Characterizing Multiparticle Entanglement in Symmetric N-Qubit States via Negativity of Covariance Matrices

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We show that higher order intergroup covariances involving even number of qubits are necessarily positive semidefinite for N-qubit separable states, which are completely symmetric under permutations of the qubits. This identification leads to a family of sufficient conditions of inseparability based on the negativity of $2k$th order intergroup covariance matrices ($2k \le N$) of symmetric N-qubit systems. These conditions have a simple structure and detect entanglement in all even partitions of the symmetric multiqubit system. The observables involved are feasible experimental quantities and do not demand full state determination through quantum state tomography.

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An important problem in quantum information theory [1,2] is the formulation of appropriate methods for detecting entanglement and then finding measures that quantify the degree of entanglement in multipartite systems. These two issues are difficult to deal with in their full generality for examining multipartite systems and therefore, a strategy in their understanding is to focus on certain special symmetric states [3]. The choice of the states with specific symmetry is based both on feasible experimental possibilities and on mathematical considerations [4]. In this communication, we examine entanglement properties of even number of qubits in quantum states obeying permutation symmetry. Symmetric multiqubit states form an important class due to their experimental significance [5,6]. Taking advantage of the elegant mathematical structure associated with symmetric states, we propose a set of sufficient but not necessary conditions to detect entanglement via experimentally amenable interparticle covariance matrix. The inseparability conditions obtained here are generalizations of our earlier result [7] for pairwise entanglement in symmetric multiqubit states. It is important to point out that in Ref. [7] these conditions are shown to exhibit a similar structure, involving the qubit cross correlations, like those for the Gaussian states [8] and thus our approach reveals a structural parallelism between the continuous variable states and multiqubits considered here.

Symmetric multiqubit states remain invariant under any permutation of the qubits and are therefore restricted to a $(N+1)$-dimensional subspace of the entire 2^N-dimensional Hilbert space—allowing for a substantial reduction in the state space. This is the maximal multiplicity space of collective angular momentum, $\hat{J}_i = \frac{1}{2} \times \sum_{\mu=1}^{N} \sigma_{\mu i}$; $i = x, y, z$, ($\sigma_{\mu i}$ is the Pauli operator of μth qubit) and is spanned by the eigenstates $\{|\frac{N}{2}, M\rangle; -\frac{N}{2} \le M \le \frac{N}{2}\}$ of $\hat{J}^2$ (with maximum eigenvalue $J = N/2$) and $\hat{J}_z$.

An arbitrary N-qubit system is characterized by the density matrix

$$\rho = \frac{1}{2^N} \sum T_{\alpha_1 \alpha_2 \ldots \alpha_N} (\sigma_{1\alpha_1} \sigma_{2\alpha_2} \ldots \sigma_{N\alpha_N}), \tag{1}$$

where $\sigma_{\mu\alpha} = (I \otimes I \otimes \ldots \otimes \sigma_\alpha \otimes I \otimes \ldots)$—with σ_α appearing in the μth position—denotes the Pauli operator of the μth qubit; $\alpha_1, \alpha_2, \ldots, \alpha_N = 0, x, y, z$ and

$$\sigma_0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

the real coefficients $T_{\alpha_1 \alpha_2 \ldots \alpha_N}$ are the averages

$$\begin{aligned} T_{\alpha_1 \alpha_2 \ldots \alpha_N} &= \mathrm{Tr}[\rho(\sigma_{1\alpha_1} \sigma_{2\alpha_2} \ldots \sigma_{N\alpha_N})] \\ &= \langle \sigma_{1\alpha_1} \sigma_{2\alpha_2} \ldots \sigma_{N\alpha_N} \rangle, \end{aligned} \tag{2}$$

and $T_{00\ldots0} = 1$ gives the normalization condition. Note that the total number of independent parameters in this trace-class density matrix is

$$\sum_r \binom{N}{r} 3^r - 1 = 2^{2N} - 1.$$

For multiqubit states obeying exchange symmetry, the state parameters $T_{\alpha_1 \alpha_2 \ldots \alpha_N}$ are symmetric under interchange of any pair of indices (corresponding to swapping of the qubits). So, the total number of parameters reduces to $(N+1)^2 - 1$. Setting $N - l$ indices equal to 0 and remaining l indices taking values x, y, z, we obtain moments of lth order ($l \le N$):

