r \geq r_0$. Finally, we note that given $t_0 < T$, since the curvature is uniformly bounded by some $\alpha(t_0)$ in $\mathbb{R}^{n+1} \times [0, t_0]$, then

$$\partial_t \phi \geq -\alpha(t_0)\phi,$$

which gives the desired lower bound after integration. □

For any $t < T$ there exists $\alpha(t) > 0$ such that $|\phi_{ss}| \leq \alpha\phi$. Thus ϕ and ϕ_{ss} are exponentially bounded, which implies that ϕ_s is exponentially bounded. In particular, given $t_0 < T$ there exist $M = M(t_0)$ and $\alpha = \alpha(t_0)$ such that $|A(s,t)| \leq M\exp(\alpha s)$ for any $t \in [0, t_0]$. Let ε, η and γ be positive constants to be chosen below. We define the lower barrier

$$(s,t) \mapsto W(s,t) \doteq \varepsilon \exp\left(\frac{s^2}{1-\eta t} + \gamma t\right). \tag{1.17}$$

Using (1.15) we can write the evolution equation of $\hat{A} \doteq A + \beta + W$ for $0 \leq t \leq 1/2\eta$ as

$$\begin{aligned}\hat{A}_t &\geq (\hat{A})_{ss} + (n-4)\frac{\phi_s}{\phi}(\hat{A})_s - 4(n-1)\frac{\phi_s^2}{\phi^2}\hat{A} \\ &+ \frac{W}{(1-\eta t)^2}\left(s^2(\eta - 4) + \gamma(1-\eta t)^2 + (1-\eta t)(2s\partial_t s - 2 - (n-4)\frac{\phi_s}{\phi}2s)\right).\end{aligned}$$

By distortion estimates of the distance function there exists $C = C(t_0)$ such that $\partial_t s \geq -Cs$ in $\mathbb{R}^{n+1} \times [0, \min\{t_0, (2\eta)^{-1}\}]$. From the boundary conditions we derive that there exists a neighbourhood of the origin where $s\phi_s/\phi$ is uniformly bounded. By the boundedness of the curvature we get $\phi_s^2 \leq 1 + C(t_0)\phi^2$ in $\mathbb{R}^{n+1} \times [0, \min\{t_0, (2\eta)^{-1}\}]$. Thus, by Claim 1.11 we deduce that away from the origin the following estimate is satisfied

$$\left(\frac{\phi_s}{\phi}\right)^2 \leq C(t_0) + \frac{1}{\phi^2} \leq C(t_0) + \frac{1}{\delta^2}.$$

Therefore $s\phi_s/\phi \leq C(t_0)(1+s)$ in $\mathbb{R}^{n+1} \times [0, \min\{t_0, (2\eta)^{-1}\}]$, up to renaming $C(t_0)$. We conclude that we can always pick $\eta = \eta(t_0)$ and $\gamma = \gamma(t_0)$ such that in $\mathbb{R}^{n+1} \times [0, \min\{t_0, (2\eta)^{-1}\}]$ we have

$$\hat{A}_t > (\hat{A})_{ss} + (n-4)\frac{\phi_s}{\phi}(\hat{A})_s - 4(n-1)\frac{\phi_s^2}{\phi^2}\hat{A}.$$

Since A is exponentially bounded and $A(\boldsymbol{o}, t) = 0$ we see that if $\hat{A} < 0$ somewhere in $\mathbb{R}^{n+1} \times [0, \min\{t_0, (2\eta)^{-1}\}]$, then there exist $z > 0$ sufficiently small, $\bar{r}$ and $\bar{t}$ such that $\hat{A}(\cdot, \bar{t})$ has a negative minimum at $\bar{r}$ where $\hat{A}(\bar{r}, \bar{t}) = -z$ for the first time. However, we have shown that $\partial_t \hat{A}(\bar{r}, \bar{t}) > 0$, which gives a contradiction. We then obtain that $A(\cdot, t) \geq -\beta$ for any $t \in [0, \min\{t_0, (2\eta)^{-1}\}]$ once we let ε in (1.17) go to zero. We may finally iterate the step and conclude that A remains bounded from below by $-\beta$ in $\mathbb{R}^{n+1} \times [0, t_0]$. Since $t_0 < T$ was arbitrary, the proof is complete. □

1.3 Analysis of Ricci flow with no minimal hyperspheres

We consider the maximal Ricci flow solution $(\mathbb{R}^{n+1}, g(t))_{0 \leq t < T}$ evolving from a complete, bounded curvature rotationally symmetric metric g_0 such that $(\mathbb{R}^{n+1}, g_0)$ does not contain minimal embedded hyperspheres. We first check that the last condition persists in time meaning that minimal hyperspheres cannot appear along the Ricci flow solution if none existed at the initial time. Since the Euclidean hypersphere of radius r is *minimal* in $(\mathbb{R}^{n+1}, g(t))$ when $\phi_s(r, t) = 0$, similarly to Angenent and Knopf [2004] we derive the control on the formation of minimal hyperspheres from applying the Sturmian theorem to the evolution equation of ϕ_s. In the following estimates C always denotes a uniform constant that may change from line to line while $\boldsymbol{o}$ denotes the origin of $\mathbb{R}^{n+1}$.

Lemma 1.12. *Let $(\mathbb{R}^{n+1}, g(t))_{0 \leq t < T}$ be the complete, bounded curvature Ricci flow solution evolving from a rotationally symmetric metric g_0. If $\phi_s(\cdot, 0) > 0$ then $\phi_s(\cdot, t) > 0$ for any $t \in [0, T)$.*

Proof. Using (1.7) and (1.16) we can write the evolution equation of ϕ_s as

$$(\phi_s)_t = \Delta \phi_s + \frac{R_{g(t)}}{n} \phi_s.$$

Assume for a contradiction that $\phi_s(p_0, t_0) < 0$ at some space-time point. By Shi [1989] given $T' \in (t_0, T)$ there exists $C = C(T')$ such that $|R_{g(t)}(\cdot)| \leq nC$ in $\mathbb{R}^{n+1} \times [0, T']$. Therefore, at any space-time point in $\mathbb{R}^{n+1} \times [0, T']$ where ϕ_s is negative the time derivative $(\phi_s)_t$ satisfies

$$(\phi_s)_t \geq \Delta \phi_s + C\phi_s. \tag{1.18}$$

We have already seen in the proof of Lemma 1.10 that $|\phi_s(p, t)| \leq \exp(C(d_{g(t)}(\boldsymbol{o}, p) + 1))$ for any $(p, t) \in \mathbb{R}^{n+1} \times [0, T']$. We can apply the maximum principle to (1.18) and conclude that $\phi_s(\cdot, t) \geq 0$ in $\mathbb{R}^{n+1} \times [0, T)$ because $T' < T$ was arbitrary.

In fact, the inequality is strict for all positive times. Indeed, if there exist $p_0 \in \mathbb{R}^{n+1}$ and $t_0 > 0$ such that $\phi_s(p_0, t_0) = 0$, then by the previous derivations we see that p_0 must be a minimum point for $\phi_s(\cdot, t_0)$. The strong maximum principle implies that ϕ_s must vanish in the space-time region $\mathbb{R}^{n+1} \times [0, t_0]$, thus violating the boundary conditions (1.2). $\square$

Next we need to control the curvature of the Ricci flow solution only in terms of lower bounds for ϕ. For if the latter condition holds, then by (1.2) and Lemma 1.12 any solution with a nonempty singular set must in particular become singular *around the origin*. This geometric property guarantees that there always exist singularity models that are not shrinking cylinders. Thus in the following we identify under which assumptions on the behaviour of g_0 at spatial infinity the evolving solution $g(t)$ satisfies the estimate

$$\phi^2 |\mathrm{Rm}_{g(t)}|_{g(t)} \leq C, \tag{1.19}$$

for some uniform constant $C = C(g_0)$. The strategy consists in proving that such bound holds outside a sufficiently large ball and then using Lemma 1.8 to deduce that the same control must extend to the ball.

Let g_0 be a complete, bounded curvature rotationally symmetric metric without minimal hyperspheres so that the radius $\phi(\cdot, 0)$ is increasing and admits a limit at infinity.

1.3.1 Ricci flow with bounded radius

When the limit $\lim_{r\to\infty} \phi(r, 0)$ is finite, the initial metric is asymptotic in C^0 to a round cylinder. In this case the solution to the Ricci flow is controlled by a shrinking cylinder at infinity.

Lemma 1.13. *Let $(\mathbb{R}^{n+1}, g(t))_{0\leq t<T}$ be the complete, bounded curvature Ricci flow solution evolving from a rotationally symmetric metric g_0 without minimal embedded hyperspheres. If g_0 is asymptotic in C^0 to the round cylinder of radius ρ at infinity, then*

(i) The solution becomes singular at a finite time satisfying $2T(n-1) \leq \rho^2$.

(ii) There exists $C > 0$ such that $\phi^2 |\mathrm{Rm}_{g(t)}|_{g(t)} \leq C$ in $\mathbb{R}^{n+1} \times [\frac{T}{2}, T)$.

Proof. The evolution equation of ϕ^2 is given by

$$\phi_t^2 = \Delta\phi^2 - 4\phi_s^2 - 2(n-1).$$

By a standard application of the maximum principle we find $\phi^2(r,t) \leq \rho^2 - 2(n-1)t$ as long as the solution exists, which then implies (i).

In order to prove (ii) we consider $t \in [T/2, T)$. From the estimate above and Lemma 1.12 we derive that $\phi(r,t)$ admits a positive finite limit as $r \to \infty$, which also implies that ϕ_s is integrable in $(0,\infty)$, once we regard ϕ_s as a function of s. Since the curvature is bounded at time t by some constant C we deduce that $|\phi_{ss}| = |K|\phi \leq C\phi \leq C$. Thus we find $\phi_s(r,t) \to 0$ at infinity. Therefore ϕ_s is uniformly controlled at the origin and at spatial infinity in $[T/2, T)$. The same argument in Lemma 1.8 shows that $|\phi_s| \leq C$ in $\mathbb{R}^{n+1} \times [T/2, T)$.

Similarly, by Shi's derivative estimates, the Koszul formula and the uniform bound on ϕ_s we obtain that

$$|\phi_{sss}|(\cdot,t) \leq \left(\phi|K_s| + |K||\phi_s|\right)(\cdot,t) \leq C(t)|\nabla \mathrm{Rm}_{g(t)}| + C(t) \leq C(t),$$

which then implies that $\phi_{ss}(r,t) \to 0$ as $r \to \infty$, being the integral of $\phi_{ss}(\cdot,t)$ on $(0,\infty)$ convergent. We conclude that $\phi^2|\mathrm{Rm}_{g(t)}|_{g(t)}$ is uniformly controlled at the origin and at spatial infinity for any $t \in [T/2, T)$. We may then apply Lemma 1.8. □

We finally consider a subclass of solutions with bounded radius defined by requiring g_0 to further satisfy the following scale-invariant pinching condition:

$$A(\cdot,0) \equiv \phi^2(L-K)(\cdot,0) = (\phi_{ss}\phi + 1 - \phi_s^2)(\cdot,0) \geq 0. \tag{1.20}$$

The constraint (1.20) implies that g_0 has a tip located at the origin. Moreover, this tip persists along the solution evolving from g_0.

Lemma 1.14. *Let $(\mathbb{R}^{n+1}, g(t))_{0\leq t<T}$ be the complete, bounded curvature Ricci flow solution evolving from a rotationally symmetric metric g_0 without minimal embedded hyperspheres. Assume that $A(\cdot,0) \geq 0$ and that g_0 is asymptotic in C^0 to a round cylinder at infinity. Then for any $p \in \mathbb{R}^{n+1}$ and $t \in [0,T)$ the following holds*

$$R_{g(t)}(\boldsymbol{o}) \geq R_{g(t)}(p).$$

Proof. By Lemma 1.10 we know that $A(\cdot,t)\geq 0$ along the flow. For any $r>0$ and for any $t\in[0,T)$ we have

$$\partial_s\left(\frac{1-\phi_s^2}{\phi^2}\right)(r,t)=-2\frac{\phi_s}{\phi^3}A(r,t)\leq 0,$$

where we have also used that $\phi_s(\cdot,t)\geq 0$. From (1.2) we derive that $\phi_{sss}(\boldsymbol{o},t)$ exists finite for any $t\in[0,T)$. We may thus apply l'Hôpital's rule to find that

$$\begin{aligned}R_{g(t)}(\boldsymbol{o})&=n(n+1)(-\phi_{sss})(\boldsymbol{o},t)\\&=n(n+1)\lim_{y\to 0}\left(\frac{1-\phi_s^2}{\phi^2}\right)(y,t)\geq n(n+1)\left(\frac{1-\phi_s^2}{\phi^2}\right)(r,t)\end{aligned}$$

for any $(r,t)\in(0,\infty)\times[0,T)$. Therefore, since the condition $A(\cdot,t)\geq 0$ also implies $-\phi_{ss}/\phi\leq(1-\phi_s^2)/\phi^2$ in $\mathbb{R}^{n+1}\setminus\{\boldsymbol{o}\}\times[0,T)$, we finally derive

$$R_{g(t)}(p)=n\left(-2\frac{\phi_{ss}}{\phi}+(n-1)\frac{1-\phi_s^2}{\phi^2}\right)(p,t)\leq n(n+1)\left(\frac{1-\phi_s^2}{\phi^2}\right)(p,t)\leq R_{g(t)}(\boldsymbol{o}).$$

□

1.3.2 Ricci flow with unbounded radius

When the radius $\phi(r,0)$ diverges as $r\to\infty$ we generally have a weaker control on the geometry at infinity. For example, there exist initial data with exponential volume growth where the curvature stays away from zero at infinity as in the bounded radius case but the scale invariant quantity $\phi^2|\mathrm{Rm}|$ diverges at spatial infinity. In order to avoid such cases, we require the curvature of the initial data to decay as $r\to\infty$.

We note that if g_0 is complete rotationally symmetric with no minimal hyperspheres and $|\mathrm{Rm}_{g_0}|_{g_0}\to 0$ at infinity then $\phi(r,0)\to\infty$, for if $\phi(r,0)\to\rho<\infty$ then $|L|\to 0$ if and only if $\phi_s^2\to 1$ which is not possible. Therefore, the decay of the curvature at infinity implies that the radius must be unbounded. In the next Lemma we show that this is enough to control the flow outside a ball uniformly in time.

Lemma 1.15. *Let $(\mathbb{R}^{n+1},g(t))_{0\leq t<T}$, with $T<\infty$, be the complete, bounded curvature Ricci flow solution evolving from a rotationally symmetric metric g_0 with no minimal* *embedded hyperspheres and curvature decaying at infinity. Then for any* $\epsilon > 0$ *there exist* $\rho = \rho(\epsilon) > 0$ *and* $C = C(\epsilon)$ *such that*

$$\sup_{(\mathbb{R}^{n+1}\setminus B(\boldsymbol{o},\rho))\times[0,T)} |\mathrm{Rm}_{g(t)}|_{g(t)} \leq \epsilon, \qquad \sup_{B(\boldsymbol{o},\rho))\times[0,T)} \phi^2 |\mathrm{Rm}_{g(t)}|_{g(t)} \leq C.$$

Proof. From the rotational symmetry we derive that the geodesic equation for $\gamma : [0,1] \to \mathbb{R}^{n+1}$ in the radial component can be written as

$$\gamma_r''(t) + \frac{\partial_r \xi}{\xi}(\gamma_r')^2(t) - \frac{\phi\, \partial_r \phi}{\xi^2}||\gamma_{S^n}'||^2_{g_{S^n}}(t) = 0,$$

with γ_{S^n}' the spherical velocity vector and ξ and ϕ the warping coefficients as in (1.8). If $\partial_r \phi > 0$ for any r, then at any stationary point of γ_r we find $\gamma_r'' > 0$ unless γ_{S^n}' vanishes as well. We may then conclude that given g_0 as in the statement, there are no (non-trivial) closed geodesics and hence there exists $\iota = \iota(g_0) > 0$ such that $\mathrm{inj}(g_0) \geq \iota$ being the curvature bounded. Therefore from Chau et al. [2011] we get that for any $\epsilon > 0$ there exists a radius ρ sufficiently large such that the curvature stays bounded by ϵ in the complement of the Euclidean ball $B(\boldsymbol{o}, \rho)$ uniformly in $[0, T)$. The second estimate in the statement follows from Lemma 1.8 once we know that the curvature and hence the radius are uniformly bounded along the hypersphere of radius 2ρ. □

Remark 1.1. *In Lemma 1.15 we consider a much larger set of initial data than that analysed in Oliynyk and Woolgar [2007]. In our setting we only require the curvature to decay at spatial infinity at some rate, without prescribing it to be stronger than quadratic. As a consequence of that, Lemma 1.15 applies, for example, to initial metrics that open up to infinity either logarithmically or polynomially.*

1.4 Blow-up of Ricci flow with no minimal hyperspheres

Throughout this section we consider the maximal Ricci flow solution $(\mathbb{R}^{n+1}, g(t))$ evolving from a rotationally symmetric metric g_0 satisfying either the assumptions in Lemma 1.13 or those in Lemma 1.15. Moreover, we assume that the maximal time of existence T is *finite*. By analysing the possible singularity models of the flow we prove Theorem 1.1 and Theorem 1.2.

1.4.1 Singularity models of Ricci flows with no minimal hyperspheres

As observed above, the lack of minimal hyperspheres implies that $(\mathbb{R}^{n+1}, g_0)$ does not contain closed geodesics so that the injectivity radius of g_0 is bounded away from zero. We can thus apply the adaptation of Perelman's no-local collapsing theorem Perelman [2002] to complete, bounded curvature Ricci flows as in Theorem 0.3: the solution $g(t)$ is weakly $\kappa-$non-collapsed in $\mathbb{R}^{n+1} \times (T/2, T)$ at any scale $r \in (0, \sqrt{T/2})$, with κ some positive constant only depending on g_0 and T. By Hamilton's compactness theorem (see Theorem 0.4) there exist blow-up sequences (p_j, t_j) such that $\lambda_j \doteq |\text{Rm}_{g(t_j)}|_{g(t_j)}(p_j) \to \infty$ and the rescaled Ricci flows $(\mathbb{R}^{n+1}, g_j(t), p_j)$ defined by $g_j(t) \doteq \lambda_j g(t_j + t/\lambda_j)$ converge in the pointed Cheeger-Gromov sense to an ancient solution $(M_\infty, g_\infty(t), p_\infty)_{-\infty<t\leq\omega}$, with $\omega \geq 0$, satisfying

(i) $g_\infty(t)$ is complete,

(ii) $\sup_{M_\infty\times(-\infty,\omega]} |\text{Rm}_{g_\infty(t)}|_{g_\infty(t)} < \infty$,

(iii) $g_\infty(t)$ is non-flat,

(iv) $g_\infty(t)$ is (weakly) κ-non-collapsed.

We call any limit ancient solution $(M_\infty, g_\infty(t), p_\infty)$ a *singularity model* for the flow.
By spherical symmetry, given a blow-up sequence (p_j, t_j) we take $p_j = (r_j, \bar{\theta})$ for some $\bar{\theta} \in S^n$. Without loss of generality we can set $r_j < \rho$ whenever there exists ρ defined as in Lemma 1.15, otherwise the curvature would stay bounded along the blow-up sequence. According to [Chow et al., 2007, Chapter 4]), we can explicitly take an exhaustion $\{U_j\}$ of M_∞ and diffeomorphisms $\Phi_j : U_j \to B_{g_j(0)}(p_j, 2^j)$ arising from the Cheeger-Gromov-Hamilton convergence. For any $\nu > 0$ we define

$$V(j, \nu) \doteq \bigcup_{\theta\in S^n} B_{g(t_j)}((r_j, \theta), \frac{\nu}{\sqrt{\lambda_j}}) = \bigcup_{\theta\in S^n} B_{g_j(0)}((r_j, \theta), \nu). \tag{1.21}$$

The rotational symmetry of the solutions ensures that $V(j, \nu)$ are annular regions in $\mathbb{R}^{n+1}$. An argument similar to the one below for a rotationally invariant Kähler Ricci flow was discussed in Song [2014].

Lemma 1.16. *Any singularity model is simply connected.*

Proof. We first prove a preliminary property.

Claim 1.17. *There exists a positive radius $\bar{\nu}$ independent of j such that*

$$B_{g_j(0)}(p_j, \nu) \subset V(j, \nu) \subset B_{g_j(0)}(p_j, 2\nu),$$

for any $\nu \geq \bar{\nu}$.

Proof of Claim 1.17. It suffices to show that $d_{g_j(0)}((r_j, \bar{\theta}), (r_j, \theta)) \leq C < \infty$, for any $\theta \in S^n$ and uniformly in j. The bound follows from Lemma 1.13 and Lemma 1.15. Namely, we can find some positive constant α only depending on the dimension such that

$$d_{g_j(0)}((r_j, \bar{\theta}), (r_j, \theta)) = \sqrt{\lambda_j}\, d_{g(t_j)}((r_j, \bar{\theta}), (r_j, \theta)) \leq \alpha \sqrt{\lambda_j}\, \phi(r_j, t_j) \leq C$$

□

We note that by Claim 1.17 the maps Φ_j^{-1} given by Hamilton's Compactness theorem are well defined on $V(j, 2^{j-1})$ for $j > j_0$, for some j_0. We can pick $\tilde{j_0} > j_0$ sufficiently large such that for any $q \in \overline{U_{j_0}}$ we have

$$d_{\Phi^*_{\tilde{j_0}+2} g_{\tilde{j_0}+2}(0)}(p_\infty, q) \leq 1 + d_{\Phi^*_{\tilde{j_0}} g_{\tilde{j_0}}(0)}(p_\infty, q) \leq 1 + d_{g_{\tilde{j_0}}(0)}(p_{\tilde{j_0}}, \Phi_{\tilde{j_0}}(q)) \leq 1 + 2^{\tilde{j_0}} < 2^{\tilde{j_0}+1}.$$

We thus obtain the following inclusions

$$\overline{U_{j_0}} \subset \Phi^{-1}_{\tilde{j_0}+2}\left(B_{g_{\tilde{j_0}+2}(0)}(p_{\tilde{j_0}+2}, 2^{\tilde{j_0}+1})\right) \subset \Phi^{-1}_{\tilde{j_0}+2}\left(V(\tilde{j_0}+2, 2^{\tilde{j_0}+1})\right) \doteq \tilde{V}_1 \subset M_\infty.$$

Since we can iterate the method by replacing $\overline{U_{j_0}}$ with $\overline{U_{\tilde{j_0}+2}}$, we may conclude that M_∞ admits an exhaustion $\{\tilde{V}_j\}$ of simply connected open sets. This completes the proof. □

Next we characterize the geometry of the possible singularity models.

Lemma 1.18. *Any singularity model is a locally conformally flat κ-solution.*

Proof. If $n = 2$ (i.e. the three-dimensional case) then the Cotton tensor of the singularity model is identically zero. Moreover, from Chen [2009] it follows that the curvature operator of the singularity model is nonnegative. Similarly, when $n \geq 3$ the Weyl tensor of the singularity model is identically zero. Accordingly, by Zhang [2008] we also derive that the curvature operator is nonnegative. In particular, since any singularity model is weakly κ-non collapsed at all scales, we find that any singularity model is a κ-solution. □

1.4.2 Type-II singularities

We may now address the proof of Theorem 1.1. The argument relies on the characterization of Type-I singularities described in Enders et al. [2011]. Similarly to Enders et al. [2011], we introduce the singular set $\Sigma \subset \mathbb{R}^{n+1}$, which is defined by the following property: given a point $p \in \mathbb{R}^{n+1}$, then $|\mathrm{Rm}_{g(t)}|_{g(t)}$ stays bounded in some neighbourhood of p as $t \nearrow T$ if and only if $p \in \mathbb{R}^{n+1} \setminus \Sigma$.

Proof of Theorem 1.1. Assume for a contradiction that $g(t)$ is a Type-I Ricci flow. If the origin is not in Σ then by definition of singular set we can find some small $\delta > 0$ such that $|\mathrm{Rm}_{g(t)}|_{g(t)} \leq C < \infty$ uniformly in $B(\boldsymbol{o}, 2\delta) \times [0, T)$. Therefore there exists $\nu > 0$ such that $\phi(\delta, t) \geq \nu > 0$ for any $t \in [0, T)$. By Lemma 1.12 we get $\phi(r, t) \geq \phi(\delta, t) \geq \nu$ for any $r \geq \delta$ and for any $t \in [0, T)$. From (ii) of Lemma 1.13 we derive that $|\mathrm{Rm}| \leq C$ outside $B(\boldsymbol{o}, 2\delta)$ uniformly in time. By Shi [1989] the last condition implies that the flow smoothly extends to time T, which is a contradiction.

We may thus consider the case $\boldsymbol{o} \in \Sigma$. By [Enders et al., 2011, Theorem 1.1] we can parabolically dilate the solution at the origin and obtain, up to passing to a subsequence, a *non-flat* gradient shrinking soliton in canonical form $(M_\infty, g_\infty(t))$. Since the soliton is non-compact, simply connected (Lemma 1.16) and locally conformally flat (Lemma 1.18), from the classification in [Zhang, 2008, Theorem 1.2] we derive that $(M_\infty, g_\infty(t))$ must be a shrinking cylinder. By the Cheeger-Gromov-Hamilton convergence and the rotational symmetry we conclude that the cylinder $\mathbb{R} \times S^n$ is exhausted by open sets diffeomorphic to $\mathbb{R}^{n+1}$. This is a contradiction[1]. Since by (i) of Lemma 1.13 $g(t)$ develops a finite-time singularity, we have just shown that this singularity must be Type-II.

[1]Explicitly, if the cylinder admitted an exhaustion by open sets diffeomorphic to $\mathbb{R}^{n+1}$, then its rank 1 compactly supported de Rham cohomology group would be trivial.

Once we know that $g(t)$ is a Type-II flow we can pick a blow-up sequence whose associated singularity model $(M_\infty, g_\infty(t))$ is an *eternal* solution to the Ricci flow with $|\mathrm{Rm}_\infty|$ attaining its supremum in the space-time (see, e.g., [Hamilton, 1995, Section 16]). By Lemma 1.18 we derive that $g_\infty(t)$ is a κ-solution with nonnegative curvature operator. Therefore the scalar curvature and the Riemann curvature are comparable up to the singular time and we can hence adapt the argument in Hamilton [1995] to extract a space-time sequence (p_j, t_j), with $t_j \nearrow T$, such that if we set $\lambda_j \doteq R_{g(t_j)}(p_j)$, then the rescaled Ricci flows $(\mathbb{R}^{n+1}, g_j(t), p_j)$ defined by $g_j(t) \doteq \lambda_j g(t_j + (\lambda_j)^{-1} t)$ (sub)converge in the pointed Cheeger-Gromov sense to a κ-solution whose scalar curvature attains its supremum in the space-time. By Hamilton's rigidity result Hamilton [1993a] we deduce that $(M_\infty, g_\infty(t))$ is a steady gradient Ricci soliton. Finally by Cao and Chen [2012] we conclude that this steady soliton is isometric to the Bryant soliton (up to scaling) Bryant [2005]. □

Remark 1.2. *One can improve the result in Theorem 1.1 and obtain that due to Brendle [2020] in dimension three and Catino et al. [2015] and Li and Zhang [2018] in dimension greater than three, any blow-up sequence whose associated singularity model is not a shrinking cylinder gives rise to the Bryant soliton in the limit.*

Remark 1.3. *The argument above highlights that the lack of minimal spheres guarantees that the curvature does not concentrate locally around some neck-region, the singularity being* slowly *forming. We also point out that the non-compactness of the underlying manifold played a crucial role. In the analogous case of S^{n+1} one has to take into account global Type-I singularities where the volume of the manifold approaches zero in the limit.*

We finally show that the Bryant soliton has to appear at the origin $\boldsymbol{o} \in \mathbb{R}^{n+1}$ if the pinching condition (1.20) is satisfied.

Corollary 1.19. *Under the same hypotheses as Theorem 1.1, if further the initial metric satisfies $K_{g_0} \leq L_{g_0}$ then there exists a sequence $t_j \nearrow T$ such that the rescaled Ricci flows $(\mathbb{R}^{n+1}, g_j(t), \boldsymbol{o})$ defined by $g_j(t) = R_{g(t_j)}(\boldsymbol{o}) g(t_j + t/R_{g(t_j)}(\boldsymbol{o}))$ on $[-R_{g(t_j)}(\boldsymbol{o}) t_j, 0]$ converge to the Bryant soliton (up to scaling).*

Proof. By Theorem 1.1 we know that the flow develops a Type-II singularity at some $T < \infty$ and that there exist rescaled Ricci flows $(\mathbb{R}^{n+1}, g_j(t), p_j)$ defined by $g_j(t) =$

$R_{g(t_j)}(p_j)g(t_j+t/R_{g(t_j)}(p_j))$ smoothly converging to the Bryant soliton, for some $t_j \nearrow T$. Since by Lemma 1.14 the scalar curvature attains its maximum at the origin, we conclude that the rescaled sequence $(\mathbb{R}^{n+1}, g_j(t), \boldsymbol{o})$ defined by $g_j(t) = R_{g(t_j)}(\boldsymbol{o})g(t_j + t/R_{g(t_j)}(\boldsymbol{o}))$ converges to the Bryant soliton as well. □

1.4.3 Immortal solutions

We now analyse the maximal Ricci flow solution $g(t)$ evolving from a rotationally symmetric metric g_0 with no minimal embedded hyperspheres and curvature decaying to zero at infinity. We assume that such solution develops a singularity at some $T < \infty$ and we aim to exhibit a contradiction, hence proving Theorem 1.2. We first show that ϕ_s admits a uniform positive lower bound in the compact region where singularities may form, then we use the exhaustion constructed in Lemma 1.16 to prove that the condition about ϕ_s implies that the singularity model $(M_\infty, g_\infty(t))$ has positive asymptotic volume ratio.

Proof of Theorem 1.2. By Lemma 1.15 we deduce that there exists $\rho_1 > 0$ such that $|\mathrm{Rm}_{g(t)}|_{g(t)} \leq C$ on $\mathbb{R}^{n+1} \setminus B(\boldsymbol{o}, \rho_1)$ uniformly with respect to time, for some $C > 0$. Furthermore, the estimate (1.19) holds in $B(\boldsymbol{o}, \rho_1) \times [0, T)$. In particular, from Lemma 1.9 and Lemma 1.12 it follows that there exists $\rho_0 < \rho_1$ satisfying $\lim_{t\nearrow T} \phi(r,t) = 0$ for any $r < \rho_0$ while $\lim_{t\nearrow T} \phi(r,t) > 0$ for all $r \in (\rho_0, \rho_1]$.
Given any $r \in (\rho_0, \rho_1)$ by Lemma 1.15 we find that

$$(|1 - \phi_s^2| + |\phi_{ss}|)(r,t) = (\phi^2|L| + \phi|K|)(r,t) \leq C(r) < \infty,$$

for all $t \in [0, T)$. Therefore both ϕ_s and ϕ_{ss} are uniformly bonded at any radius $r \in (\rho_0, \rho_1)$. Once we choose ρ_1 large enough, we let $\rho_0 + 1 < r_0 < r_1 < \rho_1$ and $\delta > 0$ satisfy

$$\phi(r_1, t) - \phi(r_0, t) \geq \phi(r_1, 0)e^{-CT} - \phi(r_0, 0)e^{CT} \geq \delta > 0,$$

where we have used that given $\delta > 0$ we can always find r_1 and r_0 as above because $\phi(r, 0) \to \infty$ as observed in Section 1.3.2.

Claim 1.20. *There exists $\tilde{r} > \rho_0$ and $\tilde{\beta} > 0$ such that $\phi_s(\tilde{r}, t) \geq \tilde{\beta} > 0$ for any $t \in [0, T)$.*

Proof of Claim 1.20. Let $r > \rho_0$. Once we fix an angle $\theta \in S^n$ we can extend a local g_{S^n}-orthonormal frame $\{e_i\}$ on S^n to a $g(t)$-orthonormal frame around $p = (r, \theta)$ of the form $\{\partial_s, e_i/\phi\}$ for any $t \in [0, T)$. Using the commutator formula (1.11) and the Koszul formula we find

$$\begin{aligned}\partial_t(\partial_s\phi)(p,t) &= \partial_s\left(-\mathrm{Ric}_{g(t)}\left(\frac{e_i}{\phi},\frac{e_i}{\phi}\right)\phi\right)(p,t) + nK\phi_s(p,t)\\ &= -\mathrm{Ric}_{g(t)}\left(\frac{e_i}{\phi},\frac{e_i}{\phi}\right)\phi_s(p,t) - \left(\mathrm{Ric}_{g(t)}\left(\frac{e_i}{\phi},\frac{e_i}{\phi}\right)\right)_s\phi(p,t) + nK\phi_s(p,t)\\ &= -\mathrm{Ric}_{g(t)}\left(\frac{e_i}{\phi},\frac{e_i}{\phi}\right)\phi_s(p,t) - \nabla_{g(t)}\mathrm{Ric}_{g(t)}\left(\partial_s,\frac{e_i}{\phi},\frac{e_i}{\phi}\right)\phi(p,t) + nK\phi_s(p,t).\end{aligned} \tag{1.22}$$

Since $r > \rho_0$ there exists $\gamma = \gamma(r) > 0$ such that $\phi(r,t) \geq \gamma > 0$ as long as the solution exists. From Lemma 1.15 and Shi's derivative estimates we deduce that $(|\mathrm{Rm}_{g(t)}| + |\nabla_{g(t)}\mathrm{Rm}_{g(t)}|)(r,t) \leq C(r) < \infty$ uniformly in $[0,T)$. Therefore we can bound the right hand side of (1.22) by a uniform positive constant only depending on r, thus obtaining that $\phi_s(r,\cdot)$ is a Lipschitz function of time on $[0,T)$. A consequence of this fact is that $\phi_s(r,\cdot)$ admits a (finite) limit as $t \nearrow T$ for any $r > \rho_0$.
Assume for a contradiction that any such limit is zero. Since the curvature is controlled in the annular region $(r_0, r_1) \times S^n$, by standard distortion estimates of the distance Hamilton [1995] we get

$$\begin{aligned}\delta &\leq \phi(r_1,t) - \phi(r_0,t) \leq \sup_{[r_0,r_1]}\phi_s(\cdot,t)(s(r_1,t) - s(r_0,t))\\ &\leq C\sup_{[r_0,r_1]}\phi_s(\cdot,t)(s(r_1,0) - s(r_0,0)) \leq C\sup_{[r_0,r_1]}\phi_s(\cdot,t),\end{aligned}$$

for any $t \in [0,T)$. Since by Lemma 1.15 ϕ_{ss} is uniformly bounded in $[r_0,r_1] \times [0,T)$ we conclude that $\sup_{[r_0,r_1]}\phi_s(\cdot,t) \to 0$ as $t \nearrow T$ as long as $\phi_s(r,t) \to 0$ for any $r > \rho_0$. This is a contradiction. Thus there exists $\tilde{r} > \rho_0$ such that $\lim_{t\nearrow T}\phi_s(\tilde{r},t) \neq 0$. By Lemma 1.12 we deduce that there exists $\tilde{\beta} > 0$ as in the statement of the Claim. $\square$

From the boundary conditions (1.2) and Claim 1.20 we derive that if ϕ_s approaches zero in $B(\boldsymbol{o}, \tilde{r})$ as $t \nearrow T$, then this must happen along a sequence of interior minima.

However, the maximum principle applied to the evolution equation (1.12) shows that this is not possible. We thus find $\beta > 0$ such that for any $t \in [0, T)$ the following holds:

$$\inf_{B(\boldsymbol{o},\tilde{r})} \phi_s(\cdot, t) \geq \beta. \tag{1.23}$$

Consider a standard parabolic rescaling $g_j(t)$ along a blow-up sequence (p_j, t_j), with $p_j = (r_j, \bar{\theta})$, converging smoothly on compact sets to a singularity model $(M_\infty, g_\infty(t), p_\infty)_{t\in(-\infty,\omega]}$. We need to show that the rescaled geodesic balls stay inside $B(\boldsymbol{o}, \tilde{r})$ for j large enough.

Claim 1.21. *For any $\nu > 0$ there exists $j_0 = j_0(\nu)$ such that for all $j \geq j_0$ the following holds:*

$$B_{g_j(0)}(p_j, \nu) \subset B(\boldsymbol{o}, \tilde{r}).$$

Proof of Claim 1.21. Assume for a contradiction that there exist $\nu > 0$ and a subsequence $q_j = (y_j, \theta_j)$ such that $q_j \in B_{g_j(0)}(p_j, \nu)$ and $y_j > \tilde{r}$. By the Cheeger-Gromov-Hamilton convergence $\Phi_j^{-1}(q_j) \in B_{g_\infty(0)}(p_\infty, 2\nu)$ for j large enough. Therefore by Lemma 1.15 we find that $R_{g_\infty(0)}$ vanishes at some $q_\infty \in B_{g_\infty(0)}(p_\infty, 2\nu)$. Since the singularity model has nonnegative curvature operator (Lemma 1.18) by a standard application of the maximum principle we deduce that $g_\infty(0)$ is flat. □

Let $\nu > \bar{\nu}$ with $\bar{\nu}$ given in Claim 1.17. Consider the annular region $V(j, \nu)$ defined as in (1.21). We let $\mu_j > 0$ be the positive quantity satisfying

$$s(r_j + \mu_j, t_j) - s(r_j, t_j) \equiv \int_{r_j}^{r_j+\mu_j} \xi(r, t_j)dr = \frac{\nu}{\sqrt{\lambda_j}}.$$

Assume that j is large enough such that the inclusion in Claim 1.21 is verified. Equivalently, we have $r_j + \mu_j \leq \tilde{r}$. From Lemma 1.12 and the lower bound (1.23) it follows that

$$\phi(r, t_j) = \int_0^r \phi_r(y, t_j)dy \geq \int_{r_j}^r (\phi_s \xi)(y, t_j)dy \geq \beta(s(r, t_j) - s(r_j, t_j)),$$

for any $r \in (r_j, r_j + \mu_j)$. We thus obtain

$$\begin{aligned} \mathrm{Vol}_{g(t_j)}(V(j,\nu)) &\geq \mathrm{Vol}_{g(t_j)}(S^n \times (r_j, r_j+\mu_j)) = C(n)\int_{r_j}^{r_j+\mu_j} (\phi^n\xi)(r,t_j)dr \\ &\geq C(n)\int_{r_j}^{r_j+\mu_j} \beta^n(s(r,t_j)-s(r_j,t_j))^n\xi(r,t_j)dr \\ &= C(n)(s(r_j+\mu_j,t_j)-s(r_j,t_j))^{n+1} = C(n)\frac{\nu^{n+1}}{\lambda_j^{\frac{n+1}{2}}}, \end{aligned} \tag{1.24}$$

for some positive constant C independent of j that we have renamed from line to line. We finally conclude that for any $\nu \geq \bar{\nu}$, with $\bar{\nu}$ defined in Claim 1.17, by the Cheeger-Gromov-Hamilton convergence and (1.24) there exists j large enough satisfying

$$\mathrm{Vol}_{g_\infty(0)}B_{g_\infty(0)}(p_\infty, 4\nu) \geq \mathrm{Vol}_{g_j(0)}B_{g_j(0)}(p_j, 2\nu) \geq \mathrm{Vol}_{g_j(0)}V(j,\nu) \geq C(n)\nu^{n+1}.$$

The last inequality implies that $g_\infty(0)$ has Euclidean volume growth, meaning that the asymptotic volume ratio of $g_\infty(0)$ is positive. Since we have already shown that any singularity model is a κ-solution (see Lemma 1.18), by [Perelman, 2002, Proposition 11.4] we derive the contradiction. Therefore the solution is immortal. □

1.5 Rotationally symmetric Ricci flow with nonnegative curvature

In this section we classify rotationally symmetric Ricci flows with bounded nonnegative Ricci tensor. We also comment on some properties satisfied by standard solutions in the sense of Lu and Tian.

Let g_0 be a complete rotationally symmetric metric on $\mathbb{R}^{n+1}$ with bounded nonnegative Ricci curvature and consider the maximal Ricci flow solution $(\mathbb{R}^{n+1}, g(t))_{0\leq t<T}$ evolving from g_0. Since by rotational symmetry the Ricci tensor is nonnegative if and only if the curvature operator is nonnegative, we can apply Hamilton's strong maximum principle Hamilton [1986] and rely again on the rotational symmetry to conclude that $\mathrm{Ric}_{g(t)} > 0$ for all $t \in (0,T)$ because the curvature operator of g_0 has non trivial kernel at any point if and only if g_0 is flat. We first report the following result, which allows to compute the maximal time of existence for positively curved solutions that are asymptotic to a round cylinder at spatial infinity.

Proposition 1.22 (Chen and Zhu [2006a], Theorem A.1)**.** *Let* $(\mathbb{R}^{n+1}, g(t))_{0\leq t<T}$ *be the maximal Ricci flow solution evolving from a complete metric* g_0 *with bounded nonnegative curvature operator, positive scalar curvature and which is asymptotic to a round cylinder of radius* ρ_0 *at spatial infinity. Then the solution satisfies:*

$$\inf_{p\in\mathbb{R}^{n+1}} R_{g(t)}(p) \geq \frac{C}{T-t},$$

for some $C > 0$*, where* $T = \frac{\rho_0^2}{2(n-1)}$.

We may now address the proof of Corollary 1.3.

Proof of Corollary 1.3. Given $t > 0$ we have $K(\cdot, t) > 0$, which is equivalent to $\phi_{ss}(\cdot, t) < 0$ on $\mathbb{R}^{n+1} \setminus \{\boldsymbol{o}\}$. The last condition implies $0 < \phi_s(\cdot, t) \leq 1$, for if $\phi_s(r_0, t) = 0$, at some r_0, then we would have $\phi_s(r, t) < 0$ for any $r > r_0$ meaning that the solution is not smooth on $\mathbb{R}^{n+1}$ because the radial function has to vanish at some finite radial coordinate. Therefore $(\mathbb{R}^{n+1}, g(t))$ does not contain minimal embedded hyperspheres for any $t \in (0, T)$. We then need to consider two different cases, depending on whether the radius $\phi(\cdot, t)$ is bounded or unbounded.
The radius is bounded. Suppose that $\phi(r, 0) \to r_0 < \infty$. The same argument for the proof of (ii) in Lemma 1.13 shows that the solution is smoothly asymptotic to a round cylinder at spatial infinity for any $t \in (0, T)$. From Proposition 1.22 we derive that the solution develops a global singularity at $T = \rho_0^2/2(n-1)$. We finally apply Theorem 1.1 to deduce that the singularity is Type-II and is modelled on the Bryant soliton once suitably dilated.
The radius is unbounded. If $\phi(r, 0) \to \infty$ as $r \to \infty$ then $\phi(r, t) \to \infty$ for any $t \in [0, T)$ because the curvature stays bounded until time T by Shi [1989]. Let us fix $t_0 > 0$. Since $0 < \phi_s(\cdot, t_0) \leq 1$ we get $|L(\cdot, t_0)| \to 0$ as $r \to \infty$. Furthermore $\phi_s(\cdot, t_0)$ has a (finite) limit at infinity being $\phi_{ss}(\cdot, t_0) \leq 0$. It follows that $\phi_{ss}(\cdot, t_0)$ and hence $K(\cdot, t_0)$ are integrable. Since by Shi's derivative estimates $|K_s(\cdot, t_0)| \leq C$ we obtain $K(r, t_0) \to 0$ as

$r \to \infty$. We have thus shown that $|\mathrm{Rm}_{g(t_0)}|_{g(t_0)} \to 0$ at infinity. By applying Theorem 1.2 we conclude that the solution is immortal. □

1.5.1 Standard solutions to the Ricci flow

In the following we relate the classification in Corollary 1.3 to the family of standard solutions introduced by Lu and Tian, where they generalized Perelman's class of special solutions discussed in Perelman [2003b]. According to Lu and Tian, we have the following:

Definition 1.1. *We let $\mathfrak{S}_{n+1}$ be the set of (smooth) complete rotationally symmetric metrics g on $\mathbb{R}^{n+1}$ satisfying*

(i) There exists a sequence of points $p_j \to \infty$ in $\mathbb{R}^{n+1}$ such that $(\mathbb{R}^{n+1}, g, p_j)$ converges to the round cylinder $(\mathbb{R} \times S^n, (dr)^2 + \rho^2 g_{S^n}, p^*)$ in pointed C^3 Cheeger-Gromov topology, for some $\rho > 0$.

(ii) $\mathrm{Rm}_g \geq 0$ everywhere and $\mathrm{Rm}_g(p) > 0$ for some $p \in \mathbb{R}^{n+1}$.

(iii) There exists $\alpha > 0$ such that on $\mathbb{R}^{n+1}$

$$|\mathrm{Rm}_g|_g + \sum_{k=1}^{4} |\nabla^k \mathrm{Rm}_g|_g \leq \alpha.$$

Remark 1.4. *To prove that $\mathfrak{S}_{n+1}$ is non-empty and to get a feeling for the metrics contained in this set, we consider g of the form* (1.3) *with $\phi(s) = \arctan(s)$. We first verify that ϕ satisfies* (1.2) *so that g is smooth, complete, and with bounded curvature on any compact region. Conditions (ii) and (iii) of Definition 1.1 follow from the formulas for the sectional curvatures* (1.5). *For what concerns the convergence to the round cylinder of radius $\rho = \pi/2$, we introduce the exhaustion $U_j \doteq (-j, \infty) \times S^n$ and the family of translations $\Phi_j : U_j \to (j, \infty) \times S^n \doteq V_j$ defined by $s \mapsto s + 2j$. Once we fix an angle $\theta \in S^n$, the embeddings satisfy $\Phi_j(0, \theta) = (2j, \theta)$ for any j. If we denote the points $(2j, \theta)$ by p_j and $(0, \theta)$ by p^*, property (i) in Definition 1.1 is then equivalent to showing that*

$$g|_{V_j} \xrightarrow{C^3} g_{cyl}, \qquad g_{cyl} = (dr)^2 + (\pi^2/4) g_{S^n}.$$

Finally such convergence follows from the fact that $|\partial_s^k \arctan(s)| \to 0$ *uniformly in* V_j *as* $j \to \infty$*, for any* $k = 1, 2, 3$.

Since any $g_0 \in \mathfrak{S}_{n+1}$ is complete with bounded curvature there exists a solution to the Ricci flow starting at g_0 Shi [1989] and such solution is unique in the class of complete solutions with bounded curvature on compact subintervals Chen and Zhu [2006b]. As a consequence of that, we may give the following definition, due to Lu and Tian.

Definition 1.2. *Let* $g_0 \in \mathfrak{S}_{n+1}$. *The maximal Ricci flow solution starting at* g_0 *is called a* standard solution.

We first report a result by Lu and Tian which shows that $\mathfrak{S}_{n+1}$ is closed with respect to the Ricci flow problem. In the following we let p_j and ρ be as in Definition 1.1.

