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Some applications of the spectral theory of automorphic forms

Research in Mathematics M.Phil

Thesis

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I, Francois Nicolas Bernard Crucifix, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

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Introduction

The aim of this M.Phil thesis is to present my research throughout the past academic year. My topics of interest ranged over a fairly wide variety of subjects. I started with the study of automorphic forms from an analytic point of view by applying spectral methods to the Laplace operator on Riemann hyperbolic surfaces. I finished with a focus on modular knots and their linking numbers and how the latter are related to the theory of well-known analytic functions. My research took many more directions, and I would rather avoid stretching the extensive list of applications, papers and books that attracted my attention. I decided to present here some of the topics I favored and parts of my research that proved or could prove fruitful. There is a significant disadvantage to proceeding in this fashion. The discussion might look rambling as the topics covered are well diversified.

The structure of the thesis is pretty standard. We start with a brief overview of general theory about automorphic forms and include or reference all the additional results we need in the next sections.

The first of these sections is about *Farey fractions*. After a straightforward introduction and usual considerations about Farey sets, we discuss their distribution. We give an alternative proof that the Farey fractions are equidistributed by means of nice arithmetic identities. Then, we present a paper on correlation measures for Farey fractions [3] and sketch the arguments that lead to two interesting theorems. To conclude the section, we show how one can recover the uniform distribution of Farey fractions using methods borrowed from the theory of automorphic forms. This emphasizes well how powerful and natural this approach can be.

The next section is dedicated to multiplier systems. We explain how they are used to generalize automorphic forms to the non-integral weights case and give two famous examples of such forms. Finally, we construct a series whose analytic properties could yield interesting information about the image structure of a given multiplier system. Our approach is inspired from techniques worked out in [19].

In the final section, we briefly outline Ghys' result that linking numbers of modular knots are given by the *Rademacher function*. We dwell on these two notions beforehand.

Some familiarity with analytic number theory, automorphic forms and spectral methods is assumed. The reader should also feel confortable with elementary number theory. No extra knowledge is required. The listed topics should cover most of our discussions and suffice to go through the present thesis. The interested reader will find appropriate complementary material in the references listed at the end of the thesis.

To conclude this introduction, I would like to thank Yiannis Petridis, for being an amazing supervisor, without whom this thesis would not exist. His assistance, availability, advice, instructive discussions and encouragements were extremely precious throughout my year of research.

C.F. Gauß
Mathematics is the queen of sciences and number theory the queen of mathematics.

1 Brief overview of general theory

1.1 Hyperbolic geometry and Möbius transformations

This overview takes most of its inspiration from [9], [10] and [11].

Throughout the chapters, the complex upper half-plane $\{x+iy \,:\, x, y \in \mathbb{R},\, y > 0\}$ is denoted by $\mathbb{H}$. It is left invariant under the action of $SL_2(\mathbb{R})$, as

$$\Im(\gamma z) = \frac{y}{|cz+d|^2},$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts on $\mathbb{H}$ by

$$\gamma z = \frac{az+b}{cz+d}.$$

Notice that γ and $-\gamma$ have the same action on $\mathbb{H}$. The factor group of transformations

$$PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\{I, -I\}$$

is called the group of *linear fractional transformations* or *Möbius transformations.* These transformations leave invariant another subset of the *Riemann sphere*, namely the real line $\mathbb{R} \cup \{\infty\}$. Möbius transformations have many nice properties, *e.g.* they map circles and lines to circles and lines, they are conformal in the Riemann sphere, and they define isometries for the *hyperbolic metric* derived from the Poincaré differential

$$ds^2(z) = \frac{|dz|^2}{\Im(z)^2} = \frac{dx^2+dy^2}{y^2}, \qquad z \in \mathbb{H}.$$

The corresponding hyperbolic distance between the points z, w in $\mathbb{H}$ is explicitly given by

$$\cosh d(z,w) = 1 + 2u(z,w), \qquad u(z,w) = \frac{|z-w|^2}{4\Im(z)\Im(w)}.$$

The hyperbolic measure is given by $d\mu(z) = dxdy/y^2$. We distinguish between three different kinds of linear fractional transformations. We say that $\gamma \in PSL_2(\mathbb{R})$ is *parabolic* if it has one fixed point in $\mathbb{R} \cup \{\infty\}$, *hyperbolic* if it has two such fixed points and *elliptic* if it has one fixed point in $\mathbb{H}$.

Discrete subgroups of $SL_2(\mathbb{R})$ play a crucial role in the theory of automorphic forms. The norm considered is the one inherited from $\mathbb{R}^4$,*i.e.*

$$\|\gamma\|^2 = a^2 + b^2 + c^2 + d^2.$$

A group $\Gamma \subset SL_2(\mathbb{R})$ is discrete if the sets $\{\gamma \in \Gamma \,:\, \|\gamma\| < \rho\}$ are finite for all $\rho > 0$.

We say that Γ acts discontinuously on $\mathbb{H}$ if the orbits $\Gamma z = \{\gamma z \,:\, \gamma \in \Gamma\}$ have no limit points in $\mathbb{H}$. In particular, the stability group of a point $z \in \mathbb{H}$,

$$\Gamma_z = \{\gamma \in \Gamma \,:\, \gamma z = z\}$$

is finite. It is a famous Poincaré theorem that a subgroup Γ of $SL_2(\mathbb{R})$ is discrete *if and only if* Γ acts discontinuously on $\mathbb{H}$. Discrete subgroups of Möbius transformations are called *Fuchsian* *groups*. There are many characterisations for such groups. We say that a Fuchsian group Γ is of the first kind if every point on the real line $\mathbb{R} \cup \{\infty\}$ is a limit point of an orbit Γz for some $z \in \mathbb{H}$.

A fundamental domain for a group $\Gamma \subset PSL_2(\mathbb{R})$, is a domain F in $\mathbb{H}$ such that distinct points in F are not equivalent under the action of Γ and any orbit of Γ contains at least one point in the closure[1] of F. There are many ways of constructing fundamental domains. For instance, the *normal polygon* $P(w)$ of a Fuchsian group $\Gamma \subset PSL_2(\mathbb{R})$ of the first kind,

$$P(w) = \{z \in \mathbb{H} \,:\, d(z, w) < d(z, \gamma w),\ \ \forall \gamma \in \Gamma,\, \gamma \neq I\}$$

is a fundamental domain for Γ if $w \in \mathbb{H}$ is not fixed by any motion in $\Gamma \setminus \{I\}$. We could dedicate hundreds of pages discussing discrete subgroups and related properties. We assume that the reader is already familiar with these notions. All the details on that matter can be found in [11].

The set of orbits $\Gamma\backslash\mathbb{H}$ is equipped with the topology that makes the quotient map $\mathbb{H} \to \Gamma\backslash\mathbb{H}$ continuous. This yields a connected *Hausdorff space* that can be seen[2] as a *Riemann surface*.

We say that a Fuchsian group is *cofinite* if there is a corresponding fundamental domain in $\mathbb{H}$ with finite hyperbolic area. As $d\mu$ is Γ-invariant, it is easy to show[3] that this area does not depend on the choice of a fundamental domain. We denote this area by $\mathrm{vol}(\Gamma\backslash\mathbb{H})$.

Clearly $\mathrm{vol}(\Gamma\backslash\mathbb{H}) < \infty$ if there exists a compact fundamental domain for Γ. However there are cofinite Fuchsian groups Γ with a non-compact fundamental polygon. This implies that some of the vertices of the polygon lie on $\mathbb{R} \cup \{\infty\}$. We call these points *cuspidal vertices* or *cusps*. If in addition, Γ is of the first kind, we can construct a fundamental polygon all of whose cuspidal vertices are inequivalent under the action of Γ. Cusps are precisely the fixed points of the parabolic motions of Γ.

Cusps will be denoted by bold letters $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots, \infty$. The stability group $\Gamma_{\mathbf{a}}$ is cyclic infinite (easy exercise). It is generated by a parabolic motion $\gamma_{\mathbf{a}}$. The *scaling matrix* $\sigma_{\mathbf{a}} \in SL_2(\mathbb{R})$ is such that

$$\sigma_{\mathbf{a}}\infty = \mathbf{a}, \qquad \text{and} \qquad \sigma_{\mathbf{a}}^{-1}\gamma_{\mathbf{a}}\sigma_{\mathbf{a}} = \pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Conjugating $\Gamma_{\mathbf{a}}$ by the scaling matrix for the cusp $\mathbf{a}$, we get that

$$\sigma_{\mathbf{a}}^{-1}\Gamma_{\mathbf{a}}\sigma_{\mathbf{a}} = B, \qquad \text{where} \qquad B = \Gamma_\infty = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Z} \right\}.$$

Most of the interesting Fuchsian groups are cofinite and of the first kind. We end this section with a focus on *congruence subgroups*. Let n be a positive integer, we define the principal congruence group of level n to be

$$\Gamma(n) = \left\{ \gamma \in SL_2(\mathbb{Z}) \,:\, \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod n \right\}.$$

[1] With respect to the euclidean topology on the Riemann sphere $\mathbb{C} \cup \{\infty\}$.
[2] This means that one can consider analytic charts for which $\Gamma\backslash\mathbb{H}$ becomes a Riemann surface.
[3] This is theorem 3.1.1 in [11].

Any subgroup of the modular group $\Gamma(1) = SL_2(\mathbb{Z})$ which contains $\Gamma(n)$ for some n is called a congruence subgroup of level n. The two other classical examples of congruence subgroups are

$$\Gamma_0(n) = \{\gamma \in SL_2(\mathbb{Z}) \,:\, c \equiv 0 \mod n\} \quad \text{and} \quad \Gamma_1(n) = \{\gamma \in \Gamma_0(n) \,:\, a \equiv d \equiv 1 \mod n\},$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. All these congruence subgroups are cofinite and of the first kind. They have finitely many cusps that can be explicitly computed by means of elementary number theory. For example, the number of inequivalent cusps of $\Gamma_0(n)$ is given by

$$h = \sum_{ab=n} \varphi[(a,b)],$$

where φ is the *Euler's totient function* and (a,b) is the largest common divisor of a and b.

1.2 Laplace operator and automorphic forms

In the present section, we follow closely the discussion in [9], [10] and [23]. The *Laplace operator* derived from the Poincaré differential is given by

$$\Delta = y^2(\partial_x^2 + \partial_y^2) = -(z - \bar{z})^2 \partial_z \partial_{\bar{z}},$$

where $\partial_z = (\partial_x - i\partial_y)/2$ and $\partial_{\bar{z}} = (\partial_x + i\partial_y)/2$. A straightforward computational exercise shows that Δ is $SL_2(\mathbb{R})$-invariant, that is

$$\Delta[f(\gamma z)] = (\Delta f)(\gamma z),$$

for all $\gamma \in SL_2(\mathbb{R})$ and all $f \in \mathcal{C}^2(\mathbb{H}, \mathbb{C})$. The determination of the spectrum and eigenfunctions of Δ is of prime importance for the sequel. We quote an important result (proposition 1.5 in [9]) which is easily obtained from the Fourier expansion theorem for periodic functions.

Proposition. *Set* $\beta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, *and consider* f, *an eigenfunction of* Δ *with eigenvalue* $\lambda = s(1-s)$. *Assume that* f *is such that* $f(\beta z) = f(z)$, *for all* $z \in \mathbb{H}$ *and that*[4]

$$f(z) = o[e(-iy)],$$

as y *tends to infinity. Then* f *has the following Fourier expansion,*

$$f(z) = \widehat{f}_0(y) + \sum_{n \in \mathbb{Z}\setminus\{0\}} \widehat{f}(n) W_s(nz),$$

where $\widehat{f}_0(y)$ *is a linear combination of the functions* y^s *and* y^{1-s} *if* $s \neq 1/2$ *and of* $\sqrt{y}$ *and* $\sqrt{y}\log(y)$ *if* $s = 1/2$. *Here the function* W_s *is derived from the Bessel function* $K_{s-1/2}$,

$$W_s(z) = 2\sqrt{y} K_{s-1/2}(2\pi y) e(x).$$

[4] We write $e(x) = \exp(2\pi i x)$.

For the rest of the section, Γ will be a cofinite discrete subgroup of $SL_2(\mathbb{R})$ of the first kind. A function $f : \mathbb{H} \to \mathbb{C}$ is said to be Γ-*automorphic* if it lives on $\Gamma\backslash\mathbb{H}$, *i.e.*

$$f(\gamma z) = f(z), \qquad \text{for all } \gamma \in \Gamma.$$

The set of all such functions is denoted by $\mathcal{A}(\Gamma\backslash\mathbb{H})$, as it is in [9]. An automorphic function which is also an eigenfunction of the Laplace operator Δ is called an *automorphic form.* The space of automorphic forms with eigenvalue $\lambda = s(1-s)$ is denoted by $\mathcal{A}_s(\Gamma\backslash\mathbb{H})$.

Assume now that $\mathbf{a}$ is a cusp for Γ. Then, if $f \in \mathcal{A}(\Gamma\backslash\mathbb{H})$, we have

$$f(\sigma_{\mathbf{a}}\beta z) = f(\gamma_{\mathbf{a}}\sigma_{\mathbf{a}} z) = f(\sigma_{\mathbf{a}} z)$$

for all $z \in \mathbb{H}$, so that $f(\sigma_{\mathbf{a}} z)$ has the following Fourier expansion,

$$f(\sigma_{\mathbf{a}} z) = \sum_{n=-\infty}^{\infty} f_{\mathbf{a}n}(y) e(nx),$$

where the n-th Fourier coefficient is given by

$$f_{\mathbf{a}n}(y) = \int_0^1 f(\sigma_{\mathbf{a}} z) e(-nx) dx, \qquad n \in \mathbb{Z}.$$

Automorphic forms whose zero-th Fourier coefficients $f_{\mathbf{a}0}$ vanish identically at all cusps $\mathbf{a}$ of Γ are called cusp forms. The space of such functions is denoted by $\mathcal{C}(\Gamma\backslash\mathbb{H})$, and we set

$$\mathcal{C}_s(\Gamma\backslash\mathbb{H}) = \mathcal{C}(\Gamma\backslash\mathbb{H}) \cap \mathcal{A}_s(\Gamma\backslash\mathbb{H}).$$

By the proposition, we know that any cusp form $f \in \mathcal{C}_s(\Gamma\backslash\mathbb{H})$ expands as

$$f(\sigma_{\mathbf{a}} z) = \sum_{n \in \mathbb{Z}\backslash\{0\}} \widehat{f_{\mathbf{a}}}(n) W_s(nz)$$

at every cusp $\mathbf{a}$ for Γ.

1.3 The spectral theorem

Let Γ be a Fuchsian group of the first kind and consider the space of smooth and bounded functions in $\mathcal{A}(\Gamma\backslash\mathbb{H})$ whose Laplacian is also bounded. We denote this space by $\mathcal{B}(\Gamma\backslash\mathbb{H})$. It is easy to show that $\mathcal{B}(\Gamma\backslash\mathbb{H})$ is dense in $L^2(\Gamma\backslash\mathbb{H})$. The next theorem is lemma 4.1 in [9].

Theorem. *The Laplacian Δ is symmetric and non-negative on $\mathcal{B}(\Gamma\backslash\mathbb{H})$ with respect to the inner product*

$$\langle f, g \rangle = \int_{\Gamma\backslash\mathbb{H}} f\overline{g} d\mu.$$

Therefore, by the Friedrichs extension criterion, it has a unique self-adjoint extension to some subspace $H(\Gamma\backslash\mathbb{H})$ of $L^2(\Gamma\backslash\mathbb{H})$, that we write again Δ.

Proof. Using Stokes' theorem, we get that, for any two functions f and g in $\mathcal{B}(\Gamma\backslash\mathbb{H})$,

$$\int_P \overline{g} \Delta f d\mu(z) = -\int_P \nabla f \cdot \overline{\nabla g}\; dxdy + \int_{\partial P} \overline{g} \partial_n f dl,$$

where P is a normal polygon with respect to Γ. In particular, P is a bounded domain and ∂P is a hyperbolic polygon whose sides can be arranged in pairs of Γ-equivalent sides. The differential operator ∇ is the eucidean gradient, $\nabla = [\partial_x, \partial_y]$, ∂_n is the outer normal euclidean derivative and dl is the euclidean length element. Note that

$$\partial_n dl = \partial_{\mathbf{n}} d\mathbf{l},$$

where $\partial_{\mathbf{n}} = y\partial_n$ and $d\mathbf{l} = y^{-1}dl$ are the associated hyperbolic operators. One can check that they are Γ-invariant. This implies that

$$\int_{\partial P} \overline{g}\partial_n f dl = \int_{\partial P} \overline{g}\partial_{\mathbf{n}} f d\mathbf{l} = 0$$

as f and g live on $\Gamma\backslash\mathbb{H}$ so that integrals along equivalent sides cancel out. Hence,

$$\langle \Delta f, g\rangle = -\int_P \nabla f \cdot \overline{\nabla g}\; dxdy = \langle f, \Delta g\rangle\,,$$

and

$$\langle \Delta f, f\rangle = -\int_P |\nabla f|^2 dxdy \leq 0,$$

as desired. □

To understand how important is the spectral resolution of Δ on $L^2(\Gamma\backslash\mathbb{H})$, we introduce the *hyperbolic lattice counting problem* following the discussion in [17]. It consists in giving asymptotics for the function counting points in a given orbit within a certain distance from a fixed point. More precisely, suppose that $\Gamma \subset PSL_2(\mathbb{R})$ is a Fuchsian group of the first kind and that z and w are fixed points in the upper half-plane $\mathbb{H}$. We are looking for estimations of the function p, defined for $x \geq 2$ by,

$$p(x) = \sum_{\substack{\gamma\in\Gamma \\ 4u(\gamma z, w)\leq x-2}} 1.$$

For Γ the modular group, one can show with elementary methods that $p(x) \sim 6x$ asymptotically. Actually, we have

$$p(x) = 6x + \mathcal{O}(x^{2/3}), \qquad \text{as } x \text{ tends to } \infty.$$

The easiest known way to generalize this result to more complicated groups, such as congruence subgroups, is to use *spectral methods.* This is illustrated in the following theorem (theorem 12.1 in [9]).

