THE NUMBER OF NEIGHBOURLY d-POLYTOPES WITH $d+3$ VERTICES

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Abstract. In this paper is proved a formula for the number of neighbourly d-polytopes with $d + 3$ vertices, when d is odd.

§1. *Introduction.* In an earlier paper (Altshuler–McMullen [1973]), a formula was found for the number of simplicial neighbourly d-polytopes with $d + 3$ vertices, in the case when d is odd. (We refer the reader to that paper for background information on neighbourly polytopes relating to the problem considered here, and to Grünbaum [1967] or McMullen–Shephard [1971] for more general information on polytopes.) In this paper, we shall solve the more general problem of enumerating all the neighbourly d-polytopes with $d + 3$ vertices. Specifically, letting $b(v, d, k)$ denote the number of combinatorial types of k-neighbourly d-polytopes with v vertices, we shall prove:

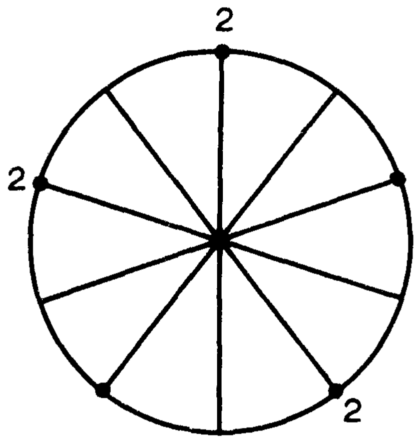
THEOREM. *For* $n \geqslant 1$,

$$b(2n + 4, 2n + 1, n) = \tfrac{1}{4}\{(5 + (-1)^n)\, 3^{[(n+1)/2]} + 6\} + \frac{1}{4(n + 2)} \sum_{\substack{h \mid n+2 \\ h \text{ odd}}} \phi(h)\,(3^{(n+2)/h} - 1),$$

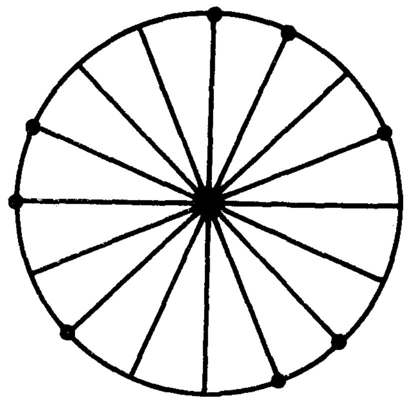
where ϕ *is Euler's function.*

§2. *Gale diagrams.* Let P be a neighbourly $(2n + 1)$-polytope with $2n + 4$ vertices. Since the combinatorial properties of P are faithfully reflected by those of its Gale diagram $\hat{P}$ (see Grünbaum [1967, §§5.4 and 6.3] and McMullen–Shephard [1971, §3.4]), it will be enough for our purposes to enumerate the isomorphism classes of the corresponding Gale diagrams.

We first remind the reader of the characterization of the Gale diagrams of simplicial neighbourly $(2m + 1)$-polytopes with $2m + 4$ vertices ($m \geqslant 1$). Since we shall need to vary m (and even allow, formally, $m = 0$ or -1), we introduce the following terminology. The points of the diagrams $\hat{Q}$ (say) which we consider occur on, or at the centre of the unit circle S in E^2; a *diameter* of $\hat{Q}$ is a diameter of S containing points of $\hat{Q}$ on S. We say $\hat{Q}$ is *balanced* if each diameter L of $\hat{Q}$ contains one or two points only, and the remaining points of $\hat{Q}$ ($2m + 3$ or $2m + 2$ in number) are equally distributed on each side of L (that is, at least $m + 1$ on each side). Then the Gale diagrams of simplicial neighbourly $(2m + 1)$-polytopes with $2m + 4$ vertices are the balanced diagrams with $2m + 4$ points, with no points at the centre of S, and such that no diameter carries points at opposite ends. In contracted form, adjacent diameters carry their points (one or two in number) at opposite ends. There are an odd number of diameters which carry two points, and adjacent diameters carrying two points are separated by an even number of diameters which carry one point. In distended form, we replace each diameter carrying two points by a pair of adjacent



Contracted



Distended

Fig. 1

diameters carrying one point at the same end. (Figure 1 illustrates this situation, with $m = 2$.) We may thus group the $2m + 4$ diameters (each carrying a single point) in adjacent pairs, k (which is odd) of which carry their points at the same end, and the remaining $m + 2 - k$ of which carry their points at opposite ends.

