# THE NUMBER OF SIMPLICIAL NEIGHBOURLY $d$-POLYTOPES WITH $d+3$ VERTICES

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*Abstract.* In this paper is proved a formula for the number of simplicial neighbourly $d$-polytopes with $d+3$ vertices, when $d$ is odd.

§1. *Introduction.* A $d$-polytope $P$ is *$k$-neighbourly* if every subset of $k$ vertices of $P$ is the set of vertices of a face of $P$. (We refer the reader to Grünbaum [1967, particularly §5.4 and §6.3] or McMullen–Shephard [1971, Chapter 3] for the terminology employed here.) The cyclic polytopes $C(v, d)$ ($v \geqslant d+1$) provide examples of polytopes which are neighbourly (that is, $n$-neighbourly, $n = [\frac{1}{2}d]$); the only $d$-polytopes which are $k$-neighbourly for $k > n$ are the simplices. The importance of the simplicial neighbourly $d$-polytopes with $v$ vertices lies in the fact, conjectured by Motzkin [1957] and proved by McMullen [1970], that, among all $d$-polytopes with $v$ vertices, they, and only they, have the maximal number of faces of each dimension.

The cyclic polytopes, with their relatively simple structure, have been thoroughly investigated. For the literature about cyclic polytopes prior to 1967, the reader should consult Grünbaum [1967]; for more recent results, we mention Altshuler [1971; 1973] and Shephard [1968] (see also McMullen–Shephard [1971, §2.3 (vi)]). Rather surprisingly, however, very little has been published about the structure of the other neighbourly polytopes; almost all the known information can be found in Grünbaum [1967, §7.2]. In particular, it is shown there that, if $d$ is even, all neighbourly $d$-polytopes are simplicial, while if $d$ is odd, for each $v \geqslant d+2$ there are neighbourly $d$-polytopes with $v$ vertices which are not simplicial.

Let $b_s(v, d, k)$ denote the number of simplicial $k$-neighbourly $d$-polytopes with $v$ vertices (so that $c_s(v, d) = b_s(v, d, 1)$). The known information about the most interesting case $k = n$ ($= [\frac{1}{2}d]$) can be summarized as follows:

If $d = 2n$, $b_s(v, d, n) = 1$ if and only if $d+1 \leqslant v \leqslant d+3$, and if $d = 2n+1$, $b_s(v, d, n) = 1$ if and only if $d+1 \leqslant v \leqslant d+2$ (Grünbaum [1967, §§7.2 and 7.3]); $b_s(8, 4, 2) = 3$ (Grünbaum–Sreedharan [1967]); and $b_s(9, 4, 2) = 23$ (Altshuler–Steinberg [1973]).

The main purpose of this paper is to prove:

THEOREM 1. *For* $n \geqslant 1$,

$$b_s(2n+4, 2n+1, n) = 2^{[(n-1)/2]} + \frac{1}{4(n+2)} \sum_{\substack{h \mid n+2 \\ h \text{ odd}}} \phi(h) . 2^{(n+2)/h},$$

*where* $\phi$ *is Euler's function.*

§2. *Gale diagrams.* Let $P$ be a simplicial neighbourly $(2n+1)$-polytope with $2n+4$ vertices. The combinatorial properties of $P$ are faithfully reflected by those of its contracted standard Gale diagram $\hat{P}$. (In matters of terminology concerning

Gale diagrams, we shall follow McMullen–Shephard [1971, §3.4].) $\hat{P}$ consists of $2n + 4$ points (counted according to multiplicity) distributed on the unit circle $S$ in $E^2$. $\hat{P}$ has an odd number, $k$ say, of diameters (that is, diameters of $S$ containing points of $\hat{P}$), and points of $\hat{P}$ on adjacent diameters occur at opposite ends.

The neighbourliness of $P$ is equivalent to there being at least $n + 1$ points of $\hat{P}$ on each side of every diameter (Grünbaum [1967, Exercise 7.3.7]). Thus, each diameter of $\hat{P}$ contains either one or two points of $\hat{P}$, and (since the total number of points is even), the number, $l$ say, of 2-diameters (that is, with two points of $\hat{P}$) is also odd. Moreover, since equal numbers of points of $\hat{P}$ on 2-diameters must lie on either side of each 2-diameter, we see that adjacent 2-diameters carry their points of $\hat{P}$ at opposite ends (compare the description of the Gale diagram of the unique neighbourly $2n$-polytope $C(2n + 3, 2n)$ with $2n + 3$ vertices). Finally, this implies that adjacent 2-diameters are separated by an even number of 1-diameters.

The converse is clear: any Gale diagram with $2n + 4$ points with an odd number of 2-diameters, of which adjacent pairs are separated by an even number of 1-diameters, is the Gale diagram of a simplicial neighbourly $(2n + 1)$-polytope with $2n + 4$ vertices.