Theorem. *As x tends to infinity, we have,*

$$p(x) = \sum_{s_j\in(1/2,1]} 2\pi^{1/2}\frac{\Gamma(s_j - 1/2)}{\Gamma(s_j+1)} u_j(z)\overline{u_j}(w)x^{s_j} + \mathcal{O}(x^{2/3}),$$

where j labels the spectral[5] *parameters s_j of Δ on $L^2(\Gamma\backslash\mathbb{H})$ that lie in $(1/2, 1]$ and $u_j \in \mathcal{A}_{s_j}(\Gamma\backslash\mathbb{H})$ are the corresponding square integrable eigenfunctions.*

[5] The eigenvalue is $\lambda_j = s_j(1 - s_j)$. Note that eigenvalues with double or higher multiplicity are repeated.

Hence, the determination of the *small* eigenvalues of Δ provides asymptotics for hyperbolic lattice counting functions!

Before giving an outline of the proof, we describe the structure of the spectrum σ_Γ of a Fuchsian group Γ.

Definition. Let $\lambda \in \mathbb{C}$. The corresponding *resolvent* associated to Δ on $H(\Gamma\backslash\mathbb{H})$ is the operator $R_\lambda = (\Delta + \lambda)^{-1}$. The point λ is *regular* if R_λ is a bounded operator defined on the whole space $H(\Gamma\backslash\mathbb{H})$. The spectrum of Γ is

$$\sigma_\Gamma = \{\lambda \in \mathbb{C} \,:\, \lambda \text{ not regular}\}.$$

The spectrum decomposes as $\sigma_\Gamma = \sigma_\Gamma^d \cup \sigma_\Gamma^c$.

- The *discrete* spectrum σ_Γ^d is the collection of points λ for which R_λ is defined on a set not dense in $H(\Gamma\backslash\mathbb{H})$;
- The *continuous* spectrum σ_Γ^c is the collection of points λ for which R_λ is unbounded.

Notice that σ_Γ^d and σ_Γ^c are not necessarily disjoint!

The discrete spectrum σ_Γ^d can be written as $\{0 < \lambda_1 \leq \lambda_2 \leq \cdots\}$ and is countable (not necessarily finite). On the other hand, if Γ has $m \geq 1$ cusps, $\sigma_\Gamma^c = [1/4, \infty)$ and each of these spectral points appear with multiplicity m. See [15] p.206 for a proof and a more detailed discussion.

The complete set of eigenfunctions associated to $\lambda = s(1-s) \in \sigma_\Gamma^c$, *i.e.*, $\Re(s) = 1/2$ or equivalently $\lambda \geq 1/4$, is given by analytic continuations of the *Eisenstein series* $E_{\mathbf{a}}(z,s)$.

Definition. Let $\mathbf{a}$ be a cusp for Γ. The corresponding *Eisenstein series* is defined for $\Re(s) > 1$ by

$$E_{\mathbf{a}}(z,s) = \sum_{\gamma \in \Gamma_{\mathbf{a}}\backslash\Gamma} \left[\Im\left(\sigma_{\mathbf{a}}^{-1}\gamma z\right)\right]^s.$$

It is in $\mathcal{A}_s(\Gamma\backslash\mathbb{H})$ as a function of z, but not in $L^2(\Gamma\backslash\mathbb{H})$.

It is well-known (chapter 6 in [9]) that these series extend meromorphically to $\mathbb{C}$ and that its poles are contained in the union of the half-plane $\Re(s) < 1/2$ and the interval $(1/2, 1]$.

We are now ready to sketch the proof of the previous theorem.

Outline of proof. The idea is to look at the spectral expansion of a particular *automorphic kernel.* A *point-pair invariant* kernel k is defined on $\mathbb{H}^2$ so that

$$k(z,w) = k[u(z,w)],$$

i.e. k depends only on the hyperbolic distance between its two arguments. Such a kernel yields an automorphic kernel K given by the series

$$K(z,w) = \sum_{\gamma \in \Gamma} k(z, \gamma w).$$

If k decreases *rapidly enough*, this series converges absolutely. The control over the growth of k is best expressed in terms of regularity assumptions on the *Selberg/Harish-Chandra transform* h of k defined by

$$\begin{aligned} q(v) &= \int_v^\infty k(u)(u-v)^{-1/2}du \\ g(r) &= 2q\left[\left(\sinh\frac{r}{2}\right)^2\right] \\ h(t) &= \int_{-\infty}^\infty e^{irt}g(r)dr. \end{aligned}$$

From now on, we will say that k is *regular*, if h is even, holomorphic in the strip $|\Im t| < 1/2+\epsilon$, for some $\epsilon > 0$ and

$$h(t) \ll \frac{1}{(|t|+1)^{2+\epsilon}} \qquad \text{in the strip.}$$

En passant, a regular kernel k defines an *invariant integral operator* L given by[6]

$$Lf(z) = \int_{\mathbb{H}} k(z,w)f(w)d\mu(w).$$

Generalising the spectral theorem provided at the end of the present section, one can get a spectral expansion for K. If k is regular, we have that[7]

$$K(z,w) = \sum_{1/2<s_j\le 1} h(t_j)u_j(z)\overline{u_j}(w) + \sum_{\mathbf{a}} \frac{1}{4\pi}\int_{-\infty}^\infty h(r)E_{\mathbf{a}}(z,1/2+ir)\overline{E_{\mathbf{a}}}(w,1/2+ir)dr,$$

where $s_j = 1/2+it_j$, $t_j \in \mathbb{C}$. The sum converges absolutely and uniformly on compacta.

Using the *Bessel* and *Cauchy-Schwarz* inequalities as well as estimations on the growth of Eisenstein series, one can show (see chapter 7 in [9] for details) that for any regular kernel k, we have

$$K(z,w) = \sum_{1/2<s_j\le 1} h(t_j)u_j(z)\overline{u_j}(w) + \mathcal{O}\left[\int_0^\infty (t+1)H(t)dt\right],$$

where H is any decreasing majorant of $|h|$ and the implied constant depends on Γ, z and w.

Now, the most natural approach would be to consider the kernel k for which $k(u) = 1$ if $u \le (x-2)/4$ and $k = 0$ elsewhere. However, this does not yield satisfactory results. As in [9], we rather take $k(u) = 1$ if $0 \le u \le (x-2)/4$, $k(u) = 0$ if $u \ge (x+y-2)/4$ and k linear continuous on $[(x-2)/4, (x+y-2)/4]$. Here $y = y(x)$ is to be chosen later to in order optimize the error term. For now, we only require that $x \ge 2y \ge 2$. Clearly, $p(x) \le 2K(z,w)$ and if $1/2 < s_j \le 1$,

$$h(t_j) = \pi^{1/2}\frac{\Gamma(s_j-1/2)}{\Gamma(s_j+1)}x^{s_j} + \mathcal{O}(y+x^{1/2}).$$

Also, if $t \ge 0$, $\Re(s) = 1/2$ and

$$h(t) \ll |s|^{-5/2}\left[\min(|s|, xy^{-1}) + \log x\right]x^{1/2},$$

[6] We omit here the discussion on the regularity assumptions on the admissible test functions f.

[7] This is theorem 7.4 in [9]

where the implied constant is absolute. Therefore, we get the following upper bound for p,

$$p(x) \leq 2K(z,w) = \sum_{s_j \in (1/2,1]} 2\pi^{1/2} \frac{\Gamma(s_j - 1/2)}{\Gamma(s_j + 1)} u_j(z)\overline{u_j}(w)x^{s_j} + \mathcal{O}(y + xy^{-1/2}).$$

The optimal choice for y is when $y = xy^{-1/2}$, that is for $y = x^{2/3}$. A similar lower bound is obtained by applying the above result with x replaced by $x - y$. Putting these two bounds together, we get the desired estimation on p.

As Δ is non-negative, all its eigenvalues are non-negative. Its first non-zero eigenvalue will be denoted by $\lambda_1(\Gamma\backslash\mathbb{H})$. The *min-max principle* applies here. We have

$$\lambda_1(\Gamma\backslash\mathbb{H}) = \inf_{f \in \mathcal{E}(\Gamma\backslash\mathbb{H})} \frac{\int_{\Gamma\backslash\mathbb{H}} |\nabla f|^2 d\mu}{\int_{\Gamma\backslash\mathbb{H}} |f|^2 d\mu},$$

where $\mathcal{E}(\Gamma\backslash\mathbb{H})$ is the space of all smooth functions f compactly supported in $\Gamma\backslash\mathbb{H}$ and orthogonal to the space of constant functions, *i.e.* such that $\int_{\Gamma\backslash\mathbb{H}} f d\mu = 0$. It can be very hard to compute the first eigenvalue for particular groups. It is less complicated to give non-trivial lower bounds. For example, we can show that $\lambda_1(\Gamma\backslash\mathbb{H}) \geq 1/4$ if the corresponding eigenfunction vanishes on the boundary of a normal polygon P.

Proposition. *Suppose that $f \in \mathcal{A}_s(\Gamma\backslash\mathbb{H})$, for some parameter s, is a solution to the Dirichlet problem*

$$\begin{cases} \Delta f = -\lambda f = s(s-1)f & \text{in } P \\ f = 0 & \text{on } \partial P, \end{cases}$$

where P is a normal polygon for Γ. Then, $\lambda \geq 1/4$.

Proof. Clearly

$$-\langle \Delta f, f \rangle = \int_P |\nabla f|^2 \geq \int_P |\partial_y f|^2 dxdy,$$

and using partial integration, we have that, for all x,

$$\int f^2 y^{-2} dy = 2 \int f(\partial_y f) y^{-1} dy$$

so that after integrating over x, we get by *Cauchy-Schwarz* inequality that

$$\int_P f^2 d\mu \leq 4 \int_P (\partial_y f)^2 dxdy.$$

Putting these two inequalities together yields

$$4\lambda \langle f, f \rangle = -4 \langle \Delta f, f \rangle \geq \langle f, f \rangle,$$

as desired. □

The distribution of eigenvalues has been extensively studied. Though, the following conjecture has been left unproved for over forty years.

Selberg's eigenvalue conjecture. *We have*

$$\lambda_1(\Gamma\backslash\mathbb{H}) \geq 1/4$$

whenever Γ is a congruence subgroup.

This is the best estimate possible on λ_1 for congruence subgroups. Indeed, the *continuous spectrum* of $\Gamma(n)$ begins at 1/4.If true, this sharp inequality would have many applications to classical number theory. This conjecture is equivalent to the Riemann hypothesis for the *Selberg zeta-function* (defined in section 4.5), $\mathcal{Z}_\Gamma$, when Γ is a congruence subgroup. Indeed, one can show that the non-trivial zeros of $\mathcal{Z}_\Gamma$ are precisely the points $1/2 \pm i\sqrt{\lambda - 1/4}$, where λ ranges over the discrete spectrum of Γ. If true, the conjecture would therefore yield the best error term in the prime geodesic theorem (see section 4.5).

The conjecture can be interpreted in many other ways; it is a consequence of the more general Ramanujan conjecture and is closely related to properties of expanding graphs.

Currently, the best known lower bound is $975/4096 \simeq 0.238$. It was obtained in 2002 by Kim and Sarnak (see appendix 2 in [12]), where bounds on parameters of automorphic cusp forms on $GL_2/\mathbb{Q}$ are explicitly related to lower bounds on $\lambda_1(\Gamma\backslash\mathbb{H})$.

We can infer from the theory of integral operators, the following discrete resolution of Δ in $\mathcal{C}(\Gamma\backslash\mathbb{H})$. See theorem 4.7 in [9] for example.

Spectral theorem. *The automorphic Laplacian Δ has pure point spectrum in $\mathcal{C}(\Gamma\backslash\mathbb{H})$,* i.e. *$\mathcal{C}(\Gamma\backslash\mathbb{H})$ is spanned by cusp forms. The eigenspaces are finite dimensional. For any complete system of cusp forms $\{u_j\}_j$, every $f \in \mathcal{C}(\Gamma\backslash\mathbb{H})$ expands as*

$$f(z) = \sum_j \langle f, u_j \rangle \, u_j(z),$$

converging in the norm topology. If, in addition, $f \in \mathcal{B}(\Gamma\backslash\mathbb{H})$, then the series converges absolutely and locally uniformly.

The eigenspaces corresponding to points in the continuous spectrum are fairly easy to determine. As we have seen, the eigenfunctions are just Eisenstein series. However, a complete system of cusp forms $\{u_j\}_j$ can be very hard to construct.

2 Farey sequence

2.1 Farey sets and growth of size

Definition. The Farey sequence $(\mathcal{F}_Q)_{Q\geq 1}$ is an increasing sequence of finite sets of rational numbers lying in $(0,1]$ defined for all $Q \geq 1$ by

$$\mathcal{F}_Q = \{p/q \,:\, 1 \leq p \leq q,\, q \leq Q,\, (p,q) = 1\},$$

where (p,q) stands for the greatest common divisor of p and q.

Hence, the greatest element of each set $\mathcal{F}_Q$ is the integer 1. The first seven Farey sets are given by

$$\begin{array}{lcll}
\mathcal{F}_1 & = & \{1\} & |\mathcal{F}_1| = 1, \\
\mathcal{F}_2 & = & \left\{\frac{1}{2}, 1\right\} & |\mathcal{F}_2| = 2, \\
\mathcal{F}_3 & = & \left\{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\right\} & |\mathcal{F}_3| = 4, \\
\mathcal{F}_4 & = & \left\{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, 1\right\} & |\mathcal{F}_4| = 6, \\
\mathcal{F}_5 & = & \left\{\frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, 1\right\} & |\mathcal{F}_5| = 10, \\
\mathcal{F}_6 & = & \left\{\frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, 1\right\} & |\mathcal{F}_6| = 12, \\
\mathcal{F}_7 & = & \left\{\frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, 1\right\} & |\mathcal{F}_7| = 18.
\end{array}$$

For $Q \geq 2$, the set $\mathcal{F}_Q$ is the union of $\mathcal{F}_{Q-1}$ with the set

$$\{p/Q \,:\, 1 \leq p \leq Q,\, (p,Q) = 1\}.$$

The latter set has cardinality $\varphi(Q)$, where φ is the usual *Euler's totient function.* As a consequence, $|\mathcal{F}_Q| = |\mathcal{F}_{Q-1}| + \varphi(Q)$ and since $|\mathcal{F}_1| = 1$, we deduce that[8]

$$|\mathcal{F}_Q| = \sum_{q=1}^{Q} \varphi(q).$$

To understand the distribution of Farey fractions, we need to study the asymptotic behavior of $|\mathcal{F}_Q|$ as Q tends to infinity. We do this this by using a straightforward application of the *Wiener-Ikehara* tauberian theorem. This result has proved very useful to get growth estimations of arithmetic functions[9]. Unfortunately this technique does not provide any upper bound on the corresponding error term. However, as it is a standard and efficient tool in analytic number theory, we will cover the elegant arguments leading to the determination of the asymptotic growth of $|\mathcal{F}_Q|$. The generalized version of the theorem we will use is as follows.

Wiener-Ikehara theorem. *Given a sequence of non negative real numbers a_n, $n \geq 1$, and a real number $b > 0$, consider the corresponding Dirichlet series f defined as*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

[8]Some mathematicians use a different convention and include 0 to the Farey sets. Using this definition, the cardinality of each set would be *shifted* by 1. We decide to discard the value 0, as it has no *arithmetic inverse.*

[9]For example, it provides the most direct way to prove the *prime number theorem.*

and assume that f converges absolutely for $\Re(s) > b$, has a meromorphic continuation on $\Re(s) \geq b$ with a simple pole at $s = b$ with residue R and is holomorphic elsewhere. Then

$$\sum_{n \leq x} a_n \sim Rx^b/b$$

as x tends to infinity.

In order to apply the previous theorem for our purposes, we need the following identity that can be found in [16] for example.

Proposition 2.1. *The Dirichlet series*

$$f(s) = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s}$$

converges absolutely for $\Re(s) > 2$, defining a holomorphic function in that region. Moreover, for the same values of s, we have

$$f(s) = \frac{\zeta(s-1)}{\zeta(s)},$$

where ζ is the well-know Riemann zeta function.

