A continuous network design model in stochastic user equilibrium based on sensitivity analysis

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Abstract

The continuous network design problem (CNDP) is known to be difficult to solve due to the property of non-convexity and nonlinearity. Such kind of CNDP can be formulated as a bi-level programme, which may be classified into Stackelberg approach and Nash one according to the relationship between the upper level and the lower level parts. This paper formulates the CNDP for road expansion based on Stackelberg game where leader and follower exist, and allows for errors of travelers' behavior in choosing their routes. In order to solve the problem by Stackelberg approach, we need a relation between link flow and design parameter. For the purpose of that, we use logit route choice model, in which there exists an explicit closed-form function between them. The developed model will be applied to two example road networks for test and compared the results between the Stackelberg and Nash approaches to emphasise their differenc between them.

1. INTRODUCTION

The network design problem (NDP) is to determine a set of design parameters that leads the road network to an optimal state after allowing for travellers' responses. NDP includes traffic signal control, traffic information provision, congestion charge and new transportation modes as well as road expansion. In general, the NDP can be formulated as a bi-level problem, which has an upper level part that represents system design and a lower level one that represents travellers' responses. According to the strategy between them, we may classified the bi-level problem into Stackelberg game and Nash game. The Stackelbeg game is different from the Nash game in that the upper level decision maker knows how the lower level decision maker will respond to an upper-level decision. Although he can not intervene in the lower level decision maker's decision, he can consider the lower level decision maker's reaction in his own decision making. Since it has been difficult in solving the problem as Stackelberg game, most of conventional NDP models have been formulated as a Nash game, in which each decision maker acts unilaterally and without consideration of the response of others. However. the Stackelberg game provides a preferable model for decision making because the system designer anticipates the responses of others. For example, traffic operator in expressway agency sets the ramp metering rates under considering the route change behaviors of drivers corresponding to the changed metering rates. Thus the design variables of upper level problem should be specified under such circumstance taking into consideration the travellers' response. This corresponds to a Stackelberg game condition.

We have long list of network design problems, which can be classified into two classes such as discrete network design problem (DNDP) and continuous network design problem (CNDP). A DNDP defines the design parameter as discrete variable, while CNDP does it as continuous one. For solving such CNDP as Stackelberg game, the sensitivity analysis of user equilibrium was introduced by Tobin and Friesz (1988) and has been used for the static network design problem byYang (1995; 1997) and in the dynamic case by Heydecker (2002). Various sensitivity analysis-based heuristic algorithms are also proposed for the CNDP and relevant problems (Friesz et al,1990; Yang and Yagar, 1994;Yang et al, 1994). Recently Maher et al (2001) proposed a bi-level problem for trip matrix estimation and traffic control problem with stochastic user equilibrium (SUE), and their solution algorithms in which SUE assignment map was approximated as a linear relationship. More detail and wider literature reviews and their algorithms are described in the paper of Yang et al (1998).

This paper formulates the CNDP for road network design based on a Stackelberg game formulation where leader and follower are identified respectively as the designer and the travelers. The present formulation allows for errors of travelers' perceptions of costs in choosing their routes, which can be described by stochastic traffic assignment. We can easily formulate the relationship between link flow and design parameters, because there exist an explicit closed-form function between them in logit-type stochastic user equilibrium assignment, while there does not exist such a function in deterministic user equilibrium assignment.

The bi-level CNDP is intrinsically nonlinear, non-convex, and hence it might be difficult to solve. We therefore suggest a heuristic solution algorithm, which makes use of derivative information of link flow with respect to the design parameters. Compared to the method of Maher et al. (2001), this paper has no linear approximation of SUE in the solution process. We directly use the derivative deduced from logit route choice model. We also compare the results between the Stackelberg approach and the Nash one to emphasise the differences between them. The developed model will be applied to two example road networks for test.

