Generalized No-Broadcasting Theorem

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We prove a generalized version of the no-broadcasting theorem, applicable to essentially *any* nonclassical finite-dimensional probabilistic model satisfying a no-signaling criterion, including ones with "superquantum" correlations. A strengthened version of the quantum no-broadcasting theorem follows, and its proof is significantly simpler than existing proofs of the no-broadcasting theorem.

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The no-cloning theorem [1], as sharpened in [2], states that there is no way of blindly copying a pair of nonorthogonal pure states. More precisely, for any pair of nonorthogonal pure states ρ_i, $i \in \{1, 2\}$, there is no trace-preserving completely positive map $\mathcal{E}$ such that $\forall\ i$, $\mathcal{E}(\rho_i) = \rho_i \otimes \rho_i$. In a classical context, with probability distributions replacing density operators, universal cloning of pure states is possible (though in the infinite-dimensional case, this may be an unphysical idealization [3,4]), but even here, universal cloning producing *independent* copies is impossible if we require mixed states to be cloned also. A set of mixed states that *can* be cloned, in classical or quantum mechanics, must be mutually orthogonal—their density matrices or probability distributions must have nonoverlapping support [5].

Broadcasting is a weaker notion than cloning. A map $\mathcal{E}$ that takes states on $\mathcal{H}$ to states on $\mathcal{H}_A \otimes \mathcal{H}_B$ *broadcasts* a state ρ if $\mathrm{Tr}_A(\mathcal{E}(\rho)) = \mathrm{Tr}_B(\mathcal{E}(\rho)) = \rho$; i.e., we do not require that the final state be a product state. A set of states is *broadcastable* if and only if there is a map $\mathcal{E}$ that broadcasts each state in the set [6]. This generalizes pure-state cloning to mixed states in a way that picks out "classical" sets of states better than requiring independent copies does: the no-broadcasting theorem [5] says not only that universal broadcasting is impossible but also that a set of states is broadcastable if and only if they commute pairwise. For the analogous classical notions, universal broadcasting can be achieved (with minor caveats about the infinite-dimensional case [4]).

Cloning and broadcasting are two of the most elementary information-theoretic tasks, and the impossibility of universal cloning and broadcasting are two of the more significant ways in which quantum theory differs from classical theory. In order to better understand these differences, and how they may underpin the better-than-classical performance of quantum information processing, it is helpful to consider the information-theoretic properties of probabilistic theories that are neither classical nor quantum. This has lately been a very active area of investigation in quantum information theory [8–12]. Some have considered whether quantum theory can be derived from information-theoretic axioms [13,14].

It is thus natural to ask whether the no-broadcasting theorem is a special feature of quantum theory, i.e., something that identifies it uniquely from a space of surrounding alternative theories, or generic, so that the *possibility* of universal broadcasting is special to *classical* theories. We show that the latter is the case. Working in a very general framework that assumes little apart from the convexity and finite dimension of state spaces, and the no-signaling principle for bipartite systems, we show that a set of states is broadcastable if and only if it is contained in a simplex generated by states that are jointly distinguishable via a single-shot measurement. This reduces to the standard no-broadcasting theorem in the quantum case and implies that universal broadcasting is possible only in classical theories. The resulting proof of the quantum no-broadcasting theorem is significantly simpler than the original proof of [5], and more intuitive and self-contained than that based on Lindblad's theorem [15] (although the latter provided useful ideas).

An operational framework.—Before proving the main theorem, we describe a mathematical formalism involving only minimal constraints that accommodates a wide range of possible probabilistic theories. Our approach is based on convex sets and has a long pedigree [8,16,17], although the investigation of information processing in this context is more recent (see [18] and in particular [10], where a no-cloning theorem is derived for all nonclassical theories). The framework is broad enough to include theories with "Popescu-Rohrlich" or "nonlocal boxes" [9,11], exhibiting stronger-than-quantum correlations.

The basic approach is operational, meaning that notions such as *system*, *state*, and *measurement* are among its fundamental concepts. We associate with a given type of system a set Ω of possible states. If it is possible to prepare a system in a state ω_1 or ω_2, then it should be possible to prepare any probabilistic mixture of the two (say by tossing a biased coin, preparing one or the other according to the outcome, and then forgetting the outcome) so we assume that Ω is convex.