Lemma 1.23 (Lu and Tian, Lemma 1)**.** *Let* $(\mathbb{R}^{n+1}, g(t))_{0 \leq t < T}$ *be the maximal solution to the Ricci flow starting at some* $g_0 \in \mathfrak{S}_{n+1}$. *For any* $T' \in (0, T)$ *there exists a subsequence of* $(\mathbb{R}^{n+1}, g(t), p_j)_{0 \leq t \leq T'}$ *which converges in* C^3 *Cheeger-Gromov pointed topology to a self-similar shrinking cylinder*

$$g_{\text{cyl}}(t) = (ds)^2 + (\rho^2 - 2(n-1)t) g_{S^n}.$$

The previous Lemma, Shi's derivative estimates Shi [1989] and Hamilton's strong maximum principle for systems Hamilton [1986] imply the following

Corollary 1.24. *Let* $(\mathbb{R}^{n+1}, g(t))_{0 \leq t < T}$ *be the maximal solution to the Ricci flow starting at some* $g_0 \in \mathfrak{S}_{n+1}$. *Then* $g(t) \in \mathfrak{S}_{n+1}$ *for any* $t \in [0, T)$.

By Proposition 1.22 we know that standard solutions survive until some finite time T which only depends on the dimension and the asymptotic round cylinder at infinity. At time $t = T$ the solution extinguishes globally. In order to fully understand the nature of this singularity one needs to classify its type and the possible limits of blow-ups. In this regard, item (i) of Corollary 1.3 immediately gives us the following, which also proves the conjecture by Chow and Tian in Corollary 1.4.

Theorem 1.25. *Any standard solution in the sense of Lu and Tian develops a global Type-II singularity at some finite time $T < \infty$. Moreover, the singularity is modelled on the Bryant soliton once suitably dilated.*

According to Lu and Tian, in the previous result we do not require standard solutions to have a well-defined tip. However we can provide a simple characterization of those standard solutions whose curvature concentrates at the origin: the conclusions of Corollary 1.19 can be strengthened in the case of standard solutions by using the trace of the Harnack estimate Hamilton [1993b] and the results in Brendle [2020] and Li and Zhang [2018].

Proof of Corollary 1.5. Given a sequence $t_j \nearrow T$, since the solution has nonnegative curvature and hence the flow is only controlled by the scalar curvature, by Lemma 1.14 and the trace of the Harnack estimate we conclude that the rescaled Ricci flows $(\mathbb{R}^{n+1}, g_j(t), \boldsymbol{o})$ defined on $[-R_{g(t_j)}(\boldsymbol{o})t_j, 0]$ by $g_j(t) \doteq R_{g(t_j)}(\boldsymbol{o})g(t_j + t/R_{g(t_j)}(\boldsymbol{o}))$ (sub)converge in the pointed Cheeger-Gromov topology. Lemma 1.18 then implies that any limit must be a locally conformally flat κ-solution with nonnegative curvature operator. If $n = 2$, then we can apply Brendle [2020], while if $n > 2$ we can first deduce that singularity model is rotationally symmetric by Catino et al. [2015] and then apply Li and Zhang [2018] to conclude that the singularity model is isometric to the Bryant soliton otherwise we would get an exhaustion of the cylinder by open sets diffeomorphic to $\mathbb{R}^{n+1}$. □

1.6 Ricci flow with necks

In this section we adapt the analysis in Angenent and Knopf [2004] to address the expectations in Oliynyk and Woolgar [2007] discussed in the introduction. Accordingly, we consider rotationally invariant asymptotically flat Ricci flows containing minimal hyperspheres. We explicitly describe a characterization of a *sufficiently pinched* initial neck leading to the formation of a Type-I singularity and we provide examples of initial necks disappearing in finite time along the Ricci flow. For simplicity, in the following we discuss the case where the solution has only one neck, but the conclusions easily generalize to the case of multiple necks.

1.6.1 Type-I neckpinches

Throughout this subsection we consider the maximal Ricci flow solution $g(t)$ defined on $[0, T)$, for some $T < \infty$, evolving from a rotationally symmetric asymptotically flat metric g_0. We refer to Oliynyk and Woolgar [2007] for a detailed analysis of this condition along the flow. For our purposes it suffices to note that there exists $\epsilon > 0$ such that

$$r^{2+\epsilon}|\mathrm{Rm}_{g(t)}|_{g(t)} \leq C(t) < \infty \tag{1.25}$$

for some positive constant depending continuously on $t \in [0, T)$, with r the fixed radial coordinate on $\mathbb{R}^{n+1}$. Indeed, the stronger than quadratic decay of the curvature is preserved along the flow Oliynyk and Woolgar [2007]. In the following we call such flow an ϵ-*asymptotically flat* Ricci flow.

Lemma 1.26. *Let* $(\mathbb{R}^{n+1}, g(t))_{0 \leq t < T}$ *be a rotationally symmetric* ϵ*-asymptotically flat Ricci flow, for some* $\epsilon > 0$ *and* $T < \infty$*. Then there exist* $\rho > 0$ *and* $C > 0$ *such that*

$$\sup_{(\mathbb{R}^{n+1}\setminus B(\boldsymbol{o},\rho))\times[0,T)} \phi^{2+\frac{\epsilon}{2}}|\mathrm{Rm}_{g(t)}|_{g(t)} \leq C.$$

Proof. Since the curvature is decaying to zero at infinity and the metric is close to the Euclidean metric, in a precise way, outside a compact region, we can apply Chau et al. [2011] and deduce that there exists $\rho > 0$ such that $|\mathrm{Rm}_{g(t)}|_{g(t)} \leq 1$ in $\mathbb{R}^{n+1} \setminus B(\boldsymbol{o}, \rho)$ uniformly in time. Define $f \doteq \phi^{\alpha}|\mathrm{Rm}_{g(t)}|^2_{g(t)}$ on the complement of $B(\boldsymbol{o}, \rho)$ with $\alpha = 4 + \epsilon$. From (1.25) it follows that given $t \in [0, T)$ we have $\phi(r, t) \sim r$ for r large enough Oliynyk and Woolgar [2007]. Thus we derive that f is uniformly bounded along the parabolic boundary of the region. It then suffices to show that f cannot diverge along a sequence of interior maxima. The evolution equation of f is given by

$$f_t \leq \Delta f - 4\alpha\phi^{\alpha-1}\phi_s|\mathrm{Rm}|(|\mathrm{Rm}|)_s + |\mathrm{Rm}|^2\phi^{\alpha}\left(\frac{\alpha}{\phi^2}(-(n-1) - \alpha\phi_s^2) + C|\mathrm{Rm}|\right)$$

where we have used a standard estimate for the evolution of the curvature along the Ricci flow. At any interior maximum point (p_0, t_0) we find

$$f_t(p_0, t_0) \leq |\mathrm{Rm}|^2 \phi^\alpha \left(\frac{\alpha}{\phi^2}(-(n-1) + \alpha\phi_s^2) + C|\mathrm{Rm}| \right)(p_0, t_0).$$

Therefore, as long as $T < \infty$ we find

$$\dot{f}_{\max} \leq C f_{\max}$$

where the inequality follows again from the curvature being uniformly bounded in the region[2]. By integrating we obtain that f is uniformly bounded as long as $T < \infty$. □

Let g_0 be rotationally symmetric and ϵ-asymptotically flat. Adopting the same notations as in Angenent and Knopf [2004], we call a local maximum(minimum) for $\phi(\cdot, 0)$ a bump(neck) for g_0. In the case of the minimum we never consider the origin. We say that a bump(neck) is degenerate when the spatial second derivative of ϕ vanishes at the maximum(minimum) point. Otherwise, the bump(neck) is referred to as non-degenerate. We note that from the boundary conditions (1.2) and the asymptotics (1.25) the existence of a neck always implies the existence of a bump and vice versa. In the case of a single bump(neck) we are dealing with, one can use Lemma 1.26 to generalize [Angenent and Knopf, 2004, Lemma 5.5] to our setting:

Corollary 1.27. *Let $(\mathbb{R}^{n+1}, g(t))_{0 \leq t < T}$ be a rotationally symmetric ϵ-asymptotically flat Ricci flow evolving from g_0, for some $\epsilon > 0$ and $T < \infty$. Then the number of necks is non-increasing. In particular, all necks and bumps are non-degenerate except when they annihilate each other.*

Proof. According to Lemma 1.26, given $T < \infty$ we get

$$\phi^2|L| = |1 - \phi_s^2|(r, t) \leq \frac{C}{\phi^{\frac{\epsilon}{2}}}(r, t) \tag{1.26}$$

for some constant $C > 0$ and for any $r \geq \rho$, with L the sectional curvature defined as in (1.5). Since $\phi(r, 0) \to \infty$ at infinity, by (1.26) we may choose $r_0 > \rho$ large enough

[2] Explicitly, once we know that $\phi_s \to 1$ at spatial infinity as long as the solution exists (Oliynyk and Woolgar [2007]), we can then apply the maximum principle to the evolution equation of ϕ_s as in Lemma 1.8.

such that $\phi_s(r,t) \geq \beta > 0$ for any $r \geq r_0$ and for any $t \in [0,T)$ with β as close as we ask to 1. We can then apply the Sturmian theorem [Angenent, 1988, Theorem D] to the evolution equation of ϕ_s in $B(\boldsymbol{o}, r_0) \times [0,T)$ and conclude that the number of zeroes is non-increasing in time and drops whenever ϕ_s has a multiple zero. □

Suppose ϕ_0 has one isolated maximum at $r_1(0)$ and one isolated minimum at $r_2(0)$. We denote the radius of the bump by $\phi_{\text{max}}(0) \doteq \phi(r_1(0), 0)$ and the radius of the neck by $\phi_{\text{min}}(0) \doteq \phi(r_2(0), 0)$. The ratio $\lambda \doteq \phi_{\text{max}}(0)/\phi_{\text{min}}(0)$ provides then a measure for the initial pinching of the neck-like region. In the following we let $r_1(t)$ and $r_2(t)$ denote the radial coordinates of the bump and of the neck along the flow respectively.
By Corollary 1.27, $\phi(\cdot, t)$ is a Morse function except at the time where the bump and the neck annihilate each other. Thus both ϕ_{max} and ϕ_{min} are smooth functions of time until either they become equal or the flow develops a singularity.
The main idea for proving the existence of local Type-I singularities consists in choosing the initial ratio between ϕ_{max} and ϕ_{min} (i.e. the initial pinching of the neck) larger than some lower bound depending on the scale invariant difference between spherical sectional curvature and radial sectional curvature so that the radius of the neck vanishes at some finite time before that of the bump does. Namely, we aim to prove the following:

Theorem 1.28. *Let $(\mathbb{R}^{n+1}, g(t))$, with $n \geq 2$, be the maximal solution to the Ricci flow evolving from an asymptotically flat rotationally symmetric metric g_0 containing a neck region $(r_1(0), r_2(0)) \times S^n$. Assume that $\text{Ric}_{g_0} > 0$ on the closed Euclidean ball $B(\boldsymbol{o}, r_1(0))$ and that $R_{g_0} \geq 0$ on $\mathbb{R}^{n+1}$. Let β be defined as*

$$\beta \doteq \inf_{\mathbb{R}^{n+1}} \phi_0^2(L_{g_0} - K_{g_0}),$$

and let $\lambda > 0$ satisfy

$$\lambda^2 > \frac{n+1-2\beta}{n-1} + 1.$$

If $\phi_0(r_1(0)) \geq \lambda\phi_0(r_2(0))$ then the following are satisfied:

(i) The Ricci flow solution develops a local Type-I singularity at some $T < \infty$.

(ii) If we set $\sigma \doteq S/\sqrt{T-t}$, with S the distance from the neck, we can write the

following cylindrical asymptotics for some uniform constants $C > c > 0$:

$$\frac{\phi}{\sqrt{2(n-1)(T-t)}} \leq 1 + C\frac{\sigma^2}{|\log(T-t)|}$$

for $|\sigma| \leq c\sqrt{|\log(T-t)|}$, *and*

$$\frac{\phi}{\sqrt{2(n-1)(T-t)}} \leq C\frac{|\sigma|}{\sqrt{|\log(T-t)|}}\sqrt{\log\left(\frac{|\sigma|}{|\log(T-t)|}\right)}$$

whenever $c\sqrt{|\log(T-t)|} \leq |\sigma| \leq (T-t)^{-\frac{\varepsilon}{2}}$, *for* $\varepsilon \in (0,1)$.

We first restate a result proved in Angenent and Knopf [2004] and we omit the proof because it dose not require modifications.

Lemma 1.29 (Angenent and Knopf [2004], Lemma 5.6)**.** *Let $g(t)$ be a Ricci flow defined on $[0,T)$ starting at g_0 as above. If $\phi_{ss}(r,0) \leq 0$ for $0 \leq r \leq r_1(0)$ then $\phi_{ss}(r,t) \leq 0$ for all $0 \leq r \leq r_1(t)$ for any $t \in [0,T)$.*

We note that given g_0 as above the condition $\phi_{ss}(r,0) \leq 0$ for $0 \leq r \leq r_1(0)$ is equivalent to requiring the Ricci tensor of g_0 to be nonnegative on the Euclidean ball centred at the origin of radius $r_1(0)$.
From Lemma 1.26 we deduce that we can apply Lemma 1.8 to the Ricci flow $g(t)$ and thus obtain that the estimate (1.19) holds on $\mathbb{R}^{n+1} \times [0,T)$. Therefore, as shown in Lemma 1.9, we have $|(\phi^2)_t| \leq C$ uniformly in $[0,T)$. We get that the following limit exists

$$D \doteq \lim_{t \nearrow T} \phi(r_1(t),t) = \lim_{t \nearrow T} \phi_{\max}(t).$$

If $D > 0$ then there is no singularity forming around the origin. This was again proved in Angenent and Knopf [2004] using Lemma 1.29 and the estimate (1.19), which in particular implies that $A = \phi^2(L-K)$ is uniformly bounded. As above, we only adapt the statement to the current setting.

Lemma 1.30 (Angenent and Knopf [2004], Lemma 7.2)**.** *If $D > 0$ the cap $((0,r_1(t)) \times S^n) \cup \{\boldsymbol{o}\}$ stays smooth.*

We are now ready to address the proof of (i) of Theorem 1.28.

Proof of (i) of Theorem 1.28. We first note that the scalar curvature is positive for any $t \in (0, T)$. From (1.25) we deduce that $A(r, t) \to 0$ at infinity for any time Oliynyk and Woolgar [2007]. Therefore $A(\cdot, 0)$ has a nonpositive finite infimum $\beta \leq 0$. According to Lemma 1.10 such lower bound β is preserved along the flow.
Suppose that the neck disappears at some time $T' \in (0, T)$. From the discussion above it follows that ϕ_{max} and ϕ_{min} are hence smooth functions in $[0, T')$. By the implicit function theorem we find that the evolution equation of ϕ_{min} is given by (see also [Angenent and Knopf, 2004, Lemma 6.1]):

$$\dot{\phi}_{\text{min}}(t) = \phi_t(r_2(t), t) = \phi_{ss}(r_2(t), t) - \frac{n-1}{\phi_{\text{min}}}. \tag{1.27}$$

Since the scalar curvature is nonnegative we can bound the right hand side by

$$\dot{\phi}_{\text{min}}(t) \leq -\frac{n-1}{2\phi_{\text{min}}},$$

which upon integration yields

$$\phi^2_{\text{min}}(t) \leq \phi^2_{\text{min}}(0) - (n-1)t. \tag{1.28}$$

From (1.28) we deduce that $T' < \phi^2_{\text{min}}(0)/(n-1)$ otherwise the solution becomes singular before it loses its neck. We can similarly estimate the evolution equation of ϕ_{max} using the lower bound for A. We obtain

$$\phi^2_{\text{max}}(t) \geq \phi^2_{\text{max}}(0) + 2(\beta - n)t.$$

We finally conclude that for any $0 \leq t < T' < \phi^2_{\text{min}}(0)/(n-1)$ we have

$$\phi^2_{\text{max}}(t) - \phi^2_{\text{min}}(t) \geq \phi^2_{\text{max}}(0) - \phi^2_{\text{min}}(0) - (n-2\beta+1)t \geq \phi^2_{\text{min}}(0)\left(\lambda^2 - 1 - \frac{n-2\beta+1}{n-1}\right).$$

By choosing the pinching of the neck λ as in the statement of Theorem 1.28 we obtain that the difference between $\phi_{\text{max}}(t)$ and $\phi_{\text{min}}(t)$ stays bounded away from zero until

$\phi^2_{\min}(0)/(n-1)$. Thus the radius of the bump is positive until $T \leq \phi^2_{\min}(0)/n-1$ and by Lemma 1.30 the cap around the origin stays smooth. In particular from the proof of Lemma 1.30 in Angenent and Knopf [2004] it follows that there exists a radial coordinate $\bar{r} > 0$ such that the curvature is uniformly bounded in the Euclidean ball $B(\boldsymbol{o}, \bar{r})$. Since by Lemma 1.26 we can apply Lemma 1.8 and hence obtain that (1.19) holds in $\mathbb{R}^{n+1} \times [0, T)$, we get

$$\sup_{\mathbb{R}^{n+1}} |\mathrm{Rm}_{g(t)}|_{g(t)} \leq \frac{C}{\phi^2_{\min}}, \tag{1.29}$$

for any time sufficiently close to T. Therefore the solution develops a singularity precisely when the neck collapses while the radius of the bump stays positive. In particular, by integrating (1.27) using the condition $R_{g(t)} \geq 0$ and the crude estimate $\phi_{ss}(r_2(t), t) \geq 0$ we get

$$(n-1)(T-t) \leq \phi^2_{\min}(t) \leq 2(n-1)(T-t)$$

which along with (1.29) show that the singularity is Type-I. □

1.6.2 Convergence to shrinking cylinders

In this subsection we adapt the analysis in Isenberg et al. [2016] to show that the asymptotically flat Type-I flows constructed above satisfy the same cylindrical asymptotics.

By the Sturmian theorem we can define a time-dependent neighbourhood of the neck as follows:

$$\Omega \doteq \left\{ \phi_{ss} \log\left(\frac{\phi}{\delta}\right) < 0 \right\}$$

for some $\delta > 0$ sufficiently small. We need a preliminary estimate, which in the asymptotically flat case holds on the entire space-time.

Lemma 1.31. *Let* $(\mathbb{R}^{n+1}, g(t))_{0 \leq t < T}$ *be a rotationally symmetric* ϵ*-asymptotically flat Ricci flow with* $T < \infty$. *There exists* $\alpha > 0$ *such that*

$$\frac{1}{\phi}(\phi_s^2 - 1) \leq \alpha, \qquad \phi_{ss}\phi \log(\phi) \geq -\alpha > -\infty,$$

uniformly in $\mathbb{R}^{n+1} \times [0, T)$.

Proof. Define $f \doteq (\phi_s^2 - 1)/\phi$. From (1.2) and Lemma 1.26 we derive that if f is not uniformly bounded from above in $[0, T)$ then for any large value M there exists a first maximum in the space-time such that $f(r_0, t_0) = M$. A simple computation gives

$$\partial_t f(r_0, t_0) \leq -\frac{(\phi_s^2 - 1)^2}{2\phi^3} + \frac{(\phi_s^2 - 1)}{\phi^3} \left(\phi_s^2(-2(n-1) - 1) + n - 1\right).$$

Since $\phi_s^2(r_0, t_0) > 1$ we get

$$\partial_t f(r_0, t_0) < -n\frac{M}{\phi^2} < 0,$$

which proves the first inequality.
For the second estimate we proceed similarly. The boundary conditions and the asymptotic flatness imply that $\psi \doteq \phi_{ss}\phi \log(\phi)$ is uniformly bounded from below at the origin and at spatial infinity. Let (r_0, t_0) be the first minimum point such that $\psi(r_0, t_0) = -M$ for some large M. By the estimate (1.19), which holds in the asymptotically flat case by Lemma 1.26, we can assume without loss of generality that $\phi(r_0, t_0) < 1$ and hence that $\phi_{ss}(r_0, t_0) > 0$. A long but straightforward computation yields

$$\begin{aligned}\partial_t \psi(r_0, t_0) \geq \frac{1}{\phi} &\left(-2\psi\phi_{ss} + \phi_{ss}\log(\phi)\phi_s^2(-4n+8) + \phi_{ss}(-(n-1) + 4\phi_s^2)\right) \\ + \frac{1}{\phi} &\left(2\phi_s^2 \frac{\phi_{ss}}{\log(\phi)} + \log(\phi)\frac{1-\phi_s^2}{\phi}\phi_s^2(-2(n-1))\right).\end{aligned}$$

By using that $(\phi_s^2 - 1)/\phi$ is uniformly controlled from above we can bound the right hand side from below as follows:

$$\partial_t \psi(r_0, t_0) \geq \frac{\phi_{ss}}{\phi}\left(2M - n + 1 - 2\frac{\alpha}{|\log(\phi)|} - \frac{\alpha|\log(\phi)|}{\phi_{ss}}\right).$$

Since by (1.19) $\phi\phi_{ss}$ is uniformly bounded, by taking M large we make ϕ as small as we need. Therefore $|\log(\phi)|/\phi_{ss} = M^{-1}\phi(\log(\phi))^2$ is small and the right hand side is positive for M large enough. □

Since the neck is shrinking at a Type-I rate one can rewrite the second inequality in the previous Lemma in a more geometric way.

Corollary 1.32. *In the region Ω where $K \leq 0$ the following holds:*

$$(T-t)|K| \leq \frac{C}{|\log(T-t)|}.$$

One can then rely on the argument in [Isenberg et al., 2016, Lemma 16] which extends to our setting up to taking $f = g$ and generalizing it to any dimension $n+1 \geq 3$. That completes the proof of (ii) of Theorem 1.28.

1.6.3 Initial data leading to Type-I neckpinches

In this subsection we sketch how by adapting the analogous construction in Angenent and Knopf [2004] one can find initial data satisfying the assumptions of Theorem 1.28. We explicitly consider the case $n \geq 4$. The cases $n = 2, 3$ only require a further smoothing step where by perturbing the metrics described in Angenent and Knopf [2004] one can obtain initial data that are again asymptotically flat. Given $0 < \alpha < 1$ we define

$$f : r \mapsto \begin{cases} \sin(r) & 0 \leq r \leq r_\alpha \\ W_\alpha(r) \equiv \sqrt{\alpha + (r - \frac{\pi}{2})^2} & r \geq r_\alpha \end{cases}$$

where r_α is the unique intersection between W_α and $\sin$ in $(0, \pi/2)$. By the analysis in Section 8 of Angenent and Knopf [2004] we can smooth f so that we obtain a radius ϕ that satisfies the following properties:

(i) $\phi(r)$ coincides with $\sin(r)$ in a radial neighbourhood of the origin so that the rotationally symmetric metric $g_0 = (dr)^2 + \phi^2(r) g_{S^n}$ is a smooth metric on $\mathbb{R}^{n+1}$ by (1.2).

(ii) $R_{g_0} \geq 0$ by [Angenent and Knopf, 2004, Lemma 8.1].

(iii) There exists $\tilde{\alpha} > 0$ such that for any $\alpha < \tilde{\alpha}$ g_0 has a bump of radius $\phi_{\text{max}}(0) \geq \tilde{\alpha}$ and has a neck of radius $W_\alpha(\pi/2) = \sqrt{\alpha}$.

(iv) The scale invariant quantity A_{g_0} satisfies $A_{g_0} \geq \beta$ with β independent of α.

(v) By direct computation one can check that g_0 satisfies (1.25) for any $\epsilon \leq 2$ and hence is asymptotically flat.

Therefore, by choosing α small enough we obtain a metric g_0 for which Theorem 1.28 applies. Equivalently, the Ricci flow evolving from g_0 develops a local Type-I singularity where the radius of the neck vanishes.

1.6.4 Immortal Ricci flows with an initial neck

It remains to show that there exist rotationally symmetric (asymptotically flat) Ricci flows on $\mathbb{R}^{n+1}$ that have an initial mild neck which then disappears in finite-time. The evolution equations of the scalar curvature and the scale-invariant quantity A prevent us from deriving general conditions which may describe the situation where the neck is *sufficiently mild* similarly to Theorem 1.28 and in fact it seems much harder to check whether the assumptions in Theorem 1.28 are in some sense optimal. However we are still able to show that there exist examples of Ricci flows evolving an initial metric with a neck to a metric with no minimal embedded hyperspheres in finite time.
We recall that given two Riemannian metrics g_1, g_2 we say that g_1 is ε-close to g_2 if

$$(1+\varepsilon)^{-1}g_2 \leq g_1 \leq (1+\varepsilon)g_2.$$

Proof of Proposition 1.7. Let $\varepsilon = 1$ and choose ε_0 to satisfy [Schnürer et al., 2008, Theorem 1.2]. The existence of an initial neck-like region is compatible with g_0 being ε_0-close to the Euclidean metric. In particular, the difference between the radius of the bump and the radius of the neck is smaller than $2\varepsilon_0$. Corollary 1.27 then implies that the critical values of the radius ϕ are smooth functions of time. By the implicit function theorem and the evolution equation (1.10) we find

$$\phi_{\max}\dot{\phi}_{\max}(t) \leq -(n-1).$$

We conclude that the neck must disappear at some finite time $\hat{t} < \phi^2_{\max}(0)/2(n-1)$, otherwise the flow would develop a finite-time singularity hence contradicting that $g(t)$ is immortal due to [Schnürer et al., 2008, Theorem 1.2]. $\square$

Remark 1.5. *We point out that in the examples given by Proposition 1.7 one can control from above the time elapsed along the flow before the neck disappears by* $\phi^2_{max}(0)/2(n-1)$.

Chapter 2

Ricci flow of warped Berger metrics on $\mathbb{R}^4$

In this chapter we study the Ricci flow on $\mathbb{R}^4$ starting at an $SU(2)$-cohomogeneity one metric g_0 whose restriction to any hypersphere is a Berger metric. We prove that if g_0 has no necks and is bounded by a cylinder, then the solution develops a global Type-II singularity and converges to the Bryant soliton when suitably dilated at the origin. This is the first example in dimension $n > 3$ of a non-rotationally symmetric Type-II flow converging to a rotationally symmetric singularity model. Next, we show that if instead g_0 has no necks, its curvature decays and the Hopf fibres are not collapsed, then the solution is immortal. Finally, we prove that if the flow is Type-I, then there exist minimal 3-spheres for times close to the maximal time.

2.1 Introduction

In Chapter 1 we have studied $SO(n+1)$-Type-II singularities on $\mathbb{R}^{n+1}$. In this chapter instead we focus on a family of 4-dimensional cohomogeneity one Ricci flows with smaller isometry group. We point out that our construction of non-rotationally symmetric Type-II Ricci flows in Di Giovanni [2020] was conducted at the same time and independently of works of Appleton [2019] and Stolarski [2019], where they obtained Ricci flat singularity models for Type-II Ricci flows.

The Ricci flow on 4-dimensional cohomogeneity one manifolds has been recently studied on various topologies Bettiol and Krishnan [2016], Isenberg et al. [2016, 2019], Appleton [2019]. Isenberg et al. [2016] showed that the Ricci flow starting at a family of *generalized warped Berger metrics* on $S^1 \times S^3$ is Type-I and becomes rotationally symmetric around any singularity. This behaviour is regarded as a Type-I example of *symmetry enhancement* along the Ricci flow.

In this chapter we study the Ricci flow evolving from a generalized warped Berger metric on $\mathbb{R}^4$. Namely, consider a metric g_0 invariant under the cohomogeneity one left-action of $SU(2)$ on $\mathbb{R}^4 = \mathbb{C}^2$ that can be written in Bianchi IX form as Gibbons and Pope [1979]

$$g_0 = (ds)^2 + a^2(s)\,\sigma_1 \otimes \sigma_1 + b^2(s)\,\sigma_2 \otimes \sigma_2 + c^2(s)\,\sigma_3 \otimes \sigma_3,$$

where $\{\sigma_i\}_{i=1}^3$ is a coframe dual to some Milnor frame $\{X_i\}_{i=1}^3$ on $SU(2)$, with X_3 tangent to the Hopf fibres. We further assume that g_0 is invariant under rotations of the Hopf fibres. The last condition means that the left-invariant vector field X_3 is Killing thus extending the Lie algebra of g_0-Killing vectors to $\mathfrak{u}(2)$. In particular, we can write g_0 as

$$g_0 = (ds)^2 + b^2(s)\,(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2) + c^2(s)\,\sigma_3 \otimes \sigma_3.$$

In analogy with Isenberg et al. [2016] we finally assume that $c \leq b$ so that each non-degenerate fiber $\{s\} \times S^3$ is a Berger sphere. We call such metric a *warped Berger* metric on $\mathbb{R}^4$. As already observed in Isenberg et al. [2016], if instead we have $c \geq b$, then the ordering is still preserved along the solution but would probably lead to different qualitative behaviours. In fact, without the warped Berger condition our argument for the conservation of the monotonicity of the warping coefficients along the solution is no longer valid. Moreover, it seems that singularities where b goes to zero - hence collapsing the base space S^2 of the Hopf-fibration - while the Hopf-fiber stays positive may occur.

We first focus on initial data with linear volume growth.

Definition 2.1. *We let $\mathcal{G}$ be the set of complete, bounded curvature warped Berger metrics g_0 on $\mathbb{R}^4$ satisfying the following conditions:*

(i) $b_s \geq 0$, $H \geq 0$, *where $H(r)$ is the* mean curvature *of the centred Euclidean sphere of Euclidean radius r with respect to g_0.*

(ii) $\sup_{p\in\mathbb{R}^4} b(p) < \infty$.

The control on the sign of H amounts to ruling out the existence of necks Angenent and Knopf [2004]. We also note that the condition in (i) is weaker than asking for both b and c to be monotone. We prove that any Ricci flow starting in $\mathcal{G}$ converges to the Bryant soliton once suitably rescaled. This provides Type-II examples of symmetry enhancement and constitutes the first explicit case in dimension higher than three of a non-locally conformally flat flat Type-II Ricci flow converging to a rotationally symmetric singularity model.

Theorem 2.1. *Let $(\mathbb{R}^4, g(t))_{0\leq t<T}$ be the maximal complete, bounded curvature solution to the Ricci flow starting at some $g_0 \in \mathcal{G}$. The solution develops a Type-II singularity at some $T < \infty$ which is modelled on the Bryant soliton once suitably dilated.*

Theorem 2.1 resembles an analogous result recently derived in Appleton [2019], where Appleton studied the Ricci flow on the blow-up of $\mathbb{C}^2/\mathbb{Z}_k$ starting from a subclass of warped Berger metrics. In particular, they showed that when $k = 2$ if the initial metric is bounded by a cylinder at infinity and both b and c are increasing and satisfy a differential inequality, then the flow is Type-II and converges to the Eguchi-Hanson metric once suitably dilated.

Theorem 2.1 characterizes the Type-II singularity only partially. Indeed, while the Type-II singularity is not isolated, being the Bryant soliton asymptotically cylindrical (see, e.g., Brendle [2014]), in general there is no control on the blow-up sequence giving rise to a family of shrinking cylinders. Moreover, the symmetries and the lack of necks suggest that the curvature ought to become large locally around the singular orbit (see also Appleton [2019]). Equivalently, one should detect the Bryant soliton when dilating the flow at the origin $\boldsymbol{o}$ by suitable factors. In our next result we address these issues hence providing a much clearer picture of the Type-II singularity developed by Ricci flows in $\mathcal{G}$. In the following statement $R_{g(t)}$ represents the *scalar curvature* of the Ricci flow solution.

Theorem 2.2. *Let $(\mathbb{R}^4, g(t))_{0\leq t<T}$ be the maximal complete, bounded curvature solution to the Ricci flow starting at some $g_0 \in \mathcal{G}$. Then the following conditions hold:*

(i) (The Bryant soliton appears at the origin.) *There exists* $t_j \nearrow T$ *such that the rescaled Ricci flows* $(\mathbb{R}^4, g_j(t), \boldsymbol{o})$ *defined by* $g_j(t) \doteq R_{g(t_j)}(\boldsymbol{o})g(t_j+(R_{g(t_j)}(\boldsymbol{o}))^{-1}t)$ *converge to the Bryant soliton in the Cheeger-Gromov sense.*

(ii) (The singularity is global.) *For any* $p \in \mathbb{R}^4$ *we have*

$$\limsup_{t \nearrow T} \left(|\mathrm{Rm}_{g(t)}|_{g(t)}(p)\right) = \infty.$$

(iii) (Type-I blow-up at infinity.) *For any* $t_j \nearrow T$ *there exist a sequence* $\{p_j\}$ *and* $\alpha > 0$ *such that* $d_{g_0}(\boldsymbol{o}, p_j) \to \infty$, $(T - t_j)R_{g(t_j)}(p_j) \leq \alpha$, *and the rescaled Ricci flows* $(\mathbb{R}^4, g_j(t), p_j)$ *defined by* $g_j(t) \doteq R_{g(t_j)}(p_j)g(t_j + (R_{g(t_j)}(p_j))^{-1}t)$ *converge to the self-similar shrinking cylinder in the Cheeger-Gromov sense.*

(iv) (Classification of singularity models.) *Any non-trivial singularity model is isometric to either the self-similar shrinking cylinder or the Bryant soliton.*

We note that as an immediate consequence of (i) the scalar curvature and the full curvature are comparable in certain regions up to the singular time. We also point out that the phenomenon of Type-II enhancement of symmetries along the Ricci flow is intrinsic to the classification of 3-dimensional κ-solutions obtained by Brendle [2020]. Item (iv) in Theorem 2.2 relies on the recent extension of Brendle's work to higher dimensions by Li and Zhang [2018].

Next, we show that the long-time property is satisfied by a class of warped Berger metrics whose curvature decays at infinity. General long-time existence results on non-compact manifolds usually rely on controlling the sign of the curvature and the volume growth Cabezas-Rivas and Wilking [2015]. From a different perspective, similar conclusions may be achieved when the analysis is restricted to families of homogeneous Riemannian metrics Lafuente [2015]. In this case the behaviour of the flow for long times is also understood Böhm and Lafuente [2018]. Instead of assuming a transitive action of a Lie group, one may study cohomogeneity one manifolds. In this direction, Oliynyk and Woolgar [2007] proved that the Ricci flow on $\mathbb{R}^n$ starting at an asymptotically flat spherically symmetric metric without necks is immortal. We improved this result by allowing any decay of the curvature in Theorem 1.2.

In our setting we consider the following set, whose intersection with $\mathcal{G}$ is empty.

Definition 2.2. *We let $\mathcal{G}_\infty$ be the set of complete warped Berger metrics g on $\mathbb{R}^4$ with* positive injectivity radius *and satisfying the following conditions:*

(i) $b_s \geq 0$, $H \geq 0$.

(ii) $|\mathrm{Rm}_g|_g(s) \to 0$ *as* $s \to \infty$ *and there exist* $\mu > 0$ *and* $s_0 > 0$ *such that* $c(s) \geq \mu$ *for any* $s \geq s_0$.

We prove the following:

Theorem 2.3. *Any complete, bounded curvature Ricci flow starting in $\mathcal{G}_\infty$ is immortal.*

The long-time property may in general fail if we omit the requirement on the monotonicity of b and H. Indeed by the adaptation of Angenent and Knopf [2004] to $\mathbb{R}^{n+1}$ derived in Chapter 1, we deduce that if g_0 is asymptotically flat with $b = c$ and $(\mathbb{R}^4, g_0)$ contains a neck which is sufficiently pinched (in a precise way), then the Ricci flow is Type-I. It therefore remains to address the relation between Type-I singularities and existence of minimal hyperspheres for Berger Ricci flows. We recall that Angenent and Knopf [2004] constructed the first examples of non-degenerate neckpinches by evolving rotationally invariant metrics on S^n containing minimal (stable) hyperspheres. Later the link between Type-I singularities and minimal spheres has been explored for Ricci flows on closed 3-manifolds by Song [2019]. In our setting we prove the following:

Theorem 2.4. *Let $(\mathbb{R}^4, g(t))_{0 \leq t < T}$ be the maximal complete, bounded curvature solution to the Ricci flow evolving from a complete warped Berger metric g_0 with positive injectivity radius and curvature decaying at infinity. If $g(t)$ develops a Type-I singularity at $T < \infty$, then there exists $\delta > 0$ such that $(\mathbb{R}^4, g(t))$ contains minimal embedded* 3-*spheres for any $t \in [T - \delta, T)$.*

We may also apply Theorem 2.1 and Theorem 2.3 to derive two simple corollaries. First we immediately deduce that

Corollary 2.5. *Neither $\mathcal{G}$ nor $\mathcal{G}_\infty$ contain shrinking Ricci solitons.*

In the second application we classify warped Berger Ricci flows with bounded nonnegative curvature. In particular we show that for positively curved warped Berger Ricci flows bounded by a cylinder at infinity, parabolic dilations at the origin along any sequence of times give rise to the Bryant soliton.

Corollary 2.6. *Let* $(\mathbb{R}^4, g(t))_{0\leq t<T}$ *be the maximal complete, bounded curvature Ricci flow starting at some complete warped Berger metric* g_0 *with bounded nonnegative curvature. Then* T *is finite if and only if* $b(\cdot, 0)$ *is bounded. If* T *is finite, then for any* $t_j \nearrow T$ *the rescaled Ricci flows* $(\mathbb{R}^4, g_j(t), \boldsymbol{o})$ *defined by* $g_j(t) \doteq R_{g(t_j)}(\boldsymbol{o})g(t_j + (R_{g(t_j)}(\boldsymbol{o}))^{-1}t)$ *(sub)converge to the Bryant soliton.*

Outline

We briefly describe the organization of the chapter.
In Section 2.2 we discuss warped Berger metrics on $\mathbb{R}^4$ and we comment on the main assumptions. In Section 2.3 we show that the condition on the lack of necks persists along the Ricci flow. The main step consists in adapting the analogous argument adopted in Appleton [2019], which relies on the application of a general maximum principle for systems of parabolic equations. In Section 2.4 we study warped Berger Ricci flows evolving from initial data either in $\mathcal{G}$ or in $\mathcal{G}_\infty$. Similarly to Isenberg et al. [2016] we show that the curvature is controlled by the size of the principal orbits and that the solution becomes rotationally symmetric around any singularity at some rate that breaks scale-invariance. An important ingredient, for the case of $\mathcal{G}_\infty$, is also given by the application of the Pseudolocality formula in Chau et al. [2011]. In Section 2.5 we prove that any singularity model is rotationally symmetric by showing that the *left-invariant* Milnor frame diagonalizing the metric generates a copy of $\mathfrak{su}(2)$ in the Lie algebra of Killing fields acting on the singularity model. We then apply the rigidity result obtained by Zhang [2008] to classify these singularity models. In Section 2.6 we prove the main results. Theorem 2.1 heavily relies on the characterization of Type-I singularities in Naber [2010] and Enders et al. [2011]. The appearance of the Bryant soliton follows from a result of Hamilton [1993a] once we know that the singularity is Type-II. The localization of the Bryant soliton in (i) of Theorem 2.2 is a direct consequence of the convergence of left-invariant vector fields obtained in Section 2.5. The property that the singularity is global depends on the monotonicity

assumption ($b_s \geq 0$, $H \geq 0$), which allows us to control the space-time region where the flow stays smooth. The Type-I blow-up at infinity follows once we know that the solution becomes singular everywhere at some finite time T. We then obtain the classification of singularity models by combining the characterization of singularity models in Section 2.5 with the analysis in Li and Zhang [2018]. The proof of Theorem 2.3 follows from a contradiction argument. We show that if a Ricci flow in $\mathcal{G}_\infty$ develops a finite-time singularity, then *any* singularity model is a non-compact κ-solution with Euclidean volume growth. However, Perelman [2002] showed that this is not possible. We finally address the proof of Theorem 2.4, which again depends on the characterization of Type-I Ricci flows obtained in Enders et al. [2011]. Section 2.7 is devoted to deriving some easy applications of the main results.

2.2 Setting

2.2.1 Warped Berger metrics on $\mathbb{R}^4$

Let (M, g) be a non-compact Riemannian manifold and let G be a compact Lie group acting on (M, g) with cohomogeneity one. Assume that there exists a singular orbit Σ_{sing}, alternatively the orbit space is homeomorphic to $\mathbb{R}$ and M is hence foliated by G/H, with H the principal isotropy group (see, e.g., Grove and Ziller [2002]). Given $q \in \Sigma_{\text{sing}}$ we consider a minimal geodesic γ starting at q and meeting all the principal orbits orthogonally. Away from the singular orbit, we can write g along γ as

$$g = ds^2 + g_s,$$

for some 1-parameter family of homogeneous metrics g_s on G/H. We may then use the action to extend such form on any orbit and therefore on the entire principal part of the manifold. We aim to describe those metrics for which $\mathsf{G} = U(2)$ and $M = \mathbb{R}^4$. Before we do that, let us first recall that we can identify $S^3 \subset \mathbb{C}^2$ and $SU(2)$ via the map f defined in Euler coordinates by

$$f : (e^{\frac{i}{2}(\theta+\psi)}\cos(\frac{\phi}{2}), e^{\frac{i}{2}(\theta-\psi)}\sin(\frac{\phi}{2})) \mapsto \begin{bmatrix} e^{\frac{i}{2}(\theta+\psi)}\cos(\frac{\phi}{2}) & -e^{-\frac{i}{2}(\theta-\psi)}\sin(\frac{\phi}{2}) \\ e^{\frac{i}{2}(\theta-\psi)}\sin(\frac{\phi}{2}) & e^{-\frac{i}{2}(\theta+\psi)}\cos(\frac{\phi}{2}) \end{bmatrix},$$

where $\phi \in [0, \pi), \psi \in [0, 4\pi), \theta \in [0, 2\pi)$. A left-invariant coframe $\{\sigma_1, \sigma_2, \sigma_3\}$ on $SU(2)$ is given by the Maurer-Cartan form $f^{-1}df = \sigma_1 h_1 + \sigma_2 h_2 + \sigma_3 h_3$ with

$$h_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad h_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad h_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

and σ_i defined in Euler coordinates by (see also Gibbons and Pope [1979])

$$\begin{aligned} \sigma_1 &= -\sin\psi d\phi + \cos\psi \sin\phi d\theta \\ \sigma_2 &= \cos\psi d\phi + \sin\psi \sin\phi d\theta \\ \sigma_3 &= d\psi + \cos\phi d\theta. \end{aligned} \tag{2.1}$$

In particular, it follows that $d\sigma_i = 1/2 \sum_{j,k} \epsilon_{ijk}\sigma_j \wedge \sigma_k$. We also let $\{X_i\}$ denote the left-invariant frame dual to $\{\sigma_i\}$ and we take X_3 to be tangent to the Hopf-fibres. Consider a metric g on $\mathbb{R}^4$ invariant under the isometric cohomogeneity-one action of $SU(2)$. By introducing a radial coordinate r and the Euler angles we can write g in Bianchi-IX form Gibbons and Pope [1979] as:

$$\begin{aligned} g &= \xi^2(r) dr \otimes dr + g_r \\ &= \xi^2(r) dr \otimes dr + a^2(r)\, \sigma_1 \otimes \sigma_1 + b^2(r)\, \sigma_2 \otimes \sigma_2 + c^2(r)\, \sigma_3 \otimes \sigma_3, \end{aligned} \tag{2.2}$$

where $\xi, a, b, c : (0, +\infty) \to (0, +\infty)$ are smooth radial functions. If we introduce the geometric quantity $s(\cdot) = d_g(\boldsymbol{o}, \cdot)$, then we may rewrite (2.2) as

$$g = ds^2 + a^2(s)\, \sigma_1 \otimes \sigma_1 + b^2(s)\, \sigma_2 \otimes \sigma_2 + c^2(s)\, \sigma_3 \otimes \sigma_3. \tag{2.3}$$

If we also assume g to be invariant under the U(1) action on the Hopf-fibres, then the vector field X_3 is Killing, thus enlarging the Lie algebra of Killing vectors to $\mathfrak{u}(2)$. This is equivalent to requiring $a = b$ on $\mathbb{R}^4$. Therefore, the Hopf-fibration allows us to write

$$g = ds^2 + b^2(s)\, \pi^* g_{S^2(\frac{1}{2})} + c^2(s)\, \sigma_3 \otimes \sigma_3, \tag{2.4}$$

where $g_{S^2(\frac{1}{2})}$ is the Fubini-Study metric and σ_3 is the one-form dual to the vector field tangent to the Hopf-fibres. In fact, once we enlarge the isometry group to $U(2)$ we are describing the entire class of $U(2)$-invariant metrics on $\mathbb{R}^4$ being any $U(2)$-left-invariant metric on S^3 diagonal - as follows from the fact that $SU(2)$-left-invariant metrics on S^3 are in 1:1 with positive definite symmetric 3-matrices and those that are $U(2)$-invariant are further invariant under the conjugation action of $SO(2)$ that fix a vector in $\mathbb{R}^3$.

In the following we usually refer to the warping coefficient b as the coefficient of g along the S^2-direction. Similarly, we often say that the factor c constitutes the size of the Hopf-fiber. According to (2.4), an $U(2)$-invariant metric g on $\mathbb{R}^4$ is given by the formula:

$$g = g_{\mathbb{R}^3} + c^2(s)\,\sigma_3 \otimes \sigma_3, \tag{2.5}$$

where $g_{\mathbb{R}^3}$ is the projection of g on the base via the Hopf-fibration $\mathbb{R}^4 \setminus \{\boldsymbol{o}\} \to \mathbb{R}^3 \setminus \{\boldsymbol{o}\}$.

We finally focus on those $U(2)$-invariant metrics g on $\mathbb{R}^4$ satisfying the *warped Berger* condition (see also Isenberg et al. [2016]):

$$c \leq b.$$

Accordingly, any homogeneous metric g_s on the principal orbit $\{s\} \times S^3$ is a Berger metric with squashing factor $c/b \leq 1$. Since such squashing factor plays an important role in the analysis and also appears in the hyperkähler quantities characterizing the Taub-NUT metric, we make the following:

Definition 2.3. *Given a warped Berger metric g, the scale-invariant roundness ratio c/b : $\mathbb{R}^4 \to (0, 1]$ is denoted by u.*

We observe that for any radial map f we think of $f = f(s) = f(s(r))$ as a function of r unless otherwise stated. From (2.2) and (2.3) we have the following relation between the two radial derivatives:

$$\partial_s = \frac{1}{\xi(r)}\partial_r. \tag{2.6}$$

The metric g in (2.4) defines a smooth metric on $\mathbb{R}^4$ if and only if b and c are smooth odd functions of the radial variable r and the condition below holds:

$$\lim_{s\to 0}\frac{db}{ds}(s)=\lim_{s\to 0}\frac{dc}{ds}(s)=1. \tag{2.7}$$

It is worth noting that the smoothness conditions reflect the underlying topology and hence lead to significant variations, both in terms of results and approach, when comparing the study of $U(2)$-invariant Ricci flows on different manifolds Isenberg et al. [2016], Appleton [2019].