2. STOCHASTIC NDP FORMULATION AND ALGORITHM

Firstly let some variables denote as

A : Set of links in the network

$\overline{A}$: Subset of links considering design parameter

x : Column vector of link flows { x_a }, $a \in A$

$\mu(c)$: Matrix of link choice proportion as a function of the vector c of link costs

T : Column vector of O-D demand levels { T_{rs} }

T_{rs} : travel demand for OD pair $r-s$

p : Vector of design parameters (variables) { p_a }, $a \in \overline{A}$

In the Stackelberg game, there is a relationship between link flow x and design variable p as

$$x = x(p)$$

By expansion around point p_0, we have the following linear approximate expression.

$$x(p) = x(p_0) + \left(\frac{\partial x(p_0)}{\partial p}\right)(p - p_0) \tag{1}$$

In case of deterministic user equilibrium assignment, closed-form functions between x and p are available only for simple networks (Heydecker, 2002). Thus it is difficult to evaluate the derivative of $\frac{\partial x(p_0)}{\partial p}$ and so it is not easy to calculate the exact solution of bi-level problem formulated as a Stackelberg game. For this problem, Yang (1995), Freisz et al (1990) Heydecker (2002) proposed a sensitivity analysis-based (SAB) method for attain the derivative, but it is difficult to evaluate the functions and to attain derivative information. While, in case of logit-type stochastic user equilibrium assignment, we do have a closed-form function between x and p, so we can calculate the derivative by direct manipulation. The bi-level network design model based on Stackelberg game in this paper may be written as

[Upper level problem]

$$\min_{p,x} \quad Z\big(p, x, c(p)\big) = \sum_a x_a \; c_a\big(x_a, p\big) + u\big(p\big) \tag{2a}$$

[Lower level problem]

$$x - \mu(c(x, p))T = 0 \tag{2b}$$

where $u(p)$ is construction cost for improving network. The upper level problem is to minimize total travel cost of road network less construction costs, and the lower level is stochastic user equilibrium assignment. In the lower level problem, let $x^*(p)$ denote the SUE solution at a given value p of the parameter vector, so that

$f(x^*(p), p) = 0$ for any given p.

If we assume the function $f(x, p)$ to be differentiable, then the first-order expansion of $f(x, p)$ in the neighbourhood of $(x, p) = (x^*(p_0), p_0)$ is;

$$f(x, p) \approx f(x^*(p_0), p_0) + \left.\frac{\partial f}{\partial x}\right|_{(x^*(p_0), p_0)} (x - x^*(p_0)) + \left.\frac{\partial f}{\partial p}\right|_{(x^*(p_0), p_0)} (p - p_0),$$

Where the derivative terms are the Jacobian matrices of $f(x, p)$ with respect to x and p respectively, evaluated at $(x^*(p_0), p_0)$, which here denote J_x and J_p.

Since $f(x^*(p_0), p_0) = 0$ by SUE at p_0, and we determine p_0, $x^*(p_0), J_x, J_p$, then for some other $p \neq p_0$ we can approximately solve the equilibrium condition $f(x(p), p) = 0$ for $x(p)$ as

$0 \approx 0 + J_x(x(p) - x^*(p_0)) + J_p(p - p_0)$.

Thus we have following equation.

$$x(p) = x^*(p_0) - J_x^{-1} J_p (p - p_0), \qquad \text{where } J_x^{-1} = \left[\left.\frac{\partial f}{\partial x}\right|_{(x^*(p_0), p_0)} \right]^{-1}, J_p = \left.\frac{\partial f}{\partial p}\right|_{(x^*(p_0), p_0)} \tag{3}$$

so that the sensitivity of equilibrium link flow with respect to design parameter is expressed in the form of the implicit function theorem as

$$\frac{dx}{dp} = -J_x^{-1} \, J_p.$$

The bi-level problem can be solved by iterative process between the upper level problem and lower level one with the equation (3). The only difference between Stackelberg game and Nash is whether they consider the equation or not.

To specify equation (3), this paper uses logit route choice model, which can be given as an explicit function of path cost as

$$\mu_k(c) = \frac{\exp(-\theta c_k)}{\sum_{i \in K} \exp(-\theta c_i)} \tag{4}$$

where, c_k is the route cost defined in equation (5) and θ is a parameter of the route perception error, K is path set for connecting each origin-destination pair.

$$c_k = \sum_a c_a \delta_{ak} \tag{5}$$

c_a is a cost for link a and δ_{ak} is a dummy variable that 1 if the link a is on the route k, 0 otherwise. We have also a relation between $\mu_k(c)$ and link choice probability $\mu_a(c)$ as follows.

