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A zero–infinity law for well-approximable points in Julia sets

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In memory of Noel Baker

Abstract. Let $T : J \to J$ be an expanding rational map of the Riemann sphere acting on its Julia set J and $f : J \to \mathbb{R}$ denote a Hölder continuous function satisfying $f(x) > \log|T'(x)|$ for all x in J. Then for any point z_0 in J define the set $D_{z_0}(f)$ of 'well-approximable' points to be the set of points in J which lie in the Euclidean ball

$$B\bigg(y, \exp\bigg(-\sum_{i=0}^{n-1} f(T^i y)\bigg)\bigg)$$

for infinitely many pairs (y, n) satisfying $T^n(y) = z_0$. In our 1997 paper, we calculated the Hausdorff dimension of $D_{z_0}(f)$. In the present paper, we shall show that the Hausdorff measure $\mathcal{H}^s$ of this set is either zero or infinite. This is in line with the general philosophy that all 'naturally' occurring sets of well-approximable points should have zero or infinite Hausdorff measure.

1. *Introduction*

In [**4**], we formulated the following general problem. Consider a metric space J equipped with a Borel probability measure m. If $T : J \to J$ is measure preserving and ergodic, we know by the ergodic theorem that for any ball B of positive m-measure, the subset

$$\{z \in J : T^n(z) \in B \text{ for infinitely many } n \in \mathbb{N}\}$$

of J has full m-measure. This means that the trajectories of m-almost all points will go through the ball B infinitely often. A natural question to ask is what happens if the ball B

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shrinks with time. More precisely, if at time n we have a ball $B(z_0, \mathrm{rad}(n))$ centred at a point $z_0 \in J$ of radius $\mathrm{rad}(n)$ ($\mathrm{rad}(n) \to 0$ as $n \to \infty$), then what kind of properties does the set W of points z have, whose images $T^n(z)$ are in $B(z_0, \mathrm{rad}(n))$ for infinitely many n? These points can be thought of as trajectories which hit a shrinking target infinitely often and are called 'well approximable' with respect to the function 'rad', in analogy to those in the classical theory of Diophantine approximation.

In [**4**], we considered a special case of the above general 'shrinking target' problem in which T is an *expanding* rational map of the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and $J = J(T)$ is its Julia set. By the definition of expanding, there exists a constant $\lambda > 1$ and an integer $p \geq 1$ such that

$$|(T^p)'(z)| \geq \lambda \quad \text{for all } z \in J,$$

where T' is the derivative of T. For such maps, it is known (see [**8**]) that J is not the whole of $\overline{\mathbb{C}}$ and we may and will assume that $\infty \notin J$. Thus we can think of J as a metric space with the usual metric on $\mathbb{C}$. Specifically, for any $\tau > 0$ and $z_0 \in J$ we considered the sets

$$W^{\bullet}_{z_0}(\tau) := \{z \in J : T^n(z) \in B(z_0, |(T^n)'(z)|^{-\tau}) \text{ for infinitely many } n \in \mathbb{N}\}$$

and

$$W_{z_0}(\tau) := \{z \text{ in } J : T^n(z) \in B(z_0, e^{-n\tau}) \text{ for infinitely many } n \in \mathbb{N}\},$$

which we referred to as the 'local' and 'global' well-approximable sets, respectively. Here the backward orbit of a selected point z_0 in J corresponds to the rationals in the classical set $W(\tau) := \{x \in \mathbb{R} : |x - p/q| \leq q^{-\tau}$ for infinitely many rationals $p/q\}$ of well approximable numbers. For Julia sets associated with rational maps, we proved [**4**] the following analogue of the Jarník–Besicovitch Theorem in the classical theory of metric Diophantine approximation.

THEOREM 1. *For the local set one has* $\dim W^{\bullet}_{z_0}(\tau) = \delta/(1+\tau)$*, where* δ *is the Hausdorff dimension of* J.

In [**4**], we also obtained a partial result on the dimension of the global set $W_{z_0}(\tau)$. However, the breakthrough in calculating the dimension of this set came by considering the following generalization. Let $f : J \to \mathbb{R}^{\geq 0}$ denote a Hölder continuous function satisfying

$$f(z) > \log|T'(z)| \quad \text{for all } z \in J,$$

and write $f_n(x)$ for the nth ergodic sum, that is

$$f_n(x) := \sum_{i=0}^{n-1} f(T^i x).$$

Then define the set $D_{z_0}(f)$ of well-approximable points to be the set of points in J which lie in the ball

$$B(y, \exp(-f_n(y)))$$

for infinitely many pairs (y, n) with $T^n(y) = z_0$. In [**5**] we proved the following.