2.2.2 Curvature terms

If g is a warped Berger metric on $\mathbb{R}^4$, then from the Koszul formula we derive the vertical sectional curvatures

$$k_{12}=\frac{1}{b^2}\left(4-3u^2-b_s^2\right), \tag{2.8}$$

$$k_{13}=k_{23}=\frac{1}{b^2}\left(u^2-b_s c_s u^{-1}\right), \tag{2.9}$$

and the mixed sectional curvatures

$$k_{01}=k_{02}=-\frac{b_{ss}}{b}, \tag{2.10}$$

$$k_{03}=-\frac{c_{ss}}{c}. \tag{2.11}$$

Moreover, unless the isometry group extends to $SO(4)$, we also have a *non-trivial* curvature term which is not the sectional curvature of a 2-plane:

$$\mathrm{Rm}_{0123}=\frac{1}{b^2}\left(c_s-b_s u\right)=\frac{u_s}{b}. \tag{2.12}$$

We finally report the formula for the scalar curvature:

$$R_g=2(k_{01}+k_{02}+k_{03}+k_{12}+k_{13}+k_{23}). \tag{2.13}$$

2.2.3 Initial data for the Ricci flow

In this chapter we study the Ricci flow problem on $\mathbb{R}^4$ with initial condition given by a warped Berger metric g_0. We first assume that g_0 is bounded by a cylinder at infinity so that the Ricci flow evolving from g_0 always encounters a finite-time singularity.

According to Angenent and Knopf [2004] and the results in Chapter 1, if $(\mathbb{R}^4, g_0)$ contains necks, then the Ricci flow solution may be Type-I and converge to a shrinking cylinder once rescaled. In order to construct Type-II singularities we thus need to exclude these initial geometries. A generalization of the notion of neck discussed in Angenent and Knopf [2004] to the $U(2)$-invariant setting consists in considering whether the mean curvature of embedded hyperspheres changes sign. Namely, we introduce the quantity $H : r \to (2b_s/b + c_s/c)(r)$ representing the mean curvature of the centred Euclidean sphere of Euclidean radius r with respect to g_0. We say that g_0 does not have necks when the mean curvature H is nonnegative on $\mathbb{R}^4 \setminus \{\boldsymbol{o}\}$.

While in the rotationally symmetric setting a Sturmian type of argument guarantees that minimal hyperspheres cannot appear along the flow, one might expect that in the $U(2)$-case the mean curvature could generally change sign along the flow. In order to prevent the latter phenomenon from happening, we require the spatial derivative b_s to be nonnegative as well.

Definition 2.4. *We let $\mathcal{G}$ be the set of complete* bounded curvature *warped Berger metrics on $\mathbb{R}^4$ satisfying the following conditions:*

(i) $b_s \geq 0$, $H \geq 0$.

(ii) $\sup_{p\in\mathbb{R}^4} b(p) < \infty$.

Remark 2.1. *From the formula for the mean curvature of the embedded hyperspheres we immediately derive that the assumption (i) in Definition 2.4 is* weaker *than asking for the monotonicity of both b and c.*

In the second class of initial data for the Ricci flow we consider warped Berger metrics without necks but whose behaviour at infinity is not controlled by that of a cylinder. Namely, we require the curvature to decay to zero and the Hopf fibres to be not collapsed.

Definition 2.5. *We let $\mathcal{G}_\infty$ be the set of complete warped Berger metrics g on $\mathbb{R}^4$ with* positive injectivity radius *and satisfying the following conditions:*

(i) $b_s \geq 0$, $H \geq 0$.

(ii) $|\mathrm{Rm}_g|_g(s) \to 0$ *as* $s \to \infty$ *and there exist* $\mu > 0$ *and* $s_0 > 0$ *such that* $c(s) \geq \mu$ *for any* $s \geq s_0$.

Remark 2.2. *We point out that the sets $\mathcal{G}$ and $\mathcal{G}_\infty$ are disjoint. For if $g_0 \in \mathcal{G} \cap \mathcal{G}_\infty$, then $b < m$, for some $m > 0$, and hence by* (2.8) *we find*

$$|4 - 3u^2 - b_s^2| = b^2|k_{12}| < m^2|k_{12}|.$$

Since the curvature is decaying to zero at infinity and $u \leq 1$ we see that $b_s \geq 1/2$ outside some Euclidean ball $B(\boldsymbol{o}, r)$, for r large enough. Therefore $b(s) \to \infty$ and this is a contradiction. We conclude that if $g_0 \in \mathcal{G}_\infty$, then $b(s) \to \infty$ being $b_s \geq 0$.

Remark 2.3. *As we will see in Chapter 3, the well known Taub-NUT metric on $\mathbb{R}^4$ Hawking [1977] is a hyperkähler metric belonging to $\mathcal{G}_\infty$ since the curvature decays to zero at cubic rate while both b and c are increasing.*

2.2.4 The Ricci flow equations

Given a complete, bounded curvature warped Berger metric g_0, there exists a unique maximal, complete, bounded curvature solution to the Ricci flow $(\mathbb{R}^4, g(t))_{0\leq t<T}$ starting at g_0 Shi [1989],Chen and Zhu [2006b]. From now on we omit to specify each time that any Ricci flow solution we consider is meant to be the unique complete, bounded curvature one evolving from some initial metric g_0.

If $(\mathbb{R}^4, g(t))_{0\leq t<T}$ is the maximal Ricci flow starting at some complete, bounded curvature warped Berger metric g_0, then the diffeomorphism invariance and the uniqueness property of the problem in the class of complete, bounded curvature solutions ensure that $g(t)$ is still an $U(2)$-invariant metric for all $t \in [0, T)$. Therefore, we can argue as in Section 2.2.1 to derive that $g(t)$ can be diagonalized with respect to a time-independent

fixed frame. Namely, the solution has the form

$$g(t) = ds \otimes ds + b^2(s,t)\,(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2) + c^2(s,t)\,\sigma_3 \otimes \sigma_3, \tag{2.14}$$

where $s = s(r,t)$ is the distance from the origin with respect to the solution and hence is time-dependent. Such geometric coordinate allows us to write the Ricci flow equations as

$$b_t = b_{ss} + \left(\frac{c_s}{c} + \frac{b_s}{b}\right) b_s + 2\frac{u^2}{b} - \frac{4}{b} \tag{2.15}$$

$$c_t = c_{ss} + 2\frac{b_s c_s}{b} - 2\frac{u^3}{b}. \tag{2.16}$$

Since the coordinate s depends on time, there is a non trivial commutator between ∂_t and ∂_s given by

$$\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right] = -(\ln(\xi))_t \frac{\partial}{\partial s} = -\left(2\frac{b_{ss}}{b} + \frac{c_{ss}}{c}\right)\frac{\partial}{\partial s}, \tag{2.17}$$

where ξ is defined as in (2.2). We finally write down the formula for the time-dependent Laplacian along a solution of the Ricci flow. Given a smooth radial map $f : \mathbb{R}^4 \to \mathbb{R}$ we have

$$\Delta f \equiv f_{ss} + \left(2\frac{b_s}{b} + \frac{c_s}{c}\right) f_s. \tag{2.18}$$

We dedicate the end of this subsection to proving that the Ricci flow solution $g(t)$ starting at g_0 remains a warped Berger metric until its maximal time of existence $T \leq \infty$. More precisely, we show the following:

Lemma 2.7. *Let* $(\mathbb{R}^4, g(t))_{0 \leq t < T}$ *be the complete, bounded curvature solution to the Ricci flow starting at some warped Berger metric* g_0 *and let* $\varepsilon \doteq \inf_{p \in \mathbb{R}^4} u(p,0)$. *Then for any* $(p,t) \in \mathbb{R}^4 \times [0,T)$ *we have*

$$\varepsilon \leq u(p,t) \leq 1.$$

Proof. We first verify that the ordering $u \leq 1$ is preserved along the flow for any $t \in [0,T)$. By Shi [1989] the curvature is bounded at any time slice $\mathbb{R}^4 \times \{t\}$, with $t \in [0,T)$. Thus from the Ricci flow equations we see that

$$|\partial_t u| \equiv \left|\partial_t \frac{c}{b}\right| = \left|-\mathrm{Ric}(X_3/c, X_3/c)\frac{c}{b} + \mathrm{Ric}(X_2/b, X_2/b)\frac{c}{b}\right|, \tag{2.19}$$

which means that there exists some time dependent positive constant $\alpha(t)$ such that $u(\cdot, t) \leq \alpha(t) < \infty$ on $\mathbb{R}^4$. As long as a (smooth) solution exists the boundary conditions (2.7) are satisfied, which then imply that the function $f \doteq \log(u)$ is smoothly defined on $\mathbb{R}^4$ and equal to zero at the origin for any time. From the evolution equations (2.15), (2.16) and the formula for the Laplacian (2.18) we get

$$f_t = \Delta f + \frac{4}{b^2}\left(1 - u^2\right). \tag{2.20}$$

Therefore whenever $u > 1$ we find

$$f_t < \Delta f.$$

We can then apply the maximum principle [Chow et al., 2006, Corollary 7.45] and conclude that since $u(\cdot, 0) \leq 1$, the same bound persists along the flow. In fact, once we know that $u \leq 1$ is preserved in time, a standard application of the strong maximum principle shows that if $u = 1$ at some $(p_0, t_0) \in \mathbb{R}^4 \setminus \{\boldsymbol{o}\} \times (0, T)$, then $u = 1$ in a space-time neighbourhood of the point and thus $u = 1$ everywhere for all earlier times by real analyticity of solutions to the Ricci flow Bando [1987].
We now let $\varepsilon \in [0, 1)$ be defined as in the statement. If $\varepsilon = 0$ there is nothing to show so we can take $\varepsilon > 0$. By using (2.19) it follows that $u(\cdot, t) \geq \alpha(t) > 0$ because $u(\cdot, 0) \geq \varepsilon > 0$. If we define $f \doteq \log(\varepsilon^{-1} u)$, since we have just shown that $u \leq 1$ along the solution, we obtain

$$f_t = \Delta f + \frac{4}{b^2}\left(1 - u^2\right) \geq \Delta f.$$

We can apply the maximum principle and conclude that $u(\cdot, t) \geq \varepsilon$ for any $t \in [0, T)$. $\square$

2.3 Ricci flow without necks

In this section we show that the monotonicity assumptions $b_s \geq 0$ and $H \geq 0$ are preserved along the Ricci flow solution. The main ingredient is given by a maximum principle for systems of parabolic equations [Protter and Weinberger, 2012, Theorem 13, p. 190] that recently Appleton [2019] used to derive similar conclusions for a family of $U(2)$-invariant Ricci flows with cylindrical asymptotics. In the following we mainly adapt their argument to the topology of $\mathbb{R}^4$, i.e. to the boundary conditions given in (2.7).

2.3.1 Basic estimates

Let g_0 be a complete, *bounded curvature* warped Berger metric on $\mathbb{R}^4$. We collect a few preliminary bounds that are necessary to apply the maximum principle for systems to the evolution equations of ub_s and cH.

Lemma 2.8. *For any $r_0 > 0$ there exists $\delta > 0$ such that $b(r) \geq \delta > 0$ for all $r \geq r_0$.*

Proof. Let us first check that b cannot be bounded from above by some $\delta > 0$ small enough for any r large. Namely, by assumption there exists $\alpha > \sup_{\mathbb{R}^4}|\text{Rm}_{g_0}|_{g_0}$. Suppose for a contradiction that there exists r_0 such that $b(r) \leq \delta$ for any $r \geq r_0$, with $\delta^2\alpha < 1/2$. From (2.8) we derive

$$|4 - 3u^2 - b_s^2| \leq \alpha b^2 \leq \frac{1}{2}$$

for all $r \geq r_0$. Since $u \leq 1$, we see that $b_s^2(r) \geq 1/2$ for any $r \geq r_0$ which contradicts the fact that b is bounded. Therefore there exists a sequence of points $p_j \to \infty$ such that $b(p_j) > \delta$, with δ given above. Assume that there exists a sequence $q_j \to \infty$ such that $b(q_j) \leq \delta$. It follows that there exists a sequence of minima $\tilde{q}_j \to \infty$ such that $b(\tilde{q}_j) \leq \delta$. From (2.8) we get

$$|4 - 3u^2 - b_s^2|(\tilde{q}_j) \equiv |4 - 3u^2|(\tilde{q}_j) \leq \alpha b^2(\tilde{q}_j) \leq \frac{1}{2},$$

which is not possible. The proof is then complete. □

A simple consequence of the previous Lemma is the following

Corollary 2.9. *Given $r_0 > 0$ there exists $\alpha > 0$ such that*

$$\sup_{\mathbb{R}^4 \setminus B(\boldsymbol{o}, r_0)} \left|\frac{b_s}{b}\right| \leq \alpha.$$

Proof. From (2.8) we derive

$$\frac{b_s^2}{b^2} \leq \frac{1}{b^2}\left(4 - 3u^2\right) + \alpha,$$

with α an upper bound for the curvature. Given $r_0 > 0$ we may apply Lemma 2.8 and conclude the proof. □

We also need to check that both b_s and c_s are exponentially bounded at spatial infinity.

Lemma 2.10. *There exist $M > 0$ and $\alpha > 0$ such that*

$$|c_s| + |b_s| \leq M \exp(\alpha s).$$

Proof. According to (2.10) and the uniform bound on the curvature we see that b and hence b_{ss} are exponentially bounded. Thus the same holds for b_s by integrating b_{ss}. Similar conclusions are satisfied by c_s. □

Finally a bound similar to Corollary 2.9 is satisfied by c_s/c as well.

Lemma 2.11. *Given $r_0 > 0$ there exists $\alpha > 0$ such that*

$$\sup_{\mathbb{R}^4 \setminus B(\boldsymbol{o}, r_0)} \left| \frac{c_s}{c} \right| \leq \alpha.$$

Proof. [Appleton, 2019, Lemma 3.4]. □

2.3.2 Maximum principle for systems

We consider the maximal Ricci flow solution $(\mathbb{R}^4, g(t))_{0 \leq t < T}$ evolving from a complete, bounded curvature warped Berger metric g_0. We note that given $t_0 < T$ then the estimates above hold uniformly for any $t \in [0, t_0]$ being the curvature uniformly bounded in the space-time region $\mathbb{R}^4 \times [0, t_0]$. From the evolution equations (2.15), (2.16), the commutator formula (2.17) and the expression for the mean curvature of embedded hyperspheres $H : (r, t) \to (2b_s/b + c_s/c)(r, t)$, we compute

$$(ub_s)_t = (ub_s)_{ss} + (ub_s)_s \left(2\frac{b_s}{b} - \frac{c_s}{c} \right) + \frac{1}{b^2} (ub_s) \left(8 - 10u^2 - 2b_s^2 \right) + 4\frac{u^2}{b^2} c_s, \quad (2.21)$$

$$(cH)_t = (cH)_{ss} + (cH)_s \left(2\frac{b_s}{b} - \frac{c_s}{c} \right) + 2\frac{cH}{b^2} \left(u^2 - b_s^2 \right) + \frac{16}{b^2} (ub_s) \left(1 - u^2 \right). \quad (2.22)$$

We may now prove the main result of this section.

Lemma 2.12. *Let $(\mathbb{R}^4, g(t))_{0 \leq t < T}$ be the maximal Ricci flow solution starting at a complete, bounded curvature warped Berger metric g_0. If $ub_s(\cdot, 0) \geq 0$ and $cH(\cdot, 0) \geq 0$ then $ub_s(\cdot, t) > 0$ and $cH(\cdot, t) > 0$ for any $t \in (0, T)$.*

Proof. Given $t_0 \in (0,T)$, by the boundary conditions there exists $\delta = \delta(t_0) > 0$ such that

$$\inf_{B(\boldsymbol{o},\delta)\times[0,t_0]} ub_s(r,t) \geq \frac{1}{2}, \qquad \inf_{B(\boldsymbol{o},\delta)\times[0,t_0]} (cH)(r,t) \geq \frac{1}{2}.$$

Using the commutator formula (2.17) we may rewrite the evolution equations (2.21) and (2.22) in the space-time region $(\mathbb{R}^4 \setminus B(\boldsymbol{o},\delta)) \times [0,t_0]$ as

$$\begin{aligned}(ub_s)_t = {} & \frac{1}{\xi^2}\left(ub_s\right)_{rr} + \frac{1}{\xi}\left(2\frac{b_s}{b} - \frac{c_s}{c} - \frac{\xi_r}{\xi^2}\right)\left(ub_s\right)_r \\ & + \frac{1}{b^2}\left(ub_s\right)\left(8 - 18u^2 - 2b_s^2\right) + 4\frac{u^2}{b^2}(cH)\end{aligned}$$

and

$$\begin{aligned}(cH)_t = {} & \frac{1}{\xi^2}\left(cH\right)_{rr} + \frac{1}{\xi}\left(2\frac{b_s}{b} - \frac{c_s}{c} - \frac{\xi_r}{\xi^2}\right)\left(cH\right)_r \\ & + \frac{16}{b^2}\left(ub_s\right)\left(1 - u^2\right) + \frac{2}{b^2}(cH)\left(u^2 - b_s^2\right).\end{aligned}$$

From Lemma 2.8 and Corollary 2.9 we derive that the zero order coefficients are uniformly bounded in $(\mathbb{R}^4 \setminus B(\boldsymbol{o},\delta)) \times [0,t_0]$. Moreover, by Lemma 2.7 we know that the ordering $u \leq 1$ is preserved along the flow, therefore the coupling coefficients $4u^2/b^2$ and $16/b^2(1-u^2)$ are both nonnegative. Similarly to Appleton [2019] we can introduce a barrier function

$$W : (r,t) \to \exp\left(\frac{s^2(r,t)}{1-\beta t} + \lambda t\right)$$

for $t \leq \min\{t_0, (2\beta)^{-1}\}$ and compute the evolution equations of $ub_s + \epsilon W$ and $cH + \epsilon W$ for any $\epsilon > 0$. Using Corollary 2.9, Lemma 2.11 and standard distortion estimates of the distance function it is straightforward to check that there exist $\beta = \beta(t_0)$ and $\lambda = \lambda(t_0)$ such that

$$\begin{aligned}(ub_s + \epsilon W)_t > {} & \frac{1}{\xi^2}\left(ub_s + \epsilon W\right)_{rr} + \frac{1}{\xi}\left(2\frac{b_s}{b} - \frac{c_s}{c} - \frac{\xi_r}{\xi^2}\right)\left(ub_s + \epsilon W\right)_r \\ & + \frac{1}{b^2}\left(ub_s + \epsilon W\right)\left(8 - 18u^2 - 2b_s^2\right) + 4\frac{u^2}{b^2}(cH + \epsilon W)\end{aligned}$$

and similarly for the evolution equation of $cH + \epsilon W$. By assumption $(ub_s + \epsilon W)(\cdot, 0) > 0$

and $(ub_s + \epsilon W)(\delta, t) > 0$ for any $t \in [0, \min\{t_0, (2\beta)^{-1}\}]$. Since $u \leq 1$, by Lemma 2.10 ub_s is exponentially bounded in space uniformly in the time interval $[0, t_0]$ and hence $(ub_s + \epsilon W)(r, t) \to \infty$ as $r \to \infty$ for any $t \in [0, \min\{t_0, (2\beta)^{-1}\}]$. The same conclusions are satisfied by $cH + \epsilon W$. We can then apply [Protter and Weinberger, 2012, Theorem 13, p.190] and conclude that ub_s and cH stay nonnegative along the flow. The strict inequality in the statement then follows from using the maximum principle [Friedman, 1964, Theorem 3, p.38] and the real analyticity of the Ricci flow solutions Bando [1987] once we know that b_s and H are nonnegative. □

2.4 Analysis of the Ricci flow

In this section we derive the main curvature estimates for the Ricci flow solution $(\mathbb{R}^4, g(t))$ evolving from a warped Berger metric metric g_0. In the first part we focus on the case $g_0 \in \mathcal{G}$. Similarly to the analyses in Isenberg et al. [2016] and Isenberg et al. [2019] (which are performed on $S^1 \times S^3$ and $S^2 \tilde{\times} S^2$ respectively) we prove that away from the origin the Ricci flow is controlled by the size of the principal orbits. In particular, we show that the formation of a singularity at some positive r (i.e. along the Euclidean hypersphere of radius r) is equivalent to $b(r, t)$ converging to zero as $t \to T$. We also describe the behaviour of the flow as the time approaches T. Analogously to Isenberg et al. [2016], we prove that around any singularity the solution becomes rotationally symmetric at some rate that breaks scale-invariance.

In the second part we extend the previous estimates to Ricci flows starting at some $g_0 \in \mathcal{G}_\infty$. Moreover, for this class of solutions we also prove that (a scale-invariant version of) the mean curvature of minimal hyperspheres admits a uniform positive lower bound in the compact region where singularities may form.

For notational reasons we always let α denote a positive constant only depending on g_0 that may change from line to line.

2.4.1 Curvature estimates in $\mathcal{G}$

Throughout this section we let $(\mathbb{R}^4, g(t))_{0 \leq t < T}$ be the maximal complete, bounded curvature Ricci flow solution evolving from some $g_0 \in \mathcal{G}$. Since $b(\cdot, 0)$ is bounded from above and $cb^2(\cdot, 0)$ is increasing, because we have $(cb^2)_s = cb^2 H$, we deduce that there exists

$\varepsilon > 0$ such that $u(\cdot, 0) \geq \varepsilon$. By Lemma 2.7 we obtain

$$\varepsilon \leq u(\cdot, t) \leq 1, \tag{2.23}$$

uniformly in the space-time $\mathbb{R}^4 \times [0, T)$. We observe that (2.23) is not available for the topologies analysed in Isenberg et al. [2019] and Appleton [2019].

Next, we show that the maximal time of existence T is finite.

Lemma 2.13. *Let $(\mathbb{R}^4, g(t))_{0 \leq t < T}$ be the Ricci flow solution starting at $g_0 \in \mathcal{G}$. Then*

$$\sup_{p \in \mathbb{R}^4} b(p, t) \leq \sup_{p \in \mathbb{R}^4} b(p, 0)$$

for any $t \in [0, T)$. Moreover, we have $T \leq \frac{\sup b^2(\cdot, 0)}{4}$.

Proof. From the boundary conditions we deduce that $b^2(\cdot, t)$ is a smooth function on $\mathbb{R}^4$ as long as the solution exists. By (2.15) and (2.18) we get

$$\partial_t b^2 = \Delta b^2 - 4b_s^2 + 4u^2 - 8 \leq \Delta b^2 - 4.$$

The conclusions then follow from the maximum principle [Chow et al., 2008, Theorem 12.14]. □

Remark 2.4. *From Lemma 2.12 and Lemma 2.13 we derive that the set $\mathcal{G}$ is preserved along the Ricci flow.*

Next, we prove that b_s and c_s are uniformly bounded in the space-time. The evolution equations of the first order spatial derivative are given by

$$(b_s)_t = \Delta(b_s) - 2\frac{b_s}{b}(b_s)_s + \left(\frac{4}{b^2} - \frac{b_s^2}{b^2} - \frac{c_s^2}{c^2} - 6\frac{u^2}{b^2}\right) b_s + 4\frac{u}{b^2}c_s \tag{2.24}$$

and

$$(c_s)_t = \Delta(c_s) - 2\frac{c_s}{c}(c_s)_s - \left(6\frac{u^2}{b^2} + 2\frac{b_s^2}{b^2}\right) c_s + 8\frac{u^3}{b^2}b_s. \tag{2.25}$$

Lemma 2.14. *There exists $\alpha > 0$ such that $|b_s| \leq \alpha$ and $|c_s| \leq \alpha$ in $\mathbb{R}^4 \times [0, T)$.*

Proof. From Lemma 2.12 and Lemma 2.13 we derive that $b_s(\cdot, t)$ is integrable for any $t \in [0, T)$. Moreover, since $b_{ss}/b = -k_{01}$ as reported in (2.10), by Shi [1989] we see that for any $t \in [0, T)$ there exists $\alpha(t) < \infty$ such that $|b_{ss}/b| \leq \alpha(t)$. Since by Lemma 2.13 b is uniformly bounded from above we deduce that b_{ss} is bounded on any time-slice and hence we get that $b_s(r, t) \to 0$ as $r \to \infty$ for any $t \in [0, T)$[1]. Since $b_s \to 0$ at spatial infinity on any time-slice, if b_s becomes unbounded as $t \nearrow T$, then there exists a critical point p_0 for $b_s(\cdot, t_0)$ where $b_s = \bar{\alpha}$ for the first time, for some $\bar{\alpha}$ large to be chosen below and for some $t_0 > 0$. Evaluating (2.24) at (p_0, t_0) we get

$$(b_s)_t(p_0, t_0) \leq \frac{1}{b^2}\left(4\bar{\alpha} - \bar{\alpha}^3 - \bar{\alpha}c_s^2 - 6\bar{\alpha}u^2 + 4uc_s\right).$$

By choosing $\bar{\alpha} > \max\{\sup|b_s|(\cdot, 0), 2\}$ one checks that the c_s- quadratic polynomial in the brackets does not admit roots, thus proving $(b_s)_t(p_0, t_0) < 0$. The exact same argument works for the lower bound of b_s. In fact, the lower bound for b_s also follows from Lemma 2.12.

We now adapt the argument for c_s. Since $b^2cH = (b^2c)_s$ and b^2c is bounded from above, we see that $b^2cH(\cdot, t)$ is integrable. By differentiating we find

$$(b^2c)_{ss} = 2bcb_{ss} + 2b_s^2c + 4bb_sc_s + b^2c_{ss}.$$

From (2.9) we derive $|b_sc_s|(\cdot, t) \leq \alpha(t)bc(\cdot, t) + u^3(\cdot, t) \leq \alpha(t)$ being the curvature bounded at any time slice $\mathbb{R}^4 \times \{t\}$ for $t \in [0, T)$. Similarly $|c_{ss}|(\cdot, t) \leq \alpha(t)$. Therefore, since b and b_s are uniformly bounded in the space-time we conclude that $|(b^2c)_{ss}|(\cdot, t) \leq \alpha(t)$, which implies $b^2cH(r, t) \to 0$ as $r \to \infty$ for any $t \in [0, T)$. In particular $c_s(r, t) \to 0$ as $r \to \infty$ because b (and hence c) is uniformly bounded from above, $b_s(r, t) \to 0$ and b is bounded away from zero at spatial infinity - being b monotone increasing. One can then argue as above that if c_s becomes unbounded as $t \nearrow T$, then there exists a first maximum p_0 where c_s attains a sufficiently large value $\bar{\alpha}$ at some $t_0 > 0$ for the first time. It follows

[1] The argument that f, f_{ss} bounded and $f_s \geq 0$ imply that $f_s \to \infty$ at spatial infinity will be used in other occasions below.

that

$$(c_s)_t(p_0, t_0) \leq \frac{1}{b^2}\left(\bar{\alpha}\left(-6u^2 - 2b_s^2\right) + 8u^3 b_s\right).$$

By Lemma 2.7 we know that the ordering $u \leq 1$ is preserved. Therefore for $\bar{\alpha}$ large enough the right hand side is strictly negative. The same conclusion holds for the lower bound. □

From the previous Lemma and the condition $u \leq 1$ we immediately derive the following bounds for the vertical sectional curvatures. From now on any estimate is satisfied in the space-time $\mathbb{R}^4 \times [0, T)$ unless otherwise stated.

Corollary 2.15. *There exists $\alpha > 0$ such that*

$$|k_{12}| + |k_{13}| \leq \frac{\alpha}{b^2}.$$

The following estimate is a necessary step to prove that the solution to the Ricci flow becomes spherically symmetric at any singularity forming away from the origin.

Lemma 2.16. *The following holds as long as the solution exists:*

$$\sup_{\mathbb{R}^4}\left(\frac{1}{b}\left(b_s^2 - 4\right)\right)_+(\cdot, t) \leq \sup_{\mathbb{R}^4}\left(\frac{1}{b}\left(b_s^2 - 4\right)\right)_+(\cdot, 0).$$

Proof. Let us denote the quantity $(b_s^2 - 4)/b$ by ϕ. By the boundary conditions ϕ is uniformly bounded from above as $r \to 0$. Moreover, as we have already argued in the proof of Lemma 2.14, we find that $\phi(r, t)$ becomes negative for r large enough. We may then let (p_0, t_0) be the maximum point among prior times where ϕ attains some positive value α. A direct computation gives

$$\phi_t = \Delta\phi - 2\frac{b_{ss}^2}{b} - \frac{\phi^2}{b} - \phi\left(2\frac{b_s^2}{b^2} + 2\frac{u^2}{b^2}\right) - 12\frac{u^2}{b^3}b_s^2 + 2c_s b_s\left(4\frac{u}{b^3} - \frac{c_s b_s}{bc^2}\right).$$

Evaluating the evolution equation at (p_0, t_0) we get

$$\phi_t(p_0, t_0) \leq \frac{1}{b^3}\left(-\frac{7}{2}b_s^4 + b_s^2\left(20 - 14u^2\right) - 2b_s^2 c_s^2 u^{-2} + 8b_s c_s u - 24 + 8u^2\right).$$

We now regard the term in the brackets as a quadratic polynomial in c_s. Chosen $\alpha > 0$, we can find $\epsilon > 0$ such that $b_s^2 = 4 + \epsilon$. The discriminant of the polynomial is given by

$$8\frac{b_s^2 b^2}{c^2}\left(-8\epsilon - \frac{7}{2}\epsilon^2 - 14\epsilon u^2 - 48u^2 + 8u^4\right) < 0,$$

where we have again used that the ordering $u \leq 1$ is preserved by Lemma 2.7. Therefore, the quantity $\sup_{\mathbb{R}^4} \phi_+(\cdot, t)$ is non-increasing along the solution. □

In the next Lemma we prove that if $c(r,t)$ converges to zero as $t \to T$ (along some sequence of times) for some $r > 0$, then the metric becomes rotationally symmetric at r.

Lemma 2.17. *There exists $\alpha > 0$ such that*

$$\varphi \doteq \frac{1}{b}\left(u^{-1} - 1\right) \leq \alpha.$$

Proof. We first prove a useful characterization of the behaviour of the second order spatial derivatives at infinity.

Claim 2.18. *For any $t > 0$ we have*

$$b_{ss}(r,t) \xrightarrow{r\nearrow\infty} 0, \qquad c_{ss}(r,t) \xrightarrow{r\nearrow\infty} 0.$$

Proof of Claim 2.18. From the proof of Lemma 2.14 we see that $b_s(\cdot, t) \to 0$ at infinity, which implies that the integral of $b_{ss}(\cdot, t)$ has a finite limit for any $t \geq 0$. By Shi's derivative estimates and the Koszul formula we find that for any $t > 0$ there exists $\alpha(t) > 0$ such that

$$\alpha(t) \geq |\nabla \mathrm{Rm}_{g(t)}(\partial_s, \partial_s, X_1/b, \partial_s, X_1/b)| = |\partial_s k_{01}| = \left|\partial_s\left(\frac{b_{ss}}{b}\right)\right| = \left|\frac{b_{sss}}{b} - \frac{b_{ss}b_s}{b^2}\right|.$$

Since b is uniformly bounded from above and $b_{ss}/b = -k_{01}$ is bounded, we see that $|b_{sss}|(\cdot, t) \leq \alpha(t)$. Therefore b_{ss} is integrable and has a bounded derivative for any $t > 0$, meaning that $b_{ss}(r,t) \to 0$ as $r \to \infty$ for any $t > 0$.

Again from the proof of Lemma 2.14 we derive that the integral of $(b^2 c)_{ss}(\cdot, t)$ is convergent for any $t \in [T/2, T)$. By computing the derivative $(b^2 c)_{sss}$ and using again Shi's derivative estimates we obtain that $(b^2c)_{ss} \to 0$, which also implies $c_{ss}(r,t) \to 0$ as $r \to \infty$ for any $t \in [T/2, T)$. □

By the boundary conditions the function $\varphi = 1/c - 1/b$ is continuously defined at the origin and identically zero. From (2.23) we see that φ is bounded along any time slice $\mathbb{R}^4 \times \{t\}$, for $t \in [0,T)$. The evolution equation of φ is given by

$$\varphi_t = \Delta\varphi + \frac{b_s^2}{b^3} - \frac{c_s^2}{c^3} + \frac{1}{b^3}\left(-4 + 2u + 2u^2\right). \tag{2.26}$$

We note that

$$\Delta\varphi = \varphi_{ss} + \varphi_s\left(2\frac{b_s}{b} + \frac{c_s}{c}\right) = -\frac{c_{ss}}{c^2} + 2\frac{c_s^2}{c^3} + \frac{b_{ss}}{b^2} - 2\frac{b_s^2}{b^3}.$$

The argument in Lemma 2.14 shows that b_s and c_s converge to zero at spatial infinity on any time-slice and similarly for the second order derivatives according to Claim 2.18. Since b and c are both uniformly bounded from above, we see that for any $\delta > 0$ and $t > 0$ there exists $r_0 = r_0(\delta, t)$ such that the time derivative of φ can be bounded for r larger than r_0 as

$$\varphi_t \leq \delta + \frac{1}{b^3}\left(-4 + 2u + 2u^2\right) \leq \delta - \frac{2u}{b^2}\varphi \leq \delta - \frac{2\varepsilon}{b^2}\varphi \leq \delta - \eta\varphi,$$

for some $\eta > 0$, where we have used Lemma 2.7 and Lemma 2.13. Therefore, if φ does not stay bounded as $t \nearrow T$, then there exists a sequence of maxima diverging as the solution approaches its maximal time of existence.

We introduce the quantity $\bar{\alpha} \doteq \lambda\varepsilon^{-1}(\varepsilon^{-1} - 1)$, where $\lambda = \sup_{\mathbb{R}^4}(b_s^2 - 4)/b(\cdot, 0)$ and ε is chosen to satisfy Lemma 2.7. Suppose that (p_0, t_0) is a space-time maximum point among prior times where φ attains some value greater than $\bar{\alpha}$. By evaluating (2.26) at (p_0, t_0) we see that

$$\varphi_t(p_0, t_0) \leq \frac{1}{b^3}\left(b_s^2\,(1-u) - 4 + 2u + 2u^2\right).$$

Using Lemma 2.16, we can estimate the time derivative from above as

$$\begin{aligned}\varphi_t(p_0,t_0) &\leq \frac{1}{b^3}\left((\lambda b+4)(1-u)-4+2u+2u^2\right)\\ &= \frac{1}{b^3}\left(-2u+2u^2+\lambda(b-c)\right)\\ &= \frac{u}{b^2}\varphi\left(-2u+\lambda b\right) \leq \frac{u}{b^2}\varphi\left(-2\varepsilon+\lambda b\right).\end{aligned}$$

From the definition of φ we derive

$$b \leq \frac{1}{\varphi}\left(\varepsilon^{-1}-1\right),$$

which then yields

$$\varphi_t(p_0,t_0) \leq \frac{u}{b^2}\varphi\left(-2\varepsilon+\frac{\lambda}{\varphi}\left(\varepsilon^{-1}-1\right)\right) \leq \frac{u}{b^2}\varphi\left(-2\varepsilon+\varepsilon\right) < 0.$$

□

An analogous bound holds for the first order spatial derivatives. Namely, we have the following

Lemma 2.19. *There exists $\alpha > 0$ such that*

$$\left|\frac{c_s}{c}-\frac{b_s}{b}\right| \leq \alpha.$$

Proof. Define $\psi \doteq c_s/c - b_s/b$. The function ψ extends to zero at the origin due to the boundary conditions. As argued before $\psi(r,t) \to 0$ as $r \to \infty$ for any $t \in [0,T)$. Consider the upper bound. Suppose that there exists a large value $\bar{\alpha}$ which ψ attains for the first time at some maximum space-time point (p_0,t_0). The evolution equation of ψ is

$$\psi_t = \Delta\psi - \psi\left(\frac{c_s^2}{c^2}+8\frac{u^2}{b^2}+2\frac{b_s^2}{b^2}\right) - 8\frac{b_s}{b^3}\left(1-u^2\right).$$

We can then evaluate both sides at (p_0, t_0) and use Lemma 2.17 to get

$$\begin{aligned}\psi_t(p_0,t_0) &\le \frac{1}{b^2}\left(-\bar\alpha\left(u^{-2}c_s^2+8u^2+2b_s^2\right)+8u|b_s|\left(1+u\right)\left(\frac{1}{c}-\frac{1}{b}\right)\right)\\ &\le \frac{8}{b^2}\left(-\bar\alpha u^2+2\alpha|b_s|\right)\le \frac{8}{b^2}\left(-\bar\alpha\varepsilon^2+2\alpha^2\right)<0,\end{aligned}$$

for $\bar\alpha$ sufficiently large, with ε satisfying Lemma 2.7. The same argument shows the existence of a uniform lower bound. □

We may now improve Lemma 2.16.

Lemma 2.20. *There exists* $\alpha > 0$ *such that*

$$\frac{1}{b}\left(b_s^2-1\right)\le\alpha.$$

Proof. Let us denote $(b_s^2-1)/b$ by φ. A direct computation gives

$$\varphi_t=\Delta\varphi-\frac{2b_{ss}^2}{b}-3\frac{b_s^4}{b^3}+\frac{b_s^2}{b^3}\left(13-14u^2\right)-2\frac{b_s^2c_s^2}{bc^2}+8\frac{ub_sc_s}{b^3}-\frac{4}{b^3}+2\frac{u^2}{b^3}.$$

Since φ is uniformly bounded from above at the origin and at spatial infinity we take (p_0, t_0) to be the maximum space-time point where $\varphi = \bar\alpha$ for the first time, for some positive $\bar\alpha$ to be chosen below. We have

$$\varphi_t(p_0,t_0)\le\frac{1}{b^2}\left\{-2\frac{u^{-1}b_s^2c_s^2}{c}+8\frac{ub_sc_s}{b}-\frac{7b_s^4}{2b}+14\frac{b_s^2}{b}\left(1-u^2\right)-\frac{9}{2b}+2\frac{u^2}{b}\right\}.$$

According to Lemma 2.19 we can bound c_s in terms of b_s. It follows that there exists some positive constant $\alpha > 0$ such that

$$\begin{aligned}\varphi_t(p_0,t_0)&\le\frac{1}{b^2}\left\{\alpha-\frac{11b_s^4}{2b}+\frac{b_s^2}{b}\left(14-6u^2\right)-\frac{9}{2b}+2\frac{u^2}{b}\right\}\\ &\le\frac{1}{2b^2}\left\{\alpha-\frac{11b_s^2}{b}\left(b_s^2-1\right)+\frac{5}{b}\left(b_s^2-1\right)+\frac{12b_s^2}{b}\left(1-u^2\right)-\frac{4}{b}\left(1-u^2\right)\right\}.\end{aligned}$$

We finally use Lemma 2.14, Lemma 2.17 and the fact that $\varphi > 0$ implies $b_s^2 > 1$ to derive

$$\varphi_t(p_0,t_0)\le\frac{1}{2b^2}\left(\alpha-6\bar\alpha\right)<0,$$

for $\bar{\alpha}$ large enough. $\square$

Next, we extend the previous arguments to the second spatial derivatives. We show that away from the origin the mixed sectional curvatures are controlled by c and hence by b as in Corollary 2.15. The flow is singular at some radius r if and only if both $c(r,t)$ and $b(r,t)$ converge to zero as t approaches T.
The evolution equations of the mixed sectional curvatures (2.10) and (2.11) are given by

$$\begin{aligned}(k_{01})_t = \Delta(k_{01}) + 2k_{01}^2 + k_{01}\left(\frac{8}{b^2} - \frac{8u^2}{b^2} - \frac{2c_s^2}{c^2} - \frac{4b_s^2}{b^2}\right) + k_{03}\left(\frac{4u^2}{b^2} - \frac{2b_sc_s}{bc}\right) \\ - \frac{4c_s^2}{b^4} + \frac{24ub_sc_s}{b^4} - \frac{2b_sc_s^3}{bc^3} - \frac{24u^2b_s^2}{b^4} + \frac{8b_s^2}{b^4} - \frac{2b_s^4}{b^4}\end{aligned} \tag{2.27}$$

and

$$\begin{aligned}(k_{03})_t = \Delta(k_{03}) + 2k_{03}^2 - 4k_{03}\left(\frac{b_s^2}{b^2} + \frac{u^2}{b^2}\right) + 4k_{01}\left(\frac{2u^2}{b^2} - \frac{b_sc_s}{bc}\right) \\ + \frac{12c_s^2}{b^4} + \frac{40u^2b_s^2}{b^4} - \frac{48ub_sc_s}{b^4} - \frac{4b_s^3c_s}{b^3c}.\end{aligned} \tag{2.28}$$

Lemma 2.21. *There exists $\alpha > 0$ such that*

$$|k_{01}| + |k_{03}| \leq \frac{\alpha}{b^2}.$$

Proof. By Shi [1989] there exists $\alpha > 0$ such that $|k_{01}| \leq \alpha$ in $\mathbb{R}^4 \times [0, T/2]$. Since by Lemma 2.13 b is uniformly bounded from above, we deduce that $b^2|k_{01}| \leq \alpha$ and similarly for $b^2|k_{03}|$ using Lemma 2.7. We may hence consider the case $t \in [T/2, T)$. Define the map $\psi \doteq bb_{ss} - \mu b_s^2 - \nu c_s^2$ in $\mathbb{R}^4 \times [T/2, T)$, for some μ and ν positive constants we will determine below. According to Claim 2.18 $\psi(r,t) \to 0$ as $r \to \infty$ for any $t \in [T/2, T)$. We then adapt the argument in [Isenberg et al., 2019, Lemma 7] to show that ψ is uniformly bounded in the space-time region. Explicitly, at any stationary point of $\psi(\cdot, t)$ we have

$$\begin{aligned}\psi_t = \Delta\psi - bb_{ss}\left(\frac{4u^2}{b^2} + \frac{2c_s^2}{c^2} + \frac{4\mu b_s^2}{b^2}\right) - \frac{16\nu u^3 b_s c_s}{b^2} - (24+8\mu)\frac{ub_s c_s}{b^2} + \frac{2u^{-1}b_s c_s^3}{c^2} \\ - \frac{8b_s^2(\mu+1)}{b^2} + \frac{4uc_{ss}}{b} + \frac{4\nu c_s^2 c_{ss}}{c} - \frac{8\nu b_s c_s c_{ss}}{b} - \frac{2u^{-1}b_s c_s c_{ss}}{c} + \frac{12\nu u^2 c_s^2}{b^2} + \frac{4c_s^2}{b^2} \\ + \frac{24u^2 b_s^2}{b^2} + \frac{12\mu u^2 b_s^2}{b^2} + \frac{2\mu b_s^2 c_s^2}{c^2} + \frac{4\nu b_s^2 c_s^2}{b^2} + \frac{2b_s^4}{b^2} + 2\nu c_{ss}^2 + 2(\mu-1)b_{ss}^2.\end{aligned}$$

Suppose ψ attains some negative value $-\bar{\alpha}$ for the first time at $t_0 \in [T/2, T)$. By Lemma 2.14 we can choose $\bar{\alpha}$ sufficiently large such that $bb_{ss} \leq -\bar{\alpha}/4$. Therefore we get

$$-bb_{ss}\left(\frac{4u^2}{b^2} + \frac{2c_s^2}{c^2} + \frac{4\mu b_s^2}{b^2}\right) \geq \bar{\alpha}\left(\frac{u^2}{2b^2} + \frac{c_s^2}{2c^2} + \frac{\mu b_s^2}{b^2}\right).$$

Evaluating the evolution equation of ψ at (p_0, t_0), we have (provided we set $\mu \geq 1$)

$$\begin{aligned}\psi_t(p_0, t_0) \geq 2\nu c_{ss}^2 + \bar{\alpha}\left(\frac{u^2}{2b^2} + \frac{c_s^2}{2c^2} + \frac{\mu b_s^2}{b^2}\right) - \frac{16\nu u^3 b_s c_s}{b^2} - (24+8\mu)\frac{ub_s c_s}{b^2} \\ + \frac{2u^{-1}b_s c_s^3}{c^2} - \frac{8b_s^2(\mu+1)}{b^2} + \frac{4uc_{ss}}{b} + \frac{4\nu c_s^2 c_{ss}}{c} - \frac{8\nu b_s c_s c_{ss}}{b} - \frac{2u^{-1}b_s c_s c_{ss}}{c}.\end{aligned}$$

One can then estimate the remaining terms exactly as in Isenberg et al. [2019] by using the uniform boundedness of the first spatial derivatives and Lemma 2.7. Namely, we can use the weighted Cauchy-Schwarz inequality to get

$$\left|\frac{2u^{-1}b_s c_s c_{ss}}{c}\right| \leq \frac{c_{ss}^2}{2} + 2\frac{u^{-2}b_s^2 c_s^2}{c^2} \leq \frac{c_{ss}^2}{2} + 2\frac{\alpha^2 u^{-2} c_s^2}{c^2} \leq \frac{c_{ss}^2}{2} + 2\frac{\alpha^2 \varepsilon^{-2} c_s^2}{c^2},$$

and similarly for the others. Therefore there exists a uniform constant α such that the time derivative at (p_0, t_0) is bounded from below as

$$\psi_t(p_0, t_0) \geq (\nu-1)c_{ss}^2 + \bar{\alpha}\left(\frac{u^2}{2b^2} + \frac{c_s^2}{2c^2} + \frac{\mu b_s^2}{b^2}\right) - \alpha(\mu+\nu+1)\left(\frac{u^2}{2b^2} + \frac{c_s^2}{2c^2} + \frac{\mu b_s^2}{b^2}\right) > 0,$$

once we choose $\mu = 1, \nu = 2$ and $\bar{\alpha}$ sufficiently large. The existence of a uniform upper bound follows from a similar argument by considering $\tilde{\psi} \doteq bb_{ss} + \mu b_s^2 + \nu b_s^2$. The very same proof applies for k_{03}. □

We can finally show that both c and b admit limits as the flow approaches its maximal time of existence $T < \infty$.

Corollary 2.22. *For any $r \geq 0$ the limits $\lim_{t \nearrow T} c(r,t)$ and $\lim_{t \nearrow T} b(r,t)$ exist and are finite.*

Proof. By applying Lemmas 2.14 and 2.21 we get

$$|(c^2)_t| \leq 2|cc_{ss}| + 4\left|ub_sc_s - u^3\right| \leq \alpha.$$

The same argument works for b as well. □

The curvature is hence uniformly controlled along any Euclidean hypersphere where the components b and c do not converge to zero as $t \nearrow T$. Namely, from Corollary 2.15 and Lemma 2.21 it follows that there exists a positive constant α such that

$$\sup_{\mathbb{R}^4 \times [0,T)} b^2 |\mathrm{\mathbf{Rm}}_{g(t)}|_{g(t)} \leq \alpha. \tag{2.29}$$

Next, we show that around a singularity rotational symmetry type of bounds hold for the second spatial derivatives as well.