Let us now consider the general situation of the Gale diagram of a not necessarily simplicial neighbourly $(2n + 1)$-polytope with $2n + 4$ vertices. The fact that the diagram is balanced implies, as before, that each of its diameters contains at most two points. But now a diameter may have a point at each end, or a point at one end, and one at the centre of the circle S.

The latter case is easy to deal with. The Gale diagram is that of a pyramid, whose basis, being a neighbourly $2n$-polytope with $2n + 3$ vertices, is the cyclic polytope $C(2n + 3, 2n)$.

Another case is also easily added to our list, the case when every diameter has a point at each end. Thus there are $n + 2$ diameters in all. This diagram is also essentially unique.

From now on, then, we shall suppose our Gale diagram $\hat{P}$ has no point at the centre of S, and not all the diameters of $\hat{P}$ have a point at each end. Let us also suppose that $\hat{P}$ is in distended form, so that each diameter carrying two points at the same end is replaced by a pair of adjacent diameters, carrying a single point at the same end.

We first observe that, if we now remove a diameter containing a point at each end, the new diagram is still balanced. So, if there are, say, p of these diameters, and we remove them all, we are left with a balanced diagram containing $2(n - p + 2)$ points, each point on a distinct diameter. Using the notation introduced above, let us suppose there are k (odd) pairs of adjacent diameters carrying their points at the same end, say $K_j{}^i (i = 0, 1; j = 1, \ldots, k)$, and $l = (n - p + 2) - k$ pairs carrying their points at opposite ends, say $L^i_j (i = 0, 1; j = 1, \ldots, l)$. Moreover, let us suppose that, for $j = 1, \ldots, k$, there are l_j pairs L^i_s separating $K_j{}^i$ and K^i_{j+1} $(K^i_{k+1} = K_1{}^i)$.

We now attempt to replace the diameters carrying points at each end. As far as the diameters of type $K_j^{\,i}$ or $L^i_{\,j}$ are concerned, inserting such a diameter does not affect the property of being balanced. So, we are only concerned to ensure that equal numbers of the $2(n-p+2)$ original points lie on each side of the new diameters. Clearly, we may separate two diameters $K_j^{\,0}$ and $K_j^{\,1}$. But as we rotate the new diameter away from this initial separating position, we see that the difference between the numbers of points on each side is alternately 2 or 0, changing each time a diameter $K_j^{\,i}$ or $L^i_{\,j}$ is crossed. In other words, the allowable positions to insert the new diameters are separating a pair $\{K_j^{\,0}, K_j^{\,1}\}$ or $\{L^0_{\,s}, L^1_{\,s}\}$; we may not separate adjacent diameters of different pairs.

Suppose, then, that for $j = 1, \ldots, k, q_j - 1$ new diameters separate $K_j^{\,0}$ and $K_j^{\,1}$, and for $j = 1, \ldots, l$, $r_j - 1$ separate $L^0_{\,j}$ and $L^1_{\,j}$. Changing the notation slightly, we see that to our diagram corresponds the sequence of positive integers,

$$(q_1,\ r_{11}, \ldots, r_{1l_1},\ q_2,\ r_{21}, \ldots, r_{2l_2}, \ldots, q_k,\ r_{kl}, \ldots, r_{kl_k}),$$

where

$$\sum_j q_j + \sum_j \sum_s r_{js} = k + l + p = n + 2.$$

Conversely, for any odd k, such a sequence (with varying q_j, l_j and r_{js}) gives rise to a balanced Gale diagram with $2n + 4$ points.

So, to find $b(2n+4, 2n+1, n)$, we must enumerate these sequences. However, we must bear in mind that the sequences should really be regarded as cyclic, and so two such sequences must be identified if one can be obtained from the other by cyclic permutation or reversal of order, as long as the special rôles of $q_1, \ldots, q_k$ are preserved. We shall solve this enumeration problem in the next section.