Proof. Clearly $\varphi(n) \leq n$ and the series f converges absolutely in the region $\Re(s) > 2$ as it is dominated by $\zeta(s-1)$. Fixing now s in the same region, we get that

$$\begin{aligned} \zeta(s)f(s) &= \left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right)\left(\sum_{m=1}^{\infty} \frac{\varphi(m)}{m^s}\right) = \sum_{\ell=1}^{\infty}\left(\sum_{mn=\ell} \varphi(m)\right)\frac{1}{\ell^s} \\ &= \sum_{\ell=1}^{\infty}\left(\sum_{m|\ell} \varphi(m)\right)\frac{1}{\ell^s} = \sum_{\ell=1}^{\infty} \frac{\ell}{\ell^s} = \zeta(s-1), \end{aligned}$$

where we used the well known arithmetic identity $\sum_{m|\ell} \varphi(m) = \ell$. □

Putting the last two results together, we get the desired estimation.

Theorem 2.2. *As Q tends to infinity, we have*

$$|\mathcal{F}_Q| \sim \frac{3Q^2}{\pi^2}.$$

Proof. Since ζ does not vanish[10] on $\Re(s) = 1$, and has a simple pole at $s = 1$ with residue 1 and no other poles on that line, the Dirichlet series f defined in the previous proposition extends meromorphically on $\Re(s) \geq 2$. It has a simple pole at $s = 2$ with residue $1/\zeta(2)$. Moreover, the extended function f is holomorphic on $\Re(s) = 2$, $s \neq 2$. The Wiener-Ikehara theorem implies that

$$|\mathcal{F}_Q| \sim \sum_{q=1}^{Q} \varphi(q) \sim \frac{Q^2}{2\zeta(2)},$$

and the proof follows from the famous equality $\zeta(2) = \pi^2/6$.

□

[10] Accurate determinations of zero-free regions for ζ have been extensively investigated. However, the Riemann conjecture has not been proved yet. The interested reader will find in [24] more information about these matters, as well as short proof of the stated property.

There is an interesting probalistic interpretation of this result. If one defines p_Q to be the probability that two integers, chosen randomly in the set $\{1, 2, \ldots, Q\}$, are coprime, then

$$\lim_{Q\to\infty} p_Q = 6/\pi^2 = 0.607927102\cdots .$$

There are better estimations on the growth of $|\mathcal{F}_Q|$, *i.e.*, involving an explicit error term, but they require a lot more technical machinery. For example, one can prove, using a *Korobov-Vinogradov* exponential sum estimates, that the following sharp improvement holds,

$$\frac{|\mathcal{F}_Q|}{Q^2} = \frac{3}{\pi^2} + \mathcal{O}\left[\frac{(\log Q)^{2/3}(\log\log Q)^{4/3}}{Q}\right],$$

as Q tends to infinity. See [25] for a proof.

2.2 Distribution

Let's now focus on the distribution of these fractions. We want to prove that they appear uniformly at infinity, *i.e.* that, given $0 \le a \le b \le 1$,

$$\frac{|\{r \in \mathcal{F}_Q \,:\, a \le r \le b\}|}{|\mathcal{F}_Q|} \sim (b-a)$$

as Q tends to infinity. A natural and direct approach is to apply *Weyl's criterion* for the uniform distribution of a sequence. The statement is as follows.

Weyl's criterion. *Let $(u_n)_{n\ge1}$ be a sequence of real numbers contained in $[0,1]$. The following three conditions are equivalent,*

- *the sequence is equidistributed, i.e., for all $0 \le a \le b \le 1$, we have*

$$\frac{|\{n \le N \,:\, a \le u_n \le b\}|}{N} \sim (b-a)$$

  as N tends to infinity;

- *for all $m = 1, 2, \ldots$,*

$$\lim_{N\to\infty} \frac{\sum_{n\le N} e(mu_n)}{N} = 0;$$

- *for any continuous function $g : [0,1] \to \mathbb{C}$,*

$$\lim_{N\to\infty} \frac{\sum_{n\le N} g(u_n)}{N} = \int_0^1 g(t)dt.$$

Here, we use the standard notation $e(x) = e^{2\pi i x}$. To prove that Farey fractions are equidistributed, we would like to check that the second condition in Weyl's criterion holds. Therefore we introduce a particular type of exponential sums.

Definition. Let m and q be two positive integers. The corresponding *Ramanujan sum* is denoted by $c_q(m)$ and defined by

$$c_q(m) = \sum_{\substack{1 \le p \le q \\ (p,q)=1}} e\left(\frac{mp}{q}\right).$$

Therefore we need to investigate the asymptotic behavior of the sums

$$\sum_{r \in \mathcal{F}_Q} e(mr) = \sum_{q \le Q} c_q(m).$$

As we have already introduced and used the Wiener-Ikehara theorem, it is tempting to study the analyticity of the generating function for Ramanujan sums given by the Dirichlet series

$$g_m(s) = \sum_{q=1}^{\infty} \frac{c_q(m)}{q^s}.$$

We need the next lemma.

Lemma 2.3. *Let m and n be positive integers. Then*

$$\sum_{q|n} c_q(m) = \begin{cases} n & \text{if } n|m \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For q a positive integer, let $\xi_q = e(1/q)$. The q complex roots of the polynomial $x^q - 1$ are precisely given by $1, \xi_q, \xi_q^2, \dots, \xi_q^{q-1}$. The root ξ_q is called primitive because its integral powers generate all the roots of the polynomial. It is an easy exercise to deduce that the set of all the primitive roots is given by $\{\xi_q^p : (p,q) = 1\}$. Therefore,

$$c_q(m) = \sum_{\substack{1 \le p \le q \\ (p,q)=1}} e\left(\frac{mp}{q}\right) = \sum_{\substack{1 \le p \le q \\ (p,q)=1}} (\xi_q^p)^m$$

is the sum of the m-th powers of the primitive q-th roots of unity. On the other hand, the collection of the integral powers of ξ_n, *i.e.* ξ_n^a, $a = 1, ..., n$, coincides exactly with the collection of the primitive roots of all the divisors of n. As a consequence,

$$\sum_{q|n} c_q(m) = \sum_{a=1}^{n} \xi_n^{ma},$$

since both sums are equal to the sum of the m-th powers of the primitive n-th roots of unity. If n divides m, then $\xi_n^m = 1$ and the right hand side of the preceding equation equals n. If n does not divide m, then summing the geometric series implies that

$$\left(\sum_{q|n} c_q(m)\right)\left[e\left(\frac{m}{n}\right) - 1\right] = e\left(\frac{m}{n}\right)[e(m) - 1] = 0,$$

and the conclusion follows since $e(m/n) \neq 1$. $\square$

There are plenty of other beautiful arithmetic identities involving Ramanujan sums. The interested reader is invited to take a look at [7], chapter XVII. For our purposes, the previous lemma is all we need. We are now ready to factor nicely the Dirichlet series g_m. Details can be found in [24].

Proposition 2.4. *Let m be a positive integer. The series g_m is analytic on $\Re(s) > 1$, and satisfies, on the same region, the identity,*

$$g_m(s) = \frac{\sigma_{1-s}(m)}{\zeta(s)},$$

where[11] *the function $\sigma_{1-s}(m)$ is the sum of the $(1-s)$-th powers of the divisors of m.*

Proof. It is a straightforward rearrangement of the terms arising from the product of the series ζ and g_m. We have

$$\begin{aligned}\zeta(s)g_m(s) &= \left(\sum_{n=1}^{\infty}\frac{1}{n^s}\right)\left(\sum_{q=1}^{\infty}\frac{c_q(m)}{q^s}\right) = \sum_{\ell=1}^{\infty}\left(\sum_{nq=\ell} c_q(m)\right)\frac{1}{\ell^s} \\ &= \sum_{\ell=1}^{\infty}\left(\sum_{q\mid\ell} c_q(m)\right)\frac{1}{\ell^s} = \sum_{\ell\mid m}\frac{\ell}{\ell^s} = \sigma_{1-s}(m),\end{aligned}$$

using previous lemma. □

Theorem 2.5. *The sequence of Farey fractions is equidistributed.*

Proof. Fix $m \geq 1$ an integer. From the last proposition, we see that g_m extends to an holomorphic function on the whole of the region $\Re(s) \geq 1$ because ζ does not vanish on this region. We deduce from the Wiener-Ikehara theorem that

$$\sum_{q\leq Q} c_q(m) = o(Q),$$

as Q tends to infinity, which clearly implies that

$$\frac{\sum_{r\in\mathcal{F}_Q} e(mr)}{|\mathcal{F}_Q|} \sim \frac{\pi^2\sum_{q\leq Q} c_q(m)}{3Q^2} \to 0,$$

as Q tends to infinity. The conclusion follows from Weyl's criterion. □

2.3 Correlations of Farey fractions

From our estimations, we see that actually

$$\frac{\sum_{r\in\mathcal{F}_Q} e(mr)}{|\mathcal{F}_Q|} = o(1/Q),$$

as Q tends to infinity. This sharp bound is better than the one needed in order to verify the second condition in Weyl's criterion. This is an encouraging result and it suggests to investigate further the distribution of Farey fractions. Our goal in the upcoming section is to discuss such features and present recent studies concerning spacing statistics of the Farey fractions. We will do so by giving an outline of the results in Florin Boca and Alexandru Zaharescu's paper about correlation measures [3]. We first recall some notions and explain our notation.

[11] Not to be confused with 'bold' σ!

Notation. Let $I \subset \mathbb{R}$ be a closed interval. Then $I = [a, b]$ for some real numbers $a \leq b$. We denote by $\ell(I) = b - a$ its length and by

$$I + \mathbb{Z} = \bigcup_{n \in \mathbb{Z}} [a + n, b + n]$$

the collection of its $\mathbb{Z}$-translated intervals. More generally, if $\mu \geq 1$ is an integer and $A \subset \mathbb{R}^\mu$, we write

$$A + \mathbb{Z}^\mu = \bigcup_{\bar{n} \in \mathbb{Z}^\mu} (A + \bar{n}),$$

where $A + \bar{n} = \{\bar{a} \in \mathbb{R}^\mu : \bar{a} - \bar{n} \in A\}$.

Definitions. Let $\nu \geq 1$ be an integer and consider $\mathcal{F} = (F_n)_{n \geq 1}$ an increasing sequence of finite subsets $F_n \subset [0, 1]$, $n \geq 1$. The *ν-level correlation measure* of the sequence $\mathcal{F}$ is defined on the set of boxes $B \subset \mathbb{R}^{\nu-1}$ by

$$R^\nu(\mathcal{F}, B) = \lim_{n \to \infty} R^\nu_{F_n}(B),$$

provided that the limit exists, where, for all finite subsets F contained in $[0, 1]$, we set

$$R^\nu_F(B) = \frac{1}{|F|} \left| \left\{ (f_i)_{i=1}^\nu \in F^\nu : f_i \neq f_j \text{ for } i \neq j,\, (f_1 - f_2, \ldots, f_{\nu-1} - f_\nu) \in \frac{B}{|F|} + \mathbb{Z}^{\nu-1} \right\} \right|.$$

The 2-level correlation measure is most usually referred to as the *pair correlation measure.* Suppose that there is a measurable function $g_\nu(\mathcal{F}) : \mathbb{R}^{\nu-1} \to \mathbb{R}\,;\, \bar{x} \mapsto g_\nu(\mathcal{F}, \bar{x})$ such that

$$R^\nu(\mathcal{F}, B) = \int_B g_\nu(\mathcal{F}, \bar{x}) d\bar{x},$$

for all boxes $B \subset \mathbb{R}^{\nu-1}$. Then $g_\nu(\mathcal{F})$ is called the *ν-level correlation function of* $\mathcal{F}$. Also, $g_2(\mathcal{F})$ is rather called *pair correlation function.*

The rest of this section is dedicated to explaining results in [3] about the correlation measures and functions of the Farey sequence $\mathcal{F} = (\mathcal{F}_Q)_{Q \geq 1}$. Although the proofs in the paper are elementary, they are lengthy and technically heavy. Therefore, we only sketch the proof of the main result here and detail its structure. Complete arguments can be found in [3].

Fix a box B in $\mathbb{R}^{\nu-1}$. A natural approach to estimate $R^\nu(\mathcal{F}, B)$ is to approximate $R^\nu_{\mathcal{F}_Q}(B)$ using a *smoothing* argument. Denote by χ_B the charasteristic function of the set B, and consider the function

$$c_Q(B) : \mathbb{R}^{\nu-1} \to \mathbb{Z}_{\geq 0}\,;\, \bar{x} \mapsto c_Q(B, \bar{x}) = \sum_{\bar{r} \in \mathbb{Z}^{\nu-1}} \chi_B\left(|\mathcal{F}_Q|(\bar{x} + \bar{r})\right).$$

Now, as $|\mathcal{F}_Q| \to \infty$ as $Q \to \infty$, and B is bounded, we know that $c_Q(B)$ has image in $\{0, 1\}$ for Q large enough. Actually, it is easy to see that, for these precise values of Q,

$$c_Q(B; \bar{x}) = \begin{cases} 1 & \text{if } \bar{x} \in \frac{B}{|\mathcal{F}_Q|} + \mathbb{Z}^{\nu-1} \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence,

$$R^\nu(\mathcal{F}, B) = \lim_{Q\to\infty} \frac{1}{|\mathcal{F}_Q|} \sum_{\substack{r_1, r_2, \ldots, r_\nu \in \mathcal{F}_Q \\ r_i \neq r_j \text{ for } i\neq j}} c_Q(B, (r_1 - r_2, \ldots, r_{\nu-1} - r_\nu)).$$

As the numbers r_i are all distinct, $(r_1 - r_2, \ldots, r_{\nu-1} - r_\nu)$ has no zero coordinates, and stays away from the origin. As a consequence we restrict our attention to boxes B that do not contain the origin. Hence, χ_B can be uniformly approximated from above and below by smooth functions $H_{B,b,\epsilon}$ and $H_{B,a,\epsilon}$, $\epsilon > 0$ with the following properties,

- $H_{B,b,\epsilon}$ and $H_{B,a,\epsilon}$ are compactly supported in sets of the type $(0, \Lambda)^{\nu-1}, \Lambda > 0$;
- $H_{B,b,\epsilon} \leq \chi_B \leq H_{B,a,\epsilon}$, and

$$\lim_{\epsilon\to 0} \sup_{\bar{x}\in\mathbb{R}^{\nu-1}} [\chi_B(\bar{x}) - H_{B,b,\epsilon}(\bar{x})] = 0 \quad \text{and} \quad \lim_{\epsilon\to 0} \sup_{\bar{x}\in\mathbb{R}^{\nu-1}} [H_{B,a,\epsilon}(\bar{x}) - \chi_B(\bar{x})] = 0.$$

So, if $H_{B,\epsilon} = H_{B,b,\epsilon}$ or $H_{B,a,\epsilon}$, the function

$$f_Q(H_{B,\epsilon}, \bar{x}) = \sum_{\bar{r}\in\mathbb{Z}^{\nu-1}} H_{B,\epsilon}\left(|\mathcal{F}_Q|(\bar{x} + \bar{r})\right)$$

tends uniformly in $\bar{x} \in \mathbb{R}^{\nu-1}$ to $c_Q(B, \bar{x})$ as ϵ tends to zero. Therefore, it turns out that it is enough to study the asymptotic behavior of the quantities

$$S_Q^\nu(H) = \frac{1}{|\mathcal{F}_Q|} \sum_{\substack{r_1, r_2, \ldots, r_\nu \in \mathcal{F}_Q \\ r_i \neq r_j,\, i\neq j}} f_Q(H, (r_1 - r_2, \ldots, r_{\nu-1} - r_\nu)).$$

The whole point in smoothing χ_B is that we have now enough regularity to apply efficient analysis tools. We want to express $S_Q^\nu(B, H)$ in a way that would reveal the distribution of Farey fractions. It is obvious that if we replace H by its Fourier expansion, exponential sums would appear. Rearranging the terms, we can get Ramanujan sums which are easy to handle and exhibit the structure of Farey fractions. Doing this carefully, and using the identity[12] $c_q(n) = \sum_{d|(q,n)} \mu(q/d)d,$ where μ is the usual Mobius function, we get equation (2.11) in [3]. We will reproduce the statement hereunder, as in our opinion, this is the key identity leading to the two main theorems in the paper. It is an easy enough expression to handle and it discloses explicitly the arithmetic nature of the Farey fractions.

Proposition 2.6. *For $Q \geq 1$ an integer, set $\Omega_Q^{\nu-1} = \{1, 2, \ldots, Q\}^{\nu-1}$. Assume that H is a smooth function with support in $(0, \Lambda)^{\nu-1}$, $\Lambda > 0$. Writing $M(x) = \sum_{n\leq x} \mu(n),$ we have that*

$$S_Q^\nu(H) = \frac{1}{|\mathcal{F}_Q|} \sum_{\bar{d}\in\Omega_Q^{\nu-1}} d_1 d_2 \cdots d_{\nu-1} M\left(\frac{Q}{d_1}\right) \cdots M\left(\frac{Q}{d_{\nu-1}}\right) u\left(H, \bar{d}\right),$$

where

$$u\left(H, \bar{d}\right) = \sum_{\bar{\ell}\in\mathbb{Z}^{\nu-1}} a_{(d_1\ell_1, d_1\ell_1 + d_2\ell_2, \ldots, d_1\ell_1 + \cdots + d_{\nu-1}\ell_{\nu-1})} \sum_{d_\nu | d_1\ell_1 + \cdots + d_{\nu-1}\ell_{\nu-1}} d_\nu M\left(\frac{Q}{d_\nu}\right),$$

and

$$a_{\bar{r}} = \frac{1}{|\mathcal{F}_Q|^{\nu-1}} \widehat{H}\left(\frac{\bar{r}}{|\mathcal{F}_Q|}\right), \qquad \widehat{H}(\bar{y}) = \int_{\mathbb{R}^{\nu-1}} H(\bar{x}) e(-\bar{x}\cdot\bar{y}) d\bar{x}.$$

[12]This identity results from lemma 2.3 after a Möbius inversion.

