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CONTINUITY OF EIGENVALUES AND SHAPE OPTIMISATION FOR LAPLACE AND STEKLOV PROBLEMS

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Abstract. We associate a sequence of variational eigenvalues to any Radon measure on a compact Riemannian manifold. For particular choices of measures, we recover the Laplace, Steklov and other classical eigenvalue problems. In the first part of the paper we study the properties of variational eigenvalues and establish a general continuity result, which shows for a sequence of measures converging in the dual of an appropriate Sobolev space, that the associated eigenvalues converge as well. The second part of the paper is devoted to various applications to shape optimization. The main theme is studying sharp isoperimetric inequalities for Steklov eigenvalues without any assumption on the number of connected components of the boundary. In particular, we solve the isoperimetric problem for each Steklov eigenvalue of planar domains: the best upper bound for the k-th perimeter-normalized Steklov eigenvalue is $8\pi k$, which is the best upper bound for the k^{th} area-normalised eigenvalue of the Laplacian on the sphere. The proof involves realizing a weighted Neumann problem as a limit of Steklov problems on perforated domains. For $k = 1$, the number of connected boundary components of a maximizing sequence must tend to infinity, and we provide a quantitative lower bound on the number of connected components. A surprising consequence of our analysis is that any maximizing sequence of planar domains with fixed perimeter must collapse to a point.

1 Introduction

For a compact, connected Riemannian manifold (M, g) of dimension d, with or without C^1 boundary ∂M, the Laplace eigenvalue problem consists in determining all $\lambda \in \mathbb{R}$ for which the following eigenvalue problem admits a nontrivial solution:

$$\begin{cases} -\Delta_g u = \lambda u & \text{in } M, \\ \partial_n u = 0 & \text{on } \partial M, \text{ when } \partial M \neq \varnothing, \end{cases}$$

where $\partial_n u$ is the outward normal derivative of u. Similarly, when $\partial M \neq \varnothing$ the Steklov problem consists in determining all $\sigma \in \mathbb{R}$ such that the following boundary value problem admits a nontrivial solution:

$$\begin{cases} \Delta_g u = 0 & \text{in } M, \\ \partial_n u = \sigma u & \text{on } \partial M. \end{cases}$$

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The eigenvalues of these problems form nondecreasing sequences

$$0 = \lambda_0(M,g) < \lambda_1(M,g) \leqslant \lambda_2(M,g) \leqslant \dots \nearrow \infty,$$
$$0 = \sigma_0(M,g) < \sigma_1(M,g) \leqslant \sigma_2(M,g) \leqslant \dots \nearrow \infty,$$

where each eigenvalue is repeated according to multiplicity. For each $k \in \mathbb{N}$, we investigate sharp upper bounds for $\lambda_k(M,g)$ and $\sigma_k(M,g)$. To that end, we define the scale invariant quantities

$$\overline{\lambda}_k(M,g) := \lambda_k(M,g)\,\mathrm{Vol}_g(M)^{\frac{2}{d}} \qquad \text{and} \qquad \overline{\sigma}_k(M,g) := \sigma_k(M,g)\frac{\mathscr{H}^{d-1}(\partial M)}{\mathrm{Vol}_g(M)^{\frac{d-2}{d}}}.$$

Here, $\mathrm{Vol}_g(M)$ is the Riemannian volume of M and $\mathscr{H}_g^{d-1}$ is the associated $(d-1)$-dimensional Hausdorff measure. These normalisations are natural for both problems, see e.g. [GNY04, CEG11, FS11, GL21].

In the present paper we study the relation between $\overline{\sigma}_k(M,g)$ and $\overline{\lambda}_k(M,g)$. In order to do so, we introduce the unifying framework of *variational eigenvalues associated with a Radon measure*. Given a Radon measure μ on M, we define

$$\lambda_k(M,g,\mu) := \inf_{F_{k+1}} \sup_{f\in F_{k+1}\setminus\{0\}} \frac{\int_M |\nabla_g f|_g^2 \,\mathrm{d}v_g}{\int_M f^2 \,\mathrm{d}\mu}, \tag{1.1}$$

where F_{k+1} is a $(k+1)$-dimensional subspace of $\mathrm{C}^\infty(M) \cap \mathrm{L}^2(M,\mu)$. To the best of our knowledge, variational eigenvalues for Radon measures were first defined to describe Laplacians on fractal sets, see e.g. the survey of Triebel [Tri08]. In the context of spectral bounds and shape optimisation, they first appeared in the work of Kokarev [Kok14] as a relaxation of the optimisation constraint. One should also see the influential work of Korevaar [Kor93] and especially of Grigor'yan–Netrusov–Yau [GNY04] where the spectrum of energy forms is investigated. The variational eigenvalues admit a natural normalisation

$$\overline{\lambda}_k(M,g,\mu) := \lambda_k(M,g,\mu)\frac{\mu(M)}{\mathrm{Vol}_g(M,g)^{\frac{d-2}{d}}}.$$

One of the main interest of introducing these variational eigenvalues is that they unify the presentation of several eigenvalue problems. For instance, for $\mu = \mathrm{d}v_g$ the volume measure associated to metric g, $\overline{\lambda}_k(M,g,\mu) = \overline{\lambda}_k(M,g)$, while for $\mu = \iota_*\mathrm{d}A_g$, the pushforward by inclusion of the boundary measure, $\overline{\lambda}_k(M,g,\mu) = \overline{\sigma}_k(M,g)$.

We present results of two types. On one hand, we study variational eigenvalues on their own. In Sect. 3 we establish the necessary functional analysis preliminaries. We define p-admissible measures, which are essentially measures that can be viewed as elements of the dual space $(\mathrm{W}^{1,p}(M))^*$ with certain compactness properties, and give various examples of p-admissible measures. Section 4 is concerned with the properties of variational eigenvalues. For example, we show that the eigenvalues associated with a 2-admissible measure form a discrete unbounded sequence accumulating only at $+\infty$. The main result of this section is Proposition 4.11, which states that convergence of measures in $(\mathrm{W}^{1,p}(M))^*$ for appropriate values of p implies convergence of the eigenvalues.

On the other hand, we apply this continuity result to obtain the aforementioned relations between $\overline{\sigma}_k$ and $\overline{\lambda}_k$. We start with isoperimetric inequalities when $d = 2$, where the form of the results are cleaner. We also obtain quantitative bounds for the first non-trivial eigenvalue in terms of the number of boundary components and describe applications to the free boundary minimal surfaces. Convergence results and isoperimetry for an arbitrary $d \geqslant 2$ are formulated in Theorem 1.11 below. We finish the introduction by stating some results in spectral flexibility which follow from our results on approximations.

1.1 Optimal isoperimetric inequalities for surfaces. Maximisation of Steklov eigenvalues normalised by perimeter goes back to the work of Weinstock [Wei54]. He proved that for simply-connected planar domains, $\overline{\sigma}_1(\Omega) \le 2\pi$, with equality if and only if Ω is a disk. This was followed by works of Hersch–Payne–Schiffer [HPS75], then later Girouard–Polterovich [GP12] and Karpukhin [Kar17] who proved that

$$\overline{\sigma}_k(M) \le 2\pi(k + \gamma + b - 1),$$

this time for compact surfaces M of genus γ with b connected boundary components. It follows from Girouard–Polterovich [GP10] that for $\gamma = 0$ and $b = 1$, this bound is saturated by a sequence of simply-connected domains $\Omega^\varepsilon \subset \mathbb{R}^2$ that degenerates to a union of k identical disks as $\varepsilon \to 0$. Bounds for $\overline{\sigma}_k$ which do not depend on the number of boundary components are notably more elusive. For M a compact orientable surface of genus γ with boundary, it was proved by Kokarev [Kok14] that

$$\overline{\sigma}_1(M) \leqslant 8\pi(\gamma + 1). \tag{1.2}$$

The bound (1.2) was later generalized in [KS20] in the following way. The *conformal eigenvalues* of a compact Riemannian manifold (M, g) are defined as

$$\Lambda_k(M, [g]) = \sup_{h \in [g]} \overline{\lambda}_k(M, h).$$

By the work of Korevaar [Kor93] $\Lambda_k(M, [g]) < +\infty$. See also Hassannezhad [Has11] and Colbois–El Soufi [CE03].

Theorem 1.1 (Karpukhin–Stern [KS20]). *Given any closed Riemannian surface (M, g) and any C^1 domain $\Omega \subset M$, one has*

$$\overline{\sigma}_1(\Omega, g) < \Lambda_1(M, [g]) \qquad \textit{and} \qquad \overline{\sigma}_2(\Omega, g) < \Lambda_2(M, [g]). \tag{1.3}$$

One obtains (1.2) from (1.3) by using the Yang–Yau bound $\overline{\lambda}_1(M, g) \leqslant 8\pi(\gamma + 1)$ [YY80]. The first result of the present paper is a non-strict version of (1.3) valid for all values of k.

Theorem 1.2. *Let (M,g) be a compact Riemannian surface and let $\Omega \subset M$ be a smooth domain such that $\partial\Omega \cap \partial M$ is either empty or equal to ∂M. Then one has*

$$\overline{\sigma}_k(\Omega,g) \leqslant \Lambda_k(M,[g]). \tag{1.4}$$

This inequality is sharp for each k.

Remark 1.3. The corresponding inequality with $\Lambda_k(M,g)$ replaced with $\overline{\lambda}_k(M,g)$ is not true in general. This will be made explicit in Theorem 1.14. The proof of Theorem 1.2 is very different from that of Theorem 1.1. It is simpler, works for all values of k as well as for surfaces with boundary, but does not imply the strict inequality.

The proof of Theorem 1.2 has two main parts. First, to prove inequality (1.4), we construct a sequence of conformal metrics $e^{f_n}g$ concentrating near the boundary $\partial\Omega$. By concentrating the metric on M close to the boundary we can approximate transmission eigenvalues on M accross $\partial\Omega$ with the corresponding Laplace eigenvalues in any given conformal class. This procedure is reminiscent of the construction of Lamberti–Provenzano [LP15] and Arrieta–Jiménez-Casas–Rodríguez-Bernal [AJCRB08]. Those transmission eigenvalues are always larger than the Steklov eigenvalues. The assumption on $\partial\Omega$ is purely technical: it ensures that a tubular neighbourhood of $\partial\Omega$ is a smooth domain in M, which in turn greatly simplifies the convergence estimates.

On the other hand, sharpness of inequality (1.4) follows from the next theorem for closed surfaces.

Theorem 1.4 (Girouard–Lagacé [GL21]). *For any closed Riemannian surface (M,g), any $f \in C^\infty(M)$ and any $k \geqslant 0$ there exists a sequence of domains $\Omega^\varepsilon \subset M$ such that*

$$\overline{\sigma}_k(\Omega^\varepsilon,g) \xrightarrow{\varepsilon\to 0} \overline{\lambda}_k(M,e^f g).$$

The domains Ω^ε are obtained via homogenisation, by removing small disks from M, in such a way that the boundary measure of Ω^ε converges to the volume measure of $(M,e^f g)$. While the first process involved concentrating all the weight of the domain at the boundary, here we equidistribute boundary weight inside the manifold to approximate the interior measure. To see that this theorem implies the sharpness of inequality (1.4), one can repeatedly apply it to a sequence of conformal metrics $g_n = e^{f_n}g$ such that $\overline{\lambda}_k(M,g_n) \to \Lambda_k(M,[g])$. We extend Theorem 1.4 to manifolds with boundary below in Theorem 1.12.

Both parts of the proof of Theorem 1.2 have a very similar structure. With that in mind, we develop general results about continuity of eigenvalues with respect to measures used in the definition of their Rayleigh quotient. The continuity criteria that we develop in this paper are natural and flexible. In addition to being used in the proof of Theorem 1.2, we also use them to stufy various spectral convergence problems that had not been considered in the litterature so far.

The following notations allows us to clarify the statement of our results:

$$\Sigma_k(M,g) = \sup_{\Omega\subset M} \overline{\sigma}_k(\Omega,g),$$
$$\Lambda_k(M) = \sup_{g} \overline{\lambda}_k(M,g),$$
$$\Sigma_k(M) = \sup_{g,\,\Omega\subset M} \overline{\sigma}_k(\Omega,g),$$

where M is a compact surface and Ω is a C^1 domain. For these optimal eigenvalues the results of this section can be summarized as follows.

Corollary 1.5. *For any compact surface M, any conformal class $[g]$ on M and any $k \geqslant 0$ one has*

$$\Sigma_k(M,g) = \Lambda_k(M,[g]), \quad \Sigma_k(M) = \Lambda_k(M).$$

In particular, using the known results on the exact values of $\Lambda_k(M)$ obtained in [KNPP21, Kar21] respectively,

$$\Sigma_k(\mathbb{S}^2) = 8\pi k,$$
$$\Sigma_k(\mathbb{RP}^2) = 4\pi(2k+1). \tag{1.5}$$

1.2 Optimal isoperimetric inequalities for planar domains. Because any domain $\Omega \subset \mathbb{R}^2$ is diffeomorphic to a domain in the sphere $\mathbb{S}^2$, it follows from (1.5) that $\overline{\sigma}_k(\Omega,g_0) \leq 8\pi k$. Following the ideas of [GL21] we show that this inequality remains sharp for planar domains.

Theorem 1.6. *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply-connected domain with C^1 boundary. There exists a sequence $\Omega^\varepsilon \subset \Omega$ of subdomains, with $\partial\Omega \subset \partial\Omega^\varepsilon$, such that*

$$\overline{\sigma}_k(\Omega^\varepsilon,g_0) = \sigma_k(\Omega^\varepsilon,g_0)\mathscr{H}^1(\partial\Omega^\varepsilon) \xrightarrow{\varepsilon\to 0} 8\pi k.$$

In particular,

$$\Sigma_k(\mathbb{R}^2) := \sup_{\Omega\subset\mathbb{R}^2} \overline{\sigma}_k(\Omega,g_0) = 8\pi k.$$

The domains Ω^ε are obtained by removing small disks from Ω. In particular, this solves [GP17, Open problem 2] for $d = 2$.

1.3 Quantitative isoperimetric bounds for $\overline{\sigma}_1$. Following [FS16, Theorem 4.3], it was suggested in [GP17] that the number of boundary components in a maximizing sequence for $\Sigma_1(\mathbb{S}^2)$ needs to be unbounded. Indeed, let $M_{0,b}$ be a a compact orientable surface of genus 0 with b boundary components and define

$$\Sigma_1(0,b) = \sup_{g} \overline{\sigma}_1(M_{0,b},g).$$

The monotonicity results of [FS16, Theorem 4.3] and [MP20, Theorem 1.3] imply that $\Sigma_1(0,b)$ is strictly monotone in b. Thus, Theorem 1.4 yields

$$\Sigma_1(0,1) < \cdots < \Sigma_1(0,b) < \Sigma_1(0,b+1) < \ldots \nearrow 8\pi,$$

which confirms the claim, yielding the direct corollary.

Corollary 1.7. *Any sequence of surfaces M^ε of genus 0 such that $\overline{\sigma}_1(M^\varepsilon) \xrightarrow{\varepsilon\to 0} 8\pi$ has unbounded number of connected boundary components.*

We refine Corollary 1.7 and obtain at the same time a quantitative improvement to Kokarev's bound (1.2).

Theorem 1.8. *For every $\varepsilon > 0$ there exists $C > 0$ such that for every $b \geqslant 1$ and every metric g*

$$\overline{\sigma}_1(M_{0,b}, g) \leqslant 8\pi - C\exp(-(1+\varepsilon)b). \tag{1.6}$$

This theorem follows from the more general Theorem 2.1 and the constant C is explicitly computable in terms of ε. It seems unlikely that this bound is sharp, yet there does not seem to be any obvious candidate for the sharp bound. We discuss in more details in Sect. 2 the parts of the proof where sharpness may be lost.

For planar domains, Theorem 2.1 also leads to the following bound, which implies that any $\Sigma_1(\mathbb{R}^2)$-maximising sequence of domains with fixed perimeter shrinks to a point.

Theorem 1.9. *For every $\varepsilon > 0$ there is $C > 0$ such that for every connected bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary,*

$$\overline{\sigma}_1(\Omega) \leqslant 8\pi - C\exp\left(\frac{-(1+\varepsilon)}{2}\frac{\mathscr{H}^1(\partial\Omega)}{\operatorname{diam}(\Omega)}\right).$$

Remark 1.10. It follows from the seminal work of Fraser–Schoen [FS16] that free boundary minimal surfaces in the unit ball are intimately related to the maximal Steklov eigenvalues. In particular, given an embedded free boundary minimal surface in the unit ball, its coordinates are Steklov eigenfunctions with eigenvalue $\sigma = 1$. In [FL14, Conjecture 3.3], Fraser and Li conjectured that for each free boundary minimal surfaces M properly embedded in $\mathbb{B} \subset \mathbb{R}^3$, this Steklov eigenvalue is always the first one, so that in such a case $2\,\mathrm{Area}_g(M) = \mathscr{H}^1(\partial M) = \overline{\sigma}_1(M, g)$. We can read Theorem 2.1 in this setting. Let $M_{0,b}$ be a free boundary minimal surface of genus 0 with b boundary components properly embedded in $\mathbb{B} \subset \mathbb{R}^3$ by its first Steklov eigenfunctions. Then, for every $\varepsilon > 0$, with the constant $C > 0$ given by Theorem 1.8,

$$\mathrm{Area}_g(M_{0,b}) \leqslant 4\pi - \frac{C}{2}\exp(-(1+\varepsilon)b).$$

Under the Fraser–Li conjecture, this holds for all free boundary minimal surfaces of genus 0 with b connected boundary component that are properly embedded in the unit ball $\mathbb{B} \subset \mathbb{R}^3$. In other words, if the Fraser–Li conjecture is true properly embedded free boundary minimal surfaces of genus 0 with area close to 4π must have a large number of boundary components.

1.4 Isoperimetry and homogenisation for domains in $\mathbb{R}^d$. Theorem 1.6 is a consequence of a more general result valid for domains in $\mathbb{R}^d$. Let $\Omega \subset \mathbb{R}^d$ be a domain, that is a connected bounded open set with C^1 boundary. For $\beta : \Omega \to \mathbb{R}$ nonnegative and non trivial, consider the weighted Neumann problem

$$\begin{cases} -\Delta f = \lambda \beta f & \text{in } \Omega, \\ \partial_n f = 0 & \text{on } \partial\Omega. \end{cases}$$

We assume that $\beta \in \mathrm{L}^{d/2}(\Omega)$ if $d \geqslant 3$, and $\beta \in \mathrm{L}^1(\log \mathrm{L})^1(\Omega)$ if $d = 2$ (see p. 9 for the definition of this space which contains L^p, $p > 1$). If the flat metric on $\mathbb{R}^d$ is denoted by g_0, then the eigenvalues of this problem can be understood in the weak sense as in (1.1), as the variational eigenvalues $\lambda_k(\Omega, g_0, \beta \mathrm{d}v_{g_0})$.

Theorem 1.11. *For any domain $\Omega \subset \mathbb{R}^d$, and any nonnegative $0 \not\equiv \beta \in \mathrm{L}^{d/2}(\Omega)$ ($d \geqslant 3$) or $\beta \in \mathrm{L}^1(\log \mathrm{L})^1(\Omega)$ ($d = 2$), there exists a family $\Omega^\varepsilon \subset \Omega$ of domains such that for each $k \in \mathbb{N}$,*

$$\overline{\sigma}_k(\Omega^\varepsilon, g_0) \xrightarrow{\varepsilon \to 0} \overline{\lambda}_k(\Omega, g_0, \beta \mathrm{d}v_{g_0}).$$

For the same family Ω^ε,

$$\lambda_k(\Omega^\varepsilon, g_0) \xrightarrow{\varepsilon \to 0} \lambda_k(\Omega, g_0), \qquad \text{and} \qquad \mathrm{Vol}_{g_0}(\Omega^\varepsilon) \xrightarrow{\varepsilon \to 0} \mathrm{Vol}_{g_0}(\Omega).$$

We note that combining the methods of [GHL21, GL21], we could have proved a weaker form of this result, i.e. with β continuous, using an intermediate dynamical boundary value problem. The proof that we present here is more direct, allows for a more singular β, and gives more information on domains that are nearly maximizing $\overline{\lambda}_k$.

