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E. R. Johnson

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Topographic waves and the evolution of coastal currents

By E. R. JOHNSON
Department of Mathematics, University College London, Gower Street, London WC1E 6BT

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An initial-value problem is considered for the oceanographically relevant case of slow flow over obstacles of small height and horizontal scale of order the fluid depth or larger. Previous work on starting flow over obstacles whose contours are closed (Johnson 1984) is extended to flow forced by a source–sink pair to cross a step change in depth bounded by a vertical sidewall. Bottom contours thus end abruptly and the near-periodic solutions of the earlier work are no longer possible. The relevant timescale for the motion is again the topographic vortex-stretching time $h/2\Omega h_0$, where h is the fluid depth, Ω the background rotation rate and h_0 the step height. This time is taken to be long compared with the inertial period but short compared with the advection time. It is shown that if shallow water lies to the right (looking away from the wall) a wavefront moves outwards exponentially fast leaving behind a flow equivalent to that obtained by replacing the step by a rigid wall. If shallow water lies to the left the wavefront approaches the wall, forming at the wall–step junction an unsteady, exponentially thinning, singular region that transports the whole flux. The relevance of these solutions to experiments and steady solutions for free-surface and two-layer flows in Davey, Gill, Johnson & Linden (1984, 1985) is discussed.

1. Introduction

The problem of starting flow over an obstacle moving transversely in a rapidly rotating fluid has been considered recently (Johnson 1984, hereinafter referred to as I) for the the limit where the inertial period is short compared with the topographic vortex-stretching time, which is in turn short compared with the advection time. Solutions in I for flow over axisymmetric bodies set impulsively in motion show topographic waves, of the form discussed by Rhines (1969), cycling clockwise round closed isobaths. In the absence of dissipation the forces on the obstacle and the streamline patterns are non-decaying and almost periodic. It is the purpose of the present work to apply similar analysis to flow over a step bounded by a sidewall. In this geometry isobaths are not closed but end abruptly. Unidirectional topographic waves, termed double Kelvin waves in Longuet-Higgins (1968), propagating towards the wall cannot be reflected and so a periodic motion is not possible.

This problem is closely related to the evolution of coastal currents forced initially to cross bottom contours. Mysak (1969) discusses the double Kelvin wave generated when a wind-stress-forced homogeneous flow crosses an infinitely long step, applying the results to currents crossing the Mendocino escarpment, and Willmott (1984) discusses the modification of these solutions in a two-layer fluid. The effect of the coastal boundary, neglected in these studies, is retained in Davey, Gill, Johnson & Linden (1984, 1985) who examine experimentally and theoretically the adjustment of two-layer and free-surface currents crossing a step bounded by a sidewall. The experimental results show flow from shallow into deep water turning to the left to run along the step before crossing, and flow from deep to shallow turning to the right. These are the directions of propagation of topographic waves above the step (Longuet-Higgins 1968; Rhines 1977) and closed-form, dispersive-wave solutions are given for the time development of these flows in the absence of the sidewall. Analytic solutions for the asymptotically steady flow in the presence of a sidewall are obtained by arguing that topographic waves carry information unidirectionally even when a sidewall is present. This leads to solutions in which there is no flux across the step at finite non-zero distances from the wall. When topographic waves approach the wall the entire flux crosses in a singular region at the origin. When topographic waves travel away from the wall the fluid crosses in an unsteady region at infinity. By presenting a closed-form solution for the initial-value problem, the present analysis verifies this argument for the special case of flows bounded by a rigid lid.

Section 2 poses a simple initial-value problem for flow driven by a source–sink pair on the sidewall of a semi-infinite domain. In the present limit, the flow set up on the inertial period timescale is irrotational, depth independent and has Coriolis force exactly balanced by pressure gradient. It can thus be termed geostrophic. This flow is not steady over the longer, topographic vortex-stretching timescale as streamlines cut isobaths. The flow evolves to a steady state almost everywhere over this longer scale. Since vortex stretching causes flow from deep to shallow to turn left and flow from shallow to deep to turn right, the evolution and final state are determined solely by the geometry of the flow, i.e. whether shallow water lies to the right or left (looking away from the wall), and are independent of flow direction. For flow driven across a vertical step, solutions are symmetric about the step and attention can be confined to a quadrant. In §3 the quadrant is mapped conformally to an infinite strip and topographic wave solutions found for the unforced problem. These free modes show that waves approaching the wall decelerate and decrease in wavelength whereas waves leaving the wall accelerate and increase in wavelength. The waves are unidirectional, propagating with shallow water to their right even in the presence of the wall. In the forced problem this implies that the wall boundary condition propagates to infinity if shallow water lies to the right and the infinity boundary condition propagates to the wall if shallow water lies to the left. The simple time-independent problem thus obtained is solved in §4 for the asymptotic steady states. In both cases there is no flux across the step at finite non-zero distances from the wall. The entire flux crosses at infinity in the former case and in a singular region at the wall–step junction in the latter. The full solution to the initial-value problem is presented in §5, where it is shown that a wavefront matching initial to final conditions advances along the step exponentially rapidly. When shallow water lies to the left a singular region develops in the neighbourhood of the origin. The thickness of the region decreases and the velocities in the region increase exponentially with time, so that the flux transported remains constant. A brief discussion of neglected effects and the modifications caused by a free surface is given in §6.