Lemma 2.23. *There exists $\alpha > 0$ such that*

$$|k_{01} - k_{03}| \leq \frac{\alpha}{b}.$$

Proof. We adapt the proof in Isenberg et al. [2016], whose argument works for a compact Type-I Ricci flow setting. Once we define $\psi \doteq c_s/c - b_s/b$, we consider the map

$$\varphi \doteq (\psi_s b)^2 \equiv \left(\frac{c_{ss}}{c} - \frac{c_s^2}{c^2} - \frac{b_{ss}}{b} + \frac{b_s^2}{b^2}\right)^2 b^2.$$

The boundary conditions (2.7) ensure that $\varphi(\boldsymbol{o}, t) = 0$ for any t. From the proof of Lemma 2.14 and Claim 2.18 we derive that if φ is not bounded, then for any sufficiently large value $\bar{\alpha}$ there exist $t_0 \geq T/2$ and a maximum point p_0 such that $\varphi(p_0, t_0) = \bar{\alpha}$ for the first time. The evolution equation for φ is

$$\varphi_t = \Delta\varphi - 2(b^2)_s(\psi_s^2)_s - 2b^2\psi_{ss}^2 - 2F\varphi - 2Gb^2\psi\psi_s + 2Hb^2\psi_s - 4b_s^2\psi_s^2 - 8\psi_s^2 + 4\psi_s^2u^2,$$

where

$$F \doteq 4\frac{b_s^2}{b^2} + 2\frac{c_s^2}{c^2} + 8\frac{u^2}{b^2}, \quad G \doteq \left(2\frac{b_s^2}{b^2} + \frac{c_s^2}{c^2} + 8\frac{u^2}{b^2}\right)_s \quad H \doteq -\left(8\frac{b_s}{b^3}\left(1-u^2\right)\right)_s.$$

Evaluating φ at the maximum point (p_0, t_0) we get

$$\begin{aligned}\varphi_t(p_0,t_0) &\leq 2b_s^2\psi_s^2 - \psi_s^2\left(8-4u^2\right) + -2F\varphi - 2Gb^2\psi\psi_s + 2Hb^2\psi_s \\ &\leq -\psi_s^2\left(8+12u^2\right) - 2Gb^2\psi\psi_s + 2Hb^2\psi_s.\end{aligned}$$

From (2.29) it easily follows that there exists some uniform constant $\alpha > 0$ such that $|G| \leq \alpha/b^3$. Being ψ uniformly bounded (Lemma 2.19), we have

$$-2Gb^2\psi\psi_s \leq \alpha\frac{|\psi_s|}{b}.$$

According to Lemma 2.17 and Lemma 2.19 an analogous estimate can be found for $|Hb^2\psi_s|$. Then

$$\varphi_t(p_0,t_0) \leq -\psi_s^2\left(8+12u^2\right) + \alpha\frac{|\psi_s|}{b} \equiv \frac{|\psi_s|}{b}\left(\left(-8-12u^2\right)\sqrt{\bar\alpha} + \alpha\right) < 0,$$

for $\bar\alpha$ sufficiently large. Therefore φ is uniformly bounded and we get

$$\begin{aligned} b|k_{01} - k_{03}| &\leq \alpha + \left|\frac{b_s^2}{b} - u^{-1}\frac{c_s^2}{c}\right| \\ &\leq \alpha + |b_s|\left|\frac{b_s}{b} - \frac{c_s}{c}\right| + |c_s|u^{-1}\left|\frac{b_s}{b} - \frac{c_s}{c}\right| \leq \alpha\left(1 + \alpha + |c_s|u^{-1}\right) \leq \alpha\left(1+\varepsilon^{-1}\right)\right), \end{aligned}$$

where we have used Lemma 2.19 and Lemma 2.7. □

We finally discuss the existence of lower bounds for the mixed sectional curvatures.

Lemma 2.24. *There exists* $\alpha > 0$ *such that*

$$c_{ss} c \log c \geq -\alpha, \quad b_{ss} b \log b \geq -\alpha.$$

Proof. We adapt the analogous argument in Isenberg et al. [2016]. Consider the map $f \doteq c_{ss} c \log c$, which is smooth in $\mathbb{R}^4 \setminus \{\boldsymbol{o}\} \times [0, T)$. Moreover f extends continuously to the origin and $f(\boldsymbol{o}, t) = 0$ as long as a solution exists. By Claim 2.18 and the fact that c is uniformly bounded from above, we deduce that $f(r, t) \to 0$ at spatial infinity for any $t \in [T/2, T)$. Suppose that there exist $(p_0, t_0) \in (\mathbb{R}^4 \setminus \{\boldsymbol{o}\}) \times [T/2, T)$ and $\bar{\alpha}$ large to be chosen below such that $f(p_0, t_0) = -\bar{\alpha}$ for the first time. From (2.11) and (2.29) it follows

$$\bar{\alpha} = |f(p_0, t_0)| \leq \alpha |\log c(p_0, t_0)|.$$

Since by Lemma 2.13 c is uniformly bounded from above the last inequality implies $\log c(p_0, t_0) < 0$ and

$$c_{ss}(p_0, t_0) = \frac{\bar{\alpha}}{c|\log c|}(p_0, t_0) \geq \bar{\alpha}. \tag{2.30}$$

for $\bar{\alpha}$ large enough. By direct computation we get

$$\begin{aligned} f_t = \Delta f &- 2\left(2 + \frac{1}{\log c}\right)\frac{c_s}{c} f_s - c \log c \left(12 \frac{u c_s^2}{b^3} - 48 \frac{u^2}{b^3} b_s c_s + 40 \frac{u^3}{b^3} b_s^2 - 4 \frac{b_s^3 c_s}{b^3}\right) \\ &- 8u^3 \log c \left(\frac{c_{ss}}{c} - \frac{b_{ss}}{b}\right) - 2\frac{c_{ss}}{c}\left(u^3 + f\right) + 2\frac{c_s^2 c_{ss}}{c}\left(2 + \frac{1}{\log c}\right) \\ &- 4c \log c \left(\frac{c_{ss} b_s^2}{b^2} + \frac{b_{ss} b_s c_s}{b^2} - \frac{c_{ss} c_s^2}{c^2}\right). \end{aligned}$$

By Lemma 2.19 we have

$$12 \frac{u c_s^2}{b^3} - 48 \frac{u^2}{b^3} b_s c_s + 40 \frac{u^3}{b^3} b_s^2 - 4 \frac{b_s^3 c_s}{b^3} \geq \frac{4}{b^2}\left(-\alpha - b_s^2 \frac{u}{b}\left(b_s^2 - u^2\right)\right) \geq -\frac{\alpha}{c^2},$$

where we have used Lemma 2.17 and Lemma 2.20 to derive the last inequality. According to Lemma 2.23 it holds

$$-8u^3 \log c \left(\frac{c_{ss}}{c} - \frac{b_{ss}}{b}\right) \geq -\alpha \frac{|\log c|}{c}.$$

By choosing $\bar{\alpha}$ large enough (and hence $c(p_0,t_0)$ small) it follows that $2(c_s c_{ss}/c)(2 + 1/\log c)(p_0,t_0) \geq 0$. Finally, Lemma 2.17, Lemma 2.19, and Lemma 2.23 yield the bounds

$$\left(\frac{c_{ss}b_s^2}{b^2} + \frac{b_{ss}b_s c_s}{b^2} - \frac{c_{ss}c_s^2}{c^2}\right)(p_0,t_0) \geq \left(\frac{b_{ss}b_s c_s}{b^2} - \frac{c_{ss}c_s^2}{c^2}\right)(p_0,t_0) =$$
$$\left(\frac{b_s c_s}{b}\left(\frac{b_{ss}}{b} - \frac{c_{ss}}{c}\right) + \frac{c_{ss}}{c}\frac{c_s(c(b_s - c_s) + c_s(c-b))}{bc}\right)(p_0,t_0) \geq \left(-\frac{\alpha}{c^2} - \alpha\frac{c_{ss}}{c}\right)(p_0,t_0).$$

Evaluating the evolution equation of f at (p_0,t_0) and using (2.30) we get the lower bound

$$\begin{aligned} f_t(p_0,t_0) &\geq -\alpha\left(\frac{|\log c|}{c} + |\log c| c_{ss}\right) + 2\frac{c_{ss}}{c}(\bar{\alpha}-1) = \frac{1}{c}\left(-\alpha(|\log c| + \bar{\alpha}) + 2c_{ss}(\bar{\alpha}-1)\right) \\ &= \frac{1}{c}\left(-\alpha|\log c| + \bar{\alpha}(c_{ss} - \alpha) + c_{ss}\frac{\bar{\alpha}}{2} + \frac{c_{ss}}{2}(\bar{\alpha}-4)\right) \geq \frac{1}{c}\left(-\alpha|\log c| + c_{ss}\frac{\bar{\alpha}}{2}\right) \\ &= \frac{1}{2|\log c|c^2}\left(-2\alpha|\log c|^2 c + \bar{\alpha}^2\right) > 0, \end{aligned}$$

for $\bar{\alpha}$ sufficiently large (i.e. c small enough). The case of $b_{ss} b \log b$ does not require modifications. □

2.4.2 Curvature estimates in $\mathcal{G}_\infty$

We consider $(\mathbb{R}^4, g(t))_{0\leq t<T}$ a maximal complete, bounded curvature Ricci flow solution evolving from some $g_0 \in \mathcal{G}_\infty$. If the solution develops a finite-time singularity at some $T < \infty$, then we can apply [Chau et al., 2011, Theorem 1.1] and conclude that there exists $\rho > 0$ such that

$$\sup_{(\mathbb{R}^4 \setminus B(\boldsymbol{o},\rho))\times[0,T)} \left|\mathrm{Rm}_{g(t)}\right|_{g(t)} \leq 1. \tag{2.31}$$

Remark 2.5. *We note that $\mathcal{G}_\infty$ is preserved along the Ricci flow. Consider the maximal Ricci flow evolving from some $g_0 \in \mathcal{G}_\infty$. By Lemma 2.12 condition (i) in Definition 2.5 persists in time. From Hamilton [1995] we also derive that $|\mathrm{Rm}|(s,t) \to 0$ as $s \to \infty$ for all $t \in [0,T)$. If $T < \infty$, then given ρ as in* (2.31)*, we can integrate the Ricci flow equation for c and use that by assumption c is bounded away from zero for all radii r sufficiently large, and find $\mu > 0$ such that $c(\cdot,t) \geq \mu > 0$ in $\mathbb{R}^4 \setminus B(\boldsymbol{o},\rho) \times [0,T)$. If* *instead $T = \infty$, then c is uniformly bounded from below away from the origin in any compact interval of existence being the curvature bounded.*

An immediate consequence of (2.31) is that for any radial coordinate $r_1 > \rho$ the spatial derivatives (up to second order) of b and c are uniformly bounded in time along the hypersphere of radius r_1. In particular, for any $r_1 > \rho$ there exists $\varepsilon = \varepsilon(r_1) > 0$ satisfying

$$\inf_{p \in B(\boldsymbol{o}, r_1)} u(p,t) \geq \varepsilon > 0, \tag{2.32}$$

for any $t \in [0,T)$. For the function u is uniformly bounded from below at the origin and along the hypersphere of radius x_1 and the evolution equation (2.20) prevents u from attaining interior minima approaching zero.
By inspection one can check that given $r_1 > \rho$ any bound derived in Subsection 2.4.1 extends to the space-time region $B(\boldsymbol{o}, r_1) \times [0,T)$. For any argument relies on a maximum principle which still applies to this setting once we know that any relevant quantity is uniformly bounded along the parabolic boundary of the region $B(\boldsymbol{o}, r_1) \times [0,T)$. Explicitly, we have the following

Lemma 2.25. *Let $(\mathbb{R}^4, g(t))_{0 \leq t < T}$, with $T < \infty$, be the maximal Ricci flow solution evolving from some $g_0 \in \mathcal{G}_\infty$ and let $\rho > 0$ satisfy*

$$\sup_{(\mathbb{R}^4 \setminus B(\boldsymbol{o},\rho)) \times [0,T)} \left|\mathrm{Rm}_{g(t)}\right|_{g(t)} \leq 1.$$

Then for any $r_1 > \rho$ there exists $\alpha = \alpha(r_1) > 0$ such that

$$\sup_{B(\boldsymbol{o},r_1) \times [0,T)} b^2 |\mathrm{Rm}_{g(t)}|_{g(t)} \leq \alpha,$$

and

$$\sup_{B(\boldsymbol{o},r_1) \times [0,T)} \left(\frac{1}{b}\left(u^{-1} - 1\right) + \left|\frac{c_s}{c} - \frac{b_s}{b}\right| + b|k_{01} - k_{03}| \right) \leq \alpha.$$

Remark 2.6. *One can verify that Lemma 2.25 holds for a larger class of Ricci flows than $\mathcal{G}_\infty$. Indeed, it suffices to control the flow uniformly along the parabolic boundary of some space-time region and then apply maximum principle arguments without relying on the quantities b_s and H being nonnegative.*

Next, we prove that b_s and H remain positive along a hypersphere of sufficiently large radius. In the following ρ still denotes the radius defined by (2.31).

Lemma 2.26. *Let* $(\mathbb{R}^4, g(t))_{0\leq t<T}$*, with* $T < \infty$*, be the maximal Ricci flow solution evolving from a warped Berger metric* $g_0 \in \mathcal{G}_\infty$*. There exist* $\tilde{r}_2 \geq \tilde{r}_1 > \rho$*,* $\delta > 0$ *and* $\tilde{t} \in [0, T)$ *such that*

$$b_s(\tilde{r}_1, t) \geq \delta, \qquad H(\tilde{r}_2, t) \geq \delta \qquad \forall t \in [\tilde{t}, T).$$

Proof. We have already shown that $b(r, 0) \to \infty$ as $r \to \infty$. Since the curvature is uniformly bounded in the complement of the Euclidean ball $B(\boldsymbol{o}, \rho)$, we can pick $\rho < r_0 < r_1$ such that

$$b(r_1, t) - b(r_0, t) \geq \varsigma > 0,$$

for some $\varsigma > 0$ and for any $t \in [0, T)$. We can use the Koszul formula to write the evolution equation of b_s as

$$\begin{aligned} \partial_t(b_s) &= \partial_s\left(-\mathrm{Ric}_{g(t)}\left(\frac{X_1}{b}, \frac{X_1}{b}\right) b\right) + \mathrm{Ric}_{g(t)}(\partial_s, \partial_s) b_s \\ &= -\mathrm{Ric}_{g(t)}\left(\frac{X_1}{b}, \frac{X_1}{b}\right) b_s - \nabla_{g(t)}\mathrm{Ric}_{g(t)}\left(\partial_s, \frac{X_1}{b}, \frac{X_1}{b}\right) b + \mathrm{Ric}_{g(t)}(\partial_s, \partial_s) b_s. \end{aligned} \tag{2.33}$$

Therefore given $r > \rho$, by (2.31) and Shi's derivative estimates there exists $\alpha(r) > 0$ such that $|(b_s)_t(r, t)| \leq \alpha$ uniformly in time. Since $T < \infty$ the last property implies that $b_s(r, \cdot)$ is Lipschitz and hence admits a finite limit as $t \nearrow T$, which we know to be nonnegative according to Lemma 2.12. Let us assume for a contradiction that any such limit is zero. Since $b_{ss}/b = -k_{01}$ is bounded in the annular region $(r_0, r_1) \times S^3$ uniformly in time and we also have that b is bounded in the same space-time region as follows from integrating the Ricci flow equation for b using that the curvature is uniformly bounded, we deduce that $\sup_{[r_0, r_1]} b_s(\cdot, t) \to 0$ as $t \nearrow T$. On the other hand, being the curvature controlled in the annular region $(r_0, r_1) \times S^3$, we get

$$\begin{aligned}\varsigma \leq b(r_1,t) - b(r_0,t) &\leq \sup_{[r_0,r_1]} b_s(\cdot,t)(s(r_1,t) - s(r_0,t)) \\ &\leq \alpha \sup_{[r_0,r_1]} b_s(\cdot,t)(s(r_1,0) - s(r_0,0)) \leq \alpha \sup_{[r_0,r_1]} b_s(\cdot,t),\end{aligned}$$

for any $t \in [0,T)$, which gives a contradiction. Therefore, there exists $\tilde{r_1} \in [r_0, r_1]$ as in the statement. The proof for H is similar. Indeed we can write $H = (\log(b^2c))_s$ and then adapt the argument above noting that by Definition 2.5 $\log(b^2c))(r,0) \to \infty$ as $r \to \infty$. In particular we can always pick $\tilde{r}_2 \geq \tilde{r}_1$. □

Next we show that cH stays away from zero in the compact region $B(\boldsymbol{o}, \rho)$ for times close to the maximal time of existence T. In the following we let $\tilde{r}_2$ and $\tilde{t}$ be defined as in the previous Lemma.

Corollary 2.27. *There exists $\mu > 0$ such that $cH \geq \mu$ in $B(\boldsymbol{o}, \tilde{r}_2) \times [\tilde{t}, T)$.*

Proof. By the boundary conditions we have $cH(\boldsymbol{o}, t) = 3$ as long as the solution exists. According to Lemma 2.12 and Lemma 2.26 there exists $\delta > 0$ such that $cH(\tilde{r}_2, t) \geq \delta$ for any $t \in [\tilde{t}, T)$ and $cH(r, \tilde{t}) \geq \delta$ for any $0 \leq r \leq \tilde{r}_2$. Suppose that cH gets smaller than $\min\{\delta, 3\}$ in $B(\boldsymbol{o}, \tilde{r}_2) \times [\tilde{t}, T)$. Then there exists a minimum point (p_0, t_0) and from (2.22), (2.32) and Lemma 2.12 we get

$$(cH)_t(p_0,t_0) \geq 2\frac{uH}{b}\left(u^2 - b_s^2\right)(p_0,t_0) \geq 2\frac{uH}{b}\left(\varepsilon^2 - b_s^2\right)(p_0,t_0).$$

From Lemma 2.19 it follows that

$$b_s = \frac{1}{2}\left(u^{-1}cH - u^{-1}c_s\right) \leq \frac{1}{2}\left(u^{-1}cH - b_s + \alpha b\right),$$

for some $\alpha = \alpha(\tilde{r}_2) > 0$. Thus we can find a uniform constant α only depending on $\tilde{r}_2$ such that

$$(cH)_t(p_0,t_0) \geq \frac{2cH}{b^2}\left(\varepsilon^2 - \alpha cH(1 + cH) - \alpha b^2\right)(p_0,t_0).$$

Therefore, whenever $cH \leq \tilde{\delta}$, for some $\tilde{\delta}$ only depending on $\tilde{r}_2$ and $\tilde{t}$, the function $t \mapsto$

$\min_{B(\boldsymbol{o},\tilde{r}_2)}(cH)(\cdot,t)$ is Lipschitz and satisfies

$$\frac{d(cH)_{\min}}{dt} \geq -\alpha(cH)_{\min}.$$

We conclude that cH cannot approach zero in the interior of $B(\boldsymbol{o},\tilde{r}_2)$ as $t \nearrow T$. □

2.5 Singularity models of warped Berger Ricci flows

In this section we perform a blow-up analysis of warped Berger Ricci flows. We first recall the following general notion.

Definition 2.6. *A complete, bounded curvature ancient solution to the Ricci flow* $(M_\infty, g_\infty(t))_{-\infty<t\leq 0}$ *is a* singularity model *for a warped Berger Ricci flow* $(\mathbb{R}^4, g(t))_{0\leq t<T}$ *if* $T < \infty$ *and there exists a sequence of space-time points* (p_j, t_j) *with* $t_j \nearrow T$ *such that* $\lambda_j \doteq |\mathrm{Rm}_{g(t_j)}|_{g(t_j)}(p_j) \to \infty$ *and the rescaled Ricci flows* $(\mathbb{R}^4, g_j(t), p_j)$ *defined by*

$$g_j(t) \doteq \lambda_j g\left(t_j + \frac{t}{\lambda_j}\right)$$

converge to $(M_\infty, g_\infty(t), p_\infty)$ *in the pointed Cheeger-Gromov sense for* $t \in (-\infty, 0]$.

Remark 2.7. *We note that by the Cheeger-Gromov convergence any singularity model* $(M_\infty, g_\infty(t))$ *of a warped Berger Ricci flow is* non-compact *and* non-flat.

The main goal of this section consists in classifying the singularity models of warped Berger Ricci flows. Namely, we show the following result

Proposition 2.28. *Let* $(\mathbb{R}^4, g(t))_{0\leq t<T}$*, with* $T < \infty$*, be the maximal Ricci flow solution evolving from a warped Berger metric* g_0 *belonging to either* $\mathcal{G}$ *or* $\mathcal{G}_\infty$*. Then any singularity model is either the self-similar shrinking soliton on the cylinder or a positively curved rotationally symmetric* κ*-solution.*

We prove the characterization of singularity models by first showing that the $U(2)$-symmetries of warped Berger Ricci flows pass to the limit and then using the rotational-symmetry type of bounds in Lemma 2.17 and Lemma 2.19 to show that the symmetries of the singularity model in fact enhance to $SO(4)$.

Given $g_0 \in \mathcal{G}$ there exists $\varepsilon > 0$ such that $u(\cdot, 0) \geq \varepsilon$. Therefore g_0 is bounded between two round cylinders outside some compact region and there exists $\alpha > 0$ such that $\mathrm{Vol}_{g_0}(B_{g_0}(p, 1)) \geq \alpha$ for any $p \in \mathbb{R}^4$. The latter condition is satisfied by any $g_0 \in \mathcal{G}_\infty$ being the injectivity radius positive and the curvature bounded. Thus if $(\mathbb{R}^4, g(t))_{0 \leq t < T}$, with $T < \infty$, is the maximal Ricci flow solution evolving from some g_0 which belongs to either $\mathcal{G}$ or $\mathcal{G}_\infty$, then by Theorem 0.3 there exists $\kappa > 0$ such that $g(t)$ is (weakly) $\kappa-$non-collapsed in $\mathbb{R}^4 \times (T/2, T)$ at any scale $\rho \in (0, \sqrt{T/2})$. Accordingly, there exist blow-up sequences satisfying Definition 2.6 and hence any warped Berger Ricci flow evolving from either $\mathcal{G}$ or $\mathcal{G}_\infty$ admits singularity models ([Hamilton, 1995, Section 16]). In particular, any singularity model of a warped Berger Ricci flow is (weakly) κ-non-collapsed at all scales.

We first consider a maximal Ricci flow solution $(\mathbb{R}^4, g(t))_{0 \leq t < T}$, with $T < \infty$, starting at some warped Berger metric $g_0 \in \mathcal{G}$. Later we check that the same conclusions are satisfied by Ricci flows in $\mathcal{G}_\infty$. We let (p_j, t_j) be a blow-up sequence of space-time points giving rise to a singularity model $(M_\infty, g_\infty(t), p_\infty)$ as in Definition 2.6 and we denote the rescaling factors $|\mathrm{Rm}_{g(t_j)}|_{g(t_j)}(p_j)$ by λ_j. Due to the $U(2)$-symmetry we may fix $\bar{\theta} \in S^3$ and we may set $p_j = (r_j, \bar{\theta})$. We also let (Φ_j) be the diffeomorphisms given by the Cheeger-Gromov-Hamilton convergence (see [Chow et al., 2007, Chapter 4]).

We first provide a topological characterization of the limit manifold.

Lemma 2.29. *Let $(M_\infty, g_\infty(t), p_\infty)_{-\infty < t \leq 0}$ be a singularity model for a Ricci flow solution $(\mathbb{R}^4, g(t))_{0 \leq t < T}$ starting at some $g_0 \in \mathcal{G}$. Then $\pi_1(M_\infty) = 0$.*

Proof. The proof follows from adapting the argument in Lemma 1.16 which extends to the $U(2)$-invariant case due to (2.29). □

Next, we prove that the symmetries of the flow are enhanced when dilating. To this aim, we first show that the Milnor frame passes to the singularity model. Since the proof of that relies on an Ascoli-Arzelà argument, we need C^3-bounds with respect to the rescaled solutions.

Lemma 2.30. *There exists a continuous function $f : (-\infty, 0] \times (0, +\infty) \to \mathbb{R}_{\geq 0}$ such* *that*

$$\sup_{B_{g_j(t)}(p_j,\nu)} \sum_{k=0}^{3} |\nabla^k_{g_j(t)} X_i|_{g_j(t)} \leq f(t,\nu),$$

for any $t \in (-\infty, 0]$ *and* $\nu > 0$ *and for* $i = 1,2,3$. *Furthermore, there exists* $\alpha > 0$ *such that for* $i = 1,2,3$

$$|\nabla_{g_j(0)} X_i|_{g_j(0)}(p_j) \geq \alpha, \tag{2.34}$$

up to passing to a subsequence.

Proof. We fix $t = 0$ and $\nu > 0$ and we let $q \in B_{g_j(0)}(p_j,\nu)$. In the following we only analyse the case of X_1 since the others are proved similarly. We deal with the bounds for $|\nabla^k_{g_j(0)} X_1|_{g_j(0)}$ with $k = 0,1,2,3$ separately.

Case $k = 0$. We consider a $g(t_j)$-unit speed geodesic from p_j to q. From Lemma 2.14 we get

$$\sqrt{\lambda_j}(b(q,t_j) - b(p_j,t_j)) \leq \sqrt{\lambda_j} \left(\sup_{B_{g(t_j)}(p_j,\frac{\nu}{\sqrt{\lambda_j}})} |b_s| \right) d_{g(t_j)}(p_j,q) \leq \alpha\, \nu.$$

The desired estimate then follows from (2.29) which gives $\lambda_j b^2(p_j,t_j) \leq \alpha$.

Case $k = 1$. By direct computation we get

$$|\nabla_{g(t)} X_1|^2(\cdot,t) = 2\left(b_s^2 + 2u^{-2} + u^2 - 2\right)(\cdot,t),$$

for any $t \in [0,T)$. Lemma 2.7 and Lemma 2.14 imply that $|\nabla_{g(t)} X_1|(\cdot,t)$ is uniformly bounded and that the estimate (2.34) is satisfied.

Case $k = 2$. We analyse in detail only one exemplificative instance. One of the terms appearing in the computation of the norm $(\lambda_j)^{-\frac{1}{2}} |\nabla^2_{g(t_j)} X_1|$ is

$$(\lambda_j)^{-\frac{1}{2}} |\nabla^2_{g(t_j)} X_1(\partial s, \partial s, \sigma_1)| b(q,t_j) \equiv (\lambda_j)^{-\frac{1}{2}} |b_{ss}|(q,t_j) \leq \alpha \sqrt{\lambda_j} b(q,t_j),$$

where we have used (2.10). The last term is then bounded because it coincides with the case $k = 0$ we have already discussed.

Case $k = 3$. One of the terms appearing in the computation of $(\lambda_j)^{-1}|\nabla^3_{g(t_j)} X_1|$ is

$$(\lambda_j)^{-1} b |\nabla^3_{g(t_j)} X_1(\partial_s, \partial_s, \partial_s, \sigma_1)|(q, t_j) = (\lambda_j)^{-1} b \left| \frac{b_{sss}}{b} \right| (q, t_j). \tag{2.35}$$

According to Shi's first derivative estimate the covariant derivatives of the curvature are bounded on the singularity models, therefore there exists a uniform constant α such that

$$|\nabla_{g(t_j)} \mathrm{Rm}_{g(t_j)}|_{g(t_j)} \leq \alpha(\lambda_j)^{\frac{3}{2}}.$$

Thus we have

$$\left| \frac{b_{sss}}{b} \right| (q, t_j) \leq \left(\left| \frac{b_{ss} b_s}{b^2} \right| + |(k_{01})_s| \right) (q, t_j) \leq \left(\alpha \frac{\lambda_j}{b} + \alpha(\lambda_j)^{\frac{3}{2}} \right) (q, t_j).$$

We can then bound the right hand side of (2.35) as

$$(\lambda_j)^{-1} b \left| \frac{b_{sss}}{b} \right| (q, t_j) \leq \alpha(1 + \sqrt{\lambda_j} b)(q, t_j) \leq f(\nu),$$

where the last inequality follows again from the case $k = 0$. The other terms are dealt with similarly.

Let now $t \in (-\infty, 0]$. By Corollary 2.22 we get

$$\lambda_j \left| b^2(p_j, t_j) - b^2(p_j, t_j + \frac{t}{\lambda_j}) \right| \leq \alpha \lambda_j \left| \frac{t}{\lambda_j} \right| \leq \alpha |t|.$$

We may then extend the proof of the bound for the case $k = 0$ for any $t \in (-\infty, 0]$. The cases $k = 1, 2, 3$ generalize easily. □

Since the rescaled Ricci flows converge to the limit ancient flow in the pointed Cheeger-Gromov sense, from Lemma 2.30 it follows that the sequence $(\Phi_j^{-1})_* X_1$ is uniformly C^3-bounded in $B_{g_\infty(0)}(p_\infty, 1)$ with respect to $g_\infty(0)$. We can then apply the Ascoli-Arzelà theorem and obtain the following

Corollary 2.31. *There exists a subsequence $(\Phi_j^{-1})_* X_1$ that converges in C^2 to a vector field $X_{1,\infty}$ on $B_{g_\infty(0)}(p_\infty, 1)$.*

From now on we re-index the subsequence given by the previous Corollary. In order to prove that $X_{1,\infty}$ is actually a Killing vector field for $g_\infty(0)$ we need a preliminary result. The following shows that the singularity model cannot be Ricci flat.

Lemma 2.32. *For any $q \in M_\infty$ and for any $t \in (-\infty, 0]$ the following is satisfied:*

$$\lim_{j\to\infty} b(\Phi_j(q), t_j + t\lambda_j^{-1}) = 0.$$

Proof. Suppose for a contradiction that there exist $q \in M_\infty$, $t \in (-\infty, 0]$, a subsequence (which we still denote by j) and $\mu > 0$ such that $b(\Phi_j(q), t_j + (\lambda_j)^{-1}t) \to \mu$. By (2.29) we immediately derive that $R_{g_\infty(t)}(q) = 0$. Since any complete ancient solution to the Ricci flow has nonnegative scalar curvature Chen [2009], a standard application of the maximum principle and the uniqueness of the flow among complete and bounded curvature solutions yield $\mathrm{Ric}_\infty \equiv 0$ everywhere in the space-time. We then fix the time to be 0 and assume that $g_\infty(0)$ is not flat. By the uniform C^1-lower bound in (2.34) there exists an open subset $U \subset B_{g_\infty(0)}(p_\infty, 1)$ where $|X_{1,\infty}|_{g_\infty(0)}|_U > 0$, with $X_{1,\infty}$ given by Corollary 2.31. From the real analyticity of the ancient limit flow Bando [1987] it follows that there exists $\bar{q} \in U$ such that $|\mathrm{Rm}_{g_\infty(0)}|_{g_\infty(0)}(\bar{q}) > 0$, otherwise the limit would be flat. Moreover, $b(\Phi_j(\bar{q}), t_j) \to 0$ as $j \to \infty$. For if such condition did not hold, then by (2.29) the Riemann tensor would vanish at $\bar{q}$. Since $g_\infty(0)$ is Ricci flat we get

$$\begin{aligned}
0 &= \left|\mathrm{Ric}_{g_\infty(0)}\right|^2_{g_\infty(0)}(\bar{q}) \\
&= \lim_{j\to\infty} \frac{1}{\lambda_j^2}\left((k_{01}+k_{02}+k_{03})^2 + 2(k_{01}+k_{12}+k_{13})^2 + (k_{03}+k_{13}+k_{23})^2\right)(\Phi_j(\bar{q}), t_j) \\
&= \lim_{j\to\infty} \frac{1}{\lambda_j^2 b^4}\left(b^4\left((2k_{01}+k_{03})^2 + 2(k_{01}+k_{12}+k_{13})^2 + (k_{03}+2k_{13})^2\right)\right)(\Phi_j(\bar{q}), t_j).
\end{aligned}$$

By the Cheeger-Gromov convergence we get $\Phi_j(\bar{q}) \in B_{g_j(0)}(p_j, 2)$ for j large enough. Therefore, from Lemma 2.30 (the case of $k = 0$) it follows that $\lambda_j b^2(\Phi_j(\bar{q}), t_j) \leq \alpha$ for j large and for some positive α. From the estimate in Lemma 2.23 we finally derive that the limit above is zero if and only if

$$\lim_{j\to\infty}\left(b^2|\mathrm{sec}_{g(t_j)}|\right)(\Phi_j(\bar{q}), t_j) = 0,$$

with $\sec_{g(t_j)}$ the maximal sectional curvature of $g(t_j)$. Therefore by Corollary 2.31 and the choice of $\bar{q}$, up to passing to a diagonal subsequence, we conclude that

$$\begin{aligned} 0 < |X_{1,\infty}|^2_{g_\infty(0)} |\mathrm{Rm}_{g_\infty(0)}|_{g_\infty(0)}(\bar{q}) &= \lim_{j\to\infty} |X_1|^2_{g_j(0)} |\mathrm{Rm}_{g_j(0)}|_{g_j(0)}(\Phi_j(\bar{q})) \\ &= \lim_{j\to\infty} \left(b^2 |\mathrm{Rm}_{g(t_j)}|_{g(t_j)}\right)(\Phi_j(\bar{q})), \end{aligned}$$

which is a contradiction because we have just proved that the right hand side must vanish. □

We can now show that $X_{1,\infty}$ is a Killing vector field on the limit manifold for any time.

Lemma 2.33. *There exists a unique smooth extension of $X_{1,\infty}$ to the limit manifold M_∞ such that $(\Phi_j^{-1})_* X_1$ converges in C^2 to $X_{1,\infty}$ on compact sets. Moreover $X_{1,\infty}$ is a $g_\infty(t)$-Killing vector field for any $t \in (-\infty, 0]$.*

Proof. We first prove that $X_{1,\infty}$ is a Killing vector field in $B_{g_\infty(0)}(p_\infty, 1)$ with respect to $g_\infty(0)$. Suppose for a contradiction that there exist $q \in B_{g_\infty(0)}(p_\infty, 1)$, $\delta > 0$ and $Z, W \in C^\infty(TM_\infty)$ such that

$$g_\infty(0)\left(\nabla_Z^{g_\infty(0)} X_{1,\infty}, W\right) + g_\infty(0)\left(Z, \nabla_W^{g_\infty(0)} X_{1,\infty}\right) \geq \delta > 0$$

in some compact neighbourhood $\overline{\Omega}$ of q. By Corollary 2.31 and the Cheeger-Gromov convergence we get

$$\left(g_j(0)\left(\nabla^{g_j(0)}_{(\Phi_j)_* Z} X_1, (\Phi_j)_* W\right) + g_j(0)\left((\Phi_j)_* Z, \nabla^{g_j(0)}_{(\Phi_j)_* W} X_1\right)\right)(\Phi_j(q)) \geq \frac{\delta}{3},$$

for some j large enough. If at $\Phi_j(q)$ we write $(\Phi_j)_* Z = z_j^0 \partial_{s(t_j)} + \sum_{k=1}^3 z_j^k X_k$ and similarly for $(\Phi_j)_* W$, then by the Koszul formula we get

$$\frac{\delta}{3} \leq \left(2\lambda_j b^2 \left(1 - u^2\right)\left(z_j^2 w_j^3 + z_j^3 w_j^2\right)\right)(\Phi_j(q)).$$

Since $|Z|_{g_\infty(0)}(q, 0) \geq \lim_{j\to\infty} \sqrt{\lambda_j} |z_j^k| b(\Phi_j(q), t_j)$, for $k = 1, 2$ and similarly for W, we can use Lemma 2.7 (with $\varepsilon > 0$) for the case $k = 3$ and conclude that there exists a positive constant β depending on q such that

$$\frac{\delta}{3} \leq 2\beta\left(1-u^2\right)(\Phi_j(q),t_j) \leq \alpha b\,(\Phi_j(q),t_j)\,,$$

where we have used Lemma 2.17. According to Lemma 2.32 we can choose j sufficiently large such that the right hand side is as small as we need, thus obtaining the contradiction. Since the limit ancient flow is real analytic Bando [1987] and by Lemma 2.29 M_∞ is simply connected, it is a classic result that $X_{1,\infty}$ extends uniquely to a global Killing vector field on $(M_\infty, g_\infty(0))$ Nomizu [1960]. Being $g_\infty(0)$ complete, we also get that $X_{1,\infty}$ is smooth.

Given $\nu > 1$, Lemma 2.30 implies that for any subsequence of $(\Phi_j^{-1})_* X_1$ there exists a sub-subsequence that converges in C^2 to some vector field on $B_{g_\infty(0)}(p_\infty, \nu)$. The argument above shows that the limit vector field must be a Killing field for $g_\infty(0)$. By the uniqueness result in Nomizu [1960] we conclude that such limit vector field is indeed $X_{1,\infty}$. The statement is then proved when $t = 0$. The very same proof for the case $t = 0$ works when $t \in (-\infty, 0]$. □

The lower bound (2.34) and the previous Lemma extend to the sequences $(\Phi_j^{-1})_* X_2$ and $(\Phi_j^{-1})_* X_3$ which then define analogous Killing vector fields $X_{2,\infty}$ and $X_{3,\infty}$ for the singularity model. Moreover, from the Cheeger-Gromov-Hamilton convergence we derive that the system $\{X_{i,\infty}\}_{i=1}^3$ is an orthogonal frame with respect to $g_\infty(t)$ for any $t \in (-\infty, 0]$. We can now prove that this frame of Killing fields implies that the singularity model is spherically symmetric.

Lemma 2.34. *The metric $g_\infty(t)$ is rotationally symmetric for any $t \in (-\infty, 0]$. Moreover $M_\infty = \mathbb{R}^4$ or $M_\infty = \mathbb{R} \times S^3$.*

Proof. According to (2.34) and the orthogonality of the vector fields $X_{i,\infty}$ there exists at least a point $q \in M_\infty$ where this frame spans a 3-dimensional subspace of $T_q M_\infty$. Therefore, since the Lie brackets are preserved in the limit, Lemma 2.33 implies that there exists a (non-trivial) copy of $\mathfrak{su}(2)$ in the Lie algebra of Killing fields $\mathfrak{iso}(M_\infty, g_\infty(t))$. By integrating the Killing fields we derive that $SU(2)$ acts isometrically with cohomogeneity one on $(M_\infty, g_\infty(t))$ for any $t \in (-\infty, 0]$. In particular, by the Lie algebra constants we see that $\{X_{i,\infty}\}_{i=1}^3$ is a Milnor frame for $g_\infty(t)$. In fact, since $||X_{1,\infty}||_{g_\infty(t)} = ||X_{2,\infty}||_{g_\infty(t)}$ on M_∞ it follows that the singularity model is $U(2)$-invariant. We can then write $M_\infty = \Sigma_{sing} \cup M_{\text{prin}}$, where M_{prin} is an open dense submanifold foliated by maximal orbits. We note that when $\Sigma_{sing} = \emptyset$ Lemma 2.29 implies that $M_\infty = \mathbb{R} \times S^3$.

All the information about $g_\infty(t)$ can be obtained by restricting it to a geodesic starting at the singular orbit and meeting the principal orbits orthogonally. Namely, once we denote the dual coframe associated with $\{X_{i,\infty}\}_{i=1}^3$ by $\{\sigma_{i,\infty}\}_{i=1}^3$, we have

$$g_\infty(t)|_{M_{\text{prin}}} = (dy_t)^2 + \phi_{1,\infty}^2(y,t)\,(\sigma_{1,\infty}^2 + \sigma_{2,\infty}^2) + \phi_{3,\infty}^2(y,t)\,\sigma_{3,\infty}^2,$$

with $\phi_{i,\infty}(y,t) \doteq |X_{i,\infty}|_{g_\infty(t)}(y)$ for any $y > 0$, $t \leq 0$ and $i = 1, 3$.

Let $q \in M_{\text{prin}}$. By the convergence of $(\Phi_j^{-1})_* X_i$ to $X_{i,\infty}$ on compact sets, we get

$$\begin{aligned}\frac{1}{\phi_{1,\infty}}\left(\frac{\phi_{1,\infty}}{\phi_{3,\infty}} - 1\right)(q,t) &\equiv \frac{1}{|X_{1,\infty}|_{g_\infty(t)}}\left(\frac{|X_{1,\infty}|_{g_\infty(t)}}{|X_{3,\infty}|_{g_\infty(t)}} - 1\right)(q) \\ &= \lim_{j\to\infty} \frac{1}{\sqrt{\lambda_j}b}\left(u^{-1} - 1\right)\left(\Phi_j(q), t_j + \frac{t}{\lambda_j}\right) \leq 0,\end{aligned}$$

where we have used the estimate in Lemma 2.17. Since the ratio $b/c \geq 1$ is scale invariant we obtain $\phi_{1,\infty} = \phi_{3,\infty}$. Thus $g_\infty(t)$ is $SO(4)$-invariantf. Furthermore, if there exists a singular orbit, then $\phi_{i,\infty} \equiv \phi_\infty$ is an odd function with $\partial_{y_t}\phi_\infty(\Sigma_{sing}, t) = 1$ (see, e.g., Grove and Ziller [2002]). From the boundary conditions (2.7) we deduce that $M_\infty = \mathbb{R}^4$. We may finally conclude that $M_\infty = \mathbb{R}^4$ or $M_\infty = \mathbb{R} \times S^3$ with

$$g_\infty(t) = (dy_t)^2 + \phi_\infty^2(y_t, t) g_{S^3}, \tag{2.36}$$

where g_{S^3} is the standard constant curvature 1 metric on S^3 and

$$\phi_\infty(q,t) = \lim_{j\to\infty} \sqrt{\lambda_j}\, b\left(\Phi_j(q), t_j + \frac{t}{\lambda_j}\right),$$

for any $(q,t) \in M_\infty \times (-\infty, 0]$. □

We now show that Lemma 2.34 actually extends to any singularity model of a warped Berger Ricci flow in $\mathcal{G}_\infty$. Indeed, given a blow-up sequence (p_j, t_j) as above and a radial coordinate $r_1 > \rho$, with ρ satisfying (2.31), then by Lemma 2.25 it suffices to prove that any rescaled geodesic ball $B_{g_j(t)}(p_j, \nu)$ lies in $B(\boldsymbol{o}, r_1)$ for j sufficiently large.

Lemma 2.35. *Let $(M_\infty, g_\infty(t), p_\infty)_{-\infty<t\leq 0}$ be a singularity model for a warped Berger Ricci flow $(\mathbb{R}^4, g(t))_{0\leq t<T}$, with $T < \infty$, starting at some $g_0 \in \mathcal{G}_\infty$. For any $t \leq 0$, for any $\nu > 0$ and for any $r_1 > \rho$, with ρ satisfying* (2.31)*, there exists $j_0 = j_0(t, \nu, r_1)$ such that for all $j \geq j_0$ we have*

$$B_{g_j(t)}(p_j, \nu) \subset B(\boldsymbol{o}, r_1).$$

Proof. Given a blow-up sequence (p_j, t_j) with $p_j = (r_j, \theta)$ for some $\theta \in S^3$, we observe that up to a finite number of indices we have $r_j < \rho$ otherwise $|\mathrm{Rm}_{g(t_j)}|_{g(t_j)}(p_j)$ would be bounded. Suppose for a contradiction that there exist a time t, a radius ν, a coordinate $r_1 > \rho$ and a subsequence $q_{j_k} = (y_{j_k}, \theta_{j_k}) \subset B_{g_j(t)}(p_j, \nu)$ such that $y_{j_k} > r_1$. Then by (2.31) and standard distortion estimates of the Riemannian distance we get

$$\begin{aligned}\frac{\nu}{\sqrt{\lambda_{j_k}}} &> d_{g(t_{j_k}+\frac{t}{\lambda_{j_k}})}(p_{j_k}, q_{j_k}) \geq \inf_{\substack{y\in S(\boldsymbol{o},\rho)\\ z\in S(\boldsymbol{o},r_1)}} d_{g(t_{j_k}+\frac{t}{\lambda_{j_k}})}(y, z)\\ &\geq \alpha \inf_{\substack{y\in S(\boldsymbol{o},\rho)\\ z\in S(\boldsymbol{o},r_1)}} d_{g_0}(y, z),\end{aligned}$$

which then gives us a contradiction for k large enough. □

We may then adapt all the arguments above and conclude that any singularity model of a warped Berger Ricci flow in $\mathcal{G}_\infty$ is rotationally symmetric. We finally address the proof of the classification result in Proposition 2.28.

Proof of Proposition 2.28. Lemma 2.34 implies that any singularity model is $SO(4)$-invariant and hence in particular locally conformally flat. Thus by Zhang [2008] we derive that any singularity model has nonnegative curvature operator. Since we have shown that singularity models are weakly κ-non-collapsed at all scales, we find that any singularity model is a κ-solution to the Ricci flow. If the curvature operator is not strictly positive at some point in the space-time, then the rotational symmetry and the classification in Hamilton [1986] shows that $(M_\infty, g_\infty(t))$ must be isometric to the self-similar shrinking soliton on the cylinder $\mathbb{R} \times S^3$.

Conversely, if the curvature operator is strictly positive at a point, then by the strong maximum principle we conclude that the singularity model is positively curved. □

Remark 2.8. *We point out that Proposition 2.28 extends to warped Berger Ricci flows for which both the estimate* (2.29) *and the rotational symmetry type of bounds in Lemmas 2.17, 2.19 and 2.23 are satisfied.*

2.6 Proofs of the main results

2.6.1 Bryant soliton singularities

In this subsection we show that any Ricci flow in $\mathcal{G}$ encounters a Type-II singularity and that Theorem 2.2 is satisfied.

Proof of Theorem 2.1. According to Lemma 2.13 the Ricci flow develops a finite-time singularity at some $T < \infty$. Suppose that the Ricci flow is Type-I and let Σ be the singular set defined as in [Enders et al., 2011, Definiton 1.5].

If the origin $\boldsymbol{o}$ does not belong to Σ, then the flow stays smooth on $B(\boldsymbol{o}, 2\rho)$ for some $\rho > 0$. Thus there exists $\delta > 0$ such that $b(\rho, t) \geq \delta > 0$ for any $t \in [0, T)$. Lemma 2.12 then implies $b(r, t) \geq \delta$ for any $r \geq \rho$ and for all $t \in [0, T)$. From the estimate (2.29) we finally deduce that the curvature stays uniformly bounded outside $B(\boldsymbol{o}, \rho)$ and hence on $\mathbb{R}^4$. The latter condition contradicts that the flow develops a singularity at T Shi [1989].

If $\boldsymbol{o} \in \Sigma$, then we can apply [Enders et al., 2011, Theorem 1.1] and derive that any parabolic dilation of the flow at $\boldsymbol{o}$ (sub)converges to a *non-flat* shrinking soliton. By the classification in Proposition 2.28 we get that any such singularity model is a shrinking cylinder Kotschwar [2008]. However by the $SU(2)$ symmetry and the Cheeger-Gromov convergence we have just shown that the cylinder $\mathbb{R} \times S^3$ is exhausted by open sets diffeomorphic to $\mathbb{R}^4$, which is not possible. Therefore, the singularity is Type-II.

Since the flow is Type-II and non-collapsed we can choose a blow-up sequence giving rise to a singularity model which consists of an *eternal* solution [Hamilton, 1995, Section 16]). By the classification in Proposition 2.28 we deduce that such eternal solution is rotationally symmetric and *positively* curved[2]. Therefore the scalar curvature and

[2] At this point, one can also rely on the recent classification of rotationally symmetric κ-solutions in Li

the Riemann curvature are comparable up to the singular time and we can hence adapt the argument in Hamilton [1995] to extract a space-time sequence (p_j, t_j), with $t_j \nearrow T$, such that if we set $\lambda_j \doteq R_{g(t_j)}(p_j)$, then the rescaled Ricci flows $(\mathbb{R}^4, g_j(t), p_j)$ defined by $g_j(t) \doteq \lambda_j g(t_j + (\lambda_j)^{-1} t)$ (sub)converge in the pointed Cheeger-Gromov sense to a κ-solution whose scalar curvature attains its supremum in the space-time. According to Hamilton [1993a] the singularity model is then a gradient steady soliton. Since by Proposition 2.28 such gradient steady soliton is complete and rotationally symmetric, it must be isometric to the Bryant soliton [Chow et al., 2007, Theorem 1.35]. □

Proof of Theorem 2.2. We prove the four points in Theorem 2.2 separately.