To *clarify* the expression, we introduce a map T to reorder the indices involved in the above summation. The action of T is as follows,

$$T(x_1, x_2, \ldots, x_{\nu-1}) = (x_1 + x_2 + \cdots + x_{\nu-1}, x_2 + \cdots + x_{\nu-1}, \ldots, x_{\nu-2} + x_{\nu-1}, x_{\nu-1}).$$

Inverting this map and using *Poisson summation formula*, we get after a lot of tedious and brute force calculations theorem 1 in [3]. Here is its statement.

Theorem 2.7. *All ν-level correlation measures of the Farey sequence $\mathcal{F} = (\mathcal{F}_Q)_{Q\geq 1}$ exist. Moreover, for any box $B \subset (0, \Lambda)^{\nu-1}$,*

$$R^\nu(\mathcal{F}, B) = 2 \sum_{\substack{\bar{a},\bar{b}\in\mathbb{Z}^{\nu-1} \\ (a_i,b_i)=1,\, 1\leq i\leq \nu-1}} \operatorname{vol}\left[\Omega_{\bar{a},\bar{b},\Lambda} \cap T^{-1}_{\bar{a},\bar{b}}(TB)\right].$$

In the above finite sum,

$$T_{\bar{a},\bar{b}}(x,y) = \frac{3}{\pi^2}\left(\frac{b_1}{y(a_1 y - b_1 x)}, \ldots, \frac{b_{\nu-1}}{y(a_{\nu-1} y - b_{\nu-1} x)}\right),$$

and

$$\Omega_{\bar{a},\bar{b},\Lambda} = \left\{(x,y)\in\mathbb{R}^2 \,:\, 0 < x \leq y \leq 1,\, y \geq \frac{3}{\pi^2\Lambda},\, 0 < a_i y - b_i x \leq 1,\, 1 \leq i \leq \nu - 1\right\}.$$

This result is impressive as it is not often the case that a sequence of interest has a *workable* correlation measure!

We can say even more about the case $\nu = 2$. The reason for this is that the expansion in proposition 1.6. simplifies nicely. We easily compute that

$$\begin{aligned} S_Q^2(H) &= \frac{1}{|\mathcal{F}_Q|}\sum_{d_1\leq Q} d_1 M\left(\frac{Q}{d_1}\right)\sum_{\ell\in\mathbb{Z}} a_{d_1\ell}\sum_{d_2|d_1\ell} d_2 M\left(\frac{Q}{d_2}\right) \\ &= \frac{1}{|\mathcal{F}_Q|}\sum_{d_1,d_2\leq Q} d_1 d_2 M\left(\frac{Q}{d_1}\right) M\left(\frac{Q}{d_2}\right)\sum_{\ell\in\mathbb{Z}} a_{\ell[d_1,d_2]}. \end{aligned}$$

Applying more or less the same technical machinery, one can get theorem 2 in [3].

Theorem 2.8. *The pair correlation function of the Farey sequence $\mathcal{F} = (\mathcal{F}_Q)_{Q\geq 1}$ is given by*

$$g_2(\mathcal{F}, x) = \frac{6}{\pi^2 x^2}\sum_{1\leq k<\pi^2 x/3} \varphi(k)\log\left(\frac{\pi^2 x}{3k}\right).$$

The function is such that

$$g_2(\mathcal{F}, x) = 1 + \mathcal{O}\left(\frac{1}{x}\right),$$

as x tends to infinity.

To conclude this section, we mention equivalent formulations of the Riemann hypothesis in terms of distribution properties of the Farey fractions. Set $r(Q,k)$ to be the k-th element in $\mathcal{F}_Q$, $1 \leq k \leq |\mathcal{F}_Q|$ when the elements are increasingly ordered. Define

$$d(Q,k) = \left| r(Q,k) - \frac{k}{|\mathcal{F}_Q|}\right|.$$

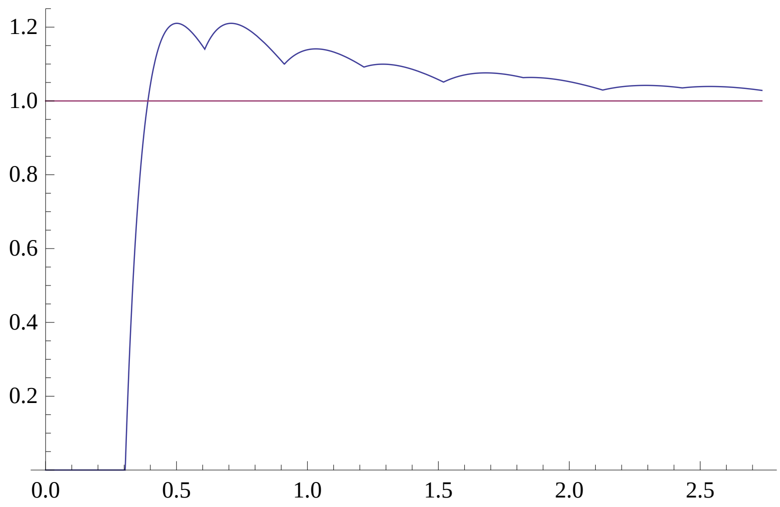

FIG. 1 : GRAPH OF THE PAIR CORRELATION FUNCTION $g_2(\mathcal{F})$ OF THE FAREY SEQUENCE

The *decay*[13] of $d(Q,k)$ measures how *strong* is the uniform distribution of the Farey sequence. Indeed, $d(Q,k)$ is the absolute difference of the k-th term of the Q-th Farey set and the k-th number of the set containing $|\mathcal{F}_Q|$ points evenly distributed in $[0,1]$. The mathematicians Franel and Landau proved that the Riemann conjecture is equivalent to the estimation

$$\sum_{k\leq|\mathcal{F}_Q|} d(Q,k) \ll_\epsilon Q^{1/2+\epsilon},$$

for all $\epsilon > 0$, as Q tends to infinity, which is itself equivalent to

$$\sum_{k\leq|\mathcal{F}_Q|} d(Q,k)^2 \ll_\epsilon \frac{1}{Q^{1-\epsilon}},$$

for all $\epsilon > 0$, as Q tends to infinity.

2.4 Good's result

The aim of this section is to show how one can deduce distribution properties about Farey fractions following a different approach, using results arising from the theory of automorphic forms and spectral methods.

Throughout the section, we will assume that $\Gamma \subset SL_2(\mathbb{R})$ is a cofinite discrete group. The bold letters $\mathbf{a}, \mathbf{b}, \ldots$ will refer to cusps for Γ.

Definition. Let I and J be two closed intervals contained in $[0,1]$. Given $x \geq 0$, we define the integer

$${}_{\mathbf{a}}\#_{\mathbf{b}}(I,J;x)$$

to be the number of double cosets $\Gamma_{\mathbf{a}}\sigma_{\mathbf{a}}\gamma\sigma_{\mathbf{b}}^{-1}\Gamma_{\mathbf{b}}$ in Γ such that

$$\gamma(\infty) \in I + \mathbb{Z}, \qquad -\gamma^{-1}(\infty) \in J + \mathbb{Z} \qquad \text{and} \qquad 0 \neq |c| \leq x,$$

[13] Intentionally vague.

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

As $\gamma(\infty) = a/c$, and $\gamma^{-1}(\infty) = -d/c$ whenever $c \neq 0$, and since $B = \sigma_{\mathbf{a}}^{-1}\Gamma_{\mathbf{a}}\sigma_{\mathbf{a}}$, we see that both

$$\gamma(\infty) \mod 1, \qquad \text{and} \qquad \gamma^{-1}(\infty) \mod 1$$

do not depend on the choice of the double coset representative γ.

Good's result p.119 in [6] transposed to the parabolic case yields the following theorem.

Theorem 2.9. *Suppose $I, J \subset [0,1]$ are given closed intervals. Then, as x tends to infinity, we have*

$$_{\mathbf{a}}\#_{\mathbf{b}}(I, J; x) \sim \frac{\ell(I)\ell(J)x^2}{\pi \operatorname{vol}(\Gamma\backslash\mathbb{H})}.$$

The corollary stated in [6] is actually much more general. But as it stands in [6], the notation is slightly confusing and since we only need to deal with cusps, we decided to simplify the statement.

Sketch of proof. To prove this theorem, Good considers generalized *Kloosterman sums* (equation 5.10 p.43 in [6]) that turn into usual Kloosterman sums in the parabolic case. Following Good's definition, we write for three integers m, n and $\nu \geq 1$,

$$_{\mathbf{a}}S_{\mathbf{b}}(m, n, \nu) = \sum_{\gamma} e\left[m\gamma(\infty) - n\gamma^{-1}(\infty)\right] = \sum_{\gamma} e\left(\frac{am + dn}{c}\right),$$

where γ runs through a complete set of representatives γ for $\Gamma_\infty\backslash\sigma_{\mathbf{a}}^{-1}\Gamma\sigma_{\mathbf{b}}/\Gamma_\infty$ such that $|c| = \nu$. For instance, if $\Gamma = SL_2(\mathbb{Z})$ is the modular group and $\mathbf{a} = \mathbf{b} = \infty$, using theorem 2.10, one can show that

$$_{\infty}S_{\infty}(m, n, \nu) = S(m, n, \nu) = \sum_{\substack{1 \leq a < c = \nu \\ ad \equiv 1 \mod c}} e\left(\frac{am + dn}{c}\right).$$

The key result in order to prove the stated theorem is the following asymptotic spectral decomposition,

$$\sum_{\nu \leq x} {_{\mathbf{a}}S_{\mathbf{b}}}(m, n, \nu) = \frac{\delta_{0m}\delta_{0n}x^2}{\pi \operatorname{vol}(\Gamma\backslash\mathbb{H})} + 2\sum_{1/2 < s_j < 1} x^{2s_j} B(3/2, s_j - 1/2)\overline{\alpha_{j\mathbf{a}}}(m)\alpha_{j\mathbf{b}}(n) + \mathcal{O}(x^{4/3}),$$

as x tends to infinity, where B is the *Beta function*, defined by

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)},$$

and for $j \geq 0$, the complex coefficients $\alpha_{j\mathbf{a}}(n)$, $n \in \mathbb{Z}$ are uniquely determined as follows. The Laplacian in $L^2(\Gamma\backslash\mathbb{H})$ has a non-empty discrete spectrum $\{s_j(1 - s_j) : j \geq 0\}$ and a corresponding maximal system $\{e_j : j \geq 0\}$ of square integrable orthonormal eigenfunctions e_j. The spectral expansion of e_j determines the numbers $\alpha_{j\mathbf{a}}(n)$. We have

$$e_j[\sigma_{\mathbf{a}}(z)] = \sum_{n \in \mathbb{Z}} \alpha_{j\mathbf{a}}(n) U_{\mathbf{a}}(z, s_j, n),$$

where $U_{\mathbf{a}}(z,s,n)$ are known special functions (see equation 4.7 p.28 in [6]).

Theorem 2.9. then follows from Weyl's criterion applied to the given expansion of $\sum_{\nu\le x}{}_{\mathbf{a}}S_{\mathbf{b}}(m,n,\nu)$.

Now, the idea is to look at $\Gamma = SL_2(\mathbb{Z})$ to get some information about Farey fractions. The group $SL_2(\mathbb{Z})$ has only one cusp at infinity, that we denote ∞. As σ_∞ is the identity transformation and as

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Z} \right\} = B,$$

we seek some information about

$$B \backslash SL_2(\mathbb{Z}) / B$$

in order to compute explicitly ${}_\infty\#_\infty$.

This information can be obtained from [9], theorem 2.7. The next theorem is a simplified version of it.

Theorem 2.10. *Assume $\Gamma = SL_2(\mathbb{Z})$. Then*

$$\Gamma_\infty \backslash \Gamma / \Gamma_\infty = \Gamma_\infty \cup \bigcup_{c=1}^{\infty} \bigcup_{d \mod c} \Gamma_\infty \omega_{d,c} \Gamma_\infty,$$

where $\omega_{d,c} = \begin{pmatrix} a_{d,c} & b_{d,c} \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. The union is disjoint and does not depend on the choice of the top two entries $a_{d,c}$ and $b_{d,c}$. Moreover, $a_{d,c} \mod c$ is also independent of that choice.

Definition. If $r \in \mathcal{F}_Q$, then $r = p/q$ with $(p,q) = 1$ and $1 \le q \le Q$. Then there exists a unique $1 \le p^* \le q$ such that $pp^* \equiv 1 \mod q$. We define the *arithmetic inverse* of r to be $r^* = p^*/q \in \mathcal{F}_Q$.

It is easy to check that the function $\mathcal{F}_Q \to \mathcal{F}_Q\,;\, r \mapsto r^*$ is an involution. We are now ready to prove our result.

Theorem 2.11. *Let I and J be two closed intervals contained in $[0,1]$. Then,*

$$|\{r \in \mathcal{F}_Q \,:\, r \in I,\, r^* \in J\}| \sim \frac{3\ell(I)\ell(J)Q^2}{\pi^2},$$

as Q tends to infinity, or equivalently,

$$\frac{|\{r \in \mathcal{F}_Q \,:\, r \in I,\, r^* \in J\}|}{|\mathcal{F}_Q|} \sim \ell(I)\ell(J),$$

as Q tends to infinity.

Proof. The decomposition theorem allows us to compute ${}_\infty\#_\infty$ explicitly. We have

$$\begin{aligned} {}_\infty\#_\infty(I,J;x) &= \left| \left\{ (c, d \mod c) \,:\, c,d \in \mathbb{Z},\, (c,d)=1,\, 1 \le c \le x,\, \frac{a_{d,c}}{c} \in I + \mathbb{Z},\, \frac{d}{c} \in J + \mathbb{Z} \right\} \right| \\ &= \left| \left\{ (c,d) \,:\, c,d \in \mathbb{Z},\, (c,d)=1,\, 1 \le c \le x,\, 1 \le d \le c,\, \frac{a}{c} \in I,\, \frac{d}{c} \in J \right\} \right|, \end{aligned}$$

where a is the unique integer such that $1 \leq a \leq c$ and $a \equiv a_{d,c} \mod c$. This is well defined since both d and $a_{d,c}$ are determined modulo c. On the other hand,

$$ad \equiv 1 \mod c, \qquad \text{so that} \qquad \left(\frac{a}{c}\right)^* = \left(\frac{d}{c}\right).$$

From all this, we deduce that, given $Q \geq 1$ an integer,

$${}_\infty\#_\infty(I, J; Q) = |\{r \in \mathcal{F}_Q \,:\, r \in I,\, r^* \in J\}|\,.$$

The conclusion is an immediate application of Good's result on the surface $SL_2(\mathbb{Z})\backslash\mathbb{H}$ whose volume is $\pi/3$. $\square$

Corollary 2.12. *The sequence of Farey fractions is equidistributed.*

Proof. It is a direct application of previous theorem with $J = [0, 1]$. $\square$

A more general interpretation of the result is that the distributions of Farey fractions and their inverses are uniform and independent.

3 Multiplier systems

3.1 Definitions and properties

We introduce multiplier systems of complex weight following the discussion in [10]. Conceptually, a multiplier system is a factor associated with some modular forms that transform in a certain way under the action of a given group. It can be viewed as a *generalized character,* in the sense that it verifies multiplicative conditions and it involves an extra parameter, the weight[14]. Equivalently, multiplier systems can be defined independently of modular forms. We consider this second approach first.

For $z \neq 0$,

$$\log z = \ln |z| + i \arg z, \qquad -\pi < \arg z \leq \pi,$$

is the *principal branch* of the complex logarithm. It is such that $z = \exp(\log z)$, $z \neq 0$. This branch of logarithm allows us to define complex powers for $z \neq 0$. Throughout the remaining of the section, we write

$$z^s = \exp(s \log z),$$

whenever $s \in \mathbb{C}$ and $z \neq 0$.

For $z \in \mathbb{C}$, and $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, we set $j_\alpha(z) = cz + d$. This function is such that

$$j_{\alpha\beta}(z) = j_\alpha(\beta z) j_\beta(z),$$

for all $\alpha, \beta \in SL_2(\mathbb{R})$, and $z \in \mathbb{H}$.

Definition. For $\alpha, \beta \in SL_2(\mathbb{R})$, define

$$\omega(\alpha, \beta) = \frac{1}{2\pi} \left[-\arg j_{\alpha\beta}(i) + \arg j_\alpha(\beta i) + \arg j_\beta(i) \right].$$

The choice of i is arbitrary. Indeed,

$$\omega(\alpha, \beta)(z) = \frac{1}{2\pi} \left[-\arg j_{\alpha\beta}(z) + \arg j_\alpha(\beta z) + \arg j_\beta(z) \right] = \omega(\alpha, \beta)(i) = \omega(\alpha, \beta)$$

for all $z \in \mathbb{H}$. The reason for this is that $|\omega(\alpha, \beta)(z) - \omega(\alpha, \beta)(i)| < 1$ and $\omega(\alpha, \beta)(z)$ is an integer[15]. Actually, as $|\omega(\alpha, \beta)(z)| < 3/2$, we see that $\omega(\alpha, \beta)$ takes values in $\{-1, 0, 1\}$. By examining numerous cases, we can get useful identities.