In order to prove this result, we realise the domains Ω^ε by removing tiny balls from Ω whose centres are periodically distributed. The construction is in the spirit of homogenisation theory, with the distinction that the radius of the balls removed is not uniform, but rather varies according to the a continuous approximation of the function β, and is chosen so that the total boundary area tends to ∞ in a controlled way as $\varepsilon \to 0$. Variation within periods in homogenisation theory has also been explored in [Pta15]. In our method of proof, the number of boundary components tends to ∞. By Theorem 1.8, this is unavoidable in dimension 2 since we can obtain planar domains with $\overline{\sigma}_1$ as close to 8π as we want. In higher dimension, it is possible to achieve the same result with only one boundary component, see [FS13, GL21].

Finally, we remark that a straightforward modification of our method yields an analogous result for compact Riemannian manifolds, see Remark 6.1 and a similar result for β continuous on closed Riemannian manifolds in [GL21, Theorem 1.1].

Theorem 1.12. *For any compact Riemannian manifold (M, g) of dimension d, and any nonnegative $0 \not\equiv \beta \in \mathrm{L}^{d/2}(M)$ ($d \geqslant 3$) or $\beta \in \mathrm{L}^1(\log \mathrm{L})^1(M)$ ($d = 2$), there exists a family $\Omega^\varepsilon \subset M$ of domains such that for each $k \in \mathbb{N}$,*

$$\overline{\sigma}_k(\Omega^\varepsilon, g) \xrightarrow{\varepsilon \to 0} \overline{\lambda}_k(M, g, \beta \mathrm{d}v_g).$$

Harmonic extensions of the associated eigenfunctions from Ω *to* M *converge strongly to eigenfunctions of the limit problem in* $\mathrm{W}^{1,2}(M)$.

For the same family Ω^ε,

$$\lambda_k(\Omega^\varepsilon, g) \xrightarrow{\varepsilon\to 0} \lambda_k(M,g), \qquad \text{and} \qquad \mathrm{Vol}_g(\Omega^\varepsilon) \xrightarrow{\varepsilon\to 0} \mathrm{Vol}_g(M).$$

Here, harmonic extensions of the associated eigenfunctions converge weakly to eigenfunctions of the limit problem in $\mathrm{W}^{1,2}(M)$.

1.5 Flexibility in the prescription of the Steklov spectrum. Bucur–Nahon [BN21] have recently shown that the Weinstock and Hersch–Payne–Schiffer inequalities are unstable, in the sense that there are simply-connected domains that are very far from the disk—or from a union of k identical disks—with their kth normalised eigenvalue arbitrarily close to $2\pi k$. In fact, they prove the following result.

Theorem 1.13. (Bucur–Nahon, [BN21, Theorem 1.1]) *Let* $\Omega_1, \Omega_2 \subset \mathbb{R}^2$ *be two bounded, conformally equivalent domains with smooth boundary. Then, there exists a sequence of open domains* Ω^ε *with uniformly bounded perimeter such that*

$$d_{Haus}(\partial\Omega^\varepsilon, \partial\Omega_1) \xrightarrow{\varepsilon\to 0} 0, \quad \text{and,} \quad \text{for all } k\in\mathbb{N},\ \overline{\sigma}_k(\Omega^\varepsilon) \xrightarrow{\varepsilon\to 0} \overline{\sigma}_k(\Omega_2).$$

The domains Ω^ε constructed in [BN21] are diffeomorphic to the original domains. They are obtained by a local perturbation of the boundary. We remark that a similar result can be obtained as an application of Theorem 1.11, see Remark 5.4 for details. However, the domains Ω^ε obtained this way have many small holes concentrated near the boundary $\partial\Omega_1$.

We further investigate flexibility results for the Steklov spectrum of domains in Euclidean space. In many ways, the Neumann and Steklov problems have similar features. This has led to an investigation of bounds for one eigenvalue problem in terms of the other, see e.g. [HS20, KS68]. It was previously thought that some universal inequalities between perimeter-normalised Steklov eigenvalues and area-normalised Neumann eigenvalues could exist. It is known from [GP10, Section 2.2] that normalised Steklov eigenvalues can be arbitrarily small while keeping the normalised Neumann eigenvalues bounded away from zero. We use Theorem 1.11 to prove that there are also domains with arbitrarily small area-normalised Neumann eigenvalues $\overline{\lambda}_k(\Omega, g_0)$, for which the Steklov eigenvalues are bounded away from zero.

Theorem 1.14. *There exists a sequence of planar domains* Ω^ε *such that the normalised Steklov eigenvalue* $\overline{\sigma}_1(\Omega^\varepsilon) \xrightarrow{\varepsilon\to 0} 8\pi$ *while for each* $k\in\mathbb{N}$, *the normalised Neumann eigenvalues satisfy* $\overline{\lambda}_k(\Omega^\varepsilon) \xrightarrow{\varepsilon\to 0} 0$.

The reader should compare with the results of [BHM21, Section 5], where another family of examples where $\overline{\sigma}_1(\Omega) \le \overline{\lambda}_1(\Omega)$ fails is constructed.

Remark 1.15. Similarly to Theorem 1.14, on any closed Riemannian surface (M,g) there exists a sequence of conformal metrics $g_n = e^{f_n} g$ and a sequence of domains $\Omega_n \subset M$ such that $\overline{\sigma}_1(\Omega_n, g_n) \xrightarrow{\varepsilon\to 0} \Sigma_1(M,g)$ while for each $k\in\mathbb{N}$, the normalised Laplace eigenvalues satisfy $\overline{\lambda}_k(M, g_n) \xrightarrow{\varepsilon\to 0} 0$.

1.6 Plan of the paper, heuristics, and strategies. The majority of the paper is centred around Theorems 1.2 and 1.11; we either discuss their applications, develop the theory towards their proof and justify some constraints that become apparent in the proof. We note that the proof of both of these theorems are very similar in nature under the scheme that we develop.

In Sect. 2, we start by presenting applications of Theorem 1.11, including the proofs of Theorems 1.6 and 1.14. The proofs of Theorems 1.8, 1.9 are independent of the rest of the paper and are also presented there.

In Sect. 3, we present the general framework of variational eigenvalues associated to a Radon measure. This is a unifying framework which allows one to compare different, seemingly unrelated, eigenvalue problems on a manifold. We start with a general description of the setup and give conditions on and examples of measures giving rise to eigenvalues behaving like Laplace eigenvalues. Finally, we obtain continuity of the eigenvalues and eigenfunctions with respect to convergence of the measures in the dual of some appropriate Sobolev space.

An immediate application of the framework presented in Sect. 3, is given in Sect. 5. In particular, we prove that on any surface we can approximate Steklov-type eigenvalues with Laplace eigenvalues associated with a degenerating sequence of metrics, giving as an application a proof of Theorem 1.2.

1.7 Notation. We make here a list of notation that is explicitly reserved throughout the paper.

Manifolds and their domains. Whenever we mention a manifold or a surface without qualification, it may have nonempty boundary, which is always assumed to be C^1. In any PDE written in strong form, the boundary term may be ignored when the manifold has empty boundary. Closed manifolds and surfaces are compact and without boundary. We reserve the letter M for manifolds. When M has nonempty boundary, we denote by $\mathrm{int}(M)$ the set $M \setminus \partial M$.

A domain in a manifold M is a bounded open connected subset of M with C^1 boundary if its boundary is nonempty. We reserve the letters Ω and Υ for domains.

Standard measures and metrics. Let (M, g) be a Riemannian manifold. We denote the volume measure $\mathrm{d}v_g$. If there is a canonical metric on M, it is denoted by g_0. This could be the flat metric on $\mathbb{R}^d$ or the round metric on the sphere. It is usually a constant curvature metric. If M has a boundary, we write $\mathrm{d}A_g$ for the boundary measure induced by the metric. It is often useful to recall that $\mathrm{d}A_g := \mathscr{H}^{d-1}\llcorner\partial M$, where $\mathscr{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure induced by the metric g on M. We abuse notation and make no distinction between $\mathrm{d}A_g$ as a measure on ∂M, and the pushforward by inclusion $\iota_*\mathrm{d}A_g$ which is a measure on M.

In cases where confusion may arise, if we want to explicitly distinguish the restriction of $\mathrm{d}v_g$ to a domain $\Omega \subset M$ we write

$$\mathrm{d}v_g^\Omega := (\mathrm{d}v_g)\llcorner\Omega,$$

and similarly for $\Sigma \subset \partial M$,

$$\mathrm{d}A_g^\Sigma := (\mathrm{d}A_g)\lfloor\Sigma := \mathscr{H}^{d-1}\lfloor\Sigma.$$

Standard function spaces and capacity. Every vector space under consideration is defined over $\mathbb{R}$. For X a topological vector space, $\xi \in X^*, x \in X$ we denote by $\langle \xi, x\rangle_X := \xi(x)$ the duality pairing. Since all vector spaces are real, we use this notation to denote an inner product as well, without confusion.

For $p \in [1,\infty]$ we let p' be its Hölder conjugate and for $p \in [1,d)$ we let $p^\star$ be its Sobolev conjugate, given respectively by

$$p' = \frac{p}{p-1} \qquad \text{and} \qquad p^\star = \frac{pd}{d-p}.$$

In order to characterise critical scenarios in dimension 2, we will require a generalisation of the usual Lebesgue L^p and Sobolev $\mathrm{W}^{1,p}$ spaces. The first spaces we introduce are the Orlicz spaces $\mathrm{L}^p(\log \mathrm{L})^a$, for $p \geqslant 1$ and $a \in \mathbb{R}$ and $\exp \mathrm{L}^a$ for $a > 0$. For a reference on Orlicz space, see e.g. [BS88, Chapters 4.6–4.8]. The space $\mathrm{L}^p(\log \mathrm{L}^a)(M)$ consists of all measurable functions f such that

$$\int_M [|f|\,(\log(2+|f|))^a]^p \,\mathrm{d}v_g < \infty.$$

For $p > 1$ and $a \in \mathbb{R}$, or $p = 1, a \geqslant 0$, it can be endowed with the Luxemburg norm

$$\|f\|_{\mathrm{L}^p(\log L)^a(M)} = \inf\left\{\eta > 0 : \int_\Omega \left[|f/\eta|\,(\log(2+|f/\eta|))^a\right]^p \mathrm{d}v_g \leqslant 1\right\},$$

under which it is a Banach space. For $a > 0$, we also define the Orlicz spaces $\exp \mathrm{L}^a$ to be

$$\exp \mathrm{L}^a := \left\{ f : M \to \mathbb{R} \text{ measurable } : \exists \eta > 0 \text{ s.t. } \int_M \exp\left(|f/\eta|^a\right)\,\mathrm{d}v_g < \infty \right\}.$$

Just like the spaces $\mathrm{L}^p(\log \mathrm{L})^a$, they can be endowed with the Luxemburg norm

$$\|f\|_{\exp \mathrm{L}^a} = \inf\left\{\eta > 0 : \int_M \exp\left(|f/\eta|^a\right)\,\mathrm{d}v_g \leqslant 1\right\},$$

under which it is also a Banach space. The space $\exp L^1$ serves as a pairing space for $\mathrm{L}^1(\log \mathrm{L})^1$, see [BS88, Theorem 4.6.5], in the sense that there is $C > 0$ so that for $f \in \mathrm{L}^1(\log \mathrm{L})^1$, $\varphi \in \exp L^1$,

$$\int f\varphi \,\mathrm{d}v_g \leqslant C\,\|f\|_{\mathrm{L}^1(\log \mathrm{L})^1}\,\|\varphi\|_{\exp \mathrm{L}^1}.$$

We identify $\exp \mathrm{L}^1$ with the dual of $\mathrm{L}^1(\log \mathrm{L})^1$. For every $p \geqslant 1$ and $a, \varepsilon > 0$, we have the relations

$$\mathrm{L}^\infty(M) \subset \exp \mathrm{L}^a(M) \subset \mathrm{L}^p(M)$$

and

$$\mathrm{L}^{p+\varepsilon}(M) \subset \mathrm{L}^p(\log \mathrm{L})^a(M) \subset \mathrm{L}^p(M) \subset \mathrm{L}^p(\log \mathrm{L})^{-a}(M) \subset \mathrm{L}^{p-\varepsilon}(M).$$

We also define for $p \geqslant 1$ and $a \in \mathbb{R}$ the Orlicz–Sobolev spaces $\mathrm{W}^{1,p,a}(M)$ as

$$\mathrm{W}^{1,p,a}(M) := \{f \in \mathrm{L}^p(\log \mathrm{L})^a(M) : \nabla f \in \mathrm{L}^p(\log \mathrm{L})^a(M)\} \tag{1.7}$$

with the gradient being understood in the weak sense, see [Cia96, Section 2] for this definition. We note that for every $p \geqslant 1, a \geqslant 0, \varepsilon > 0$ we have the relations

$$\mathrm{W}^{1,p+\varepsilon}(M) \subset \mathrm{W}^{1,p,a}(M) \subset \mathrm{W}^{1,p}(M) \subset \mathrm{W}^{1,p,-a}(M) \subset \mathrm{W}^{1,p-\varepsilon}(M).$$

Finally, we will make use of the notion of p-capacity. Given two sets $\Upsilon \subset\subset \Omega \subsetneq M$, we write

$$\mathrm{C}_0^\infty(\Omega) := \{f \in \mathrm{C}^\infty(\Omega) \,:\, f \equiv 0 \text{ on } \partial\Omega \cap \mathrm{int}(M)\}\,.$$

The p-capacity of Υ with respect to Ω is defined as

$$\mathrm{cap}_p(\Upsilon, \Omega) := \inf \left\{ \int_M |\nabla f|^p \, \mathrm{d}v_g : f \in \mathrm{C}_0^\infty(\Omega), f \equiv 1 \text{ on } \Upsilon \right\},$$

and the p-capacity of Υ as

$$\mathrm{cap}_p(\Upsilon) := \inf \left\{ \mathrm{cap}_p(\Upsilon, \Omega) : \Omega \subsetneq M, 0 < \mathrm{Vol}_g(\Omega) \leqslant \mathrm{Vol}_g(M)/2 \right\}.$$

We note that if $\Omega \cap \partial M$ is not empty, we do not require in the definition of the capacity that f vanishes on that set.

Asymptotic notation. We make extensive use throughout the paper of the so-called Landau asymptotic notation. We write

- without distinction, $f_1 = O(f_2)$ or $f_1 \ll f_2$ to mean that there exists $C > 0$ such that $|f_1| \leqslant C f_2$;
- $f_1 \asymp f_2$ to mean that $f_1 \ll f_2$ and $f_2 \ll f_1$;
- $f_1 \sim f_2$ to mean that $\frac{f_1}{f_2} \to 1$;
- $f_1 = o(f_2)$ to mean that $\frac{f_1}{f_2} \to 0$.

The limits in the last two bullet points will either be as some parameter tends to 0 or ∞ and will be clear from context. The use of a subscript in the notation, e.g. $f_1 \ll_M f_2$ or $f_1 = o_k(f_2)$, indicates that the constant C, or the quantities involved in the definition of the limit may depend on the subscript.

2 Applications and Motivation.

In this section, we give application of Theorem 1.11 to shape optimisation for the Steklov problem in $\mathbb{R}^2$, and to spectral flexibility. We also provide the proofs of Theorems 1.8 and 1.9.

2.1 Approximation by Steklov eigenvalues. We start by proving Theorem 1.6 from Theorems 1.2 and 1.11.

Proof of Theorem 1.6. Let $\Omega \subset \mathbb{R}^2$ be a simply-connected C^1 domain. We know from [Her70, Pet14, KNPP21] that $\Lambda_k(\mathbb{S}^2) = 8\pi k$. Let $\delta > 0$, and g be a smooth metric on $\mathbb{S}^2$ such that such that

$$\overline{\lambda}_k(\mathbb{S}^2, g) > \Lambda_k(\mathbb{S}^2) - \delta = 8\pi k - \delta.$$

Let Υ be $\mathbb{S}^2$ with a small disk removed. It is well known that as the radius of that disk goes to 0, the Neumann eigenvalues $\lambda_k(\Upsilon, g)$ converge to $\lambda_k(\mathbb{S}^2, g)$, see [Ann86, Théorème 2]. Thus, removing a small enough disk ,

$$\overline{\lambda}_k(\Upsilon, g) > \overline{\lambda}_k(\mathbb{S}^2, g) - \delta.$$

Let $\Phi : \Omega \to \Upsilon$ be a conformal diffeomorphism. Since Dirichlet energy is a conformal invariant, the kth Neumann eigenvalue of Υ is equal to the variational eigenvalue $\lambda_k(\Omega, g_0, \Phi^*(\mathrm{d}v_g))$. The homogenisation Theorem 1.11 guarantees the existence of $\Omega^\varepsilon \subset \Omega$ such that

$$\sigma_k(\Omega^\varepsilon)\mathscr{H}^1(\partial\Omega^\varepsilon) > \lambda_k(\Omega, g_0, |\mathrm{d}\Phi|^2\,\mathrm{d}x) \int_\Omega \Phi^*(\mathrm{d}v_g) - \delta. \tag{2.1}$$

Putting this all back together yields the bound $\sigma_k(\Omega^\varepsilon)\mathscr{H}^1(\partial\Omega^\varepsilon) > 8\pi k - 3\delta$. Since $\delta > 0$ is arbitrary $\Sigma_k(\mathbb{R}^2) \geqslant 8\pi k$, and by Theorem 1.2 this is in fact an equality. □

The exact same proof can be used to obtain the comparison between Steklov and Neumann eigenvalues.

Proof of Theorem 1.14. For $\delta > 0$, proceed as in the proof of Theorem 1.6, but start with $\Omega \subset \mathbb{R}^2$ such that $\overline{\lambda}_k(\Omega) < \frac{\delta}{2}$, for instance a very thin rectangle. By Theorem 1.11, one can choose ε in (2.1) small enough that $\overline{\lambda}_k(\Omega^\varepsilon, g_0) < \delta$. This concludes the proof. □

2.2 Geometric and topological properties of maximising sequences. In the present section we prove Theorems 1.8 and 1.9

The domains Ω^ε constructed in Theorem 1.11, are obtained by removing many tiny balls whose total boundary length tends to $+\infty$. In particular, the length of each boundary component relative to the total length of the boundary tends to zero. We show that *any* maximizing sequence of domains for $\Sigma_1(\mathbb{S}^2)$ or $\Sigma_1(\mathbb{R}^2)$ exhibits this behaviour. Moreover, for any metric on $M_{0,b}$ one has the following quantitative relation between the relative length of the longest boundary component and the *Steklov spectral defect*

$$\mathrm{def}(M_{0,b}, g) := 8\pi - \overline{\sigma}_1(M_{0,b}, g).$$

Theorem 2.1. *Let* (M, g) *be a compact Riemannian surface of genus* 0 *with* b *boundary components and let* L *be the length of its longest boundary component. Then, whenever* $\operatorname{def}(M, g) \leqslant 4\pi$,

$$b \geqslant \frac{\mathscr{H}^1(\partial M)}{L} \geqslant \left(1 - \frac{\operatorname{def}(M,g)}{4\pi}\right) \log\left(\frac{8\pi}{\operatorname{def}(M,g)} - 1\right). \tag{2.2}$$

The first inequality is the trivial statement that $\mathscr{H}^1(\partial M_{0,b}) \leqslant Lb$, the main content of this theorem is the second inequality. One can interpret this result as a quantitative improvement of Kokarev's estimate (1.2). This is the essence of Theorems 1.8 and 1.9 which we now prove using Theorem 2.1.