(i) The Bryant soliton appears at the origin. Let (p_j, t_j) and λ_j be defined as in the proof of Theorem 2.1, let Φ_j be the family of diffeomorphisms given by the Cheeger-Gromov convergence and let $(\mathbb{R}^4, g_\infty(t), p_\infty)$ be the Bryant soliton arising as limit singularity model. By the $SU(2)$ symmetry we may choose p_j of the form (r_j, θ) for some $\theta \in S^3$. Suppose for a contradiction that there exists $\delta > 0$ such that for j sufficiently large we have $d_{g_j(0)}(\boldsymbol{o}, p_j) \geq 2\delta > 0$. We may then find points $q_j \equiv (\tilde{r}_j, \theta)$, with $\tilde{r}_j < r_j$, such that, up to passing to a subsequence, $\Phi_j^{-1}(q_j) \to q_\infty$ for some q_∞ satisfying $d_{g_\infty(0)}(p_\infty, q_\infty) = \delta$. Since the scalar curvature of the Bryant soliton $g_\infty(t)$ attains its maximum at the centre of symmetry, i.e. at the origin of $\mathbb{R}^4$, we deduce that $p_\infty = \boldsymbol{o} \in \mathbb{R}^4$ and therefore that the Killing vectors $\{X_{i,\infty}\}$ constructed above need to vanish at p_∞. Equivalently, from the argument in Lemma 2.34 we derive that

$$0 = |X_{1,\infty}|_{g_\infty(0)}(p_\infty) = \lim_{j \to \infty} \sqrt{\lambda_j} b(r_j, t_j).$$

Since the warping coefficient b is monotone in space and $\tilde{r}_j < r_j$ we have

$$|X_{1,\infty}|_{g_\infty(0)}(q_\infty) = \lim_{j \to \infty} \sqrt{\lambda_j} b(\tilde{r}_j, t_j) \leq 0,$$

which is not possible because the Killing fields generating the rotational symmetry cannot

and Zhang [2018] to conclude that such eternal solution must be isometric to the Bryant soliton. However, we chose to present a more self-contained argument which is sufficient to complete the proof of Theorem 1.

vanish along a principal orbit, i.e. away from the origin. Therefore, up to choosing a subsequence, we have $d_{g_j(0)}(\boldsymbol{o}, p_j) \to 0$. In particular, we may pick a subsequence such that $R_{g(t_j)}(\boldsymbol{o}) \geq (1 - \delta_j)\lambda_j$ for some $\delta_j \to 0$. If we then dilate the Ricci flow by factors $R_{g(t_j)}(\boldsymbol{o})$ we still obtain the Bryant soliton as pointed Cheeger-Gromov limit.

(ii) The singularity is global. Consider the set of points where the flow becomes singular as $t \nearrow T$:

$$\Omega \doteq \left\{ p \in \mathbb{R}^4 : \lim_{t \nearrow T} b(p,t) = \lim_{t \nearrow T} c(p,t) = 0. \right\}.$$

We note that the previous definition makes sense due to Corollary 2.22 and the estimate (2.29). Part (ii) in the statement of Theorem 2.2 is equivalent to showing $\Omega = \mathbb{R}^4$. Indeed we have proved above that the curvature cannot stay uniformly bounded at the origin, while away from the origin the estimate (2.29) implies that both b and c need to converge to zero as $t \nearrow T$ for the curvature to blow-up. We assume for a contradiction that $\Omega \neq \mathbb{R}^4$. By Lemma 2.12 there exists $\bar{r} \geq 0$ satisfying $\Omega = B(\boldsymbol{o}, \bar{r})$. We may always take the Euclidean ball $B(\boldsymbol{o}, \bar{r})$ to be *closed* because by Corollary 2.22 there exists a uniform constant $\alpha > 0$ such that $b^2(r,t) \leq \alpha(T - t)$ for all $r < \bar{r}$.

Claim 2.36. *Let* $(\mathbb{R}^4, g(t))_{0 \leq t < T}$ *be the Ricci flow starting at some* $g_0 \in \mathcal{G}$. *Then* $\lim_{t \nearrow T} cH(r,t) = 0$ *for any* $r > \bar{r}$.

Proof of Claim 2.36. We prove the Claim by a blow-up argument. Namely, we show that if the statement was false, then any singularity model would have Euclidean volume growth, thus leading to a contradiction.

Since by (2.29) the curvature is uniformly controlled in time for any radius $r > \bar{r}$ by some positive constant only depending on r, the same argument in the proof of Lemma 2.26 shows that the limit $\lim_{t \nearrow T} cH(r,t)$ is well defined and finite (and nonnegative by Lemma 2.12) for any $r > \bar{r}$. Suppose that there exists $r_0 > \bar{r}$ such that $\lim_{t \nearrow T} cH(r_0,t) > 0$. Then the same argument in Corollary 2.27 implies that cH is uniformly bounded from below by some $\mu > 0$ on the ball $B(\boldsymbol{o}, r_0)$ for times close to T. In particular, by Lemma

2.25 there exists $\alpha > 0$ satisfying

$$b_s \geq \frac{1}{3}\left(\mu u^{-1} - \alpha b\right)$$

on $B(\boldsymbol{o}, r_0)$ for times close enough to T. Let us rescale the solution along a blow-up sequence and let $(M_\infty, g_\infty(t), p_\infty)$ be the associated singularity model. We note that by Lemma 2.32 if $q \in M_\infty$, then $\Phi_j(q) \in B(\boldsymbol{o}, r_0)$ for j large enough. Moreover, we have

$$b_s(\Phi_j(q), t_j) \geq \frac{1}{3}\left(\mu u^{-1} - \alpha b\right)(\Phi_j(q), t_j) \geq \frac{1}{6}\mu.$$

Thus, from Corollary 2.31 and Lemma 2.34 we derive that

$$\begin{aligned}(1 - (\partial_y \phi_\infty)^2)(q, 0) = (\phi_\infty^2 k_{12}^\infty)(q, 0) &= \lim_{j\to\infty}\left(4 - b_s^2 - 3u^2\right)(\Phi_j(q), t_j) \\ &= \lim_{j\to\infty}\left(1 - b_s^2\right)(\Phi_j(q), t_j) \leq 1 - \frac{1}{36}\mu^2,\end{aligned}$$

which then implies $\partial_y \phi_\infty(q, 0) \geq \mu/6$ for any $q \in M_\infty$. Since the limit is rotationally symmetric we obtain

$$\mathrm{Vol}_{g_\infty(0)}(B_{g_\infty(0)}(p_\infty, \rho)) \geq \alpha \rho^4, \tag{2.37}$$

for any $\rho \geq 1$ and for some $\alpha > 0$. By Proposition 2.28 and the bound (2.37) we conclude that $(M_\infty, g_\infty(t))$ is a non-compact κ-solution with *positive asymptotic volume ratio*. According to a rigidity property proved by [Perelman, 2002, Proposition 11.4] $g_\infty(t)$ must then be flat, which is a contradiction. □

Since by Claim 2.36 $(b^2 c)_s(r, t) = b^2 c H(r, t) \to 0$ as $t \nearrow T$ for all $r > \bar{r}$ we can argue as in the proof of Lemma 2.26 and deduce that there exists $\gamma > 0$ such that

$$\lim_{t\nearrow T} b^2 c(r, t) = \gamma, \quad \forall r > \bar{r}. \tag{2.38}$$

We now show that if b is small at $\bar{r}$ for times close to T, then b cannot jump to some positive quantity $\gamma^{1/3}$ for all $r > \bar{r}$ when $t \nearrow T$. Let $\varepsilon < 1$ and $T_\varepsilon < T$ to be chosen below such that $b^2(\bar{r}, t) \leq \varepsilon/2$ for all $t \in [T_\varepsilon, T)$. We let $r_\varepsilon > \bar{r}$ be such that $b^2(r_\varepsilon, T_\varepsilon) \leq \varepsilon$

and $\tilde{T}$ be the first time larger than T_ε such that $b(r_\varepsilon, \tilde{T}) = 1$, if such time exists. By Lemma 2.20 and Lemma 2.24 we have

$$\begin{aligned}\partial_t b^2(r_\varepsilon, t) &= 2\left(bb_{ss} + u^{-1}b_s c_s + b_s^2 + 2u^2 - 4\right)(r_\varepsilon, t) \le 2(bb_{ss} + \alpha b)(r_\varepsilon, t) \\ &\le \alpha\left(\frac{1}{|\log b|} + b\right)(r_\varepsilon, t) \le \frac{\alpha}{|\log b|}(r_\varepsilon, t),\end{aligned}$$

for some $\alpha > 0$ independent of ε and t and for all $t \in [T_\varepsilon, \tilde{T}]$. Thus, we can integrate the previous inequality and obtain

$$b^2(r_\varepsilon, t)\left(2|\log(b(r_\varepsilon, t))| + 1\right) \le \alpha(t - T_\varepsilon) + b^2(r_\varepsilon, T_\varepsilon)\left(2|\log(b(r_\varepsilon, T_\varepsilon))| + 1\right).$$

Therefore, since $b^2(r_\varepsilon, T_\varepsilon) \le \varepsilon < 1$ we get

$$b^2(r_\varepsilon, t) \le \alpha(T - T_\varepsilon) + 3\varepsilon.$$

Once we choose ε and T_ε accordingly, we derive that $\tilde{T}$ does not exist and hence that $b^2(r_\varepsilon, t) \le \gamma^{\frac{2}{3}}/4$ for all $t \in [T_\varepsilon, T)$. We then find

$$b^2 c(r_\varepsilon, t) \le b^3(r_\varepsilon, t) \le \gamma/8,$$

which contradicts (2.38). Therefore $\Omega = \mathbb{R}^4$.

(iii) Type-I blow-up at infinity. Once we know that the singularity is global it is natural to expect shrinking cylinders to appear when dilating the solution at infinity.
Let $t_j \nearrow T$, $\delta > 0$ arbitrary and $\varepsilon > 0$ be a positive quantity to be chosen below. Since the spatial derivatives of b and c are decaying to zero at infinity for any $t \ge T/2$, we may always pick points p_j such that $d_{g_0}(\boldsymbol{o}, p_j) \to \infty$ and

$$\sup_{B_{g(t)}(p_j,\delta)}\left(|k_{01}| + |k_{03}| + \left|\frac{b_s}{b}\right| + \left|\frac{c_s}{c}\right|\right) \le \varepsilon, \tag{2.39}$$

for all $t \in [T/2, T_j \doteq (T + t_j)/2]$. Let us denote the factors $R_{g(t_j)}(p_j)$ by λ_j. From (2.39) we derive $\lambda_j b^2(p_j, t_j) \ge \beta > 0$ uniformly with respect to j. Since by Corollary 2.22 and part (ii) of Theorem 2.2 $b^2(\cdot, t) \le \alpha(T - t)$ for some uniform constant $\alpha > 0$, we also see that $\lambda_j \to \infty$. Similarly, by (2.39) we find that $\partial_t b^2(\cdot,t) \leq -\alpha < 0$ in $B_{g(t)}(p_j,\delta)$ for all $t \in [T/2, T_j]$. Therefore we have

$$(T - t_j)\lambda_j = 2(T_j - t_j)\lambda_j \leq \frac{\alpha}{b^2(p_j,t_j)}(T_j - t_j) \leq \frac{\alpha(T_j - t_j)}{b^2(p_j,T_j) + \alpha(T_j - t_j)} \leq \alpha,$$

where we have used (2.29). Analogously, given $\nu > 0$, $t \leq 0$ and $p \in B_{g(t_j)}(p_j, \nu(\lambda_j)^{-1/2})$ we see that

$$\lambda_j b^2(p, t_j + (\lambda_j)^{-1}t) \geq \lambda_j b^2(p,t_j) + \alpha|t|,$$

for j large enough. From (2.39) we also derive the following spatial control:

$$\left|\log\left(\frac{b(p,t_j)}{b(p_j,t_j)}\right)\right| \leq \varepsilon\frac{\nu}{\sqrt{\lambda_j}}.$$

We may finally estimate the curvature of the rescaled Ricci flows as

$$\begin{aligned}\frac{1}{\lambda_j}|\mathrm{Rm}_{g(t_j+(\lambda_j)^{-1}t)}|(p) &\leq \frac{\alpha}{\lambda_j b^2(p, t_j + (\lambda_j)^{-1}t))} \leq \frac{\alpha}{\lambda_j b^2(p,t_j) + \alpha|t|} \\ &\leq \frac{\alpha}{\beta\exp(-2\frac{\varepsilon\nu}{\sqrt{\lambda_j}}) + \alpha|t|}.\end{aligned}$$

Since the flow is weakly κ-non-collapsed for some $\kappa > 0$ we may apply Hamilton's compactness theorem and conclude that the sequence of rescaled Ricci flows converge in the pointed Cheeger-Gromov sense to a singularity model $(M_\infty, g_\infty, p_\infty)_{-\infty<t\leq 0}$ to which the classification in Proposition 2.28 applies. In particular, $g_\infty(t)$ is of the form (2.36). Arguing as in the proof of Claim 2.36 and using (2.39) we see that

$$\begin{aligned}(1 - (\partial_y\phi_\infty)^2)(q,0) &= (\phi_\infty^2 k_{12}^\infty)(q,0) = \lim_{j\to\infty}\left(4 - b_s^2 - 3u^2\right)(\Phi_j(q),t_j) \\ &= \lim_{j\to\infty}\left(1 - b_s^2\right)(\Phi_j(q),t_j) \geq 1 - (\sup_{\mathbb{R}^4} b^2(\cdot,0))\varepsilon^2 \geq \frac{1}{2},\end{aligned}$$

for ε small enough, where we have used the fact that b is uniformly bounded from above. We finally conclude that $|\partial_y\phi_\infty| < 1/2$ which by the boundary conditions (2.7) and the classification in Proposition 2.28 implies that $(M_\infty, g_\infty(t))$ is the self-similar shrinking soliton on $\mathbb{R} \times S^3$.

(iv) Classification of singularity models. According to Proposition 2.28 if the singularity model is not a family of shrinking cylinders, then it must be a positively curved rotationally symmetric κ-solution. By the recent classification in Li and Zhang [2018] we conclude that in this case the singularity model is isometric to the Bryant soliton (up to scaling). □

2.6.2 Immortal warped Berger Ricci flows

Let $(\mathbb{R}^4, g(t))_{0\leq t<T}$ be the maximal Ricci flow starting at some $g_0 \in \mathcal{G}_\infty$ and suppose that $T < \infty$.

Proof of Theorem 2.3. Let $\tilde{r}_2$, $\tilde{t}$ and μ be given by Corollary 2.27 and consider a blow-up sequence giving rise to a singularity model $(M_\infty, g_\infty(t), p_\infty)$. Since by Lemma 2.35 the rescaled geodesic balls are included in $B(\boldsymbol{o}, \tilde{r}_2)$ for j large enough, we can argue exactly as in the proof of Claim 2.36 and deduce that any singularity model for the flow is in fact flat. This shows that the maximal time of existence cannot be finite. □

2.6.3 Type-I Berger Ricci flows contain minimal 3-spheres

The existence of sufficiently pinched minimal embedded hyperspheres gives rise to Type-I singularities for (asymptotically flat) rotationally symmetric Ricci flows on $\mathbb{R}^n$ as shown in Chapter 1. Thus, in general we cannot extend the conclusions of Theorem 2.1 and Theorem 2.3 to include initial data containing minimal 3-spheres.

While in the $SO(n)$-invariant case no minimal spheres can appear along the flow, in the $SU(2)$-cohomogeneity one setting an analogous property might fail. On the other hand, minimal spheres can disappear in finite time (see Proposition 1.7).

In the following we consider a Type-I warped Berger Ricci flow whose curvature is controlled at spatial infinity uniformly in time. A priori one might expect that there exist examples of Type-I singularities where both b and c have local minima while the mean curvature of the embedded hyperspheres remains positive. The next result rules out this possibility. We prove that for times close to the maximal time of existence a Type-I warped Berger Ricci flow solution $(\mathbb{R}^4, g(t))$ must contain minimal 3-spheres.

Proof of Theorem 2.4. The decay of the curvature and the lower bound for the injectivity radius ensure that (2.31) holds for some sufficiently large radius ρ. Therefore Lemma 2.25 and hence the classification of singularity models in Proposition 2.28 are satisfied in this setting (see also Remark 2.8). A first consequence of this fact is that the same argument in the proof of Theorem 2.1 shows that if the origin is in the singular set (as defined in Enders et al. [2011]), then the singularity cannot be Type-I.

Therefore we only need to consider the case where the curvature stays uniformly bounded in a Euclidean ball $B(\boldsymbol{o}, 2\bar{r})$ for some $\bar{r} > 0$. Accordingly, there exists $\varepsilon > 0$ such that $b(\bar{r}, t) \geq c(\bar{r}, t) \geq \varepsilon$ for any $t \in [0, T)$.

Assume for a contradiction that there exists a sequence $t_j \nearrow T$ such that the mean curvature of hyperspheres $H(\cdot, t_j)$ is strictly positive on the time slices $\mathbb{R}^4 \setminus \{\boldsymbol{o}\} \times \{t_j\}$ - recall that $H(r, t) \to \infty$ as $r \to \boldsymbol{o}$. From the identity $H = (\log(b^2 c))_s$ we deduce that

$$b^2 c(r, t_j) \geq b^2 c(\bar{r}, t_j) \geq \varepsilon^3, \tag{2.40}$$

for any j and for any $r \geq \bar{r}$. Since Corollary 2.22 holds in this setting, the singular set contains a point $p \in \mathbb{R}^4 \setminus B(\boldsymbol{o}, 2\bar{r})$ such that $b(p, t) \to 0$ as $t \nearrow T$ and similarly for c by Lemma 2.17. For if such p did not exist, then by the first estimate in Lemma 2.25 applied to the region $B(\boldsymbol{o}, \rho) \setminus B(\boldsymbol{o}, \bar{r}) \times [0, T)$, the curvature would be bounded as $t \nearrow T$ which is not possible by Shi [1989]. This contradicts the inequalities in (2.40). □

Remark 2.9. *The proof actually shows that H has to change sign for times close to the maximal time of existence. Equivalently, the Ricci flow solution contains neck-like regions that pinch off in finite time at a Type-I rate.*

2.7 Some applications

In this section we provide two simple applications of the main results. On the one hand we rule out the existence of Taub-NUT like shrinking solitons on $\mathbb{R}^4$. On the other hand, we completely classify Ricci flows of nonnegatively curved warped Berger metrics.

2.7.1 Non existence of Taub-NUT like shrinking solitons

Theorem 2.1 immediately implies that there are no warped Berger shrinking solitons with no necks and bounded by a round cylinder at infinity. Namely, we have the following

Corollary 2.37. *The set $\mathcal{G}$ does not contain shrinking Ricci solitons.*

Recently, Appleton found non-collapsed Taub-NUT like gradient *steady* solitons on $\mathbb{R}^4$ Appleton [2018]. It is straightforward to check that these solitons belong to $\mathcal{G}_\infty$. Indeed the curvature decays linearly at spatial infinity and both the warping functions b and c are increasing in space. According to Theorem 2.3 there are no shrinking solitons on $\mathbb{R}^4$ analogous to the steady ones constructed in Appleton [2018]. More precisely, we have shown the following:

Corollary 2.38. *The set $\mathcal{G}_\infty$ does not contain shrinking Ricci solitons.*

We note that by Kotschwar [2008] it is known that there do not exist complete non-trivial *rotationally invariant* shrinking soliton structures on $\mathbb{R}^4$.

2.7.2 Ricci flow of Berger metrics with nonnegative curvature

By combining Theorem 2.1 and Theorem 2.3 we are able to classify Ricci flows evolving from complete warped Berger metrics with bounded nonnegative curvature operator. We recall that by Hamilton [1986] the curvature operator stays nonnegative along the Ricci flow in any dimension.

Proof of Corollary 2.6. If g_0 is a complete warped Berger metric with bounded nonnegative curvature, then the injectivity radius of g_0 is positive and $b_{ss} \leq 0$ and $c_{ss} \leq 0$. By completeness the latter condition implies that both b_s and c_s are nonnegative. Thus there exists a positive (possibly infinite) quantity $\mu \doteq \lim_{r\to\infty} b(r, 0)$.

If $\mu < \infty$, i.e. $b(\cdot, 0)$ is bounded on $\mathbb{R}^4$, then g_0 belongs to $\mathcal{G}$ and the conclusions of Theorem 2.1 and Theorem 2.2 apply. In particular, there exists a sequence $t_j \nearrow T$ such that the rescaled Ricci flows $(\mathbb{R}^4, g_j(t), \boldsymbol{o})$ defined by $g_j(t) \doteq R_{g(t_j)}(\boldsymbol{o})g(t_j + (R_{g(t_j)}(\boldsymbol{o}))^{-1}t)$ converge to the Bryant soliton in the Cheeger-Gromov sense. Given any other sequence $\tilde{t}_j \nearrow T$, from the trace of the Harnack estimate Hamilton [1993b] we derive that

$$R(\boldsymbol{o}, \tilde{t}_j) \geq \frac{t_j}{\tilde{t}_j} R(\boldsymbol{o}, t_j),$$

up to reordering the indices. Therefore we conclude that dilations of the Ricci flow by factors $R(\boldsymbol{o}, \tilde{t}_j)$ still give rise to the Bryant soliton.

If $\mu = \infty$, then consider the Ricci flow solution starting at g_0 and pick $0 < t_0 < T$. The vertical sectional curvatures decay to zero at infinity being the spatial derivatives $b_s(\cdot, t_0)$ and $c_s(\cdot, t_0)$ decreasing and nonnegative. In particular $b_{ss}/b(\cdot, t_0)$ (and $c_{ss}/c(\cdot, t_0)$ as well) is integrable. By the same argument we used to prove Claim 2.18 we get $b_{ss}/b(r, t_0) \rightarrow 0$ at infinity and similarly for $c_{ss}/c(r, t_0)$. Therefore $g(t_0) \in \mathcal{G}_\infty$ and we can apply Theorem 2.3.

□

Chapter 3

Convergence of Ricci flow solutions to Taub-NUT

In this chapter we study the Ricci flow starting at an $SU(2)$ cohomogeneity one metric g_0 on $\mathbb{R}^4$ with monotone warping coefficients and whose restriction to any hypersphere is a Berger metric. If g_0 has bounded Hopf-fiber, curvature controlled by the size of the orbits and opens faster than a paraboloid in the directions orthogonal to the Hopf-fiber, then the flow converges to the Taub-NUT metric g_{TNUT} in the Cheeger-Gromov sense in infinite time. We also classify the long-time behaviour when g_0 is asymptotically flat. In order to identify infinite-time singularity models we obtain a uniqueness result for g_{TNUT}.

3.1 Introduction

By Shi [1989] we know that a solution to the Ricci flow exists smoothly for all positive times if the curvature is bounded on any time slice. Since the flow is a heat-type evolution problem for Riemannian metrics, it is tempting to expect immortal solutions to approach more regular structures in infinite time. We recall that, as reported in the introductory chapter, the behaviour of a solution existing for all positive times has been classified depending on whether the curvature decays at least as fast as t^{-1} Hamilton [1995]:

$$\begin{aligned} \textit{Type-II(b)}: &\quad \limsup_{t\nearrow\infty}\left(\sup_M\, t|\mathrm{Rm}_{g(t)}|_{g(t)}\right) = \infty, \\ \textit{Type-III}: &\quad \limsup_{t\nearrow\infty}\left(\sup_M\, t|\mathrm{Rm}_{g(t)}|_{g(t)}\right) < \infty. \end{aligned}$$

Several examples of Type-III singularities for the Ricci flow have been found, both in the compact setting Lott and Šešum [2014] and in the non-compact one Oliynyk and Woolgar [2007]. In fact, some of these cases have been shown to be occurrences of more general phenomena related to either the dimension or the existence of many symmetries: Bamler [2018] proved that any closed 3-dimensional immortal Ricci flow encounters a Type-III singularity in infinite time, while Böhm [2015] showed that the same conclusion applies to any immortal *homogeneous* Ricci flow.

If a solution develops a Type-III singularity and converges smoothly *without rescaling* in the Cheeger-Gromov sense, meaning that some control on the injectivity radius is available, then the limit is flat. Since Ricci-flat metrics constitute fixed points for the flow, it is natural to search for immortal solutions converging to Ricci-flat *non*-flat metrics in infinite time, thus encountering Type-II(b) singularities. In this sense, only few results are known and most of them are stability properties: the initial condition needs to be *sufficiently close* to the Ricci-flat metric for the Ricci flow solution to converge. For such results, whether the underlying topology is compact or not plays a key role in the analysis. Using Perelman's λ-functional, Haslhofer and Müller [2014] proved stability properties for closed Ricci-flat spaces, generalizing earlier work of Šešum [2006]. In the non-compact setting, Deruelle and Kröncke [2020] derived a stability result for a class of ALE Ricci flat manifolds.

Since the Ricci flow preserves isometries, one might consider looking for solutions converging to Ricci-flat fixed points when symmetries are present. In this direction, Marxen [2019] recently generalized earlier work of Hamilton to prove that if (N, g_N) is closed and Ricci-flat, then a class of warped product solutions to the Ricci flow $(\mathbb{R}\times N, g(t))$, of the form $g(t) = k^2(r,t)dr^2 + f^2(r,t)g_N$, converge to $(\mathbb{R}\times N, dr^2 + c^2 g_N)$, for some $c > 0$, whenever the initial condition is asymptotic to the target Ricci-flat metric. On the other hand, in the maximally symmetric case of homogeneous Ricci flows, convergence to Ricci-flat non-flat spaces is not possible due to a classic result of Alekseevskii and Kimelfeld [1975]. One of the main contributions of this chapter consists in proving that a large family of cohomogeneity one metrics on $\mathbb{R}^4$ converge to the Ricci-flat Taub-NUT metric in infinite time along the Ricci flow.

More precisely, we focus on a subset of warped Berger metrics in $\mathcal{G}_\infty$ as defined in Chapter 2. We first recall that a well-known warped Berger metric on $\mathbb{R}^4$ is given by the Taub-NUT metric g_{TNUT}, which can be written explicitly as

$$g_{\mathsf{TNUT}} = \frac{1}{16}\left(1+\frac{2m^{-1}}{r}\right)dr^2 + \frac{r^2}{4}\left(1+\frac{2m^{-1}}{r}\right)\pi^* g_{S^2(\frac{1}{2})} + \frac{m^{-2}}{1+\frac{2m^{-1}}{r}}\,\sigma_3\otimes\sigma_3,$$

for some parameter m which we call the *mass* of g_{TNUT} and which measures the inverse of the *finite* size of the Hopf-fiber at spatial infinity. The Taub-NUT metric is a gravitational instanton found on $\mathbb{R}^4$ by Hawking [1977]: it is a *hyperkähler* and thus *Ricci-flat* asymptotically flat metric. We point out that the stability result in Deruelle and Kröncke [2020] does *not* apply to the Taub-NUT metric which is not ALE being $(\mathbb{R}^4\setminus\{\boldsymbol{o}\}, g_{\mathsf{TNUT}})$ the total space of a circle fibration with fibres approaching constant length at spatial infinity.

In Theorem 2.3 we proved that if g_0 is a complete warped Berger metric with monotone coefficients and curvature decaying at spatial infinity, then the maximal Ricci flow solution starting at g_0 is immortal. In light of such result and being g_{TNUT} an asymptotically flat metric, we first focus on the following family of initial data:

Definition 3.1. *The class $\mathcal{G}_{\mathsf{AF}}$ consists of all complete warped Berger metrics g on $\mathbb{R}^4$ with monotone warping coefficients satisfying*

$$\sup_{p\in\mathbb{R}^4}\,\left(d_g(\boldsymbol{o},p)\right)^{2+\epsilon}|\mathrm{Rm}_g|_g(p) < \infty, \tag{3.1}$$

for some $\epsilon > 0$. A metric $g\in\mathcal{G}_{\mathsf{AF}}$ is called asymptotically flat.

The class $\mathcal{G}_{\mathsf{AF}}$ divides in two categories (see Lemma 3.5): metrics with cubic volume growth, for which b opens up linearly and the size of the Hopf-fiber approaches a positive finite quantity m^{-1}, and metrics with Euclidean volume growth. Consistently with the Taub-NUT construction we say that a metric $g\in\mathcal{G}_{\mathsf{AF}}$ has positive *mass* m in the first case and zero mass in the second case respectively. We prove that for solutions evolving from initial data in $\mathcal{G}_{\mathsf{AF}}$, the long-time behaviour only depends on the mass.

In the following we say that a Ricci flow solution converges to a Ricci-flat metric g_∞ on $\mathbb{R}^4$ in the pointed Cheeger-Gromov sense as $t\nearrow\infty$ if for any $t_j\nearrow\infty$ the sequence $(\mathbb{R}^4, g_j(t), \boldsymbol{o})$, defined by $g_j(t) = g(t_j + t)$, converges to $(\mathbb{R}^4, g_\infty, \boldsymbol{o})$ in the pointed Cheeger-Gromov sense. In particular, there is no rescaling of the solution.

Theorem 3.1. *Let $(\mathbb{R}^4, g(t))_{t\geq 0}$ be the maximal solution to the Ricci flow starting at some $g_0 \in \mathcal{G}_{\mathsf{AF}}$. Either one of the following conditions is satisfied:*

(i) If g_0 has positive mass m, then $g(t)$ encounters a Type-II(b) singularity and converges to the Taub-NUT metric of mass m in the pointed Cheeger-Gromov sense as $t \nearrow \infty$.

(ii) If g_0 has zero mass, then the solution encounters a Type-III singularity and converges to the Euclidean metric in the pointed Cheeger-Gromov sense as $t \nearrow \infty$.

We note that an analogous Type-III result for $SO(n)$-invariant Ricci flows without minimal hyperspheres was obtained in Oliynyk and Woolgar [2007]. Moreover, a numerical investigation on the stability of the Taub-NUT metric for warped Berger Ricci flows in $\bar{B}^4$ was conducted in Holzegel et al. [2007]: Theorem 3.1 and its generalization below provide a rigorous frame for addressing the questions raised in Holzegel et al. [2007] on the $\mathbb{R}^4$-topology.

Appleton [2018] proved that on $\mathbb{R}^4$ there exists a warped Berger gradient steady soliton with monotone coefficients, bounded Hopf-fiber and coefficient b in the directions orthogonal to the Hopf-fiber opening as fast as a paraboloid in $\mathbb{R}^3$. Namely, the soliton satisfies the asymptotics:

$$c(s) \sim \text{constant}, \quad b(s) \sim \sqrt{s}.$$

Consequently, we cannot expect initial data opening with arbitrary speed to converge to g_{TNUT} along the flow. The paraboloid growth rate plays a role in Ivey [1994], where they found a family of positively curved, pinched $SO(3)$-invariant immortal Ricci flows on $\mathbb{R}^3$ opening (at least) as fast as a paraboloid that do converge in subsequences in the pointed Cheeger-Gromov sense as $t \nearrow \infty$. Partly motivated by such analysis, we investigate whether a similar convergence property holds for warped Berger Ricci flows opening *faster* than a paraboloid, thus ruling out Appleton's soliton, without restricting to positively curved pinched solutions. With that in mind, we give the following:

Definition 3.2. *For all $0 \leq k < 1$, the class $\mathcal{G}_k$ consists of all complete warped Berger metrics g with monotone coefficients satisfying:*

(i) $0 < \liminf_{s\to\infty} b_s\, b^k(s) \leq \limsup_{s\to\infty} b_s\, b^k(s) < \infty$,

(ii) $\sup_{p\in\mathbb{R}^4} \left(b^2|\mathrm{Rm}_g|_g\right)(p) < \infty$,

(iii) $\sup_{p\in\mathbb{R}^4} c(p) < \infty$.

We note that the assumptions in Definition 3.2 are independent and that properties (i), (ii) cannot be replaced by requiring a suitable rate of curvature decay (a thorough discussion is provided in Section 3.2.6). In particular, we point out that metrics in $\mathcal{G}_{\mathsf{AF}}$ with positive mass belong to $\mathcal{G}_0$. By integrating (i) we see that if $g \in \mathcal{G}_k$, then the warping coefficient b grows like $s^{\frac{1}{k+1}}$, meaning that the projection of g on the base space via the Hopf-map opens faster than a paraboloid in $\mathbb{R}^3$. The first order constraints in (i) and the decay in (ii) allow us to apply a maximum principle argument to show (we refer to the Outline for more details) that Ricci flow solutions starting in $\mathcal{G}_k$ have a well defined behaviour at spatial infinity on any time-slice, meaning that (i) is preserved - not uniformly though, in fact according to the convergence to Taub-NUT we deduce that there will be a jump in infinite time.

We still call *mass* the inverse of the size of the Hopf-fiber at spatial infinity. We prove that any maximal Ricci flow solution starting in $\mathcal{G}_k$ develops a Type-II(b) singularity modelled by an ancient solution satisfying the conditions below.

Definition 3.3. *Let $m > 0$. The class $\mathcal{A}$ consists of all complete, warped Berger ancient solutions to the Ricci flow on $\mathbb{R}^4$ with monotone coefficients and curvature uniformly bounded in the space-time, satisfying*

$$b_s \geq \frac{f\left(\frac{b}{c}\right)}{\frac{b}{c}}$$

$$\sup_{\mathbb{R}^4\times(-\infty,0]} c = m^{-1}$$

for some continuous positive function f such that $f(z) \to \infty$ as $z \to \infty$.

We point out that the class $\mathcal{A}$ describes warped Berger ancient solutions opening faster than a paraboloid in the directions orthogonal to the Hopf-fiber. Our second main result is a rigidity property.

Theorem 3.2. *The only ancient solution in $\mathcal{A}$ is the Taub-NUT metric.*

First, we observe that the result is optimal, for the existence of the gradient steady soliton found by Appleton highlights that we cannot drop the requirement on f to diverge in space-time regions where the roundness ratio c/b becomes degenerate. Moreover, the Euclidean metric would also be included in the class $\mathcal{A}$ if we allowed the size of the Hopf-fiber to be unbounded.

We emphasize that the rigidity result applies to possible *collapsed* infinite-time singularity models. Indeed, since the Taub-NUT metric is asymptotically flat with bounded Hopf-fiber, we see that for any $\kappa > 0$ there exist $p \in \mathbb{R}^4$ and $r > 0$ such that g_{TNUT} is κ-strongly collapsed at p for all scales larger than r. It is also worth comparing Theorem 3.2 with a quantization result obtained by Minerbe [2010], where they proved that a class of hyperkähler 4-manifolds must have cubic volume growth. Our rigidity result may then be interpreted in terms of quantization of the volume growth as well, for in the definition of $\mathcal{A}$ we, *a priori*, allow for ancient solutions with volume growth faster than quadratic.

As a consequence of the previous rigidity result we show the following:

Theorem 3.3. *Let $(\mathbb{R}^4, g(t))_{t\geq 0}$ be the maximal solution to the Ricci flow starting at some $g_0 \in \mathcal{G}_k$ with mass $m > 0$. Then $g(t)$ converges to the Taub-NUT metric of mass m in the pointed Cheeger-Gromov sense as $t \nearrow \infty$.*

Again, the result is in some sense optimal because from the existence of the soliton we derive that we cannot extend the convergence to initial data in $\mathcal{G}_1$. Theorem 3.3 is not a stability property: metrics in $\mathcal{G}_k$ are, with the exception of a subclass in $\mathcal{G}_0$, not asymptotically flat and indeed they have different volume growth and rate of decay of the curvature with respect to the Taub-NUT metric. In fact, we can find initial data with nonnegative sectional curvature flowing to the Ricci-flat Taub-NUT metric. While the fact that positive sectional curvature is not preserved along the flow in dimension higher than three is well known, even in the cohomogeneity one setting Bettiol and Krishnan [2016], in the result below we prove that negative sectional curvature terms not only appear along the solution but also balance out the positive terms to yield a Ricci-flat limit in infinite time.

Corollary 3.4. *There exists a complete, bounded curvature warped Berger metric* g_0 *with* $\sec(g_0) \geq 0$ *such that the maximal Ricci flow solution starting at* g_0 *is immortal and converges to* g_{TNUT} *in the pointed Cheeger-Gromov sense as* $t \nearrow \infty$.

Outline

In Section 3.2 we describe the class of initial data and we comment on the assumptions. In particular, we recap a few key properties of g_{TNUT}. In Section 3.3 we focus on Ricci flows starting in $\mathcal{G}_{\mathsf{AF}}$. In the asymptotically flat setting one can control the solution at spatial infinity in a precise way and hence maximum principle arguments follow. Similarly to other cohomogeneity one scenarios Oliynyk and Woolgar [2007], Isenberg et al. [2016], Appleton [2019], Di Giovanni [2020], we prove that the curvature is uniformly controlled whenever the principal orbits are non-degenerate. More importantly, we show that if the Hopf-fiber is bounded, then the solution always opens faster than a paraboloid in $\mathbb{R}^3$ in any space-time region where the roundness ratio c/b gets small. We dedicate Section 3.4 to extending the analysis for asymptotically flat Ricci flows with positive mass to solutions starting in $\mathcal{G}_k$. In this regard, a few extra-steps are needed to prove that the initial assumptions in the definition of $\mathcal{G}_k$ do imply that the behaviour of the warping coefficients at spatial infinity along the solution is known: an important ingredient is the preservation of the power law decay of the curvature along Ricci flow solutions derived in Lott and Zhang [2016]. Once we can control the solution on the parabolic boundary, we then rely on a maximum principle argument to prove that the *faster than a paraboloid*-growth condition holds uniformly in any space-time region where the roundness ratio c/b is small. One may then concentrate on compact time-dependent regions where the squashing factor c/b is non-degenerate and, analogously to the asymptotically flat case, we prove that there are no space-time regions resembling necks in infinite-time: this is the key result to show that the curvature is uniformly bounded in time. In Section 3.5 we present a compactness result for a class of warped Berger solutions of the Ricci-flow on $\mathbb{R}^4$. Such property has an analogous counterpart in Appleton [2019], where they formulate the compactness theorem under a different set of assumptions. In particular, they focus on non-collapsed sequences of Ricci flows, being interested in applying the result to the analysis of finite-time singularities. However, in our setting such assumption is not available for we wish to study infinite-time singularity models of (non-rescaled) Ricci-flows. Therefore, we prove that one can still pass to a pointed Cheeger-Gromov limit which not only preserves the symmetries but whose warping coefficients are smooth limits of the warping coefficients along the sequence, provided that the roundness ratio c/b is non-degenerate at the given origins we center the solutions at. As a first application of the compactness result, we show that the curvature of any Ricci flow solution in $\mathcal{G}_k$ and $\mathcal{G}_{\mathsf{AF}}$ is uniformly bounded in time so that we never need to rescale for obtaining smooth limits at infinite time. Section 3.6 is devoted to proving that the only complete warped Berger ancient solution with monotone coefficients, bounded curvature, bounded Hopf-fiber and opening faster than a paraboloid along the directions orthogonal to the Hopf-fiber is g_{TNUT}. The argument follows a similar approach used by Appleton [2019] to derive a uniqueness result for the Eguchi-Hanson metric: we rely on the Compactness result in Section 3.5 to show that relevant geometric quantities always attain their critical values in the space-time, up to passing to a pointed Cheeger-Gromov limit sharing the same features of the given ancient solution. In particular, we prove that one of the hyperkähler first-order quantities which vanishes identically for g_{TNUT} is always nonnegative on the class of ancient solutions described above: this yields that the ancient solution is Ricci-flat and hence homothetic to g_{TNUT}. We point out that differently from the case discussed by Appleton, we *cannot* use the κ-non-collapsedness of the ancient solutions, which plays an important role in their analysis. Therefore, one of the main difficulties here consists in ensuring that the roundness ratio c/b stays positive along any space-time sequence we use to approximate critical values of some given geometric quantity so that the compactness result can indeed be applied. In fact, we know that on the soliton the hyperkähler quantity mentioned before approaches its *negative* infimum in space-time regions where the roundness ratio becomes degenerate. Finally, we rely on the uniqueness result in Section 3.6 to prove the convergence of immortal Ricci flows in $\mathcal{G}_k$ and $\mathcal{G}_{\mathsf{AF}}$.

3.2 Initial data for the Ricci flow

3.2.1 Warped Berger metrics with monotone coefficients

Since we are interested in studying the long-time behaviour of the Ricci flow, we always consider maximal solutions evolving from warped Berger metrics (2.4) with coefficients b and c increasing in space. Namely, we make the following:

Definition 3.4. *A warped Berger metric has* monotone coefficients *if*

$$b_s \geq 0, \quad c_s \geq 0. \tag{3.2}$$

The reason we restrict our analysis to this subclass is twofold. From Theorem 1.6 we know that there exist spherically symmetric asymptotically flat initial data containing minimal 3-spheres leading to the formation of finite-time Type-I singularities along the Ricci flow. The monotonicity condition is meant to generalise the lack of minimal embedded spheres for the SO(n)-invariant setting and is hence natural when the emphasis is on investigating the long-time behaviour of the Ricci flow. Indeed, in Chapter 2 we proved that the maximal complete, bounded curvature Ricci flow solution starting at some warped Berger metric with monotone coefficients and curvature decaying at spatial infinity is immortal. In fact, the result holds with assumptions weaker than the spatial monotonicity of both the coefficients b and c. However, the stronger requirement provided in Definition 3.4 allows us to control the injectivity radius of the solution only in terms of upper bounds of the curvature.

Since by Theorem 2.3 we have a large family of immortal solutions, we wish to determine for which subclass it is possible to classify the infinite-time singularity models. In particular, we aim to identify a class of initial data giving rise to solutions encountering a Type-II(b) singularity at infinite time modelled by the Taub-NUT metric. In order to do that, we first recollect a few properties of the Ricci-flat Taub-NUT metric.

3.2.2 The Taub-NUT metric

The Taub-NUT metric is a complete gravitational instanton found on $\mathbb{R}^4$ by Hawking [1977]. Following Gibbons and Pope [1979], Atiyah and Hitchin [1985] we describe the

Taub-NUT metric g_{TNUT} as the complete, non-flat, warped Berger metric on $\mathbb{R}^4$ of the form (2.4), whose warping coefficients b and c satisfy the differential equations below:

$$\mathsf{J}_1 \doteq c_s - u^2 = 0, \tag{3.3}$$

and

$$\mathsf{J}_2 \doteq b_s + u - 2 = 0. \tag{3.4}$$

The first-order conditions define a *hyperkähler* structure on $(\mathbb{R}^4, g_{\mathsf{TNUT}})$, so that g_{TNUT} is in particular *Ricci-flat*. One may solve explicitly the equations and write g_{TNUT} as (see also Fine et al. [2017]):

$$g_{\mathsf{TNUT}} = \frac{1}{16}\left(1 + \frac{2m^{-1}}{r}\right) dr^2 + \frac{r^2}{4}\left(1 + \frac{2m^{-1}}{r}\right) \pi^* g_{S^2(\frac{1}{2})} + \frac{m^{-2}}{1 + \frac{2m^{-1}}{r}}\, \sigma_3 \otimes \sigma_3, \tag{3.5}$$

where m is a positive parameter quantifying the *mass* of the magnetic monopole giving rise to the Taub-NUT metric Hawking [1977]. Since m^{-1} measures the size of the Hopf-fiber at spatial infinity, we see that g_{TNUT} has *cubic volume growth*, meaning that there exist $A \geq \alpha > 0$ such that

$$\alpha\, r^3 \leq \mathrm{Vol}_{g_{\mathsf{TNUT}}}\left(B_{g_{\mathsf{TNUT}}}(\boldsymbol{o}, r)\right) \leq A\, r^3, \quad \forall r \geq 1.$$

From the formulas of the curvature terms given above we also derive that g_{TNUT} is an *asymptotically flat* metric satisfying

$$\sup_{p \in \mathbb{R}^4} \left(d_{g_{\mathsf{TNUT}}}(\boldsymbol{o}, p)\right)^3 |\mathrm{Rm}_{g_{\mathsf{TNUT}}}|_{g_{\mathsf{TNUT}}}(p) < \infty.$$

Since by (3.4) the coefficient b along the S^2-direction orthogonal to the Hopf-fiber grows linearly in the distance, we get

$$b^3(s)|k_{12}|(s) \geq \delta > 0,$$

for all $s \geq 1$. Namely, we find that $b^3|\mathrm{Rm}_{g_{\mathsf{TNUT}}}|_{g_{\mathsf{TNUT}}} \geq \delta$ away from the unit ball with respect to g_{TNUT} centred at the origin. By the latter property and the uniform boundedness of the Hopf-fiber we derive that for any $\kappa > 0$ there exist $p \in \mathbb{R}^4$ and $r > 0$ such that g_{TNUT} is strongly κ*-collapsed* at p for all scales larger than r. According to Perelman's analysis, we can rule out the Taub-NUT metric as a possible *finite-time* singularity model for the Ricci flow. One of the main goals of this chapter consists in showing that g_{TNUT} can actually appear as an *infinite-time* singularity model for immortal Ricci flow solutions.

3.2.3 Asymptotically flat initial data

Since the curvature of the Taub-NUT metric decays at a cubic rate at spatial infinity, it is worth investigating the Ricci flow starting at a warped Berger asymptotically flat metric. It turns out that, as long as we restrict our analysis to asymptotically flat metrics with monotone coefficients, the long-time behaviour of the flow for those initial data can be entirely classified and only depends on the length of the Hopf-fiber at spatial infinity. First, we set the following:

Definition 3.5. *The class $\mathcal{G}_{\mathsf{AF}}$ consists of all complete warped Berger metrics g on $\mathbb{R}^4$ with monotone coefficients satisfying*

$$\sup_{p\in\mathbb{R}^4} \left(d_g(\boldsymbol{o},p)\right)^{2+\epsilon} |\mathrm{Rm}_g|_g(p) < \infty, \tag{3.6}$$

for some $\epsilon > 0$. A metric $g \in \mathcal{G}_{\mathsf{AF}}$ is called asymptotically flat.

Below we provide a simple characterization of $\mathcal{G}_{\mathsf{AF}}$. In fact, for the next result we may also drop the assumption on the monotonicity of the warping coefficients.