2. Governing equations

Consider a horizontally semi-infinite layer of inviscid, incompressible fluid of average depth h and constant density ρ, rotating as a rigid body with constant angular frequency Ω about a vertical axis Oz^*. Take Cartesian axes $Ox^*y^*z^*$ so the vertical sidewall of the fluid is given by $y^* = 0$, the upper rigid boundary by $z^* = h$ and the lower boundary by $z^* = h_0 f(x^*/l)$ for $y^* > 0$. Suppose that at $t^* = 0$ a vertical line source–sink pair of volume flux per unit length $8\pi Ul$, is switched on, at $x^* = \pm l$, such that the Rossby number $Ro = U/2\Omega l$ is small. Then, as noted in I, topographic compression of vortex filaments generates vorticity of order $2\Omega h_0/h$ and so introduces a topographic vortex-stretching timescale $h/2\Omega h_0$ in addition to the advection time l/U and the inertial period $(2\Omega)^{-1}$. Following I, introduce the vortex-stretching time $t = 2\Omega h_0 t^*/h$ and consider the limit $Ro \ll h_0/h \ll 1$, to obtain the non-dimensional geostrophic relations

$$u = -p_y, \quad v = p_x, \quad p_z = 0, \tag{2.1}$$

and field equation,

$$\nabla^2 p_t + \partial(p,f) = 0, \tag{2.2}$$

where $\partial(p,f) = p_x f_y - f_x p_y$, relating vorticity generation to topographic compression of vortex lines. The boundary conditions on the flow are

$$p = \operatorname{sgn}(|x|-1) \quad (y = 0,\ t \geqslant 0), \tag{2.3}$$

$$\nabla p \to 0 \quad (x^2+y^2 \to \infty,\ t > 0), \tag{2.4}$$

where, for definiteness, the source has been taken to be at $x = 1$ and the sink at $x = -1$. As (2.2) is linear in p, reversing the flow direction leaves the streamline patterns and flow evolution unaltered. The initial condition required by (2.2) is obtained by noting that the flow set up on the shorter, inertial-period, timescale is unaffected by topography and hence irrotational. Thus

$$\nabla^2 p = 0 \quad (y \geqslant 0, t = 0). \tag{2.5}$$

These equations take a particularly simple form for a step change in depth, i.e. $f = \gamma \operatorname{sgn} x$, where $\gamma = \pm 1$ depending on whether shallow water lies to the left or right looking away from the bounding wall. Equation (2.2) can then be combined with (2.5) to give

$$\nabla^2 p = 0 \quad (x \neq 0, y \geqslant 0, t \geqslant 0). \tag{2.6}$$

The motion remains irrotational away from the step. Integrating (2.2) by parts across the step from $-\epsilon$ to ϵ with $0 < \epsilon \ll 1$ gives

$$[p_{xt} - fp_y]_{-\epsilon}^{\epsilon} + \int_{-\epsilon}^{\epsilon} (p_{yyt} + fp_{xy})\, \mathrm{d}x = 0.$$

Since f is bounded, p_x is bounded and so p is continuous at the step. Furthermore, as the integrand is bounded, the integral vanishes as $\epsilon \to 0$. The matching conditions across the step are thus

$$[p] = 0, \quad [p_{xt}] - 2\gamma p_y = 0 \quad (x = 0, y \geqslant 0, t \geqslant 0), \tag{2.7a,b}$$

where [ ] denotes the jump in the enclosed quantity in passing from $x = 0^-$ to $x = 0^+$. The problem may be further simplified by letting $p = \eta(|x|, y, t)$, where η is defined solely in the first quadrant. Then

$$\nabla^2 \eta = 0 \quad (x \geqslant 0, y \geqslant 0, t \geqslant 0), \tag{2.8}$$

$$\eta = \operatorname{sgn}(x-1) \quad (y = 0, t \geqslant 0), \tag{2.9}$$

$$\nabla \eta \to 0 \quad (x^2+y^2 \to \infty, t \geqslant 0), \tag{2.10}$$

$$\eta_{xt} - \gamma \eta_y = 0 \quad (x = 0, y \geqslant 0, t \geqslant 0), \tag{2.11}$$

$$\eta_x = 0 \quad (x = 0, y \geqslant 0, t = 0), \tag{2.12}$$

where the required initial value of η_x at the step follows from noting that, subject to the condition (2.3), the initial irrotational flow given by (2.5) is even in x.

