Nonlinear Quantum Critical Transport and the Schwinger Mechanism for a Superfluid-Mott-Insulator Transition of Bosons

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Scaling arguments imply that quantum-critical points exhibit universal nonlinear responses to external probes. We investigate the origins of such nonlinearities in transport, which is especially problematic since the system is necessarily driven far from equilibrium. We argue that for a wide class of systems the new ingredient that enters is the Schwinger mechanism—the production of carriers from the vacuum by the applied field—which is then balanced against a scattering rate that is itself set by the field. We show by explicit computation how this works for the case of the symmetric superfluid-Mott insulator transition of bosons.

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The notion of quantum criticality provides one of the few general approaches to the study of strongly correlated quantum many-body systems [1]. The scale invariance that characterizes the zero-temperature critical point leads to characteristic universal power law dependences for various quantities in its proximity; these dependences can be computed within a continuum field theory.

While the power law dependences can generally be related to an underlying set of (possibly unknown) exponents by scaling arguments, establishing the actual mechanism that gives rise to them—in the sense of prescribing a controlled computation—is not always a trivial task. For example, much attention has focused on understanding the real-time dynamics at nonzero temperatures [2] where the textbook procedure of analytic continuation from Matsubara expressions typically yields incorrect answers and no insight.

Here, we address another such question, that of transport at finite fields. Specifically, consider a quantum-critical point between an insulator and a metal or superconductor/superfluid characterized by a correlation length that diverges as $\xi \sim \delta^{-\nu}$ and an energy scale that vanishes as $\Delta \sim \delta^{\nu z}$, where δ is a measure of the distance to the transition. Dimensional analysis implies that the zero-temperature conductivity obeys the scaling form

$$\sigma(\delta, E) = E^{(d-2)/(z+1)} \Sigma\left(\frac{\delta}{E^{\nu(z+1)}}\right) \tag{1}$$

in d spatial dimensions. Thus, generically, the system exhibits a nonlinear current-voltage characteristic at criticality. The question of interest to us is how the system might produce such a response. Evidently, linear response theory or naive perturbation theory to higher orders is no help. Physically, the system must set up a steady state whose properties depend in singular fashion on E with no expectation that it resembles the steady state obtained in thermal equilibrium.

We show that the properties of such weakly interacting fixed points may be understood as follows: The application of an electric field leads to a biased growth of current carrying fluctuations by an analogue of the Schwinger mechanism [3], by which an electric field produces electron-hole pairs from the vacuum. This process is nonperturbative in the electric field. The fluctuations produced in this way scatter from one another due to the interactions at the fixed point, thus producing current relaxation, which is again nonperturbative in the electric field at the fixed point. Together, these effects establish a steady-state distribution that carries a current [4]. We implement this idea for the simplest quantum phase transition with singular transport—the symmetric superfluid to Mott insulator transition of bosons in a periodic potential—and recover the dependence (1) with $z = 1$. In $d = 2$ we find a linear conductivity $\sigma = (\pi/4)e^{*2}/h$, different from those calculated previously [2,5]. The difference between these results arises due to the different regimes of frequency, temperature, and electric field (ω, T, E) to which they apply and the different physical process important in each case. The linear response calculated at zero temperature [5] amounts to the limit ($\omega \to 0$, $T = 0$, $E = 0$), while the finite-temperature, linear response [2] corresponds to ($\omega = 0$, $T \neq 0$, $E = 0$). Our result follows from taking the frequency and temperature to zero at finite field ($\omega = 0$, $T = 0$, $E \neq 0$). The noncommutativity of the limits $\omega \to 0$ and $T \to 0$ was first recognized in [2]. Our result suggests that similar care should be taken with the electric field. Next, we outline our computation, which builds upon the important work by Damle and Sachdev [2] on the finite-temperature transport of the same model; see also the interesting, but rather different, work of Dalidovich and Phillips [6].