THEOREM 2. *The set $D_{z_0}(f)$ has Hausdorff dimension α, where α is the unique positive number satisfying the pressure equation*

$$P(T, -\alpha \cdot f) = 0.$$

Here $P(-,-)$ denotes the topological pressure.

It is easy to verify (see §1.3 in [**5**]) that the 'local' set corresponds to $D_{z_0}(f)$ with $f(z) = (1+\tau)\log|T'(z)|$ whilst the 'global' set corresponds to $D_{z_0}(f)$ with $f(z) = \log|T'(z)| + \tau$. This solves the problem of calculating $\dim W_{z_0}(\tau)$. The theorem may be viewed as an extension of the Bowen–Manning–McCluskey formula, which states that $P(T, -\delta \log|T'|) = 0$. In [**5**], we also demonstrated an unexpected link between the dimension results for $D_{z_0}(f)$ and the dimension of exceptional sets arising from points with 'badly behaved' ergodic averages. In short, given an ergodic measure m on J, these are points z in J at which the ergodic average $\lim_{n\to\infty} n^{-1} f_n(z)$ of f does not tend to the expected limit $\int_J f\, dm$. In turn, Falconer [**3**] has recently shown a rather elegant connection between the dimension results for $D_{z_0}(f)$ and the multifractal spectrum associated with the dynamical system $T : J \to J$. Also, the dimension result for the local set has recently been extended to parabolic rational maps [**7**]. This concludes our brief overview of recent developments and various connections. Returning to the main theme, it follows from the definition of s-dimensional Hausdorff measure $\mathcal{H}^s$ that

$$\mathcal{H}^s(D_{z_0}(f)) = \begin{cases} 0 & \text{if } s > \alpha \\ \infty & \text{if } s < \alpha. \end{cases}$$

However, if $s = \alpha$ then $\mathcal{H}^s(D_{z_0}(f))$ may be zero or infinite, or may satisfy

$$0 < \mathcal{H}^s(D_{z_0}(f)) < \infty.$$

In this paper, we shall prove that the latter is impossible.

THEOREM 3. *Let α be Hausdorff dimension of $D_{z_0}(f)$. Then the α-dimensional Hausdorff measure of $D_{z_0}(f)$ is either zero or infinity.*

Note that if one sets $f = \log|T'|$, the set $D_{z_0}(f)$ has full δ-dimensional Hausdorff measure in J, where δ denotes the Hausdorff dimension of J. However, for expanding rational maps, J has finite positive δ-dimensional Hausdorff measure and so clearly the theorem does not extend to this case. In fact, the condition $f > \log|T'|$ everywhere on J guarantees that $\mathcal{H}^\delta(D_{z_0}(f)) = 0$—see the Appendix.

In the language of geometric measure theory, Theorem 3 simply states that the sets $D_{z_0}(f)$ are not s-sets. This is a well-known fact for the analogous, classical sets of well-approximable real numbers. In fact, in the classical set-up the result is a consequence of a much stronger statement whose proof is very technical and rather intricate (see [**2**] and references within)—there seems to be no direct approach. However, by making use of the geometry of the Julia set and the existence of generalized conformal measures, we are able to give a direct and, in some sense, a 'natural' proof of the statement for expanding rational maps.

2. *Conformal measures*

We shall write $\mathcal{H}^s$ for the s-dimensional Hausdorff measure. Note that the s-dimensional Hausdorff measure is s-conformal; that is, if T is injective on some set $X \subset J$ then

$$\mathcal{H}^s(TX) = \int_X |T'(x)|^s \, d\mathcal{H}^s(x).$$

Recall that a function $f : J \to \mathbb{R}$ is said to be Hölder continuous if and only if there is a constant C satisfying the following condition. For any ball B in J and any natural number n such that T^n is injective on B, one has for all x, y in $B \cap J$

$$|f_n(x) - f_n(y)| \leq C.$$

The constant C will be referred to as a distortion constant for the function f.

We need the following powerful result from complex analysis (see [**6**]).