Proof of Theorem 1.8. We may assume without loss of generality that $\overline{\sigma}_1 > 4\pi$ and therefore that $b \geqslant 3$, and that ε is sufficiently small. Exponentiating the leftmost and rightmost side in (2.2) and rearranging yields

$$\overline{\sigma}_1 \leqslant \frac{8\pi}{1 + \exp\left(\frac{-4\pi b}{(\overline{\sigma}_1 - 4\pi)}\right)}.$$

We see that when $8\pi > \overline{\sigma}_1 \geqslant 4\pi \frac{2+\varepsilon}{1+\varepsilon}$, then we get the upper bound

$$\overline{\sigma}_1 \leqslant \frac{8\pi}{1 + \exp\left(\frac{-4\pi b}{(\overline{\sigma}_1 - 4\pi)}\right)} \leqslant \frac{8\pi}{1 + \exp(-(1+\varepsilon)b)}.$$

To arrive at (1.6) we use the inequality $(1+x)^{-1} < 1 - x + x^2$ and choose C depending on ε to be such that

$$C \exp(-3(1+\varepsilon)) \leqslant \frac{4\pi\varepsilon}{1+\varepsilon} \leqslant \exp(-3) - \exp(-6)$$

for ε small enough. ◻

Proof of Theorem 1.9. Let Ω be a connected bounded domain in $\mathbb{R}^2$, and $\mathscr{C} \subset \partial\Omega$ be the boundary of the unbounded connected component of $\mathbb{R}^2 \setminus \Omega$. Then, we have that $2 \operatorname{diam}(\Omega) \leqslant \mathscr{H}^1(\mathscr{C}) \leqslant L$, where L is the length of the longest boundary component. The proof is completed in exactly the same way as above. ◻

The proof of Theorem 2.1 is based on Hersch's renormalisation scheme [Her70], as well as on a quantitative version of Kokarev's no atom lemma [Kok14, Lemma 2.1].

Let $\mathbb{B}$ be the unit ball in $\mathbb{R}^3$. For $\xi \in \mathbb{B}$, Hersch's conformal diffeomorphism $\Psi_\xi : \mathbb{S}^2 \to \mathbb{S}^2$ is defined as

$$\Psi_\xi(x) := \frac{(1 - |\xi|^2)x + 2(1 + \xi \cdot x)\xi}{|\xi + x|^2}.$$

Lemma 2.2. (Hersch's renormalisation scheme, see [GNP09, Lau21]). *Let μ be a measure on $\mathbb{S}^2$ such that for all $x \in \mathbb{S}^2$, $\mu(\{x\}) \leqslant \frac{1}{2}\mu(\mathbb{S}^2)$. Then, there exists a unique $\xi \in \mathbb{B}$ such that the pushforward measure $(\Psi_\xi)_*\mu$ has its center of mass at the origin. In other words, for $j \in \{1,2,3\}$, the coordinate functions $x_j : \mathbb{S}^2 \to \mathbb{R}$ satisfy*

$$\int_{\mathbb{S}^2} x_j \, \mathrm{d}(\Psi_\xi)_*\mu = \int_{\mathbb{S}^2} x_j \circ \Psi_\xi \, \mathrm{d}\mu = 0. \tag{2.3}$$

Remark 2.3. In the classical formulation of Hersch's scheme as in e.g. [GNP09] the measure μ is precluded from having points of non-zero mass. In the form presented here the measure μ is allowed to have point masses. The proof is different from the classical topological arguments and can be found in [Lau21].

Given $y \in \mathbb{S}^2$, we define the closed hemisphere

$$\mathbb{S}^2_y := \left\{x \in \mathbb{S}^2 : x \cdot y \geqslant 0\right\}.$$

For $\Omega \subset \mathbb{S}^2_y$, recall that we define the capacity of Ω in $\mathbb{S}^2_y$ as

$$\operatorname{cap}_2(\Omega, \mathbb{S}^2_y) = \inf\left\{\int_{\mathbb{S}^2_y} |\nabla f|^2 \, \mathrm{d}v_g : f \in \mathrm{C}^\infty_0(\mathbb{S}^2_y),\ f\big|_\Omega \equiv 1\right\}.$$

Lemma 2.4. *Let $K_a \subset \mathbb{S}^2_y$ be a closed spherical cap of area $a < 2\pi$ centred at $y \in \mathbb{S}^2$. The capacity of K_a in $\mathbb{S}^2_y$ is given by*

$$\operatorname{cap}_2(K_a, \mathbb{S}^2_y) = \frac{4\pi}{\log(\frac{4\pi}{a} - 1)}.$$

Proof. Let $\Phi : \mathbb{D} \to \mathbb{S}^2$ be the stereographic parametrisation of $\mathbb{S}^2_y$. By elementary trigonometry, $\Phi^{-1}(K_a) = B(0, r_a) \subset \mathbb{D}$, where

$$r_a = \sqrt{\frac{a}{4\pi - a}}.$$

Let $\chi_a : \mathbb{D} \to \mathbb{R}$ be the capacitary potential for $B(0, r_a)$, i.e. the radial function defined in polar coordinates (t, θ) as

$$\chi_a(t) := \begin{cases} \frac{\log t}{\log r_a} & \text{for } r_a < t \leqslant 1, \\ 1 & \text{for } 0 \leqslant t \leqslant r_a. \end{cases}$$

It follows by invariance of the Dirichlet energy under conformal transformations that $(\Phi^{-1})^*\chi_a$ is the capacitary potential of K_a, and thus that

$$\operatorname{cap}_2(K_a, S^2_y) = \int_{\mathbb{D}} |\nabla \chi_a|^2 \, dv_{g_0} = \int_{r_a}^1 \left(\partial_t \chi_a(t)^2\right) t \, \mathrm{d}t = \frac{4\pi}{\log(\frac{4\pi}{a} - 1)}. \qquad \square$$

Proof of Theorem 2.1. The proof is based on Kokarev's proof of (1.2), keeping a precise track of all quantities involved. Note that the theorem is trivially true when $\operatorname{def}(M,g) \geqslant 4\pi$, so we assume that $\operatorname{def}(M,g) < 4\pi$. Let $\mathscr{C} \subset \partial M$ be the longest connected component of the boundary and fix $y \in \mathbb{S}^2$. It follows from the Koebe uniformization theorem that there exists a diffeomorphism $\Phi : M \to \Omega \subset \mathbb{S}^2_y$, conformal in the interior of M, sending $\mathscr{C}$ to the equator, i.e $\Phi(\mathscr{C}) = \partial\mathbb{S}^2_y$. Let $\mu := \Phi_* \mathrm{d}s$. be the pushforward of the boundary measure by Φ. The equator carries the length of $\mathscr{C}$:

$$\mu(\partial\mathbb{S}^2_y) = \mathscr{H}^1(\mathscr{C}) \geq \frac{\mathscr{H}^1(\partial M)}{b}.$$

We apply the Hersch renormalisation scheme to the measure μ. By Lemma 2.2, there is a unique $\xi \in \mathbb{B}$ so that the measure $\zeta := (\Psi_\xi)_*\mu$ has its center of mass at the origin. In other words, we can read from (2.3) that for $j \in \{1,2,3\}$, the functions $x_j \circ \Psi_\xi \circ \Phi$ are trial functions for σ_1 on M. Thus, by conformal invariance of the Dirichlet energy,

$$\sum_{j=1}^3 \sigma_1(M,g) \int_{\partial M} x_j^2 \circ \Psi_\xi \circ \Phi \,\mathrm{d}s \leqslant \sum_{j=1}^3 \int_{\Psi_\xi(\mathbb{S}^2_y)} |\nabla_{g_0} x_j|^2 \,\mathrm{d}A_{g_0}.$$

Using the pointwise identities $\sum_{j=1}^3 x_j^2 = 1$ and $\sum_{j=1}^3 |\nabla_{g_0} x_j|^2 = 2$, this leads to a strict form of Kokarev's bound from [Kok14]:

$$\overline{\sigma}_1(\Omega,g) \leqslant 2\,\mathrm{Area}_{g_0}(\Psi_\xi(\mathbb{S}^2_y)) < 8\pi.$$

Because the total area of $\mathbb{S}^2$ is 4π, it follows that the opposite hemisphere $\mathbb{S}^2_{-y}$ is mapped by Ψ_ξ to a spherical cap with small area:

$$\mathrm{Area}_{g_0}\left(\Psi_\xi(\mathbb{S}^2_{-y})\right) \leq \frac{1}{2}\left(8\pi - \overline{\sigma}_1(\Omega,g)\right) = \frac{\operatorname{def}(\Omega,g)}{2}. \tag{2.4}$$

Let $z \in \mathbb{S}^2$ be the center of the spherical cap $K_a = \Psi_{\boldsymbol{\xi}}(\mathbb{S}^2_{-y})$, where $a = \mathrm{Area}_{g_0}(K_a)$. The center of the circle ∂K_a is $\kappa z \in \mathbb{B}$, where $2\pi(1-\kappa) = a < \operatorname{def}(\Omega,g)/2$. The spectral defect is smaller than 4π by hypothesis. Hence,

$$\kappa > 1 - \frac{\operatorname{def}(\Omega,g)}{4\pi} > 0.$$

Let $\pi_z : \mathbb{R}^3 \to \mathbb{R}$ correspond to the projection on the subspace $\mathbb{R}z$. That is, $\pi_z(x) := (x \cdot z)$. Then the measure $\rho := (\pi_z)_*\zeta = (\pi_z \circ \Psi_\xi)_*\mu$ is supported in the interval $(-1,1)$ and has an atom of weight $\mu(\partial S^2_y) = \mathscr{H}^1(\mathscr{C})$ located at $\kappa \in (0,1)$. Because the center of mass of ζ is the origin $0 \in \mathbb{B}$, we have

$$0 = \int_{-1}^1 t\,\mathrm{d}\rho \geq \int_{-1}^0 t\,\mathrm{d}\rho + \kappa\rho(\{\kappa\}) = \int_{-1}^0 t\,\mathrm{d}\rho + \kappa\mathscr{H}^1(\mathscr{C}).$$

In particular

$$\kappa \mathscr{H}^1(\mathscr{C}) \le \int_{-1}^{0} -t \,\mathrm{d}\rho < \rho(-1,0) = \zeta(\mathbb{S}^2_{-z}).$$

It follows that

$$\zeta(\mathbb{S}^2_{-z}) \ge \left(1 - \frac{\operatorname{def}(\Omega, g)}{4\pi}\right) \mathscr{H}^1(\mathscr{C}) > 0. \tag{2.5}$$

Let $\chi_a \in \mathrm{W}^{1,2}(\mathbb{S}^2)$ be the capacitary potential of $K_a \subset \mathbb{S}^2_z$, and $m_{\chi_a} = \frac{1}{\mathscr{H}^1(\partial M)} \int_{\mathbb{S}^2} \chi_a \,\mathrm{d}\zeta$. We can thus use $\chi_a - m_{\chi_a} \in \mathrm{W}^{1,2}(\mathbb{S}^2)$ as a trial function for $\sigma_1(M) = \lambda_1(\Omega, g_0, \mu)$. By Lemma 2.4

$$\sigma_1(M) \int_{\mathbb{S}^2} (\chi_a - m_{\chi_a})^2 \,\mathrm{d}\zeta \le \frac{4\pi}{\log(\frac{4\pi}{a} - 1)}.$$

Now, using that $\chi_a \equiv 1$ on K_a together with (2.5) we get

$$\begin{aligned}
\int_{\mathbb{S}^2} (\chi_a - m_{\chi_a})^2 \,\mathrm{d}\zeta &\ge \int_{K_a} (\chi_a - m_{\chi_a})^2 \,\mathrm{d}\zeta + \int_{\mathbb{S}^2_{-z}} (\chi_a - m_{\chi_a})^2 \,\mathrm{d}\zeta \\
&\ge \left((1 - m_{\chi_a})^2 + m_{\chi_a}^2 \left(1 - \frac{\operatorname{def}(M,g)}{4\pi}\right)\right) \mathscr{H}^1(\mathscr{C}) \\
&= \left(\left(2 - \frac{\operatorname{def}(M,g)}{4\pi}\right) m_{\chi_a}^2 - 2m_{\chi_a} + 1\right) \mathscr{H}^1(\mathscr{C}) \qquad (2.6) \\
&= \left(\frac{\overline{\sigma}_1(M,g)}{4\pi} m_{\chi_a}^2 - 2m_{\chi_a} + 1\right) \mathscr{H}^1(\mathscr{C}) \\
&\ge \left(\frac{\overline{\sigma}_1(M,g) - 4\pi}{\overline{\sigma}_1(M,g)}\right) \mathscr{H}^1(\mathscr{C}),
\end{aligned}$$

where in the last step we have minimized the quadratic form. Putting all of this together leads to

$$\frac{\mathscr{H}^1(\partial M)}{\mathscr{H}^1(\mathscr{C})} \geqslant \left(1 - \frac{\operatorname{def}(\Omega, g)}{4\pi}\right) \log\left(\frac{4\pi}{a} - 1\right).$$

Recall that $a = \mathrm{Area}_{g_0}(K_a) \le \operatorname{def}(\Omega, g)/2$ to finish the proof. □

Remark 2.5. As was mentioned in the introduction, it is unlikely that the inequality in Theorem 2.1 is sharp. In its proof, we identify two main arguments where a loss of sharpness may have occured. First, in (2.4), we bound the deficit from below by the area of a single disk, whereas it could be bounded from below by the total area of all b disks in the complement of $\Psi_\xi(\Omega)$. This would lead to an improvement if all of those disks have comparable size. The arguments of [GL21] suggest that for sequences maximising the first Steklov eigenvalues all disks in the complement will have comparable size. However, we do not have a proof and it might not hold for all domains whose first normalised eigenvalue is close to the maximum. Another loss of sharpness is that the capacitary potential may not be the best trial function for σ_1. Finding a better trial function would improve the bounds obtained in (2.6).

3 Admissible Measures and Associated Function Spaces

The goal of this section is to properly define which measures allow for the definition of variational eigenvalues, and to define associated Sobolev-type spaces appropriate for our purpose. At the end of this section, we will provide explicit examples of admissible measures.

3.1 Sobolev-type spaces.

DEFINITION 3.1. *For $1 \leqslant p < \infty$, M a compact Riemannian manifold and μ a Radon measure on M, we define $\mathscr{W}^{1,p}(M,\mu)$ to be the completion of $\mathrm{C}^\infty(M)$ with respect to the norm*

$$\|u\|^p_{\mathscr{W}^{1,p}(M,\mu)} = \int_M |u|^p \,\mathrm{d}\mu + \int_M |\nabla u|_g^p \,\mathrm{d}v_g = \|u\|^p_{\mathrm{L}^p(M,\mu)} + \|\nabla u\|^p_{\mathrm{L}^p(M,g)}. \quad (3.1)$$

This completion (3.1) gives rise to an embedding $\tau_p^\mu : \mathscr{W}^{1,p}(M,\mu) \to \mathrm{L}^p(M,\mu)$ of norm 1.

In the classical setting where μ is the volume measure associated to g, the map τ_p^μ is the natural embedding of the Sobolev space $\mathrm{W}^{1,p}(M) \subset \mathrm{L}^p(M)$. If we want to make M explicit, we denote the embedding operator $\tau^\mu_{p,M}$. Since M is compact $\mathscr{W}^{1,p}(M,\mu) \subset \mathscr{W}^{1,q}(M,\mu)$ whenever $p \geqslant q$. For $1 < p < \infty$, the closed unit ball in $\mathscr{W}^{1,p}(M,\mu)$ is clearly weakly compact so that $\mathscr{W}^{1,p}(M,\mu)$ is a reflexive Banach space.

Convention. *We adopt the following conventions in order to make the notation a bit lighter for spaces and operators that appear often. We write $\mathrm{L}^p(M)$ for $\mathrm{L}^p(M,\mathrm{d}v_g)$, $\mathrm{L}^p(\partial M)$ for $\mathrm{L}^p(M,\mathrm{d}A_g)$ and $W^{1,p}(M) := \mathscr{W}^{1,p}(M,\mathrm{d}v_g)$. In general, the measure μ may be omitted from the notation when it is the natural volume measure given by the Riemannian metric, for instance as $\lambda_k(M,g) := \lambda_k(M,g,\mathrm{d}v_g)$.*

Denote the average of a function $f \in \mathrm{L}^1(M,\mu)$ by

$$m_{f,\mu} := \frac{1}{\mu(M)} \int_M f \,\mathrm{d}\mu.$$

DEFINITION 3.2. *We say that a Radon measure μ supports a p-Poincaré inequality if there is $K > 0$ such that for all $f \in \mathscr{W}^{1,p}(M,\mu)$*

$$\int_M (f - m_{f,\mu})^p \,\mathrm{d}\mu \leqslant K \int_M |\nabla f|^p \,\mathrm{d}v_g.$$

We denote by $K_{p,\mu}$ the smallest such number K.

For general measures, the space $\mathscr{W}^{1,p}(M,\mu)$ could be very different from the Sobolev space $\mathrm{W}^{1,p}(M)$ and solutions to (weak) elliptic PDEs in those spaces could lack the natural properties one expects from them. For that reason we restrict ourselves to a particular class of admissible measures, first introduced in [KS20] for $d = p = 2$, see also [Kok14] for a similar definition.

Definition 3.3. *Let M be a Riemannian manifold, $p \in (1,\infty)$, and μ be a Radon measure on M not supported on a single point. The measure μ is called p-admissible if it supports a p-Poincaré inequality and the operator τ_p^μ is compact. For $p = 2$, we simply say that μ is admissible.*

It is clear from that definition that $\mathrm{d}v_g$ and the boundary measure $\mathrm{d}A_g$ are p-admissible for all $p \in (1,\infty)$. The aim of the rest of this subsection is to prove the following two theorems. The first one gives a characterisation of p-admissible measures. The second one essentially says that when μ is a p-admissible measure there is an isomorphism between $\mathscr{W}^{1,p}(M,\mu)$ and $\mathrm{W}^{1,p}(M)$. Their proofs are intertwined but they are better stated separately for ease of reference.

Theorem 3.4. *Let μ be a Radon measure and $p \in (1,\infty)$. Then, μ is p-admissible if and only if the identity map on $\mathrm{C}^\infty(M)$ extends to a compact operator T_p^μ : $\mathrm{W}^{1,p}(M) \to \mathrm{L}^p(M,\mu)$.*

Theorem 3.5. *Let $p \in (1,\infty)$ and suppose that μ is not supported on a single point and supports a p-Poincaré inequality. There exists $c_{p,\mu}$, $C_{p,\mu} > 0$ so that for every $f \in \mathrm{C}^\infty(M)$*

$$c_{p,\mu} \|f\|_{\mathscr{W}^{1,p}(M,\mu)} \leqslant \|f\|_{\mathrm{W}^{1,p}(M)} \leqslant C_{p,\mu} \|f\|_{\mathscr{W}^{1,p}(M,\mu)} .$$

In particular, the completions $\mathscr{W}^{1,p}(M,\mu)$ and $\mathrm{W}^{1,p}(M)$ of $\mathrm{C}^\infty(M)$ are isomorphic.

We start by proving the first inequality in Theorem 3.5.

Proposition 3.6. *Let $p \in (1,\infty)$ and μ be a Radon measure on M supporting a p-Poincaré inequality. Suppose furthermore that μ is not supported on a single point. Then, there is $c_{p,\mu} > 0$ such that for all $f \in \mathrm{C}^\infty(M)$*

$$c_{p,\mu} \|f\|_{\mathscr{W}^{1,p}(M,\mu)} \leqslant \|f\|_{\mathrm{W}^{1,p}(M)} .$$

In particular, the identity map on $\mathrm{C}^\infty(M)$ extends to a bounded operator T_p^μ : $\mathrm{W}^{1,p}(M) \to \mathrm{L}^p(M,\mu)$.

Proof. If $p > d$, this follows from the boundedness of the map $\mathrm{W}^{1,p}(M) \to \mathrm{C}(M) \to \mathrm{L}^p(M,\mu)$. Suppose then that $p \leqslant d$. Since μ is not supported on a single point, supports a p-Poincaré inequality and points have vanishing p-capacity, this means that μ has no point masses.