$$T^{(l)}_{i_1 i_2 \ldots i_l} = \langle \sigma_{1 i_1} \sigma_{2 i_2} \ldots \sigma_{l i_l} \rangle = T_{i_1 i_2 \ldots i_l 00\ldots}, \tag{3}$$

where $i_1, i_2, \ldots, i_l = x, y, z$. It is convenient to introduce collective multi-indices $i \equiv \{i_1 i_2 \ldots i_k\}$, $j \equiv \{j_1 j_2 \ldots j_k\}$, so that the moments $T^{(2k)}$ of even order $2k$ (with $k = 1, 2, \ldots, [N/2]$) may be arranged as $3^k \times 3^k$ real symmetric matrices and moments $T^{(k)}$ of kth order are arranged as

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3^k componental columns:

$$T_{ij}^{(2k)} = T_{i_1 i_2 \ldots i_k; j_1 j_2 \ldots j_k}^{(2k)}, \quad \text{and} \quad T_i^{(k)} = T_{i_1 i_2 \ldots i_k}^{(k)}. \tag{4}$$

Let us consider k qubit operators $\hat{\mathbf{A}}^{(k)}$ and $\hat{\mathbf{B}}^{(k)}$, associated with two different groups a and b:

$$\hat{A}_i^{(k)} = \sigma_{a_1 i_1} \sigma_{a_2 i_2} \ldots \sigma_{a_k i_k}, \qquad \hat{B}_j^{(k)} = \sigma_{b_1 j_1} \sigma_{b_2 j_2} \ldots \sigma_{b_k j_k}. \tag{5}$$

Arranging $\hat{\mathbf{A}}^{(k)}$ and $\hat{\mathbf{B}}^{(k)}$ as a column $\hat{\xi}^{(k)}$ of $2 \cdot 3^k$ operators [correspondingly, $\hat{\xi}^{(k)\dagger} = (\hat{\mathbf{A}}^{(k)\dagger}, \hat{\mathbf{B}}^{(k)\dagger})$, as a row of operators], we define the $2k$th order variance matrix, as in Ref. [7]

$$\mathcal{V}^{(2k)} = \tfrac{1}{2}[\langle \hat{\Delta} \xi^{(k)} \hat{\Delta} \xi^{(k)\dagger} \rangle + \text{H.c.}], \tag{6}$$

where $\hat{\Delta} \xi^{(k)} = \hat{\xi}^{(k)} - \langle \hat{\xi}^{(k)} \rangle$. Note that by construction (6), $\mathcal{V}^{(2k)}$ is $(2 \cdot 3^k \times 2 \cdot 3^k)$-dimensional real symmetric positive semidefinite matrix. The elements of the variance matrix are

$$\mathcal{V}_{ij}^{(2k)} = \tfrac{1}{2}(\langle \{\hat{\xi}_i^{(k)}, \hat{\xi}_j^{(k)}\} \rangle - \{\langle \hat{\xi}_i^{(k)} \rangle, \langle \hat{\xi}_j^{(k)} \rangle\}) \tag{7}$$

(where $\{\hat{\xi}_i^{(k)}, \hat{\xi}_j^{(k)}\} = \hat{\xi}_i^{(k)} \hat{\xi}_j^{(k)} + \hat{\xi}_j^{(k)} \hat{\xi}_i^{(k)}$) and $\mathcal{V}^{(2k)}$ is cast in a $(3^k \times 3^k)$ block form,

$$\mathcal{V}^{(2k)} = \begin{pmatrix} \mathcal{A}^{(2k)} & \mathcal{C}^{(2k)} \\ \mathcal{C}^{(2k)T} & \mathcal{B}^{(2k)} \end{pmatrix}. \tag{8}$$