KÖBE DISTORTION THEOREM. *Let $\Delta \subset \overline{\mathbb{C}}$ be a topological disc with boundary containing at least two points and let $V \subset \Delta$ be compact. Then there exists a constant $C(\Delta, V)$ such that for any univalent holomorphic function $U : \Delta \to \mathbb{C}$ the following inequality is satisfied,*

$$\sup_{x,y \in V} \frac{|U'(x)|}{|U'(y)|} \leq C(\Delta, V).$$

This implies that the function $\log |T'|$ on J is Hölder continuous.

A more general class of conformal measures was introduced by Denker and Urbański in [**1**] and numerous other papers. If $f : J \to \mathbb{R}$ is Hölder continuous, then a measure ν on J is said to be f-conformal if and only if the following holds. For all measurable subsets X of J on which T is injective:

$$\nu(TX) = \int_X \exp(f(x)) \, d\nu(x). \tag{1}$$

In this notation, $\mathcal{H}^s$ is $s \log |T'|$-conformal. Denker and Urbański have proved (see [**1**]) the following theorem.

THEOREM 4. *Let T, f and α be as in Theorem 2. Then there is a unique non-atomic $\alpha \cdot f$-conformal probability measure on J.*

From now on we shall refer to the unique $\alpha \cdot f$-conformal probability measure as ν. We recall the following fact concerning conformal measures.

LEMMA 1. *Let T^n be injective on a ball B in J. Then one has*

$$\nu(T^n B) \asymp \exp(\alpha f_n(B)) \nu(B)$$

where $f_n(B)$ is the value of f_n at any point of B and the implied constants are independent of B and n. Similarly one has for any measurable subset $A \subset B$

$$\mathcal{H}^\alpha(T^n A) \asymp |(T^n)'(B)|^\alpha \mathcal{H}^\alpha(A).$$

Proof. This follows from the transformation formula (1) iterated n times, combined with the Hölder continuity of the functions f and $\log |T'|$. □

Next, we state a useful formula for the ν measure of an arbitrary ball. For any ball B in J, we shall write $n_0(B)$ for the largest natural number n for which T^n is injective on B. Using the fact that T is expanding one may show (see, for example, Lemma 4 of [**5**]) that there is an $N \in \mathbb{N}$ such that for any ball B in J,

$$J \subset T^{n_0(B)+N} B.$$

This fact together with the previous lemma gives the following lemma.

LEMMA 2. *For any ball B one has $\nu(B) \asymp \exp(-\alpha f_{n_0(B)}(B))$.*

3. *Proof of Theorem 3*

Define the Hölder continuous function $g : J \to \mathbb{R}^{>0}$ by

$$g(x) := f(x) - \log |T'(x)|.$$

For means of calculation we shall introduce the following sets for $C > 0$:

$$E(C) := \{x \in J : T^n(x) \in B(z_0, C \exp(-g_n(x))) \text{ for infinitely many } n \in \mathbb{N}\}$$

where g_n is the nth ergodic sum of g. It follows from the Köbe Distortion Theorem that there is a constant $C > 1$ such that

$$E(C^{-1}) \subset D_{z_0}(f) \subset E(C).$$

Therefore, in order to prove Theorem 3 it is sufficient to show that either $E(C)$ has zero Hausdorff measure for all C, or that $E(C)$ has infinite Hausdorff measure for all C.

LEMMA 3. *For $x \in J$, one has $x \in E(C)$ if and only if $T(x) \in E(Ce^{-g(x)})$.*

Proof. Let $x \in E(C)$. This means that $T^n(x) \in B(z_0, C \exp(-g_n(x)))$ for infinitely many natural numbers n. This is equivalent to $T^{n-1}(Tx) \in B(z_0, C \exp(-g_{n-1}(Tx) - g(x)))$ for infinitely many natural numbers n. Replacing $n - 1$ by n in this we obtain $T(x) \in E(C \exp(-g(x)))$. □

LEMMA 4. *For $x \in J$ one has $x \in E(C)$ if and only if $T^n(x) \in E(\exp(-g_n(x))C)$.*

Proof. This follows from Lemma 3 by induction on n. □

LEMMA 5. *For any $a > 0$ we have $\mathcal{H}^\alpha(E(aC)) \asymp \mathcal{H}^\alpha(E(C))$, where the implied constants depend on a, but not on C.*