We proceed in a similar manner to the proof of [Kok14, Lemma 2.2] where $d = p = 2$ and μ is a probability measure. For any $\Omega \subset M$ with $\mu(\Omega) > 0$, define $K_{p,*}(\Omega)$ via

$$\frac{1}{K_{p,*}(\Omega)} := \inf_{\substack{\mathrm{supp}(f)\subset\Omega \\ f\not\equiv 0}} \frac{\int_\Omega |\nabla f|^p \,\mathrm{d}v_g}{\int_\Omega |f|^p \,\mathrm{d}\mu}.$$

Let f be a smooth function supported on Ω and assume that $\mu(\Omega)^{p/p'}\mu(M)^{-p} \leqslant 2^{-p}$. From this assumption and Hölder's inequality,

$$\begin{aligned}\int_M |f - m_{f,\mu}|^p \, \mathrm{d}\mu &\geqslant 2^{1-p} \int_M |f|^p \, \mathrm{d}\mu - \int_M |m_{f,\mu}|^p \, \mathrm{d}\mu \\ &\geqslant \left(2^{1-p} - \frac{\mu(\Omega)^{p/p'}}{\mu(M)^{p-1}}\right) \int_\Omega |f|^p \, \mathrm{d}\mu \\ &\geqslant 2^{-p} \int_\Omega |f|^p \, \mathrm{d}\mu.\end{aligned}$$

We therefore have that for such Ω

$$\frac{1}{K_{p,\mu}} \leqslant \inf_{\substack{\mathrm{supp}(f)\subset\Omega \\ f\not\equiv 0}} \frac{\int_\Omega |\nabla f|^p \, \mathrm{d}v_g}{\int_\Omega |f - m_{f,\mu}|^p \, \mathrm{d}\mu} \leqslant \frac{2^p}{K_{p,*}(\Omega)}.$$

Since μ has no point masses there is a finite covering of M with domains $\{\Omega_j\}$ such that

$$0 < \mu(\Omega_j) < \frac{\mu(M)^{p'}}{2^{p'}}.$$

with associated smooth partition of unity $\left\{\rho_j^p\right\}$. Then, for all $f \in \mathrm{C}^\infty(M)$,

$$\int_M |f\rho_j|^p \, \mathrm{d}\mu \leqslant 2^{2p-1} K_{p,\mu} \left(\int_M |\nabla f|^p \, \rho_j^p + |\nabla \rho_j|^p \, |f|^p \, \mathrm{d}v_g\right).$$

Summing up those inequalities proves as we claimed that

$$\|f\|^p_{\mathscr{W}^{1,p}(M,\mu)} \leqslant (1 + 2^{2p-1}) K_{p,\mu} \sup_j \|\rho_j\|^p_{\mathrm{C}^1(M)} \|f\|^p_{\mathrm{W}^{1,p}(M)}. \qquad \square$$

As an immediate corollary, we get the necessity in Theorem 3.4.

Corollary 3.7. *Let $p \in (1,\infty)$ and μ be a p-admissible measure on M. Then, the identity map on $\mathrm{C}^\infty(M)$ extends to a compact operator $T_p^\mu : \mathrm{W}^{1,p}(M) \to \mathrm{L}^p(M,\mu)$.*

Proof. Proposition 3.6 implies that the identity map on $\mathrm{C}^\infty(M)$ extends to a bounded map $j : \mathrm{W}^{1,p}(M) \to \mathscr{W}^{1,p}(M,\mu)$.

But then $T_p^\mu = \tau_p^\mu \circ j$ is an extension of the identity which is the composition of a compact and bounded operator, hence itself compact. $\square$

One of our main tools going forward is estimates on (weak) solutions to the differential equation

$$\begin{cases} -\Delta\varphi_{\xi,\mu} = \mu - \frac{\mu(M)}{\xi(M)}\xi & \text{in } M, \\ \partial_\nu \varphi_{\xi,\mu} = 0 & \text{on } \partial M, \end{cases}$$

for measures ξ and μ. Note that $\mu - \frac{\mu(M)}{\xi(M)}\xi$ vanishes on constant functions, if they are shown to be in $\mathrm{W}^{1,p}(M)^*$ existence of a solution is easily guaranteed; we are specifically interested in estimating its norm in terms of trace operators and the Poincaré constants K_p. We require a generalisation of the Lax–Milgram theorem to Banach spaces.

Theorem 3.8 (Banach–Nečas–Babuška Theorem, [EG04, Theorem 2.6]). *Let X and Y be real Banach spaces, with Y being reflexive. Let a be a bilinear form on $X \times Y$. Then, for every $L \in Y^*$ there is a unique $x \in X$ such that for all $y \in Y$,*

$$a(x, y) = \langle L, y \rangle$$

if and only if a satisfies the Brezzi condition*, i.e. there exists $\kappa > 0$ such that*

$$\forall x \in X, \quad \kappa \left\| x \right\|_X \leqslant \sup_{y \in Y} \frac{a(x, y)}{\left\| y \right\|_Y}$$

and a is weakly nondegenerate, i.e. if $a(x, y) = 0$ for all $x \in X$, then $y = 0$.

Our goal is to use the Banach–Nečas–Babuška Theorem with $a : \mathscr{W}^{1,p'}(M, \mu) \times \mathscr{W}^{1,p}(M, \mu) \to \mathbb{R}$ given by

$$a(\varphi, f) = \int_M \nabla\varphi \cdot \nabla f \, \mathrm{d}v_g.$$

It is clearly weakly nondegenerate if we restrict ourselves to functions of zero mean. The following lemma establishes the Brezzi condition.

Lemma 3.9. *Let M be a Riemannian manifold, $p \in (1, \infty)$ with Hölder conjugate $p' = p/(p-1)$ and μ a Radon measure supporting a p-Poincaré inequality and such that $\tau_{p'}^{\mu}$ is compact. Then, there exists $\kappa > 0$ such that for all $\varphi \in \mathscr{W}^{1,p'}(M, \mu)$,*

$$\kappa \left\| \varphi - m_{\varphi,\mu} \right\|_{\mathscr{W}^{1,p'}(M,\mu)} \leqslant \sup_{0 \not\equiv f \in \mathscr{W}^{1,p}(M,\mu)} \frac{\int_M \nabla\varphi \cdot \nabla f \, \mathrm{d}v_g}{\left\| f \right\|_{\mathscr{W}^{1,p}(M,\mu)}}.$$

Proof. Towards a contradiction, we assume that such a κ does not exist. This implies the existence of a sequence $\varphi_n \in \mathscr{W}^{1,p'}(M, \mu)$ such that

$$\left\| \varphi_n - m_{\varphi_n,\mu} \right\|_{\mathrm{L}^{p'}(M,\mu)}^{p'} + \left\| \nabla\varphi_n \right\|_{\mathrm{L}^{p'}(M)}^{p'} = 1$$

and

$$\sup_{0 \not\equiv f \in \mathscr{W}^{1,p}(M,\mu)} \frac{\int_M \nabla\varphi_n \cdot \nabla f \, \mathrm{d}v_g}{\left\| f \right\|_{\mathscr{W}^{1,p}(M,\mu)}} \xrightarrow{n \to \infty} 0. \tag{3.2}$$

We first prove that if (3.2) goes to 0, then $|\nabla\varphi_n|_{\mathrm{L}^{p'}(M)}$ does as well. Since μ supports a p-Poincaré inequality, we have that

$$\begin{aligned}\sup_{0\not\equiv f\in\mathscr{W}^{1,p}(M,\mu)}\frac{\int_M\nabla\varphi_n\cdot\nabla f\,\mathrm{d}v_g}{\|f\|_{\mathscr{W}^{1,p}(M,\mu)}}&\geqslant\sup_{\substack{0\not\equiv f\in\mathscr{W}^{1,p}(M,\mu)\\ m_{f,\mu}=0}}\frac{\int_M\nabla\varphi_n\cdot\nabla f\,\mathrm{d}v_g}{\|f\|_{\mathscr{W}^{1,p}(M,\mu)}}\\ &\geqslant\sup_{0\not\equiv f\in\mathscr{W}^{1,p}(M,\mu)}\frac{\int_M\nabla\varphi_n\cdot\nabla f\,\mathrm{d}v_g}{(1+K_{p,\mu})\,\|\nabla f\|_{\mathrm{L}^p(M)}}.\end{aligned}\tag{3.3}$$

By density of smooth vector fields and duality, we have that

$$\|\nabla\varphi\|_{\mathrm{L}^{p'}(M)}=\sup_{F\in\Gamma(TM)}\frac{\int_M\nabla\varphi\cdot F\mathrm{d}v_g}{\|F\|_{\mathrm{L}^p(M)}},$$

where $\Gamma(TM)$ is the set of smooth vector fields on M.

By the L^p-Helmholtz decomposition of vector fields on C^1 domains, see e.g. [SSV14], we can write $F=F_1+F_2$ where $\operatorname{div}F_1=0$, $F_1\cdot\nu\big|_{\partial M}=0$, $F_2=\nabla f$ and $\|F_1\|_{\mathrm{L}^p(M)}+\|F_2\|_{\mathrm{L}^p(M)}\leqslant C_p\|F\|_{\mathrm{L}^p(M)}$. Here ν is the normal vector to the boundary. By the divergence theorem

$$\int_M\nabla\varphi\cdot F_1\mathrm{d}v_g=\int_M\operatorname{div}(\varphi F_1)-\varphi\operatorname{div}F_1\,\mathrm{d}v_g=\int_{\partial M}\varphi F_1\cdot\nu\,\mathrm{d}A_g=0.$$

As a result, one has

$$\|\nabla\varphi\|_{\mathrm{L}^{p'}(M)}\leqslant C_p\sup_{f\in\mathrm{C}^\infty(M)}\frac{\int_M\nabla\varphi\cdot\nabla f\,\mathrm{d}v_g}{\|\nabla f\|_{\mathrm{L}^p(M)}}.$$

Thus, from (3.3), we see that if (3.2) holds then $|\nabla\varphi_n|_{\mathrm{L}^{p'}}\to0$. Therefore, we have that φ_n is a sequence in $\mathscr{W}^{1,p'}(M,\mu)$ so that

$$\|\varphi_n-m_{\varphi_n,\mu}\|_{\mathrm{L}^{p'}(M,\mu)}\xrightarrow{n\to\infty}1\qquad\text{and}\qquad\|\nabla\varphi_n\|_{\mathrm{L}^{p'}(M)}\xrightarrow{n\to\infty}0.$$

By compactness of $\tau_{p'}^\mu$ there is $\varphi\in\mathscr{W}^{1,p}(M,\mu)$ such that φ_n converges to φ weakly in $\mathscr{W}^{1,p'}(M,\mu)$ and strongly in $\mathrm{L}^{p'}(M)$. In other words, φ is such that

$$\|\varphi-m_{\varphi,\mu}\|_{\mathrm{L}^{p'}(M,\mu)}=1\quad\text{and}\quad\|\nabla\varphi\|_{\mathrm{L}^{p'}(M)}=0.$$

This means that φ is constant a.e., and since $\tau_{p'}^\mu$ extends the identity on $\mathrm{C}^\infty(M)$, φ is also μ-a.e. constant. But then, $\|\varphi-m_{\varphi,\mu}\|_{\mathrm{L}^{p'}(M,\mu)}=0$, a contradiction. □

Lemma 3.10. *Let M be a compact Riemannian manifold, $p\in(1,\infty)$, ξ a Radon measure that supports a p-Poincaré inequality and such that $\tau_{p'}^\xi$ is compact; and μ be a Radon measure such that the identity on $\mathrm{C}^\infty(M)$ extends to a bounded operator*

$T_p^{\xi,\mu} : \mathscr{W}^{1,p}(M,\xi) \to \mathrm{L}^p(M,\mu)$. *Then, there exists a unique* $\varphi_{\xi,\mu} \in \mathscr{W}^{1,p'}(M,\xi)$ *with* $m_{\varphi_{\xi,\mu},\xi} = 0$ *and such that for all* $f \in \mathscr{W}^{1,p}(M,\xi)$

$$\int_M \nabla f \cdot \nabla \varphi_{\xi,\mu} \, \mathrm{d}v_g = \int_M f \, \mathrm{d}\mu - \frac{\mu(M)}{\xi(M)} \int_M f \, \mathrm{d}\xi. \tag{3.4}$$

Moreover, if μ *supports a p-Poincaré inequality,* $\varphi_{\xi,\mu}$ *satisfies*

$$\|\nabla \varphi_{\xi,\mu}\|_{\mathrm{L}^{p'}(M)} \leqslant (1 + K_{p,\mathrm{d}v_g}) \mu(M)^{1/p'} \left\| T_p^{\mu} \right\|. \tag{3.5}$$

Remark 3.11. The condition on the existence of $T_p^{\xi,\mu}$ is later shown to always be satisfied for p-admissible measures.

Proof. Let

$$X_p := \left\{ f \in \mathscr{W}^{1,p}(M,\xi) : m_{f,\xi} = 0 \right\}$$

and consider the bilinear form $a : X_{p'} \times X_p \to \mathbb{R}$ given by

$$a(\varphi, f) = \int_M \nabla \varphi \cdot \nabla f \, \mathrm{d}v_g.$$

It follows from Lemma 3.9 that a satisfies the Brezzi condition, and it is weakly nondegenerate on $X_{p'} \times X_p$. Furthermore, since μ has finite volume $1 \in \mathrm{L}^{p'}(M,\mu)$. This means that $L := (T_p^{\xi,\mu})^* 1 \in X_p^*$ and for $f \in X_p$

$$\langle L, f \rangle := \int_M f \, \mathrm{d}\mu. \tag{3.6}$$

By the Banach–Nečas–Babuška theorem there exists a unique $\varphi_{\mu,\xi} \in X_{p'}$ so that for all $f \in X_p$, $a(\varphi_{\xi,\mu}, f) = L(f)$. For $f \in \mathscr{W}^{1,p}(M,\xi)$, we obtain the identity (3.4) by noticing that formula (3.6) extends from X_p to $\mathscr{W}^{1,p}(M,\xi)$ and computing

$$\langle L, f \rangle = \langle L, f - m_{f,\xi} \rangle + \langle L, m_{f,\xi} \rangle = a(\varphi_{\mu,\xi}, f) + \frac{\mu(M)}{\xi(M)} \int_M f \, \mathrm{d}\xi.$$

We turn our attention to estimate (3.5). As in the proof of Lemma 3.9, we have that there exists $C_p > 0$ so that

$$\|\nabla \varphi_{\xi,\mu}\|_{\mathrm{L}^{p'}(M)} \leqslant C_p \sup_{f \in \mathrm{C}^\infty(M)} \frac{\int_M \nabla \varphi_{\xi,\mu} \cdot \nabla f \, \mathrm{d}v_g}{\|\nabla f\|_{\mathrm{L}^p(M)}}. \tag{3.7}$$

From the weak characterisation of $\varphi_{\xi,\mu}$ that for any $f \in \mathrm{C}^\infty(M)$,

$$\left| \int_M \nabla f \cdot \nabla \varphi_{\xi,\mu} \, \mathrm{d}v_g \right| = \left| \int_M f \, \mathrm{d}\mu - \frac{\mu(M)}{\xi(M)} \int_M f \, \mathrm{d}\xi \right|. \tag{3.8}$$

Since the left-hand side is invariant under addition of a constant to f, we may assume that $\int_M f \, \mathrm{d}\xi = 0$. By Hölder's inequality, if μ supports a p-Poincaré inequality we have that

$$\left| \int_M f \, \mathrm{d}\mu \right| \leqslant \mu(M)^{1/p'} \|f\|_{\mathrm{L}^p(M,\mu)} \quad \leqslant \mu(M)^{p'} (1 + K_{p,\mathrm{d}v_g}) \left\| T_p^\mu \right\| \|\nabla f\|_{\mathrm{L}^p(M)} ,$$

where T_p^μ is bounded from Proposition 3.6. Inserting this estimate into (3.8) and (3.7) completes the proof. □

We can now prove that the spaces $\mathrm{W}^{1,p}(M)$ and $\mathscr{W}^{1,p}(M,\mu)$ are isomorphic.

Proposition 3.12. *Suppose that the identity on* $\mathrm{C}^\infty(M)$ *extends to a bounded operator* $T_p^\mu : \mathrm{W}^{1,p}(M) \to \mathrm{L}^p(M,\mu)$*. Then, there is* $C_{p,\mu}$ *such that*

$$\|f\|_{\mathrm{W}^{1,p}(M)} \leqslant C_{p,\mu} \|f\|_{\mathscr{W}^{1,p}(M,\mu)} .$$

If moreover μ *supports a* p*-Poincaré inequality, then we can take*

$$C_{p,\mu} = \left(1 + K_{p,\mathrm{d}v_g}\right) \left(1 + \frac{\mathrm{Vol}_g(M)^{1+\frac{1}{p}}}{\mu(M)^{1-\frac{1}{p'}}} \right) \left(1 + \left\|T_p^\mu\right\|\right) .$$

Before carrying on with the proof, we note that we have proved in Proposition 3.6 that supporting a p-Poincaré inequality implies that T_p^μ is bounded, so that this proposition implies the second bound in Theorem 3.5.

Proof. We have that

$$\begin{aligned} \|f\|_{\mathrm{L}^p(M)} &\leqslant \|f - m_f\|_{\mathrm{L}^p(M)} + \|m_f\|_{\mathrm{L}^p(M)} \\ &\leqslant K_{p,\mathrm{d}v_g}^{1/p} \|\nabla f\|_{\mathrm{L}^p(M)} + \mathrm{Vol}_g(M)^{1/p} |m_f| . \end{aligned}$$

From Lemma 3.10 with $\xi = \mathrm{d}v_g$, there is $\varphi \in \mathrm{W}^{1,p'}(M)$ such that

$$\begin{aligned} \mathrm{Vol}_g(M)^{1/p} |m_f| &\leqslant \frac{\mathrm{Vol}_g(M)^{1+\frac{1}{p}}}{\mu(M)} \left[\left| \int_M \nabla f \cdot \nabla \varphi \, \mathrm{d}v_g \right| + \left| \int_M f \, \mathrm{d}\mu \right| \right] \\ &\leqslant \frac{\mathrm{Vol}_g(M)^{1+1/p}}{\mu(M)^{1-\frac{1}{p'}}} \left[\|\nabla\varphi\|_{\mathrm{L}^{p'}(M)} \|\nabla f\|_{\mathrm{L}^p(M)} + \|f\|_{\mathrm{L}^p(M,\mu)} \right] . \end{aligned}$$

The estimate on $C_{p,\mu}$ can be then be read from the bound on $\|\nabla\varphi\|_{\mathrm{L}^{p'}(M)}$ obtained in Lemma 3.10 under the p-Poincaré inequality condition. □

We can now prove sufficiency in Theorem 3.4.

Proposition 3.13. *Let* μ *be a Radon measure on* M *and suppose that the identity map on* $\mathrm{C}^\infty(M)$ *extends to a compact operator* $T_p^\mu : \mathrm{W}^{1,p}(M) \to \mathrm{L}^p(M,\mu)$*. Then,* μ *is* p*-admissible.*

Proof. Proposition 3.12 implies that the identity map on $\mathrm{C}^\infty(M)$ extends to a bounded map $j\colon \mathscr{W}^{1,p}(M,\mu) \to \mathrm{W}^{1,p}(M)$ Thus, since it is the composition of a compact and a bounded operator, $\tau_p^\mu = T_p^\mu \circ j$ is also compact. We now prove that the Poincaré inequality holds. Assume otherwise, then there exists a sequence of smooth functions f_n such that

$$\int_M f_n^p \,\mathrm{d}\mu = 1, \quad \int_M |\nabla f_n|^p \,\mathrm{d}v_g \to 0, \quad \int_M f_n \,\mathrm{d}\mu = 0.$$

By Proposition 3.12, the functions f_n are uniformly bounded in $\mathrm{W}^{1,p}(M)$. Since T_p^μ is compact, there is $f \in \mathrm{W}^{1,p}(M)$ such that, up to choosing a subsequence, $f_n \rightharpoonup f$ weakly in $\mathrm{W}^{1,p}(M)$ and $f_n \to T_p^\mu f$ strongly in $\mathrm{L}^p(M,\mu)$. By lower semicontinuity, $\|\nabla f\|_{\mathrm{L}^p(M)} = 0$, therefore, f is $\mathrm{d}v_g$-a.e. constant. Since T_p^μ extends the identity on $\mathrm{C}^\infty(M)$, $T_p^\mu f$ is a μ-a.e. constant, which contradicts the fact that

$$\int_M (T_p^\mu f)^p \,\mathrm{d}\mu = 1, \qquad \int_M T_p^\mu f \,\mathrm{d}\mu = 0. \qquad \square$$

We finally write the two following propositions that allow us to rewrite Lemma 3.10 with weaker conditions, for ease of reference. The first proposition indicates that p-admissibility is a monotone condition.