3. Free modes

Before considering the forced problem, it is informative to examine free modes of the system. Consider (2.8)–(2.12) with (2.9) replaced by the homogeneous condition

$$\eta = 0 \quad (x \geqslant 0,\ y = 0,\ t > 0). \tag{3.1}$$

Reduce this system to a more standard form by the conformal mapping

$$\xi + \mathrm{i}\theta = \log (x + \mathrm{i}y), \tag{3.2}$$

where θ is the usual polar angle and $\xi = \log r$ for radius $r = (x^2+y^2)^{\frac{1}{2}}$. The first quadrant maps to the semi-infinite strip $-\infty < \xi < \infty$, $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. Equations (2.8) and (2.10) are invariant, and (3.1) and (2.11) become

$$\eta = 0 \quad (\theta = 0), \quad \eta_{\theta t} + \gamma \eta_{\xi} = 0 \quad (\theta = \tfrac{1}{2}\pi) \quad (t > 0). \tag{3.3 a, b}$$

The transformed system admits propagating-wave solutions of the form

$$\eta = \mathrm{Re}\,\{A \sinh k\theta \exp (\mathrm{i}k\xi - \mathrm{i}\gamma\omega t)\},$$

provided that ω and k satisfy the dispersion relation

$$\omega = \tanh \tfrac{1}{2}\pi k. \tag{3.4}$$

The phase and group speeds are

$$c_{\mathrm{p}} = \frac{1}{k} \tanh \tfrac{1}{2}\pi k, \quad c_{\mathrm{g}} = \tfrac{1}{2}\pi\, \mathrm{sech}^2\, \tfrac{1}{2}\pi k. \tag{3.5 a, b}$$

The waves are unidirectional, travelling at all wavelengths with shallow water to their right and so moving towards the wall for $\gamma = -1$ and away for $\gamma = 1$. In the transformed plane, long waves travel fastest with group and phase speeds both $\frac{1}{2}\pi$. Both speeds decrease monotonically to zero with decreasing wavelength. In the original coordinates the local wavenumber of a disturbance of a given frequency increases with decreasing distance from the origin. Defining the local radial wavenumber k_{ℓ} as the radial derivative of the phase gives

$$k_{\ell} = \frac{k}{r}. \tag{3.6}$$

The wavelength and speed decrease to zero for waves near the origin and become infinite for waves far away.

This behaviour contrasts with the wavelike solutions possible above an infinitely long step in the absence of a sidewall (Longuet-Higgins 1968; Rhines 1969, 1977; LeBlond & Mysak 1978 chapter 4). System (2.8), (2.10), (2.11) admits solutions of the form

$$\eta = \mathrm{Re}\,\{A \exp (\mathrm{i}ky - \mathrm{i}\gamma\omega t - |k|\, x)\},$$

provided $\omega = \mathrm{sgn}\, k$, giving phase and group speeds

$$c_{\mathrm{p}} = \frac{1}{|k|}, \quad c_{\mathrm{g}} = 2\delta(k), \tag{3.7 a, b}$$

where $\delta(k)$ is the Dirac delta function. The waves are unidirectional with phase speed increasing indefinitely with increasing wavelength and group velocity non-zero solely for infinitely long waves, for which it is infinite. An initial distribution of η drives a standing-wave pattern with period 2π, transporting no energy.