Proof. Suppose $a < 1$. Then clearly $\mathcal{H}^\alpha(E(aC)) \leq \mathcal{H}^\alpha(E(C))$. On the other hand, by the conformality of Hausdorff measure (see Lemma 1) and the fact that T is expanding, we have that $\mathcal{H}^\alpha(T^m(E(C))) \gg \mathcal{H}^\alpha(E(C))$—here the implied constant depends on m. As $g(z) > 0$ everywhere on the compact set J, there is an $\epsilon > 0$ such that $g(z) > \epsilon$. Thus $g_m(z) > m\epsilon$, which together with Lemma 4 implies that $T^m(E(C)) \subset E(C \exp(-m\epsilon))$. Hence

$$\mathcal{H}^\alpha(E(C)) \ll \mathcal{H}^\alpha(E(C \exp(-m\epsilon))),$$

and choosing m such that $\exp(-m\epsilon) \leq a \leq \exp(-(m-1)\epsilon)$ completes the proof when $a < 1$. For the case when $a > 1$, start by considering $\mathcal{H}^\alpha(T^m(E(aC)))$ and repeat the above argument with obvious modifications. □

LEMMA 6. *For any ball $B = B(z, r)$ centred on J we have*

$$\mathcal{H}^\alpha(B \cap E(C)) \asymp |(T^{n_0(B)})'(z)|^{-\alpha} \mathcal{H}^\alpha(E(\exp(-g_{n_0(B)}(z))C)).$$

The implied constants are independent of B and C.

Proof. Let $n_0 = n_0(B(z, r))$. By Lemma 1 we have

$$\mathcal{H}^\alpha(B \cap E(C)) \asymp |(T^{n_0})'(z)|^{-\alpha} \mathcal{H}^\alpha(T^{n_0}(B \cap E(C))).$$

By Lemma 4 we have

$$T^{n_0}(B \cap E(C)) \subset E(\exp(-g_{n_0}(z) + a)C),$$

where a is a distortion constant for g. Therefore,

$$\mathcal{H}^\alpha(B \cap E(C)) \ll |(T^{n_0})'(z)|^{-\alpha} \mathcal{H}^\alpha(E(\exp(-g_{n_0}(z) + a)C)).$$

By Lemma 5 this implies

$$\mathcal{H}^\alpha(B \cap E(C)) \ll |(T^{n_0})'(z)|^{-\alpha} \mathcal{H}^\alpha(E(\exp(-g_{n_0}(z))C)).$$

On the other hand, there is a constant N such that $T^{n_0+N}(B) \supset J$. This implies

$$\begin{aligned}\mathcal{H}^\alpha(B \cap E(C)) &\gg |(T^{n_0+N})'(z)|^{-\alpha} \mathcal{H}^\alpha(T^{n_0+N}(B \cap E(C))) \\ &\gg |(T^{n_0})'(z)|^{-\alpha} \mathcal{H}^\alpha(E(\exp(-g_{n_0+N}(z) - a)C)),\end{aligned}$$

and again by Lemma 5 we have

$$\mathcal{H}^\alpha(B \cap E(C)) \gg |(T^{n_0})'(z)|^{-\alpha} \mathcal{H}^\alpha(E(\exp(-g_{n_0}(z))C)). \qquad \square$$

To prove the next lemma, we will use a carefully chosen cover of the Julia set J which we now describe. Let I be the set of pairs $(y, n) \in J \times \mathbb{N}$ such that $T^n(y) = z_0$. Suppose we have $c_1, C > 0$ and set $I(C, c_1) = \{(y, n) \in I : |C - \log g_n(y)| < c_1\}$. It is known that if c_1 is sufficiently large (depending only on J and f) then for any $C > 0$ the following is a cover of J:

$$\mathcal{C}(C) = \{B(y, c_1|(T^n)'(y)|^{-1}) : (y, n) \in I(C, c_1)\}.$$

We shall fix c_1 sufficiently large so that $\mathcal{C}(C)$ is a cover. There is a constant N depending only on J, f and c_1 such that no point of J is in more than N of the balls in the cover $\mathcal{C}(C)$. These statements are contained in Lemma 8 of [**5**]. The upshot of this is that for any measure μ on J and any measurable subset $A \subseteq J$, we have

$$\mu(A) \asymp \sum_{(y,n)\in I(C,c_1)} \mu(A \cap B(y, c_1|(T^n)'(y)|^{-1})). \tag{2}$$

The implied constants are independent of μ and C (in fact, the implied constants are 1 and N).