PROPOSITION 3.14. *Suppose that $T_p^\mu : \mathrm{W}^{1,p}(M) \to \mathrm{L}^p(M,\mu)$ is bounded. Then for all $q > p$, T_q^μ is compact. In particular, μ is q-admissible and admissibility is a monotone condition.*

Proof. If $q > d$, T_q^μ is compact since the embedding $\mathrm{W}^{1,q}(M) \to \mathrm{C}(M)$ is compact, so we suppose now that $p < q \leqslant d$. Compactness of T_q^μ follows from general interpolation theory. Given two compatible normed vector spaces, i.e. spaces X_0, X_1 that are both subspaces of a larger topological vector space V, Peetre's K-functional is defined on $f \in X_0 + X_1$ as

$$K(f,t,X_0,X_1) := \inf\left\{\|f_0\|_{X_0} + t\,\|f_1\|_{X_1} : f = f_0 + f_1,\ f_0 \in X_0,\ f_1 \in X_1\right\}.$$

For $0 < \theta < 1$ and $1 \leqslant q < \infty$, let $(X_0,X_1)_{\theta,q}$ be the interpolation space between X_0 and X_1 (see [BS88, Chapter 5]):

$$(X_0,X_1)_{\theta,q} := \left\{ f \in X_0 + X_1 : \|f\|_{\theta,q} := \left(\int_0^\infty \left(t^{-\theta} K(f,t,X_0,X_1)\right)^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} < \infty \right\}.$$

We use the interpolation theorem found in [CF89, Theorem 2.1] which states the following. Given Y_0, Y_1 compatible Banach spaces; X_0, X_1 Banach spaces such that X_1 is continuously embedded in X_0 and T is a linear operator such that $T : X_0 \to Y_0$ is bounded and the restriction $T : X_1 \to Y_1$ is compact. Then, for $0 < \theta < 1$ and $1 \leqslant q < \infty$, the operator

$$T : (X_0,X_1)_{\theta,q} \to (Y_0,Y_1)_{\theta,q} \qquad \text{is compact.}$$

Let $r > d$ and $\theta = \frac{r(q-p)}{q(r-p)} < 1$. We take Y_0 to be $\mathrm{L}^p(M,\mu)$ and Y_1 to be $\mathrm{L}^r(M,\mu)$, it follows from [BS88, Theorems 5.1.9 and 5.2.4] that $(Y_0, Y_1)_{\theta,q} = \mathrm{L}^q(M,\mu)$.

On the other hand, taking $X_0 = \mathrm{W}^{1,p}(M)$ and $X_1 = \mathrm{W}^{1,r}(M)$ it follows from [Bad09, Theorem 6.2, Corollary 1.3 and Remark 4.3], the later remark treating the case with C^1 boundary, that

$$(X_0, X_1)_{\theta,q} = \mathrm{W}^{1,q}(M).$$

Therefore, the interpolation theorem tells us that $T_q^\mu : \mathrm{W}^{1,q}(M) \to \mathrm{L}^q(M,\mu)$ is indeed compact. □

Proposition 3.15. *Let μ, ξ be two admissible measures. Then, the identity on $\mathrm{C}^\infty(M)$ extends to a compact operator $T_p^{\xi,\mu} : \mathscr{W}^{1,p}(M,\xi) \to \mathrm{L}^p(M,\mu)$.*

Proof. Define $T_p^{\xi,\mu}$ as the composition

$$\begin{array}{ccc} \mathscr{W}^{1,p}(M,\xi) & \xrightarrow{\ j\ } & \mathrm{W}^{1,p}(M) \\ & {\scriptstyle T_p^{\xi,\mu}} \searrow & \downarrow {\scriptstyle T_\mu} \\ & & \mathrm{L}^p(M,\mu) \end{array}$$

where by Theorem 3.5 j is bounded since ξ is p-admissible. By Theorem 3.4, T_p^μ is compact. Thus, $T_p^{\xi,\mu}$ is compact as the composition of a compact and a bounded operator, and it is an extension of the identity on $\mathrm{C}^\infty(M)$. □

We therefore rewrite the statement of Lemma 3.10 in the following way.

Lemma 3.16. *Let M be a compact Riemannian manifold, $p \in (1,2]$, ξ and μ both p-admissible measures. Then, there exists a unique $\varphi_{\xi,\mu} \in \mathrm{W}^{1,p'}(M)$ with $m_{\varphi_{\xi,\mu},\xi} = 0$ and such that for all $f \in \mathrm{W}^{1,p}(M)$,*

$$\int_M \nabla f \cdot \nabla \varphi_{\xi,\mu} \, \mathrm{d}v_g = \int_M f \, \mathrm{d}\mu - \frac{\mu(M)}{\xi(M)} \int_M f \, \mathrm{d}\xi.$$

Moreover,

$$\|\nabla \varphi_{\xi,\mu}\|_{\mathrm{L}^{p'}(M)} \leqslant (1 + K_{p,\mathrm{d}v_g}) \mu(M)^{1/p'} \left\| T_p^\mu \right\|.$$

3.2 Examples and admissibility criteria. Let us now give a few examples of admissible measures, as well as a local criterion that characterises them. We start with basic examples.

Example 3.17. On a smooth compact manifold M with C^1 boundary, the volume measure $\mathrm{d}v_g$ is p-admissible for every $p \in (1,\infty)$, as is the pushforward by inclusion of the boundary measure $\iota_* \, \mathrm{d}A_g$. Any linear combination of them is also p-admissible.

Example 3.18. It follows from the definition of the capacity that measures supported on a set of p-capacity zero do not support a p-Poincaré inequality, and as such are not admissible.

We now explore the edge cases of admissibility. We provide those examples for $p = 2$ since that is the context where they will be relevant. This last example allows us to obtain the weakest integrability condition on β in Theorem 1.11. We need to introduce the following characterisation of compactness beforehand. Maz'ya's compactness criterion [Maz11, Section 11.9.1] states that T_p^μ is compact if and only if

$$\lim_{r \to 0^+} \sup \left\{ \frac{\mu(\Upsilon)}{\operatorname{cap}_p(\Upsilon)} : \Upsilon \subset M,\ \operatorname{diam}(\Upsilon) \leqslant r \right\} = 0. \tag{3.9}$$

The isocapacitary inequality [Maz11, Equations 2.2.11 and 2.2.12] states that for every $\Upsilon \subset M$, with $\operatorname{Vol}_g(\Upsilon) \leqslant \operatorname{Vol}_g(M)/2$ that

$$\operatorname{cap}_p(\Upsilon) \gg_M \begin{cases} \log(1/\operatorname{Vol}_g(\Upsilon))^{1-d} & \text{if } p = d, \\ \operatorname{Vol}_g(\Upsilon)^{\frac{d-p}{d}} & \text{if } d > p. \end{cases} \tag{3.10}$$

Example 3.19. Let $0 \leqslant \beta \in \mathrm{L}^1(\log \mathrm{L})^1(M)$ (for $d = 2$) or $0 \leqslant \beta \in \mathrm{L}^{d/2}(M)$ (for $d > 2$) be a positive density and $\mu = \beta \mathrm{d}v_g$. Then, μ is admissible. For $1 \leqslant p < d/2$ (for $d \geq 3$), and for $p = 1$ (when $d = 2$), there exists $\beta \in \mathrm{L}^p(M)$ such that $\beta \mathrm{d}v_g$ is not an admissible measure. We split the proof of these claims in a few cases.

Case (i): $\mathbf{p < d/2}$, $\mathbf{d \geqslant 3}$. Consider any $x \in M$, $r_y = \operatorname{dist}(x, y)$ and

$$\beta(y) = \max \left\{ \frac{1}{r_y^{d/p} \log(1/r_y)}, 1 \right\}.$$

It is easy to see that $\beta \in \mathrm{L}^p$ and that the measure $\beta \mathrm{d}v_g$ fails Maz'ya's compactness criterion, hence T_μ is not compact and μ is not admissible.

Case (ii): $\mathbf{p = 1}$, $\mathbf{d = 2}$. Similarly to the previous case, for some $0 < \delta < 1$ let

$$\beta(y) = \max \left\{ \frac{1}{r_y^2 \log(1/r_y)^{1+\delta}, 1} \right\}.$$

This time, $\beta \in \mathrm{L}^1(M)$ but we can see that T_μ is not even bounded on $\mathrm{W}^{1,2}(M)$, so certainly not admissible. Indeed, consider the function $f(y) = -\log(r_y)^a$. It is a standard computation to see that $f \in \mathrm{W}^{1,2}(M)$ when $a < 1/2$. On the other hand, choosing $a = \delta/2$, for some $\varepsilon > 0$

$$\int_M f^2 \beta \, \mathrm{d}v_g \gg_M \int_0^\varepsilon \frac{\mathrm{d}r}{r \log(1/r)} = +\infty.$$

Case (iii): $\mathbf{d=2}$ and $\boldsymbol{\beta} \in \mathbf{L^1(\log L)^1}(M)$ or $\mathbf{d \geqslant 3}$ and $\mathbf{p \geqslant d/2}$. Suppose without loss of generality that $|\beta| \geqslant 1$ a.e. For $d=2$, it follows from Jensen's inequality with the convex function $\varphi(x) = x\log x$ that for any $\Upsilon \subset M$,

$$\log\left(\frac{1}{\mathrm{Vol}_g(\Upsilon)}\int_\Upsilon \beta \,\mathrm{d}v_g\right)\int_\Upsilon \beta\,\mathrm{d}v_g \leqslant \int_\Upsilon \beta\log\beta\,\mathrm{d}v_g.$$

In other words,

$$\log\left(\frac{1}{\mathrm{Vol}_g(\Upsilon)}\right)^{-1} \geqslant \frac{\mu(\Upsilon)}{\|\beta\|_{\mathrm{L}^1(\log \mathrm{L})^1(\Upsilon)} - \mu(\Upsilon)\log(\mu(\Upsilon))}.$$

Here, we supposed that Υ is chosen with diameter small enough that $\log(\mu(\Upsilon)) < 0$, ensuring that all quantities involved are positive. Inserting into Maz'ya's compactness criterion (3.9) along with the isocapacitary inequality (3.10) gives

$$\begin{aligned}\lim_{r\to0^+} &\sup\left\{\frac{\mu(\Upsilon)}{\mathrm{cap}_2(\Upsilon)} : \mathrm{diam}(\Upsilon) \leqslant r\right\}\\ &\leqslant \lim_{r\to0^+}\sup\left\{\|\beta\|_{\mathrm{L}^1(\log \mathrm{L})^1(\Upsilon)} - \mu(\Upsilon)\log(\mu(\Upsilon)) : \mathrm{diam}(\Upsilon)\leqslant r\right\}.\end{aligned}$$

Note that if $\|\beta\|_{\mathrm{L}^1(\log \mathrm{L})^1(\Upsilon)}$ goes to 0 uniformly in Υ, then so does $\mu(\Upsilon)$, hence $\mu(\Upsilon)\log(\mu(\Upsilon))$ as well.

For every Υ, let $\Upsilon_m = \Upsilon \cap \{\beta\log\beta < m\}$, and observe that $\|\beta\|_{\mathrm{L}^1(\log \mathrm{L})^1(\Upsilon)} = \|\beta\|_{\mathrm{L}^1(\log \mathrm{L})^1(\Upsilon_m)} + \|\beta\|_{\mathrm{L}^1(\log \mathrm{L})^1(\Upsilon\setminus\Upsilon_m)}$. It follows from density of smooth functions in L^1 that for every $\varepsilon > 0$, there is m_ε large enough so that for every $m > m_\varepsilon$,

$$\|\beta\|_{\mathrm{L}^1(\log \mathrm{L})^1(\Upsilon\setminus\Upsilon_m)} \leqslant \|\beta\|_{\mathrm{L}^1(\log \mathrm{L})^1(M\setminus M_m)} \leqslant \varepsilon,$$

which hence holds uniformly in Υ. On the other hand, if $\mathrm{diam}\,\Upsilon \leqslant r$,

$$\|\beta\|_{\mathrm{L}^1(\log \mathrm{L})^1(\Upsilon_m)} \leqslant m\log mr^2.$$

Taking $m = r^{-1}$, we have that for r small enough $m \geqslant m_\varepsilon$ and we deduce that for every $\varepsilon > 0$,

$$\lim_{r\to0^+}\sup\left\{\|\beta\|_{\mathrm{L}^1(\log \mathrm{L})^1(\Upsilon)} : \mathrm{diam}\,\Upsilon \leqslant r\right\} \leqslant \varepsilon$$

so that by Maz'ya's compactness criterion T_μ is compact and μ is admissible.

For $d > 2$, it follows from Hölder's inequality that

$$\mathrm{Vol}_g(\Upsilon)^{\frac{2-d}{d}} \leqslant \frac{\|\beta\|_{\mathrm{L}^{d/2}(\Upsilon)}}{\mu(\Upsilon)}.$$

Therefore, inserting in Maz'ya's criterion along with the isocapacitary inequality

$$\lim_{r\to0^+}\sup\left\{\frac{\mu(\Upsilon)}{\mathrm{cap}_2(\Upsilon)} : \mathrm{diam}\,\Upsilon\leqslant r\right\} \leqslant \lim_{r\to0^+}\sup\left\{\|\beta\|_{\mathrm{L}^{d/2}(\Upsilon)} : \mathrm{diam}\,\Upsilon \leqslant r\right\}.$$

The same argument as earlier but with the sets $\Upsilon_m = \Upsilon \cap \left\{\beta^{d/2} < m\right\}$ shows that this limit converges to 0, so that μ is admissible.

4 Variational Eigenvalues

Eigenvalue convergence results are ubiquitous in the literature and the proofs of a large number of them follow similar steps. In the present section we formulate these steps explicitly in sufficient generality to allow direct application to many natural eigenvalue problems, including both the Steklov and Laplace problems.

4.1 Variational eigenvalues associated to a Radon measure. We generalise to higher dimension the definition of eigenvalues associated to a measure, introduced in [Kok14] for surfaces. Let (M, g) be a compact Riemannian manifold.

For a Radon measure μ on M, we define the variational eigenvalues $\lambda_k(M, g, \mu)$ in the following way. For any $f \in \mathrm{C}^\infty(M)$ such that $f \not\equiv 0$ in $\mathrm{L}^2(M, \mu)$, we define the Rayleigh quotient $R_g(f, \mu)$ by

$$R_g(f, \mu) := \frac{\displaystyle\int_M |\nabla f|_g^2 \,\mathrm{d}v_g}{\displaystyle\int_M f^2 \,\mathrm{d}\mu}.$$

The eigenvalues $\lambda_k(M, g, \mu)$ are then given by

$$\lambda_k(M, g, \mu) := \inf_{F_{k+1}} \sup_{f \in F_{k+1} \setminus \{0\}} R_g(f, \mu), \tag{4.1}$$

where the infimum is taken over all $(k + 1)$-dimensional subspaces $F_{k+1} \subset \mathrm{C}^\infty(M)$ that remain $(k + 1)$-dimensional in $\mathrm{L}^2(M, \mu)$. A natural normalisation for these eigenvalues is

$$\overline{\lambda}_k(M, g, \mu) := \lambda_k(M, g, \mu) \frac{\mu(M)}{\mathrm{Vol}_g(M)^{\frac{d-2}{d}}},$$

see e.g. [GNY04].

The following proposition states that the eigenvalues of admissible measures possess all the natural properties one expects from eigenvalues of an operator of Laplace-type.

Proposition 4.1. *Let μ be an admissible measure. Then one has*

$$0 = \lambda_0(M, g, \mu) < \lambda_1(M, g, \mu) \leqslant \lambda_2(M, g, \mu) \leqslant \ldots \nearrow \infty;$$

i.e. the first eigenvalue is positive, the multiplicity of each eigenvalue is finite, and the eigenvalues tend to $+\infty$. Moreover, there exists an orthogonal basis of eigenfunctions $f_j \in \mathscr{W}^{1,2}(M, \mu)$ satisfying

$$\int_M \nabla f_j \cdot \nabla u \,\mathrm{d}v_g = \lambda_j(M, g, \mu) \int_M f_j u \,\mathrm{d}\mu$$

for all $u \in \mathscr{W}^{1,2}(M, \mu)$.

Proof. That $\lambda_1(M,g,\mu) > 0$ is readily seen to be equivalent to μ supporting a 2-Poincaré inequality. The rest of the proof is standard. The bilinear form

$$a(f,\varphi) = \int_M \nabla f \cdot \nabla\varphi \, \mathrm{d}v_g$$

is bounded and coercive on the set of functions of μ-average 0 in $\mathscr{W}^{1,2}(M,\mu)$. The statement is therefore a direct application of [BB92, Theorem 6.3.4] and the Courant–Fischer–Weyl minmax principle. □

We revisit the examples from the previous section and how they give rise to natural eigenvalues.

EXAMPLE 4.2. If M is closed and $\mu = \mathrm{d}v_g$, the volume measure associated to g, then $\lambda_k(M,g,\mathrm{d}v_g)$ are eigenvalue of the Laplace operator. In this case T_μ is the usual embedding $\mathrm{W}^{1,2}(M) \subset \mathrm{L}^2(M)$. If M is a compact manifold with boundary, then $\lambda_k(M,g,\mathrm{d}v_g)$ are Neumann eigenvalues.

EXAMPLE 4.3. If M is a compact manifold with boundary and $\mu = \iota_*\mathrm{d}A_g$, the pushforward by inclusion of the induced volume measure on ∂M, then $\lambda_k(M,g,\mu)$ are Steklov eigenvalues.

EXAMPLE 4.4. If M is a compact manifold, $\Sigma \subset M$ is a closed smooth hypersurface in the interior of M, and $\mu = \iota_*\mathrm{d}A_g^\Sigma$ is the pushforward by inclusion of the induced volume measure on Σ, then $\lambda_k(M,g,\mu)$ are the eigenvalues of the transmission problem

$$\begin{cases} \Delta u = 0 & \text{in } M \setminus \Sigma, \\ (\partial_{n^+} + \partial_{n^-})u = \lambda u & \text{on } \Sigma. \end{cases}$$

where $\partial_{n^\pm}$ are normal derivatives in opposite directions on Σ.

EXAMPLE 4.5. For $\beta > 0$, if $\mu = \iota_*\mathrm{d}A_g + \beta\,\mathrm{d}v_g$, then $\lambda_k(M,g,\mu)$ are eigenvalues associated with a dynamical boundary value problem, see [BF05, GHL21], given by

$$\begin{cases} -\Delta f = \lambda\beta f & \text{in } M, \\ \partial_n f = \lambda f & \text{on } \partial M. \end{cases}$$

The corresponding Laplace-type operator acts in $\mathrm{L}^2(M) \oplus \mathrm{L}^2(\partial M, \mathrm{d}A_g)$ and is not densely defined. From the perspective of variational eigenvalues, this does not cause any problem.

EXAMPLE 4.6. For $0 \leqslant \beta \in \mathrm{L}^{d/2}(M)$ $(d=3)$, or $0 \leqslant \beta \in \mathrm{L}^1(\log \mathrm{L})^1(M)$ $(d=2)$ and $\mu = \beta\,\mathrm{d}v_g$, then $\lambda_k(M,g,\mu)$ are the eigenvalues of the weighted problem

$$\begin{cases} -\Delta f = \lambda\beta f & \text{in } M, \\ \partial_n f = 0 & \text{on } \partial M. \end{cases}$$

By Example 3.19 μ is admissible, so that the spectrum is indeed discrete.

4.2 Continuity of eigenvalues. While the eigenvalues $\lambda_k(M, g, \mu)$ may not necessarily be continuous under weak-$*$ convergence of measures, they are always upper-semicontinuous, see [Kok14, Proposition 1.1] for $d = 2$. We include the proof in this context for completeness, but it is the same in essence.