Clearly, the off-diagonal block $\mathcal{C}^{(2k)}$ corresponds to $2k$th order covariances among the intergroup of multiqubits

$$\mathcal{C}_{ij}^{(2k)} = \langle \hat{A}_i^{(k)} \hat{B}_j^{(k)} \rangle - \langle \hat{A}_i^{(k)} \rangle \langle \hat{B}_j^{(k)} \rangle = T_{ij}^{(2k)} - T_i^{(k)} T_j^{(k)}. \tag{9}$$

In the second line of (9), we have used (3) and (4). The diagonal blocks $\mathcal{A}^{(2k)}$ and $\mathcal{B}^{(2k)}$ are identical for a symmetric intragroup multiqubit system: $\mathcal{A}_{ij}^{(2k)} = \mathcal{B}_{ij}^{(2k)} = \langle \hat{A}_i^{(k)} \hat{A}_j^{(k)} \rangle - \langle \hat{A}_i^{(k)} \rangle \langle \hat{A}_j^{(k)} \rangle$ because the intragroup averages are the same, viz., $\langle \hat{A}_i^{(k)} \hat{A}_j^{(k)} \rangle = \langle \hat{B}_i^{(k)} \hat{B}_j^{(k)} \rangle$ and $\langle \hat{A}_i^{(k)} \rangle = \langle \hat{B}_i^{(k)} \rangle$. Under identical local unitary transformations $U \otimes U \otimes \ldots \otimes U$ on the qubits—which preserve the symmetric space structure—the blocks of the variance matrix change as

$$\begin{aligned} \mathcal{A}^{(2k)} &\to \mathcal{A}^{(2k)'} = \mathcal{R} \mathcal{A}^{(2k)} \mathcal{R}^T, \\ \mathcal{C}^{(2k)} &\to \mathcal{C}^{(2k)'} = \mathcal{R} \mathcal{C}^{(2k)} \mathcal{R}^T, \end{aligned} \tag{10}$$

with $\mathcal{R} = \underbrace{R \otimes R \otimes \ldots \otimes R}_{k\text{times}}$, a $3^k \times 3^k$ real orthogonal matrix, comprised of direct products (containing k factors) of three-dimensional rotations $R \in \text{SO}(3)$—corresponding uniquely to 2×2 unitary matrices $U \in SU(2)$.

We now focus on the question: how would multiqubit entanglement manifest itself under different partitioning of a symmetric system? Our identification here is that the intergroup covariance matrix $\mathcal{C}^{(2k)}$ holds a key to symmetric multiqubit entanglement, coming from various even partitioning of the system. It is worth mentioning at this juncture the important difference between the recent Letter of Korbicz *et al.* [9] from our present work. These authors have proposed necessary and sufficient conditions for entanglement, reflected through two and three qubit partitions of a symmetric multiqubit system. Strikingly, the two-qubit result is shown [7] to be captured by the off-diagonal block of the variance matrix. An important open problem, concerning the inseparability features hidden in all the even qubit reduced systems of a symmetric N-qubit state, is what we are addressing here, by generalizing our approach outlined in Ref. [7].

First of all, we note that positivity of the variance matrix $\mathcal{V}^{(2k)}$ demands that the diagonal blocks $\mathcal{A}^{(2k)}$ be positive semidefinite. However, there are no constraints of positivity on the off-diagonal block $\mathcal{C}^{(2k)}$ as such, though separable symmetric states carry a distinguishing feature:

Theorem.—For every separable symmetric multiqubit state, intergroup covariance matrices $\mathcal{C}^{(2k)}$ of various order $2k \leq N$ are necessarily positive semidefinite.

Proof.—Consider a separable symmetric state of $2k$ qubits, which is decomposable as a convex sum of direct products of k-qubit density matrices $\rho_w^{(k)}$:

$$\rho_{\text{sep}}^{(2k)} = \sum_w p_w \rho_w^{(k)} \otimes \rho_w^{(k)}; \quad \sum_w p_w = 1; \quad 0 \leq p_w \leq 1. \tag{11}$$