Whatever else a state is, it should define probabilities for measurement outcomes. We write the probability of getting outcome e when the state is ω as $e(\omega)$. Suppose that $\omega = p\omega_1 + (1-p)\omega_2$ and that this mixed state is prepared by tossing a biased coin, as above. The probability of outcome e should be the weighted average of the probabilities of e given ω_1 and ω_2, i.e., $e(\omega) = pe(\omega_1) + (1-p)e(\omega_2)$. Therefore we identify e with an affine functional from $\Omega \rightarrow [0,1]$. We refer to any such functional as an *effect*. The *unit effect* u is defined by $u(\omega) = 1$ for all $\omega \in \Omega$ and represents a measurement outcome that is certain to occur no matter what the state. The set of all effects is denoted $[0, u]$. A measurement corresponds to a set of effects $\{e_i\}$ such that $\sum_i e_i = u$, that is $\sum_i e_i(\omega) = 1$ for all $\omega \in \Omega$.

In quantum theory, Ω is the set of density operators on some Hilbert space. A particular measurement outcome is associated with a positive operator E bounded by 0 and the identity I, such that the probability of getting this outcome for state ρ is given by $\mathrm{Tr}(E\rho)$. This does define an affine linear functional on the state space since if $\rho = p\rho_1 + (1-p)\rho_2$, then $\mathrm{Tr}(E\rho) = p\mathrm{Tr}(E\rho_1) + (1-p)\mathrm{Tr}(E\rho_2)$. The unit effect corresponds to I, and a measurement as a whole corresponds to a set of positive operators summing to I, i.e., a discrete POVM.

A transformation in quantum theory corresponds to a linear trace-preserving completely positive map taking states on a Hilbert space $\mathcal{H}$ into states on a Hilbert space $\mathcal{H}'$ [19]. As with the rule for measurement outcomes, linearity ensures that transformations respect probabilistic mixtures of states. In the generalized framework, a transformation corresponds to an affine mapping $T: \Omega \rightarrow \Omega'$, where Ω is the state space of the system prior to the transformation, and Ω' is the post-transformation state space. One should not assume that *all* such affine maps correspond to allowed transformations in a particular theory. For example, in quantum theory only completely positive maps (not arbitrary positive maps) do.

The set of all affine functionals from $\Omega \rightarrow \mathbb{R}$ is a vector space denoted $A(\Omega)$. There is a natural embedding of Ω in $A(\Omega)^*$ [the dual space of $A(\Omega)$], given by $\omega \mapsto \hat{\omega}$, where $\hat{\omega}(a) = a(\omega)$ for all $a \in A(\Omega)$. This enables us to identify ω with $\hat{\omega}$, writing either $\omega(a)$ or $a(\omega)$ as convenient. Let $V(\Omega)$ be the linear span of Ω in $A(\Omega)^*$. Then, Ω is finite dimensional iff $V(\Omega)$ is finite dimensional. We assume state spaces are finite dimensional and compact, which guarantees that Ω is the closed convex hull of its extreme points (referred to as *pure* states).

A d-dimensional system is *classical* iff Ω is the convex hull of $d+1$ linearly independent pure states (a simplex), in which case Ω can be thought of as the set of probability distributions over $d+1$ distinct possibilities. Only in such systems can the extremal points be perfectly distinguished from each other by a single measurement, a point discussed in the proof of Theorem 2 below. Classical systems are also characterized by the fact that each state has a unique decomposition into extremal states. A theory is classical iff each system in the theory is classical.

Joint systems.—Suppose systems A and B have state spaces Ω_A and Ω_B. The joint system AB will have its own state space, Ω_{AB}, but how are Ω_A, Ω_B, and Ω_{AB} related? Assume the following: (i) a joint state defines a joint probability for each pair of effects (e_A, e_B), where $e_A \in A(\Omega_A)$ and $e_B \in A(\Omega_B)$; (ii) these joint probabilities respect the no-signaling principle; i.e., the marginal probabilities for the outcomes of a measurement on B do not depend on which measurement was performed on A and vice versa; (iii) if the joint probabilities for all pairs of effects (e_A, e_B) are specified, then the joint state is specified.