LEMMA 7. *Let B be a ball in J. Then $\mathcal{H}^\alpha(B \cap E(C)) \asymp C^\alpha \mathcal{H}^\alpha(B \cap E(1))$.*

Proof. By Lemma 6 it is sufficient to prove the lemma in the case $B = J$. By (2) we have

$$\mathcal{H}^\alpha(E(C)) \asymp \sum_{(y,n)\in I(C,c_1)} \mathcal{H}^\alpha(B(y, c_1|(T^n)'(y)|^{-1}) \cap E(C)).$$

For the moment we shall concentrate on one of the balls $B = B(y, c_1|(T^n)'(y)|^{-1})$ in the cover $\mathcal{C}(C)$. Note that $n_0(B) = n + O(1)$ (see Lemma 5 of [**5**]). Therefore, by Lemma 6 we have

$$\mathcal{H}^\alpha(B \cap E(C)) \asymp |(T^n)'(y)|^{-\alpha}\mathcal{H}^\alpha(E(Ce^{-g_n(y)})).$$

Now by Lemma 5 as $\exp(g_n(y)) \asymp C$, we have

$$\mathcal{H}^\alpha(E(C) \cap B) \asymp |(T^n)'(y)|^{-\alpha}\mathcal{H}^\alpha(E(1)).$$

Again since $\exp(g_n(y)) \asymp C$, by Lemma 2 we have

$$|(T^n)'(y)|^{-\alpha} \asymp C^\alpha \exp(-\alpha f_n(y)) \asymp C^\alpha \nu(B).$$

Summing this over the balls B in our cover $\mathcal{C}(C)$ we obtain

$$\mathcal{H}^\alpha(E(C)) \asymp C^\alpha \mathcal{H}^\alpha(E(1)) \sum_{(y,n)\in I(C,c_1)} \nu(B(y, c_1|(T^n)'(y)|^{-1})).$$

Now, using (2) in the opposite direction we have

$$\mathcal{H}^\alpha(E(C)) \asymp C^\alpha \mathcal{H}^\alpha(E(1))\nu(J) = C^\alpha \mathcal{H}^\alpha(E(1)).$$

This proves the lemma. □

For a ball B in J we shall use the notation

$$m(B) := \mathcal{H}^\alpha(B \cap E(1)).$$

LEMMA 8. *For any ball B in J we have $m(B) \asymp \mathcal{H}^\alpha(E(1))\nu(B)$.*

Proof. Given $B = B(z, r)$, let $n_0 = n_0(B(z, r))$. By Lemma 6 with $C = 1$ and Lemma 7 with $B = J$, we have

$$\begin{aligned} m(B) &\asymp |(T^{n_0})'(z)|^{-\alpha}\mathcal{H}^\alpha(E(\exp(-g_{n_0}(z)))) \\ &\asymp \exp(-\alpha f_{n_0}(z))\mathcal{H}^\alpha(E(1)). \end{aligned}$$

The lemma now follows on applying Lemma 2. □

During the proof of the next result we will require the following fact (see Corollary 1 of [**5**]):

$$\sum_{y:T^n(y)=z_0} \exp(-\alpha f_n(y)) \asymp 1.$$

PROPOSITION 1. $\nu(E(1)) = 0$.

Proof. For C sufficiently large the set $E(1)$ is contained in the limsup of the sets $A(n)$ defined by

$$A(n) := \bigcup_{y:T^n(y)=z_0} B(y, C\exp(-f_n(y))).$$

Therefore, in view of the Borel–Cantelli Lemma it is sufficient to show that $\sum_n \nu(A(n))$ converges. By Lemma 1 we have

$$\nu(A(n)) \ll \sum_{y:T^n(y)=z_0} \exp(-\alpha f_n(y))\nu(B(z_0, C^2 \exp(-g_n(y)))).$$

As $g(z) > 0$ everywhere on the compact set J, there is an $\epsilon > 0$ such that $g(z) > \epsilon$. Thus $g_n(z) > n\epsilon$. This together with the above fact implies that

$$\nu(A(n)) \ll \nu(B(z_0, C^2 \exp(-n\epsilon))) \sum_{y:T^n(y)=z_0} \exp(-\alpha f_n(y)) \ll \nu(B(z_0, C^2 \exp(-n\epsilon))).$$