In this state the interqubit averages $\langle \hat{A}_i^{(k)} \hat{B}_j^{(k)} \rangle$ are also separable:

$$\begin{aligned} \langle \hat{A}_i^{(k)} \hat{B}_j^{(k)} \rangle_{\text{sep}} &= \sum_w p_w \langle \hat{A}_i^{(k)} \rangle_w \langle \hat{B}_j^{(k)} \rangle_w = \sum_w p_w \langle \hat{A}_i^{(k)} \rangle_w \langle \hat{A}_j^{(k)} \rangle_w \\ &= \sum_w p_w T_i^{(k)}(w) T_j^{(k)}(w), \end{aligned} \tag{12}$$

where we have denoted $\text{Tr}\,(\rho_w^{(k)} \hat{A}_i^{(k)}) = \langle \hat{A}_i^{(k)} \rangle_w$ and used the fact $\langle \hat{A}_i^{(k)} \rangle_w = \langle \hat{B}_i^{(k)} \rangle_w$ and the notation $\langle \hat{A}_i^{(k)} \rangle_w = T_i^{(k)}(w)$. It is also clear that

$$\langle \hat{A}_i^{(k)} \rangle_{\text{sep}} = \langle \hat{B}_i^{(k)} \rangle_{\text{sep}} = \sum_w p_w T_i^{(k)}(w). \tag{13}$$

The real quadratic form $Q^{(2k)} = X^T \mathcal{C}^{(2k)} X = \sum_{i,j} \mathcal{C}_{ij}^{(2k)} X_i X_j$, with an arbitrary real 3^k componental column X, when evaluated in the separable state (11) gives

$$Q_{\text{sep}}^{(2k)} = \sum_w p_w [T_i^{(k)}(w) X_i]^2 - \left\{ \sum_w p_w [T_i^{(k)}(w) X_i] \right\}^2, \tag{14}$$

which is necessarily a positive semidefinite quantity, implying in turn that $\mathcal{C}^{(2k)} \geq 0$; $k = 1, 2, \ldots, [N/2]$, in a separable symmetric multiqubit state (11). This proves our theorem. □

The above theorem leads to sufficient conditions for entanglement associated with even number of qubits in a symmetric state: if the intergroup covariance matrix $\mathcal{C}^{(2k)}$ is negative, then the symmetric multiqubit state exhibits $2k$-qubit entanglement for $k = 1, 2, \ldots$ with $2k \leq N$.

This leads to a hierarchy of inseparability conditions, which test entanglement in even partitioning. For $k = 1$, the condition $C^{(2)} < 0$ has been shown in Ref. [7] to be a direct consequence of Peres-Horodecki partial transpose criterion [10] on two-qubit partitions of a symmetric multiqubit state. Thus, negativity of the covariance matrix $C^{(2)}$ serves as both necessary and sufficient for pairwise entanglement in the symmetric N-qubit system. Any test which confirms the negativity of the real symmetric $3^k \times 3^k$ covariance matrix $\mathcal{C}^{(2k)}$ is sufficient to assert the inseparability of the symmetric multiqubit state. In order to establish the negativity of $\mathcal{C}^{(2k)}$, the Sylvester criterion [11] may be used: negative value assumed by any of the principal minors of a Hermitian matrix implies that the matrix is not positive semidefinite. Thus a series of sufficient conditions for entanglement of $2k$ qubits could be extracted from negative principal minors (of various orders) [12] of the corresponding covariance matrix $\mathcal{C}^{(2k)}$. This brings out inseparability constraints involving a few correlation observables, making our criterion experimentally amenable. It may be noted that a series of inseparability conditions, resulting from negative principal minors of various orders, demonstrate [13] negativity of the (infinite-dimensional) partial transpose of a bipartite continuous variable density matrix, which is the Peres-Horodecki criterion [10] for infinite-dimensional states.

We now test our inseparability conditions $\mathcal{C}^{(2k)} < 0$ by considering some well-known examples of symmetric N-qubit states, such as GHZ and W type states, which have attracted experimental focus [5].