$$\mu_a(c) = \sum_k \mu_k(c)\delta_{ak}, \qquad k \in K$$

For evaluating the equation (3) two derivatives of f with respect to x, p are required. The first derivative is given by

$$\begin{aligned}\frac{\partial f_b}{\partial x_a} &= \frac{\partial}{\partial x_a}(x_b - \mu_b(c)T) \\ &= \delta_{ba} - \left(\frac{\partial \mu_b(c)}{\partial c_a}\frac{\partial c_a}{\partial x_a}\right)T\end{aligned} \qquad a, b \in A$$

where, δ_{ba} is Kronecker delta and $\dfrac{\partial c_a}{\partial x_a}$ may be easily determined when the link cost function is specified. $\dfrac{\partial \mu_b(c)}{\partial c_a}$ can be rewritten as following form.

$$\frac{\partial \mu_b(c)}{\partial c_a} = \frac{\partial}{\partial c_a}\left(\sum_k \mu_k(c)\delta_{bk}\right) = \sum_k \frac{\partial \mu_k}{\partial c_a}\delta_{bk} \qquad k \in K$$

With the equation (4) of logit model and equation (5), $\dfrac{\partial \mu_k}{\partial c_a}$ may be converted and summarized as

$$\frac{\partial \mu_k}{\partial c_a} = -\theta\left[\mu_k \delta_{ak} - \mu_k \left(\sum_i \mu_i \delta_{ai}\right)\right] \qquad i \in K$$

Therefore

$$\frac{\partial \mu_b(c)}{\partial c_a} = \sum_k \frac{\partial \mu_k}{\partial c_a}\delta_{bk} = -\theta \sum_k \left[\mu_k \delta_{ak} - \mu_k \left(\sum_i \mu_i \delta_{ai}\right)\right]\delta_{bk}$$

Thus, we finally get equation (6)

$$\frac{\partial f_b}{\partial x_a} = \kappa_{ba} - \theta\left(\frac{\partial c_a}{\partial x_a}\right)\sum_k \left[\mu_k \delta_{ak} - \mu_k \left(\sum_i \mu_i \delta_{ai}\right)\right]\delta_{bk} T \qquad k, i \in K \tag{6}$$

where $\delta_{ak}, \delta_{bk}, \delta_{ai}$ are dummy variables that 1 if $a, b \in k, a \in i$, 0 otherwise.

Following the same way described above, we can get $\dfrac{\partial f_b}{\partial p_a}$ as,

$$\frac{\partial f_b}{\partial p_a} = \theta\left(\frac{\partial c_a}{\partial p_a}\right)\sum_k \left[\mu_k \delta_{ak} - \mu_k \left(\sum_i \mu_i \delta_{ai}\right)\right]\delta_{bk} T \qquad k, i \in K \tag{7}$$

So, we can determine the equation (3) by using the equation (6) and (7). The equation (6) and (7) are similar to the results of Davis (1994).

Through equation (3), now we can solve the bi-level problem of equation (2). The solution algorithm for the problem can be listed as follows.

(0) Initialization : $n = 0, \quad p^0$

(1) $n = n+1$

(2) Solve lower level problem with p^{n-1} and yield $x(p^{n-1})$

(3) Calculate derivative information : J_x^{-1}, J_p and yield $x(p^n, p^{n-1})$ by using following equation,

$$x(p^n, p^{n-1}) = x(p^{n-1}) - J_x^{-1} J_p (p^n - p^{n-1})$$

(4) Solve upper level problem with $x(p^n, p^{n-1})$ and yield { p^n }

(5) Convergence check

When criterion is meet, stop

Otherwise, goto step (1)

In the algorithm, the upper level problem can be solved by unconstrained nonlinear programming method such as Newton-Rapson method, Davidon-Fletcher-Powell (DFP) method and Broyden-Fletcher-Goldfarb-Shanno (BFGS) method. While the lower level problem may be easily evaluated by conventional stochastic user equilibrium model given design parameter p^{n-1}. Several convergence criteria can be used for stopping solution process. This paper uses the difference between current value of design parameter and previous one.