Lemma 3.5. *Let g be an asymptotically flat warped Berger metric on $\mathbb{R}^4$. Then either one of the following conditions is satisfied:*

(i) There exist the limits

$$\lim_{s(p)\to\infty} b_s(p) = 2, \qquad \lim_{s(p)\to\infty} c_s(p) = 0, \qquad \lim_{s(p)\to\infty} c(p) \doteq m_g^{-1} \in (0,\infty).$$

(ii) There exist the limits

$$\lim_{s(p)\to\infty} b_s(p) = 1, \qquad \lim_{s(p)\to\infty} c_s(p) = 1.$$

Proof. In the following we always take $s \geq 1$ and we let ϵ and α be the positive number appearing in (3.6) and a uniform constant that may change from line to line respectively. We first note that the asymptotic behaviour of the derivatives is a known fact Unnebrink [1996] under slightly weaker assumptions. We now explicitly check that under the requirement of faster than quadratic curvature decay, in case (i) the warping coefficient c admits a finite positive limit at spatial infinity. Since c_s is decaying to zero at spatial infinity there exists $\gamma > 0$ such that $c(s) \leq \gamma\, s$. Consider the quantity $\nu = \min\{\epsilon, 3/4\}$. From (2.11) and (3.6) we derive

$$|s^{1+\nu} c_{ss}| \leq \gamma |s^{2+\nu} \frac{c_{ss}}{c}| \leq \alpha.$$

We can thus apply l'Hôpital formula and conclude that $s^{\frac{2}{3}\nu} c_s$ is bounded for all $s \geq 1$. It follows that

$$c(s) \leq \alpha(1 + s^{1-\frac{2}{3}\nu}),$$

for all $s \geq 1$. Since b grows linearly with respect to the geometric coordinate s, we also have

$$s^{2+\nu} \frac{u^2}{b^2} \leq \alpha s^{-2+\nu} c^2 \leq \alpha\, s^{-\frac{\nu}{3}}.$$

The previous estimate, the condition $b_s \to 2$ and formula (2.9) yield

$$\left| s^{1+\nu} \frac{c_s}{c} \right| \leq \alpha.$$

By integrating we conclude that there there exist $0 < \delta < M < \infty$ such that

$$\delta \leq c \leq M$$

for any $s \geq 1$. The uniform upper bound for c and (2.11) give $|c_{ss}| \leq \alpha s^{-2-\nu}$. Integrating and using that $c_s \to 0$ at infinity we obtain $|c_s| \leq \alpha s^{-1-\nu}$. Therefore c admits a limit at infinity, which by the previous analysis needs to be positive and finite. □

Remark 3.1. *From the classification result in Lemma 3.5 we see that any metric* $g \in \mathcal{G}_{\mathsf{AF}}$ *with vanishing asymptotic volume ratio behaves like the Taub-NUT metric at spatial infinity, in the sense that* $\mathrm{Vol}_g B_g(\boldsymbol{o}, r) \sim r^3$*, for* $r \geq 1$*. In particular, for any such* g *the Hopf-fiber has a well defined positive and finite length at infinity. In analogy with the magnetic monopole construction of the Taub-NUT metric, we refer to the quantity* $(\lim_{s(p)\to\infty} c(p))^{-1} \equiv m_g$ *as the* mass *of* g*. Accordingly, Lemma 3.5 implies that the class* $\mathcal{G}_{\mathsf{AF}}$ *is the union of cubic volume growth metrics with bounded Hopf-fiber -i.e.* positive mass - *and of Euclidean volume growth metrics with unbounded Hopf-fiber - i.e.* zero mass.

3.2.4 Initial data opening faster than a paraboloid

Appleton proved that on $\mathbb{R}^4$ there exists a warped Berger gradient steady soliton with monotone coefficients which is characterized by the following asymptotics at spatial infinity Appleton [2018]:

$$c(s) \sim \text{constant}, \quad b(s) \sim \sqrt{s}.$$

Therefore, the length of the Hopf-fiber approaches a positive finite quantity at spatial infinity, while the projection on the base space $\mathbb{R}^3 \setminus \{\boldsymbol{o}\}$

$$g_{\mathbb{R}^3} = ds^2 + b^2(s)\, \pi^* g_{S^2(\frac{1}{2})}$$

opens as fast as a paraboloid on $\mathbb{R}^3$. Thus, we derive that initial data opening at spatial infinity with arbitrary speed may fail to converge to the Taub-NUT metric in infinite time, the soliton being an explicit example for that.

Ivey [1994] showed that a family of positively curved, pinched SO(3)-invariant immortal Ricci flows on $\mathbb{R}^3$ opening (at least) as fast as a paraboloid do converge along subsequences in the pointed Cheeger-Gromov sense as $t \nearrow \infty$. In line with this result, one is tempted to ask whether an analogous property holds in our setting. Accordingly, we aim to determine whether the soliton provides a sort of lower barrier for the convergence property, in the sense that any solution with bounded Hopf-fiber and warping coefficient along the S^2-direction growing faster than a paraboloid in $\mathbb{R}^3$ does flow to the Taub-NUT metric in infinite time. From a slightly different angle, we investigate whether the Taub-NUT metric is the only complete, bounded curvature warped Berger ancient Ricci flow with monotone coefficients, bounded Hopf-fiber and opening faster than the soliton.

By the previous observations we need to characterize the property of a warped Berger metric opening faster than a paraboloid in a way that would be meaningful and hence preserved along a Ricci flow solution. The following definition is equivalent to the one given in the Introduction because the Hopf-fiber is uniformly bounded: we prefer the formulation below because it is invariant under rescaling. We recall that u is the roundness ratio c/b.

Definition 3.6. *For all $0 \leq k < 1$, the class $\mathcal{G}_k$ consists of all complete warped Berger metrics g with monotone coefficients satisfying:*

$$0 < \liminf_{s\to\infty} (b_s u^{-k})(s) \leq \limsup_{s\to\infty} (b_s u^{-k})(s) < \infty, \tag{3.7}$$

$$\sup_{p\in\mathbb{R}^4} \left(b^2|\mathrm{Rm}_g|_g\right)(p) < \infty, \tag{3.8}$$

$$\sup_{p\in\mathbb{R}^4} c(p) < \infty. \tag{3.9}$$

Since from (3.9) we see that $u^{-1} \sim b$ away from the origin, we can integrate (3.7) and derive that for any metric $g \in \mathcal{G}_k$ the warping coefficient b along the S^2-direction satisfies $b(s) \sim s^{\frac{1}{k+1}}$ for all s large enough, meaning that g opens faster than a paraboloid. In particular, for any warped Berger metric $g \in \mathcal{G}_k$ the volume of geodesic balls of radius r centred at the origin grows as

$$\mathrm{Vol}_g B_g(\boldsymbol{o}, r) \sim r^{\frac{2}{k+1}+1}.$$

By combining (3.7) and (3.8) we find that the curvature of a metric $g \in \mathcal{G}_k$ decays at a rate

$$\sup_{p\in\mathbb{R}^4} \left(d_g(\boldsymbol{o}, p)\right)^{\frac{2}{k+1}} |\mathrm{Rm}_g|_g(p) < \infty. \tag{3.10}$$

Remark 3.2. *It is worth determining which of the previous conditions are related and how.*

(i) (3.7), (3.8) *and* (3.9) $\Rightarrow$ (3.10)*: this easily follows from integration.*

(ii) (3.8) $\not\Rightarrow$ (3.7)*: it suffices to consider the following example*

$$g = ds^2 + \arctan^2(s)\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) = ds^2 + \arctan^2(s) g_{S^3},$$

which satisfies the condition in (3.8)*, yet the metric has cylindrical asymptotics. Indeed, the maximal complete, bounded curvature Ricci flow solution starting at such g encounters a finite-time Type-II singularity according to Theorem 2.1.*

(iii) (3.10) $\not\Rightarrow$ (3.7)*: one can take a warped Berger metric with $c(s) = \arctan(s)$ and $b(s) = s\log(s)$ for all $s \geq 1$ and find that* (3.10) *holds with $k = 0$ while the warping coefficient b grows faster than a linear function of the distance.*

(iv) (3.7) $\not\Rightarrow$ (3.8)*: the first-order constraint given by* (3.7) *does not rule out second order terms which are not controlled by the size of the principal orbit b.*

By the existence of the steady soliton found by Appleton we know that Ricci flow solutions starting at initial data as in Definition 3.6 with $k = 1$ might in general fail to converge to the Taub-NUT metric. We also note that the Euclidean metric would be included in the class $\mathcal{G}_0$ if we dropped the requirement on the size of the Hopf-fiber in (3.9).

The class of asymptotically flat warped Berger metrics with positive mass - i.e. bounded Hopf-fiber - is contained in $\mathcal{G}_0$. The sets $\mathcal{G}_k$ though allow for initial data with geometric features different from the Taub-NUT metric, beyond the rates of both decay of the curvature and growth of the volume of geodesic balls. Indeed, we now describe a metric $g \in \mathcal{G}_0$ with nonnegative sectional curvature.

Lemma 3.6. *There exists $g \in \mathcal{G}_0$ satisfying* $\sec(g) \geq 0$.

Proof. Consider the warped Berger metric g on $\mathbb{R}_+ \times S^3$ defined by

$$g = ds^2 + s^2\,\pi^* g_{S^2(\frac{1}{2})} + c^2(s)\,\sigma_3 \otimes \sigma_3,$$

where $c(s) = \int_0^s \frac{1}{1+y^4}\, dy$. By the smoothness conditions in (2.7) we see that g extends to a complete warped Berger metric on $\mathbb{R}^4$ with monotone coefficients. Since b is linear and c is concave we have $k_{01} = 0$ and $k_{03} \geq 0$. Moreover

$$k_{12} = \frac{1}{b^2}\left(4 - 3u^2 - b_s^2\right) = \frac{3}{b^2}\left(1 - u^2\right) \geq 0.$$

Finally, by direct computation we check that $(c - s(1+s^4)^{-1/3})_s \geq 0$, hence yielding $k_{13} \geq 0$. Therefore, we have shown that $\mathrm{sec}(g) \geq 0$. In order to prove that $g \in \mathcal{G}_0$ it suffices to show that $b^2|\mathrm{sec}(g)|$ is bounded since (3.7) and (3.9) are satisfied with $k = 0$. To this aim, we find that there exists $\alpha > 0$ such that

$$\begin{aligned}
&b^2(s)|k_{01}|(s) = 0,\\
&b^2(s)|k_{03}|(s) = s^2\left|\frac{1}{c(s)}\left(-\frac{4s^3}{(1+s^4)^2}\right)\right| \leq \alpha,\\
&b^2(s)|k_{12}|(s) = 3\left|1 - u^2(s)\right| \leq \alpha,\\
&b^2(s)|k_{13}|(s) = \left|u^2 - \frac{s}{c(s)(1+s^4)}\right| \leq \alpha.
\end{aligned}$$

□

3.2.5 The Ricci flow equations

We recall that the maximal complete, bounded curvature Ricci flow solution starting at a warped Berger metric g_0 has the form (2.14)

$$g(t) = ds \otimes ds + b^2(s,t)\,(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2) + c^2(s,t)\,\sigma_3 \otimes \sigma_3.$$

Throughout this chapter we often compute the evolution equation of geometric quantities at stationary points. To this aim, the commutator formula (2.17) plays an important role. Say that we are interested in studying the sign of $\partial_t f$ at a local minimum point, then we report the evolution equation of f at such minimum point after using the conditions $f_{ss} \geq 0$ and $f_s = 0$.

We note that the Ricci flow preserves the class of warped Berger metrics with monotone coefficients.

Lemma 3.7. *Let $(\mathbb{R}^4, g(t))_{0\leq t<T}$ be the maximal Ricci flow solution starting at some complete, bounded curvature warped Berger metric g_0 with monotone coefficients. Then the following properties hold:*

(i) $u(\cdot,t) \leq 1$ *for all* $t \in [0,T)$.

(ii) $b_s(\cdot,t) > 0$, $c_s(\cdot,t) > 0$ *for all* $t \in (0,T)$.

(iii) $u(\cdot,t) \geq \inf_{\mathbb{R}^4} u(\cdot,0)$, *for all* $t \in [0,T)$.

Proof. The proof of (i) and (iii) follows from the same arguments in Lemma 2.7. Similarly, one can easily adapt the proof of Lemma 2.12 by replacing the evolution equation of cH with that of c_s to show that (ii) is satisfied as long as the solution exists. □

Remark 3.3. *In the following we always implicitly use that for warped Berger Ricci flows the roundness ratio u is bounded by 1.*

From the analysis in Chapter 2 (see Theorem 2.3) we finally derive that any Ricci flow solution we consider below is in fact immortal.

Corollary 3.8. *Let $(\mathbb{R}^4, g(t))_{0\leq t<T}$ be the maximal Ricci flow solution starting at some g_0 belonging to either $\mathcal{G}_k$ or $\mathcal{G}_{\mathsf{AF}}$. Then the solution is immortal.*

Proof. By direct computation one may check that $(\mathbb{R}^4, g)$ does not contain closed geodesics when g is a warped Berger metric with warping coefficients b and c strictly increasing in space. Therefore, given a maximal Ricci flow solution as in the statement, by Lemma 3.7 we see that we can find $t_0 \in (0,T)$ such that $(\mathbb{R}^4, g(t_0))$ does not contain closed geodesics. Since the curvature is bounded, we deduce that inj$(g(t_0)) > 0$. We may then apply Theorem 2.3 to the initial condition $(\mathbb{R}^4, g(t_0))$, being the decay of the curvature preserved along the flow Hamilton [1995], and conclude that the Ricci flow solution exists smoothly for all positive times.

□

3.3 The Ricci flow in $\mathcal{G}_{\mathsf{AF}}$

In this section we study the Ricci flow problem in $\mathcal{G}_{\mathsf{AF}}$. According to the characterization of asymptotically flat warped Berger metrics provided in Lemma 3.5, given $g \in \mathcal{G}_{\mathsf{AF}}$ we refer to the inverse of $\sup_{\mathbb{R}^4} c$ as the *mass* of g. In particular, we recall that the set $\mathcal{G}_{\mathsf{AF}}$ decomposes in the union of metrics with zero mass, or equivalently Euclidean volume growth, and of metrics with positive mass, or equivalently cubic volume growth. The key results of this section consist in showing that for any Ricci flow solution in $\mathcal{G}_{\mathsf{AF}}$ the curvature is controlled by the size of the principal orbits uniformly in time and the spatial derivative b_s is bounded away from zero in any space-time region where the roundness ratio u is not degenerate, once we let the flow start.

We point out that most of the analysis could in fact be performed on a more general level, however we prefer to focus first on the asymptotically flat case which is easier to deal with - and also includes metrics with positive asymptotic volume ratio - before discussing the problem for metrics in $\mathcal{G}_k$, for which the behaviour of the flow at spatial infinity is a priori less rigid.

We first verify that the Ricci flow acts on the class $\mathcal{G}_{\mathsf{AF}}$ preserving the curvature decay of the initial metric (3.6).

Lemma 3.9. *Let $(\mathbb{R}^4, g(t))_{t\geq 0}$ be the maximal Ricci flow solution starting at some $g_0 \in \mathcal{G}_{\mathsf{AF}}$ and let $\epsilon > 0$ be such that $\sup_{\mathbb{R}^4}(d_{g_0}(\boldsymbol{o}, \cdot))^{2+\epsilon}|\mathrm{Rm}_{g_0}|_{g_0}(\cdot) < \infty$. For any $T' < \infty$ there exists $\alpha(T')$ such that*

$$\sup_{p\in\mathbb{R}^4} (d_{g_0}(\boldsymbol{o}, p))^{2+\epsilon}|\mathrm{Rm}_{g(t)}|_{g(t)}(p) \leq \alpha(T'),$$

for all $t \in [0, T']$.

Remark 3.4. *According to [Chow et al., 2008, Lemma 12.5], if ϕ is a distance-like function with respect to g_0, meaning that*

$$\alpha^{-1}(s_0(p) + 1) \leq \phi(p) \leq \alpha(s_0(p) + 1),$$
$$|\nabla_{g_0}\phi|_{g_0} \leq \alpha,$$
$$\nabla^2_{g_0}(\phi) \leq \alpha g_0,$$

then for any $T' < T$ it follows that there exists $\alpha_{T'}$ such that for all $t \in [0, T']$ we get

$$\alpha_{T'}^{-1}(s_t(p) + 1) \leq \phi(p) \leq \alpha(s_t(p) + 1),$$
$$|\nabla_{g_t}\phi|_{g(t)} \leq \alpha_{T'},$$
$$\nabla^2_{g_t}(\phi) \leq \alpha_{T'} g(t).$$

Therefore, the notion of power-law decay of the curvature can equivalently be measured in terms of g_0 or $g(t)$.

Proof. We let s_0 be the geometric coordinate representing the distance function from the origin induced by g_0 so that we can write the initial metric g_0 as in (2.4). We consider the smooth function $\phi : s_0 \mapsto \sqrt{s_0^2 + 1}$. From the connection terms, we see that

$$|\nabla_{g_0}\phi|_{g_0} = |\partial_{s_0}\phi| \leq 1,$$

and

$$\nabla^2_{g_0}\phi(\partial_{s_0}, \partial_{s_0}) = \partial^2_{s_0}\phi.$$

Moreover, whenever b is positive we have

$$\nabla^2_{g_0}\phi(X_1/|X_1|_{g_0}, X_1/|X_1|_{g_0}) = \frac{b_{s_0}}{b}\frac{s_0}{\sqrt{1 + s_0^2}}.$$

From Corollary 2.9 we derive that the previous quantity is bounded away from the origin. Since by the boundary conditions $b_{s_0}(\boldsymbol{o}) = 1$ we may conclude that the bound extends at the origin as well. Similar arguments work when evaluating the Hessian of ϕ along X_3. Therefore we have just shown that ϕ is a smooth distance-like function on $(\mathbb{R}^4, g_0)$ in the sense of Chow et al. [2008][Lemma 12.30]. Since $b_{s_0} \geq 0$ and $c_{s_0} \geq 0$, the same computations yield

$$\nabla^2_{g_0}\phi \geq 0.$$

We may finally apply Proposition B.10 in Lott and Zhang [2016] to our setting, thus proving that the power law decay of the curvature in (3.6) persists along the flow. □

A simple consequence of the power law decay being preserved along the Ricci flow is the *conservation of mass* along the flow.

Corollary 3.10. *Let* $(\mathbb{R}^4, g(t))_{t\geq 0}$ *be the maximal Ricci flow solution starting at some* $g_0 \in \mathcal{G}_{\mathsf{AF}}$ *with positive mass* m_{g_0}*. Then* $m_{g(t)} = m_{g_0}$ *for any* $t \geq 0$*.*

Proof. According to Corollary 3.8 we define $\alpha : [0, \infty) \to \mathbb{R}$ by $\alpha(t) = \sup_{\mathbb{R}^4} |\mathrm{Rm}|(\cdot, t)$. Given $t > 0$, since by Lemma 3.7 we know that c_s stays nonnegative along the flow, we deduce that the quantity $m^{-1}_{g(t)} := \lim_{s(t)\to\infty} c(s(t), t)$ exists and is well defined. Moreover, the curvature is uniformly bounded in $\mathbb{R}^4 \times [0, t]$ by $\alpha(t)$ and we can then rely on standard distortion estimates of the distance function along bounded curvature Ricci flow to derive

$$m^{-1}_{g(t)} = \lim_{s(t)\to\infty} c(s(t), t) = \lim_{s_0(p)\to\infty} c(p, t),$$

where s_0 is the g_0-distance from the origin. Suppose for a contradiction that there exists $t_1 > 0$ such that $m_{g(t_1)} \neq m_{g_0}$. By the Ricci flow equations and Lemma 3.9 we obtain

$$|\partial_t \log(c(p, \cdot))| \leq \frac{\alpha(t_1)}{s_0(p)^{2+\epsilon}}$$

in $\mathbb{R}^4 \times [0, t_1]$ up to modifying $\alpha(t_1)$ by a constant factor. Once we integrate we find

$$\frac{1}{t_1}\left|\log\left(\frac{c(p, t_1)}{c(p, 0)}\right)\right| \leq \frac{\alpha(t_1)}{s_0^{2+\epsilon}(p)}.$$

We may finally let $s_0(p) \to \infty$ and get a contradiction. □

Remark 3.5. *A consequence of Corollary 3.10 is given by the fact that the size of the Hopf-fiber stays uniformly bounded in the positive-mass case.*

We may now focus on first order estimates. We recall that the evolution equations for the first spatial derivatives are (2.24)

$$\partial_t b_s = \Delta b_s - 2\frac{b_s b_{ss}}{b} + \frac{1}{b^2}\left(b_s\left(4 - b_s^2 - (c_s u^{-1})^2 - 6u^2\right) + 4c_s u\right)$$

and (2.25)

$$\partial_t c_s = \Delta c_s - 2\frac{c_s c_{ss}}{c} + \frac{1}{b^2}\left(c_s\left(-6u^2 - 2b_s^2\right) + 8b_s u^3\right),$$

where the Laplacian formula is given in (2.18). Similarly to the analysis for the finite-time case in Chapter 2, we show that the derivatives stay uniformly bounded and that the flow becomes rotationally symmetric in space-time regions where the orbits get degenerate. In the following estimates α always denotes a uniform, space-time independent constant that may change from line to line, unless otherwise stated.

Lemma 3.11. *If* $(\mathbb{R}^4, g(t))_{t\geq 0}$ *is the maximal Ricci flow solution evolving from some* $g_0 \in \mathcal{G}_{\mathsf{AF}}$*, then the following conditions hold:*

$$\sup_{\mathbb{R}^4\times[0,+\infty)} (b_s + c_s) < \infty, \tag{3.11}$$

$$\sup_{\mathbb{R}^4\setminus\{\boldsymbol{o}\}\times[0,+\infty)} \frac{1}{c}(b_s^2 - 4) < \infty, \tag{3.12}$$

$$\sup_{\mathbb{R}^4\times[0,+\infty)} \frac{1}{c}(1-u) < \infty, \tag{3.13}$$

$$\sup_{\mathbb{R}^4\times[0,+\infty)} \frac{1}{c}|c_s - b_s u| < \infty. \tag{3.14}$$

Proof. Estimate (3.11). Consider the upper bound for b_s. Since $b_s(\boldsymbol{o},t) = 1$ and by Lemmas 3.5 and 3.9 we see that $b_s(s,t)$ converges to either 1 or 2 at infinity for any $t \geq 0$, we deduce that if b_s attains a value $\bar{\alpha} > \sup b_s(\cdot, 0)$, then there exists a maximum point (p_0, t_0) among prior times where $b_s(p_0,t_0) = \bar{\alpha}$ for the first time. We can then argue as in Lemma 2.14. The same argument works for c_s as well.

Estimate (3.12). Set $\varphi \doteq c^{-1}(b_s^2 - 4)$. From the boundary conditions we see that φ diverges to minus infinity at the origin uniformly in time. On the other hand, from Lemma 3.9 and Remark 3.4 we know that the stronger than quadratic decay of the curvature persists along the solution and hence Lemma 3.5 applies to the solution $g(t)$, for any $t \geq 0$. In particular we see that $\varphi \to 0$ at spatial infinity as long as the solution exists. At any positive interior maximum point (p_0, t_0) we have

$$\begin{aligned}\varphi_t(p_0,t_0) \leq \frac{1}{b^2 c}\left(-(c_s u^{-1})^2\left(b_s^2+4\right) + b_s c_s\left(8u^{-1} + 8u - 2b_s^2 u^{-1}\right)\right) \\ + \frac{1}{b^2 c}\left(8b_s^2 - 2b_s^4 - 10u^2 b_s^2 - 8u^2\right).\end{aligned}$$

Since $b_s^2 > 4$ at any positive value of φ, we get

$$\varphi_t(p_0, t_0) \leq \frac{1}{b^2 c}\left(-(c_s u^{-1})^2\left(b_s^2 + 4\right) + 8ub_s c_s - 10u^2 b_s^2 - 8u^2\right).$$

Being the c_s-quadratic above always negative, we conclude that φ is uniformly bounded *from above* in the space-time.

Estimate (3.13) We first prove that $\psi \doteq c^{-1/2} - b^{-1/2}$ is uniformly bounded in the space-time. Since the curvature is bounded, by (2.12) we see that $b^{-1}u_s = O(1)$ as $s \to 0$ uniformly in time. Therefore $\psi(\boldsymbol{o}, t) = 0$ as long as the solution exists. Moreover, from Lemma 3.9 we also derive that ψ is uniformly bounded at spatial infinity by the inverse of the size of the Hopf-fiber. At any interior maximum point (p_0, t_0) we have

$$\begin{aligned}
\psi_t(p_0, t_0) &\leq \frac{1}{b^{\frac{5}{2}}}\left(\frac{b_s^2}{4}\left(1 - \sqrt{u}\right) + u^2 + u^{\frac{3}{2}} - 2\right) \\
&\leq \frac{1}{b^{\frac{5}{2}}}\left(\frac{b_s^2}{4}\left(1 - \sqrt{u}\right) - 2\left(1 - \sqrt{u}\right)\right) \\
&\leq \frac{1}{b^{\frac{5}{2}}}\left(1 - \sqrt{u}\right)\left(\alpha c - 1\right),
\end{aligned}$$

where $\alpha > 0$ is a uniform constant given by the estimate (3.12). Say that $\psi(p_0, t_0) = M$. By choosing M large enough we can make c as small as we ask. Therefore, the right hand side of the evolution equation becomes strictly negative, hence showing that ψ is uniformly bounded in the space-time. We may now consider $f \doteq c^{-1}(1 - u)$. Similarly to the case of ψ above, f is uniformly bounded both at the origin and at spatial infinity. At any maximum point we have (see Lemma 2.17)

$$\begin{aligned}
f_t &\leq \frac{1}{b^3}\left(b_s^2\left(1 - u\right) + 2u + 2u^2 - 4\right) \\
&\leq \frac{1}{b^3}\left(\left(4 + \alpha c\right)\left(1 - u\right) + 2u + 2u^2 - 4\right) \\
&\leq \frac{1}{b^3}\left(-2u + 2u^2 + \alpha c\left(1 - u\right)\right) = \frac{cf}{b^3}\left(-2u + \alpha c\right).
\end{aligned}$$

We have shown that $u^{1/2} \geq 1 - \alpha c^{1/2}$. Therefore, if we pick the value attained by f large enough, we see that $(-2u + \alpha c)(p_0, t_0) \leq -1$. That completes the proof.

Estimate (3.14) Again $c_s/c - b_s/b$ is uniformly bounded at the origin and at spatial infinity. Once the quantity is controlled along the parabolic boundary of the space-time, one can then argue as in Lemma 2.19. □

Before we prove analogous second order estimates, we first show that in the cubic volume growth case the spatial derivative c_s decays at some specific rate in space-time regions where u is small. We recall that for the Taub-NUT metric we have $c_s = u^2$.

Lemma 3.12. *Let $(\mathbb{R}^4, g(t))_{t\geq 0}$ be the maximal Ricci flow starting at some $g_0 \in \mathcal{G}_{\mathsf{AF}}$ and let $\epsilon > 0$ satisfy $\sup_{\mathbb{R}^4}(d_{g_0}(\boldsymbol{o},\cdot))^{2+\epsilon}|\mathrm{Rm}_{g_0}|_{g_0}(\cdot) < \infty$. For any $1 < k < \min\{1+\epsilon, \sqrt{2}\}$ there exists $\alpha > 0$ independent of k such that*

$$\sup_{\mathbb{R}^4\times[0,\infty)} c_s u^{-k} < \alpha.$$

Proof. From (2.9) and Lemma 3.9 we derive that for any $t \geq 0$ there exists $\alpha = \alpha(t)$ such that

$$b^{2+\epsilon}\left|\frac{u^2}{b^2} - \frac{b_s c_s}{bc}\right| \leq \alpha.$$

Since b is linear at infinity, we get that for s large

$$|c_s u^{-1}|b^{\epsilon} \leq \alpha + u^2 b^{\epsilon}.$$

Therefore, for any $1 < k < \min\{1+\epsilon, \sqrt{2}\}$ we have

$$c_s u^{-k} \leq \frac{1}{c^{\epsilon}}\left(\alpha + u^2 b^{\epsilon}\right) u^{1+\epsilon-k} \leq \frac{\alpha}{c^{\epsilon}} u^{1+\epsilon-k} + u^{3-k},$$

which is uniformly bounded at spatial infinity by Lemma 3.9. In particular, we see that $(c_s u^{-k})(s,t)$ either converges to zero (in the cubic volume growth case) or to 1 (in the Euclidean volume growth case) for any $1 < k < \min\{1+\epsilon, \sqrt{2}\}$. By the boundary conditions we derive that if $c_s u^{-k}$ becomes unbounded as $t \nearrow \infty$ then there exists a sequence of maxima diverging. The evolution equation of $c_s u^{-k}$ at a maximum point is

$$\begin{aligned}\partial_t\left(c_s u^{-k}\right)|_{\max} &\leq \frac{c_s u^{-k}}{b^2}\left(b_s^2(k^2-2)-4k+u^2(-6+4k)+(c_s u^{-1})^2(k^2-2k)\right)\\&+\frac{1}{b^2}\left(c_s u^{-k}\left(b_s c_s u^{-1}(-2k^2+2k)\right)+8u^{3-k}b_s\right)\\&\leq \frac{1}{b^2}\left(-4k(c_s u^{-k})+\alpha\right)<\frac{1}{b^2}\left(-4(c_s u^{-k})+\alpha\right),\end{aligned}$$

where we have used that $1<k<\sqrt{2}$ and the estimate (3.11). We conclude that for any $k\in(1,\min\{1+\epsilon,\sqrt{2}\})$ the function $c_s u^{-k}$ admits a uniform upper bound independent of k.

□

Once we control the ratio u from below by c_s, we can prove the following:

Lemma 3.13. *If $(\mathbb{R}^4, g(t))_{t\geq 0}$ is the maximal Ricci flow solution starting at some $g_0\in\mathcal{G}_{\mathsf{AF}}$, then*

$$\sup_{\mathbb{R}^4\times[0,+\infty)} c^2\left(|k_{01}|+|k_{03}|\right)<\infty.$$

Proof. First, we prove that $-bck_{01}=cb_{ss}$ has a uniform lower bound in the space-time. In analogy with Isenberg et al. [2019], we consider the quantity $f\doteq cb_{ss}-2b_s^2-c_s^2$ which we see to be uniformly bounded at the origin and at spatial infinity by the boundary conditions and Lemma 3.9 respectively. A long yet straightforward computation yields that whenever f attains some negative minimum value its evolution equation becomes (see also (2.27))

$$\begin{aligned}\partial_t f|_{f_{\min}<0} &\geq 2b_{ss}^2(2-u)+2c_{ss}^2+c_{ss}\left(4\frac{u^2}{b}-2\frac{b_s c_s}{c}-4\frac{b_s c_s}{b}\right)+\\&+\frac{ub_{ss}}{b}\left(4-3b_s^2-8u^2-c_s^2+2b_s c_s u^{-1}\left(1-4u^{-1}\right)\right)\\&+\frac{1}{c^2}\left(2b_s c_s^3+4c_s^2u^3(1+3u)+4b_s^2c_s^2(1+u^2)-24b_s c_s u^4+2b_s^4u^2(2+u)\right)\\&+\frac{1}{c^2}\left(24b_s^2u^4(1+u)-8b_s^2u^2(2+u)-8b_s c_s u^3(2+3u)\right).\end{aligned}$$

Since the first order spatial derivatives are uniformly bounded we may assume that $cb_{ss}<f_{\min}/2$ provided that $|f_{\min}|$ is large enough. By applying Cauchy-Schwarz to the

coefficients of c_{ss} and again using (3.11) we get

$$\partial_t f|_{f_{\min}<0} \geq 2b_{ss}^2 + \frac{c_{ss}^2}{2} + \frac{ub_{ss}}{b}\left(4 - 3b_s^2 - 8u^2 - c_s^2 + 2b_s c_s u^{-1}\left(1 - 4u^{-1}\right)\right) - \frac{\alpha}{c^2},$$

for some uniform constant $\alpha > 0$. By the monotonicity of b_s and c_s we finally obtain

$$\partial_t f|_{f_{\min}<0} \geq \frac{1}{c^2}\left(2(b_{ss}c)^2 + 4u^2 b_{ss}c - \alpha\right) \geq \frac{1}{c^2}\left(\frac{f_{\min}^2}{2} + 4f_{\min} - \alpha\right) > 0,$$

for $|f_{\min}|$ large enough. The existence of a uniform upper bound for cb_{ss} follows from the similar arguments.

We now show that $-c^2k_{03} = cc_{ss}$ has a uniform lower bound as long as the solution exists. We proceed as before. We define $h = cc_{ss} - 2c_s^2 - b_s^2$, which by the boundary conditions and Lemma 3.9 is uniformly bounded at the origin and at spatial infinity. Suppose that h attains a negative minimum. According to (3.11) we find that $cc_{ss} \leq h_{\min}/2$, whenever $|h_{\min}|$ is sufficiently large. At such point we can write the evolution equation of h as (see also (2.28))

$$\partial_t h|_{h_{\min}<0} \geq \frac{1}{c^2}\left(2(cc_{ss})^2 + \alpha cc_{ss} + 2b_{ss}^2c^2 - \alpha|b_{ss}c| - \alpha\right),$$

where α is a uniform positive constant given by b_s and c_s being positive and bounded along the flow - and by u being bounded by 1. Since we have just checked that $|cb_{ss}|$ is uniformly bounded, the right hand side is positive once we pick $|h_{\min}|$ and hence $|cc_{ss}|$ large enough. Analogously one may check that cc_{ss} is uniformly bounded from above in the space-time. □

From the estimates (3.11) and (3.13) and the previous Lemma we deduce that the curvature is uniformly controlled in time in regions where the orbits do not become degenerate.

Corollary 3.14. *If $(\mathbb{R}^4, g(t))_{t\geq 0}$ is the maximal Ricci flow solution starting at some $g_0 \in \mathcal{G}_{\mathsf{AF}}$, then*

$$\sup_{\mathbb{R}^4\times[0,+\infty)} \left(c^2|\mathrm{Rm}_{g(t)}|_{g(t)}\right) < \infty.$$

We dedicate the end of this section to proving that the spatial derivative b_s has a uniform lower bound in the space-time domain where the squashing factor u stays positive. We start by showing that in the bounded Hopf-fiber setting $b_s u^{-1}$ always diverges when u^{-1}, and hence b by (3.13), is large. This estimate will play a key role in characterizing the possible infinite-time singularity models and turns out to be satisfied by Ricci flows in $\mathcal{G}_k$ as well.

Lemma 3.15. *Let $(\mathbb{R}^4, g(t))_{t\geq 0}$ be the maximal Ricci flow solution starting at some $g_0 \in \mathcal{G}_{\mathsf{AF}}$ with bounded Hopf-fiber. There exist $\alpha, \lambda > 0$ such that*

$$b^{\lambda}(b_s u^{-1} - \log(b)) \geq -\alpha,$$

uniformly in the space-time.

Proof. We let $\chi \doteq b^{\lambda}(b_s u^{-1} - \log(b))$ be defined smoothly on $\mathbb{R}^4 \setminus \{\boldsymbol{o}\} \times [0, +\infty)$ and we extend it continuously at the origin. From the boundary conditions and Lemma 3.9 we see that $\chi(\boldsymbol{o}, t) = 0$ and $\chi(s, t) \to \infty$ as $s \to \infty$ for all positive times. Assume that χ attains some large negative value at a minimum point (p_0, t_0) among prior times. The evolution of χ at (p_0, t_0) becomes

$$\partial_t \chi(p_0, t_0)|_{\chi_{\min}<0} \geq \frac{1}{b^2}\left(\chi\left(b_s^2(\lambda^2 + 4\lambda) - 2\lambda b_s c_s u^{-1} + 2\lambda u^2 - 4\lambda\right) + 4c_s b^{\lambda}\right) \tag{3.15}$$

$$+ \frac{1}{b^2}\left(b^{\lambda} b_s u^{-1}(2b_s^2 - 4b_s c_s u^{-1} - 2u^2) + b^{\lambda}(4 - 2u^2 - 4b_s^2 + 2b_s c_s u^{-1})\right). \tag{3.16}$$

Since $|\chi_{\min}| = b^{\lambda}(\log(b) - b_s u^{-1}) \leq b^{\lambda}\log(b)$, we see that b can be taken as large as we want once we pick $|\chi_{\min}|$ large. Similarly, at any negative minimum of χ the derivative b_s is small whenever the value of b is sufficiently large, being c uniformly bounded from above. Thus, whenever $|\chi_{\min}|$ is large enough, we may write the evolution equation of χ as

$$\partial_t \chi(p_0, t_0)|_{\chi_{\min}} \geq \frac{1}{b^2}\left(\lambda|\chi_{\min}| + b^{\lambda} b_s u^{-1}(-4b_s c_s u^{-1} - 2u^2) + b^{\lambda}\right). \tag{3.17}$$

Finally, we note that according to Lemma 3.12 we can find $k > 1$ such that

$$4b^{\lambda}b_s^2 c_s u^{-2} \leq 4\alpha b^{\lambda} b_s^2 u^{-2+k} \leq \alpha b^{\lambda}(\log(b))^2 u^k,$$

where again we have used that $\chi(p_0, t_0) < 0$. We may then choose $\lambda \leq 1$ and conclude that

$$\partial_t \chi(p_0, t_0)|_{\chi_{\min} < 0} \geq \frac{1}{b^2}\left(\lambda|\chi_{\min}| + \frac{b^{\lambda}}{2}\right) > 0.$$

□

We now show that b_s cannot become degenerate in space-time regions where the quantity u is bounded away from zero. On the one hand this control is necessary for the compactness result we rely on for proving that symmetries are preserved on any pointed Cheeger-Gromov limit. On the other, we see that if we were in a rotationally-symmetric setting, the solution would have positive asymptotic volume ratio. The latter observation will be crucial when showing that any Ricci flow in $\mathcal{G}_{\mathsf{AF}}$ has curvature uniformly bounded in the space-time.

We recall that the mean curvature of the Euclidean 3-sphere $S(\boldsymbol{o}, z)$ with respect to the solution $g(t)$ is given by

$$H(z,t) = \left(2\frac{b_s}{b} + \frac{c_s}{c}\right)(z,t).$$

Lemma 3.16. *If* $(\mathbb{R}^4, g(t))_{t\geq 0}$ *is the maximal Ricci flow solution starting at some* $g_0 \in \mathcal{G}_{\mathsf{AF}}$*, then there exists* $\beta > 0$ *such that*

$$\inf_{\mathbb{R}^4}\left(b_s u^{-1}\right)(\cdot, t) \geq \beta,$$

for all times $t \geq 1$.

Proof. Case (i): Positive mass. We consider the maximal immortal Ricci flow solution evolving from g_0 for times $t \geq 1$ so that $b_s(\cdot, t)$ is positive everywhere by the strong maximum principle (see Lemma 3.7). Given $\alpha, \lambda > 0$ as in Lemma 3.15, we see that $b_s u^{-1} \geq 1$ in the time-dependent region $V(t) = \{p \in \mathbb{R}^4 : \log(b(p,t)) - \alpha/b^{\lambda}(p,t) \geq 1\}$.

We note that since b is monotone we may identify $V(t)$ with the complement of some time-dependent Euclidean ball $B(\boldsymbol{o}, r(t))$, with $t \mapsto r(t)$ a continuous function. From the estimate (3.13) we derive that $u(\cdot, t) \geq \varepsilon$ in $B(\boldsymbol{o}, r(t))$ for all $t \geq 1$ for some $\varepsilon > 0$, being $b(\cdot, t)$ uniformly bounded in $B(\boldsymbol{o}, r(t))$. In particular, we deduce that

$$cH(r(t), t) \geq 2b_s u(r(t), t) \geq 2\varepsilon^2 > 0,$$

for all $t \geq 1$. Similarly, $cH(\boldsymbol{o}, t) = 3$ for all times according to the boundary conditions. Therefore, if cH attains some value $\tilde{\varepsilon} > 0$ small enough in $B(\boldsymbol{o}, r(t))$ for the first time, then this must happen at an interior minimum point (p_0, t_0) and we have

$$\partial_t(cH)(p_0, t_0)|_{\tilde{\varepsilon}} \geq \frac{1}{b^2}\left(2cH\left(u^2 - b_s^2\right) + 16 b_s u(1 - u^2)\right).$$

Since $u \geq \varepsilon$ in $B(\boldsymbol{o}, r(t))$, we see that if $\tilde{\varepsilon}$ is small enough, then $b_s u^{-1}(p_0, t_0) \leq 1$, which hence yields $\partial_t(cH)(p_0, t_0) > 0$. We conclude that cH is uniformly bounded from below in $B(\boldsymbol{o}, r(t))$, for all times $t \geq 1$. Since $b_s u^{-1} \geq 1$ in $V(t)$, if the quantity attains some value β sufficiently small for some time $t_1 > 1$, then there exists an interior minimum point (p_0, t_0) in $B(\boldsymbol{o}, r(t_0))$ among times $t \in (1, t_1]$. The evolution equation of $b_s u^{-1}$ at such minimum point is

$$\partial_t\left(b_s u^{-1}\right)(p_0, t_0) \geq \frac{1}{b^2}\left(b_s u^{-1}(2b_s^2 - 4b_s c_s u^{-1} - 2u^2) + 4c_s\right).$$

From the estimate $cH \geq \tilde{\varepsilon}$ we conclude that $c_s(p_0, t_0) \geq \tilde{\varepsilon}/2$ whenever β is small enough. Therefore, the right hand side of the evolution equation is positive and hence $b_s u^{-1} \geq \beta > 0$ for all times $t \geq 1$.

Case (ii): Zero mass. In this case $cH(\cdot, t) \to 3$ at spatial infinity as long as the solution exists. Thus, we can argue as above using that $u \geq \delta$ in the space-time, for some $\delta > 0$, as follows from (iii) in Lemma 3.7.

□

3.4 The Ricci flow in $\mathcal{G}_k$

In this section we extend the analysis of asymptotically flat warped Berger Ricci flows to solutions in $\mathcal{G}_k$. One of the main difficulty consists in controlling the flow in the space-time region where the roundness ratio u is small. In the asymptotically flat case the stronger than quadratic decay of the curvature determines the behaviour of the warping coefficients at spatial infinity precisely. Once such decay is preserved along the flow, one can then rely on maximum principle arguments to derive time-independent bounds. On the contrary, for the case of $\mathcal{G}_k$ some extra work is needed to control the solution along the parabolic boundary of the space-time and hence ensure that the condition of opening faster than a paraboloid is indeed preserved.

First, we note that one can argue as in Lemma 3.9 to prove that the power law decay of the curvature in $\mathcal{G}_k$ persists along the solution.

Lemma 3.17. *Let $(\mathbb{R}^4, g(t))_{t\geq 0}$ be the maximal Ricci flow solution starting at some $g_0 \in \mathcal{G}_k$. For any $T' < \infty$ there exists $\alpha(T') > 0$ such that*

$$\sup_{p\in\mathbb{R}^4} (d_{g_0}(\boldsymbol{o}, p))^{\frac{2}{k+1}} |\mathrm{Rm}_{g(t)}|_{g(t)}(p) \leq \alpha(T'),$$

for all $t \in [0, T']$.

As a simple consequence of Lemma 3.17 we derive that the volume growth rate of metrics in $\mathcal{G}_k$ is preserved along the solution as well as a conservation mass principle. We recall that given $g \in \mathcal{G}_k$ we call *mass* (of g) the quantity $(\lim_{s\to\infty} c(s))^{-1}$ and we denote such positive finite number by m_g.

Corollary 3.18. *Let $(\mathbb{R}^4, g(t))_{t\geq 0}$ be the maximal Ricci flow solution starting at some $g_0 \in \mathcal{G}_k$. For any $t \geq 0$ there exist $B(t) > \beta(t) > 0$ such that*

$$\beta(t) s_0^{\frac{1}{k+1}} \leq b(s_0, t) \leq B(t) s_0^{\frac{1}{k+1}}.$$

Moreover, we have $m_{g(t)} = m_{g_0}$ for all $t \geq 0$.

Proof. Suppose for a contradiction that there exist a sequence p_j and t_0 such that $s_0(p_j) \to \infty$ and $b(p_j, t_0)(s_0(p_j))^{-\frac{1}{k+1}} \to 0$. Then, from the decay of the curvature in Lemma 3.17

we get

$$\frac{\log\left(\frac{b(p_j,t_0)}{b(p_j,0)}\right)}{t_0} \leq \alpha(t_0)s_0^{-\frac{2}{k+1}}.$$

Since by integrating (3.7) we see that $b(s_0,0) \geq \beta_0 s_0^{\frac{1}{k+1}}$ for s_0 large enough, the contradiction follows. Similar arguments work for the upper bound while for the conservation of mass the proof is the same as in the asymptotically flat case (see Lemma 3.10). □

Remark 3.6. *According to Lemma 3.17 and Corollary 3.18 we deduce that for any $t \geq 0$ there exists some positive constant $\alpha(t)$ such that $b^2|Rm|(\cdot,t) \leq \alpha(t)$ on the time-slice $\mathbb{R}^4 \times \{t\}$.*

Next, we show that the first order derivatives are uniformly bounded in the space-time. Since we cannot a priori control the behaviour of b_s at spatial infinity on any time-slice, the proof requires an extra step when compared to its asymptotically flat counterpart. We recall that by Lemma 3.7 the derivatives b_s and c_s are positive as soon as the flow starts.