These assumptions do not determine Ω_{AB} uniquely in general, but they do imply [10,17,20,21] that it must be a convex set whose span can be identified with the vector space $V(\Omega_A) \otimes V(\Omega_B)$. Further, it must lie between two extremes, the *maximal* and the *minimal tensor product*. The maximal tensor product, $\Omega_A \otimes_{\max} \Omega_B$, is the set of all bilinear functionals $\phi: A(\Omega_A) \times A(\Omega_B) \rightarrow \mathbb{R}$ such that (i) $\phi(e, f) \geq 0$ for all pairs of effects (e, f), and (ii) $\phi(u_A, u_B) = 1$, where u_A and u_B are the unit effects for systems A and B. The maximal tensor product has an important operational characterization: it is the largest set of states in $(A(\Omega_A) \otimes A(\Omega_B))^*$ assigning probabilities to all product measurements but not allowing signaling [17,20]. The minimal tensor product, $\Omega_A \otimes_{\min} \Omega_B$, is the convex hull of the product states, where a product $\omega_A \otimes \omega_B$ is defined by $(\omega_A \otimes \omega_B)(a, b) = \omega_A(a)\omega_B(b)$ for all pairs $(a, b) \in A(\Omega_A) \times A(\Omega_B)$. These appeared in [22] in the context of abstract compact convex sets.

Joint states in $\Omega_A \otimes_{\min} \Omega_B$ are *separable* and those in $\Omega_A \otimes_{\max} \Omega_B$ but not in $\Omega_A \otimes_{\min} \Omega_B$ are *entangled*.

A particular theory should specify, besides Ω_A and Ω_B, a set of joint states Ω_{AB} such that $\Omega_A \otimes_{\min} \Omega_B \subseteq \Omega_{AB} \subseteq \Omega_A \otimes_{\max} \Omega_B$. We call this $\Omega_A \otimes \Omega_B$, keeping in mind that this may be any convex set bounded by the minimal and maximal tensor products. If either A or B is classical, then $\Omega_A \otimes_{\min} \Omega_B = \Omega_A \otimes_{\max} \Omega_B$ and there is no entanglement. In particular, if both are classical, then both $\Omega_A \otimes_{\min} \Omega_B$ and $\Omega_A \otimes_{\max} \Omega_B$ are the simplex whose vertices are ordered pairs of an extremal point of Ω_A and one of Ω_B. For quantum theory $\Omega_A \otimes_{\min} \Omega_B \subset \Omega_{AB} \subset \Omega_A \otimes_{\max} \Omega_B$, where the inclusions are strict [17,20,21,23,24].

This treatment is easy to generalize to multipartite systems, by allowing A and B themselves to be composite. Also, we can define unambiguously the notion of a reduced (or marginal) state. Any state $\omega_{AB} \in \Omega_A \otimes_{\max} \Omega_B$ has reduced states ω_A and ω_B defined such that $\omega_A(a) = \omega_{AB}(a, u_B)$ and $\omega_B(b) = \omega_{AB}(u_A, b)$. It is easy to show that if either reduced state is pure, then $\omega_{AB} = \omega_A \otimes \omega_B$.

Cloning and broadcasting.—We can generalize the definitions of cloning and broadcasting given at the beginning. Consider a state space Ω and a transformation T: $\Omega \rightarrow \Omega \otimes \Omega$. Denote the reduced states of $T(\omega)$ by $(T(\omega))_A$ and $(T(\omega))_B$. We say T *clones* a state ω iff $T(\omega) = \omega \otimes \omega$. A set of states is *cloneable* iff there is a single map T such that T clones each state in the set. We say T *broadcasts* a state ω iff $(T(\omega))_A = (T(\omega))_B = \omega$. And a set of states is *broadcastable* iff there is a single map T such that T broadcasts each state in the set. In addition, we say a set of states $\{\omega_1, \ldots, \omega_n\}$ is *jointly distinguishable* iff they can be distinguished with certainty with a single-shot measurement; i.e., there exists a measurement E with outcomes $e_1, \ldots, e_n$ such that $\omega_i(e_j) = \delta_{ij}$.

Theorem 1.—For any finite-dimensional state space Ω and any choice of tensor product $\Omega \otimes \Omega$, a set of states is cloneable iff it is jointly distinguishable.

A rigorous proof of this theorem is given in [25]. Intuitively, if a set of states is cloneable, then they may be distinguished by making many clones and then identifying the state with suitable measurements on the copies. Conversely, if the states are jointly distinguishable, then one way of cloning is to perform the measurement that distinguishes them and then to prepare two copies.

Theorem 2.—Universal cloning is possible only for classical systems.