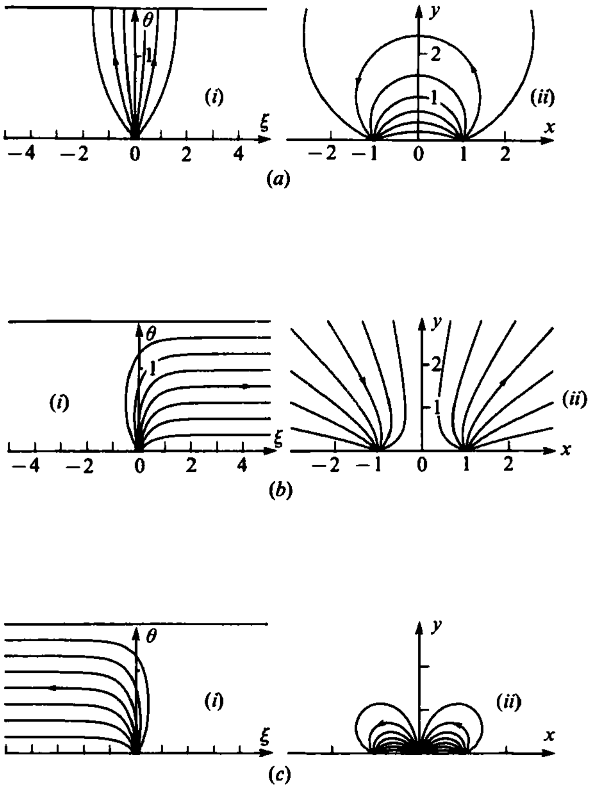

FIGURE 1. Streamline patterns for flow driven over a step change in depth by a source–sink pair situated on a bounding wall. The step is at $x = 0$ and the source and sink at $x = \pm 1$ on $y = 0$. In this and the following figure the source is taken to be at $x = 1$ and the sink at $x = -1$. Reversing the flow direction, however, simply reverses the direction of the arrow heads: the patterns and evolution are unaltered. The streamline interval is 0.25. (*a*) The initial, irrotational flow set up on the inertial timescale. (*b*) The final state when the deeper fluid lies to the left, looking away from the wall (i.e. $\gamma = 1$). (*c*) The final state when the deeper fluid lies to the right (i.e. $\gamma = -1$). In each case diagram (*i*) gives the solution in the (ξ, θ)-strip (see text) and (*ii*) the solution in the original coordinates. The scale for each figure is given by (*a*).

4. The initial-value problem – the long-time solution

The initial-value problem may be solved similarly. However, the considerations of the previous section enable the long-time solution to be obtained without solving the full time-dependent problem. The flow set up on the inertial timescale is two-dimensional irrotational flow from a source–sink pair,

$$\eta = 1 + \frac{2}{\pi}\left\{\arctan\frac{y}{x+1} - \arctan\frac{y}{x-1}\right\} \quad (t = 0). \tag{4.1}$$

Figure 1 (*a*) gives the streamlines for this in the (ξ, θ)-strip and (x, y)-half-plane. Since information at all wavelengths propagates with shallow water on the right, the value $\eta = -1$ propagates along the $\theta = \frac{1}{2}\pi$ boundary from the right for $\gamma = 1$ and the value $\eta = 1$ propagates from the left for $\gamma = -1$. Hence the large-time boundary condition is

$$\eta = -\gamma \quad (\theta = \tfrac{1}{2}\pi,\ t \to \infty). \tag{4.2}$$

The large-time solutions are thus

$$\eta \to 1-\frac{2}{\pi}\left\{\arctan\frac{y}{x-1}+\arctan\frac{y}{x+1}\right\} \quad (\gamma = 1,\, t\to\infty), \tag{4.3a}$$

$$\eta \to 1+\frac{2}{\pi}\left\{2\arctan\frac{y}{x}-\arctan\frac{y}{x+1}-\arctan\frac{y}{x-1}\right\} \quad (\gamma = -1,\, t\to\infty). \tag{4.3b}$$

The flows are symmetric about $x = 0$ and consist, in the first quadrant, of a solid wall at $x = 0$ and, paired with the original source at $x = 1$, a sink of equal strength at the origin (for $\gamma = -1$) or infinity (for $\gamma = 1$). Figure 1(*b*, *c*) give the large-time flow patterns in the transformed strip, and the corresponding patterns in the original coordinates. In both cases there is no flow across $x = 0$ for $y > 0$, consistent with the requirement that steady geostrophic flow cannot cross bottom contours. The step acts as a solid boundary even though it occupies a vanishingly small fraction of the fluid depth. The height of the step affects solely the adjustment time and not the form of the final state. For $\gamma = -1$ the fluid travels from the source to the sink via an infinitesimal singular region at the origin. For $\gamma = 1$ the fluid crosses the step in an unsteady region arbitrarily far from the origin.