Lemma 3.19. *Let $(\mathbb{R}^4, g(t))_{t\geq 0}$ be the maximal Ricci flow solution starting at some $g_0 \in \mathcal{G}_k$. Then*

$$\sup_{\mathbb{R}^4\times[0,\infty)} (b_s + c_s) < \infty.$$

Proof. Since by Lemmas 3.7 and 3.18 c is uniformly bounded and spatially increasing, we see that $c_s(\cdot,t)$ is integrable for all $t \geq 0$. Moreover, from (2.11) we derive that $|c_{ss}| \leq \alpha(t)c(s,t) \leq \alpha(t)m_{g_0}^{-1}$ in the space-time being the flow smooth for all positive times. Therefore $c_s(s,t) \to 0$ as $s \to \infty$ for all $t \geq 0$. We can then argue exactly as in Lemma 2.14 to prove that c_s is uniformly bounded.
For what concerns b_s, we note that the evolution equation (2.24) can be written as

$$\partial_t\, b_s = \Delta b_s + \frac{1}{b^2}\left(b_s(4 - b_s^2 - (c_s u^{-1})^2 - 6u^2 - 2b_{ss}b) + 4c_s u\right).$$

From the boundary conditions and the curvature being bounded we derive that given $t_0 > 0$ there exist r_{t_0} and $\alpha(t_0)$ positive such that

$$\partial_t\, b_s \leq \Delta b_s + \alpha(t_0),$$

in $B_{g_0}(\boldsymbol{o}, r_{t_0}) \times [0, t_0]$. Since the curvature is uniformly bounded for all times in $[0, t_0]$, in the complement region $\mathbb{R}^4 \setminus B_{g_0}(\boldsymbol{o}, r_{t_0}) \times [0, t_0]$ we can rely on Corollary 2.9 and Lemma 2.11 to bound the evolution equation of b_s by

$$\partial_t\, b_s \leq \Delta b_s + \frac{1}{b^2}\left(4b_s + 4c_s u\right) - 2\frac{b_s b_{ss}}{b} \leq \Delta b_s + \frac{\alpha(t_0)}{b} - 2\frac{b_s b_{ss}}{b} \leq \Delta b_s + \frac{\alpha(t_0)}{b} + \alpha(t_0) b_s,$$

where we have also used that $|k_{01}|$ is uniformly bounded for $0 \leq t \leq t_0$. Finally, since b is monotone and the flow is smooth we can combine the estimates in the two space-time regions and conclude that there exists $\alpha(t_0) > 0$ such that

$$\partial_t\, b_s \leq \Delta b_s + \alpha(t_0)(1 + b_s),$$

in $\mathbb{R}^4 \times [0, t_0]$. Thus, since b_s is exponentially bounded as we derive from $|k_{01}|$ being bounded, we may apply the maximum principle in [Chow et al., 2008, Theorem 12.14] to deduce that for any $t_0 > 0$ there exists $A_{t_0} > 0$ such that

$$b_s(\cdot, t) \leq \sup_{\mathbb{R}^4} b_s(\cdot, 0) + A_{t_0}$$

in $\mathbb{R}^4 \times [0, t_0]$. From Shi's derivative estimates [Chow et al., 2008, Theorem 14.13] and the decay of the curvature in Lemma 3.17 it follows that $|\nabla \mathrm{Rm}|(s_0, t) = \mathcal{O}(s_0^{-2/k+1})$ for all $t > 0$. Therefore, from the commutator formula we get

$$|\partial_t\, b_s| = |\partial_s(-\mathrm{Ric}_{11} b) + \mathrm{Ric}_{ss} b_s| \leq \alpha\left(|\nabla \mathrm{Rm}| b + |\mathrm{Rm}| b_s\right).$$

Since we have previously shown that $b_s(\cdot, t)$ is bounded on any time-slice we may apply Corollary 3.18 and derive that for any $T' > 1$ there exists $\alpha(T')$ such that

$$|\partial_t\, b_s| \leq \alpha(T')(s_0 + 1)^{-\frac{1}{k+1}},$$

in $\mathbb{R}^4 \times [1, T']$. Therefore we have proved that for any $\varepsilon > 0$ and for any $t > 1$ there exists $r(t, \varepsilon)$ such that

$$b_s(s_0, t) \leq \Big(\sup_{\mathbb{R}^4 \times [0,1]} b_s\Big) + \varepsilon,$$

whenever $s_0 \geq r(t, \varepsilon)$. Once we know that b_s is uniformly bounded at spatial-infinity on any time-slice we can rely on the same argument in Lemma 2.14 to prove that in fact b_s is uniformly bounded everywhere in the space-time. $\square$

Thanks to Corollary 3.18 and Lemma 3.19 we can immediately extend the rotational symmetry type of bounds to Ricci flow solutions starting in $\mathcal{G}_k$. Namely, we have:

Corollary 3.20. *If $(\mathbb{R}^4, g(t))_{t\geq 0}$ is the maximal Ricci flow solution evolving from some $g_0 \in \mathcal{G}_k$, then the following conditions hold:*

$$\sup_{\mathbb{R}^4\setminus\{o\}\times[0,+\infty)} \frac{1}{c}(b_s^2 - 4) < \infty,$$
$$\sup_{\mathbb{R}^4\times[0,+\infty)} \frac{1}{c}(1 - u) < \infty,$$
$$\sup_{\mathbb{R}^4\times[0,+\infty)} \frac{1}{c}|c_s - ub_s| < \infty.$$

Similarly, the decay of the curvature being preserved as in Lemma 3.17 and the control on the asymptotic behaviour of the warping coefficients as in Corollary 3.18 ensure that second order estimates analogous to the asymptotically flat case still hold for Ricci flows evolving from initial data in $\mathcal{G}_k$. For example, since $b_{ss}/b = \mathcal{O}(s_0^{-2/k+1})$ we see that $b_{ss}c$ decays as $s_0^{-1/k+1}$ and one can hence apply maximum principle arguments as in Lemma 3.13 once we know that the first order derivatives are uniformly bounded. In particular, the curvature of the solution is again controlled by the size of the Hopf-fiber:

Corollary 3.21. *If $(\mathbb{R}^4, g(t))_{t\geq 0}$ is the maximal Ricci flow solution starting at some $g_0 \in \mathcal{G}_k$, then*

$$\sup_{\mathbb{R}^4\times[0,+\infty)} \left(c^2|\mathrm{Rm}_{g(t)}|_{g(t)}\right) < \infty.$$

Next, we prove that for Ricci flows starting in $\mathcal{G}_k$ the quantity $b_s u^{-1}$ is controlled from below in any region where u becomes degenerate exactly as for the asymptotically flat case. If a Ricci flow solution in $\mathcal{G}_k$ has curvature bounded uniformly in time, then such estimate implies that any infinite-time singularity model must open up along the S^2-direction faster than a paraboloid in $\mathbb{R}^3$. However, differently from the asymptotically flat case, for solutions in $\mathcal{G}_k$ we need a preliminary bound to make sure that $b_s u^{-1}$ does indeed diverge at spatial infinity on any time-slice.

Lemma 3.22. *Let $(\mathbb{R}^4, g(t))_{t\geq 0}$ be the maximal Ricci flow solution starting at some $g_0 \in \mathcal{G}_k$ and let $k < \bar{k} < 1$ and $\delta \in (0, 1-\bar{k})$. For any $t \geq 0$ there exists $\alpha(t) > 0$ such that:*

$$u^{-\delta}\left(b_s u^{-\bar{k}} - 1\right) \geq -\alpha(t) > -\infty.$$

Proof. We set $F_{\bar{k},\delta} \doteq u^{-\delta}(b_s u^{-\bar{k}} - 1)$. By the boundary conditions we see that $F_{\bar{k},\delta}(\boldsymbol{o}, t) = 0$ for all $t \geq 0$. Moreover, from the definition of $\mathcal{G}_k$ it follows that $F_{\bar{k},\delta}(s_0, 0) \to \infty$ as $s_0 \to \infty$, meaning that $\inf F_{\bar{k},\delta}(\cdot, 0) > -\infty$. We now argue as for the proof of Lemma 3.19. First, the evolution equation of $F_{\bar{k},\delta}$ is given by

$$\begin{aligned}
\partial_t F_{\bar{k},\delta} &= \Delta F_{\bar{k},\delta} - 2\delta u^{-\delta}\left(\frac{b_s}{b} - \frac{c_s}{c}\right)\left(b_{ss}u^{-\bar{k}} + \bar{k}b_s u^{-\bar{k}}\left(\frac{b_s}{b} - \frac{c_s}{c}\right)\right)\\
&\quad + u^{-\delta}\left(-2(\bar{k}+1)\frac{b_s b_{ss} u^{-\bar{k}}}{b} + 2\bar{k}\frac{c_s b_{ss} u^{-\bar{k}}}{c}\right)\\
&\quad + \frac{u^{-\delta-\bar{k}}}{b^2}\left(b_s\left(4(1-\bar{k}) - (1+\bar{k}^2)(b_s^2 + \left(\frac{c_s}{u}\right)^2) + 2\bar{k}^2 b_s \frac{c_s}{u} + u^2(4\bar{k} - 6)\right) + 4c_s u\right)\\
&\quad + \frac{F_{\bar{k},\delta}}{b^2}\left(-\delta^2(b_s - c_s u^{-1})^2 - 4\delta(1-u^2)\right).
\end{aligned}$$

Since the curvature is bounded, from the boundary conditions one can check that for any $t_0 > 0$ there exists $r_{t_0} > 0$ and $\alpha(t_0) > 0$ such that

$$\partial_t F_{\bar{k},\delta} \geq \Delta F_{\bar{k},\delta} - \alpha(t_0),$$

in $B_{g_0}(\boldsymbol{o}, r_{t_0}) \times [0, t_0]$. For analysing the terms in the evolution equation for radii larger than r_{t_0} we first note that by Corollary 3.18

$$\frac{u^{-\delta-\bar{k}}}{b} = \mathcal{O}\left(s_0^{-\frac{1}{k+1}(1-\delta-\bar{k})}\right), \tag{3.18}$$

which hence decays at spatial infinity on any time-slice because $\bar{k} + \delta < 1$. From (3.18) and Lemma 3.17 we derive that $u^{-\bar{k}-\delta}|b_{ss}|$ decays at spatial infinity as long as the solution exists. Since by Lemma 3.19 the first derivatives b_s and c_s are bounded, we find that all the second order terms in the evolution equation of $F_{\bar{k},\delta}$ decay to zero at the rate given in (3.18) for all $t \geq 0$. Similarly, from Lemma 3.17 and Corollary 3.18 we see that

$b^2|\mathrm{Rm}|(\cdot,t)$ is bounded on any time-slice, meaning that $|b_s c_s u^{-1}| \leq \alpha(t)$. Thus any term of the form $b_s c_s u^{-\bar{k}-\delta}$ decays at the same rate given by (3.18). To sum up, in the region $\mathbb{R}^4 \setminus B_{g_0}(\boldsymbol{o}, r_{t_0}) \times [0,t_0]$ we can then write the evolution equation of $F_{\bar{k},\delta}$ as

$$\partial_t F_{\bar{k},\delta} \geq \Delta F_{\bar{k},\delta} - \alpha(t_0) + \frac{F_{\bar{k},\delta}}{b^2}\left(-\delta^2(b_s - c_s u^{-1})^2 - 4\delta(1-u^2)\right),$$

for some $\alpha(t_0)$. Finally, we note that

$$\begin{aligned}\frac{F_{\bar{k},\delta}}{b^2}\left(-\delta^2(b_s - c_s u^{-1})^2 - 4\delta(1-u^2)\right) &\geq \frac{b_s u^{-\bar{k}-\delta}}{b^2}\left(-\delta^2(b_s - c_s u^{-1})^2 - 4\delta(1-u^2)\right) \\ &\geq \frac{b_s u^{-\bar{k}-\delta}}{b^2}\left(-\delta^2(b_s^2 + (c_s u^{-1})^2) - 4\delta\right),\end{aligned}$$

and the last terms are again bounded away from the origin on any time-slice as observed above. Therefore, for any $t_0 > 0$ there exists $\alpha(t_0) > 0$ such that

$$\partial_t F_{\bar{k},\delta} \geq \Delta F_{\bar{k},\delta} - \alpha(t_0).$$

From the maximum principle [Chow et al., 2008, Theorem 12.14] we conclude that

$$F_{\bar{k},\delta}(\cdot,t) \geq \inf_{\mathbb{R}^4} F_{\bar{k},\delta}(\cdot,0) - \alpha(t) > -\infty,$$

for all positive times. □

We finally need to check that the spatial derivative c_s decays at some rate in any space-time region where u is small.

Lemma 3.23. *Let $(\mathbb{R}^4, g(t))$ be the maximal Ricci flow solution evolving from some $g_0 \in \mathcal{G}_k$. For any $\hat{k} \in (k,1)$ we have*

$$\sup_{\mathbb{R}^4 \times [0,\infty)} \left(c_s u^{-1+\hat{k}}\right) < \infty.$$

Proof. Given $\hat{k} > k$, let $\bar{k} \in (k,\hat{k})$ and δ be defined so that Lemma 3.22 holds. As observed in Remark 3.6, from (2.8) we see that for any $t \geq 0$ there exists $A(t) > 0$ such that $|b_s c_s u^{-1}| \leq A(t)$. Thus, by Lemma 3.22 we get

$$A(t) \geq b_s c_s u^{-1} \geq c_s(-\alpha(t)u^{\delta} + 1)u^{-1+\bar{k}}.$$

Therefore, we find that $\lim_{s_0 \to \infty}(c_s u^{-1+\hat{k}})(s_0, t) = 0$ for all $t \geq 0$. The evolution equation of $c_s u^{-\tilde{k}}$, with $\tilde{k} \in (0,1)$, at any positive maximum is given by

$$\partial_t \left(c_s u^{-\tilde{k}}\right)|_{\max} \leq \frac{c_s u^{-\tilde{k}}}{b^2}\left(b_s^2(\tilde{k}^2 - 2) - 4\tilde{k} + u^2(-6 + 4\tilde{k}) + (c_s u^{-1})^2(\tilde{k}^2 - 2\tilde{k})\right) + \frac{1}{b^2}\left(c_s u^{-\tilde{k}}\left(b_s c_s u^{-1}(-2\tilde{k}^2 + 2\tilde{k})\right) + 8u^{3-\tilde{k}} b_s\right)$$

Since $\tilde{k} < 1$ and b_s is uniformly bounded by Lemma 3.19, we get

$$\partial_t \left(c_s u^{-\tilde{k}}\right)|_{\max} \leq \frac{1}{b^2}\left(c_s u^{-\tilde{k}}\left(-4\tilde{k} + (c_s u^{-1})^2(\tilde{k}^2 - 2\tilde{k}) + b_s c_s u^{-1}(-2\tilde{k}^2 + 2\tilde{k})\right) + \alpha\right).$$

Finally, if the value of the maximum is large enough, then we find that

$$(c_s u^{-1})^2(\tilde{k}^2 - 2\tilde{k}) + b_s c_s u^{-1}(-2\tilde{k}^2 + 2\tilde{k}) \leq \tilde{k} c_s u^{-1}(-c_s u^{-1} + 2b_s) < 0.$$

Thus, we have shown that $\partial_t(c_s u^{-\tilde{k}}) < 0$ at any maximum value large enough. That completes the proof. □

We may now complete the section by noting that, as in the asymptotically flat case, for any Ricci flow in $\mathcal{G}_k$ the warping coefficient in the directions orthogonal to the Hopf-fiber grows faster than a paraboloid in $\mathbb{R}^3$.

Lemma 3.24. *Let $(\mathbb{R}^4, g(t))_{t \geq 0}$ be the maximal Ricci flow solution starting at some $g_0 \in \mathcal{G}_k$. There exist $\alpha, \lambda > 0$ such that*

$$b^{\lambda}(b_s u^{-1} - \log(b)) \geq -\alpha,$$

uniformly in the space-time. Moreover, there exists $\beta > 0$ such that

$$\inf_{\mathbb{R}^4} \left(b_s u^{-1}\right)(\cdot, t) \geq \beta,$$

for all times $t \geq 1$.

Proof. Given $\chi \doteq b^{\lambda}(b_s u^{-1} - \log(b))$, we note that χ vanishes at the origin for all times and that, according to Lemma 3.22, $\chi(s_0, t) \to \infty$ as $s_0 \to \infty$ for all positive times. Thus we can consider the evolution equation of χ at some negative minimum point (p_0, t_0) and arguing as in the proof of Lemma 3.15 for the asymptotically flat setting, we deduce that whenever $|\chi_{\min}|$ is large enough, then the evolution equation of χ satisfies (3.17). Namely, we have

$$\partial_t \chi(p_0, t_0)|_{\chi_{\min}} \geq \frac{1}{b^2}\left(\lambda|\chi_{\min}| + b^{\lambda} b_s u^{-1}(-4 b_s c_s u^{-1} - 2u^2) + b^{\lambda}\right)$$

From $\chi(p_0, t_0) < 0$ we derive

$$b^{\lambda}(1 - 4b_s^2 c_s u^{-2}) \geq b^{\lambda}(1 - 4(\log(b))^2 c_s).$$

If we pick $\hat{k} \in (k, 1)$ and $\alpha_{\hat{k}}$ so that Lemma 3.23 holds, then the last term can be bounded as:

$$b^{\lambda}(1 - 4(\log(b))^2 c_s) \geq b^{\lambda}(1 - 4\alpha_{\hat{k}}(\log(b))^2 u^{1-\hat{k}}),$$

which is positive whenever $|\chi_{\min}|$ and hence u^{-1} are large enough. Therefore, if we let $\lambda \leq 1$, then we obtain:

$$\partial_t \chi(p_0, t_0)|_{\chi_{\min}} > 0.$$

Once we know that χ is uniformly bounded from below in the space-time, one can argue exactly as in the positive mass-case of Lemma 3.16. □

3.5 Compactness of warped Berger Ricci flows

In this section we show that a class of complete, bounded curvature warped Berger Ricci flows with monotone coefficients is compact under the pointed Cheeger-Gromov topology. The main step in the argument consists in proving that the Killing vectors generating the $U(2)$-symmetry pass to the limit without becoming degenerate. Once we know that the Cheeger-Gromov limit is a warped Berger Ricci flow, the control on the curvature and the monotonicity conditions allow us to prove smooth convergence of the warping functions b and c up to diffeomorphisms. The next result plays a central role in the convergence argument because it allows to work directly at the level of the metric coefficients. In particular, this will be of importance when classifying ancient solutions to the Ricci flow because whenever the compactness result holds one can always apply maximum principle arguments to geometric quantities by passing to pointed Cheeger-Gromov limits where critical values are indeed achieved in the space-time.

Proposition 3.25. *Let $(\mathbb{R}^4, g_j(t), p_j)_{t\in I}$ be a sequence of pointed complete warped Berger solutions to the Ricci flow with monotone coefficients defined on $I \ni 0$. If*

$$\sup_j \left(\sup_{\mathbb{R}^4\times I} |\mathbf{Rm}_{g_j(t)}|_{g_j(t)} \right) < \infty, \tag{3.19}$$

$$\sup_j \left(\sup_{\mathbb{R}^4\times I} (b_j)_s + (c_j)_s \right) < \infty, \tag{3.20}$$

$$\liminf_{j\to\infty} (b_j(p_j, 0)) + \liminf_{j\to\infty} ((b_j)_s(p_j, 0)) > 0, \tag{3.21}$$

$$\limsup_{j\to\infty} \; (b_j(p_j, 0)) < \infty, \tag{3.22}$$

$$\liminf_{j\to\infty} \; (u_j(p_j, 0)) > 0, \tag{3.23}$$

then $(\mathbb{R}^4, g_j(t), p_j)$ subsequentially converges in the pointed Cheeger-Gromov sense to a complete $U(2)$ invariant Ricci flow solution $(M_\infty, g_\infty(t), p_\infty)_{t\in I}$ satisfying:

(i) $M_\infty = \mathbb{R}^4$ *or* $M_\infty = \mathbb{R}\times S^3$.

(ii) There exist warping coefficients $\xi_\infty, b_\infty, c_\infty$ such that $g_\infty(t)$ can be written as

$$g_\infty(t) = \xi_\infty(r_\infty, t)dr_\infty^2 + b_\infty^2(r_\infty, t)\,(\sigma_1^2 + \sigma_2^2) + c_\infty^2(r_\infty, t)\,\sigma_3^2,$$

where $r_\infty(\cdot) = d_{g_\infty(0)}(\boldsymbol{o}_\infty, \cdot)$ if $M_\infty = \mathbb{R}^4$, and $r_\infty(\cdot) = d_{g_\infty(0)}(\Sigma_{p_\infty}, \cdot)$, with Σ_{p_∞} the principal orbit passing through p_∞, if $M_\infty = \mathbb{R}\times S^3$.

(iii) There exist radial functions s_j such that $\xi_j(s_j, t)$, $b_j(s_j, t)$, $c_j(s_j, t)$ converge smoothly on compact sets to $\xi_\infty(r_\infty, t)$, $b_\infty(r_\infty, t)$, $c_\infty(r_\infty, t)$ respectively.

Remark on the assumptions. The uniform bound on the curvature and the monotonicity of the coefficients guarantee that Hamilton's compactness result can be applied to the sequence of solutions. The control on the first order derivatives along with (3.22) ensure that the Killing vectors are bounded in any geodesic ball. Proposition 3.25 has a counterpart for warped Berger Ricci flows on the blow up of $\mathbb{C}^2/\mathbb{Z}_k$ satisfying a few first and second-order estimates Appleton [2019]. However, for these topologies the compactness property for complete solutions is formulated in the case where the Ricci flows are κ-non-collapsed at some sequence of scales diverging to infinity so that the resulting limit is κ-non-collapsed (for all scales) [Appleton, 2019, Corollary 8.2]. Such assumption is natural when studying *finite-time singularity models* for the Ricci flow, which arise via a blow-up procedure. In fact, Perelman proved that any finite-time singularity model of the Ricci flow is κ-non-collapsed. On the other hand, this assumption is not available in our setting for we are interested in proving long-time convergence of the Ricci flow to *infinite-time singularity models* which are, in the case of the Taub-NUT metric, collapsed for some sufficiently large scale. The latter represents a key difference and accounts for the conditions (3.21) and (3.23), which ensure that the Killing vectors do not become degenerate when passing to the limit.

Remark 3.7. *We point out that the structure of the proof below mainly follows from adapting the analysis in Section 2.5 and especially the analogous local result in Appleton [2019]. Since the compactness property derived in Appleton [2019] relies on a different set of assumptions and works on the topology of the blow-up of $\mathbb{C}^2/\mathbb{Z}_k$, we still present a full argument in detail.*

Proof. First, we recall that the monotonicity of the warping functions implies that $(\mathbb{R}^4, g_j(0))$ does not have closed geodesics. Since the curvature is uniformly bounded, we see that $\inf_j \text{inj}(g_j(0)) > 0$ and we may then apply Hamilton's compactness theorem and extract a subsequence converging in the pointed Cheeger-Gromov sense to a Ricci flow solution $(M_\infty, g_\infty(t), p_\infty)_{t\in I}$.

In the following we denote by Φ_j the diffeomorphisms given by the Cheeger-Gromov convergence. We also observe that by (3.22) we can rely on the same argument in Lemma 1.16 to prove that the limit manifold is *simply connected*.

Consider the Killing vector fields $\{Y_1, Y_2, Y_3, X_3\}$ generating the $U(2)$ symmetries. Since we have

$$c_j^2(\cdot,t)g_{S^3} \leq b_j^2(\cdot,t)(\sigma_1^2+\sigma_2^2)+c_j^2(\cdot,t)\sigma_3^2 \leq b_j^2(\cdot,t)g_{S^3},$$

with g_{S^3} the bi-invariant constant curvature 1 metric on the 3-sphere, and $\{Y_i\}$ are orthonormal with respect to g_{S^3}, we deduce that

$$c_j(\cdot,t) \leq |Y_i|_j(\cdot,t) \leq b_j(\cdot,t), \tag{3.24}$$

for all $t \in I$ and for $i = 1,2,3$. Let $\nu > 0$ and $q \in B_{g_j(0)}(p_j,\nu)$. From the conditions (3.20) and (3.22) we derive that there exists $\alpha > 0$ such that

$$b_j(q,0) \leq b_j(p_j,0) + \left(\sup_{B_{g_j(0)}(p_j,\nu)} (b_j)_s\right)\nu \leq \alpha(1+\nu).$$

We may then extend such bounds to other times by using (3.19), for given $t \in I$ we have

$$|\partial_t \log b_j|(\cdot,t)| \leq |\mathrm{Ric}_j|_j(\cdot,t) \leq \alpha.$$

Therefore, for any $\nu > 0$ and $t \in I$ there exists a constant $\alpha = \alpha(t,\nu)$ such that

$$\sup_{B_{g_j(0)}(p_j,\nu)} |Y_i|_j(\cdot,t) \leq \alpha(t,\nu), \tag{3.25}$$

for all $i = 1,2,3$. The Killing equation also implies

$$|\nabla_j^2 Y_i|_j(\cdot,t) \leq \alpha|Y_i|_j|\mathrm{Rm}_j|_j(\cdot,t),$$

for all times $t \in I$. By the Cheeger-Gromov convergence we deduce that, up to passing to a subsequence, there exist C^1-limits $\{Y_{i,\infty}\}$ defined on $B_{g_\infty(0)}(p_\infty,1)$. The C^1-convergence implies that $\{Y_{i,\infty}\}$ are $g_\infty(t)$-Killing vector fields. We observe that in Lemma 2.33 we needed C^3 bounds because we did consider left-invariant vector fields that are not Killing vectors along the sequence. We may then proceed as in Lemma 2.33 to derive that $\{Y_{i,\infty}\}$ extend to smooth Killing vectors on M_∞. Similar conclusions apply to X_3, which converges to a $g_\infty(t)$-Killing vector $X_{3,\infty}$ up to pulling back by Φ_j.

We now show that the Killing vectors are not degenerate. Namely, we have

Claim 3.26. *$U(2)$ acts on $(M_\infty, g_\infty(t))$ with cohomogeneity-1.*

Proof of Claim 3.26. We first show that the limit Killing vectors are not trivial.

If there exists $\varepsilon > 0$ such that $c_j(p_j, 0) \geq \varepsilon$ along a subsequence, then from (3.24) we see that $Y_{i,\infty}$ do not vanish at p_∞.

We may then assume that $c_j(p_j, 0) \to 0$. According to (3.23) we also have $b_j(p_j, 0) \to 0$. Therefore, from (3.21) it follows that $(b_j)_s(p_j, 0) \geq 2\beta > 0$, for some constant β independent of j. Since the curvature is uniformly bounded we see that $|(b_j)_{ss}|(\cdot, 0) \leq \alpha_\rho$ in $B(\boldsymbol{o}, \rho)$, for $\rho < 0$, which implies $(b_j)_s(\cdot, 0) \geq \beta$ in $B_{g_j(0)}(p_j, r)$ for some r small enough. Similarly, by (2.12) and (3.23) we get that $u_j(\cdot, 0)$ is uniformly bounded from below in $B_{g_j(0)}(p_j, \tilde{r})$ for some radius sufficiently small. We can then pick $\hat{r} = \min\{r, \tilde{r}\}$ and conclude that there exist points $q_j \in B_{g_j(0)}(p_j, \hat{r})$ such that $c_j(q_j, 0)$ admits a positive lower bound. Therefore we find $q_\infty \in B_{g_\infty(0)}(p_\infty, \hat{r})$ such that $Y_{i,\infty}$ are not trivial at q_∞.

Once we know that $\{Y_{i,\infty}\}$ are non-trivial, we may deduce that there exists a non-degenerate copy of $\mathfrak{su}[2]$ in the Lie algebra of Killing vector fields $\mathfrak{iso}(M_\infty, g_\infty(t))$ because the Lie brackets pass to the limit. Since the limit is complete we can integrate the Lie algebra action and obtain that $SU(2)$ acts on $(M_\infty, g_\infty(t))$. Consider $q \in M_\infty$ such that $Y_{i,\infty}$ do not vanish at q. Suppose that there exist coefficients α_i such that the vector $Y_\infty = \sum_i \alpha_i Y_{i,\infty}$ vanishes at q. A diagonal argument yields

$$0 = |Y_\infty|^2_{g_\infty(0)}(q) = \lim_{j\to\infty} \left| \sum_i \alpha_i Y_i \right|^2_{g_j(0)} (\Phi_j(q)).$$

As observed above, for any spherical vector field Y we have $|Y|_{g_j(0)}(p) \geq c_j(p, 0)|Y|_{g_{S^3}}$. Since the right-invariant vector field Y_i are orthonormal with respect to the round metric on the 3-sphere and $c_j(\Phi_j(q), 0)$ is bounded away from zero being the Killing vectors $Y_{i,\infty}$ non-trivial at q, we conclude that the limit above is zero if and only if $\alpha_i = 0$. Thus, there exists an $SU(2)$-orbit of codimension 1. From the convergence of X_3 to $X_{3,\infty}$ we finally deduce that the limit metric is invariant under a cohomogeneity-one $U(2)$-action. $\square$

Since the limit manifold M_∞ is non-compact and simply connected, from the cohomogeneity one $U(2)$ action we derive that either M_∞ is foliated by principal orbits, in this case $M_\infty = \mathbb{R} \times S^3$, or there exists a singular orbit Σ_{sing}. From now on we assume that the action admits a singular orbit for the case of the cylinder can be dealt with similarly. As discussed in Section 2.2.1, we can diagonalize the Ricci flow limit and use the extra-degree of symmetry provided by the Killing vector field $X_{3,\infty}$ to write the solution as

$$g_\infty(t) = \xi_\infty(r_\infty, t)dr_\infty^2 + b_\infty^2(r_\infty, t)\,(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2) + c_\infty^2(r_\infty, t)\,\sigma_3 \otimes \sigma_3,$$

where $\{\sigma_i\}$ is the coframe dual to the Milnor frame $\{X_{i,\infty}\}$ of left-invariant vectors induced by the right-invariant Killing vectors $\{Y_{i,\infty}\}$, while $r_\infty(\cdot) = d_{g_\infty(0)}(\Sigma_{\text{sing}}, \cdot)$. In particular, we have $c_\infty = |X_{3,\infty}|_\infty$ in the space-time. Thus $c_j \to c_\infty$ on compact sets and one obtain an analogous conclusion for $b_j \to b_\infty$ once the points p_j are chosen of the form $((s_0)_j, e)$, with e the identity in $SU(2)$.

Let $z_\infty \in \Sigma_{\text{sing}}$ and let $z_j = \Phi_j(z_\infty)$. If, for a contradiction, there exists $\varepsilon > 0$ such that $d_{g_j(0)}(\boldsymbol{o}, z_j) \geq 2\varepsilon$, then we could find points $\tilde{z}_j$ satisfying $d_{g_j(0)}(z_j, \tilde{z}_j) = \varepsilon$ with $c_j(\tilde{z}_j, 0) \leq c_j(z_j, 0)$. By the monotonicity of the warping coefficients it follows that on M_∞ there exists a point $\tilde{z}_\infty \in S_{g_\infty(0)}(\Sigma_{\text{sing}}, \varepsilon)$ such that $c_\infty(\tilde{z}_\infty, 0) \leq 0$, which is a contradiction. Thus both b_∞ and c_∞ vanish at Σ_{sing}, meaning that $M_\infty = \mathbb{R}^4$ and hence $\Sigma_{\text{sing}} = \boldsymbol{o}_\infty$. From the same argument we deduce that the radial coordinates

$$s_j(\cdot) \doteq d_{g_j(0)}(\boldsymbol{o}, \Phi_j(\cdot)).$$

converge to r_∞ in C^0 on compact sets.

We know that b_j and c_j converge to b_∞ and c_∞ respectively in C^0 on compact sets. Consider $0 < \delta < D$. Once again by (3.20) we see that $b_j(\cdot, 0)$ is uniformly bounded from above on $B_{g_j(0)}(\boldsymbol{o}, D)$. The cohomogeneity one action implies that b_∞ is bounded away from zero in $B_{g_\infty(0)}(\boldsymbol{o}_\infty, D) \setminus B_{g_\infty(0)}(\boldsymbol{o}_\infty, \delta)$ thus yielding

$$\inf_{B_{g_j(0)}(\boldsymbol{o}, D) \setminus B_{g_j(0)}(\boldsymbol{o}, \delta)} b_j(\cdot, 0) \geq \alpha(\delta, D) > 0.$$

A similar estimate holds for c_j as well. The latter bounds, along with (3.19) and Shi's derivative estimates yield

$$\sup_{B_{g_\infty(0)}(\boldsymbol{o}_\infty,D)\setminus B_{g_\infty(0)}(\boldsymbol{o}_\infty,\delta)} \left|\nabla^k_{g_\infty(0)} s_j\right| \leq \alpha(k,\delta,D) < \infty,$$

for any positive integer k. Therefore $s_j \to r_\infty$ smoothly on compact sets away from the origin. By a similar argument the C^0-convergence of $c_j \circ s_j$ to c_∞ and $b_j \circ s_j$ to b_∞ respectively are in fact smooth on compact sets away from the origin. One can finally check that $\xi_j \circ s_j$ converges smoothly on compact sets away from the origin to $\xi_\infty = |\partial_{r_\infty}|_{g_\infty}$. The boundary conditions at the origin and the uniform bounds on the curvature allow to extend the convergence at the singular orbit $\boldsymbol{o}_\infty$ as well. □

Remark 3.8. *According to the compactness property, whenever a family of warped Berger Ricci flows satisfies the assumptions as in the statement, then the pointed Cheeger-Gromov limit is attained by parametrizing the radial distance function.*

We dedicate the end of this section to proving that as a consequence of the rotational symmetry type of estimates in Lemma 3.11, Corollary 3.20, the scale-invariant lower bounds for the spatial derivative b_s in Lemmas 3.16, 3.24 and the compactness result in Proposition 3.25 any Ricci flow solution considered so far has curvature uniformly bounded in the space-time.

Proposition 3.27. *If $(\mathbb{R}^4, g(t))_{t\geq 0}$ is the maximal Ricci flow solution evolving from some g_0 belonging to either $\mathcal{G}_k$ or $\mathcal{G}_{\mathsf{AF}}$, then*

$$\limsup_{t\nearrow\infty}\left(\sup_{\mathbb{R}^4}|\mathrm{Rm}_{g(t)}|_{g(t)}\right) < \infty.$$

Proof. In the following we provide the details for the case $g_0 \in \mathcal{G}_k$. The proof in the asymptotically flat setting only requires to replace the analysis of Section 3.4 with its counterpart in Section 3.3. Assume for a contradiction that the curvature becomes unbounded. The solution then develops a Type-II(b) singularity. Following Chow and Knopf [2004][Chapter 8] we deduce that there exists a space-time sequence (p_j, t_j), with $t_j \to \infty$, such that for all $\varepsilon > 0$ the Ricci flows $g_j(t) \doteq \lambda_j g(t_j + \lambda_j^{-1} t)$, where

$\lambda_j = |\text{Rm}|(p_j, t_j)$, satisfy

$$\sup_{\mathbb{R}^4 \times I} |\text{Rm}_{g_j(t)}|_{g_j(t)} \leq 1 + \varepsilon, \tag{3.26}$$

for all $j \geq j_0(\varepsilon, I)$, with I some interval. In particular, if the curvature diverges along some sequence of times, then we can choose space time points (p_j, t_j) such that $\lambda_j \nearrow \infty$. We now check that we can apply the compactness result to the sequence of Ricci flows on a given interval $I \ni 0$.

By Lemma 3.7 the sequence of solutions has monotone coefficients. From (3.26) and Lemma 3.19 we also see that (3.19) and (3.20) respectively are satisfied. From Corollary 3.21 it follows that $c(p_j, t_j) \to 0$ because $\lambda_j \to \infty$. Thus, we can use Corollary 3.20 to deduce that $b(p_j, t_j) \to 0$ and hence that

$$b_j^2(p_j, 0) = \lambda_j b^2(p_j, t_j) = \lambda_j c^2(p_j, t_j) u^{-2}(p_j, t_j) \leq \alpha(1 + \alpha b)^2(p_j, t_j) \leq \alpha,$$

for some $\alpha > 0$. Since the roundness ratio is scale invariant and $c(p_j, t_j)$ is converging to zero, we can again apply Corollary 3.20 to derive that $u(p_j, t_j) \to 1$. Finally, Lemma 3.24 implies that $b_s(p_j, t_j) \geq \beta/2$ for j large enough.

Therefore, the assumptions in Proposition 3.25 are satisfied and we can hence apply a diagonal argument and pick a subsequence converging to an ancient Ricci flow solution $(M_\infty, g_\infty(t), p_\infty)_{t \leq 0}$ of the form

$$g_\infty(t) = \xi_\infty(r_\infty, t) dr_\infty^2 + b_\infty^2(r_\infty, t)\,(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2) + c_\infty^2(r_\infty, t)\,\sigma_3 \otimes \sigma_3.$$

The convergence of the warping coefficients and Corollary 3.20 yield:

$$\begin{aligned} \frac{1}{c_\infty}(1 - u_\infty)(q, t) &= \lim_{j\to\infty} \frac{1}{c_j}(1 - u_j)(s_j(q), t) \\ &= \lim_{j\to\infty} \frac{1}{\sqrt{\lambda_j} c}(1 - u)(s_j(q), t_j + \lambda_j^{-1} t) \leq \lim_{j\to\infty} \frac{\alpha}{\sqrt{\lambda_j}} = 0, \end{aligned}$$

for any $q \in M_\infty$ and $t \leq 0$. Therefore $(M_\infty, g_\infty(t))$ is a κ-non collapsed - being the limit of a blow-up sequence - rotationally symmetric ancient solution to the Ricci flow and hence a κ-solution by Zhang [2008]. In particular, from Lemma 3.24 we get

$$(b_\infty)_s(q,t) = \lim_{j\to\infty}(b_s u^{-1})(s_j(q), t_j + \lambda_j^{-1}t) \geq \beta > 0,$$

where we have used that by the rotational symmetry of the limit the scale-invariant quantity $u(s_j(\cdot), t)$ is converging to 1. We conclude that $b_\infty = c_\infty$ grow at least linearly and hence that the limit ancient Ricci flow is a non-compact κ-solution with positive asymptotic volume ratio. A rigidity result of [Perelman, 2002, Proposition 11.4] implies that $g_\infty(t)$ needs to be flat, which contradicts the choice of the factors λ_j.

□

3.6 Ancient solutions opening faster than a paraboloid

In this section we prove that the only complete, warped Berger ancient solution with monotone coefficients, curvature uniformly bounded in the space-time, bounded Hopf-fiber and opening faster than a paraboloid in the directions orthogonal to the Hopf-fiber is the Taub-NUT metric. The main idea consists in showing that for any ancient solution belonging to this class, the warping coefficient b is actually a linear function of the distance in any region where the squashing factor u is small. In other words, the ancient solution behaves exactly as the Taub-NUT metric along the S^2-directions whenever b is large. Once such control is available, one can consider scale-invariant first-order quantities derived from the hyperkähler odes (3.3), (3.4), and prove that they have a sign on the ancient solution, the aim being to finally show that (3.4) is in fact satisfied in the space-time.

It is worth outlining a strategy we often use in the following which was adopted in Appleton [2019] to prove a uniqueness result for the Eguchi-Hanson metric, with a substantial difference given by the assumption of κ-non-collapsedness as we describe below.

Suppose that we are given a geometric scale-invariant quantity $\mathcal{L}$ and that we want to prove that $\mathcal{L} \geq 0$ in the space-time $\mathbb{R}^4 \times (-\infty, 0]$. Once we know that $\mathcal{L}$ is uniformly bounded and that, say, $\partial_t \mathcal{L} > 0$ at any negative minimum, one might try to apply a maximum principle argument to the evolution equation of $\mathcal{L}$. In general though, the infimum of $\mathcal{L}$ may not be achieved. In this case, one could consider a space-time sequence (p_j, t_j)

such that $\mathcal{L}(p_j, t_j) \to \inf \mathcal{L} < 0$ and define the Ricci flow sequence $g_j(t) = g(t_j + t)$, centred at p_j. *If* the compactness result in Proposition 3.25 holds, then on the limit ancient Ricci flow the analogous quantity $\mathcal{L}_\infty$ achieves its (negative) infimum in the space-time, thus allowing to rely on maximum principle arguments to obtain the contradiction.

This is exactly the strategy used in Appleton [2019]. However in that case the ancient solutions analysed are all *non*-collapsed and hence the roundness ratio is *a priori* controlled from below away from the singular orbit uniformly in time, meaning that the compactness property can be applied for any sequence of Ricci flows as above. On the contrary, in our *collapsed* setting we always need to verify that the squashing factor u stays away from zero along the space-time sequence (p_j, t_j) used to approximate the infimum of $\mathcal{L}$ so that the compactness result can indeed be used.

The latter represents the main difficulty when analysing the collapsed case and the requirement on the ancient solution to open up faster than a paraboloid along the S^2-directions allows us to bypass this issue. In fact, the application of the compactness result in Proposition 3.25 yields that for any ancient solution opening faster than a paraboloid the hyperkähler quantity J_2 given in (3.4) is nonnegative. On the gradient steady soliton found by Appleton instead we find that J_2 approaches its infimum -2 at spatial infinity on any time-slice, thus along space-time sequences where the squashing factor u becomes degenerate.

In order to ease the notations, we give the following:

Definition 3.7. *Let $m > 0$. The class $\mathcal{A}$ consists of all complete, warped Berger ancient solutions to the Ricci flow with monotone coefficients and curvature uniformly bounded in the space-time, satisfying*

$$\inf_{\mathbb{R}^4 \times (-\infty, 0]} \frac{b_s u^{-1}}{f(u^{-1})} > 0, \tag{3.27}$$

$$\sup_{\mathbb{R}^4 \times (-\infty, 0]} c = m^{-1}, \tag{3.28}$$

for some continuous positive function f such that $f(z) \to \infty$ as $z \to \infty$.

Remark 3.9. *We again point out that* (3.27) *means that the warping coefficient b opens* *faster than a paraboloid in* $\mathbb{R}^3$ *on any time-slice. In particular, the volume of geodesic balls* $B_{g(t)}(\boldsymbol{o}, r)$ *grows faster than* r^2*. However,* a priori *there is no upper bound for the volume growth.*

We start by proving that the first order derivatives are uniformly bounded. In fact, from the following estimate we also derive that c_s decays at some rate in any space-time region where the squashing factor is small.

Lemma 3.28. *If* $(\mathbb{R}^4, g(t))_{t \le 0}$ *is an ancient solution in* $\mathcal{A}$*, then*

$$2b_s + c_s u^{-1} - 4 \le 0.$$

Proof. Let h denote the quantity $2b_s + c_s u^{-1} - 4$. By the boundary conditions we see that $h(\boldsymbol{o}, t) = -1$. Thus, if h is positive somewhere in the space-time, then there exist $p \in \mathbb{R}^4$ and $t \le 0$ such that $h(p,t) > 0$ and $h_s(p,t) > 0$. The latter condition implies

$$\left(2b_{ss} + c_{ss}u^{-1} + c_s\left(\frac{b_s}{c} - \frac{c_s u^{-1}}{c}\right)\right)(p,t) > 0.$$

Therefore, the scalar curvature (2.13) is bounded from above as follows:

$$R(p,t) < \frac{2}{b^2}\left(-b_s c_s u^{-1} - (c_s u^{-1})^2 + 4 - u^2 - b_s^2\right)(p,t).$$

Since $h(p,t) > 0$, we get

$$R(p,t) < \frac{2}{b^2}\left(-4 - \frac{3}{4}(c_s u^{-1})^2 + 4 - u^2\right)(p,t) < 0.$$

However, according to Chen [2009] any complete ancient solution to the Ricci flow has nonnegative scalar curvature. We conclude that $h \le 0$ in the space-time. □

Since we aim to prove that $\mathsf{J}_2 \ge 0$ for any ancient solution in $\mathcal{A}$, let us consider the evolution equation of $\mathsf{J}_2 = b_s + u - 2$ at any negative minimum:

$$\begin{aligned}\partial_t \mathsf{J}_2|_{\min<0} \ge \frac{1}{b^2}&\left(-(c_s u^{-1})^2(2 + \mathsf{J}_2) + c_s(8 + 4\mathsf{J}_2)\right) \\ &+ \frac{1}{b^2}\left(-\mathsf{J}_2^3 - 6\mathsf{J}_2^2 - 8\mathsf{J}_2 - 3\mathsf{J}_2 u^2 - 6u^2\right).\end{aligned}$$

We simplify the evolution equation by introducing $z = \mathsf{J}_2 + 2 > 0$. Then we obtain

$$\partial_t \, \mathsf{J}_2|_{\min<0} \geq -\frac{z}{b^2}\left((c_s u^{-1})^2 - 4c_s + z^2 - 4 + 3u^2\right). \tag{3.29}$$

In order to prove that the c_s-quadratic is always negative along an ancient solution in $\mathcal{A}$, one needs to control c_s in terms of b_s and u in a precise way everywhere in the space-time. To this aim, we first show that c_s decays to zero at the same rate given by the hyperkähler quantity (3.3) for any solution in $\mathcal{A}$.

Lemma 3.29. *If* $(\mathbb{R}^4, g(t))_{t\leq 0}$ *is an ancient solution in* $\mathcal{A}$*, then*

$$c_s - 2u^2 \leq 0.$$

Proof. Let $\psi \doteq c_s - 2u^2$ and let $\Psi_\infty \doteq \sup_{\mathbb{R}^4\times(-\infty,0]} \psi$. From Lemma 3.28 it follows that Ψ_∞ is bounded and we can hence assume for a contradiction that $\Psi_\infty > 0$. By direct computation we check that the evolution equation of ψ at any positive maximum is given by

$$\partial_t \, \psi|_{\max} \leq \frac{1}{b^2}\left(-c_s\left(6u^2 + 2b_s^2\right) + 8b_s u^3 + u^2\left(8b_s^2 - 8b_s c_s u^{-1} - 16(1-u^2)\right)\right).$$

Since $c_s > 2u^2$ at any positive maximum of ψ, we find

$$\begin{aligned}\partial_t \, \psi|_{\max} &< \frac{1}{b^2}\left(4b_s^2 u^2 - 8b_s u^3 - 16u^2 + 4u^4\right) \\ &\leq \frac{4u^2}{b^2}\left(b_s^2 - 2b_s u - 4 + u^2\right).\end{aligned}$$

Finally, we note that the quadratic on the right hand side is always negative because $0 \leq b_s \leq 2$ by Lemma 3.28. Therefore, we have shown that $\partial_t \psi < 0$ at any positive maximum. Let now (p_j, t_j) be a space-time sequence satisfying $\psi(p_j, t_j) \to \Psi_\infty$. From Lemma 3.28 we derive

$$4 \geq c_s u^{-1}(p_j, t_j) \geq \frac{\Psi_\infty}{2} u^{-1}(p_j, t_j),$$

for any j large enough. Thus $u(p_j, t_j) \geq \Psi_\infty/8$ for j large enough. Since c is uniformly bounded in the space-time, the latter also yields $b(p_j, t_j) \leq 8m^{-1}/\Psi_\infty$. Moreover, by the uniform lower bound for u and (3.27) we obtain $b_s(p_j, t_j) \geq \varepsilon$, for some $\varepsilon > 0$. We can then apply the compactness result in Proposition 3.25 to the sequence $(\mathbb{R}^4, g_j(t), p_j)_{t\leq 0}$, with $g_j(t) \doteq g(t_j + t)$, and deduce that there exists a warped Berger ancient solution $(M_\infty, g_\infty(t), p_\infty)_{t\leq 0}$, with $M_\infty = \mathbb{R}^4$ or $M_\infty = \mathbb{R} \times S^3$, satisfying:

$$\psi_\infty(p_\infty, 0) = ((c_\infty)_s - 2u_\infty^2)(p_\infty, 0) = \sup_{M_\infty \times (-\infty, 0]} \psi_\infty = \Psi_\infty > 0.$$

However, from the previous calculations we see that

$$\partial_t\, \psi_\infty|_{\Psi_\infty} < 0,$$

hence arriving to a contradiction. □

While the previous Lemma gives us a precise upper bound for c_s in terms of u, in order to control (3.29) at any negative minimum we also need a lower bound for $c_s u^{-1}$. Thus, we consider the difference between J_2 and $\mathsf{J}_1 u^{-1}$.

Lemma 3.30. *If* $(\mathbb{R}^4, g(t))_{t\leq 0}$ *is an ancient solution in* $\mathcal{A}$*, then*

$$\mathsf{J}_2 - \mathsf{J}_1 u^{-1} = b_s - c_s u^{-1} - 2(1 - u) \leq 0.$$

Proof. Let ϕ denote $\mathsf{J}_2 - \mathsf{J}_1 u^{-1}$. By Lemma 3.28 we get that ϕ is uniformly bounded in the space-time. Suppose for a contradiction that $\Phi_\infty \doteq \sup_{\mathbb{R}^4\times(-\infty,0]} \phi$ is positive. Given a space-time sequence (p_j, t_j) such that $\phi(p_j, t_j) \to \Phi_\infty$, from Lemma 3.28 we find

$$\begin{aligned} \frac{\Phi_\infty}{2} \leq \phi(p_j, t_j) &= \left(b_s + \frac{c_s u^{-1}}{2} - 2 - \frac{3}{2}c_s u^{-1} + 2u\right)(p_j, t_j) \\ &\leq \left(-\frac{3}{2}c_s u^{-1} + 2u\right)(p_j, t_j) \end{aligned}$$

Thus $u(p_j, t_j)$ is uniformly bounded along the sequence. We may then use (3.27) as in the proof of Lemma 3.29 to deduce that the compactness result applies to the sequence of ancient Ricci flows translated by times t_j and centred at p_j. In particular, there exists a limit warped Berger ancient Ricci flow $(M_\infty, g_\infty(t), p_\infty)_{t\leq 0}$ such that

$$\phi_\infty(p_\infty,0) = \left((b_\infty)_s - (c_\infty)_s u_\infty^{-1} - 2(1-u_\infty)\right)(p_\infty,0) = \sup_{M_\infty\times(-\infty,0]} \phi_\infty = \Phi_\infty > 0.$$

Thus, it remains to check that ϕ_∞ cannot achieve a positive supremum along a warped Berger ancient solution as in $\mathcal{A}$. In fact, we show that for any complete, warped Berger ancient Ricci flow with monotone coefficients ϕ_∞ never attains its positive supremum in the space-time.