PROPOSITION 4.7. *Let (M, g) be a Riemannian manifold and assume $\mu_n \overset{*}{\rightharpoonup} \mu$. Then*

$$\limsup_{n\to\infty} \lambda_k(M, g, \mu_n) \leqslant \lambda_k(M, g, \mu)$$

Proof. Let $\varepsilon > 0$ be arbitrary. Let $F \subset \mathrm{C}^\infty(M)$ be a $(k+1)$-dimensional subspace that remains $(k+1)$-dimensional in $\mathrm{L}^2(M, \mu)$ and such that

$$\sup_{f\in F\setminus\{0\}} R_g(f, \mu) \leqslant \lambda_k(M, g, \mu) + \varepsilon.$$

Convergence $\mu_n \overset{*}{\rightharpoonup} \mu$ implies that for large n the subspace F is $(k+1)$-dimensional in $\mathrm{L}^2(M, \mu_n)$ and

$$\lim_{n\to\infty} \sup_{f\in F\setminus\{0\}} R_g(f, \mu_n) = \sup_{f\in F\setminus\{0\}} R_g(f, \mu).$$

As a result, for large n one has

$$\lambda_k(M, g, \mu_n) \leqslant \sup_{f\in F\setminus\{0\}} R_g(f, \mu_n) \leqslant \lambda_k(M, g, \mu) + 2\varepsilon. \qquad \square$$

For many applications it is important to establish continuity of eigenvalues. To the best of the authors' knowledge there is no sufficiently general condition that guarantees continuity of $\lambda_k(M, g, \mu)$ which can be verified in an efficient manner in our current setting. As an example, we note that all examples of convergence covered in the present paper fail the integral distance convergence criterion given in [Kok14, Section 4.2]. Many stronger convergence criteria exists, see e.g. [AP21, BLLdC08, BL07], however they generally require explicit knowledge of some transition operators between Hilbert spaces, which usually means having explicit information about the eigenfunctions. Our goal is to obtain synthetic criteria for eigenvalue and eigenfunction convergence which depends only on the measures μ_n, and potentially on domains Ω_n on which it is supported.

Let $\Omega_n \subset M$ be a sequence of domains viewed as Riemannian manifolds with the metric induced on M, and $\{\mu_n : n \in \mathbb{N}\}$, μ be Radon measures so that $\mathrm{supp}(\mu_n) \subset \overline{\Omega_n}$. We use the same notation g, μ_n for their restrictions to Ω_n. Suppose that

- **(M1)** $\mu_n \overset{*}{\rightharpoonup} \mu$ and $\mathrm{Vol}_g(M \setminus \Omega_n) \to 0$;
- **(M2)** the measures μ, μ_n are admissible for all n;
- **(M3)** there is an equibounded family of extension maps $J_n : \mathscr{W}^{1,2}(\Omega_n, \mu_n) \to \mathscr{W}^{1,2}(M, \mu_n)$.

The condition (**M2**) guarantees the existence of the μ_n-orthonormal collection of eigenfunctions $f_j^n \in \mathscr{W}^{1,2}(\Omega_n, \mu_n)$ associated with $\lambda_j(\Omega_n, g, \mu_n)$. In any situation where $\Omega_n = M$ for all n, the third condition and the volume part of the first condition are automatically satisfied. The map J_n is often built using harmonic extensions, and the collection $J_n f_j^n$ remains μ_n orthonormal.

We now describe two conditions for the eigenfunctions.

(**EF1**) For all $u \in \mathscr{W}^{1,2}(M, \mu)$, the functions f_j^n satisfy

$$\lim_{n\to\infty} \left| \langle J_n f_j^n, u \rangle_{\mathrm{L}^2(M,\mu)} - \langle f_j^n, u \rangle_{\mathrm{L}^2(\Omega_n,\mu_n)} \right| = 0.$$

(**EF2**) For every $j, k \in \mathbb{N}$, the functions f_j^n, f_k^n satisfy

$$\lim_{n\to\infty} \left| \langle J_n f_j^n, J_n f_k^n \rangle_{\mathrm{L}^2(M,\mu)} - \langle f_j^n, f_k^n \rangle_{\mathrm{L}^2(\Omega_n,\mu_n)} \right| = \lim_{n\to\infty} \left| \langle f_j^n, f_k^n \rangle_{\mathrm{L}^2(M,\mu)} - \delta_{jk} \right| = 0,$$

where δ_{jk} is the Kronecker delta.

Condition (**EF2**) implies that $\left\{ f_j^n : n \in \mathbb{N} \right\}$ is bounded in $\mathscr{W}^{1,2}(M, \mu)$, so that up to a subsequence, $f_j^n \rightharpoonup f_j$ weakly in $\mathscr{W}^{1,2}(M, \mu)$ and $\lambda_j(\Omega_n, g, \mu_n) \to \lambda_j$ for some $\lambda_j \geqslant 0$.

Condition (**EF1**) implies that the functions f_j are eigenfunctions associated with (M, g, μ) with the corresponding eigenvalues λ_j. At this point it is unclear whether λ_j is indeed the j-th eigenvalue $\lambda_j(M, g, \mu)$. This will follow from condition (**EF2**), which says essentially that the eigenfunctions do not lose mass in the limit. We formalize this procedure in the following proposition.

PROPOSITION 4.8. *Assume that the domains $\Omega_n \subset M$ and the Radon measures μ_n, μ on (M, g) satisfy conditions (**M1**)–(**M3**), and that the eigenfunctions associated with μ_n satisfy conditions (**EF1**)–(**EF2**). Then*

$$\lim_{n\to\infty} \lambda_j(\Omega_n, g, \mu_n) = \lambda_j(M, g, \mu),$$

and, up to a choice of subsequence,

$$\lim_{n\to\infty} J_n f_j^n = f_j,$$

weakly in $\mathscr{W}^{1,2}(M, \mu)$. If $\lambda_j(M, g, \mu)$ is simple, the convergence is along the whole sequence. Finally, if

$$\lim_{n\to\infty} \left\| \nabla J_n f_j^n \right\|_{\mathrm{L}^2(M\setminus\Omega_n, \mathrm{d}v_g)} = 0, \tag{4.2}$$

the convergence is strong in $\mathscr{W}^{1,2}(M, \mu)$.

Proof. From the definition via Rayleigh quotient, we see that for all j and n, $\lambda_j(\Omega_n, g, \mu_n) \leqslant \lambda_j(M, g, \mu_n)$. By Proposition 4.7 along with Condition (**M1**), we have that up to a subsequence $\lambda_j(\Omega_n, g, \mu_n) \to \lambda_j \leqslant \lambda_j(M, g, \mu)$. For each fixed j, we have

$$\left\|J_n f_j^n\right\|^2_{\mathscr{W}^{1,2}(M,\mu)} = \left\|J_n f_j^n\right\|^2_{\mathrm{L}^2(M,\mu)} + \left\|\nabla J_n f_j^n\right\|^2_{\mathrm{L}^2(M\setminus\Omega_n, \mathrm{d}v_g)}. \tag{4.3}$$

In view of condition (**M3**), the first term on the right-hand side converges to 1. By condition (**EF2**) there exists $C > 0$ such that for all n,

$$\left\|\nabla J_n f_j^n\right\|^2_{\mathrm{L}^2(M\setminus\Omega_n, \mathrm{d}v_g)} \leqslant \|J_n\|^2 \left\|f_j^n\right\|^2_{\mathscr{W}^{1,2}(\Omega_n,\mu_n)} \leqslant C(\lambda_j(\Omega_n, g, \mu_n) + 1).$$

This means that the sequence $\left\{f_j^n : n \in \mathbb{N}\right\}$ is bounded in $\mathscr{W}^{1,2}(M,\mu)$ so that up to a subsequence, there exists f_j such that $J_n f_j^n \rightharpoonup f_j$ weakly in $\mathscr{W}^{1,2}(M,\mu)$.

We now claim that f_j is an eigenfunction associated with (M, g, μ) and corresponding eigenvalue λ_j. Since all relevant quantities are equibounded in $\mathscr{W}^{1,2}(M,\mu)$, we may use smooth functions as trial functions for f_j and λ_j. By weak convergence we have that for any $u \in \mathrm{C}^\infty(M)$,

$$\begin{aligned}\lambda_j(\Omega_n, g, \mu_n) &+ \int_{M\setminus\Omega_n} \nabla J_n f_j^n \cdot \nabla u \,\mathrm{d}v_g \\ &= \int_M \nabla J_n f_j^n \cdot \nabla u \,\mathrm{d}v_g \xrightarrow{n\to\infty} \int_M \nabla f_j \cdot \nabla v \,\mathrm{d}v_g,\end{aligned} \tag{4.4}$$

and by conditions (**M1**) and (**M3**)

$$\int_{M\setminus\Omega_n} \nabla J_n f_j^n \cdot \nabla u \,\mathrm{d}v_g \leqslant \mathrm{Vol}_g(M \setminus \Omega_n)^{1/2} \|J_n\| \, \|u\|_{\mathrm{C}^1(M)} \left\|f_j^n\right\|_{\mathscr{W}^{1,2}(M,\mu_n)} \xrightarrow{n\to\infty} 0. \tag{4.5}$$

On the other hand we have that

$$\begin{aligned}\left|\langle f_j, u\rangle_{\mathrm{L}^2(M,\mu)} - \langle f_j^n, u\rangle_{\mathrm{L}^2(M,\mu_n)}\right| &\leqslant \left|\langle f_j - J_n f_j^n, u\rangle_{\mathrm{L}^2(M,\mu)}\right| + \\ &+ \left|\langle J_n f_j^n, u\rangle_{\mathrm{L}^2(M,\mu)} - \langle f_j^n, u\rangle_{\mathrm{L}^2(M,\mu_n)}\right|.\end{aligned} \tag{4.6}$$

By Condition (**M2**), $J_n f_j^n$ converges strongly in $\mathrm{L}^2(M,\mu)$ so that the first term on the right-hand side converges to 0 while Condition (**EF1**) implies that the second term converges to 0. Putting together (4.4), (4.5) and (4.6) does yield that

$$\forall u \in \mathscr{W}^{1,2}(M,\mu) \qquad \int_M \nabla f_j \nabla u \,\mathrm{d}v_g = \lambda_j \int_M f_j u \,\mathrm{d}\mu.$$

We can now prove that the limit sequence f_j is orthonormal. Indeed,

$$\langle f_j, f_k\rangle_{\mathrm{L}^2(M,\mu)} = \langle J_n f_j^n, J_n f_k^n\rangle_{\mathrm{L}^2(M,\mu)} + \langle f_j, f_k - J_n f_k^n\rangle_{\mathrm{L}^2(M,\mu)} + \langle f_j - J_n f_j^n, f_k^n\rangle_{\mathrm{L}^2(M,\mu)}.$$

Strong convergence in $\mathrm{L}^2(M,\mu)$, Conditions (**M3**) and (**EF2**) and the Cauchy-Schwarz inequality imply that the the first term on right-handside converges to δ_{jk} whereas the last two terms on the right-hand side converge to 0.

To prove that $\lambda_j(M,g,\mu) \leqslant \lambda_j$, we use the space $F_{j+1} = \mathrm{span}\{f_0,\dots,f_j\}$ as a test space in (4.1). For any $f = \sum a_i f_i \in F_j$ one has

$$\frac{\displaystyle\int_M |\nabla f|_g^2 \,\mathrm{d}v_g}{\displaystyle\int_M f^2 \,\mathrm{d}\mu} = \frac{\sum_{i=0}^j \lambda_i a_i^2}{\sum_{i=0}^j a_i^2} \leqslant \lambda_j \frac{\sum_{i=0}^j a_i^2}{\sum_{i=0}^j a_i^2} = \lambda_j,$$

where orthonormality of $\{f_j\}$ is used in the first equality. Finally, note that weak convergence and convergence of the norms implies strong convergence, and it follows from (4.3) that (4.2) implies convergence of the norms. □

Our goal is now to provide conditions that can be verified directly on the measures μ_n, μ to ensure convergence.

LEMMA 4.9. *Let $u, v \in \mathrm{W}^{1,2}(M)$. Then, if $d \geqslant 3$, then $uv \in \mathrm{W}^{1,\frac{d}{d-1}}$ and*

$$\|uv\|_{\mathrm{W}^{1,\frac{d}{d-1}}(M)} \leqslant C_d \|u\|_{\mathrm{W}^{1,2}(M)} \|v\|_{\mathrm{W}^{1,2}(M)}$$

For $d = 2$, $uv \in \mathrm{W}^{1,p}$ for every $p < 2$, with the same norm estimate.

Proof. It is sufficient to verify the claim for $\nabla(uv)$. Let $p = \frac{d}{d-1}$ ($d \geqslant 3$) or $p < 2$ ($d = 2$). By the inequality $(a+b)^p \leqslant 2^{p-1}(a^p + b^p)$ and Hölder's inequality with exponents $2/p$ and $2/(2-p)$, we have that

$$\begin{aligned} 2^{1-p} \int_M |\nabla(uv)|^p \,\mathrm{d}v_g &\leqslant \int_M |\nabla u|^p |v|^p + |\nabla v|^p |u|^p \,\mathrm{d}v_g \\ &\leqslant \left(\int_M |\nabla u|^2 \,\mathrm{d}v_g\right)^{\frac{p}{2}} \left(\int_M |v|^{\frac{2p}{2-p}} \,\mathrm{d}v_g\right)^{1-\frac{p}{2}} + \\ &\quad + \left(\int_M |\nabla v|^2 \,\mathrm{d}v_g\right)^{\frac{p}{2}} \left(\int_M |u|^{\frac{2p}{2-p}} \,\mathrm{d}v_g\right)^{1-\frac{p}{2}}. \end{aligned}$$

By our conditions on p, the Sobolev embedding $\mathrm{W}^{1,2}(M) \to \mathrm{L}^{\frac{2p}{2-p}}(M)$ is bounded, so that

$$\left(\int_M |\nabla u|^2 \,\mathrm{d}v_g\right)^{\frac{p}{2}} \left(\int_M |v|^{\frac{2p}{2-p}} \,\mathrm{d}v_g\right)^{1-\frac{p}{2}} \leqslant C \left(\int_M |\nabla u|^2 \,\mathrm{d}v_g\right)^{\frac{p}{2}} \|v\|_{\mathrm{W}^{1,2}(M)}^p,$$

and similarly swapping the roles of u and v. This is precisely our claim. □

In dimension 2, the target space for the product will be an Orlicz–Sobolev space, recall (1.7).

LEMMA 4.10. *Let $u, v \in \mathrm{W}^{1,2}(M)$ and $d = 2$. Then, $uv \in \mathrm{W}^{1,2,-1/2}(M)$ and there is C such that*

$$\|uv\|_{\exp \mathrm{L}^1(M)} \leqslant C_2 \|u\|_{\mathrm{W}^{1,2}(M)} \|v\|_{\mathrm{W}^{1,2}(M)} .$$

Proof. Just as in the previous case, it is sufficient to verify the claim for $\nabla(uv) = u\nabla v + v\nabla u$, and as such to verify that $u\nabla v \in \mathrm{L}^2(\log L)^{-1/2}$. Trudinger's theorem [Tru67] states that for $d = 2$, $\mathrm{W}^{1,2}(M)$ embeds continuously in $\exp \mathrm{L}^2(M)$, so that $u \in \exp \mathrm{L}^2(M)$, and by assumption $\nabla v \in \mathrm{L}^2(M)$. Taking

$$R(t) = \frac{t^2}{\log(2+t)}, \qquad M(t) = t^2, \qquad N(t) = \exp(t^2) - 1$$

in [And60, Theorem 1], we have the equivalence between the following statements:

- for all $u \in \exp \mathrm{L}^2(M)$ and $v \in \mathrm{L}^2(M)$, $uv \in \mathrm{L}^2(\log L)^{-1/2}$;
- there is $a, b > 0$ such that

$$\frac{(ast)^2}{\log(2+(ast))} \leqslant s^2 + \exp(t^2) - 1 \qquad \forall s, t \geqslant b. \tag{4.7}$$

We verify that this second statement holds for $b = 1$. On the one hand,

$$s^2 \geqslant \frac{\exp(2a^2t^2)}{a^2t^2} \quad \Longrightarrow \quad \frac{(ast)^2}{\log(2+(ast))} \leqslant s^2,$$

while on the other hand

$$s^2 \leqslant \frac{\exp(t^2)}{a^2t^2} \text{ and } t \geqslant 1 \quad \Longrightarrow \quad \frac{(ast)^2}{\log(2+(ast))} \leqslant \exp(t^2) - 1.$$

Choosing $a \leqslant 2^{-1/2}$ ensures that the inequality (4.7) holds for all $s, t \geqslant 1$. Therefore,

$$\|u\nabla v\|_{\mathrm{L}^2(\log L)^{-1/2}} \leqslant C \|u\|_{\exp L^2} \|\nabla v\|_{\mathrm{L}^2} \leqslant C' \|u\|_{\mathrm{W}^{1,2}} \|v\|_{\mathrm{W}^{1,2}} .$$

the same holds swapping u and v and our claim follows. ◻

Let X be a completion of $\mathrm{C}^\infty(M)$ under some norm $\|\cdot\|_X$. We can interpret measures as bounded linear functionals on X as $\langle \mu, f\rangle_X = \int_M f \,\mathrm{d}\mu$ as long as

$$\|\mu\|_{X^*} = \sup_{f \in \mathrm{C}^\infty(M)\setminus\{0\}} \frac{\left|\int_M f \,\mathrm{d}\mu\right|}{\|f\|_X}$$

is finite.

Proposition 4.11. *Suppose that $\Omega_n \subset M$ is a sequence of domains in M and μ_n, μ are Radon measures on M satisfying conditions (**M1**)–(**M3**). If $d \geqslant 3$, suppose that $\mu_n \to \mu$ in $\mathrm{W}^{1,\frac{d}{d-1}}(M)^*$. If $d = 2$, suppose that $\mu_n \to \mu$ in $\mathrm{W}^{1,2,-1/2}(M)^*$. Then, Conditions (**EF1**)–(**EF2**) are satisfied by the eigenfunctions. In particular,*

$$\lim_{n\to\infty} \lambda_j(\Omega_n, g, \mu_n) = \lambda_j(M, g, \mu).$$

Remark 4.12. Since M is compact, convergence in $\mathrm{W}^{1,p}(M)^*$ implies convergence in $\mathrm{W}^{1,q}(M)^*$ for every $q > p$, so that in practice one can verify this criterion for any $p < \frac{d}{d-1}$. Often the case $p = 1$ provides easier computations. We also remark that if $\mu_n \to \mu$ in $\mathrm{W}^{1,p}(M)^*$, then $\mu_n \overset{*}{\rightharpoonup} \mu$ weakly-$*$ in the sense of measures.

Proof. Let us first assume that $d \geqslant 3$ and put $p = \frac{d}{d-1}$, or $d = 2$, $p < 2$. We first observe that the trace operators $T_2^{\mu_n}$ are bounded, uniformly in n. Indeed, by Lemma 4.9 for every $u \in \mathrm{W}^{1,2}(M)$,

$$\begin{aligned}\int_M u^2(\mathrm{d}\mu_n - \mathrm{d}\mu) &\leqslant \left\|u^2\right\|_{\mathrm{W}^{1,p}(M)} \left\|\mu_n - \mu\right\|_{\mathrm{W}^{1,p}(M)^*} \\ &\leqslant C_p \left\|u\right\|^2_{\mathrm{W}^{1,2}(M)} \left\|\mu_n - \mu\right\|_{\mathrm{W}^{1,p}(M)^*},\end{aligned}$$

so that

$$\begin{aligned}\int_M u^2 \,\mathrm{d}\mu_n &= \int_M u^2(\mathrm{d}\mu_n - \mathrm{d}\mu) + \int_M u^2 \,\mathrm{d}\mu \\ &\leqslant \left(C_p \left\|\mu_n - \mu\right\|_{\mathrm{W}^{1,p}(M)^*} + \|T_2^\mu\|^2\right) \|u\|^2_{\mathrm{W}^{1,2}(M)}.\end{aligned}$$

To verify (**EF1**) we note that by Lemma 4.9 one has

$$\begin{aligned}\int_M (J_n f_j^n)^2 \,\mathrm{d}\mu &\leqslant \left\|(J_n f_j^n)^2\right\|_{\mathrm{W}^{1,p}(M)} \|\mu\|_{\mathrm{W}^{1,p}(M)^*} \\ &\leqslant C_p \|\mu\|_{\mathrm{W}^{1,p}(M)^*} \left\|J_n f_j^n\right\|^2_{\mathrm{W}^{1,2}(M)} \\ &\leqslant C_p \|\mu\|_{\mathrm{W}^{1,p}(M)^*} (1 + \|T_2^{\mu_n}\|) \left\|J_n f_j^n\right\|^2_{\mathscr{W}^{1,2}(M,\mu_n)}.\end{aligned}$$

We have that

$$\left|\int_M J_n f_j^n u(\mathrm{d}\mu_n - \mathrm{d}\mu)\right| \leqslant \left\|J_n f_j^n u\right\|_{\mathrm{W}^{1,p}(M)} \left\|\mu_n - \mu\right\|_{\mathrm{W}^{1,p}(M)^*}.$$

This goes to 0 by Lemma 4.9 and convergence $\mu_n \to \mu$ in $\mathscr{W}^{1,p}(M)^*$, so that Condition (**EF1**) is satisfied.