Proposition 3.1. *Let $\alpha, \beta, \gamma, \delta \in SL_2(\mathbb{R})$. Suppose that $\delta \in B$,* i.e., $\delta = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$, *for some*

[14] Warning : for non integral weights, a multiplier system ceases to be strictly multiplicative. However, a multiplier system of integral weight is just a unitary character satisfying a consistency condition.

[15] We have

$$e[\omega(\alpha, \beta)(z)] = j_\alpha(\beta z) j_\beta(z) j_{\alpha\beta}(z)^{-1} = 1,$$

so that $\omega(\alpha, \beta) : \mathbb{H} \to \mathbb{Z}$ and since $\omega(\alpha, \beta)$ is continuous, it is also constant.

$d \in \mathbb{R}$. *Then,* $\omega(\alpha, \beta) = \omega(\beta, \alpha)$, *if* $\alpha\beta = \beta\alpha$, *and*

$$\begin{aligned} \omega(\alpha\beta, \gamma) + \omega(\alpha, \beta) &= \omega(\alpha, \beta\gamma) + \omega(\beta, \gamma); \\ \omega(\delta\alpha, \beta) &= \omega(\alpha, \beta\delta) = \omega(\alpha, \beta); \\ \omega(\alpha\delta, \beta) &= \omega(\alpha, \delta\beta); \\ \omega(\alpha, \beta) &= \omega(\alpha^{-1}\delta\alpha, \beta) + \omega(\alpha, \alpha^{-1}\delta\alpha\beta); \\ \omega(\alpha, \delta) &= \omega(\delta, \alpha) = 0; \\ \omega(\alpha\delta\alpha^{-1}, \alpha) &= \omega(\alpha, \alpha^{-1}\delta\alpha) = 0. \end{aligned}$$

There are many more properties involving $\omega(\alpha, \beta)$, but we don't really need them in the sequel.

Definition. For any real number m, w_m, the *factor system of weight* m, is defined by

$$w_m(\alpha, \beta) = e[m\omega(\alpha, \beta)],$$

for all $\alpha, \beta \in SL_2(\mathbb{R})$.

Since $\omega(\alpha, \beta)$ is an integer, $w_m = w_{m+\mathbb{Z}}$, and $w_{\mathbb{Z}} = 1$. For $\alpha \in SL_2(\mathbb{R})$, the corresponding *slash* operator, $|_m\alpha$, acts on functions $f : \mathbb{H} \to \mathbb{C}$ by

$$f_{|_m\alpha}(z) = j_\alpha(z)^{-m} f(\alpha z).$$

It follows directly from the definition of w_m that

$$w_m(\alpha, \beta) j_{\alpha\beta}(z)^m = j_\alpha(\beta z)^m j_\beta(z)^m,$$

which implies that

$$f_{|_m\alpha\beta} = w_m(\alpha, \beta) f_{|_m\alpha|_m\beta}, \qquad \forall \alpha, \beta \in SL_2(\mathbb{R}).$$

Definition. Let $\Gamma \subset SL_2(\mathbb{R})$ be a discrete group containing $-I$ and m a real number[16]. A *multiplier system of weight* m for Γ is a function $\vartheta_m : \Gamma \to S^1$ such that[17] $\vartheta_m(-I) = e(-m/2)$ and

$$\vartheta_m(\alpha\beta) = w_m(\alpha, \beta)\vartheta_m(\alpha)\vartheta_m(\beta),$$

for all $\alpha, \beta \in \Gamma$.

3.2 Automorphic forms of non integral weights

From now on, $\Gamma \subset SL_2(\mathbb{R})$ is a discrete group that contains $-I$ and m is a given real number. There are plenty of groups Γ with no corresponding multiplier systems of weight m. For example (see [10] p.42), if Γ contains no parabolic elements, then Γ has a multiplier system of weight m *if and only if* $m \in u_\Gamma\mathbb{Z}$, where

$$u_\Gamma = \frac{4\pi}{\mathrm{vol}(\Gamma\backslash\mathbb{H})[n_1, n_2, \ldots, n_\ell]},$$

[16] It is possible to consider multiplier systems for groups not containing $-I$, but dealing with this extra case would unnecessarily complicate our discussion.

[17] The first condition is sometimes omitted in the definition because one can show that the other *acceptable* value for $\vartheta_m(-I)$, $-e(-m/2)$, is not compatible with the second condition.

$n_1, n_2, \ldots, n_\ell$ are orders of elliptic generators of Γ, and $[n_1, n_2, \ldots, n_\ell]$ is the least common multiple of these orders. In fact, when Γ contains a free subgroup of finite index, one can construct a multiplier system for Γ by appropriately choosing the values taken by the system on each of the generators of the free group, and extending it to the whole of the free group. However in much more general contexts, the existence of multiplier systems of non-integral weights is a deep fact.

From the preceding discussion, it is easy to see (cfr. proposition 2.1. in [8]) that a multiplier system of weight m for Γ, ϑ_m, exists *if and only if* one can find a non-zero meromorphic function $f : \mathbb{H} \to \mathbb{C}$ for which the slash operators act by

$$f_{|_m\gamma} = \vartheta_m(\gamma) f, \qquad \forall \gamma \in \Gamma.$$

We write $\mathcal{A}(\Gamma\backslash\mathbb{H}, \vartheta_m)$ for the set of functions $f : \mathbb{H} \to \mathbb{C}$ that transform as above, and

$$L^2(\Gamma\backslash\mathbb{H}, \vartheta_m) \subset \mathcal{A}(\Gamma\backslash\mathbb{H}, \vartheta_m)$$

the subset of all $d\mu$-square summable functions. We consider a last crucial set of functions.

Definition. Given a multiplier system ϑ_m, we denote by

$$\mathcal{M}_m(\Gamma, \vartheta_m) \subset \mathcal{A}(\Gamma\backslash\mathbb{H}, \vartheta_m),$$

the linear space of *automorphic forms* for Γ with multiplier ϑ_m of weight m.

As in the integral weight case, *automorphic* stands for holomorphic in $\mathbb{H}$ and at every cusp. As the action of the slash operator *twists* the function with a multiplier, we need to explain our meaning of being holomorphic at a cusp $\mathbf{a}$. If $\sigma_{\mathbf{a}}$ is the corresponding scaling matrix, then the stability group $\Gamma_{\mathbf{a}}$ is generated by $\gamma_{\mathbf{a}}$ and $-\gamma_{\mathbf{a}}$, where

$$\gamma_{\mathbf{a}} = \sigma_{\mathbf{a}} \beta \sigma_{\mathbf{a}}^{-\mathbf{1}}, \qquad \beta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Since $|\vartheta_m(\gamma_{\mathbf{a}})| = 1$, there exists $0 \leq k_{\mathbf{a}} < 1$ such that $\vartheta_m(\gamma_{\mathbf{a}}) = e(k_{\mathbf{a}})$. Given $f \in \mathcal{A}(\Gamma\backslash\mathbb{H}, \vartheta_m)$, if we set

$$g_{\mathbf{a}}(z) = e(-k_{\mathbf{a}} z) f_{|_m\sigma_{\mathbf{a}}}(z),$$

we compute that, using $j_\beta(z) = 1$,

$$\begin{aligned}
g_{\mathbf{a}}(\beta z) &= e(-k_{\mathbf{a}} \beta z) f_{|_m\sigma_{\mathbf{a}}}(\beta z) = e[-k_{\mathbf{a}}(z+1)] j_{\sigma_{\mathbf{a}}}(\beta z)^{-m} f(\sigma_{\mathbf{a}} \beta z) \\
&= e(-k_{\mathbf{a}} z) e(-k_{\mathbf{a}}) j_{\sigma_{\mathbf{a}}}(\beta z)^{-m} f(\gamma_{\mathbf{a}} \sigma_{\mathbf{a}} z) \\
&= e(-k_{\mathbf{a}} z) e(-k_{\mathbf{a}}) \vartheta_m(\gamma_{\mathbf{a}}) f(\sigma_{\mathbf{a}} z) j_{\gamma_{\mathbf{a}}}(\sigma_{\mathbf{a}} z)^m j_{\sigma_{\mathbf{a}}}(\beta z)^{-m} \\
&= e(-k_{\mathbf{a}} z) f_{|_m\sigma_{\mathbf{a}}}(z) j_{\sigma_{\mathbf{a}}}(z)^m j_{\gamma_{\mathbf{a}}}(\sigma_{\mathbf{a}} z)^m j_{\sigma_{\mathbf{a}}}(\beta z)^{-m} \\
&= g_{\mathbf{a}}(z) e[m(\omega(\gamma_{\mathbf{a}}, \sigma_{\mathbf{a}}) - \omega(\gamma_{\mathbf{a}}, \beta))] \\
&= g_{\mathbf{a}}(z),
\end{aligned}$$

where we used proposition 3.1 to get the last equality. As a consequence we can write $g_{\mathbf{a}}(z) = h_{\mathbf{a}}[e(z)]$, for some function $h_{\mathbf{a}} : \mathbb{C}\backslash\{0\} \to \mathbb{C}$. We then say that $f \in \mathcal{A}(\Gamma\backslash\mathbb{H}, \vartheta_m)$ is holomorphic at a cusp $\mathbf{a}$ for Γ if $h_{\mathbf{a}}$ can be extended to the whole plane so that it is holomorphic at the origin.

As we mentioned before, it is not obvious at all that there are some non-empty $\mathcal{M}_m(\Gamma, \vartheta_m)$ spaces. We give two examples that arise from the theory of modular forms and elliptic curves.

The first one is obtained from the *discriminant* function Δ defined on the complex upper half-plane by the infinite product

$$\Delta(z) = (2\pi)^{12} e(z) \prod_{n=1}^{\infty} [1 - e(nz)]^{24}.$$

It is well known that Δ is a cusp form of weight 12 and that it does not vanish anywhere in $\mathbb{H}$. As a consequence, the *Dedekind* η-function,

$$\eta : \mathbb{H} \to \mathbb{C}\,;\, z \mapsto \eta(z) = (2\pi)^{-1/2} \Delta(z)^{1/24} = e\left(\frac{z}{24}\right) \prod_{n=1}^{\infty} [1 - e(nz)]$$

is well defined. It was introduced by Dedekind himself in 1877. The next theorem shows that η has indeed a proper multiplier system. Details can be found in [1] and more will be said about this function in the next chapter about linking numbers of modular knots.

Theorem 3.2. *The Dedekind η-function is an automorphic*[18] *form of weight $1/2$ for the modular group. More precisely, for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, we have*

$$\eta_{|_{1/2}\gamma} = \vartheta(\gamma)\eta,$$

where $\vartheta(-\gamma) = e(1/4)\vartheta(\gamma)$ for any γ, and

$$\vartheta(\gamma) = \begin{cases} e\left(\frac{b}{24}\right) & \text{if } c = 0; \\ e\left(\frac{a+d}{24c} - \frac{s(d,c)}{2} - \frac{1}{8}\right) & \text{if } c > 0. \end{cases}$$

In the preceding theorem, $s(d, c)$ is a classical Dedekind sum. The reader will find a detailed definition and a more advanced discussion about it in the next chapter. Another closely related function is the famous θ-function, usually defined[19] by its Fourier expansion in the complex half-plane $\mathbb{H}$ as

$$\theta(z) = \sum_{n=-\infty}^{\infty} e(n^2 z).$$

One can prove (see [10]) that θ is an automorphic form of weight $1/2$ for the group $\Gamma_0(4)$. These two examples emphasize the importance of multiplier systems and the need to *understand* them.

[18] It is actually a *cusp* form!

[19] An alternative definition is given by the *Jacobi product representation*,

$$\theta(z) = \prod_{n=1}^{\infty} [1 - e(nz)][1 + e(nz + z/2)].$$

3.3 Construction of a new series

This section is about on-going research. Our results seem promising enough to be included in the present section. It is sensible to first motivate our investigations. The idea is to adapt the elegant method in Risager & Truelsen's paper [19] to prove that the angles $\varphi(\gamma z)$ between the tangent line to the $[i, \gamma z]$ hyperbolic geodesic at point i and the line $i\mathbb{R}$ are equidistributed mod π when γ ranges[20] over a discrete cofinite subgroup Γ of $SL_2(\mathbb{R})$ and $z \in \mathbb{H}$ is fixed.

In [19], the authors introduce the series S_n, defined for $n \in \mathbb{Z}$, $z \in \mathbb{H}$ and $\Re(s) > 1$, by

$$S_n(z,s) = \sum_{\gamma\in\Gamma} \frac{e[n\varphi(\gamma z)/\pi]}{\cosh[d(i,\gamma z)]^s}.$$

Theorem 6.1 in [22] yields a good enough estimation to show that S_n is well defined, that $z \mapsto S_n(z,s)$ is in $L^2(\Gamma\backslash\mathbb{H})$ for all such s and that it is holomorphic in its s-variable. In order to use Weyl's criterion (see section 2.2), and prove the uniform distribution of the angles, we need to show that

$$\lim_{\rho\to\infty} \frac{1}{\rho} \sum_{\substack{\gamma\in\Gamma \\ \cosh[d(i,\gamma z)]\leq\rho}} e[n\varphi(\gamma z)/\pi] = 0,$$

for $n \neq 0$. The main point in dealing with the series S_n is that the nature of its pole at $s = 1$ is closely related to such estimations via a modified version of the Wiener-Ikehara tauberian theorem (section 2.1). To achieve this, the strategy used in [19] is to look at the action of the hyperbolic Laplacian on S_n in order to extend meromorphically the series on $\Re(s) > 1 - \epsilon$ for some $\epsilon > 0$ thanks to known analytical properties of the resolvent.

Our plan is to mimic this approach. We want to define an appropriate series and look at how this series transforms under the action of a certain differential operator. This would help us to extend the considered series to a larger domain and a tauberian theorem could yield estimations on averaging sums for a given multiplier system.

Remark. Before doing so, we should briefly mention another direction that we investigated although it proved fruitless. Just as we did for Farey fractions, we wondered whether something could be said about correlation measures for the sequence of angles. Our best hope was to get some information about the pair correlation measure, which requires a careful study of expressions for the difference of two angles. Here is the easiest such expression that we could find. For $z, z_0 \in \mathbb{H}$ and $\gamma, \gamma' \in SL_2(\mathbb{R})$, we have

$$\sin^2[\varphi(\gamma z) - \varphi(\gamma' z_0)] = \frac{\Lambda[u(i,\gamma z), u(i,\gamma' z_0), u(\gamma z, \gamma' z_0)]}{u(i,\gamma z)u(i,\gamma' z_0)[1+u(i,\gamma z)][1+u(i,\gamma' z_0)]},$$

where Λ is the three variables symmetric real function defined by

$$\Lambda(a,b,c) = abc + \frac{ab+bc+ca}{2} - \frac{a^2+b^2+c^2}{4},$$

and u is the usual change of variable $2u(w,w')+1 = \cosh[d(w,w')]$. This equality is rather ugly and we could not exploit it to approximate, asymptotically as ρ tends to infinity, the numbers

$$\#\left\{(\gamma,\gamma') \in \Gamma^2 : \max\{d(i,\gamma z), d(i,\gamma' z_0)\} < \rho, \gamma \neq \gamma', \varphi(\gamma z) - \varphi(\gamma' z_0) \in \frac{I}{N(z;\rho)N(z_0;\rho)} + \mathbb{Z}\right\},$$

[20] We order these angles with respect to the hyperbolic cosine of the hyperbolic distance from i to γz.

where $N(z;\rho) = \#\{\gamma \in \Gamma \,:\, d(i,\gamma z) < \rho\}$ and $I \subset [0,1]$ is a fixed interval. The main difficulty is that the intervals $I/N(z;\rho)N(z_0;\rho)$ are *shrinking* as ρ increases and the methods used in [19] for instance are not designed for this purpose. Before closing this remark, we would like to mention an easy olympiad-type problem composed a few years ago and which popped in my mind as I worked out explicitly Λ. It might entertain you on a train journey:

Let a, b and c be three positive real numbers. Show that there is a triangle with sides of lengths a, b and c *if and only if*

$$\frac{v(a)^2 + v(b)^2 + v(c)^2}{2} \leq v(a)v(b) + v(b)v(c) + v(c)v(a),$$

where $v(x) = x^2$.

Written in this form, we see how similar the expression for Λ is. We deduce from our equality, that if a, b and c are the hyperbolic lengths of a hyperbolic triangle sides, then

$$\frac{u(a)^2 + u(b)^2 + u(c)^2}{2} \leq u(a)u(b) + u(b)u(c) + u(c)u(a) + u(a)u(b)u(c)/2,$$

where $u(x) = 2[\cosh(x) - 1] \sim x^2$, as x tends to zero. However, this is not an *if and only if* statement.

Back to business... We now introduce a very important operator. When functions in $\mathcal{A}(\Gamma\backslash\mathbb{H}, \vartheta_m)$ are lifted to $SL_2(\mathbb{R})$, the *Casimir* operator on $SL_2(\mathbb{R})$ becomes

$$\Delta_m = y^2\left(\partial_x^2 + \partial_y^2\right) - 2iym\partial_x.$$

This operator has many interesting properties and its spectral resolution on $L^2(\Gamma\backslash\mathbb{H}, \vartheta_m)$ plays a very important role in the theory of multiplier systems.