Field theory.—The critical region of the superfluid to Mott insulator transition with particle hole symmetry is described by a charged scalar field with a quartic interaction [1,2]:

$$\mathcal{H} = \int d^d x[\Pi^\dagger \Pi + \nabla\phi^\dagger \nabla\phi + m^2 \phi^\dagger \phi + \lambda(\phi^\dagger \phi)^2], \tag{2}$$

where ϕ is the complex scalar field and Π is its conjugate momentum. These satisfy the usual commutation relations $[\phi(\mathbf{x}, t), \Pi(\mathbf{y}, t)] = i\delta(\mathbf{x} - \mathbf{y})$. We choose the bare interaction λ to have its fixed point value $u^* \Lambda^{3-d}$ with momentum cutoff Λ [1] although we do not need the precise, regularization-dependent, value. At the zero-temperature critical point, the renormalized mass is zero corresponding to a particular choice m^* of the bare mass. The effects of applying an electric field, $\mathbf{E}$, are included by minimally coupling to a vector potential; $\nabla\phi \rightarrow D\phi = (\nabla + i\mathbf{A})\phi$. We choose the gauge $\mathbf{A} = \mathbf{E}t$ for initial convenience, and later we switch to a scalar potential.

Mode expansion.—The normal modes of this Hamiltonian (in the absence of interaction) are charge density fluctuations. These occur with positive and negative charges, corresponding to an increase or a decrease in charge density from the average. The first step in our analysis is to reexpress (2) in terms of creation and annihilation operators for these normal modes. The transformation is standard for the Klein-Gordon theory. Using $a^\dagger$ and a to represent the creation and annihilation of positively charged density fluctuations and $b^\dagger$ and b for negatively charged fluctuations, the noninteracting part of the Hamiltonian reduces to

$$\mathcal{H}_0 = \int \frac{d^2k}{(2\pi)^2}[a^\dagger(\mathbf{k}, t), b(-\mathbf{k}, t)] \times \begin{pmatrix} \epsilon_\mathbf{k} + B_\mathbf{k} & B_\mathbf{k} \\ B_\mathbf{k} & \epsilon_\mathbf{k} + B_\mathbf{k} \end{pmatrix} \begin{pmatrix} a(\mathbf{k}, t) \\ b^\dagger(-\mathbf{k}, t) \end{pmatrix}, \tag{3}$$

where $\epsilon_\mathbf{k} = \sqrt{\mathbf{k}^2 + m^2}$ is the mode energy and $B_\mathbf{k} = [(\mathbf{E}t)^2 + 2\mathbf{k} \cdot \mathbf{E}t]/2\epsilon_\mathbf{k}$. The Hamiltonian has an explicit time dependence arising from our choice of gauge. This time dependence is responsible for the production of fluctuations by the electric field.

Schwinger mechanism.—Let us ignore the interaction between the normal modes of the system. This allows us to describe the pair production process, after which we will put back the interactions. Our first step is to diagonalize the Hamiltonian (3) using a Bogoliubov transformation. Because of the time dependence of the Hamiltonian, the transformation used to carry out this diagonalization must itself be time dependent. This has important consequences for the equations of motion. In the instantaneously diagonal basis, or *adiabatic particle basis,* the Heisenberg equations of motion for operators pick up extra terms from the time dependence of the Bogoliubov transformation matrix. We are concerned with the equations of motion for the regular and anomalous distribution functions; $f(\mathbf{k}, t) = \langle a^\dagger(\mathbf{k}, t) a(\mathbf{k}, t)\rangle = \langle b^\dagger(-\mathbf{k}, t) b(-\mathbf{k}, t)\rangle$ and $g(\mathbf{k}, t) = \langle b(-\mathbf{k}, t) a(\mathbf{k}, t)\rangle$, where the angular brackets indicate averages over the state of the system [7]. After transforming to the adiabatic particle basis, the equations of motion for these reduce to