5. The initial-value problem – the full solution

A more complete description of the flow can be obtained by considering the long-time behaviour of the full solution. The system satisfied by η consists of the homogeneous system of §3, with (3.1) replaced by

$$\eta = \operatorname{sgn}\xi \quad (\theta = 0,\, t \geqslant 0), \tag{5.1}$$

and the initial condition $\qquad \eta_\theta = 0 \quad (\theta = \tfrac{1}{2}\pi,\, t = 0). \qquad (5.2)$

Introduce the Fourier integral representation

$$\eta = -\gamma+\tfrac{1}{2}\pi\int_{-\infty}^{\infty}\hat\eta(k,\theta,t)\exp(\mathrm{i}k\xi)\,\mathrm{d}k, \tag{5.3}$$

with the exact path of the integral to be determined. Then $\hat\eta$ satisfies

$$\hat\eta_{\theta\theta}-k^2\hat\eta = 0 \quad (0<\theta<\tfrac{1}{2}\pi), \tag{5.4}$$

$$\hat\eta = \frac{2}{\mathrm{i}k} \quad (\theta = 0), \tag{5.5}$$

$$\hat\eta_{\theta t}+\mathrm{i}k\gamma\hat\eta = 0 \quad (t>0), \quad \hat\eta_\theta = 0 \quad (t=0) \quad (\theta = \tfrac{1}{2}\pi). \tag{5.6a, b}$$

The solution of (5.4) and (5.5) can be written

$$\hat\eta = \frac{2\cosh k\theta}{\mathrm{i}k}+A(t)\sinh k\theta$$

provided, from (5.6*a*), $A(t)$ satisfies

$$A_t+\mathrm{i}\gamma\tanh\tfrac{1}{2}\pi k\,A = \frac{-2\gamma}{k}. \tag{5.7}$$

The complementary function for (5.7) is a negative exponential in t for k near zero provided $\operatorname{Im} k<0$ for $\gamma = 1$ and $\operatorname{Im} k>0$ for $\gamma = -1$. Then (5.7) gives the large-time particular solution

$$\eta_0(\xi,\theta) = -\gamma+\frac{1}{\mathrm{i}\pi}\int_{-\infty}^{\infty}\frac{\sinh k(\tfrac{1}{2}\pi-\theta)}{k\sinh\tfrac{1}{2}\pi k}\exp(\mathrm{i}k\xi)\,\mathrm{d}k,$$

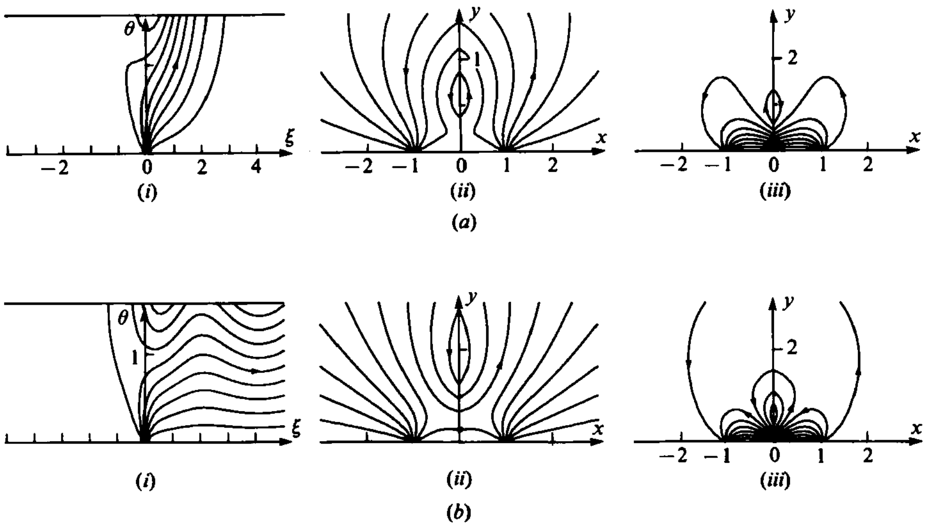

FIGURE 2. Streamline patterns at intermediate stages of the flow in figure 1. In both cases (*i*) gives the pattern in the (ξ, θ)-strip, (*ii*) the pattern in the original coordinates for $\gamma = 1$ and (*iii*) the pattern in the original coordinates for $\gamma = -1$. (*a*) $t = 2$; (*b*) $t = 10$. The scales are as in figure 1.

with the inversion contour passing above the pole at $k = 0$ for $\gamma = -1$ and below for $\gamma = 1$. In terms of integrals defined for real k,

$$\eta_0(\xi, \theta) = \frac{-2\gamma\theta}{\pi} + \frac{2}{\pi}\int_0^\infty \frac{\sinh k(\frac{1}{2}\pi - \theta)}{k \sinh \frac{1}{2}\pi k} \sin k\xi \, \mathrm{d}k. \tag{5.8}$$