For an even [14] N-qubit GHZ state [15]:

$$|\mathrm{GHZ}_N\rangle = \frac{1}{\sqrt{2}}(|0_N\rangle + |1_N\rangle) = \frac{1}{\sqrt{2}}(|00\ldots 0\rangle + |11\ldots 1\rangle) \tag{15}$$

we find that $C^{(N)}$ has one negative eigenvalue, $\lambda^{(-)} = -2^{[(N/2)-1]}$. The lower order covariances $\mathcal{C}^{(2k)}$, for $k < N/2$, are all positive semidefinite. This is obvious because GHZ state is separable with the disposal of qubits. Thus, our $\mathcal{C}$-matrix criterion is in concordance with the known result that the GHZ state is N-party entangled and is fragile under disposal of particles [15].

From an experimental point of view, it may be noted that the lowest order (see [12]) principal minor, which records negativity of $C^{(N)}$ is the diagonal element, with the index $i = \{xxx\ldots xy\}$:

$$\begin{aligned} C_{ii}^{(N)} &= T_{ii}^{(N)} - (T_i^{(N/2)})^2 \\ &= \begin{cases} -1, & \text{if } N/2 = \text{ even integer,} \\ -2, & \text{if } N/2 = \text{ odd integer.} \end{cases} \end{aligned} \tag{16}$$

More specifically, even-N-qubit entanglement in GHZ states is revealed [16] by the measurement of the N-qubit observable $\langle \sigma_{1x}\sigma_{2x}\ldots\sigma_{(N/2)y}\sigma_{(N/2)+1x}\sigma_{(N/2)+2x}\ldots\sigma_{Ny}\rangle$ (where the qubit indices may be conveniently interchanged).

Next, consider N-qubit W state [15]:

$$|W_N\rangle = \frac{1}{\sqrt{N}}(|100\ldots 0\rangle + |010\ldots 0\rangle + \ldots). \tag{17}$$

Here, the covariance matrices $\mathcal{C}^{(2k)}$, of all orders $k = 1, 2, \ldots, [N/2]$, are negative (with only one negative eigenvalue, $\lambda^{(-)} = -\frac{2k(k-1)}{N^2}$). Therefore, W state of N qubits is confirmed to exhibit $2k$-qubit entanglement for all values of k (with, of course, $2k \le N$). Here again, the $2k$-qubit entanglement is seen explicitly through the measurement of one of the diagonal elements of the covariance matrix $\mathcal{C}_{ii}^{(2k)}$, with $i = \{zzz\ldots z\}$, for which $\mathcal{C}_{ii}^{(2k)} = T_{ii}^{(2k)} = -1$. It is therefore sufficient to check that $\langle\sigma_{1z}\sigma_{2z}\ldots\sigma_{2kz}\rangle$ is negative. Thus the W state has $2k$-qubit entanglement in all the even partitions $2k = 2, 4, 6, \ldots$ of the state. Our results confirm that the W state is robust under disposal of qubits [15].

We now investigate the implications of our inseparability conditions $\mathcal{C}^{(2k)} < 0$ for mixed states: to this end, suppose that experimentally produced W and GHZ states have noiselike admixture of incoherently superposed symmetric states:

$$\rho_{\text{noisy}}^{(N)} = \frac{(1-x)}{N+1}P_N + x|\psi\rangle\langle\psi|, \qquad 0 \le x \le 1, \tag{18}$$

where $P_N = \sum_{M=-N/2}^{N/2} |\frac{N}{2}M\rangle\langle\frac{N}{2}M|$ denotes the projection operator onto the symmetric subspace $\mathrm{Sym}\,(C^2 \otimes C^2 \otimes \ldots \otimes C^2)$ of N qubits [P_N is an identity matrix in the symmetric subspace of qubits and hence $P_N/(N+1)$ corresponds to a maximally disordered separable symmetric state], and $|\psi\rangle$ is either a N-qubit $|\mathrm{GHZ}\rangle_N$ or $|\mathrm{W}\rangle_N$ state. For the least eigenvalue of the covariance matrix $C^{(N)}$ to be negative, the mixing parameter x has to be greater than a certain threshold value. We find the following range of x for which inseparability is indicated via negativity of $C^{(N)}$ for $N = 2$, 4 and 6 qubits:

$$\begin{aligned} \mathrm{GHZ} - \text{noisy state: } 0.25 < x \le 1, &\quad \text{for } N = 2. \\ 0.0625 < x \le 1, &\quad \text{for } N = 4. \\ 0.014 < x \le 1, &\quad \text{for } N = 6. \end{aligned} \tag{19}$$

$$\begin{aligned} W - \text{noisy state: } 0.25 < x \le 1, &\quad \text{for } N = 2. \\ 0.0899 < x \le 1, &\quad \text{for } N = 4. \\ 0.042 < x \le 1, &\quad \text{for } N = 6. \end{aligned} \tag{20}$$