Theorem 3.3. *Let m be a real number. For any function $f \in \mathcal{C}^2(\mathbb{H}, \mathbb{C})$ and $\gamma \in SL_2(\mathbb{R})$, we have*

$$\Delta_m\left[f(\gamma z)\left(\frac{c\bar{z}+d}{cz+d}\right)^m\right] = \left(\frac{c\bar{z}+d}{cz+d}\right)^m (\Delta_m f)(\gamma z).$$

Moreover, if Γ is a discrete subgroup of $SL_2(\mathbb{R})$ and ϑ_m is a multiplier system of weight m for Γ, the second order differential operator Δ_m is self-adjoint on a dense subset of $L^2(\Gamma\backslash\mathbb{H}, \vartheta_m)$.

The proof of the first property is straightforward, and the second one was studied by Roelcke in [20]. Let's define a new function that will be later automorphized in order to get the desired series.

Definition. Let m be real and s a complex number. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, define

$$[\gamma, m, s] \,:\, \mathbb{H} \to \mathbb{C}\,;\, z \mapsto [\gamma, m, s](z) = (cz+d)^{-m}[\cosh d(i,\gamma z)]^{-s}.$$

In order to work out the *Laplacian* Δ_m of $[\gamma, m, s]$ we need some useful expansions for $\cosh d(i,\gamma z)$ and its z and $\bar{z}$-derivatives.

Lemma 3.4. *Whenever $\gamma \in SL_2(\mathbb{R})$ and $z \in \mathbb{H}$, we have*

$$\cosh d(i, \gamma z) = \frac{(ax+b)^2 + (cx+d)^2 + (a^2+c^2)y^2}{2y} = \frac{|cz+d|^2 + |az+b|^2}{2y}$$

and

$$4y^2 \partial_z[\cosh d(i, \gamma z)]\partial_{\bar{z}}[\cosh d(i, \gamma z)] = \cosh^2 d(i, \gamma z) - 1.$$

Proof. The first equations in the general introduction about hyperbolic geometry show that

$$\cosh d(i, \gamma z) = 1 + 2u(i, \gamma z) = 1 + \frac{|i - \gamma z|^2}{2\Im(i)\Im(\gamma z)}.$$

As we mentioned before, $\Im(\gamma z) = y|cz+d|^{-2}$ and hence

$$\begin{aligned}
\cosh d(i, \gamma z) &= 1 + \frac{\left| i - \frac{az+b}{cz+d} \right|^2}{\frac{2y}{|cz+d|^2}} = 1 + \frac{|i(cz+d) - (az+b)|^2}{2y} \\
&= \frac{2y + (-cy - ax - b)^2 + (cx + d - ay)^2}{2y} \\
&= \frac{2y(1 + bc - ad) + 2x(ab + cd) + (x^2 + y^2)(c^2 + a^2) + b^2 + d^2}{2y} \\
&= \frac{(ax+b)^2 + (cx+d)^2 + (a^2+c^2)y^2}{2y},
\end{aligned}$$

as $\det(\gamma) = ad - bc = 1$. Remember that $\partial_z = (\partial_x - i\partial_y)/2$ and $\partial_{\bar{z}} = (\partial_x + i\partial_y)/2$ so that

$$4y^2 \partial_z[\cosh d(i, \gamma z)]\partial_{\bar{z}}[\cosh d(i, \gamma z)] = y^2 \left| \partial_x[\cosh d(i, \gamma z)] + i\partial_y[\cosh d(i, \gamma z)] \right|^2.$$

Since

$$\partial_x[\cosh d(i, \gamma z)] = \frac{ux + v}{y} \qquad \text{and} \qquad \partial_y[\cosh d(i, \gamma z)] = \frac{u(y^2 - x^2) - w - 2xv}{2y^2},$$

where

$$u = a^2 + c^2, \qquad v = ab + cd, \qquad \text{and} \qquad w = b^2 + d^2,$$

we get that

$$\begin{aligned}
4y^2 \partial_z[\cosh d(i, \gamma z)]\partial_{\bar{z}}[\cosh d(i, \gamma z)] &= (ux+v)^2 + \frac{[u(y^2 - x^2) - w - 2xv]^2}{4y^2} \\
&= v^2 - uw + \left[\frac{w + 2vx + u(x^2 + y^2)}{2y} \right]^2,
\end{aligned}$$

after some tedious algebraic work. Note that the term between brackets is precisely equal to $\cosh d(i, \gamma z)$ by the first part of the lemma, and because

$$v^2 - uw = (ab + cd)^2 - (a^2 + c^2)(b^2 + d^2) = -(ad - bc)^2 = -1,$$

the conclusion follows. $\square$

Lemma 3.5. *For all $z \in \mathbb{H}$,*

$$\begin{aligned}
[\Delta_m + s(1-s)][\gamma, m, s](z) &= -s(s+1)[\gamma, m, s+2](z) + 2iymc(m+s)[\gamma, m+1, s](z) \\
&\quad + 2ims(ax+b)[\gamma, m+1, s+1](z).
\end{aligned}$$

Proof. Recall Fay's decomposition [4] for the operator Δ_m,

$$\Delta_m = L_{m+1}K_m + m(1+m),$$

where $z = x+iy$, $K_m = (z-\bar{z})\partial_z + m$, and $L_{m+1} = (\bar{z}-z)\partial_{\bar{z}} - (m+1)$. We compute

$$K_m[\gamma,m,s](z) = (z-\bar{z})\partial_z[\gamma,m,s] + m[\gamma,m,s]$$

Now,

$$\begin{aligned}
\partial_z[\gamma,m,s](z) &= \partial_z\left[(cz+d)^{-m}[\cosh d(i,\gamma z)]^{-s}\right] \\
&= -mc(cz+d)^{-(m+1)}[\cosh d(i,\gamma z)]^{-s} \\
&\quad -s(cz+d)^{-m}[\cosh d(i,\gamma z)]^{-(s+1)}\partial_z[\cosh d(i,\gamma z)] \\
&= -mc[\gamma,m+1,s](z) - s[\gamma,m,s+1](z)\partial_z[\cosh d(i,\gamma z)].
\end{aligned}$$

Therefore,

$$K_m[\gamma,m,s](z) = -mc(z-\bar{z})[\gamma,m+1,s](z) - s(z-\bar{z})[\gamma,m,s+1](z)\partial_z[\cosh d(i,\gamma z)] + m[\gamma,m,s].$$

Similarly,

$$L_{m+1}[\gamma,m,s](z) = (\bar{z}-z)\partial_{\bar{z}}[\gamma,m,s](z) - (m+1)[\gamma,m,s](z).$$

Now,

$$\begin{aligned}
\partial_{\bar{z}}[\gamma,m,s](z) &= -s(cz+d)^{-m}[\cosh d(i,\gamma z)]^{-(s+1)}\partial_{\bar{z}}[\cosh d(i,\gamma z)] \\
&= -s[\gamma,m,s+1](z)\partial_{\bar{z}}[\cosh d(i,\gamma z)],
\end{aligned}$$

so that

$$L_{m+1}[\gamma,m,s](z) = -s(\bar{z}-z)[\gamma,m,s+1](z)\partial_{\bar{z}}[\cosh d(i,\gamma z)] - (m+1)[\gamma,m,s](z).$$

Putting these together, we get that

$$\begin{aligned}
\Delta_m[\gamma,m,s](z) &= -mcL_{m+1}\left\{(z-\bar{z})[\gamma,m+1,s](z)\right\} \\
&\quad -sL_{m+1}\left\{(z-\bar{z})[\gamma,m,s+1](z)\partial_z[\cosh d(i,\gamma z)]\right\} \\
&\quad -sm(\bar{z}-z)[\gamma,m,s+1](z)\partial_{\bar{z}}[\cosh d(i,\gamma z)] \\
&= -mc(\bar{z}-z)\partial_{\bar{z}}\left\{(z-\bar{z})[\gamma,m+1,s](z)\right\} \\
&\quad -s(\bar{z}-z)\partial_{\bar{z}}\left\{(z-\bar{z})[\gamma,m,s+1](z)\partial_z[\cosh d(i,\gamma z)]\right\} \\
&\quad +m(m+1)c(z-\bar{z})[\gamma,m+1,s](z) \\
&\quad +(m+1)s(z-\bar{z})[\gamma,m,s+1](z)\partial_z[\cosh d(i,\gamma z)] \\
&\quad -sm(\bar{z}-z)[\gamma,m,s+1](z)\partial_{\bar{z}}[\cosh d(i,\gamma z)] \\
&= mc(\bar{z}-z)[\gamma,m+1,s](z) + mc(z-\bar{z})^2\partial_{\bar{z}}[\gamma,m+1,s](z) \\
&\quad +s(\bar{z}-z)[\gamma,m,s+1](z)\partial_z[\cosh d(i,\gamma z)] \\
&\quad +s(\bar{z}-z)^2[\gamma,m,s+1](z)\partial_{\bar{z}}\partial_z[\cosh d(i,\gamma z)] \\
&\quad +s(\bar{z}-z)^2\partial_{\bar{z}}[\gamma,m,s+1](z)\partial_z[\cosh d(i,\gamma z)] \\
&\quad +m(m+1)c(z-\bar{z})[\gamma,m+1,s](z) \\
&\quad +(m+1)s(z-\bar{z})[\gamma,m,s+1](z)\partial_z[\cosh d(i,\gamma z)] \\
&\quad -sm(\bar{z}-z)[\gamma,m,s+1](z)\partial_{\bar{z}}[\cosh d(i,\gamma z)]
\end{aligned}$$

After expanding further the derivatives of $[\gamma, m, s]$ and noting that $(\bar{z} - z) = -2iy$, we get that

$$\begin{aligned}
\Delta_m[\gamma, m, s](z) &= -2iymc[\gamma, m+1, s](z) + 4smcy^2[\gamma, m+1, s+1](z)\partial_{\bar{z}}[\cosh d(i, \gamma z)] \\
&\quad -2iys[\gamma, m, s+1](z)\partial_z[\cosh d(i, \gamma z)] \\
&\quad -4y^2 s[\gamma, m, s+1](z)\partial_{\bar{z}}\partial_z[\cosh d(i, \gamma z)] \\
&\quad +4y^2 s(s+1)[\gamma, m, s+2](z)\partial_{\bar{z}}[\cosh d(i, \gamma z)]\partial_z[\cosh d(i, \gamma z)] \\
&\quad +2iym(m+1)c[\gamma, m+1, s](z) \\
&\quad +2iy(m+1)s[\gamma, m, s+1](z)\partial_z[\cosh d(i, \gamma z)] \\
&\quad +2iysm[\gamma, m, s+1](z)\partial_{\bar{z}}[\cosh d(i, \gamma z)].
\end{aligned}$$

Using the preceding lemma, we see that

$$4y^2[\gamma, m, s+2](z)\partial_{\bar{z}}[\cosh d(i, \gamma z)]\partial_z[\cosh d(i, \gamma z)] = [\gamma, m, s](z) - [\gamma, m, s+2](z).$$

Moreover, the operator $4y^2\partial_{\bar{z}}\partial_z = \Delta$ is the hyperbolic Laplacian which commutes with the $SL_2(\mathbb{R})$-action, so that

$$4y^2\partial_{\bar{z}}\partial_z[\cosh d(i, \gamma z)] = [\Delta(2u+1)](\gamma z),$$

where $u(z) = (\cosh d(i, z) - 1)/2$. Using equation 1.21 in [9], we see that this last expression is equal to $2\cosh d(i, \gamma z)$.

These identities yield, after simplifying and rearranging the equality above,

$$\begin{aligned}
\Delta_m[\gamma, m, s](z) &= s(s-1)[\gamma, m, s](z) - s(s+1)[\gamma, m, s+2](z) \\
&\quad 2iym^2c[\gamma, m+1, s](z) + 4smcy^2[\gamma, m+1, s+1](z)\partial_{\bar{z}}[\cosh d(i, \gamma z)] \\
&\quad 2iysm[\gamma, m, s+1](z)(\partial_z + \partial_{\bar{z}})[\cosh d(i, \gamma z)].
\end{aligned}$$

The essential part of calculus is now done. But the right hand side is not elegantly written. To get rid of the partial derivatives, we notice that

$$a = a(ad-bc) = d(a^2+c^2) - c(cd+ab) = du - vc \quad \text{and} \quad b = d(ab+cd) - c(b^2+d^2) = dv - cw,$$

so that

$$\begin{aligned}
ax + b &= (du - vc)x + (dv - cw) \\
&= -\frac{c}{2}[w + 2vx + u(x^2+y^2)] + (cx + d + icy)(ux + v) - icy(ux + v) \\
&\quad +\frac{c}{2}[u(y^2 - x^2) - w - 2xv] \\
&= -yc\cosh d(i, \gamma z) + y(cz+d)(\partial_z + \partial_{\bar{z}})[\cosh d(i, \gamma z)] - 2icy^2\partial_{\bar{z}}[\cosh d(i, \gamma z)].
\end{aligned}$$

Adding and subtracting the quantity $2iymsc[\gamma, m+1, s]$ to $\Delta_m[\gamma, m, s](z)$ yields the following synthetic identity, easier to handle than the rather lengthy expression we had obtained so far,

$$\begin{aligned}
\Delta_m[\gamma, m, s](z) &= s(s-1)[\gamma, m, s](z) - s(s+1)[\gamma, m, s+2](z) \\
&\quad +2iymc(m+s)[\gamma, m+1, s](z) + 2ims(ax+b)[\gamma, m+1, s+1](z),
\end{aligned}$$

which is the stated equality. $\square$

Definition. Let $m \leq 0$ be a real number and ϑ_m be a multiplier system of weight m for Γ. We define for $\Re(s) > 1 - m/2$

$$G_m(z; s) = \sum_{\gamma \in \Gamma} \frac{\overline{\vartheta_m}(\gamma)}{(cz+d)^m [\cosh\, d(i, \gamma z)]^s}.$$

The series G_m converges absolutely in the half-plane $\Re(s) > 1 - m/2$ and converges locally uniformly in $\mathbb{H}$ to a bounded function, in particular it is in $L^2(\Gamma\backslash\mathbb{H}, \vartheta_m)$. To show this, we use again the estimation given by theorem 6.1 in [22] and the inequality

$$|cz+d| \leq \left(|cz+d|^2 + |az+b|^2\right)^{1/2} = \sqrt{2y}[\cosh\, d(i, \gamma z)]^{1/2},$$

that follows from lemma 2.4.

The transformation rule is easily obtained from the identity

$$j_\alpha(\gamma z)^{-m}\overline{\vartheta_m}(\alpha) = j_{\alpha\gamma}(z)^{-m}\overline{\vartheta_m}(\alpha\gamma) j_\gamma(z)^m \vartheta_m(\gamma),$$

which follows from the multiplicative properties of multiplier systems and the functions j, and the fact that $\vartheta_m^{-1} = \overline{\vartheta_m}$. We find that

$$\begin{aligned}
G_m(\gamma z; s) &= \sum_{\alpha \in \Gamma} \overline{\vartheta_m}(\alpha) j_\alpha(\gamma z)^{-m} [\cosh d(i, \alpha\gamma z)]^{-s} \\
&= j_\gamma(z)^m \vartheta_m(\gamma) \sum_{\alpha \in \Gamma} j_{\alpha\gamma}(z)^{-m} \overline{\vartheta_m}(\alpha\gamma) [\cosh d(i, \alpha\gamma z)]^{-s} \\
&= j_\gamma(z)^m \vartheta_m(\gamma) \sum_{\beta \in \Gamma} j_\beta(z)^{-m} \overline{\vartheta_m}(\beta) [\cosh d(i, \beta z)]^{-s} \\
&= \vartheta_m(\gamma)(cz+d)^m G_m(z; s).
\end{aligned}$$

Also, the previous lemmas directly yield the following proposition.

Proposition 3.6. *Let $m \leq -1$ be a real number and ϑ_m be a multiplier system of weight m for Γ. If $\Re(s) > 1 - m/2$, then*

$$[\Delta_m + s(1-s)]\, G_m(z; s) = -s(s+1) G_m(z; s+2) + 2imy(m+s) H_m^1(z; s) + 2imsH_m^2(z; s),$$

where

$$H_m^1(z; s) = \sum_{\gamma \in \Gamma} \frac{c\,\overline{\vartheta_m}(\gamma)}{(cz+d)^{m+1} [\cosh\, d(i, \gamma z)]^s},\ H_m^2(z; s) = \sum_{\gamma \in \Gamma} \frac{(ax+b)\,\overline{\vartheta_m}(\gamma)}{(cz+d)^{m+1} [\cosh\, d(i, \gamma z)]^{s+1}}.$$

We can invert the previous identity applying the resolvent $R(s) = [\Delta_m - s(s-1)]^{-1}$ on both sides. The analytic properties of the resolvent have been treated in lots of papers, see [4] for example. The resolvent is holomorphic for $s(1-s)$ not in the spectrum of Δ_m. In order to extend G_m, we need to study the convergence of the series H_m^1 and H_m^2. Notice that

$$\max\{y|c|, |ax+b|\} \leq |(ax+b)^2 + (cx+d)^2 + (a^2+c^2)y^2|^{1/2} = (2y)^{1/2}[\cosh\, d(i, \gamma z)]^{1/2},$$

so that $H_m^2(z; s)$ converges absolutely for $\Re(s) > -m/2$. On the other hand, the given estimation only implies that $H_m^1(z; s)$ converges absolutely for $\Re(s) > 1 - m/2$ which is not good enough.