$$\begin{aligned} \frac{df(\mathbf{k}, t)}{dt} &= \frac{\dot{\epsilon}_\mathbf{k}(t)}{\epsilon_\mathbf{k}(t)} \mathrm{Re} g(\mathbf{k}, t), \\ \frac{dg(\mathbf{k}, t)}{dt} &= \frac{\dot{\epsilon}_\mathbf{k}(t)}{2\epsilon_\mathbf{k}(t)}[2f(\mathbf{k}, t) + 1] - 2i\epsilon_\mathbf{k}(t) g(\mathbf{k}, t), \end{aligned} \tag{4}$$

where $\epsilon_\mathbf{k}(t) = \sqrt{(\epsilon_\mathbf{k} + B_\mathbf{k})^2 - B_\mathbf{k}^2} = \epsilon_{\mathbf{k}+\mathbf{E}t}$ is the mode energy in the adiabatic particle basis. These equations contain all of the ingredients necessary to describe the Schwinger mechanism [3]. The terms proportional to $\dot{\epsilon}_\mathbf{k}(t)$ result from the time dependence of the Hamiltonian and are responsible for pair creation. The second equation can be solved explicitly for $g(\mathbf{k}, t)$ and substituted back into the first. The resulting equation contains a source term for the production of fluctuations, which includes oscillatory behavior coming from the quantum coherence of the pair production. If we ignore these transients, the source term may be replaced by a delta function with the appropriate weight. The equation of motion for $f(\mathbf{k}, t)$ then reduces to

$$\frac{df(\mathbf{k}, t)}{dt} = \delta(t + \mathbf{k} \cdot \mathbf{E}/\mathbf{E}^2) e^{-\pi(k\perp^2 + m^2)/E}. \tag{5}$$

The pair production described in this way may be understood by analogy with Landau-Zener tunneling [8] as follows: under the action of the electric field, the momentum of a charged excitation increases linearly in time as $\mathbf{k} + \mathbf{E}t$. The energy of the charged excitation changes in time accordingly; at large momenta it is proportional to $|\mathbf{k} + \mathbf{E}t|$, and at a time $t = -\mathbf{k} \cdot \mathbf{E}/E^2$ it passes though a minimum equal to m. This is very similar to the energy dependence of modes in the Landau-Zener model. Except for the Bose enhancement factor $2f(\mathbf{k}, t) + 1$ the pair production Eq. (4) has precisely the same form as in Landau-Zener tunneling.

Scattering and steady state.—The steady creation of pairs from the vacuum will, in the absence of scattering, lead to a secular divergence of the current. Thus, consideration of the scattering is essential for understanding the steady-state transport. Below $d = 3$, the critical behavior is controlled by the interacting, Wilson-Fisher fixed point. If we access the properties of this fixed point in a weak coupling expansion, such as the $1/N$ expansion, which we use in this Letter, then the leading order description of the critical, steady state can be obtained by considering the scattering of the particles produced via the Schwinger mechanism. Heuristically, the pair production in (5) leads to a growth in the current with a rate proportional to $E^{(d+1)/2}$, while scattering with a coupling of order $1/N$ to N channels is expected (on dimensional grounds) to lead to a current relaxation rate of order $(1/N)\sqrt{E}$; together these will reproduce (1) with $z = 1$. In the remainder of this Letter we shall see how this insight can be turned into a computation within the quantum Boltzmann equation framework.

Generally, $1/N$ expansions extend the number of modes of the model from its initial value to some large number N, allowing all of these modes to interact with one another. If N is taken to be very large, the interaction may be expanded perturbatively in $1/N$. Crucially, in the present case, we couple the electric field to just two modes of our extended theory (i.e., one of the $N/2$ copies of the model). The field induces fluctuations in these two modes via the Schwinger mechanism. These fluctuations scatter into the remaining $N-2$ modes, thus allowing the modes that are coupled to the electric field to reach a steady state. We discuss moving beyond leading order later in the Letter. The details of how this $1/N$ expansion is set up are given in [9].

The scattering integrals in our Boltzmann equations may be determined using several methods. In the case where only regular distribution functions are required, it is possible to use Fermi's golden rule. Since we also have to consider anomalous distributions, this simplest approach does not work. It is, however, possible to determine the scattering integrals by using Heisenberg's equations of motion with the $1/N$ Hamiltonian and time-dependent perturbation theory; this is after all how Fermi's golden rule is derived in the first place.