We observe that the x range for N-qubit entanglement is smaller for the noisy W state [see (20)], than that (19) for the noisy GHZ state. But eventually for large N, both the noisy states remain entangled for all values of x. A more general trend (but a restricted domain for x) is found by examining the lowest order principal minor: the noisy GHZ state is N (even) qubit entangled, when $\frac{1}{N^2} < x \le 1$ (verified by demanding that the diagonal element $T_{ii}^{(N)} = \frac{(1-x)}{(N^2-1)} - x < 0$; the index $i = \{xx\ldots y\}$). For mixed noisy state of W, the inseparability range—for N-qubit entanglement—is identified to be $\frac{1}{(N+2)} < x \leq 1$, resulting from the negative diagonal element $T^{(N)}_{ii} = \frac{(1-x)}{(N+1)} - x$, with $i = \{zz\ldots z\}$.

Entanglement in various even partitions of the W-noisy state (18) are examined by using the n qubit reduced W noisy state,

$$\rho^{(N-n)}_{\rm noisy}(W) = \frac{(1-x)}{(N-n+1)} P_{N-n} + x\left[\frac{(N-n)}{N}|\mathrm{W}_{N-n}\rangle\langle \mathrm{W}_{N-n}| + \frac{n}{N}|0_{N-n}\rangle\langle 0_{N-n}|\right]. \quad (21)$$

The covariance matrices of all even partitions of the W noisy state are found to be negative, in a specific inseparability range of the mixing parameter x. For example we find that a W noisy state is two-qubit entangled when $\frac{N^2}{N^2+12} < x \leq 1$. Note that this inseparability range for two-qubit entanglement is much restricted than the one realized for entanglement in the largest even partition of the state [see (20)]. As N increases $x \to 1$, indicating that in the large N limit the two-qubit partition of a noisy W state is separable throughout the range $0 \leq x < 1$. The n-qubit reduced GHZ noisy state is a convex sum of three separable states $P_{N-n}/(N-n+1)$, $|0_{N-n}\rangle\langle 0_{N-n}|$, and $|1_{N-n}\rangle\langle 1_{N-n}|$ and is thus a separable state.

In conclusion, we have here generalized our formalism of the symmetric two-qubit inseparability condition, expressed in terms of interqubit covariance matrix [7], to all even qubit partitions of symmetric N-qubit systems. This takes the form of a hierarchy of inseparability conditions on the intergroup covariance matrices of even order: $\mathcal{C}^{(2k)} < 0$, $k = 1, 2, \ldots$ with $2k \leq N$. Only for $k = 1$ (i.e., for two-qubit partitions) the inseparability condition is both necessary and sufficient, and for all other values of k, these conditions are only sufficient. We have illustrated their use for both pure and mixed states involving GHZ- and W-type states. The symmetric multiqubit system considered here facilitates a richer analysis in terms of SO(3) irreducible tensors [17]. The irreducible tensor approach leads to a family of criteria [17] for entanglement based on covariance matrices involving collective angular momentum variables and is suitable to test inseparability in macroscopic atomic ensembles [6]. Our approach suggests further generalization to d-level symmetric multiparticle systems also.

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[1] M. B. Plenio and S. Virmani, Quantum Inf. Comput. **7**, 1 (2007).

[2] K. Życzkowski and I. Bengstsson, quant-ph/0606228.

[3] K. G. H. Vollbrecht and R. F. Werner, Phys. Rev. A **64**, 062307 (2001).