Finally, using Lemma 4.9 one last time, we have that

$$\left|\int_M J_n f_j^n J_n f_k^n(\mathrm{d}\mu_n - \mathrm{d}\mu)\right| \leqslant \left\|f_k^n f_j^n\right\|_{\mathrm{W}^{1,p}(M)} \left\|\mu_n - \mu\right\|_{\mathrm{W}^{1,p}(M)^*} \xrightarrow{n\to\infty} 0,$$

so that Condition (**EF2**) is indeed satisfied.

The case $d = 2$ and $\mu_n \to \mu$ in $\mathrm{W}^{1,2,-1/2}(M)^*$ follows from replacing Lemma 4.9 by Lemma 4.10. □

Verifying that $\mu_n \to \mu \in \mathrm{W}^{1,p}(M)^*$ is in principle a global question (or at the very least the local character of it should be verified independent of n). Following the ideas set out in [GHL21, GL21] we provide the following blueprint for verifying convergence in an effective way that is based on Lemma 3.16. This lemma implies, amongst other things, that if μ is p-admissible then there exists $\varphi_n \in \mathrm{W}^{1,p'}(M)$ such that

$$\langle \mu_n - \mu, f\rangle_{\mathrm{W}^{1,p}(M)} = \left(\frac{\mu_n(M)}{\mu(M)} - 1\right)\langle \mu, f\rangle_{\mathrm{W}^{1,p}(M)} + \int_M \nabla\varphi_n \cdot \nabla f \,\mathrm{d}v_g.$$

The first term is easily seen to converge to 0 uniformly for $\|f\|_{\mathrm{W}^{1,p}(M)} \leqslant 1$. However, the estimates we obtained on φ_n in Lemma 3.16 are not on their face strong enough to guarantee convergence. In Sect. 6 we get over this hurdle by partitioning M into an almost disjoint union $M = \bigcup_{z\in I_n} Q_z^n$. Recalling that we use $\langle\cdot,\cdot\rangle_X$ to denote the duality pairing in a vector space X, this allows us to write

$$\langle \mu_n - \mu, f\rangle_{\mathrm{W}^{1,p}(M)^*} = \sum_{z\in I}\langle \mu_n - \mu, f\rangle_{\mathrm{W}^{1,p}(Q_z^n)^*}.$$

Using Lemma 3.16 in every Q_z^n provides us with an effective mean of proving that this converges to 0. In view of estimate (3.5), if $\mu_n(Q_z^n)$ is comparable for every z then by Hölder's inequality

$$\sum_{z\in I} \mu_n(Q_z^n)^{1/p'} \|f\|_{\mathrm{W}^{1,p}(Q_z^\varepsilon)} \ll \|f\|_{\mathrm{W}^{1,p}(M)}$$

so that in principle one will need to prove only that the p-Poincaré constants of Q_z^ε are uniformly bounded and that the traces $T_p^{\mu_n}$ restricted to Q_z^ε converge to 0. Exploiting the potential lack of scale invariance in the defining equation for φ_n is often key for this. For a concrete application of this scheme, see the proof of Proposition 6.5.

5 First Examples of Spectrum Convergence

In this section we collect several applications of the setup presented in the previous section. Most of the results in this section are generalisations of known results either to a manifold context or to higher dimensions.

5.1 Convergence for L^p densities. The case $d = 2$, $\beta_n \in \mathrm{L}^p$, $p > 1$ has previously appeared in [KNPP20, Lemma 6.2].

Proposition 5.1. *Let β_n be a sequence of non-negative densities converging in $\mathrm{L}^{\frac{d}{2}}(\log \mathrm{L})^a(M)$ to a non-negative density β, where $a = 0$ for $d \geqslant 3$ and $a = 1$ for $d = 2$. Then $\lambda_k(M, g, \beta_n \,\mathrm{d}v_g) \to \lambda_k(M, g, \beta \,\mathrm{d}v_g)$ as $n \to \infty$.*

Proof. Conditions (**M1**)–(**M3**) are respected, admissibility following from Example 3.19. Let $u \in \mathrm{W}^{1,\frac{d}{d-1}}(M)$ for $d \geqslant 3$. Then by the Sobolev embedding $\mathrm{W}^{1,\frac{d}{d-1}}(M) \to \mathrm{L}^{\frac{d}{d-2}}(M)$ and Hölder's inequality with exponents $\frac{d}{d-2}$ and $\frac{d}{2}$,

$$\left| \int u(\beta_n - \beta) \, \mathrm{d}v_g \right| \ll_{d,M} \|u\|_{\mathrm{W}^{1,\frac{d}{d-1}}(M)} \|\beta_n - \beta\|_{\mathrm{L}^{d/2}} .$$

We deduce that $\beta_n \mathrm{d}v_g \to \beta \mathrm{d}v_g$ in $\mathrm{W}^{1,\frac{d}{d-1}}(M)^*$, so that by Proposition 4.11 the eigenvalues converge. For $d = 2$, proceed the same way but with the pairing of the spaces $\exp \mathrm{L}^1(M)$ and $\mathrm{L}^1(\log \mathrm{L})^1(M)$, along with the optimal Sobolev embedding $\mathrm{W}^{1,2,-1/2}(M) \to \exp \mathrm{L}^1(M)$, see [Cia96, Example 1]. □

5.2 Approximation of eigenvalues of measures supported on a hypersurface. Let (M, g) be a compact Riemannian manifold. Let $\Sigma \subset M$ be a compact, not necessarily connected, codimension 1 smooth submanifold without boundary and $\rho \in C(\Sigma)$ be a non-negative density on Σ. Assume $\Sigma = \Sigma_i \sqcup \Sigma_b$, where $\Sigma_i \cap \partial M = \varnothing$ and Σ_b is either empty or coincides with ∂M. Let $N_{\varepsilon,i}$ be an ε-tubular neighbourhood of Σ_i. For sufficiently small ε the exponential map $\exp_{\Sigma_i}$ can be used to identify $N_{\varepsilon,i}$ with $N^\varepsilon \Sigma_i$, the ε-ball in the normal bundle of Σ_i. Similarly, if $\Sigma_b = \partial M$ is not empty, its ε-tubular neighbourhood $N_{\varepsilon,b}$ can be identified with $\Sigma_b \times [0, \varepsilon]$ using $\exp_{\Sigma_b}$. If n is an outward unit normal then we define

$$\rho_\varepsilon(y) = \begin{cases} \frac{1}{2\varepsilon}\rho(x) & \text{if } y = \exp_x(w) \in N_{\varepsilon,i}, \ (x, w) \in N^\varepsilon \Sigma_i, \\ \frac{1}{\varepsilon}\rho(x) & \text{if } y = \exp_x(-tn) \in N_{\varepsilon,b}, \ (x, t) \in \Sigma_b \times [0, \varepsilon], \\ 0 & \text{otherwise.} \end{cases}$$

The next theorem says that we can approximate the eigenvalues of weighted Steklov or transmission problems as in Example 4.4 using weighted Laplace eigenvalues. When $\Sigma = \Sigma_b = \partial M$, our construction is similar to the one found for domains in $\mathbb{R}^d$ in [LP15].

Theorem 5.2. *Let* $\mathrm{d}A_g^\Sigma$ *be the volume measure on* Σ. *Then one has*

$$\lambda_k(M, g, \rho_\varepsilon \mathrm{d}v_g) \to \lambda_k\left(M, g, \rho \mathrm{d}A_g^\Sigma\right)$$

as $\varepsilon \to 0$.

Proof. We give the proof in the case $\Sigma = \Sigma_i$, the other case is analogous. The conditions (**M1**)–(**M3**) are obviously satisfied. We claim that for any $u \in \mathrm{W}^{1,1}(M)$ one has

$$\left| \int_M u\rho_\varepsilon \, \mathrm{d}v_g - \int_\Sigma u\rho \, \mathrm{d}A_g^\Sigma \right| \leqslant C \|u\|_{\mathrm{W}^{1,1}(N_\varepsilon)} \tag{5.1}$$

for ε small enough. In particular, for any $p > 1$ and $u \in \mathrm{W}^{1,p}(M)$ one has by Hölder's inequality

$$\left| \langle \rho_\varepsilon \mathrm{d}v_g - \rho \mathrm{d}A_g^\Sigma, u \rangle \right| \ll \mathrm{Vol}(N_\varepsilon)^{1/p'} \|u\|_{\mathrm{W}^{1,p}(M)} ,$$

which goes to 0 since $\mathrm{Vol}(N_\varepsilon) \to 0$. Thus, proving (5.1) implies that $\rho_\varepsilon \mathrm{d}v_g \to \rho \mathrm{d}A_g^\Sigma$ in $\mathrm{W}^{1,p}(M)^*$, and Proposition 4.11 implies that the eigenvalues converge.

We use coordinates (x, w) on N_ε induced by the identification with $N^\varepsilon\Sigma$ via the exponential map. Let $\xi(x, w)\,\mathrm{d}w\,\mathrm{d}A_g^\Sigma$ be the volume measure on N_ε, where we denoted the pullback $\pi^*\,\mathrm{d}A_g^\Sigma$ simply by $\mathrm{d}A_g^\Sigma$, $\pi\colon N^\varepsilon\Sigma \to \Sigma$. Let also $\zeta(x) = \int_{N_x^\varepsilon\Sigma} \xi(x, w)\,\mathrm{d}w$ be the fiber integral of $\xi(x, w)$. Since the differential of the exponential map at the origin is equal to identity, one has that

$$|1 - \xi(x, w)| = O(\varepsilon) \qquad |1 - \zeta(x)| = O(\varepsilon) \tag{5.2}$$

as $\varepsilon \to 0$. Then one has

$$\begin{aligned}
\left|\int_M u\rho_\varepsilon\,\mathrm{d}v_g - \int_\Sigma u\rho\,\mathrm{d}A_g^\Sigma\right| &\leqslant \frac{\|\rho\|_{L^\infty}}{2\varepsilon}\int_\Sigma\int_{N_x^\varepsilon\Sigma} |u(x,w)\xi(x,w) - u(x,0)\zeta(x)|\,\mathrm{d}w\,\mathrm{d}A_g^\Sigma \\
&\leqslant \frac{C}{\varepsilon}\int_\Sigma\int_{N_x^\varepsilon\Sigma} |u(x,w) - u(x,0)|\zeta(x)\,\mathrm{d}w\,\mathrm{d}A_g^\Sigma \\
&\quad + \frac{C}{\varepsilon}\int_\Sigma\int_{N_x^\varepsilon\Sigma} |u(x,w)||\zeta(x) - \xi(x,w)|\,\mathrm{d}w\,\mathrm{d}A_g^\Sigma \\
&\leqslant \frac{C'}{\varepsilon}\int_\Sigma\int_{N_x^\varepsilon\Sigma} |u(x,w) - u(x,0)|\,\mathrm{d}w\,\mathrm{d}A_g^\Sigma \\
&\quad + C''\int_\Sigma\int_{N_x^\varepsilon\Sigma} |u(x,w)|\,\mathrm{d}w\,\mathrm{d}A_g^\Sigma,
\end{aligned} \tag{5.3}$$

where we used (5.2) in the last step. For a fixed x the inside integral in the first term is a 1-dimensional integral that can be estimated as follows,

$$\begin{aligned}
\int_{-\varepsilon}^{\varepsilon} |u(x,t) - u(x,0)|\,\mathrm{d}t &= \int_{-\varepsilon}^{\varepsilon}\left|\int_0^t u_t(x,s)\,\mathrm{d}s\right|\,\mathrm{d}t \\
&\leqslant \int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon} |u_t(x,s)|\,\mathrm{d}s\,\mathrm{d}t \\
&= 2\varepsilon\int_{-\varepsilon}^{\varepsilon} |u_t(x,s)|\,\mathrm{d}s.
\end{aligned}$$

Putting this back into (5.3) and using (5.2) completes the proof of estimate (5.1). □

5.3 Application to shape optimisation in dimension 2. We can now prove Theorem 1.2, with the following proposition.

PROPOSITION 5.3. *Let (M, g) be a compact Riemannian surface and let $\Omega \subset M$ be a smooth domain such that $\partial\Omega \cap \partial M$ is either empty or equal to ∂M. Then*

$$\overline{\sigma}_k(\Omega, g) \leqslant \Lambda_k(M, [g]).$$

Proof. We first note that in dimension 2 for any smooth $\rho > 0$ one has that $\lambda_k(M, g, \rho\, \mathrm{d}v_g)$ coincides with the classical Laplacian eigenvalues $\lambda_k(M, \rho g, \mathrm{d}v_{\rho g})$ of the conformal metric ρg. Since smooth functions are dense in L^p for every $p \in [1, \infty)$, Theorem 5.2 and Proposition 5.1 imply that

$$\overline{\lambda}_k(M, g, \mathrm{d}A_g^\Sigma) \leqslant \Lambda_k(M, [g]),$$

where $\mathrm{d}A_g^\Sigma$ is the surface measure of $\Sigma = \partial\Omega$. At the same time, a comparison of the Rayleigh quotients for $\lambda_k(M, g, \mathrm{d}A_g^\Sigma)$ and $\sigma_k(\Omega, g)$ we see that they are the same except for the fact that former integrates the Dirichlet energy over all of M whereas the latter integrates it over $\Omega \subset M$. This directly yields the inequality

$$\overline{\sigma}_k(\Omega, g) \leqslant \overline{\lambda}_k(M, g, \mathrm{d}A_g^\Sigma). \qquad \square$$

REMARK 5.4. It is clear from our constructions that as soon as a measure μ on M is limit in $\mathrm{W}^{1,\frac{d}{d-1}}(M)^*$ of measures of the form $\beta\, \mathrm{d}x$ respecting conditions (**M1**)–(**M3**), we obtain similarly to the last proposition

$$\lambda_k(\Omega, g, \mu)\mu(M) \leqslant \Lambda_k(M, [g]).$$

6 Homogenisation

In this section, we fix a bounded domain $\Omega \subset \mathbb{R}^d$ with C^1 boundary and we put $M = \overline{\Omega}$. Let $\beta \in \mathrm{C}(M)$ be nonnegative and nontrivial, g_0 be the flat metric and $\mathrm{d}A$ be the boundary measure on ∂M.

6.1 Construction of perforated sets. We construct domains $\Omega^\varepsilon \subset M$ in the spirit of deterministic homogenisation theory. For $z \in \mathbb{Z}^d$, consider the cubes

$$Z_z^\varepsilon := \varepsilon z + \left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right]^d \subset \mathbb{R}^d$$

and define

$$\mathbf{I}^\varepsilon := \left\{ z \in \mathbb{Z}^d : Z_z^\varepsilon \subset M \right\}.$$

For $\alpha > d - 1$, we set

$$r_z^\varepsilon := \left(\frac{\varepsilon^\alpha}{a_d} \beta(\varepsilon z) \right)^{\frac{1}{d-1}}, \qquad B_z^\varepsilon = B(\varepsilon z, r_z^\varepsilon), \qquad \text{and} \qquad Q_z^\varepsilon := Z_z^\varepsilon \setminus B_z^\varepsilon,$$

where a_d is the area of the unit sphere in $\mathbb{R}^d$ and where, by convention, we put $B(x, 0) = \varnothing$ for any $x \in \mathbb{R}^d$. We set as well

$$\mathbf{R}^\varepsilon := M \setminus \bigcup_{z \in \mathbf{I}^\varepsilon} Z_z^\varepsilon, \qquad \widetilde{\mathbf{I}}^\varepsilon := \{ z \in \mathbf{I}^\varepsilon : \beta(\varepsilon z) \neq 0 \} \qquad \text{and} \qquad \mathbf{B}^\varepsilon := \bigcup_{z \in \widetilde{\mathbf{I}}^\varepsilon} B_z^\varepsilon;$$

and finally

$$\Omega^\varepsilon := M \setminus \mathbf{B}^\varepsilon \qquad \text{and} \qquad \mu_\alpha^\varepsilon := \iota_* \mathrm{d}A^\varepsilon,$$

where $\iota_* \mathrm{d}A^\varepsilon$ is the pushforward by inclusion of the natural boundary measure on Ω^ε. For any measure ξ, we define the normalised measure $\overline{\xi} := \xi(M)^{-1}\xi$.

- For all ε, z,
$$(r_z^\varepsilon)^{d-1} \ll \varepsilon^\alpha \max_{x\in M} \beta(x).$$
- The number of holes satisfies $\#\widetilde{\mathbf{I}}^\varepsilon \ll_M \varepsilon^{-d}$ as $\varepsilon \to 0$.
- The total boundary area of the holes satisfies the asymptotic relationship
$$\mathscr{H}^{d-1}(\partial \mathbf{B}^\varepsilon) = \sum_{z\in\widetilde{\mathbf{I}}^\varepsilon} a_d (r_z^\varepsilon)^{d-1} \sim \varepsilon^{\alpha-d} \int_M \beta \,\mathrm{d}x.$$
while the total volume of the holes satisfies
$$\mathrm{Vol}(\mathbf{B}^\varepsilon) = \sum_{z\in\widetilde{\mathbf{I}}^\varepsilon} d a_d (r_z^\varepsilon)^d = O_{M,\beta}\left(\varepsilon^{\frac{d\alpha}{d-1}-d}\right), \tag{6.1}$$
In particular, Condition (**M1**) is satisfied with
$$\overline{\mu}_\alpha^\varepsilon \overset{*}{\rightharpoonup} \overline{\mu}_\alpha := \begin{cases} \overline{\beta \mathrm{d}v_g} & \text{if } d-1<\alpha<d, \\ \overline{\beta \mathrm{d}v_g + \iota_* \mathrm{d}A} & \text{if } \alpha = d, \\ \overline{\iota_* \mathrm{d}A} & \text{if } \alpha > d; \end{cases} \tag{6.2}$$
and
$$\mathrm{d}x\big|_{\Omega^\varepsilon} \overset{*}{\rightharpoonup} \mathrm{d}x\big|_M.$$
- It is a standard fact that on C^1 domains the trace maps $T_2^{\mu^\varepsilon}$ and the Sobolev embeddings T_2^{μ} are compact, and that the first Steklov and Neumann eigenvalues are always positive so that Condition (**M2**) is met in both cases.
- The set $\mathbf{R}^\varepsilon$ is a subset of a $\sqrt{d}\varepsilon$-collar neighbourhood of ∂M, as such $\mathrm{Vol}(\mathbf{R}^\varepsilon) = O_{d,M}(\varepsilon)$.
- Denoting by $J^\varepsilon : \mathscr{W}^{1,2}(\Omega^\varepsilon, \mu_\alpha^\varepsilon) \to \mathscr{W}^{1,2}(\Omega, \mu_\alpha^\varepsilon)$ the map extending harmonically in $\mathbf{B}^\varepsilon$, we have that J^ε is bounded independently of ε, see [RT75, Example 1, page 40]. Condition (**M3**) is therefore satisfied.

REMARK 6.1. To obtain Theorem 1.12 we note that it is possible to achieve a similar setting by removing geodesic balls of radius r_ε around a maximal ε-separated subset of a Riemannian manifold M. See [AP21] and [GL21] for similar constructions in the context of the Neumann and Steklov problems, respectively. This makes it possible to directly extend the statements to the situation where Ω is a bounded domain with C^1 boundary in a complete manifold $\widetilde{M}$, the implicit constants then depending on the metric of $\widetilde{M}$ restricted to Ω. We keep the periodic description here to emphasise the fact that we do not need the Riemannian setting in order to obtain large normalised Steklov eigenvalues.

With the notation introduced above, we may now state the main theorem of this section.