One-loop scattering integrals.—The one-loop correction is particularly simple to calculate. After calculating the Heisenberg equations of motion including the interaction Hamiltonian and then using mean field theory, one finds that the only modification is to replace $B_{\mathbf{k}}$ with $B_{\mathbf{k}} + \Sigma_{\mathbf{k}}$, where the self-energy $\Sigma_{\mathbf{k}} = m(E)^2/(2\epsilon_{\mathbf{k}})$. An expression for the one-loop renormalized mass $m(E)$ is given to lowest order in a $1/N$ expansion in Eq. (7) below [10]. By making a new Bogoliubov transformation to the *one-loop renormalized adiabatic particle basis,* one may reduce the Heisenberg equations for the distribution functions to the same form as (4), replacing the adiabatic mode energy $\epsilon_{\mathbf{k}}(t)$ everywhere with the renormalized mode energy, $\epsilon_{\mathbf{k},m}(t) = \sqrt{(\mathbf{k}+\mathbf{E}t)^2 + m(E)^2}$. This modification points to a feedback between the production of fluctuations and the scattering between them. Since the rate of production depends exponentially upon the energy gap, it is suppressed by scattering—although its functional dependence on E is not changed. Of course, this scattering is not sufficient to establish a steady state and, indeed, can be ignored at leading order in $1/N$ altogether as will be clear below.

Two-loop scattering integrals.—The next order scattering integral is deduced from the Heisenberg equations of motion for the various mode operators followed by time-dependent perturbation theory to lowest order in $1/N$. To lowest order in $1/N$ and at zero temperature, the dominant scattering is due to processes where the modes coupled to the E field scatter off one another and into the $N-2$ uncoupled modes of the extended theory. At this juncture, it is useful to make the change of variables t, $\mathbf{q} \to t$, $\mathbf{k}$, where $\mathbf{k} = \mathbf{q} + \mathbf{E}t$ is the instantaneous momentum of the quasiparticle. Rather than label the normal modes by their initial momentum and keep the label fixed in time, we choose to label them with their instantaneous momentum. This is equivalent to changing gauge to a scalar potential. With this change of variables all explicit time dependence is removed, except for in the distribution functions. In particular, $\epsilon_{\mathbf{q}}(t) = \epsilon_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$. The resulting Boltzmann equations are

$$\begin{aligned}
[\partial_t + \mathbf{E}\cdot\partial_{\mathbf{k}}]f(\mathbf{k},t) &= +\frac{\mathbf{E}\cdot\partial_{\mathbf{k}}\epsilon_{\mathbf{k}}}{\epsilon_{\mathbf{k}}}\,\mathrm{Re}g(\mathbf{k},t) \\
&\quad + \gamma_{\mathbf{k}}(t)[\mathrm{Re}g(\mathbf{k},t) - f(\mathbf{k},t)], \\
[\partial_t + \mathbf{E}\cdot\partial_{\mathbf{k}}]g(\mathbf{k},t) &= -2i\epsilon_{\mathbf{k}}g(\mathbf{k},t) + \frac{\mathbf{E}\cdot\partial_{\mathbf{k}}\epsilon_{\mathbf{k}}}{\epsilon_{\mathbf{k}}} \\
&\quad \times[2f(\mathbf{k},t)+1] - \gamma_{\mathbf{k}}(t)f(\mathbf{k},t)
\end{aligned} \tag{6}$$

with the supplementary definitions

$$\begin{aligned}
\gamma_{\mathbf{k}}(t) &= \frac{8}{N}\frac{\sqrt{|\mathbf{k}| + k_\parallel}}{\epsilon_{\mathbf{k}}}\int\frac{d\mathbf{k}'}{(2\pi)^2}\frac{\sqrt{|\mathbf{k}'| + k'_\parallel}}{\epsilon_{\mathbf{k}'}}f(\mathbf{k}',t), \\
m(E,t) &= \frac{16\pi}{N}\int\frac{d\mathbf{k}}{(2\pi)^2}\frac{f(\mathbf{k},t)}{\epsilon_{\mathbf{k}}} + \Delta.
\end{aligned} \tag{7}$$

The damping factor $\gamma(\mathbf{k},t)$ is derived allowing for the zero-temperature, critical propagation of the $N-2$ uncoupled modes.