We compute the evolution equation of ϕ_∞ at a positive maximum and we drop the ∞-subscript from the notation:

$$\begin{aligned}\partial_t\phi|_{\max} \leq\ & \frac{1}{b^2}\left(b_s\left(4 - b_s^2 - (c_s u^{-1})^2 - 14u^2\right)\right)\\ &+ \frac{1}{b^2}\left(c_s u^{-1}\left(b_s^2 + (c_s u^{-1})^2 + 4 + 6u^2\right)\right)\\ &+ \frac{1}{b^2}\left(2u\left(-3b_s^2 - (c_s u^{-1})^2 + 4b_s c_s u^{-1} + 4(1-u^2)\right)\right).\end{aligned}$$

At any positive maximum of ϕ we can bound the evolution equation by

$$\begin{aligned}\partial_t\phi|_{\max} \leq\ & \frac{1}{b^2}\left(4(1-u)\left(-\left(c_s u^{-1} - u\right)^2\right)\right)\\ &+ \frac{\phi}{b^2}\left(-\phi^2 - \phi(6 + 2c_s u^{-1}) - 2(c_s u^{-1})^2 - 8c_s u^{-1} + 4c_s - 8 - 2u^2\right) < 0.\end{aligned}$$

Therefore, given a warped Berger ancient solution with monotone coefficients the geometric quantity ϕ cannot achieve its supremum in the space-time. This completes the proof. □

We may now go back to the evolution equation of J_2 at a negative minimum (3.29). The roots of the c_s-quadratic are

$$y_\pm = u^2\left(2 \pm \sqrt{1 + u^{-2}(4 - z^2)}\right).$$

From Lemma 3.29 we immediately derive that

$$c_s \leq 2u^2 < y_+,$$

for any Ricci flow in $\mathcal{A}$. According to Lemma 3.30, in order to prove that $c_s > y_-$ everywhere in the space-time, it suffices to show that

$$y_- < b_s u - 2u(1-u). \tag{3.30}$$

The latter is equivalent to

$$1 + u^{-1}(2-z) < \sqrt{1 + u^{-2}(4-z^2)}.$$

After taking the square of the equation and rearranging the terms, we see that (3.30) holds if and only if

$$2u^{-1}(2-z)(1-zu^{-1}) < 0,$$

which is indeed satisfied because by definition $zu^{-1} = (b_s + u)u^{-1} > 1$ and at any negative minimum of J_2 we have $z < 2$. Therefore, we conclude that $c_s \in (y_-, y_+)$ in the space-time.

To sum up, we have shown that:

Lemma 3.31. *Given an ancient solution to the Ricci flow in $\mathcal{A}$, the evolution equation of J_2 at any negative minimum satisfies*

$$\partial_t \mathsf{J}_2|_{min<0} > 0.$$

The final and most difficult step consists in proving that, up to passing to a subsequence, the hyperkähler quantity J_2 always attains its infimum on any ancient solution in $\mathcal{A}$. We emphasize that any conclusion achieved so far does extend to the steady soliton found by Appleton [2019]. Explicitly, one may check that in order to apply the compactness result for proving Lemmas 3.29 and 3.30 it suffices to require $b_s u^{-1}$ to be uniformly bounded from below in the space-time, which holds along the soliton because it opens as fast as a paraboloid at spatial infinity. However, along the soliton the infimum of J_2 is -2 and is never attained in the space-time. Indeed, along any space-time sequence (p_j, t_j) satisfying $\mathsf{J}_2(p_j, t_j) \to -2$ the roundness ratio u becomes degenerate. Thus, we see once again that in the collapsed setting the main issue consists in verifying that geometric quantities do attain their infimum (supremum) in regions of the space-time where the squashing factor stays positive. This is where the condition (3.27), with f diverging to infinity when $u^{-1} \to \infty$, plays a role via a sort of approximation method to show that, in fact, for any solution in $\mathcal{A}$ we have

$$\lim_{u^{-1}\to\infty} \mathsf{J}_2 = 0.$$

Equivalently, below we prove that for any ancient solution in $\mathcal{A}$ the warping coefficient b in the directions orthogonal to the Hopf-fiber grows linearly in space, meaning that the volume of geodesic balls $B_{g(t)}(\boldsymbol{o}, r)$ behaves like r^3 for any radius r large enough and on any time-slice.

Proposition 3.32. *If* $(\mathbb{R}^4, g(t))_{t\leq 0}$ *is an ancient solution to the Ricci flow in* $\mathcal{A}$*, then there exists* $\delta > 0$ *satisfying*

$$\inf_{\mathbb{R}^4\times(-\infty,0]} u^{-\delta}(b_s - 2) > -\infty.$$

Proof. For any $\delta, \epsilon > 0$ we define

$$\omega_{\delta,\epsilon} = u^{-\delta}\left(\epsilon b_s u^{-1} + b_s - 2\right),$$

and

$$\Omega_{\delta,\epsilon} = \inf_{\mathbb{R}^4\times(-\infty,0]} \omega_{\delta,\epsilon}.$$

In the following we always take $\Omega_{\delta,\epsilon}$ to be *negative*. From (3.27) we see that there exists $\beta > 0$ such that

$$\omega_{\delta,\epsilon} \geq u^{-\delta}\left(\epsilon(\beta f(u^{-1})) + b_s - 2\right),$$

with $f(u^{-1}) \to \infty$ as $u^{-1} \to \infty$. Thus, given δ and ϵ positive constants, the quantity $\Omega_{\delta,\epsilon}$

is finite. Given a negative minimum point for $\omega_{\delta,\epsilon}$, the evolution equation becomes

$$
\begin{aligned}
\partial_t \omega_{\delta,\epsilon}|_{\min<0} \geq\ & \frac{\omega_{\delta,\epsilon}}{b^2}\left(b_s^2(\delta^2+2\delta+2\delta\frac{\epsilon u^{-1}}{1+\epsilon u^{-1}}) + (c_s u^{-1})^2(\delta^2+2\delta\frac{\epsilon u^{-1}}{1+\epsilon u^{-1}})\right)\\
&+ \frac{\omega_{\delta,\epsilon}}{b^2}\left(b_s c_s u^{-1}(-2\delta^2-2\delta-4\delta\frac{\epsilon u^{-1}}{1+\epsilon u^{-1}}) - 4\delta(1-u^2)\right)\\
&+ \frac{\epsilon b_s u^{-1-\delta}}{b^2}\left(2b_s^2\frac{\epsilon u^{-1}}{1+\epsilon u^{-1}} + (c_s u^{-1})^2(-2+2\frac{\epsilon u^{-1}}{1+\epsilon u^{-1}})\right)\\
&+ \frac{\epsilon b_s u^{-1-\delta}}{b^2}(-4\frac{\epsilon u^{-1}}{1+\epsilon u^{-1}} b_s c_s u^{-1} - 2u^2)\\
&+ \frac{1}{b^2}\left(4\epsilon c_s u^{-\delta} + 4c_s u^{1-\delta} + b_s u^{-\delta}(4-b_s^2-(c_s u^{-1})^2-6u^2)\right).
\end{aligned}
$$

Since b_s and c_s are nonnegative and $\omega_{\delta,\epsilon} < 0$, we may bound the right hand side by:

$$
\begin{aligned}
\partial_t \omega_{\delta,\epsilon}|_{\min<0} \geq\ & \frac{\delta\omega_{\delta,\epsilon}}{b^2}\left(b_s^2(\delta+2+2\frac{\epsilon u^{-1}}{1+\epsilon u^{-1}}) + (c_s u^{-1})^2(\delta+2\frac{\epsilon u^{-1}}{1+\epsilon u^{-1}}) - 4(1-u^2)\right)\\
&+ \frac{\epsilon b_s u^{-1-\delta}}{b^2}\left(-2(c_s u^{-1})^2 - 4\frac{\epsilon u^{-1}}{1+\epsilon u^{-1}} b_s c_s u^{-1} - 2u^2\right)\\
&+ \frac{1}{b^2}\left(b_s u^{-\delta}(4-b_s^2-(c_s u^{-1})^2-6u^2)\right).
\end{aligned}
$$

We note that

$$
b_s u^{-\delta}(4-b_s^2) = (b_s^2+2b_s)(-\omega_{\delta,\epsilon}+\epsilon b_s u^{-1-\delta}) \geq -(b_s^2+2b_s)\omega_{\delta,\epsilon}.
$$

Since by Lemma 3.28 $b_s \leq 2$, if we take δ positive such that $2-\delta^2-4\delta \geq 0$, then we can write

$$
\begin{aligned}
\partial_t \omega_{\delta,\epsilon}|_{\min<0} \geq\ & \frac{\delta|\omega_{\delta,\epsilon}|}{b^2}\left((c_s u^{-1})^2(-\delta-2\frac{\epsilon u^{-1}}{1+\epsilon u^{-1}}) + 4(1-u^2)\right)\\
&+ \frac{\epsilon b_s u^{-1-\delta}}{b^2}\left(-2(c_s u^{-1})^2 - 4\frac{\epsilon u^{-1}}{1+\epsilon u^{-1}} b_s c_s u^{-1} - 2u^2\right)\\
&+ \frac{1}{b^2}\left(-b_s u^{-\delta}((c_s u^{-1})^2+6u^2)\right).
\end{aligned}
$$

By Lemma 3.29 we see that $|c_s u^{-1}| \leq 2u$. Moreover, whenever $\omega_{\delta,\epsilon} < 0$ we have $\epsilon b_s u^{-1} < 2$. Since by the condition we set previously $\delta < 1$, we can bound the evolution equation by

$$\begin{aligned}\partial_t\omega_{\delta,\epsilon}|_{\min<0} \geq& \frac{\delta|\omega_{\delta,\epsilon}|}{b^2}\left(-12u^2+4(1-u^2)\right)\\ &+\frac{2u^{-\delta}}{b^2}\left(-8u^2-16u-2u^2\right)\\ &+\frac{1}{b^2}\left(-2u^{-\delta}(4u^2+6u^2)\right).\end{aligned}$$

Finally, we observe that $|\omega_{\delta,\epsilon}|\leq 2u^{-\delta}$ at any negative minimum. Therefore, from the evolution equation we deduce that if $|\omega_{\delta,\epsilon}|=\Omega_\delta$, for some Ω_δ large enough and *independent* of ϵ, then

$$\partial_t\omega_{\delta,\epsilon}|_{\min=-\Omega_\delta<0}\geq \frac{1}{b^2}\left(\delta\Omega_\delta-1\right)>0.$$

Let (p_j,t_j) be a space-time sequence satisfying $\omega_{\delta,\epsilon}(p_j,t_j)\to\Omega_{\delta,\epsilon}<0$. According to (3.27) the squashing factor is bounded away from zero by some positive quantity depending on ϵ along the given sequence. From (3.28) we then derive that $b(p_j,t_j)\leq\alpha(\epsilon)<\infty$. Again by the constraint in (3.27) the spatial derivative b_s has a uniform lower bound along the sequence because u is non-degenerate. Since the ancient solution belongs to $\mathcal{A}$ we deduce that we may apply the compactness result in Proposition 3.25 to the sequence $(\mathbb{R}^4, g(t_j+t), p_j)_{t\leq 0}$ and conclude that there exists a warped Berger ancient solution $(M_\infty, g_\infty(t), p_\infty)_{t\leq 0}$ such that

$$(\omega_{\delta,\epsilon})_\infty(p_\infty,0)=\inf_{M_\infty\times(-\infty,0]}(\omega_{\delta,\epsilon})_\infty=\Omega_{\delta,\epsilon}<0.$$

From the previous analysis we derive that there exists $\Omega_\delta>0$ such that

$$\partial_t(\omega_{\delta,\epsilon})_\infty(p_\infty,0)>0,$$

when $(\omega_{\delta,\epsilon})_\infty(p_\infty,0)<-\Omega_\delta$.

Thus, fixed δ as above, we see that $\Omega_{\delta,\epsilon}\geq-|\Omega_\delta|$, for all $\epsilon>0$. We then let $\epsilon\searrow 0$ and conclude that there exists $\delta>0$ such that

$$\omega_{\delta,0}=u^{-\delta}(b_s-2)\geq-|\Omega_\delta|>-\infty.$$

□

We now have all the ingredients to prove Theorem 3.2.

Proof of Theorem 3.2. By Proposition 3.32 we know that there exist $\delta, \Omega_\delta > 0$ such that

$$\mathsf{J}_2 \geq u - \Omega_\delta \, u^\delta. \tag{3.31}$$

Let us set the notation $\inf \mathsf{J}_2 \doteq \mathcal{J}_2$ and assume for a contradiction that $\mathcal{J}_2 < 0$. Given (p_j, t_j) such that $\mathsf{J}_2(p_j, t_j) \to \mathcal{J}_2$, , by (3.31) we see that

$$u(p_j, t_j)^\delta \geq \frac{|\mathcal{J}_2|}{2\Omega_\delta},$$

for all j large enough. We may then argue as for the proof of Proposition 3.32 and derive that there exists a limit warped Berger solution $(M_\infty, g_\infty(t), p_\infty)_{t \leq 0}$ such that

$$(\mathsf{J}_2)_\infty(p_\infty, 0) = \inf_{M_\infty \times (-\infty, 0]} (\mathsf{J}_2)_\infty = \mathcal{J}_2 < 0.$$

However, Lemma 3.31 implies that $\partial_t (\mathsf{J}_2)_\infty(p_\infty, 0) > 0$. Therefore, for any ancient solution in $\mathcal{A}$ we have $\mathsf{J}_2 \geq 0$.

Claim 3.33. *Let* $(\mathbb{R}^4, g(t))_{t \leq 0}$ *be a complete, bounded curvature warped Berger ancient solution to the Ricci flow. Assume that there exist* $r > 0$, $t_0 \leq 0$ *such that*

$$\mathsf{J}_2(\cdot, t_0)|_{B_{g(t_0)}(\boldsymbol{o}, r)} \geq 0.$$

Then $g(t) \equiv g$ *is Ricci-flat.*

Proof of Claim 3.33. Since the curvature is bounded, we may apply l'Hôpital rule and find that the scalar curvature at the origin is given by:

$$R(\boldsymbol{o}, t) = -4(c_{sss}(\boldsymbol{o}, t) + 2b_{sss}(\boldsymbol{o}, t)).$$

In particular, we derive that

$$(u)_{ss}(\boldsymbol{o},t) = \frac{1}{3}\left(c_{sss}(\boldsymbol{o},t) - b_{sss}(\boldsymbol{o},t)\right),$$

Therefore, we get

$$(\mathsf{J}_2)_s(\boldsymbol{o},t) = 0, \quad (\mathsf{J}_2)_{ss}(\boldsymbol{o},t) = \frac{1}{3}\left(2b_{sss}(\boldsymbol{o},t) + c_{sss}(\boldsymbol{o},t)\right) = -\frac{R(\boldsymbol{o},t)}{12}.$$

Since from the boundary conditions we see that $\mathsf{J}_2(\boldsymbol{o},t) = 0$, from the assumption we deduce that the origin must be a local minimum for $\mathsf{J}_2(\cdot,t_0)$, meaning that $(\mathsf{J}_2)_{ss}(\boldsymbol{o},t_0) \geq 0$ and hence $R(\boldsymbol{o},t_0) \leq 0$. By Chen [2009] and a standard application of the strong maximum principle to the evolution equation of the scalar curvature we conclude that the ancient solution is in fact a stationary Ricci flat metric. □

By Claim 3.33 any ancient solution in $\mathcal{A}$ is Ricci-flat. Accordingly, we drop the time-dependence and we simply write g. Suppose for a contradiction that $\mathsf{J}_2 > 0$ somewhere. From the boundary conditions we get that there exists $p \in \mathbb{R}^4$ such that $\mathsf{J}_2(p) > 0$ and $(\mathsf{J}_2)_s(p) > 0$. Thus

$$\left(b_{ss} + \frac{c_s}{b} - b_s\frac{u}{b}\right)(p) > 0,$$

and $(b_s + u)(p) > 2$. We then obtain

$$\begin{aligned}
\mathrm{Ric}\,(X_1,X_1)\,(p) &= (-bb_{ss} - b_s c_s u^{-1} - b_s^2 - 2u^2 + 4)(p) \\
&< (c_s - b_s u - b_s c_s u^{-1} - b_s^2 - 2u^2 + 4)(p) \\
&< (2c_s(1-u^{-1}) + 2u(1-u))(p) < 0,
\end{aligned}$$

where the last inequality follows from Lemma 3.30. Therefore $\mathsf{J}_2 = 0$ everywhere in the space-time. Again, by Lemma 3.30 we see that $\mathsf{J}_1 \geq 0$ and a similar argument yields $\mathsf{J}_1 = 0$. Since the Hopf-fiber is uniformly bounded for any ancient solution in $\mathcal{A}$ and the differential equations (3.3) and (3.4) are satisfied in the space-time, we can conclude that g is the Taub-NUT metric of mass m, with m given by (3.28). □

3.7 Long-time behaviour of the Ricci flow

In this section we apply the compactness property in Proposition 3.25 and the rigidity result in Theorem 3.2 to study the long-time behaviour of Ricci flow solutions in $\mathcal{G}_k$ and in $\mathcal{G}_{\mathsf{AF}}$. In particular, we show that any solution in $\mathcal{G}_k$ encounters a Type-II(b) singularity in infinite-time, which is modelled by the Taub-NUT metric in a precise way. Namely, any solution in $\mathcal{G}_k$ converges to g_{TNUT} in the Cheeger-Gromov sense.

We recall that an immortal Ricci flow solution $(M, g(t))_{t\geq 0}$ converges to a stationary Ricci-flat metric (M_∞, g_∞) in the pointed Cheeger-Gromov sense in infinite time if there exist $p \in M$ and $p_\infty \in M_\infty$ such that for any sequence $t_j \nearrow \infty$ the pointed sequence $(M, g(t_j + t), p)$ converges to $(M_\infty, g_\infty, p_\infty)$ in the Cheeger-Gromov sense. We point out that for this notion of convergence we do *not* rescale the immortal solution, so that if g_∞ is *non*-flat, then $g(t)$ develops a Type-II(b) singularity in infinite-time.

3.7.1 The positive mass case

In the following we focus the attention on Ricci flow solutions starting in $\mathcal{G}_k$. We recall that any metric in $\mathcal{G}_{\mathsf{AF}}$ with positive-mass belongs to $\mathcal{G}_0$. We first prove that the mass of any Ricci flow solution in $\mathcal{G}_k$ is in fact preserved in any region where b is large, uniformly in time.

Lemma 3.34. *Let $(\mathbb{R}^4, g(t))_{t\geq 0}$ be the maximal Ricci flow solution evolving from some $g_0 \in \mathcal{G}_k$ with mass m_{g_0}. There exists $\nu > 0$ such that for all $\gamma < (m_{g_0})^{-1}$ we have*

$$\inf_{\mathbb{R}^4\times[0,+\infty)} b^\nu(c-\gamma) > -\infty.$$

Proof. Given $\nu > 0$ and $\gamma < (m_{g_0})^{-1}$ we let $f = b^\nu(c-\gamma)$. From Lemma 3.18 we derive that $f(s,t) \to \infty$ at spatial infinity on any time-slice. Assume that there exist $p_0 \in \mathbb{R}^4$ and $t_0 > 0$ such that f attains a negative minimum at (p_0, t_0). The evolution equation of f is

$$\partial_t f(p_0,t_0) \geq \frac{1}{b^2}\left(f\left(\nu^2 b_s^2(1-\frac{c-\gamma}{c}) + 2\nu u^2 - 4\nu\right) - 2u^2 b^\nu c\right).$$

If $|f(p_0,t_0)|$ is large, then $b(p_0,t_0)$ is large too. Thus, from the rotational symmetry type of bounds in Corollary 3.20 we see that

$$\frac{|c-\gamma|}{c}(p_0,t_0) \leq 1 + \frac{m_{g_0}^{-1}}{c}(p_0,t_0) \leq \alpha < \infty.$$

Therefore, we obtain

$$\partial_t f(p_0,t_0) \geq \frac{|f|}{b^2}\left(-(1+\alpha)\nu^2 b_s^2 - \frac{2}{|f|}u^2 b^\nu \gamma + 2\nu\right).$$

Since by Lemma 3.19 the derivative b_s is uniformly bounded in time, we may pick $\nu > 0$ small enough such that

$$\partial_t f(p_0,t_0) \geq \frac{|f|}{b^2}\left(\nu - 2\frac{u(m_{g_0})^{-2}}{|f|}\right).$$

Finally, once we let $|f|$ be sufficiently large depending on the choice of ν and on the value of m_{g_0}, we conclude that $\partial_t f(p_0,t_0) > 0$, which completes the proof. $\square$

We may now prove that any Ricci flow in $\mathcal{G}_k$ converges to g_{TNUT} in infinite-time.

Proof of Theorem 3.3. Let $t_j \nearrow \infty$ and consider the pointed sequence of Ricci flow solutions $(\mathbb{R}^4, g_j(t), \boldsymbol{o})_{t\in[-t_j,0]}$, with $g_j(t) = g(t_j + t)$. According to Proposition 3.27, the curvature is uniformly bounded along the sequence. Moreover, the first order derivatives are controlled in the space-time by Lemma 3.19. From the boundary conditions we also derive that conditions (3.21), (3.22) and (3.23) are satisfied by the sequence given above. Therefore, after a diagonal argument we deduce that $(\mathbb{R}^4, g_j(t), \boldsymbol{o})$ converges to an ancient solution $(\mathbb{R}^4, g_\infty(t), \boldsymbol{o})_{t\leq 0}$ as in Proposition 3.25. In particular, $(\mathbb{R}^4, g_\infty(t))_{t\leq 0}$ is a complete, warped Berger ancient solution with monotone coefficients and curvature uniformly bounded in the space-time. Moreover, from the convergence of the warping coefficients given by Proposition 3.25 we find that $c_\infty \leq m_{g_0}^{-1}$. By Lemma 3.24 and the bound on the Hopf-fiber we know that b_∞ diverges at spatial infinity on any time-slice. According to Lemma 3.24 and Lemma 3.34 we may pick $\lambda > 0$ small enough and $\alpha > 0$ such that

$$(b_\infty)_s u_\infty^{-1} \geq \log(b_\infty) - \frac{\alpha}{b_\infty^\lambda}$$

and

$$c_\infty \geq \frac{m_{g_0}^{-1}}{2} - \frac{\alpha}{b_\infty^\lambda}.$$

Let V_λ be the space time region where $b_\infty^\lambda \geq 4(\alpha+1)(m_{g_0}+1)$. Thus, in V_λ we get

$$(b_\infty)_s u_\infty^{-1} \geq \log(u_\infty^{-1}) + \log(c_\infty) - \frac{1}{4} \geq \log(u_\infty^{-1}) - \log(m_{g_0}) - \log(4) - \frac{1}{4}.$$

Since by Lemma 3.24 we have $(b_\infty)_s u_\infty^{-1} \geq \beta$, for some $\beta > 0$, we conclude that there exists a continuous function $f : [1,\infty) \to \mathbb{R}_{>0}$, with $f(z) \to \infty$ as $z \to \infty$, such that

$$(b_s)_\infty u_\infty^{-1} \geq f(u_\infty^{-1})$$

in $\mathbb{R}^4 \times (-\infty, 0]$. Therefore, the limit ancient Ricci flow $(\mathbb{R}^4, g_\infty(t))$ belongs to $\mathcal{A}$ (see Definition 3.7). By the rigidity property in Theorem 3.2 we see that $g_\infty(t)$ is the Taub-NUT metric g_{TNUT} of mass *exactly* m_{g_0} as follows from Lemma 3.34. □

Remark 3.10. *We note again that the argument above works for any asymptotically flat Ricci flow with positive mass, hence proving (i) of Theorem 3.1 as well.*

We also get:

Proof of Corollary 3.4. Let $g_0 \in \mathcal{G}_0$ be as in Lemma 3.6. We can then apply Corollary 3.8 and Theorem 3.3 to derive that the maximal complete, bounded curvature Ricci flow solution starting at g_0 is immortal and converges to g_{TNUT} in the pointed Cheeger-Gromov sense as $t \nearrow \infty$. □

3.7.2 The zero mass case

In order to prove that any asymptotically flat Ricci flow with Euclidean volume growth, or equivalently zero mass, encounters a Type-III singularity in infinite-time, we first show that the roundness ratio converges to 1 uniformly in any space-time region where b is large.

Lemma 3.35. *If* $(\mathbb{R}^4, g(t))_{t\geq 0}$ *is the maximal Ricci flow solution starting at some* $g_0 \in \mathcal{G}_{\mathsf{AF}}$

with zero mass, then there exists $\nu > 0$ *such that*

$$\sup_{\mathbb{R}^4\times[0,\infty)} b^{\nu}(1-u) < \infty.$$

Proof. Let $\epsilon > 0$ satisfy $\sup_{\mathbb{R}^4}(\, d_{g_0}(\boldsymbol{o},\cdot))^{2+\epsilon}|\mathrm{Rm}|_{g_0}(\cdot) < \infty$. If we pick $0 < \nu < \epsilon$, then we may apply l'Hôpital formula to $b^{\nu}(1-u)$ and use (2.12) to derive that $b^{\nu}(1-u)$ converges to zero at spatial infinity. Since by Lemma 3.9 the decay of the curvature persists in time, such conclusion holds along the solution. At any positive maximum point we have

$$\partial_t(b^{\nu}(1-u))|_{\max} \leq \frac{b^{\nu}(1-u)}{b^2}\left(\nu^2 b_s^2 u^{-1} - 4\nu + 2\nu u^2 - 4u(1+u)\right).$$

We may take $\nu < 1$ and hence write

$$\partial_t(b^{\nu}(1-u))|_{\max} \leq \frac{b^{\nu}(1-u)}{b^2}\left(\nu^2 b_s^2 u^{-1} - 4\nu - 4u\right).$$

We note that from Lemma 3.5 it follows that $u(\cdot,0) \geq \varepsilon > 0$ in the zero-mass case. Thus, we may apply (iii) in Lemma 3.7 and Lemma 3.11 to write

$$\partial_t(b^{\nu}(1-u))|_{\max} \leq \frac{b^{\nu}(1-u)}{b^2}\left(\nu^2 b_s^2 \varepsilon^{-1} - 4\nu - 4\varepsilon\right) \leq \frac{b^{\nu}(1-u)}{b^2}\left(\alpha\nu^2 \varepsilon^{-1} - 4\nu - 4\varepsilon\right),$$

which is negative whenever ν is sufficiently small depending on ε and α. □

As a consequence of the previous result, we prove that the squashing factor u converges to 1 as time goes to infinity.

Corollary 3.36. *If* $(\mathbb{R}^4, g(t))_{t\geq 0}$ *is the maximal Ricci flow solution starting at some* $g_0 \in \mathcal{G}_{\mathsf{AF}}$ *with zero mass, then there exist* $\beta > 0$ *and* $\delta > 0$ *such that*

$$(1-u)(\cdot,t) \leq \frac{\beta}{(1+\beta t)^{\delta}},$$

for all times $t \geq 0$.

Proof. Consider the function $h = 1-u$. In the zero mass case we have $h(\boldsymbol{o},t) = 0$ and

$h(s,t) \to 0$ as $s \to \infty$, for all positive times. At any positive maximum we may compute that:

$$\partial_t(1-u)|_{\max} \leq \frac{1}{b^2}\left(-4u(1-u^2)\right).$$

From Lemma 3.35 we see that $b \leq \alpha(1-u)^{-\nu}$, yielding

$$\partial_t(1-u)|_{\max} \leq \left(-\frac{(1-u)^{\frac{2}{\nu}}}{\alpha}\right) 4u(1-u^2)$$

Since the roundness ratio is uniformly bounded from below by Lemma 3.7, we find that there exists $\varepsilon > 0$ such that

$$\partial_t(1-u)|_{\max} \leq -\left(\frac{4}{\alpha}\varepsilon(1+\varepsilon)\right)(1-u)^{\frac{2}{\nu}+1} \leq -\beta(1-u)^{\frac{2}{\nu}+1}.$$

We may then apply the maximum principle and integrate the previous relation to obtain

$$(1-u)_{\max}(t) \leq \frac{\beta}{(1+\beta t)^{\frac{\nu}{2}}},$$

up to choosing β large enough. □

We finally prove that any immortal Ricci flow in $\mathcal{G}_{\mathsf{AF}}$ with zero mass encounters a Type-III singularity at infinite time.

Proof of (ii) in Theorem 3.1. Given an immortal Ricci flow solution $(\mathbb{R}^4, g(t))_{t\geq 0}$ starting at $g_0 \in \mathcal{G}_{\mathsf{AF}}$ with $m_{g_0} = 0$, let us assume for a contradiction that there is a Type-II(b) singularity at infinite time. One can then argue as for the proof of Proposition 3.27 and deduce that there exists a space-time sequence (p_j, t_j), with $t_j \nearrow \infty$, such that the pointed sequence $(\mathbb{R}^4, g_j(t), p_j)$, defined by $g_j(t) = \lambda_j g(t_j + t/\lambda_j)$, where $\lambda_j = |\mathrm{Rm}|(p_j, t_j)$, converges to an eternal Ricci flow solution $(M_\infty, g_\infty(t), p_\infty)$. By Lemma 3.5 and Lemma 3.9 there exists $\varepsilon > 0$ such that

$$u(\cdot, t) \geq \varepsilon > 0,$$

for all times $t \geq 0$. We may then apply Lemma 3.16 and derive that (3.21), (3.23) are satisfied along the Ricci flow sequence. From Corollary 3.14 we also get

$$b_j(p_j, 0) = \sqrt{\lambda_j} b(p_j, t_j) = \sqrt{(c^2|\mathrm{Rm}|)(p_j, t_j)} u^{-1}(p_j, t_j) \leq \alpha \varepsilon^{-1},$$

for some $\alpha > 0$. Therefore, the compactness result in Proposition 3.25 holds and we may hence use Corollary 3.36 to deduce that $(M_\infty, g_\infty(t), p_\infty)_{t \in \mathbb{R}}$ is an eternal warped Berger solution to the Ricci flow such that

$$(1 - u_\infty)(q, t) = \lim_{j \to \infty} (1 - u)(s_j(q), t_j + t) = 0,$$

for all $q \in M_\infty$ and for all $t \in \mathbb{R}$. Thus $g_\infty(t)$ is rotationally symmetric. Since by Lemma 3.16 we see that $g_\infty(t)$ has Euclidean volume growth, we can follow the proof of Proposition 3.27 to conclude that $g_\infty(t)$ is flat, therefore contradicting the choice of the rescaling factors.

We have shown that for any Ricci flow solution starting at some $g_0 \in \mathcal{G}_{\mathsf{AF}}$ with zero mass, the singularity forming at infinite time is of Type-III. Given $t_j \nearrow \infty$, the Ricci flow sequence $(\mathbb{R}^4, g_j(t), \boldsymbol{o})_{t \in [-t_j, 0]}$ satisfies the assumptions in Proposition 3.25 and hence converges to a warped Berger solution $(\mathbb{R}^4, g_\infty(t), p_\infty)$. From the Type-III condition we see that $|\mathrm{Rm}_\infty|_\infty(\cdot, t) \equiv 0$, meaning that (M_∞, g_∞) is the Euclidean space. □

Conclusions and future directions

In this thesis we have analysed families of cohomogeneity one Ricci flows. We have shown that for (complete, bounded curvature) $SO(n+1)$-invariant Ricci flows on $\mathbb{R}^{n+1}$ and warped Berger Ricci flows on $\mathbb{R}^4$ the following property is satisfied:

If the length of the Killing vectors grows monotonically with the distance from the singular orbit, then the behaviour of the Ricci flow solution can be characterized in terms of the asymptotics of the initial metric at spatial infinity:

(i) *If the Killing vectors are bounded on the manifold, then the Ricci flow solution encounters a Type-II singularity with the curvature slowly concentrating at the singular orbit.*

(ii) *If instead the curvature of the input metric decays at spatial infinity and the injectivity radius is positive, then the flow is immortal.*

Comparing this pattern with analogous conclusions arising from the analysis in Appleton [2019], we expect this characterization to hold for more general cohomogeneity one Ricci flows, provided that the solutions can still be diagonalized with respect to a time-independent frame and minimal principal orbits can be avoided along the flow. In particular, for any such flow no region around the singular orbit should stay smooth if the flow becomes singular.njh The topology of the singular orbit determines then the type of singularity, as for the self-intersections of the minimal S^2 in the analysis of Máximo [2014] and Appleton [2019] respectively. Moreover, for non-trivial singular orbits we expect symmetries to not be enhanced, meaning that the singularity model would belong to the same class of cohomogeneity one manifolds according to compactness properties similar to Proposition 3.25. This would lead to investigating when non-collapsed cohomogeneity one steady solitons can appear as Type-II singularity models.

For what concerns infinite-time singularity models instead, we stress that the fact that the Taub-NUT metric is hyperkähler played a fundamental role because it allowed us to reduce the convergence to studying *first order* quantities. We expect the strategy described in Chapter 3 to apply to more arbitrary cohomogeneity one Ricci flows satisfying the conditions in (ii) above and admitting fixed points with integrable constraints. We also point out that any potential generalization of the rigidity type of result in Theorem 3.2 to more general cohomogeneity one (or higher) families of ancient Ricci flows would still need to depend on a delicate analysis of the space-time region where sizes of orthogonal Killing vectors are no longer comparable.

Bibliography

D. Alekseevskii and B. N. Kimelfeld. Structure of homogeneous Riemannian spaces with zero Ricci curvature. *Funktional. Anal. i Prilov Zen.*, 9(2):5–11, 1975.

S. Angenent. The zero set of a solution of a parabolic equation. *J. Reine Angew. Math.*, 390:79–96, 1988.

S. Angenent and D. Knopf. An example of neckpinching for Ricci flow on S^{n+1}. *Math. Res. Lett.*, 11, 07 2004.

S. Angenent and D. Knopf. Precise asymptotics of the Ricci flow neckpinch. *Comm. Anal. Geom.*, 15(4):773–844, 2007.

S. Angenent, J. Isenberg, and D. Knopf. Degenerate neckpinches in Ricci flow. *J. Reine Angew. Math.*, (709):81–117, 2015.

A. Appleton. A family of non-collapsed steady Ricci solitons in even dimensions greater or equal to four. *arXiv preprint arXiv:1708.00161*, 2018.

A. Appleton. Eguchi-Hanson singularities in U(2)-invariant Ricci flow. *arXiv preprint arXiv:1903.09936*, 2019.

M. F. Atiyah and N. Hitchin. *The geometry and dynamics of magnetic monopoles*. Princeton University Press, 1985.

R. H. Bamler. Long-time behavior of 3–dimensional Ricci flow: introduction. *Geom. Topol.*, 22(2):757–774, 2018.

R. H. Bamler and B. Kleiner. Ricci flow and contractibility of spaces of metrics. *arXiv preprint arXiv:1909.08710*, 2019.

S. Bando. Real analyticity of solutions of Hamilton's equation. *Math. Z.*, 195(1):93–97, 1987.

R. G. Bettiol and A. M. Krishnan. Four-dimensional cohomogeneity one Ricci flow and nonnegative sectional curvature. *arXiv preprint arXiv:1606.00778*, 2016.

C. Böhm. Inhomogeneous Einstein metrics on low-dimensional spheres and other low-dimensional spaces. *Invent. Math.*, 134(1):145–176, 1998.

C. Böhm. On the long time behavior of homogeneous Ricci flows. *Comm. Math. Helv*, 90(3):543–571, 2015.

C. Böhm and R. A. Lafuente. Immortal homogeneous Ricci flows. *Invent. Math.*, 212(2): 461–529, 2018.

C. Böhm, R. Lafuente, and M. Simon. Optimal Curvature Estimates for Homogeneous Ricci Flows. *Int. Math. Res. Not.*, 2019(14):4431–4468, 11 2017.

K. A. Brakke. The motion of a surface by its mean curvature. 1978.

S. Brendle. Rotational symmetry of Ricci solitons in higher dimensions. *J. Differ. Geom.*, 97(2):191–214, 2014.

S. Brendle. Ancient solutions to the Ricci flow in dimension 3. *Acta Math.*, 225(1):1 – 102, 2020.

S. Brendle and R. Schoen. Manifolds with 1/4-pinched curvature are space forms. *J. Am. Math. Soc.*, 22(1):287–307, 2009.

R. L. Bryant. Ricci flow solitons in dimension three with SO(3)-symmetries. `https://services.math.duke.edu/~bryant/3DRotSymRicciSolitons.pdf`, 2005.

R. L. Bryant and S. M. Salamon. On the construction of some complete metrics with exceptional holonomy. *Duke Math. J.*, 58(3):829–850, 1989.

E. Cabezas-Rivas and B. Wilking. How to produce a Ricci flow via Cheeger–Gromoll exhaustion. *J. Eur. Math. Soc.*, 17(12):3153–3194, 2015.

H.-D. Cao and Q. Chen. On locally conformally flat gradient steady ricci solitons. *Trans. Amer. Math. Soc.*, 364(5):2377–2391, 2012.

G. Catino, C. Mantegazza, and L. Mazzieri. Locally conformally flat ancient Ricci flows. *Anal. PDE*, 8(2):365–371, 2015.

A. Chau, L.-F. Tam, and C. Yu. Pseudolocality for the Ricci flow and applications. *Canad. J. Math.*, 63:55–85, 2011.

B.-L. Chen. Strong uniqueness of the Ricci flow. *J. Differ. Geom.*, 82(2):363–382, 06 2009.

B.-L. Chen and X.-P. Zhu. Ricci flow with surgery on four-manifolds with positive isotropic curvature. *J. Differ. Geom.*, 74(2):177–264, 2006a.

B.-L. Chen and X.-P. Zhu. Uniqueness of the Ricci flow on complete noncompact manifolds. *J. Differ. Geom.*, 74(1):119–154, 2006b.

B. Chow and D. Knopf. *The Ricci flow: an introduction*, volume 1. American Mathematical Soc., 2004.

B. Chow, P. Lu, and L. Ni. *Hamilton's Ricci flow*, volume 77 of *Graduate studies in Mathematics*. AMS, 2006.

B. Chow, S. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, and L. Ni. The Ricci flow: Techniques and applications-part I: Geometric aspects. *Mathematical Surveys and Monographs*, 135, 2007.

B. Chow, S. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, and L. Ni. The Ricci flow: Techniques and applications part II: Analytic aspects. *Mathematical Surveys and Monographs*, 144, 2008.

A. Deruelle and K. Kröncke. Stability of ALE Ricci-flat manifolds under Ricci flow. *J. Geom. Anal.*, pages 1–42, 2020.

D. M. DeTurck. Deforming metrics in the direction of their Ricci tensors. *J. Differ. Geom.*, 18(1):157–162, 1983.

F. Di Giovanni. Ricci flow of warped Berger metrics on $\mathbb{R}^4$. *Calc.Var.*, 59(162), 2020.

F. Di Giovanni. Convergence of Ricci flow solutions to Taub-NUT. *Commun. Partial Differ. Equ.*, 2021a.

F. Di Giovanni. Rotationally symmetric Ricci flow on $\mathbb{R}^{n+1}$. *Adv. Math.*, 381:107621, 2021b.

Y. Ding. A remark on degenerate singularities in three dimensional Ricci flow. *Pacific J. Math.*, 240(2):289–308, 2009.

J. Eells and J. H. Sampson. Harmonic mappings of Riemannian manifolds. *Amer. J. Math.*, 86(1):109–160, 1964.

J. Enders, R. Müller, and P. Topping. On type-I singularities in Ricci flow. *Comm. Anal. Geom.*, 19(5):905–922, 2011.

J. Fine, J. D. Lotay, and M. Singer. The space of hyperkähler metrics on 4-manifold with boundary. *Forum Math. Sigma*, 5:e6, 2017.

L. Foscolo and M. Haskins. New G_2-holonomy cones and exotic nearly Kähler structures on S^6 and $S^3 \times S^3$. *Ann. Math.*, 185(1):59–130, 2017.

A. Friedman. *Partial differential equations of parabolic type*. Prentice Hall, Inc., 1964.

G. W. Gibbons and C. N. Pope. The positive action conjecture and asymptotically Euclidean metrics in quantum gravity. *Comm. Math. Phys.*, 66(3):267–290, 1979.

G. Giesen and P. Topping. Ricci flows with unbounded curvature. *Math. Z.*, 273(1-2): 449–460, 2013.

K. Grove and W. Ziller. Cohomogeneity one manifolds with positive Ricci curvature. *Invent. Math.*, 149(3):619–646, 2002.

H.-L. Gu and X.-P. Zhu. The existence of type-II singularities for the Ricci flow on S^{n+1}. *Comm. Anal. Geom.*, 16(3):467–494, 2008.

R. S. Hamilton. Three-manifolds with positive Ricci curvature. *J. Differ. Geom.*, 17(2): 255–306, 1982.

R. S. Hamilton. Four-manifolds with positive curvature operator. *J. Differ. Geom.*, 24(2): 153–179, 1986.

R. S. Hamilton. Eternal solutions to the Ricci flow. *J. Differ. Geom.*, 38(1):1–11, 1993a.

R. S. Hamilton. The Harnack estimate for the Ricci flow. *J. Differ. Geom.*, 37(1):225–243, 1993b.

R. S. Hamilton. The formation of singularities in the Ricci flow. *Surveys in Differential Geometry*, 2:7–136, 1995.

R. Haslhofer and R. Müller. Dynamical stability and instability of Ricci-flat metrics. *Math. Ann.*, 360(1-2):547–553, 2014.

S. W. Hawking. Gravitational instantons. *Phys. Lett.*, 60A(2):81–83, 1977.

G. Holzegel, T. Schmelzer, and C. Warnick. Ricci flows connecting Taub-Bolt and Taub-NUT metrics. *Class. Quantum Gravity*, 24:6201–6217, 2007.

J. Isenberg, D. Knopf, and N. Šešum. Ricci flow neckpinches without rotational symmetry. *Comm. Part. Diff. Eq.*, 41(12):1860–1894, 2016.

J. Isenberg, D. Knopf, and N. Šešum. Non-Kähler Ricci flow singularities that converge to Kähler-Ricci solitons. *arXiv preprint arXiv:1703.02918v5*, 2019.

T. A. Ivey. The Ricci flow on radially symmetric $\mathbb{R}^3$. *Comm. Part. Diff. Eq.*, 19(9-10): 1481–1500, 1994.

B. Kotschwar. On rotationally invariant shrinking Ricci solitons. *Pacific J. Math.*, 236(1): 73–88, 2008.

R. Lafuente. Scalar curvature behavior of homogeneous Ricci flows. *J. Geom. Anal.*, 25, 12 2015.

J. Lauret. Ricci flow of homogeneous manifolds. *Math. Z.*, 274(1-2):373–403, 2013.

X. Li and Y. Zhang. Ancient solutions to the Ricci flow in higher dimensions. *arXiv preprint arXiv:1812.04156*, 2018.

J. Lott and N. Šešum. Ricci flow on three-dimensional manifolds with symmetry. *Comm. Math. Helv*, 89(1):1–32, 2014.

J. Lott and Z. Zhang. Ricci flow on quasiprojective manifolds 2. *J. Eur. Math. Soc.*, 18 (8):1813–1854, 2016.

P. Lu and G. Tian. Uniqueness of standard solutions in the work of perelman. `http://www.math.lsa.umich.edu/lott/ricciflow/perelman.html`.

T. Marxen. Convergence of Ricci Flow on a Class of Warped Product Metrics. *J. Geom. Anal.*, pages 1–35, 2019.

D. Máximo. On the blow-up of four-dimensional ricci flow singularities. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2014(692):153–171, 2014.

V. Minerbe. On the asymptotic geometry of gravitational instantons. *Ann. Sci. de l'Ecole Norm. Superieure*, Ser. 4, 43(6):883–924, 2010.

A. Naber. Noncompact shrinking four solitons with nonnegative curvature. *J. Reine Angew. Math.*, 2010(645):125–153, 2010.

K. Nomizu. On local and global existence of Killing vector fields. *Ann. of Math.*, 72(1): 105–120, 1960. ISSN 0003486X.

T. Oliynyk and E. Woolgar. Rotationally symmetric Ricci flow on asymptotically flat manifolds. *Comm. Anal. Geom.*, 15(3):535–568, 2007.

D. Page. A compact rotating gravitational instanton. *Physics Letters B*, 79(3):235–238, 1978.

D. N. Page and C. Pope. Inhomogeneous Einstein metrics on complex line bundles. *Class. Quantum Gravity*, 4(2):213, 1987.

G. Perelman. The entropy formula for the Ricci flow and its geometric applications. *arXiv preprint math/0211159*, 2002.

G. Perelman. Finite extinction time for the solutions to the ricci flow on certain three-manifolds. *arXiv preprint math/0307245*, 2003a.

G. Perelman. Ricci flow with surgery on three-manifolds. *arXiv preprint math/0303109*, 2003b.

M. H. Protter and H. F. Weinberger. *Maximum principles in differential equations*. Springer Science & Business Media, 2012.

Y. Sakane. Examples of compact Einstein Kähler manifolds with positive Ricci tensor. *Osaka J. Math.*, 23(3):585–616, 1986.

O. Schnürer, F. Schulze, and M. Simon. Stability of Euclidean space under Ricci flow. *Comm. Anal. Geom.*, 16(1):127–158, 2008.

N. Šešum. Linear and dynamical stability of Ricci-flat metrics. *Duke Math. J.*, 133(1): 1–26, 2006.

W.-X. Shi. Deforming the metric on complete Riemannian manifolds. *J. Differ. Geom.*, 30(1):223–301, 1989.

A. Song. Appearance of stable minimal spheres along the Ricci flow in positive scalar curvature. *Geom. Topol.*, 23(7):3501 – 3535, 2019.

J. Song. Some type-I solutions of Ricci flow with rotational symmetry. *Int. Math. Res. Not.*, 2015(16):7365–7381, 2014.

M. Stolarski. Curvature blow-up in doubly-warped product metrics evolving by Ricci flow. *arXiv preprint arXiv:1905.00087*, 2019.

P. Topping. *Lectures on the Ricci flow*, volume 325. Cambridge University Press, 2006.

P. Topping. Uniqueness of instantaneously complete Ricci flows. *Geom. Topol.*, 19(3): 1477–1492, 2015.

S. Unnebrink. Asymptotically flat 4-manifolds. *Differential Geom. Appl.*, 6(3):271–274, 1996.

H. Wu. On Type-II singularities in Ricci flow on $\mathbb{R}^N$. *Comm. Part. Diff. Eq.*, 39(11): 2064–2090, 2014.

Z.-H. Zhang. Gradient Shrinking Solitons with vanishing Weyl tensor. *Pacific J. Math.*, 242, 2008.