§3. *The proof of the theorem.* We have reduced the proof of the theorem to the problem of enumerating the Gale diagrams characterized in the last section. With each such Gale diagram, we associate a labelling of the vertices of a regular polygon with an odd number of sides, as follows. If there are $l$ 2-diameters, let them be numbered $D_1, \ldots, D_l$ in cyclic order. Suppose that there are $2(m_i - 1)$ 1-diameters between $D_i$ and $D_{i+1}$ ($i = 1, \ldots, l$; $D_{l+1} = D_1$). Then we label the vertices of an $l$-gon in cyclic order $m_1, \ldots, m_l$, noticing that, if the Gale diagram has $k$ diameters in all,

$$\sum_{i=1}^{l} m_i = \tfrac{1}{2} \sum_{i=1}^{l} 2(m_i - 1) + l = \tfrac{1}{2}\{(k - l) + 2l\} = n + 2.$$

To each isomorphism type of Gale diagram we are considering corresponds precisely one equivalence class of such labelled regular polygons under orthogonal transformations, and conversely.

It is therefore enough to count the number of essentially distinct ways of attaching to the vertices of regular $l$-gons ($l = 1, 3, \ldots$) positive integer labels totalling $n + 2$. But this problem is just a straight-forward application of the well-known theorem of Pôlya [1937]. Indeed, the answer is implicit in the formula for $\mathscr{D}_s$ (in case $s$ is odd), given on page 169; it is the coefficient of $x^{n+2}$ in

$$\sum_{k \text{ odd}} \left\{ \frac{1}{2} \frac{x^k}{(1 - x)(1 - x^2)^{(k-1)/2}} + \frac{1}{2k} \sum_{h|k} \phi(h) \frac{x^k}{(1 - x^h)^{k/h}} \right\}$$

$$= \frac{x(1 + x)}{2(1 - 2x^2)} + \sum_{h \text{ odd}} \frac{\phi(h)}{4h} \log(1 - 2x^h),$$

which is the number given in the statement of the theorem.

§4. *Further results and conjectures.* An immediate consequence of Theorem 1 is that

$$\lim_{n\to\infty} b_s(2n+4, 2n+1, n) = \infty.$$

It is reasonable to suppose that the following is also true.

CONJECTURE 1. *For every $r > 3$,*

$$\lim_{d\to\infty} b_s(d+r, d, [\tfrac{1}{2}d]) = \infty.$$

Another result following easily from Theorem 1 is:

THEOREM 2.

$$\lim_{n\to\infty} \frac{b_s(2n+4, 2n+1, n)}{c_s(2n+4, 2n+1)} = 0.$$

For, in Grünbaum [1967, §6.3] is given Perles' formula for $c_s(d+3, d)$ (which can be proved by a method very similar to that of §3). In case $d = 2n+1$, we have

$$c_s(2n+4, 2n+1) = 2^n - (n+2) + \frac{1}{8(n+2)} \sum_{\substack{h|n+2 \\ h \text{ odd}}} \phi(h)\, 2^{(2n+4)/h}.$$

Thus,

$$c_s(2n+4, 2n+1) \geqslant 2^n - (n+2) + \frac{1}{n+2}\, .2^{2n+1} = \alpha_n \quad \text{(say)}.$$

On the other hand, since $\sum_{h|n+2} \phi(h) = n+2$, we easily see that

$$\begin{aligned} b_s(2n+4, 2n+1, n) &< 2^{[(n-1)/2]} + \tfrac{1}{4}.2^{n+2} \\ &= 2^{[(n-1)/2]} + 2^n. \end{aligned}$$

(More careful estimates lead to the bound $2^n$.) Since

$$\lim_{n\to\infty} \frac{2^{[(n-1)/2]} + 2^n}{\alpha_n} = 0,$$

this proves the theorem.

There is an obvious conjectural generalization of Theorem 2, along the lines of Conjecture 1. However, we shall propose instead two stronger generalizations; we shall state them as conjectures, although we do not wish to commit ourselves too firmly to belief in them.

CONJECTURE 2. *For each fixed $r \geqslant 3$ and $k \geqslant 1$,*

$$\lim_{d\to\infty} \frac{b_s(d+r, d, k+1)}{b_s(d+r, d, k)} = 0.$$

CONJECTURE 3. *For each fixed $r \geqslant 3$ and $k \geqslant 0$,*

$$\lim_{d\to\infty} \frac{b_s(d+r, d, [\tfrac{1}{2}d] - k + 1)}{b_s(d+r, d, [\tfrac{1}{2}d] - k)} = 0.$$

In case $r = 2$, the appropriate limits are 1 and $k/(k+1)$ respectively.

§5. *Remarks.* Theorem 2 could be proved without using Theorem 1; from the description of star-diagrams in Grünbaum [1967, §6.2], together with the upper-bound theorem (McMullen [1970]), one can easily obtain

$$b_s(2n+4, 2n+1, n) \leqslant 2^{n+1},$$

which of course, leads to Theorem 2. However, Theorem 2 perhaps takes on a fuller significance when viewed in the light of the consequence of Theorem 1 mentioned in §4.

A brief remark about the origins of this paper is appropriate. The result of Theorem 1 (in a somewhat different formulation) is due to the first author, while the second author provided the briefer proof presented here. We wish to thank Professor C. A. Rogers for putting us in contact, and thus making our collaboration possible.

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