Scaling and current.—The above equations clearly permit a universal, steady-state solution whose properties are governed only by the electric field and the fixed point value of the coupling. Writing the scaling forms $f/g(\mathbf{k},E) = f/g(\mathbf{k}/\sqrt{E}) \equiv f/g(\tilde{k})$, $m = \tilde{m}\sqrt{E}$, and $\gamma = \tilde{\gamma}\sqrt{E}$, we see that Eq. (6) reduces to E independent equations for $f/g(\tilde{k})$. This by itself is sufficient to establish a current proportional to $E^{d/2}$ in dimensions $d<3$. For $d>3$ the fixed point is Gaussian and the necessity of including dangerously irrelevant scattering processes modifies this scaling.

We can make progress on the actual solution by making two simplifications valid for the leading order (in $1/N$) computation of the current: we can ignore the mass renormalization and eliminate the second equation in favor of a local source term [11]. With these we find the greatly simplified and soluble equation

$$\mathbf{E}\cdot\frac{\partial f(\mathbf{k})}{\partial\mathbf{k}} = -\gamma_{\mathbf{k}}f(\mathbf{k}) + e^{-\pi\epsilon_{\mathbf{k}}^2/E}\delta(\mathbf{k}\cdot\mathbf{E}/E^2) \tag{8}$$

with $\gamma_{\mathbf{k}}$ and $m(E)$ given by Eq. (6) in the case of steady-state distributions.

The current carried in the steady state is given by

$$\mathbf{j} = 2\int\frac{d\mathbf{k}}{(2\pi)^2}\frac{k_\parallel}{\epsilon_{\mathbf{k}}}f(\mathbf{k}), \tag{9}$$

with an additional term involving $g(\mathbf{k})$ being subdominant in $1/N$. Upon rescaling, this reduces to a form that, in two dimensions, is proportional to the electric field. The constant of proportionality is universal and may be calculated in the $1/N$ limit to be $\sigma = (N\pi/8)e^{*2}/h$ [12], where $N = 2$ in the physical system.

Dissipation and higher orders.—The steady state that we have described involves a balance between pair production, their acceleration by the field, and relaxation due to scattering. The latter processes need to relax the current, the number of charge carriers, as well as the energy. The first two require processes present in the Hamiltonian but energy relaxation is present only at leading order where the infinitely many orthogonal modes act as a heat sink. In order that we be able to go to higher orders in $1/N$, we need to understand how that problem is to be dealt with. Qualitatively we expect to mimic the logic of linear response theory, where Joule heating is an $O(E^2)$ process that can be dealt with without invalidating the $O(E)$ result one computes. In our case the leading current response is $O(E^{d/2})$ but the Joule heating $\mathbf{j} \cdot \mathbf{E}$ is still down by a factor of E so the same logic is *prima facie* applicable. Explicitly, this can be done by constructing a solution that carries a heat current to the boundaries of the sample, e.g., transverse to the direction of electric current flow [13]. Roughly, this can be thought of as a local "effective temperature" $T_{\rm eff}(y)$, which drives a heat current $\kappa\nabla^2 T_{\rm eff}(y)$. By scaling, $T_{\rm eff} \propto \sqrt{E}$. Requiring that the variation in $T_{\rm eff}$ be less than its mean value and factoring in the scaling form of κ leads to the conclusion that such a solution can be constructed for a system with transverse dimension, L_y, and electric fields satisfying $L_y\sqrt{E} \ll 1$. Such a restriction is not unusual, a similar construction for an ordinary metal at finite temperature also yields a bound on system size and electric field [13]. While the estimate that we give would apply to a generic critical point with $z = 1$, our particular system is an even better bet for such a construction as κ is infinite at all T due to an absence of energy current relaxation [14].