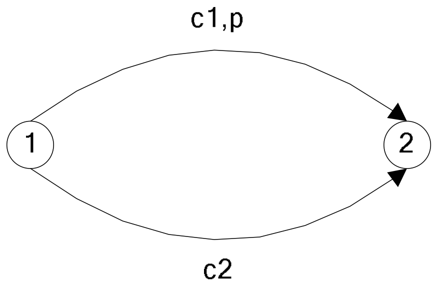
3. NUMERICAL CALCULATION

In order to illustrate use of the model and the solution algorithm suggested in the paper, two example networks are used. The first example involves a simple network with one origin-destination pair connecting 2 paths. This example network is used for comparison between Stackelberg approach and C-Nash one. The second example is a medium size network with one origin-destination pairs connecting 6 paths, with considering link capacity improvement as design parameter.

3.1 Comparison of Stackelbeg and C-Nash approach

Consider a test network consisting of two links (routes) serving a single O-D pair with $T_{12} = 1.0$ and parameter $\theta = 1.0$ in logit model. The design parameter p is only adopted on the link 1. The cost functions on the links are $c_1 = 1 + 2px_1^2$, $c_2 = 2 + x_2$ and the construction function is set to be

$$u(p) = Cw(p - p_0)^{\alpha}, \quad Cw = 20, \quad \alpha = 2.0.$$



<Figure 1> Example network 1

Table 1 and Table 2 present comparisons of the Stackelberg approach with those of C-Nash. Each approach has different optimal design parameter values such that the value of Stackelberg approach is changed from the initial value of 1.0 to optimal value of 0.989130, while C-Nash changed from the value of 1.0 to 0.851355. But both approaches converge to a stochastic user equilibrium for each case because the values of equivalent path cost (*Ec*) are the same as shown in the last column of the table, although each path cost is different. The equivalent path cost is expressed as $Ec_a = c_a + \frac{1}{\theta}\ln(x_a)$.

Table 2 shows the optimal value of design parameter, construction costs, the values of upper level problem and total costs. Note here that the values of upper level problem. As we expect, Stackelberg approach has a smaller objective value than that of Nash, which imply that Stackelberg approach gives a better solution. The results are consistent with the work of Fisk (1984) for the comparison between Stackelberg and C-Nash approach. But regarding to the total cost C-Nash approach has lower value.

<Table 1> Results of link volumes and equivalent cost (*Ec*) in initial and optimal states

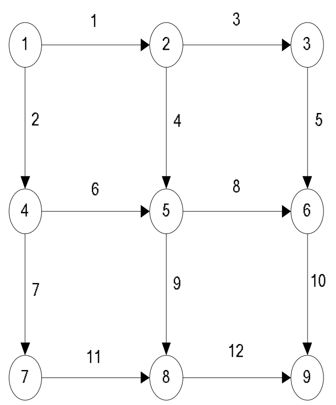
Model		p	x_1	x_2	c_1	c_2	Ec_1	Ec_2
Stackelberg	Initial	1.0	0.635614	0.364386	1.808009	2.364386	1.354845	1.354844
	optimal	0.989130	0.636734	0.363266	1.802047	2.363266	1.350644	1.350646
C-Nash	Initial	1.0	0.635614	0.364386	1.808009	2.364386	1.354845	1.354844
	optimal	0.851355	0.651519	0.348481	1.722760	2.348481	1.294311	1.294309

<Table 2> Summary of numerical results

Model		p	Construction cost	Upper level objective	Total cost $\sum_a x_a c_a$
Stackelberg	Initial	1.0	0.0	2.010746	2.010745
	optimal	0.989130	0.002368	2.008282	2.005918
C-Nash	Initial	1.0	0.0	2.010746	2.010745
	optimal	0.851355	0.441908	2.382721	1.940812

3.2 Second example for capacity expansion

The second network has 9 nodes and 12 links with one OD pair from node 1 to node 9, consisting of 6 paths. The network specifications are given in Table 3 and travel demand $T_{19} = 100$. The parameter in logit model is $\theta = 0.02$. The BPR (Bureau of Public Roads) cost function is used for the link travel cost and the expansion of link capacity is used as design parameter.