[4] J. K. Stockton, J. M. Geremia, A. C. Doherty, and H. Mabuchi, Phys. Rev. A **67**, 022112 (2003); H. P. Breuer, Phys. Rev. A **71**, 062330 (2005); J. Phys. A **38**, 9019 (2005); D. Chruściński and A. Kossakowski, Phys. Rev. A **73**, 062314 (2006); **73**, 062315 (2006); quant-ph/0606181.

[5] C. A. Sackett *et al.*, Nature (London) **404**, 256 (2000); C. F. Roos *et al.*, Science **304**, 1478 (2004); D. Leibfried *et al.*, Nature (London) **438**, 639 (2005); H. Häffner *et al.*, Nature (London) **438**, 643 (2005).

[6] A. Sørensen, L.-M. Duan, J. I. Cirac, and P. Zoller, Nature (London) **409**, 63 (2001).

[7] A. R. Usha Devi, M. S. Uma, R. Prabhu, and A. K. Rajagopal, quant-ph/0601035; Phys. Lett. A (to be published).

[8] R. Simon, Phys. Rev. Lett. **84**, 2726 (2000); L.-M. Duan, G. Giedke, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **84**, 2722 (2000).

[9] J. K. Korbicz, J. I. Cirac, and M. Lewenstein, Phys. Rev. Lett. **95**, 120502 (2005); **95**, 259901(E) (2005); J. K. Korbicz *et al.*, Phys. Rev. A **74**, 052319 (2006).

[10] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996); M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).

[11] *Matrix Theory and Applications*, edited by R. C. Johnson (American Mathematical Society, Providence, 1990).

[12] While one can directly check negativity of the finite-dimensional matrix $\mathcal{C}^{(2k)}$ by evaluating negative eigenvalues of the matrix (if they exist), using standard linear algebra packages, experimental verification of a negative eigenvalue would involve larger number of measurements. Easily implementable experimental test of entanglement results from lowest order negative principal minor.

[13] E. Shchukin and W. Vogel, Phys. Rev. Lett. **95**, 230502 (2005); A. Miranowicz and M. Piani, Phys. Rev. Lett. **97**, 058901 (2006).

[14] Though our criterion is not applicable to detect N-qubit entanglement for odd-N, we note that a $(N-1)$-qubit reduced system, with one qubit removed, is verifiable by our methods. A genuine N-party entangled state (with odd-N) is separable with the disposal of a qubit and this could be tested through positive semidefinite covariance matrices $\mathcal{C}^{(2k)}$ for all $2k = N-1, N-3, \ldots, 2$. However, verification of positivity of $\mathcal{C}^{(2k)}$ does not ensure that the state is separable, as this criterion is only sufficient.

[15] W. Dür, Phys. Rev. A **63**, 020303(R) (2001); A. K. Rajagopal and R. W. Rendell, Phys. Rev. A **65**, 032328 (2002).

[16] The diagonal element $C^{(N)}_{ii} = T^{(2k)}_{ii} - (T^{(k)}_i)^2 < 0$ whenever $T^{(2k)}_{ii}$ itself is negative. Negative value assumed by any diagonal element $T^{(2k)}_{ii}$ is therefore sufficient to check negativity of $\mathcal{C}^{(2k)}$. For example, in an even N-qubit GHZ state, $T^{(N)}_{ii} = \langle \sigma_{1x}\sigma_{2x}\ldots\sigma_{(N/2)y}\sigma_{(N/2)+1x} \times \sigma_{(N/2)+2x}\ldots\sigma_{Ny}\rangle = -1$ (where the multi-index $i = \{xx\ldots y\}$), while the $N/2$-qubit correlation observable $\langle\sigma_{1x}\sigma_{2x}\ldots\sigma_{(N/2)y}\rangle$ takes the value 0, 1 depending on whether $N/2 =$ even or odd, respectively. Measurement of N-qubit correlation $T^{(N)}_{ii} = \langle\sigma_{1x}\sigma_{2x}\ldots\sigma_{(N/2)y} \times \sigma_{(N/2)+1x}\sigma_{(N/2)+2x}\ldots\sigma_{Ny}\rangle$ is therefore sufficient to verify multiqubit entanglement.

[17] A. R. Usha Devi, R. Prabhu, and A. K. Rajagopal, quant-ph/0612210.