Theorem 6.2. *For all $j \in \mathbb{N}$, and*

$$\alpha > d-1$$

the Steklov eigenvalues of the perforated domains Ω^ε satisfy

$$\sigma_j(\Omega^\varepsilon)\frac{\mathscr{H}^{d-1}(\partial\Omega^\varepsilon)}{\operatorname{Vol}(\Omega^\varepsilon)^{\frac{d-2}{d}}} = \overline{\lambda}_j(\Omega^\varepsilon, g_0, \overline{\mu}_\alpha^\varepsilon) \xrightarrow{\varepsilon\to 0} \overline{\lambda}_j(M, g_0, \overline{\mu}_\alpha),$$

where $\overline{\mu}$ is defined in (6.2) and the associated eigenfunctions extended to M converge strongly in $\mathrm{W}^{1,2}(M)$. The Neumann eigenvalues satisfy

$$\lambda_j(\Omega^\varepsilon, g_0)\operatorname{Vol}(\Omega^\varepsilon)^{2/d} = \overline{\lambda}_j(\Omega^\varepsilon, g_0, \mathrm{d}x) \xrightarrow{\varepsilon\to 0} \overline{\lambda}_j(M, g_0, \mathrm{d}x), \tag{6.3}$$

and the associated eigenfunctions extended to M converge weakly in $\mathrm{W}^{1,2}(M)$.

The proof of Theorem 6.2 is split into two parts: Proposition 6.3 where the convergence of the Neumann eigenvalues is shown and Proposition 6.5 where we prove convergence of the Steklov eigenvalues. In both cases, we prove that the associated measures converge in $\mathrm{W}^{1,p}(M)^*$, for all $p > 1$. In the construction of perforated sets, we already have that conditions (**M1**)–(**M3**) are satisfied, so that by Proposition 4.11, this is enough to obtain eigenvalue and eigenfunction convergence. For the Steklov problem, we observe that Condition (4.2) follows directly from [GHL21, Lemma 12] so that strong convergence of the eigenfunctions also follows if we prove the appropriate convergence of the measures.

We note that (6.3) could be deduced by an appropriate modification of the proofs in [RT75] or [AP21], however this would require introducing new concepts whereas the results from Sects. 3 and 4 can prove both convergence of the Neumann and Steklov eigenpairs at the same time. This also puts an emphasis on the fact that it is achieved for the same domains.

PROPOSITION 6.3. *As $\varepsilon \to 0$, the measures $\mathrm{d}x\big|_{\Omega^\varepsilon}$ converge to $\mathrm{d}x\big|_M$ in $\mathrm{W}^{1,p}(M)^*$ for every $p \in (1,\infty)$. In particular, the Neumann eigenpairs for Ω^ε converges to those of Ω.*

Proof. For $f \in \mathrm{W}^{1,p}(M)$, we have that

$$\begin{aligned}\left|\langle \mathrm{d}x\big|_{\Omega^\varepsilon} - \mathrm{d}x\big|_M, f\rangle\right| &= \left|\int_{\mathbf{B}^\varepsilon} f\,\mathrm{d}x\right| \\ &\leqslant \operatorname{Vol}_{g_0}(\mathbf{B}^\varepsilon)^{\frac{p-1}{p}}\,\|f\|_{\mathrm{L}^p(\mathbf{B}^\varepsilon)} \\ &\leqslant \operatorname{Vol}_{g_0}(\mathbf{B}^\varepsilon)^{\frac{p-1}{p}}\,\|f\|_{\mathrm{W}^{1,p}(M)}.\end{aligned}$$

By (6.1), this last line goes to 0 as $\varepsilon \to 0$. $\square$

6.2 Convergence of the Steklov eigenpairs. Before proving convergence of the Steklov eigenpairs, we require the following useful lemma.

LEMMA 6.4. *Let $0 < r \leqslant R \leqslant 1$, and $p \in (1, d)$. Then, there exists $C_{p,d} > 0$ such that for all $f \in \mathrm{W}^{1,p}(B(0,R))$,*

$$\|f\|^p_{\mathrm{L}^p(r\mathbb{S}^{d-1})} \leqslant C_{p,d} \max\left\{r^{d-1}R^{-d}, r^{p-1}\right\} \|f\|^p_{\mathrm{W}^{1,p}(B(0,R))}.$$

Proof. By density of smooth functions in $\mathrm{W}^{1,p}(B(0,R))$ it is sufficient to prove the inequality for smooth f. Let $\widetilde{f} \in \mathrm{W}^{1,p}(B(0,R))$ be the radially constant function given by $\widetilde{f}(\rho,\theta) = f(r,\theta)$, since $p < d$ we can assign any value of $\widetilde{f}$ at 0. Set $F := f - \widetilde{f} \in \mathrm{W}^{1,p}(B(0,R))$ and observe that F vanishes on $\partial B(0,r)$, and that $\partial_\rho F = \partial_\rho f$. We directly compute that

$$\begin{aligned} \|f\|^p_{\mathrm{L}^p(r\mathbb{S}^{d-1})} &= dr^{d-1}R^{-d}\left\|\widetilde{f}\right\|^p_{\mathrm{L}^p(B(0,R))} \\ &\leqslant 2^{p-1}dr^{d-1}R^{-d}\left(\|f\|^p_{\mathrm{L}^p(B(0,R))} + \|F\|^p_{\mathrm{L}^p(B(0,R))}\right). \end{aligned} \tag{6.4}$$

To conclude, we will bound the norm of F with a radial Friedrichs' inequality. By simple integration and Hölder's inequality we have that for every $\rho \in (0,R)$ and $\theta \in \mathbb{S}^{d-1}$

$$|F(\rho,\theta)|^p = \left|\int_r^\rho \partial_s f(s,\theta)\,\mathrm{d}s\right|^p \leqslant \left|\int_r^\rho s^{\frac{1-d}{p-1}}\,\mathrm{d}s\right|^{p-1}\int_r^\rho |\partial_s f(s,\theta)|^p s^{d-1}\,\mathrm{d}s.$$

Integrating both sides of this inequality on $B(0,R)$ tells us that since $p < d$

$$\begin{aligned} \|F\|^p_{\mathrm{L}^p(B(0,R))} &\leqslant \int_0^R \rho^{d-1}\left|\int_r^\rho s^{\frac{1-d}{p-1}}\,\mathrm{d}s\right|^{p-1}\|\partial_\rho f\|^p_{\mathrm{L}^p(B(0,\rho))}\,\mathrm{d}\rho \\ &\leqslant \frac{p-1}{d-p}\|\partial_\rho f\|^p_{\mathrm{L}^p(B(0,R))}\int_0^R \rho^{p-1}\left|1-\left(\frac{r}{\rho}\right)^{\frac{p-d}{p-1}}\right|^{p-1}\,\mathrm{d}\rho \end{aligned} \tag{6.5}$$

This integral can be split into regions where $\rho \leqslant a := 2^{\frac{d-p}{p-1}}r$ and $a \leqslant \rho \leqslant R$. In the first region, the integral is bounded by 2^{p-1}. In the second region, we have that

$$\begin{aligned} \int_a^R \rho^{p-1}\left|1-\left(\frac{r}{\rho}\right)^{\frac{p-d}{p-1}}\right|^{p-1}\,\mathrm{d}\rho &\leqslant \frac{1}{2}\int_a^R \rho^{d-1}r^{p-d}\,\mathrm{d}\rho \\ &\leqslant \frac{1}{2d}R^{-d}r^{p-d}. \end{aligned}$$

Inserting this estimate into (6.5) and then (6.4) yields our claim. □

The main purpose of this section is to prove the following proposition

PROPOSITION 6.5. *As $\varepsilon \to 0$, the measures $\overline{\mu}^\varepsilon_\alpha \to \overline{\mu}_\alpha$ in $\mathrm{W}^{1,p}(M)^*$ for all $p > 1$.*

Proof. Without loss of generality, by monotonicity of the dual spaces $\mathrm{W}^{1,p}(M)^*$ we assume that $p < 2$.

For $f \in \mathrm{W}^{1,p}(M)$ we have the decomposition

$$\langle \overline{\mu}_\alpha^\varepsilon - \overline{\mu}_\alpha, f\rangle_{\mathrm{W}^{1,p}(M)} = \frac{\mathbf{1}_{\{\alpha \leqslant d\}}}{\mu_\alpha(M)} \int_{\mathbf{R}^\varepsilon} \beta f \,\mathrm{d}x + \left(\frac{1}{\mu_\alpha^\varepsilon(M)} - \frac{\mathbf{1}_{\{\alpha \geqslant d\}}}{\mu_\alpha(M)}\right) \int_{\partial M} f \,\mathrm{d}A + \sum_{z \in \mathbf{I}^\varepsilon} \langle \overline{\mu}_\alpha^\varepsilon - \overline{\mu}_\alpha, f\rangle_{\mathrm{W}^{1,p}(Z_z^\varepsilon)} \tag{6.6}$$

We first observe that

$$\int_{\mathbf{R}^\varepsilon} \beta f \,\mathrm{d}x \leqslant \|\beta\|_{\mathrm{C}^0(M)} \|f\|_{\mathrm{L}^p(M)} \operatorname{Vol}(\mathbf{R}^\varepsilon)^{\frac{p-1}{p}} \ll_{M,\beta} \varepsilon^{\frac{p-1}{p}}.$$

We also have that

$$\lim_{\varepsilon \to 0} \left(\frac{1}{\mu^\varepsilon(M)} - \frac{\mathbf{1}_{\{\alpha \geqslant d\}}}{\mu(M)}\right) \int_{\partial M} f \,\mathrm{d}A = 0;$$

for $\alpha \geqslant d$ this follows from the fact that $\mu_\alpha^\varepsilon(M) \xrightarrow{\varepsilon \to 0} \mu_\alpha(M)$ whereas for $d - 1 < \alpha < d$ this follows from $\mu_\alpha^\varepsilon(M) \xrightarrow{\varepsilon \to 0} \infty$. We are now left only with the sum term in (6.6).

In the case where $\alpha > d$, we have that $\overline{\mu}_\alpha$ is supported on ∂M so that by Hölder's inequality on sums that

$$\begin{aligned}\sum_{z \in \mathbf{I}^\varepsilon} \left|\langle \overline{\mu}_\alpha^\varepsilon - \overline{\mu}_\alpha, f\rangle_{\mathrm{W}^{1,p}(Z_z^\varepsilon)}\right| &\leqslant \frac{1}{\mu_\alpha^\varepsilon(M)} \sum_{z \in \mathbf{I}^\varepsilon} \int_{\partial B_z^\varepsilon} |f| \,\mathrm{d}A \\ &\ll \sum_{z \in \mathbf{I}^\varepsilon} \mathscr{H}^{d-1}(\partial B_z^\varepsilon)^{\frac{p-1}{p}} \|f\|_{\mathrm{L}^p(\partial B_z^\varepsilon)} \\ &\ll_{\beta,M} \varepsilon^{(\alpha-d)\frac{p-1}{p}} \left(\sum_{z \in \mathbf{I}^\varepsilon} \|f\|_{\mathrm{L}^p(\partial B_z^\varepsilon)}^p\right)^{1/p};\end{aligned}$$

this goes to 0 as $\varepsilon \to 0$. Indeed, since $\alpha > d$ we have that $(r_z^\varepsilon)^{d-1}\varepsilon^{-d}$ remains uniformly bounded as $\varepsilon \to 0$, so that by Lemma 6.4

$$\sum_{z \in \mathbf{I}^\varepsilon} \|f\|_{\mathrm{L}^p(\partial B_z^\varepsilon)}^p \leqslant \sum_{z \in \mathbf{I}^\varepsilon} \|f\|_{\mathrm{W}^{1,p}(Z_z^\varepsilon)}^p \leqslant \|f\|_{\mathrm{W}^{1,p}(M)}^p.$$

When $d - 1 < \alpha \leqslant d$, we split the sum into $z \in \mathbf{I}^\varepsilon \setminus \widetilde{\mathbf{I}}^\varepsilon$ and $z \in \widetilde{\mathbf{I}}^\varepsilon$. In the first case, we have that

$$\begin{aligned}\left|\sum_{z \in \mathbf{I}^\varepsilon \setminus \widetilde{\mathbf{I}}^\varepsilon} \langle \overline{\mu}_\alpha^\varepsilon - \overline{\mu}_\alpha, f\rangle_{\mathrm{W}^{1,p}(Z_z^\varepsilon)}\right| &= \frac{1}{\mu_\alpha(M)} \left|\sum_{z \in \mathbf{I}^\varepsilon \setminus \widetilde{\mathbf{I}}^\varepsilon} \int_{Z_z^\varepsilon} \beta f \,\mathrm{d}x\right| \\ &\leqslant \sup_{z \in \mathbf{I}^\varepsilon} \sup_{x \in Z_z^\varepsilon} |\beta(x)| \, \|f\|_{\mathrm{L}^1(M)}.\end{aligned}$$

By uniform continuity of β, and since $\beta(\varepsilon z) = 0$ for all $z \in \mathbf{I}^\varepsilon \setminus \widetilde{\mathbf{I}}^\varepsilon$, that last quantity vanishes as $\varepsilon \to 0$.

Finally, when $z \in \widetilde{\mathbf{I}}^\varepsilon$, both $\overline{\mu}_\alpha$ and $\overline{\mu}_\alpha^\varepsilon$ are p-admissible on Z_z^ε. Therefore we can apply Lemma 3.16 to obtain the existence of $\varphi_z^\varepsilon \in \mathrm{W}^{1,p'}(Z_z^\varepsilon)$ such that

$$\begin{aligned}\langle \overline{\mu}_\alpha^\varepsilon - \overline{\mu}_\alpha, f\rangle_{\mathrm{W}^{1,p}(Z_z^\varepsilon)} &= \left(\frac{\overline{\mu}_\alpha^\varepsilon(Z_z^\varepsilon)}{\overline{\mu}_\alpha(Z_z^\varepsilon)} - 1\right)\langle \overline{\mu}, f\rangle_{\mathrm{W}^{1,p}(Z_z^\varepsilon)} + \int_{Z_z^\varepsilon} \nabla\varphi_z^\varepsilon \cdot \nabla f \,\mathrm{d}x \\ &= \left(\frac{\beta(\varepsilon z)}{\varepsilon^{-d}\int_{Z_z^\varepsilon}\beta\,\mathrm{d}x} - 1\right)\langle \overline{\mu}_\alpha, f\rangle_{\mathrm{W}^{1,p}(Q_z^\varepsilon)} + \int_{Z_z^\varepsilon} \nabla\varphi_z^\varepsilon \cdot \nabla f\,\mathrm{d}x.\end{aligned} \tag{6.7}$$

By uniform continuity of β and the integral mean value theorem, since $\overline{\mu}_\alpha \in \mathrm{W}^{1,p}(M)^*$, the first term on the right-hand side vanishes in the limit. For the last term in (6.7), by Hölder's inequality we have that

$$\int_{Z_z^\varepsilon} \nabla\varphi_z^\varepsilon \cdot \nabla f\,\mathrm{d}x \leqslant \|\nabla\varphi_z^\varepsilon\|_{\mathrm{L}^{p'}(Z_z^\varepsilon)} \|\nabla f\|_{\mathrm{L}^p(Z_z^\varepsilon)}$$

The second part of Lemma 3.16 is barely too weak to show that the $\mathrm{L}^{p'}(Z_z^\varepsilon)$ norm of $\nabla\varphi_z^\varepsilon$ converges to zero fast enough. In order to prove so, we exploit the lack of scale invariance in the defining equation for φ_z^ε.

Define $\varphi_z^{\varepsilon,s} : sZ_z^\varepsilon \to \mathbb{R}$ as $\varphi_z^{\varepsilon,s} := \varphi_z^\varepsilon(x/s)$. We have that $\varphi_z^{\varepsilon,s}$ satisfies the weak differential equation

$$\forall f \in \mathrm{W}^{1,p}(sZ_z^\varepsilon), \quad \int_{sZ_z^\varepsilon} \nabla\varphi_z^{\varepsilon,s} \cdot \nabla f\,\mathrm{d}x = \frac{1}{s}\int_{s\partial B_z^\varepsilon} f\,\frac{\mathrm{d}A}{\mu^\varepsilon(M)} - \frac{1}{s^2}\frac{\overline{\mu}^\varepsilon(Z_z^\varepsilon)}{\overline{\mu}(Z_z^\varepsilon)}\int_{sZ_z^\varepsilon} f\,\mathrm{d}\overline{\mu}.$$

Furthermore, $\int_{sZ_z^\varepsilon} \varphi_z^{\varepsilon,s} d\overline{\mu} = 0$. Therefore, $\varphi_z^{\varepsilon,s}$ is the solution of the equation (3.4) for measures $(s\mu^\varepsilon(M))^{-1}\,\mathrm{d}A^{s\partial B_z^\varepsilon}$ and $\mathrm{d}\overline{\mu}|_{sZ_z^\varepsilon}$. Thus, the estimate (3.5) in Lemma 3.16 implies that

$$\|\nabla\varphi_z^{\varepsilon,s}\|_{\mathrm{L}^{p'}(sQ_z^\varepsilon)} \ll_\beta (1 + K_{s,\varepsilon}) s^{\frac{d-1}{p'}-1} \frac{\varepsilon^{\frac{\alpha}{p'}}}{1+\varepsilon^{\frac{\alpha-d}{p'}}} \left\| T_{p,B(0,sR)}^{\mathrm{d}A^{\partial B(0,sr_z^\varepsilon)}} \right\|.$$

Here, $K_{s,\varepsilon}$ is the Poincaré constant for sQ_z^ε, by scaling it is easy to see that it remains bounded as long as $s = O\left(\varepsilon^{-1}\right)$. We therefore choose $s = \varepsilon^{-1}$, and see that by Lemma 6.4 we then have that

$$\left\| T_{p,B(0,sR)}^{\mathrm{d}A^{s\partial B(0,sr_z^\varepsilon)}} \right\| \ll_{d,p} \varepsilon^{\frac{\alpha-d+1}{p}}.$$

Finally, we see by scaling that

$$\begin{aligned}\|\nabla\varphi_z^\varepsilon\|_{\mathrm{L}^{p'}(Z_z^\varepsilon)} &= \varepsilon^{\frac{d}{p'}-1}\left\|\nabla\varphi_z^{\varepsilon,\varepsilon^{-1}}\right\|_{\mathrm{L}^{p'}(Z_z^\varepsilon)} \\ &\ll_{d,p,\beta} \frac{\varepsilon^{\alpha/p'}}{1+\varepsilon^{\frac{\alpha-d}{p'}}}\varepsilon^{\frac{p+\alpha-d}{p}}.\end{aligned}$$

so that

$$\begin{aligned}\sum_{z\in\widetilde{\mathbf{I}}^\varepsilon}\left|\int_M \nabla\varphi_z^\varepsilon\cdot\nabla f\,\mathrm{d}x\right| &\leqslant \sum_{z\in\widetilde{\mathbf{I}}^\varepsilon}\|\nabla\varphi_z^\varepsilon\|_{\mathrm{L}^{p'}(Z_z^\varepsilon)}\|\nabla f\|_{\mathrm{L}^p(Z_z^\varepsilon)}\\ &\ll_{d,p,\beta}\frac{\varepsilon^{\frac{\alpha-d}{p'}}}{1+\varepsilon^{\frac{\alpha-d}{p'}}}\varepsilon^{\frac{\alpha-d+p}{p}}\|\nabla f\|_{\mathrm{L}^p(M)}\\ &=\frac{\varepsilon^{\alpha-(d-1)}}{1+\varepsilon^{\frac{\alpha-d}{p'}}}\|\nabla f\|_{\mathrm{L}^p(M)},\end{aligned}$$

where in the second step we used the inequality $\sum_{i=1}^n |a_i| \leqslant n^{\frac{1}{p'}}\left(\sum_{i=1}^n |a_i|^p\right)^{\frac{1}{p}}$. □

Proof of Theorem 6.2. We have already shown that conditions (**M1**)–(**M3**) are satisfied. By Proposition 6.3 for the Neumann problem or 6.5 for the Steklov problem, the conditions of Proposition 4.11 are satisfied so that we indeed have convergence of the eigenvalues and eigenfunctions. □

We finally have everything we need to prove Theorem 1.11.

Proof of Theorem 1.11. By density of continuous functions in either $\mathrm{L}^{d/2}(M)$ ($d \geqslant 3$) or $\mathrm{L}^1(\log \mathrm{L})^1(M)$ ($d = 2$) there is a sequence of nonnegative β_n converging to β in the relevant space. By Proposition 5.1, $\overline{\lambda}_k(M, g, \beta_n \mathrm{d}v_g) \to \overline{\lambda}_k(M, g, \beta \mathrm{d}v_g)$. Theorem 6.2 along with the volume estimate (6.1) entails that Theorem 1.11 holds for each β_n. Extracting a sequence from a diagonal argument yields the sequence Ω^ε so that Theorem 1.11 holds for β. □

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