<Figure 2> Example network 2

<Table 3> Network specification for the example

Link	Free flow travel time	Link capacity	Link	Free flow travel time	Link capacity
1	12	35	7	15	20
2	10	35	8	10	35
3	15	20	9	10	35
4	10	35	10	12	35
5	15	20	11	15	20
6	10	35	12	10	35

Table 4 and Table 5 show the sensitivity, optimal design values and the values of upper level objective functions when two design parameters (capacity expansion of link 6 and link 10) are adopted in the network. Sensitivity is used to predict changes in equilibrium link flow pattern in response to any small variance in design value. The sensitivity of each link with respect to design variables are depicted in Table 4. The sensitivities of link 6 have the same signs but somewhat different values with respect to each design parameter. This implies that the volume of link 6 is more influenced by the capacity improvement of link 10 than that of link 6. On the other hand, link 10 has negative values of sensitivity with respect to design parameter of link 10, leading to the fact that the link volumes decrease as the capacity increases. These results show that capacity improvement does not always induce increase in link volume under fixed demand.

Table 5 gives optimal increase of link capacity, objective values and construction costs. The design parameters are converged to 0.013664 and to 0.382983 from zero respectively, and the objective value of upper level problem decreases from 6,116 to 6,112.

<Table 4> Sensitivity of each link with respect to design parameter

Link	$\frac{dx_a}{dp_6}$	$\frac{dx_a}{dp_{10}}$	Link	$\frac{dx_a}{dp_6}$	$\frac{dx_a}{dp_{10}}$
1	-0.012794	-0.080154	7	0.010775	1.144179
2	0.040372	-0.211988	8	0.009549	0.044914
3	0.247200	0.880170	9	0.027923	0.134546
4	-0.036169	0.084235	10	0.003091	-0.019392
5	-0.026262	0.030326	11	-0.017086	0.348960
6	0.065149	0.139766	12	-0.007189	-0.406587

<Table 5> Optimal design parameters, objective values and construction costs

		Initial	Optimal
Link	Link 6	0.0	0.013664
	Link 10	0.0	0.382983
Objective values		6,116.1274	6,112.7075
Construction cost		0.0	2.934989

Figures in Table 6 and Table 7 show some results when 3 design parameters (improvement capacity of link 5, link 6 and link 10) are adopted. The figures of Table 6 are corresponding to those of Table 4 in two design parameters case. Note the minus sign of optimal design value of link 6 in Table 7. This sign implies that decrease in link capacity may minimize the objective value of upper level problem, which is an unexpected outcome. This phenomenon is known Braess's paradox when the road capacities are expanded.

<Tabel 6> Sensitivity of each link with respect to design parameter

Link	$\frac{dx_a}{dp_5}$	$\frac{dx_a}{dp_6}$	$\frac{dx_a}{dp_{10}}$	Link	$\frac{dx_a}{dp_5}$	$\frac{dx_a}{dp_6}$	$\frac{dx_a}{dp_{10}}$
1	-0.046033	0.146746	-0.160547	7	-0.029576	0.120788	-0.151013
2	-0.270415	1.410046	-0.018507	8	-0.011724	0.035798	0.171164
3	-0.000339	0.135086	-0.000071	9	-0.000322	-0.080295	-0.193762
4	0.000339	-0.135086	0.000071	10	-0.007020	0.154011	0.203658
5	0.040456	-0.109789	0.112646	11	0.017395	-0.231300	-0.109157
6	0.071824	-0.717401	-0.173787	12	-0.014076	0.062481	-0.147360

<Table 7> Optimal design parameters, objective values and construction costs

		Initial	Optimal
Link	Link 5	0.0	0.334750
	Link 6	0.0	-0.955765
	Link 10	0.0	1.989180
Objective values		6,116.1274	5,994.8091
Construction cost		0.0	98.7576

4. CONCLUSION

In this paper, we propose a continuous network design model in stochastic user equilibrium based on Stackelberg game. The CNDP is formulated as a bi-level problem where the upper level problem is to determine optimal road capacity able to minimize total network cost, and the lower is to depict stochastic travel behaviour according to the design parameter. Due to the existence of explicit function between link flow and design parameter in logit model, we can easily derive the derivative and introduce it to the solution procedures. The derivative information has many important implications in both network design problem and operational level.

From the numerical calculations, we calculate sensitivity, optimal design parameters and also can detect phenomenon of Braess's paradox as the road capacities are expanded. But these results are brought out under fixed OD demand, thus we expect to get somewhat different results when the demand is elastic, which remains for next work. Numerical results also show the extent to which the Stackelberg approach is better than Nash one in certain example networks.

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