This is the steady-state solution (4.3). The intermediate-time solution follows from (5.7) with initial condition (5.6*b*) as

$$\eta(\xi, \theta, t) = \eta_0(\xi, \theta) + \frac{2\gamma\theta}{\pi} + \frac{4}{\pi}\int_0^\infty \frac{\sinh k\theta}{k \sinh \pi k} \sin\left[k\xi - (\tanh \tfrac{1}{2}\pi k)\gamma t\right] \mathrm{d}k. \tag{5.9}$$

The form of (5.9) shows the solution for $\gamma = -1$ to follow from that for $\gamma = 1$ by running time backwards, i.e. replacing t by $-t$. Alternatively, the solutions may be related by reflecting about $\xi = 0$ and changing the sign of η. Streamline patterns for $\gamma = \pm 1$ at a given time are thus related in the original coordinates by inversion in the unit circle. This can be seen clearly from the fast Fourier transform (FFT) inversion of (5.9) displayed in figure 2 at times $t = 2$, 10 for $\gamma = 1$ in the (ξ, θ)-strip and the corresponding solutions for $\gamma = \pm 1$ in the original coordinates. The infinite-time solutions of figure 1 are also related through inversion in the unit circle.

The behaviour of the solution is most clearly illustrated by considering its asymptotic form above the depth discontinuity. It is useful to consider first the flow velocity perpendicular to the step, since this must approach zero for the geostrophic flow to become steady. It is given by $U(y) = -\eta_y$ $(x = 0)$ and has the initial value $U(y) = -4/\pi(1+y^2)$, decaying algebraically and monotonically from a maximum speed of $2/\pi$ at the wall. At intermediate times it is simpler to consider $-yU(y) = \eta_\xi(\theta = \frac{1}{2}\pi) = V(\xi)$ (say) where, from (5.9),

$$V(\xi) = \frac{2}{\pi}\int_{-\infty}^{\infty} \operatorname{sech} \tfrac{1}{2}\pi k \exp\{\mathrm{i}E(k)\,t\}\,\mathrm{d}k, \tag{5.10}$$

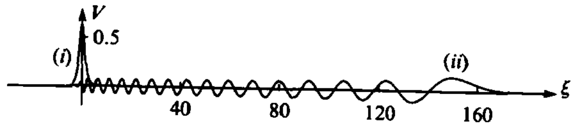

FIGURE 3. The scaled cross-step velocity, $V(\xi) = -yU(y)$, in the stretched coordinate system $\xi = \log y$. (*i*) The initial profile ($t = 0$); (*ii*) the dispersed wavetrain ($t = 100$). The enhanced amplitude in the neighbourhood of the wavefront at $\xi = \frac{1}{2}\pi t$ is clearly visible.

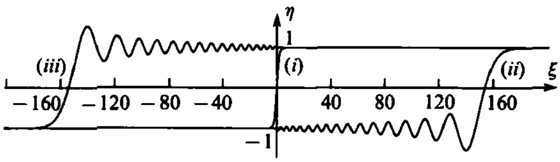

FIGURE 4. Stream-function values above the depth discontinuity in the stretched coordinate system. (*i*) The initial profile ($t = 0$; (*ii*) the profile at $t = 100$ for $\gamma = 1$, i.e. waves travelling outwards; (*iii*) the profile at $t = 100$ for $\gamma = -1$, i.e. waves travelling inwards.

for $E(k) = k\xi/t - \tanh\frac{1}{2}\pi k$. At $y = 1$, i.e. $\xi = 0$, (5.10) can be evaluated directly to give $U(1) = -V(0) = -(2/\pi)J_0(t)$, where J_0 is the zero-order Bessel function of the first kind. The velocity oscillates with zero mean, frequency approaching π, and amplitude decaying as $t^{-\frac{1}{2}}$. For other values of ξ the integral can be estimated at large time by the method of stationary phase. For $|\xi|, t \to \infty$ such that ξ/t remains constant, i.e. points moving outward (for $\gamma = 1$) with positions $y = y_0 \exp(ct)$ or inward (for $\gamma = -1$) with positions $y = y_0 \exp(-ct)$, the dominant contribution to the integrand occurs for those wavenumbers where $E'(k)$ vanishes, i.e. for $\xi/\gamma t = c_g(k)$, as expected. Outside the region $0 < \xi/\gamma t < \frac{1}{2}\pi$, the disturbance decays exponentially with increasing $|\xi|$ and as t^{-1} with increasing time. In the original coordinate system this causes the disturbance to be confined in $\exp(-\frac{1}{2}\pi t) \leqslant y \leqslant 1$ for $\gamma = -1$ and to spread exponentially in $1 \leqslant y \leqslant \exp(\frac{1}{2}\pi t)$ for $\gamma = 1$. The usual stationary-phase formula yields, for $\gamma = 1$,