Proof.—Universal cloning implies that the set of all pure states is cloneable, so any finite subset $\{\omega_1, \ldots, \omega_n\}$ is cloneable. From Theorem 1, it follows that this set is jointly distinguishable; thus, we can find affine functionals $e_1, \ldots, e_n$ such that $\omega_i(e_j) = \delta_{ij}$. It follows that the ω_i are linearly independent in $V(\Omega)$. Since this holds for any finite subset of pure states, all pure states are linearly independent, and since $V(\Omega)$ is finite dimensional, there can be only a finite number of such states. Thus, Ω is a simplex and the system is classical. □

For any kind of system, broadcasting of pure states reduces to cloning because if ω is pure and $T(\omega)$ has reduced states equal to ω, then $T(\omega) = \omega \otimes \omega$. So Theorem 2 implies that universal broadcasting is possible only for classical systems. Our main theorem goes further: it specifies exactly when a set of states is broadcastable.

Theorem 3.—A set of states is broadcastable iff it lies in a simplex generated by jointly distinguishable states.

The proof requires a definition and a lemma.

Definition.—A *compression* of Ω onto a subset Γ is an idempotent affine mapping $\Omega \rightarrow \Omega$ having range Γ.

Lemma 1.—Let T: $\Omega \rightarrow \Omega$ be any transformation taking Ω into itself. Then there exists a compression of Ω onto the set of fixed points of T.

Proof of Lemma 1.—For each $n \in \mathbb{N}$, let $P_n = \frac{1}{n} \times \sum_{k=1}^{n} T^k$: $\Omega \rightarrow \Omega$. T, and hence P_n, lift uniquely to linear maps $V(\Omega) \rightarrow V(\Omega)$. The convex span of $\Omega \cup -\Omega$ is the unit ball defining the *base norm* on V [26], with respect to which $\| T \| \leq 1$, hence $\| P_n \| \leq 1$. Since Ω and the unit ball are compact, some subsequence (P_{n_j}) converges to an affine map P: $\Omega \rightarrow \Omega$. It is easily shown that $\| P_{n+1} - P_n \| \rightarrow 0$ as $n \rightarrow \infty$. Hence (P_{n_j+1}) also converges to P. For any n, $TP_n = \frac{n+1}{n} P_{n+1} - \frac{1}{n} T$. Hence $TP = \lim_{j\rightarrow\infty} TP_{n_j} = \lim_{j\rightarrow\infty} P_{n_j+1} = P$. So $P(\phi)$ is a fixed point of T for every $\phi \in \Omega$. From the definition of P_n, if $T(\alpha) = \alpha$, then $P_n(\alpha) = \alpha$, whence $P(\alpha) = \alpha$. In particular, since $P(\phi)$ is a fixed point of T, $P(P(\phi)) = P(\phi)$ for all $\phi \in \Omega$; i.e., P is idempotent. □

Proof of Theorem 3.—For the "if" direction, it is easily verified that the map that clones the extreme points of the simplex (cf. the discussion following Theorem 1) broadcasts the entire simplex. For "only if," consider an arbitrary affine mapping B': $\Omega \rightarrow \Omega \otimes \Omega$, and denote by Γ' the set of states broadcast by B'. This is convex, since if B' broadcasts two states, it broadcasts any convex combination of them. We shall show that Γ' is contained in a simplex Γ generated by distinguishable states. Let σ: $\Omega \otimes \Omega \rightarrow \Omega \otimes \Omega$ be the swap operation, defined by $\sigma(\omega_A \otimes \omega_B) = \omega_B \otimes \omega_A$, and define the symmetrized map B: $\Omega \rightarrow \Omega \otimes \Omega$ by $B = (B' + \sigma \circ B')/2$. Denote by Γ the set of states broadcast by B and note that $\Gamma' \subseteq \Gamma$. Denote by B_A the "marginal map" obtained by applying B and then taking the reduced state on Ω_A, so that $B_A(\omega) = B(\omega)_A$, and similarly for B_B. Owing to the symmetry, $B_A = B_B$, and hence $\omega \in \Gamma$ iff ω is a fixed point of B_A. Hence, Lemma 1 provides a compression P onto Γ. (The set Γ' broadcast by B' is the *intersection* of the possibly distinct fixed-point sets of the marginal maps B'_A and B'_B, so Lemma 1 does not provide a compression onto it.) Our strategy will be to use the compression P to define a map Q: $\Gamma \rightarrow \Gamma \otimes_{\max} \Gamma$ such that Q is universally broadcasting on Γ, hence cloning for all states extremal in Γ, which implies (by Theorem 1) that the extremal states of Γ are jointly distinguishable. To apply Theorem 1 requires that Q's domain be Γ, rather than Ω, because the extremal points of Γ are not necessarily extremal in Ω.