Bose gases.—While this work was motivated by the fundamental question of understanding the nonlinear quantum-critical states, we would be remiss if we did not note that the Mott transition described by (1) has been observed, remarkably enough, in atomic Bose-Einstein condensates placed in optical lattices of variable depth [15,16]. In this system, the role of the electric field is played by an intensity gradient in the optical field, or alternatively by an acceleration of the optical lattice. The universal current predicted within our analysis amounts to a steady flow of matter proportional to the acceleration of the optical lattice. Field gradients and accelerations in these systems are easily made large in the sense of our theoretical discussion. Whether the nonlinear response discussed here can be observed given the complications of the confining potential and the differing requirements of equilibration in these systems appears to be a fit topic for further theoretical and experimental work.

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[4] This mechanism of nonlinearity generation by quantum tunneling has a classical analogue in which nonlinearities are generated by thermal activation, e.g., B. I. Halperin and D. R. Nelson, J. Low Temp. Phys. **36**, 599 (1979).

[5] M.-C. Cha *et al.*, Phys. Rev. B **44**, 6883 (1991).

[6] D. Dalidovich and P. Phillips, Phys. Rev. Lett. **93**, 027004 (2004). In that work a dissipative bath is part of the fixed point action, and so the existence of a nonlinear response is immediate, much as in stochastic models of classical critical dynamics, e.g., A. T. Dorsey, Phys. Rev. B **43**, 7575 (1991). Hamiltonian actions, such as the one we consider, present trickier questions.

[7] Strictly, the distribution functions are defined through a Wigner transformation of the regular and anomalous Green's functions; $f(\mathbf{k}, t) = \int d^2q\langle a^\dagger(\mathbf{k} + \mathbf{q}/2, t)a(\mathbf{k} - \mathbf{q}/2, t)\rangle/(2\pi)^2$ and $g(\mathbf{k}, t) = \int d^2q\langle b^\dagger(-\mathbf{k} - \mathbf{q}/2, t)a(\mathbf{k} - \mathbf{q}/2, t)\rangle/(2\pi)^2$. We assume this form implicitly in the main text, while retaining the simpler notations $f(\mathbf{k}, t) = \langle a^\dagger(\mathbf{k}, t)a(\mathbf{k}, t)\rangle$ and $g(\mathbf{k}, t) = \langle b(-\mathbf{k}, t)a(\mathbf{k}, t)\rangle$.

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[9] S. Sachdev, Phys. Rev. B **57**, 7157 (1998).

[10] The dominant scattering at this order for E-field coupled modes is from other E-field coupled modes; at $T = 0$ and to lowest order in $1/N$ there are no fluctuations in the other $N - 1$ copies. Because of this, $m(E)$ is $O(1/N)$ rather than $O(1)$ as in thermal equilibrium.

[11] The terms $\gamma\,\mathrm{Re}g$ in the first equation and γf in the second are also neglected. This is justified by considering the order of magnitude of each term in the Boltzmann Eq. (6) at differing values of $k_\parallel$. The neglected terms are subdominant in $1/N$ throughout the range of $k_\parallel$.

[12] If one calculates the response in the gapped phase $[m^2(E = 0) = \Delta^2]$, one finds an apparent universal conductivity. This was first recognized in linear response at thermal equilibrium by D. Dalidovich and P. Phillips, Phys. Rev. B **64**, 052507 (2001). It occurs due to a cancellation of the exponentially small density of carriers with the correspondingly small scattering between them.

[13] A.-M. Tremblay *et al.*, Phys. Rev. A **19**, 1721 (1979).

[14] Note that the absence of energy current and momentum current relaxation does not feed into the relaxation of electrical current. Energy and momentum currents are even under charge conjugation, whereas charge current is odd. This prevents mixing under the RG flow and permits a finite charge conductivity notwithstanding the infinite energy and momentum conductivities.

[15] M. Greiner *et al.*, Nature (London) **415**, 39 (2002).

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