$$V \sim \frac{4}{\pi}(\pi t)^{-\frac{1}{2}}\left(1-\frac{2\xi}{\pi t}\right)^{-\frac{1}{4}} \cos\left\{k_0\xi - t\left(1-\frac{2\xi}{\pi t}\right)^{\frac{1}{2}} + \tfrac{1}{4}\pi\right\}, \tag{5.11}$$

where $k_0 = (2/\pi)\,\mathrm{arcsech}\,[(2\xi/\pi t)^{\frac{1}{2}}]$. Except in the neighbourhood of the wavefront at $\xi = \frac{1}{2}\pi\gamma t$, V decays as $t^{-\frac{1}{2}}$. The singularity at $\xi = \frac{1}{2}\pi\gamma t$ corresponds to the maximum in c_g for the long waves forming the front. Retaining third-order terms in the expansion of $E(k)$ in this neighbourhood yields a standard matching in terms of the Airy function, for $\gamma = 1$,

$$V \sim \frac{4}{\pi}t^{-\frac{1}{3}}\,\mathrm{Ai}\left[\frac{2}{\pi}t^{-\frac{1}{3}}(\xi - \tfrac{1}{2}\pi t)\right]. \tag{5.12}$$

In a widening region of thickness $t^{\frac{1}{3}}$ about the front, the disturbance decays as $t^{-\frac{1}{3}}$. This slower decay manifests itself in an enhanced amplitude near the front, as can be seen from the FFT inversion of (5.10) in figure 3, showing the initial distribution and that at $t = 100$ for $\gamma = 1$ (that for $\gamma = -1$ following by reflection about $\xi = 0$).

Applying the same arguments to the stream function above the step, and including the effect of the pole at the origin, gives, to order t^{-1} for $\gamma = 1$,

$$\eta \sim \begin{cases} -1 + \dfrac{4}{\pi}(\pi t)^{-\frac{1}{2}} k_0^{-1} \sin\left\{k_0 \xi - t\left(1 - \dfrac{2\xi}{\pi t}\right)^{\frac{1}{2}} + \frac{1}{4}\pi\right\} & (0 < \xi < \frac{1}{2}\pi t), \\ \operatorname{sgn}(\xi - \frac{1}{2}\pi t) \quad (\text{otherwise}). \end{cases} \tag{5.13}$$

The behaviour at the wavefront follows from integrating V as

$$\eta \sim -1 + 2\int_{-\infty}^{X} \mathrm{Ai}\,(s)\,\mathrm{d}s, \quad X = \frac{2}{\pi}\, t(\xi - \tfrac{1}{2}\pi t). \tag{5.14}$$

Asymptotic forms for this integral in Abramowitz & Stegun (1965, chapter 10) show that η approaches 1 exponentially for $\xi > \frac{1}{2}\pi t$ and matches smoothly to the wavetrain for $\xi < \frac{1}{2}\pi t$. Figure 4, from FFT inversion of (5.9), shows stream-function values along the step at times $t = 0$ and $t = 100$ for $\gamma = \pm 1$, the solution for $\gamma = -1$ following by reflecting that for $\gamma = 1$ about $\xi = 0$ and $\eta = 0$. These patterns correspond to the free-surface or interface displacements of Davey *et al.* (1984, 1985). The advancing wavefront is clearly visible, leaving behind a wavetrain oscillating about the final η value with amplitude decaying as $t^{-\frac{1}{2}}$. Each excursion corresponds to an eddy forming over the step, at unit distance from the wall, and travelling outwards ($\gamma = 1$) or inwards ($\gamma = -1$), decaying in amplitude. These eddies may be seen in the original coordinates in figure 2. For $\gamma = -1$ the long-time value $\eta = 1$ propagates towards the wall so that the transition occurs near $y = \exp(-\frac{1}{2}\pi t)$. Conservation of flux then implies that the velocity near the wall grows exponentially to form the singular source–sink region at the origin in the long-time solution of § 4. For $\gamma = 1$ the long-time value $\eta = -1$ propagates to infinity with the transition occurring near $\exp(\frac{1}{2}\pi t)$. In both cases the mean of the motion rapidly becomes that of the long-time flow but the oscillatory tail behind the wavefront means that there is only a slow decay to the final state.