Maybe one can improve this bound... Another idea would be to consider particular groups Γ for which the paramater c vanishes for all but finitely many group elements. In that case, it would be possible to extend meromorphically the series G_m to $\Re(s) > 1 - \epsilon$, for some $\epsilon = \epsilon(m) > 0$, if we retrict ourselves to the case $-2 < m \leq -1$.

Another way to go round the difficulty is to try to bound $|cz + d|$ from behind and get some information about the series G_m for $m > -1$. Notice also that the restriction $m \leq 0$ could be relaxed. It is there in the first place only to ensure that G_m is *nicely* convergent.

A good apprehension of the analytic properties of G_m could shed light on the value distribution of a given multiplier system. In fact, a tauberian theorem could translate this information to investigate the asymptotic behaviour of *twisted* (involving $j_\gamma(z)$ factors) sums of the numbers $\vartheta_m(\gamma)$.

4 Modular knots and linking numbers

The aim of this last section is to explain a very important result of Etienne Ghys,

The linking number between the modular knot obtained from a primitive hyperbolic element $\gamma \in PSL_2(\mathbb{Z})$ and the trefoil knot is equal to the Rademacher function evaluated at γ.

To fully understand the statement above, we need to introduce and develop new notions. This will be done in the next three sections. We then provide a beautiful proof of the result and conclude with some *counting* theorems.

4.1 Modular knots

To define modular knots, we first need to study the topology of the quotient space $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$. This quotient space can be canonically identified with the space of two-dimensional lattices in $\mathbb{R}^2$ whose fundamental domain has area one. The reason for this is that for any such lattice Λ, one can find independent vectors μ and ω such that

$$\Lambda = \{n\mu + m\omega \,:\, n, m \in \mathbb{Z}\} \qquad \text{and} \qquad |\mu \times \omega| = 1,$$

and the vectors μ' and ω' also generate the lattice Λ *if and only if* there is a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ such that

$$\mu' = a\mu + b\omega \qquad \text{and} \qquad \omega' = c\mu + d\omega.$$

These identities imply that $z' = \gamma z$ where $z' = \mu'/\omega'$ and $z = \mu/\omega$. Here and in the sequel the vectors $\mu, \omega, \ldots$ are treated as complex numbers through the natural identification $\mathbb{R}^2 \cong \mathbb{C}$. For any given lattice, one can define[21] the following two Eisenstein series

$$g_2(\Lambda) = 60 \sum_{\omega \in \Lambda \setminus \{0\}} 1/\omega^4 \qquad \text{and} \qquad g_3(\Lambda) = 140 \sum_{\omega \in \Lambda \setminus \{0\}} 1/\omega^6.$$

Thanks to a very appealing theorem (see [23] for instance), we know that these numbers are the coefficients of an *elliptic curve* $y^2 = x^3 - g_2(\Lambda)x - g_3(\Lambda)$ whose *discriminant*[22]

$$D = g_2(\Lambda)^3 - 27g_3(\Lambda)^2$$

is not zero. Conversely, if the discriminant of the curve $y^2 = x^3 - g_2 x - g_3$ is not zero, then $g_2 = g_2(\Lambda)$ and $g_3 = g_3(\Lambda)$ for some lattice Λ. As a consequence, the space of two-dimensional lattices in $\mathbb{R}^2$ can be identified with $\mathbb{C}^2 \setminus \mathcal{L}$ where

$$\mathcal{L} = \{z, w \in \mathbb{C}^2 \,:\, D(z, w) = 0\}, \qquad \text{and} \qquad D(z, w) = z^3 - 27w^2.$$

This space is embedded in $\mathbb{R}^4$ and we cannot picture it. Fortunately, we deal only with unimodular lattices and this extra assumption will lead us to work with a three-dimensional object.

[21] These series converge absolutely. Notice also that with an odd exponent the series would vanish, as you can see by pairing the terms $\pm\omega$.

[22] The discriminant of an elliptic curve is usually referred to as Δ. However Δ has already been defined in the previous chapter and we choose D instead to avoid any confusion.

Given a lattice Λ, we can rescale it to get infinitely many other lattices $t\Lambda$, $t > 0$. We will consider only the rescaled lattice for which $|g_2(t\Lambda)|^2 + |g_3(t\Lambda)|^2 = 1$. There is one unique $t > 0$ for which the last equality holds.

Hence, we can identify $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$ with $\mathcal{S}\backslash\mathcal{L}$, where $\mathcal{S} = \{(z,w) \in \mathbb{C}^2 \,:\, |z|^2+|w|^2 = 1\}$. The unit sphere $\mathcal{S}$ is three-dimensional so that $\mathcal{S} \cap \mathcal{L}$ is one-dimensional and embedded in $\mathbb{R}^3$. Since it is a closed curve, it defines a *knot* in the usual three-dimensional space. Now, $\mathcal{S}$ is topologically equivalent to the three-sphere S^3 and the image of $\mathcal{S} \cap \mathcal{L}$ under the considered homeomorphism is the *trefoil knot* ℓ pictured hereunder[23]

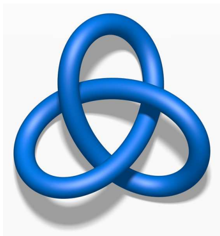

FIG. 2 : THE TREFOIL KNOT ℓ

Therefore, we have proved that there is a homeomorphism

$$Y = SL_2(\mathbb{R})/SL_2(\mathbb{Z}) \cong S^3 \setminus \ell.$$

To construct modular knots in $S^3 \setminus \ell$, we now consider a particular flow on Y, namely, the *modular flow* $\mathcal{G}_t$, defined for $t \in \mathbb{R}$ by

$$\mathcal{G}_t\left(\gamma SL_2(\mathbb{Z})\right) = \phi^t \gamma SL_2(\mathbb{Z}) \qquad \text{where} \qquad \phi^t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

The flow $\mathcal{G}_t$ is obviously well defined. Notice also that, up to conjugacy, the matrices ϕ^t range over all the hyperbolic elements of $SL_2(\mathbb{R})$ as t ranges over $\mathbb{R}$. If $\gamma \in SL_2(\mathbb{Z})$ is hyperbolic, there exists a real number $t = t(\gamma)$ and $\delta = \delta(\gamma)$ a 2×2 matrix with real entries such that $\phi^t = \pm\delta\gamma\delta^{-1}$. It is clear that $\Lambda = \delta\mathbb{Z}^2$ is fixed by ϕ^t as $\gamma\mathbb{Z}^2 = \mathbb{Z}^2$. Equivalently,

$$\mathcal{G}_{\tau+t}\left(\delta SL_2(\mathbb{Z})\right) = \mathcal{G}_{\tau}\left(\delta SL_2(\mathbb{Z})\right),$$

for all $\tau \in \mathbb{R}$. This shows that every hyperbolic matrix γ yields a periodic orbit of the modular flow of period $t(\gamma)$. Conjugated elements in $SL_2(\mathbb{Z})$ clearly share the same periodic orbit in Y. Also, we can restrict our attention to *primitive matrices* as it is enough to go once around the orbit in order to picture the corresponding modular flow. By definition, a matrix is said to be primitive in a group Γ if it is not a non-trivial power of another element in Γ. It is now easy to see that there is a bijection between periodic orbits of the modular flow and conjugacy classes of hyperbolic primitive motions in $PSL_2(\mathbb{Z})$. These periodic orbits are closed curves in Y. Therefore, each one of them yields a knot in the complement of the trefoil knot $S^3 \setminus \ell \cong Y$.

The space Y also describes the unit tangent bundle of the *modular orbifold* $\Sigma = \mathbb{H}/PSL_2(\mathbb{Z})$. From this point of view, $\mathcal{G}_t$ appears to be the geodesic flow of Σ.

[23] All our images are taken from the website of the AMS.

This geometric picture enables us to better visualize primitive hyperbolic matrices as closed geodesics of a certain length.

Definition. Given a hyperbolic motion $\gamma \in PSL_2(\mathbb{Z})$, we denote by k_γ the *modular knot* in $S^3 \setminus \ell$ associated by the above homeomorphism to the periodic orbit of the modular flow corresponding to the conjugacy class of the primitive motion of γ.

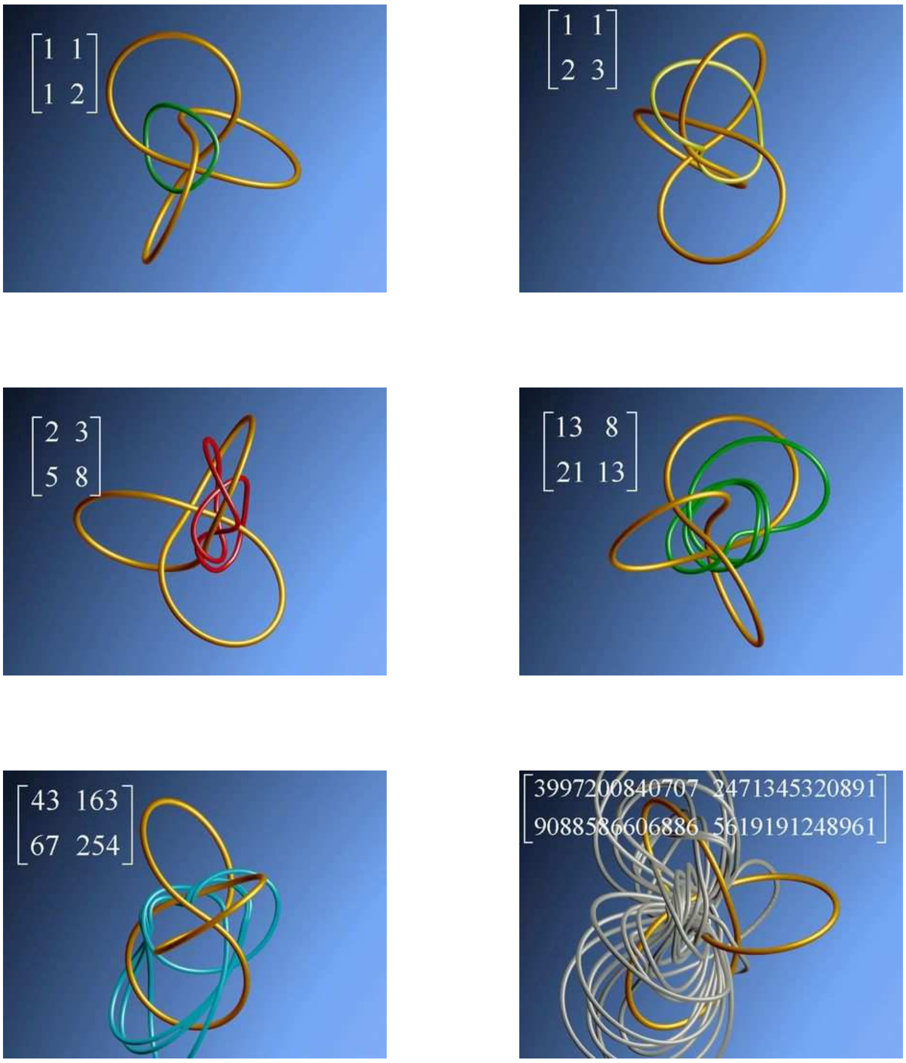

FIG. 3 : THE TREFOIL KNOT ℓ (ORANGE) AND THE KNOTS k_γ FOR γ AS INDICATED AT THE TOP LEFT CORNER OF THE PICTURES

4.2 Linking numbers

In order to investigate further properties of modular knots, we briefly examine the pictures in FIG. 3. For $\gamma = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$, the modular knots k_γ are trivial but tangled differently around the trefoil knot ℓ. A bit more exciting is the next knot, associated to $\gamma = \begin{pmatrix} 2 & 3 \\ 5 & 8 \end{pmatrix}$. In that case, k_γ is another trefoil knot, as one can easily deduce from the picture. The three last knots are more complicated. We want to investigate features of modular knots and the most natural topological property from this point of view is related to *linking numbers.* The linking number associated to two knots is a very intuitive concept. It can be understood as the number of times that each knot winds around the other. It is thus an integer, but may be positive or negative depending on the orientation of the two knots. We can give several equivalent definitions of the notion of linking number. We consider a very formal one, given by the Gauß's integral.

Definition. The linking number between two closed curves k and k' in $\mathbb{R}^3$ is denoted by $\mathrm{link}(k, k')$ and we have

$$\mathrm{link}(k, k') = \frac{1}{4\pi} \oint_k \oint_{k'} \frac{r - r'}{|r - r'|^3} \cdot (dr \times dr').$$

4.3 The Rademacher function

In section 3.2 we introduced the Dedekind η-function. It has been established that

$$\eta^{24}(\gamma z) = j_\gamma^{12}(z)\eta^{24}(z), \qquad \text{for every } \gamma \in PSL_2(\mathbb{Z}).$$

Since Δ does not vanish, neither does η, and $\log \eta$ possesses a holomorphic branch on $\mathbb{H}$. The preceding identity yields the following one,

$$24(\log \eta)(\gamma z) = 24(\log \eta)(z) + 12(\operatorname{sign} c)^2 \log\left(\frac{j_\gamma(z)}{i \operatorname{sign} c}\right) + 2i\pi\phi(\gamma),$$

for some integral valued function ϕ defined on the space of Möbius transformations $PSL_2(\mathbb{Z})$. We consider here the principal branch of the logarithm and

$$\operatorname{sign} c = \begin{cases} 0 & \text{if } c = 0; \\ 1 & \text{if } c > 0; \\ -1 & \text{otherwise.} \end{cases}$$

For our equality to be completely correct, we understand that

$$(\operatorname{sign} c)^2 \log\left(\frac{j_\gamma(z)}{i \operatorname{sign} c}\right) = 0, \qquad \text{if } c = 0.$$

The numerical determination of ϕ is quite complicated. It has been studied by many mathematicians for a long time and in a wide range of fields, including the theory of numbers, combinatorics, knot theory, topology, etc. A fairly extensive discussion about ϕ can be found in [2] and [18].

From the identity defining ϕ and theorem 3.2, we expect a *nice* expression for the multiplier system of η in terms of ϕ. The proof of the next proposition follows easily from a careful analysis of the real and imaginary parts of the quantity $(\operatorname{sign} c)^2 \log\left(\frac{j_\gamma(z)}{i \operatorname{sign} c}\right)$. Details are given in [14].

Proposition 4.1. *The multiplier system of the η-function, ϑ, is such that*

$$\vartheta(\gamma) = e[\psi(\gamma)/24], \qquad \textit{where} \qquad \psi(\gamma) = \phi(\gamma) - 3\operatorname{sign}[c(a + d)].$$

Definition. The function $\psi : PSL_2(\mathbb{Z}) \to \mathbb{Z}$ is called the *Rademacher function*.

A third characterization of the Rademacher function involves Dedekind sums. Recall that the *sawtooth* function f is defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} x - \lfloor x \rfloor - 1/2 & \text{if } x \text{ is not an integer;} \\ 0 & \text{otherwise,} \end{cases}$$

where $\lfloor \cdot \rfloor$ is the *floor* function.

Definition. Given two coprime integers a, b, $b \geq 1$ the Dedekind sum $s(a, b)$ is defined by

$$s(a,b) = \sum_{i=1}^{b} f\left(\frac{ai}{b}\right) f\left(\frac{i}{b}\right).$$

Before exhibiting the link between Dedekind sums and the Rademacher function, we mention the most famous formula about Dedekind sums. Several elementary proofs can be found in [18].

Reciprocity formula for Dedekind sums. *Let $a, b \geq 1$ be two coprime integers. Then*

$$s(a,b) + s(b,a) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{b}{a} + \frac{1}{ab}\right).$$

The proof of the next result is tricky and lengthy. We will avoid it. The interested reader will find detailed arguments in [2].

Theorem 4.2. *Let $\gamma \in SL_2(\mathbb{Z})$. Then,*

$$\phi(\gamma) = \begin{cases} b/d & \textit{for } c = 0; \\ (a+d)/c - 12(\operatorname{sign} c)s(d, |c|) & \textit{otherwise.} \end{cases}$$

It is, *a priori*, not clear that $(a+d)/c - 12(\operatorname{sign} c)s(d, |c|)$ is an integer. One can check that it is actually the case using the reciprocity formula.

4.4 Ghys' result

As we mentioned in the beginning of the present section, the linking number between k_γ and ℓ is given by the Rademacher function evaluated at γ.