§3. *The proof of the theorem.* We solve the enumeration problem of §2 by means of Burnside's Lemma. That is, for each $k = 1, 3, \ldots$ we count the number of arrangements invariant under each possible symmetry, and average this over this whole group of symmetries, which is, in this case, the dihedral group D_k.

The symmetries are of two kinds: k reflexions, and k rotations (there are $\phi(h)$ of order exactly h, for each divisor h of k, where ϕ is Euler's function). We first consider the k reflexions. The invariant arrangements are all similar, so it is enough to consider a typical example, that which leaves q_1 fixed, and interchanges the pairs $\{q_2, q_k\}, \{q_3, q_{k-1}\}, \ldots$, and so on. So, we must also have $l_1 = l_k$, $l_2 = l_{k-1}, \ldots$, and so on, and, for $j = 1, \ldots, \frac{1}{2}(k+1)$, $s = 1, \ldots, l_j$,

$$r_{js} = r_{k-j+2,\, l_j - s + 1}.$$

In the case $j = \frac{1}{2}(k+1)$, there are two cases, according as $l_{(k+1)/2}$ is even or odd. In the latter case, the term $r_{(k+1)/2,\,(l_j+1)/2}$ is also fixed by the reflexion. In the former case, there is no such invariant term; to treat the two cases together, we shall here replace $l_{(k+1)/2}$ by $1 + l_{(k+1)/2}$, relabel $r_{(k+1)/2,\,s}$ by $r_{(k+1)/2,\,s+1}$ for $s \geqslant \frac{1}{2} l_{(k+1)/2}$, and introduce the term $r_{(k+1)/2,\,m+1} = 0$, with $m = \frac{1}{2} l_{(k+1)/2}$.

We conclude, therefore, that for reflexion terms, we must enumerate the sequences

$$q_1, r_{11}, \ldots, r_{1l_1}, q_2, \ldots, q_{(k+1)/2}, r_{(k+1)/2,\,1}, \ldots, r_{(k+1)/2,\,t},$$

where all terms (except possibly the last) are positive, and

$$q_1 + 2(r_{11} + \dots + r_{(k+1)/2,\,t-1}) + r_{(k+1)/2,\,t} = n + 2.$$

We first need a lemma.

LEMMA. *The number of sequences* $(x_1, \dots, x_v)$ *of positive integers with sum* m *is*

$$\binom{m-1}{v-1}.$$

The proof is by induction on the length v of the sequence; it is easy, and is therefore left to the reader.

So, for each $m \leqslant [\frac{1}{2}(n+1)]$, we have $\binom{m-1}{v-1}$ possible sequences $(x_1, \dots, x_v)$ of length v. (To simplify certain expressions below, we adopt the special convention $\binom{-1}{-1} = 1$.) We can choose $\frac{1}{2}(k-1)$ of the x_i to play the rôles of $q_2, \dots, q_{(k+1)/2}$ in $\binom{v}{\frac{1}{2}(k-1)}$ ways; the remaining x_i's play the rôles of $r_{11}, \dots, r_{(k+1)/2,\,t-1}$. We must now account for q_1 and $r_{(k+1)/2,\,t}$ from the $n + 2 - m$ points left over; since q_1 is positive, this can be done in $n + 2 - 2m$ ways. Hence, each reflexion term contributes a total of

$$\sum_{m \leqslant [(n+1)/2]} \sum_{v \geqslant 0} (n + 2 - 2m) \binom{v}{(k-1)/2} \binom{m-1}{v-1}.$$

It follows that the total contribution from all reflexion terms (for all k) is

$$\sum_{k \text{ odd}} \tfrac{1}{2}k \left\{ k \sum_{m \leqslant [(n+1)/2]} \sum_{v \geqslant 0} (n + 2 - 2m) \binom{v}{(k-1)/2} \binom{m-1}{v-1} \right\}$$

$$= \tfrac{1}{2} \sum_{m \leqslant [(n+1)/2]} \sum_{v \geqslant 0} \left\{ (n + 2 - 2m) \binom{m-1}{v-1} \sum_{k \text{ odd}} \binom{v}{(k-1)/2} \right\}$$

$$= \tfrac{1}{2} \sum_{m \leqslant [(n+1)/2]} (n + 2 - 2m) \left\{ \sum_{v \geqslant 0} \binom{m-1}{v-1} 2^v \right\}$$