Next, note that there is a $\rho > 0$ such that T is injective on any ball B centred on J of radius $r < \rho$. Therefore,

$$n_0(B(z, r)) \geq \frac{\log(\rho/r)}{\log \|T'\|} \gg -\log(r),$$

where $\|T'\|$ is the supremum norm of T' on the δ-neighbourhood of J. Therefore, by Lemma 2 and the fact that T is expanding, we have

$$\log \nu(B(z, r)) \ll -n_0(B(z, r)) \ll \log(r).$$

We now have

$$\log \nu(A(n)) \ll \log(C^2 \exp(-n\epsilon)) \ll -n.$$

By the ratio test it follows that $\sum \nu(A(n))$ converges. □

Proof of Theorem 3. Choose $\epsilon > 0$. In view of the above proposition, there is a cover $\{B_i\}$ of $E(1)$ such that

$$\sum_i \nu(B_i) < \epsilon.$$

This implies by Lemma 8 that

$$\sum_i m(B_i) \ll \epsilon\, \mathcal{H}^\alpha(E(1)).$$

Now $\{B_i\}$ is a cover of $E(1)$, so we also have that

$$\mathcal{H}^\alpha(E(1)) \leq \sum_i m(B_i).$$

Therefore,

$$\mathcal{H}^\alpha(E(1)) \ll \epsilon \mathcal{H}^\alpha(E(1)).$$

Since $\epsilon > 0$ is arbitrary, this implies that $\mathcal{H}^\alpha(E(1))$ is either zero or infinite. By Lemma 5 we have $\mathcal{H}^\alpha(E(C)) = \mathcal{H}^\alpha(E(1))$ for all $C > 0$ which completes the proof of the theorem. □

A. *Appendix*

We end by establishing the remarks made following the statement of Theorem 3 in the introduction. As always, δ is the Hausdorff dimension of J.

LEMMA A.1.

(i) *If* $f = \log|T'|$, *then*

$$0 < \mathcal{H}^\delta(D_{z_0}(f)) < \infty.$$

(ii) *If* $f > \log|T'|$ *everywhere on* J, *then*

$$\mathcal{H}^\delta(D_{z_0}(f)) = 0.$$

The proof of the lemma makes use of certain well-known facts which we now summarize. For an expanding rational map T, the δ-conformal measure ν supported on J is equivalent to δ-dimensional Hausdorff measure $\mathcal{H}^\delta$ and so $\mathcal{H}^\delta(J)$ is positive and finite. Furthermore, ν has a unique (and hence ergodic) equivalent T-invariant probability measure, μ which is the unique equilibrium state for T and $-\delta\log|T'|$. Thus, for any measurable subset X of J we have that

$$\mu(X) \asymp \mathcal{H}^\delta(X). \tag{A.1}$$

Also

$$\mu(B(x,r)) \asymp r^\delta \tag{A.2}$$

for any ball B with centre x in J and radius $r < r_0$. For further details see [**4, 8**].

Proof. We work with the T-invariant probability measure μ. Recall, that there is a constant $C > 1$ such that

$$E(C^{-1}) \subset D_{z_0}(f) \subset E(C).$$

If $f = \log|T'|$, then $E(C^{-1})$ consists of points x in J whose forward orbit $T^n(x)$ lands in the fixed ball $B(z_0, C^{-1})$ infinitely often. Either by Poincaré recurrence or the ergodic theorem, $\mu(E(C^{-1})) = 1$. Thus $\mu(D_{z_0}(f)) = 1$, which together with (A.1) proves the first part of the lemma.

If $f > \log|T'|$ everywhere on J, then as already mentioned above $g_n(x) > n\epsilon$ everywhere on J. Thus, $E(C) \subset E^*(C) := \limsup_{n\to\infty} E_n^*(C)$ where

$$E_n^*(C) := \{x \in J : T^n(x) \in B(z_0, C\exp(-n\epsilon))\}.$$

By (A.2) and the fact that μ is T–invariant, we have

$$\mu(E_n^*(C)) = \mu(B(z_0, C\exp(-n\epsilon))) \asymp (C\exp(-n\epsilon))^\delta.$$

Hence

$$\sum_{n=1}^{\infty} \mu(E_n^*(C)) \ll \sum_{n=1}^{\infty} \exp(-n\epsilon\delta) < \infty,$$

and so, by the Borel–Cantelli Lemma $\mu(E^*(C)) = 0$. Thus $\mu(E(C)) = 0$, which together with (A.1) completes the proof of the lemma. □

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