6. Discussion

Analytic solutions have been presented for the temporal development of flow forced across a depth discontinuity by a source–sink pair on a vertical sidewall of a rapidly rotating container. An initial, unsteady, irrotational but geostrophic flow, set up in a time of order the inertial period, evolves almost everywhere to a steady state over the timescale relevant for topographic vortex stretching. On this scale a wavefront travels outwards with exponentially increasing displacement, or inwards with exponentially decreasing displacement, the direction of propagation being such that shallow water lies to the right. Ahead of the wavefront the flow deviates little from the initial conditions and behind it oscillates with small amplitude about its final state. Depending on the direction of propagation of the front, the flux forced by the source–sink pair crosses the step in an unsteady region at infinity or in an increasingly narrow, singular region in the neighbourhood of the wall.

In this singular region and above the step, higher-order terms, negligible in the remainder of the flow field, are important. Which neglected effect first becomes important depends on the relative sizes of the relevant small parameters (see I). Advective effects are measured by $S^{-1} = h\,Ro/h_0$, the inverse of the Hide (1961) parameter, giving the ratio of the time taken for the wavefront to reach the wall to that for a particle to travel from the source to the step. If S^{-1} is the largest of the neglected small parameters then the governing dynamics above the step, near the wall, and at large time are those for the nonlinear conservation of potential vorticity. The present steady solutions obey these almost everywhere and so it is likely that only in the singular region will the symmetry of the flow be broken and generation of relative vorticity by vortex stretching be important. This is not a steady process and the continuing narrowing of these regions means that eventually horizontal diffusion, measured by an Ekman number $E = \nu/2\Omega l^2$, is important. The wall-step singularity appears simpler than the closely related problem of nonlinear critical layers as the outer solution is determined entirely without knowledge of the inner region. This may not be so along the step and the previously derived jump conditions may be invalid at large time.

If diffusion is the largest of the neglected effects, or after a period of advective adjustment, $E^{\frac{1}{4}}$ and $E^{\frac{1}{3}}$ layers ensure continuity of the solution above the step and a modified $E^{\frac{1}{4}}$ layer (Greenspan 1968, chapter 2) carries the flux across the step in the singular region. Viscous dissipation destroys vorticity generated by vortex stretching and flow patterns retain their symmetry unless the sidwall boundary separates. Ekman pumping effects, measured by $\mu = (\nu/\Omega h_0^2)^{\frac{1}{2}}$, the ratio of Ekman-layer thickness to step height, can be easily incorporated even at zero order. The single modification of (2.8) to (2.12) is to replace (2.11) by

$$\eta_{xt} + \mu\eta_x - \gamma\eta_y = 0 \quad (x = 0,\, y \geqslant 0,\, t \geqslant 0), \tag{6.1}$$

causing topographic waves to decay, irrespective of wavelength, on the Ekman spin-up time. The long-time solution varies continuously from the pattern in figure 1(*a*) to those in figure 1(*b*, *c*) as μ decreases from infinity, when the flow is irrotational everywhere and the step is irrelevant, to zero when the Ekman layer is vanishingly thin.

Greenspan (1968, chapter 2) reports experiment and theory for inertial waves, depth-independent topographic waves and geostrophic modes in rapidly rotating containers with order-unity depth changes. His results make it likely that the flow patterns presented here will closely approximate those above a step whose height is a substantial fraction of the fluid depth. The vortex-stretching time in this case is of order the inertial period and so only a few rotation periods would be needed to set up the asymptotic state.

The source–sink flow presented here is a simple analytic and experimental method for forcing a cross-step flow. In oceanographic applications the cross-step flow is more likely to be forced by wind stresses, modelled in the laboratory by a differentially rotated lid. The present method of solution applies directly. Once a particular solution ignoring topography is subtracted, the remainder of the solution is forced by an inhomogeneity above the step.

The sole effect in the present limit of allowing the surface to be free or considering a two-layer fluid with a rigid lid and an inert layer (Rhines 1977; Willmott 1984 for the unbounded case) is to replace the field equation (2.8) by

$$\nabla^2\eta - a^{-2}\eta = 0, \tag{6.2}$$

where $a = (gh)^{\frac{1}{2}}/\Omega l$ is a non-dimensional Rossby radius and g is gravitational acceleration for a free surface or reduced gravity in the two-layer flow. The conformal-mapping technique is then inapplicable. However, by analogy with the effect of finite Rossby radius on waves in the absence of a bounding wall, it appears that the free surface simply introduces a maximum propagation velocity for long waves in the original coordinate system. This causes a slower evolution at large distances but does not affect the details of the singular region. It is under this assumption that the solutions of Davey *et al.* (1984, 1985) were obtained.

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