Considering P as a map $\Omega \rightarrow \Gamma$, there is a unique map $P \otimes P$: $\Omega \otimes_{\max} \Omega \rightarrow \Gamma \otimes_{\max} \Gamma$ satisfying $(P \otimes P)(\omega_A \otimes \omega_B) = P(\omega_A) \otimes P(\omega_B)$. Define Q: $\Gamma \rightarrow \Gamma \otimes_{\max} \Gamma$ by $Q(\gamma) = (P \otimes P)(B(\gamma))$. We claim that Q is universally broadcasting on Γ. For if $\gamma \in \Gamma$, we have, for all effects e_Γ on Γ,

$$\begin{aligned} Q_A(\gamma)(e_\Gamma) &= Q(\gamma)(e_\Gamma \otimes u_\Gamma) = ((P \otimes P)B(\gamma))(e_\Gamma \otimes u_\Gamma) \\ &= B(\gamma)(P^* e_\Gamma \otimes P^* u_\Gamma) = B_A(\gamma)(P^* e_\Gamma) \\ &= \gamma(P^* e_\Gamma) = P(\gamma)(e_\Gamma) = \gamma(e_\Gamma), \end{aligned}$$

where u_Γ is the unit effect on Γ and $P^* e := e \circ P$ for arbitrary effects e on Ω. Note that the last step uses the fact that $P(\gamma) = \gamma$, since $\gamma \in \Gamma$. It follows that $Q_A(\gamma) = \gamma$; similarly, $Q_B(\gamma) = \gamma$. Since Q is universally broadcasting on Γ, it broadcasts Γ's extremal states. Broadcasting reduces to cloning for extremal states, so Q is universally cloning on the set of extremal points of Γ, as promised. It follows from Theorem 1 that the extreme points of Γ are jointly distinguishable in Γ, i.e., via an observable consisting of effects in $[0, u_\Gamma]$. Using P we can extend effects $e_\Gamma \in [0, u_\Gamma]$, defined only on the span $V(\Gamma)$ of Γ, to functionals $e_\Gamma \circ P$ defined on all of $V(\Omega)$ and contained in $[0, u]$. Thus, any observable on Γ extends, via P, to one on Ω. This includes the one that distinguishes the extreme points of Γ, so Theorem 3 follows. □

Corollary (quantum no-broadcasting theorem).—Let Γ be a set of density operators on a Hilbert space $\mathcal{H}$. If there is a positive map T: $\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ broadcasting each $\rho \in \Gamma$, then the operators in Γ mutually commute.

Proof.—By Theorem 3, Γ is contained in a simplex generated by distinguishable, hence commuting, density operators. Hence the operators in Γ also commute. □

This result is stronger than that in [5], which applied to completely positive maps rather than all positive maps.

Theorem 3 tells us little about the convex structure of the set Γ of states broadcast by a map B. But Ref. [25] builds on it to show that any such Γ *is* a simplex generated by jointly distinguishable states.

Conclusion.—In order to understand the nature of information processing in quantum mechanics, it is useful to demarcate those phenomena that are *essentially* quantum from those that are more generically nonclassical. This Letter has identified an important feature of quantum information that is generic: the no-broadcasting theorem.

In [13] it was shown that the conjunction of no-signaling, no-broadcasting, and no-bit-commitment implies the existence of noncommuting observables and entangled states for theories in a C^*-algebraic framework, yielding theories quite close to quantum theory. However, this framework is already close to quantum theory, since all theories in it have Hilbert space representations and the finite-dimensional ones are just quantum theory, classical probability, and quantum theory with superselection rules. The framework adopted in this Letter is more natural for pursuing the program of deriving quantum theory from information theoretic axioms [13,14], as it is narrow enough to allow axioms to be succinctly expressed mathematically, but broad enough that the main substantive assumptions will be contained in the axioms rather than in the framework itself. The framework *assumes* no-signaling, and we have shown that no-broadcasting holds for any nonclassical model within it. Such models can be very different from quantum theory, e.g., they may support stronger-than-quantum correlations [9]. It thus seems unlikely that these three axioms alone would get one particularly close to quantum theory. Our results suggest that future progress in characterizing quantum theory in terms of information-theoretic tasks is likely to require assumptions of a less generic character.

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