$$= \tfrac{1}{2}(n+2) + \sum_{1 \leqslant m \leqslant [(n+1)/2]} (n + 2 - 2m)\, 3^{m-1}$$

$$= \tfrac{1}{2}(n+2) + (n+2) \frac{3^{[(n+1)/2]} - 1}{3 - 1}$$

$$- 2 \frac{[(n+1)/2]\, 3^{[(n+3)/2]} - [(n+3)/2]\, 3^{[(n+1)/2]} + 1}{(3-1)^2}$$

$$= \tfrac{1}{2} \{(n + 3 - 2[(n+1)/2])\, 3^{[(n+1)/2]} - 1\}$$

$$= \tfrac{1}{4} \{(5 + (-1)^n)\, 3^{[(n+1)/2]} - 2\}.$$

The rotation terms are rather easier. For each of the $\phi(h)$ rotations of order h in D_k, we must count the sequences

$$(q_1, r_{11}, \ldots, r_{1, l_1}, q_2, \ldots, q_{k/h}, r_{k/h, 1}, \ldots, r_{k/h, l_{k/h}})$$

whose sum is $(n+2)/h$ (taking, as before, a typical sequence). Of course, h must be a common divisor of k and $n+2$. The number of such sequences of total length v is

$$\binom{v-1}{(k/h)-1}\binom{((n+2)/h)-1}{v-1} = \frac{k}{n+2}\binom{v}{k/h}\binom{(n+2)/h}{v},$$

since the rôle of q_1 is already assigned, and we can choose any $(k/h)-1$ out of the remaining $v-1$ to play the rôles of $q_2, \ldots, q_{k/h}$.

Thus the total rotation term (again, for all k) is

$$\sum_{k \text{ odd}} \frac{1}{2k}\left\{\sum_{h|(n+2,\, k)} \phi(h) \sum_{v \geqslant 0} \frac{k}{n+2}\binom{v}{k/h}\binom{(n+2)/h}{v}\right\}$$

$$= \frac{1}{2(n+2)} \sum_{k \text{ odd}} \sum_{h|(n+2,\, k)} \phi(h) \sum_{v \geqslant 0} \binom{(n+2)/h}{k/h}\binom{((n+2)/h)-k/h}{((n+2)/h)-v}$$

$$= \frac{1}{2(n+2)} \sum_{\substack{h|n+2 \\ h \text{ odd}}} \phi(h) \sum_{t \text{ odd}} \binom{(n+2)/h}{t} 2^{((n+2)/h)-t}$$

$$= \frac{1}{2(n+2)} \sum_{\substack{h|n+2 \\ h \text{ odd}}} \phi(h)\, \tfrac{1}{2}(3^{(n+2)/h} - 1).$$

If we now collect these terms together, and recall the two additional exceptional cases, we obtain the formula given in the theorem.

§4. *Remarks.* In the special case $n = 1$ (so $d = 3$), we obtain

$$b(6, 3, 1) = 7.$$

Of course, this is just $c(6, 3)$, since every 3-polytope is 1-neighbourly.

In Altshuler–McMullen [1973], two conjectures were made about the behaviour of $b_s(v, d, k)$, the number of simplicial k-neighbourly d-polytopes with v vertices. We wish here to propose as problems:

CONJECTURE 1. *For each fixed $r \geqslant 3$ and $k \geqslant 1$,*

$$\lim_{d \to \infty} \frac{b(d+r, d, k+1)}{b(d+r, d, k)} = 0.$$

CONJECTURE 2. *For each fixed $r \geqslant 3$ and $k \geqslant 1$,*

$$\lim_{d \to \infty} \frac{b(d+r, d, [\tfrac{1}{2}d] - k + 1)}{b(d+r, d, [\tfrac{1}{2}d] - k)} = 0.$$

Since

$$b(d+2, d, k) = [\tfrac{1}{4}(d-2k+2)^2],$$

the appropriate limit for $r = 2$ in the first problem is 1. In the second problem, when $r = 2$, the ratio takes the value

$$\left(\frac{k}{k+1}\right)^2 \quad \text{or} \quad \frac{k}{k+2},$$

according as d is even or odd.

References

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