Theorem 4.3. *Let $\gamma \in PSL_2(\mathbb{Z})$ be a hyperbolic motion. Then*

$$\operatorname{link}(k_\gamma, \ell) = \psi(\gamma).$$

Proof. We more or less copy the *first proof* in [5]. It is based on the *monodromy* definition of the Dedekind η-function and uses *Jacobi's identity*,

$$D(\Lambda) = g_2(\Lambda)^3 - 27 g_3(\Lambda)^2 = (2\pi)^{12}\omega^{-12}\eta^{24}\left(\frac{\mu}{\omega}\right),$$

for $\Lambda = \{n\mu + n\omega : n, m \in \mathbb{Z}\}$. Now remember that $Y \cong S^3 \setminus \ell$, so that we can associate a unimodular lattice $\Lambda(z)$ to each point z in the complement of the trefoil knot in S^3. Hence, it makes sense to consider the function

$$\chi : S^3 \setminus \ell \to S^1 \subset \mathbb{C}\,;\, z \mapsto \chi(z) = \frac{D[\Lambda(z)]}{|D[\Lambda(z)]|}.$$

Let k be a closed curve in $S^3 \backslash \ell$. As the trefoil ℓ has equation $D = 0$, the linking number $\mathrm{link}(k, \ell)$ corresponds to the winding number of $\chi(k)$ around the origin. Let $\gamma = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{R})$ be a hyperbolic motion. The corresponding periodic orbit of the modular flow has period $t = t(\gamma)$. Hence, we can parametrize the modular knot k_γ by

$$p \,:\, [0, t] \to S^3 \setminus \ell \cong Y \,;\, \tau \mapsto p(\tau) = \Lambda(\tau),$$

where $\Lambda(\tau) = \{n\phi^\tau \mu + m\phi^\tau \omega \,:\, n, m \in \mathbb{Z}\}$ and

$$\phi^t \mu = a\mu + b\omega \qquad \text{and} \qquad \phi^t \omega = c\mu + d\omega,$$

for some μ and ω in $\mathbb{C}$. We want to compute the total variation of the argument of $\chi[\Lambda(\tau)]$ or $D[\Lambda(\tau)]$ as τ ranges over $[0, t]$. Remember that, if we write $D[p(\tau)] = e[r(\tau)]$ for some continuous function r defined on $[0, t]$, then

$$\mathrm{VarArg}_{\tau \in [0,t]}\, p(\tau) = 2\pi \Re \left[r(t) - r(0) \right].$$

Set $z = \mu/\omega$. By Jacobi's identity, this variation is given by

$$\begin{aligned} \mathrm{VarArg}_{\tau \in [0,t]}\, \chi[\Lambda(\tau)] &= -12\Im \left[\log \left(\frac{\phi^t \omega}{\omega} \right) \right] + 24\Im \left[(\log \eta) \left(\frac{\phi^t \mu}{\phi^t \omega} \right) - (\log \eta) \left(\frac{\mu}{\omega} \right) \right] \\ &= -12\Im \left[\log j_\gamma(z) \right] + 24\Im \left[(\log \eta)(\gamma z) - (\log \eta)(z) \right] \\ &= \Im[2\pi i \psi(\gamma)] = 2\pi \psi(\gamma), \end{aligned}$$

where we used proposition 4.1 to get the last equality.

$\square$

We now have an efficient tool to determine $\mathrm{link}(\ell, k_\gamma)$. To illutrate this, we computed the linking numbers $\psi(\gamma)$ associated to the motions γ whose corresponding modular knot k_γ is one of the first five knots pictured in FIG. 3.

γ	$s(d,c)$	$\phi(\gamma)$	$\psi(\gamma)$
$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$	0	3	0
$\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$	0	2	-1
$\begin{pmatrix} 2 & 3 \\ 5 & 8 \end{pmatrix}$	0	2	-1
$\begin{pmatrix} 13 & 8 \\ 21 & 13 \end{pmatrix}$	$-4/63$	2	-1
$\begin{pmatrix} 43 & 163 \\ 67 & 254 \end{pmatrix}$	$-51/134$	9	6

One of the research problems we focused on for a little while is the following.

Given two hyperbolic $PSL_2(\mathbb{Z})$ motions γ and γ', is there a convenient way to compute $\mathrm{link}(k_\gamma, k_{\gamma'})$*? For particular modular knots k_γ, but not as singular as ℓ, can we find a function arising from the theory of modular forms that nicely characterizes the linking numbers* $\mathrm{link}(k_\gamma, k_{\gamma'})$*?*

4.5 Sarnak and Mozzochi's work

We will focus here on Sarnak and Mozzochi's work about the statistical behavior of modular knots and their linking numbers around the trefoil knot ℓ. The main results, as well as a short outline of the proofs can be found in [21]. The details will appear[24] in [14]. We won't reproduce here the proofs of the results. The interested reader should have a look at the referenced material.

Definition. Denote by Π the set of all primitive hyperbolic conjugacy classes in $PSL_2(\mathbb{Z})$. Equivalently, as we have noted previously, Π is the set of all primitive closed geodesics on the Riemann surface $\mathbb{H}/PSL_2(\mathbb{Z})$. The hyperbolic length of a geodesic $\gamma \in \Pi$, will be denoted by $l(\gamma)$ and the *geodesic counting* function is written π, *i.e.* for $x \geq 1$, we have

$$\pi(x) = \sum_{\substack{\gamma \in \Pi \\ l(\gamma) \leq x}} 1.$$

The next result is a strong version of the prime geodesic theorem (cfr. [13]).

Theorem 4.4. *We have*

$$\pi(x) = \mathrm{Li}(e^x) + \mathcal{O}\left(e^{7x/10}\right),$$

as x tends to infinity, where

$$\mathrm{Li}(e^x) = \int_2^{e^x} \frac{dy}{\log y} \sim \frac{e^x}{x}, \qquad \textit{as } x \to \infty$$

is the usual logarithmic integral function from the theory of prime numbers.

This result resembles the *prime number theorem.* A common way to prove the latter is to study the *Riemann zeta-function* ζ defined on $\Re(s) > 1$ by

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}.$$

The key points in the proof are that ζ has a simple pole at $s = 1$ and that it has a non-zero analytic extension to a neighbourhood of $\Re(s) \geq 1$. To show this, one usually studies the analytic properties of the logarithmic derivative of ζ, namely ζ'/ζ, and then gets the estimation (without error term) using a tauberian theorem.

To prove the prime geodesic theorem (without error term), the easiest way is to mimic the preceding approach. The analogies of ζ one shall consider is the *Selberg zeta-function*, $\mathcal{Z}_\Gamma$, defined on $\Re(s) > 1$ by

$$\mathcal{Z}_\Gamma(s) = \prod_p \prod_{k=0}^{\infty} [1 - e^{-\tau(p)(s+k)}],$$

where p ranges over Π, the set of all primitive hyperbolic conjugacy classes in Γ and $\tau(p)$ is the norm of p such that

$$\mathrm{Tr}(p) = \tau(p)^{1/2} + \tau(p)^{-1/2}.$$

[24]Warning to the reader : the document, as it currently (August 2010) stands, contains several typos.

The study of the logarithmic derivative of $\mathcal{Z}_\Gamma$, or to be precise of the quantity

$$\frac{1}{2s-1}\frac{\mathcal{Z}'_\Gamma}{\mathcal{Z}_\Gamma}(s) - \frac{1}{2a-1}\frac{\mathcal{Z}'_\Gamma}{\mathcal{Z}_\Gamma}(a)$$

yields the meromorphic continuation of $\mathcal{Z}_\Gamma$ to the whole complex plane. To see this, we need the *resolvent trace formula* in its usual form (see theorem 10.1 in [9]). The estimation follows, as in the proof of the prime number theorem, from a tauberian theorem.

To investigate the statistical behavior of prime geodesics with a given linking number around the trefoil, we generalize the counting function π.

Definition. For $n \in \mathbb{Z}$ and $x \geq 1$, we write

$$\pi(x;n) = \sum_{\substack{\gamma\in\Pi \\ l(\gamma)\leq x \\ \mathrm{link}(k_\gamma,\ell)=n}} 1 = \sum_{\substack{\gamma\in\Pi \\ l(\gamma)\leq x \\ \psi(\gamma)=n}} 1.$$

A first beautiful result taken from [21] asserts that $\pi(x;n)$ has a uniform asymptotic estimation as x tends to ∞, *i.e.* with an asymptotic error term independent of n. More precisely, Sarnak proved the following estimation.

Theorem 4.5. *Given an integer n, we have*

$$\pi(x;n) \sim \frac{\pi(x)}{3x}\left[1 + \frac{2\left[1-\left(\frac{n\pi}{3}\right)^2\right]}{x^3} + \mathcal{O}(x^{-3})\right], \qquad \text{as } x \to \infty.$$

This theorem follows from the stronger estimation,

$$\sum_{\substack{\gamma\in\Pi \\ l(\gamma)\leq x \\ \psi(\gamma)=n}} l(x) = \frac{1}{3}\,\mathrm{Li}(e^x;n) + \mathcal{O}(e^{3x/4}),$$

as x tends to infinity, where $\mathrm{Li}(e^x;n)$ is the suitable variation of Li defined, for $x \geq 2$, as

$$\mathrm{Li}(e^x;n) = \int_2^{e^x} \frac{\log y}{(\log y)^2 + \left(\frac{n\pi}{3}\right)^2}\,dy.$$

Using the above estimation, one can also establish the distribution of the linking numbers $\psi(\gamma)$. It turns out that it is a *Cauchy distribution.*

Theorem 4.6. *For $-\infty \leq a \leq b \leq \infty$, we have*

$$\sum_{\substack{\gamma\in\Pi \\ l(\gamma)\leq x \\ a\leq\psi(\gamma)/l(\gamma)\leq b}} 1 \quad\to\quad \frac{\arctan\left(\frac{\pi b}{3}\right) - \arctan\left(\frac{\pi a}{3}\right)}{\pi},$$

as x tends to ∞.

The proofs use tools borrowed from the theory of automorphic forms, such as *Selberg's trace formula* (see theorem 10.2 in [9]). It is quite remarkable and surprising how these methods apply effectively to knots!

Given a knot k in $S^3 \setminus \ell$, it seems difficult to assert whether or not one can find $\gamma \in \Pi$ such that $k = k_\gamma$, *i.e.*, k is homologous to k_γ. However, once we know that this is the case, asymptotics for the function counting these conjugacy classes γ are more approachable. For example, if t is the trivial knot, we easily infer the following two estimations from Ghys' results.

Proposition 4.7. *As x tends to infinity, we have*

$$\sum_{\substack{\gamma\in\Pi\\ l(\gamma)\le x\\ k_\gamma=t}} 1 \quad\sim\quad \frac{xe^{x/2}}{2},$$

and for a fixed integer n,

$$\sum_{\substack{\gamma\in\Pi\\ l(\gamma)\le x\\ k_\gamma=t,\ \psi(\gamma)=n}} 1 \quad\sim\quad e^{x/4}.$$

The results in proposition 4.7 are surprising as they show that *thin-looking* subsets of Π grow much more quickly than the error term in theorem 4.4. It is probably worth describing the arguments yielding proposition 4.7. The approach is different from the one leading to theorems 4.4, 4.5 and 4.6. The fine asymptotics are obtained thanks to an explicit description of the set of conjugacy classes $\{\gamma \in \Pi : k_\gamma = t\}$.

Sketch of proof. We denote by $\{\alpha\}$ the conjugacy class of a primitive hyperbolic element $\alpha \in PSL_2(\mathbb{Z})$. From the free product decomposition (see [5] p.272)

$$PSL_2(\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z},$$

one can deduce that

$$(a,b) \mapsto \left\{\begin{pmatrix} 1 & -b \\ -a & ab+1 \end{pmatrix}\right\}$$

is a bijection from $\mathbb{N}^2$ to $\{\gamma \in \Pi : k_\gamma = t\}$. Hence,

$$\sum_{\substack{\gamma\in\Pi\\ l(\gamma)\le x\\ k_\gamma=t}} 1 \quad\sim\quad \sum_{\substack{\gamma\in\Pi\\ t(\gamma)\le e^{x/2}\\ k_\gamma=t}} 1 \quad\sim\quad \sum_{\substack{a,b\ge 1\\ ab+2\le e^{x/2}}} 1 \quad\sim\quad \frac{xe^{x/2}}{2},$$

as desired.

Now, to prove the second part, notice that

$$\psi\begin{pmatrix} 1 & -b \\ -a & ab+1 \end{pmatrix} = a-b.$$

Therefore,

$$\sum_{\substack{\gamma\in\Pi\\ l(\gamma)\le x\\ k_\gamma=t,\ \psi(\gamma)=n}} 1 \quad\sim\quad \sum_{\substack{\gamma\in\Pi\\ t(\gamma)\le e^{x/2}\\ k_\gamma=t,\ \psi(\gamma)=n}} 1 \quad\sim\quad \sum_{\substack{a,a-n\ge 1\\ a^2-an+2\le e^{x/2}}} 1 \quad\sim\quad e^{x/4}.$$

Conclusion

As we have seen, the spectral theory of automorphic forms plays a role in many distinct areas. First of all, it has nice applications in classical number theory as emphasized in the first section dedicated to Farey fractions and their uniform distribution. In the next section, we investigate innovative approaches to study the statistics of multiplier systems. The spectral theory of automorphic forms is actually *itself* a modern and bustling field of research with a lot of promising new ideas. Finally, the last section presents a recent theory that depicts particular knots in a clever and surprising way. The list of applications does not stop here. The spectral theory of automorphic forms is a fascinating and interdisciplinary field of study, standing at the forefront of mathematical research!

It has occupied genius minds for centuries, to name but just a few, Hardy, Ramanujan and Selberg. There are many unsolved questions that keep the field extremely hectic. We mention in the very first section the Selberg's eigenvalue conjecture. This is probably the most fundamental open problem concerning modular forms. However there are other important questions to answer in order to improve our understanding of this area of mathematics.

More than ever, spectral methods applied to automorphic forms turn out to be crucial in the development of modern mathematics.

Pure mathematics is, in its way, the poetry of logical ideas.
Albert Einstein

References

[1] T. M. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, Graduate Texts in Mathematics Volume 41, Springer-Verlag, New York, 1997

[2] M. Atiyah, *The logarithm of the Dedekind η-function,* Annals of Mathematics, 278 (1987) : 335-380, 1987

[3] F. Boca & A. Zaharescu, *The Correlations of Farey Fractions*, Journal of the London Mathematical Society 72 (2005) : 25-39, Cambridge University Press, 2005

[4] J. D. Fay, *Fourier Coefficients of the Resolvent for a Fuchsian Group*, Journal für die Reine und Angewandte Mathematik, 293-294, 1976

[5] E. Ghys, *Knots and Dynamics,* Proceedings of the International Congress of Mathematicians, Madrid, 2006

[6] A. Good, *Local Analysis Selberg's Trace Form*, Lecture Notes in Mathematics, Springer-Verlag, 1983

[7] G. H. Hardy & E. M. Wright, *An Introduction to the Theory of Numbers* (Fifth edition), Oxford University Press, 1980

[8] D. A. Hejhal, *The Selberg Trace Formula for $PSL(2,\mathbb{R})$*, Lecture Notes in Mathematics, Springer-Verlag, 1983

[9] H. Iwaniec, *Spectral Methods of Automorphic Forms*, Graduate Studies in Mathematics Volume 53, American Mathematical Society, Providence, 2002

[10] H. Iwaniec, *Topics in Classical Automorphic Forms*, Graduate Studies in Mathematics Volume 17, American Mathematical Society, Providence, 1997

[11] S. Katok, *Fuchsian Groups*, Chicago Lectures in Mathematics Series, The University of Chicago Press, 1992

[12] H. Kim, *Functoriality for the exterior square of* GL_4 *and the symmetric fourth of* GL_2*. With appendix 1 by D. Ramakrishnan and appendix 2 by H. Kim and P. Sarnak*, Journal of the American Mathematical Society, Volume 16, Number 1, 139-183, 2003

[13] W. Luo, *Quantum ergodicity of eigenfunctions on $PSL(2,\mathbb{Z})/\mathbb{H}$*, IHES, Number 81, 207-237, 1995

[14] C. J. Mozzochi, *Linking numbers of modular geodesics*, in preparation

[15] W. Müller, *Spectral theory for Riemannian manifolds with cusps and a related trace formula*, Math. Nachr. 111, 197-288, 1983

[16] M. R. Murty, *Problems in Analytic Number Theory*, Graduate Texts in Mathematics Volume 206, Springer-Verlag, 2001

[17] Y. Petridis, *L-functions*, lecture notes, course given at the London Taught Course Center, 2008-2009

[18] H. Rademacher & E. Grosswald, *Dedekind Sums*, Carus Mathematical Monographs, Number 19, Mathematical Association of America, 1972

[19] M. S. Risager & J. L. Truelsen, *Distribution of Angles in Hyperbolic Lattices*, The Quarterly Journal of Mathematics 61 (2010) : 117-133, Oxford University Press, 2008

[20] W. Roelcke, *Das Eigenwertproblem der Automorphen Formen in der Hyperbolischen Ebene*, Annals of Mathematics, 167 (1966) : 292-337 & 168 (1967) : 261-324, 1967

[21] P. Sarnak, *Linking Numbers of Modular Knots*, Communications in Mathematical Analysis, Volume 8 Number 2, 136-144, 2010

[22] A. Selberg, *Gottingen Lecture Notes in Collected Papers*, Vol. I, Springer-Verlag, Berlin, 1989

[23] J. P. Serre, *A Course in Arithmetic*, Graduate Texts in Mathematics, Volume 7, Springer-Verlag, New York, 1973

[24] E. C. Titchmarsh & D. R. Heath-Brown, *The Theory of the Riemann Zeta-Function*, Oxford University Press, 1987

[25] A. Walfisz, *Weylsche Exponentialsummen in der neueren Zahlentheorie*, VEB Deutsche Verlag der Wissenschaften, Berlin, 1963